Module – 5 Complex Integration

Cauchy's integral formulae – Problems – Taylor's expansions with simple problems – Laurent's expansions with simple problems – Singularities – Types of Poles and Residues – Cauchy's residue theorem (without proof) – Contour integration: Unit circle, semicircular contour – Application of Contour integration in Engineering.

Cauchy's Integral Theorem

If $f(z)$ is analytic at every point of the region R bounded by a simple closed curve C and if $f'(z)$ is continuous at all points inside and on C, then $\int_C f(z)\,dz = 0$

Cauchy's integral formula for n^{th} derivative

If $f(z)$ is analytic inside and on a simple closed curve C and $z = a$ is any interior point of the region R enclosed by C, then $f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

$$(i.e.)\quad \boxed{\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(a)}$$

Taylor's series

If $f(z)$ is analytic inside a circle C with centre at a then Taylor's series about $z = a$ is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots$$

Laurent's series

If C_1, C_2 are two concentric circles with centre at z = a and radii r_1 and r_2 ($r_1 < r_2$) and if f(z) is analytic inside and on the circles and within the annular region between C_1 and C_2, then for any z in the annular region, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}}dz$ and $b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{-n+1}}dz$

Cauchy's Residue theorem

If f(z) is analytic inside a closed curve C except at a finite number of isolated singular points $a_1, a_2, \ldots a_n$ inside C, then

$\int_C f(z)dz = 2\pi i \times$(sum of the residues of f(z) at these singular points).

Contour Integration

Type I:

$$\boxed{\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta}$$

Let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta = izd\theta \Rightarrow d\theta = \dfrac{dz}{iz}$

Then we have

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right); \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\cos 2\theta = \text{Real part of } z^2 \ ; \qquad \cos n\theta = \text{Real part of } z^n$$

$$\sin 2\theta = \text{Im part of } z^2; \qquad \sin n\theta = \text{Im part of } z^n$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} = \text{Real part of } \left[\frac{1+z^2}{2}\right];$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \text{Real part of } \left[\frac{1-z^2}{2}\right]$$

$\therefore$

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C f(z)\, dz$, where C *is* $|z| = 1$ and solve by known method.

Type II:

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx}$$

Using Cauchy's integral formula, find $\int_C \dfrac{z+4}{z^2+2z+5}\, dz$**, where C is** $|z+1-i| = 2$

Solution:

$|z + 1 - i| = 2$

$|x + iy + 1 - i| = 2$

$|(x+1) + i(y-1)| = 2, \qquad \sqrt{(x+1)^2 + (y-1)^2} = 2$

Squaring on both sides,

$$\left(x+1\right)^2+\left(y-1\right)^2=4$$

This is equation of circle with centre $\left(-1,1\right)$ and radius 2.

$$z^2+2z+5=0$$

$$z=\frac{-2\pm\sqrt{4-4\left(1\right)\left(5\right)}}{2\left(1\right)}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\int_C \frac{z+4}{\left[z-\left(-1+2i\right)\right]\left[z-\left(-1-2i\right)\right]}\,dz$$

Here $-1+2i$ lies inside the circle c and $-1-2i$ lies outside the circle c.

Let $a=-1+2i$

By Cauchy's integral formula, $f\left(a\right)=\frac{1}{2\pi i}\int_C \frac{f\left(z\right)}{z-a}\,dz$

Substituting for *a*, $f\left(-1+2i\right)=\frac{1}{2\pi i}\int_C \frac{f\left(z\right)}{z-\left(-1+2i\right)}\,dz$ ......$\left(1\right)$

Comparing equation (1) with given problem,

$$f\left(z\right)=\frac{z+4}{z-\left(-1-2i\right)}$$

$$f\left(-1+2i\right)=\frac{-1+2i+4}{-1+2i-\left(-1-2i\right)}=\frac{2i+3}{-1+2i+1+2i}=\frac{2i+3}{4i}$$

Substituting for $f\left(-1+2i\right)$ in $\left(1\right)$

$$\frac{2i+3}{4i}=\frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}\,dz$$

Cross multiplying

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\frac{\left(2i+3\right)\left(2\pi i\right)}{4i}=\frac{\pi}{2}\left(3+2i\right)$$

Using Cauchy's integral formula, evaluate $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)\ (z-1)}dz$**, where C is** $\left|z\right|=3$

Solution:

We know that, Cauchy's integral formula is $f\left(a\right)=\frac{1}{2\pi i}\int_C \frac{f\left(z\right)}{z-a}\,dz$

(i.e) $2\pi i\ f\left(a\right)=\int_C \frac{f\left(z\right)}{z-a}\,dz$

Given: $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz$ Here, $f(z)=\sin \pi z^2+\cos \pi z^2$

The points $a_1=1, a_2=2$ lies inside $|z|=3$

Now, $\frac{1}{(z-1)(z-2)}=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$ (by Partial fraction method)

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)}dz+\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)}dz$$
$$=-2\pi i\ f(1)+2\pi i\ f(2)$$

$f(z)=\sin \pi z^2+\cos \pi z^2$

$f(1)=\sin \pi+\cos \pi=-1$ and $f(2)=\sin 4\pi+\cos 4\pi=1$

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-2\pi i(-1)+2\pi i(1)=4\pi i$$

Using Cauchy's integral formula, evaluate $\int_C \frac{1}{(z-2)\ (z+1)^2}dz$, where C is $|z|=\frac{3}{2}$

Solution:

Here $z=-1$ is a pole lies inside the circle

$z=2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{\left(z+1\right)^2\left(z-2\right)}=\int \frac{\frac{1}{z-2}}{\left(z+1\right)^2}\,dz$$

Here $f\left(z\right)=\frac{1}{z-2}$, $f'\left(z\right)=-\frac{1}{(z-2)^2}$

Hence by Cauchy's integral formula

$$\int_C \frac{f\left(z\right)}{\left(z-a\right)^{n+1}}\,dz=\frac{2\pi i}{n!}f^{n}\left(a\right)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2}\,dz = \frac{2\pi i}{1!}\,f'(-1)$$

$$= 2\pi i\left[\frac{-1}{(-1-2)^2}\right] \quad \left(\because f'(z) = \frac{-1}{(z-2)^2}\right) = 2\pi i\left[\frac{-1}{9}\right]$$

$$\int_C \frac{1}{(z-2)\ \ (z+1)^2}dz = \frac{-2}{9}\pi i.$$

Using Cauchy's integral formula, evaluate $\int_c \frac{z}{z^2+1}dz$ where C is $|z+i|=1$.

Solution:

Consider the curve

$|z+i| = 1 \Rightarrow |x+iy+i| = 1$

$|x+i(y+1)| = 1 \Rightarrow x^2+(y+1)^2 = 1$

Which is a circle with centre (0,–1) and radius 1

The poles are obtained by $z^2+1=0$

$\Rightarrow z=i$ is a simple pole which lies outside C.

$z=-i$ is a simple pole which lies inside C.

$$\int_c \frac{z}{z^2+1}dz = \int_c \frac{z}{(z+i)(z-i)}dz = \int_c \frac{\frac{z}{(z-i)}}{(z+i)} = 2\pi i f(-i) \ldots (1)$$

$$f(z) = \frac{z}{(z-i)},\ f(-i) = \frac{-i}{(-i-i)} = \frac{-i}{-2i} = \frac{1}{2}$$

$$(1) \Rightarrow \int_c \frac{z}{z^2+1}dz = 2\pi i f(-i) = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

Expand $f(z) = \log(1+z)$ in Taylor's series about $z=0$

Solution: Let $f(z) = \log(1+z)$ $f(0) = \log 1 = 0$

$$f'(z) = \frac{1}{1+z} \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$$

$$f'''(z)=\frac{2}{(1+z)^3} \quad f'''(0)=2$$

$$f^{iv}(z)=\frac{-6}{(1+z)^4} \qquad f^{iv}(0)=-6$$

$$\log(1+z)= f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+...= z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+....$$

Find the Taylor's series expansion of f(z) = $\frac{z}{(z+1)(z-3)}$, in the region $|z|<1$

Solution:

Splitting f(z) into partial fractions, we have

$$f(z)=\frac{z}{(z+1)(z-3)}=\frac{A}{(z+1)}+\frac{B}{(z-3)}$$

$$\Rightarrow z=A(z-3)+B(z+1)$$

$$\text{put } z=-1, \text{we get } A=\frac{1}{4}$$

$$\text{put } z=3, \text{we get } B=\frac{3}{4}$$

$$\therefore \quad f(z)=\frac{1}{4}\left(\frac{1}{z+1}\right)+\frac{3}{4}\left(\frac{1}{z-3}\right)=\frac{1}{4}\left(\frac{1}{1+z}\right)+\frac{3}{4}\left(\frac{1}{-3}\right)\left(\frac{1}{1-\frac{z}{3}}\right)$$

$$=\frac{1}{4}\left[(1+z)^{-1}-\left(1-\frac{z}{3}\right)^{-1}\right]$$

$$=\frac{1}{4}\left[\left(1-z+z^2-....\right)-\left(1+\frac{z}{3}+\frac{z^2}{9}+...\right)\right]$$

$$=\frac{1}{4}\left[\left((-1)-\frac{1}{3}\right)z+\left((-1)^2-\left(\frac{1}{3}\right)^2\right)z^2+...\right]$$

$$\therefore f(z)=\frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n-\left(\frac{1}{3}\right)^n\right)z^n$$

Obtain Taylor's Series to represent the function $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ in the region | z | < 2

Solution:

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degree of the numerator and denominator are same we have to divide and apply partial fractions.

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

$$|z|<2 \quad \Rightarrow \frac{|z|}{2}<1 \text{ and } \therefore \frac{|z|}{3}<1$$

Consider

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}=\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}=\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

$$\therefore \frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Find the Laurent's series expansion of $\frac{1}{(z-2)(z-1)}$ valid in the regions $|z|>2$ and $0<|z-1|<1$

Solution:

$$f(z)=\frac{1}{(z-2)(z-1)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}=\frac{A(z-2)+B(z-1)}{(z-2)(z-1)}$$

$$\Rightarrow 1=A(z-2)+B(z-1)$$

Put $z=1,\ A=-1$

$z=2,\ B=1$

$$\therefore f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

Region1:

$$|z|>2 \Rightarrow 2<|z|$$

$$\Rightarrow \left|\frac{2}{z}\right|<1$$

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$=-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=-\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^{2}+\ldots\right)+\frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^{2}+\ldots\right)$$

$$=-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}+\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n}$$

$$=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{2^{n}}{z^{n+1}}$$

Region2:

Put $z-1=t \Rightarrow z=1+t$

$$0<|z-1|<1 \Rightarrow 0<|t|<1$$

$$\Rightarrow |t|<1$$

$$f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

$$=\frac{-1}{t}+\frac{1}{t-1}$$

$$=\frac{-1}{t}+\frac{1}{-(1-t)}$$

$$=\frac{-1}{t}-(1-t)^{-1}$$

$$=\frac{-1}{t}-\left(1+t+t^{2}+\ldots\right)$$

$$=\frac{-1}{(z-1)}-\left(1+(z-1)+(z-1)^2+\ldots\right)$$

$$=\frac{-1}{(z-1)}-\sum_{n=0}^{\infty}(z-1)^n$$

Expand f(z) = $\dfrac{z^2-1}{z^2+5z+6}$ in a Laurent's series expansion for $|z|>3$ and $2<|z|<3$

Solution:

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

Consider $\dfrac{-5z-7}{(z+3)(z+2)}$

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{A}{z+2}+\frac{B}{z+3}=\frac{A(z+3)+B(z+2)}{(z+3)(z+2)}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put $z=-2$ then $A=3$

Put $z=-3$ then $B=-8$

Substituting we get, $\dfrac{-5z-7}{(z+3)(z+2)}=\dfrac{3}{z+2}-\dfrac{8}{z+3}$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}$$

(i) **Given** $|z|>3 \quad \Rightarrow \quad \dfrac{3}{|z|}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\ldots\right)-\frac{8}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\ldots\right)$$

(ii) **Given** $2<|z|<3\Rightarrow \quad \dfrac{2}{|z|}<1$ and $\dfrac{|z|}{3}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-...\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Obtain the Laurent's series expansion for the function f(z) = $\frac{4z}{\left(z^2-1\right)\left(z-4\right)}$ **in** $|z-1|>4$ **and** $2<|z-1|<3$

Solution:

Put $z-1=u \Rightarrow z=u+1$

Now, $f(z)=\dfrac{4z}{\left(z^2-1\right)(z-4)}=\dfrac{4z}{(z-1)(z+1)(z-4)}$

Hence $f(u)=\dfrac{4(u+1)}{u(u+2)(u-3)}$

$$\frac{4(u+1)}{u(u+2)(u-3)}=\frac{A}{u}+\frac{B}{u+2}+\frac{C}{u-3}=\frac{A(u+2)(u-3)+Bu(u-3)+Cu(u+2)}{u(u+2)(u-3)}$$

$$4(u+1)=A(u+2)(u-3)+Bu(u-3)+Cu(u+2)$$

Put $u=0$ then $A=\dfrac{-2}{3}$

Put $u=-2$ then $B=\dfrac{-2}{5}$

Put $u=3$ then $C=\dfrac{16}{15}$

$$f(u)=\frac{4(u+1)}{u(u+2)(u-3)}=\frac{-2/3}{u}+\frac{-2/5}{u+2}+\frac{16/15}{u-3}$$

(i) $|u|>4 \quad \Rightarrow \quad \dfrac{4}{|u|}<1$

$$f(u)=\frac{-2/3}{u}-\frac{2/5}{u+2}+\frac{16/15}{u-3}$$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{u\left(1-\frac{3}{u}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} + \frac{16}{15}\left(\frac{1}{u}\right)\left(1-\frac{3}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right) + \frac{16}{15}\left(1+\frac{3}{u}+\frac{9}{u^2}+....\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right) + \frac{16}{15}\left(1+\frac{3}{(z-1)}+\frac{9}{(z-1)^2}+....\right)\right]$$

(ii) $2 < |u| < 3 \Rightarrow \quad \frac{2}{|u|} < 1 \text{ and } \frac{|u|}{3} < 1$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{-3\left(1-\frac{u}{3}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} - \frac{16}{45}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right) - \frac{16}{45}\left(1+\frac{u}{3}+\frac{u^2}{9}+....\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right) - \frac{16}{45}\left(1+\frac{(z-1)}{3}+\frac{(z-1)^2}{9}+....\right)\right]$$

Find the Laurent's series expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in 1 < | z + 1| < 3

Solution:

The singular points are z = 0, z = 2, z = –1

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$\Rightarrow$ 7z – 2 = A(z – 2) (z + 1) + B z (z + 1) + C z (z – 2)

Put z = 0, –2 = A(–2) $\Rightarrow$ A = 1

z = 2, 14 – 2 = B 2(2 + 1) $\Rightarrow$ B = 2

z = –1, – 7 – 2 = C(–1)(– 1 – 2) $\Rightarrow$ C = –3

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Put t = z + 1 $\Rightarrow$ z = t – 1

$\therefore$ 1 < | t | < 3

1 < | t | $\Rightarrow \left|\frac{1}{t}\right| < 1$ and $\left|\frac{t}{3}\right| < 1$

$$\begin{aligned}
f(z) &= \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1} \\
&= \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t} \\
&= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{(-3)\left(1-\frac{t}{3}\right)} - \frac{3}{t} \\
&= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t} \\
&= \frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\dots\right] - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\dots\right] - \frac{3}{t} \\
&= -\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\dots - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\dots\right] \\
&= -2(z+1)^{-1}+(z+1)^{-2}+(z+1)^{-3}+\dots - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\dots\right]
\end{aligned}$$

Evaluate $\int_C \frac{z\ dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$ by Cauchy Residue theorem.

Solution:

The poles are obtained by (z – 1) $(z-2)^2$ = 0

$\Rightarrow$ z = 1 is a simple pole and z = 2 is a pole of order 2.

C is the circle | z – 2 | = $\frac{1}{2}$

Here z = 1 lies outside C and z = 2 lies inside C.

Residue at z=2: (Pole of order 2)

$$\text{Res } f(z) = \underset{z\to 2}{lt}\frac{d}{dz}(z-2)^2\frac{z}{(z-1)(z-2)^2} = \underset{z\to 2}{lt}\frac{z-1-z}{(z-1)^2} = -1$$

By Cauchy Residue theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i(-1) = -2\pi i$$

Using Cauchy's residue theorem evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$, where C is $|z|=2$

Solution:

$|z|=2$ is the equation of the circle with centre at origin and radius 2.

$$(z^2-1)(z-3)=0$$

$$(z^2-1)=0, \quad (z-3)=0$$
$$z^2=1, \quad z=3$$
$$z=\pm 1, \quad z=3$$

$z=1,-1$ lies inside the circle and $z=3$ lies outside the circle

Residue at $z=1$ is

$$= Lt_{z\to 1}\left((z-1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to 1}\left(\frac{3z^2+z-1}{(z+1)(z-3)}\right) = -\frac{3}{4}$$

Residue at $z=-1$ is

$$= Lt_{z\to -1}\left((z+1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to -1}\left(\frac{3z^2+z-1}{(z-1)(z-3)}\right) = \frac{1}{8}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{3z^2+z-1}{\left(z^2-1\right)(z-3)}dz = 2\pi i\left(\frac{1}{8}-\frac{3}{4}\right) = -\frac{5\pi i}{4}$$

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is $|z-i|=2$ using Cauchy's residue theorem

Solution:

Let $f(z)=\dfrac{z-1}{(z+1)^2(z-2)}$

poles of $f(z)$ are z = - 1 (pole of order 2) and z = 2 (simple pole)

Given: $|z-i|=2$

$$|x+iy-i|=2 \Rightarrow |x+i(y-1)|=2$$

Squaring on both sides $\sqrt{x^2+(y-1)^2}=2 \Rightarrow x^2+(y-1)^2=4$

This is equation of circle with centre $(0,1)$ and radius 2

Hence, The pole $z=2$ lies outside C and $z=-1$ lies inside C

Residue of f(z) at $z=-1$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left((z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}\right)$$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left(\frac{(z-1)}{(z-2)}\right) = Lt_{z\to -1}\left(\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right)$$

$$= Lt_{z\to -1}\left(\frac{-1}{(z-2)^2}\right) = -\frac{1}{9}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{(z-1)}{(z+1)^2(z-2)}\,dz = 2\pi i\left(0-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Using Cauchy's residue theorem, find $\int_C \frac{z+1}{(z-3)(z-1)}\,dz$ **, where C is** $|z|=2$

Solution:

The singular points are given by (z – 1) (z – 3) = 0 $\Rightarrow$ z = 1 , 3

Given C is | z | = 2

If z = 1 then | z | = | 1 | = 1 < 2

If z = 3 then | z | = | 3 | = 3 > 2

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

Residue at z=1:

$$\text{Res}\Big|_{z=1} = \lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}(z-1)\frac{z+1}{(z-3)(z-1)} = -1$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)}\,dz = 2\pi i(-1) = -2\pi i$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **by using Contour integration.**

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+5\frac{(z^2-1)}{2iz}} = \int_C \frac{\frac{dz}{iz}}{\frac{26iz+5z^2-5}{2iz}} = 2\int_C \frac{dz}{5z^2+26iz-5}$$

Let $f(z) = \dfrac{1}{5z^2+26iz-5}$ $\qquad \therefore I = 2\int_C f(z)dz$

The poles of f(z) are given by $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{(26i)^2 - 4\ .\ 5(-5)}}{10} = \frac{-26i \pm \sqrt{-676+100}}{10} = \frac{-26i \pm \sqrt{-576}}{10} = \frac{-26i \pm 24i}{10}$$

$$z = -\frac{i}{5}\ ,\ -5i$$

which are simple poles.

Now $5z^2 + 26iz - 5 = 5\left(z+\dfrac{i}{5}\right)(z+5i)$

Since $\left|\dfrac{-i}{5}\right| = \dfrac{1}{5} < 1$, the pole $z = \dfrac{-i}{5}$ lies inside C

and $|-5i| = 5 > 1$, $\therefore$ the pole $z = -5i$ lies outside C.

Now $R\left(-\dfrac{i}{5}\right) = \lim\limits_{z\to\frac{-i}{5}}\left(z+\dfrac{i}{5}\right)f(z) = \lim\limits_{z\to\frac{-i}{5}}\left(z+\dfrac{i}{5}\right)\dfrac{1}{5\left(z+\frac{i}{5}\right)(z+5i)} = \lim\limits_{z\to\frac{-i}{5}}\dfrac{1}{5(z+5i)}$

$$= \lim_{z\to\frac{-i}{5}} \frac{1}{5\left(-\frac{i}{5}+5i\right)} = \frac{1}{24i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\therefore I = 2.\frac{\pi}{12} = \frac{\pi}{6}$$

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$ by using Contour integration.

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+12\frac{(z^2+1)}{2z}} = \int_C \frac{dz}{\frac{iz(13z+6z^2+6)}{z}} = \int_C \frac{dz}{i(6z^2+13z+6)} = \frac{1}{i6}\int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$$

Let $f(z) = \int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$ $\qquad \therefore I = \frac{1}{6i}\int_C f(z)dz$

The poles of f(z) are given by $z^2 + \frac{13}{6}z + 1 = 0$

By solving we get $z = -\frac{2}{3}, \ -\frac{3}{2}$

which are simple poles.

Now $z^2 + \frac{13}{6}z + 1 = \left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)$

Since $\left|\frac{-2}{3}\right| = \frac{2}{3} < 1$, the pole $z = \frac{-2}{3}$ lies inside C

and $\left|\frac{-3}{2}\right| = 1.5 > 1$, $\therefore$ the pole $z = \frac{-3}{2}$ lies outside C.

$$Now \quad R\left(-\frac{2}{3}\right) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)f(z) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)\frac{1}{\left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)} = \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)}$$

$$= \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)} = \frac{6}{5}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{6}{5}\right) = \frac{12\pi i}{5}, \qquad \therefore I = \frac{1}{6i}\times\left(\frac{12\pi i}{5}\right) = \frac{2\pi}{5}.$$

Evaluate $\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$ by using Contour integration

Solution:

Consider the unit circle | z | = 1 as contour C.

$Put\ z = e^{i\theta}, \ then\ \frac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$\cos3\theta$=R.P. of $e^{i3\theta}$=R.P. of $(e^{i\theta})^3$=R.P. of z^3

$$\therefore \qquad I = \int_C \frac{R.P. of\ z^3 \frac{dz}{iz}}{5-4\frac{(z^2+1)}{2z}} = R.P.\ of\ \int_C \frac{z^3 dz}{\frac{iz(5z-2z^2-2)}{z}}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{i(-2z^2+5z-2)}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{-i(2z^2-5z+2)}$$

$$= R.P. of\ \frac{-1}{2i}\int_C \frac{z^3 dz}{(2z-1)(z-2)}$$

Let $\int_C f(z)dz = \int_C \frac{z^3 dz}{(2z-1)(z-2)}$ $\qquad \therefore I = R.P.$ of $\frac{-1}{2i}\int_C f(z)dz$

The poles of f(z) are given by

$(2z-1)(z-2) = 0$

$z = \frac{1}{2},\ z = 2$

$z = \frac{1}{2},\ z = 2$ (simple poles)

$z = \frac{1}{2}$ is a pole lies inside c.

$z = 2$ is a pole lies outside c.

Now $\quad \mathrm{Res}\left(z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{\left(z-\frac{1}{2}\right)\ (z-2)} = \frac{-1}{12}$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\ \left(\frac{-1}{12}\right) = \frac{-\pi i}{6}$$

$$\therefore I = R.P.of\ \frac{-1}{2i}\ .\frac{-\pi i}{6} = R.P.of\ \frac{\pi}{12} = \frac{\pi}{12}$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2}$ **, | p | < 1**

Solution: Let $z = e^{i\theta},\ dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z^2-1}{2iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_C \frac{\left(dz/iz\right)}{1-2p\left(\frac{z^2-1}{2iz}\right)+p^2} \quad , C \text{ is } |z| = 1$$

$$= \int_C \frac{dz}{iz - p(z^2-1) + izp^2} = -\int_C \frac{dz}{pz^2 - iz(p^2+1) - p} = -\frac{1}{p}\int_C \frac{dz}{z^2 - iz\left(p+\frac{1}{p}\right) - 1}$$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} \quad ....(1)$$

The poles are given by $z = ip \;\&\; z = \frac{i}{p}$

| z | = | i p | = p < 1. $\therefore$ z= ip lies inside C and $z = \frac{i}{p}$ lies outside C.

$$\therefore \left[\text{Res of } f(z)\right]_{z=ip} = \underset{z\to ip}{\text{Lt}} (z-ip)\left[\frac{1}{(z-ip)\left(z-\frac{i}{p}\right)}\right] = \underset{z\to ip}{\text{Lt}} \left(\frac{1}{z-\frac{i}{p}}\right) = \frac{1}{i\left(p-\frac{1}{p}\right)} = \frac{ip}{1-p^2}$$

By Cauchy Residue Theorem $\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} = 2\pi i\left(\frac{ip}{1-p^2}\right) = \frac{-2\pi p}{1-p^2}$

From (1) $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\left(-\frac{2\pi p}{1-p^2}\right) = \frac{2\pi}{1-p^2}$

Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}, (a>0)$ **using contour integration**

Solution:

Let $f(z)=\dfrac{1}{(z^2+a^2)^2}$. Consider $\int_c f(z)dz$

where C is the contour consists of the upper half circle c_1 of $|z|=R$ & the real axix from –R to R.

$$\therefore \int_c f(z)dz=\int_{c_1} f(z)dz+\int_{-R}^{R} f(z)dz \text{.......... ....} \quad (1)$$

The poles of $f(z)$ are given by $(z^2+a^2)^2=0 \Rightarrow z=\pm ai$ (twice)

z = ai is a pole of order 2 & lies inside C

z = $-ai$ is a pole of order 2 & lies outside C

$$\text{Res}\left[f(z),ai\right]=\lim_{z\to ai}\frac{d}{dz}\left[(z-ai)^2\frac{1}{(z+ai)^2(z-ai)^2}\right]=\lim_{z\to ai}\frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]=\frac{-2}{(2ai)^3}=\frac{1}{4a^3i}$$

By Cauchy's Residue Theorem $\int f(z)dz=2\pi i(\dfrac{1}{4a^3i})=\dfrac{\pi}{2a^3}$

In (1) $R\to\infty$, then $\int_{c_1} f(z)dz=0$

$$\therefore (1)\Rightarrow \int_c f(z)dz=\int_{-\infty}^{\infty} f(x)dx$$

$$=\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$=2\int_0^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$\int_0^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}$$

Evaluate $\displaystyle\int_0^{\infty}\frac{\cos ax\,dx}{x^2+1}$, $a>0$, using contour integration.

Solution:

$$\int_0^{\infty}\frac{\cos ax\,dx}{1+x^2}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax\,dx}{1+x^2}$$

Now $\displaystyle\int_{-\infty}^{\infty}\frac{\cos ax\,dx}{1+x^2}=\int_{-\infty}^{\infty}\frac{\text{RP of } e^{iax}}{1+x^2}dx \qquad \left\{\because e^{i\theta}=\cos\theta+i\sin\theta\right\}$

Consider $\int\limits_c f(z)\,dz = \text{R.P}\int\limits_c \frac{e^{iaz}}{1+z^2}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [–R, R]. By Cauchy's residue theorem, we have

$$\int\limits_c f(z)\,dz = \int\limits_{-R}^{R} f(x)\,dx + \int\limits_{\Gamma} f(z)\,dz$$

The poles of f(z) are at $1+z^2=0$

$z^2=-1 \Rightarrow z=\pm i$

The point $z=i$ lies inside the semi-circle and the point $z=-i$ lies outside the semi-circle

Residue at $z=i$ is given by

$$Lt_{z\to i}(z-i)\,f(z) = Lt_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)}$$

$$=Lt_{z\to i}\frac{e^{iaz}}{(z+i)} = \frac{e^{ia(i)}}{i+i} = \frac{e^{ai^2}}{2i} = \frac{e^{-a}}{2i}$$

By Cauchy Residue theorem,

$$R.P\int\limits_c \frac{e^{iaz}\,dz}{1+z^2} = \text{R.P of } 2\pi i\left(\frac{e^{-a}}{2i}\right) = \text{R.P of } \pi e^{-a} = \pi e^{-a}$$

$$\therefore \int\limits_{-R}^{R} f(x)\,dx + \int\limits_{\Gamma} f(z)\,dz = \pi e^{-a}$$

If $R\to\infty$, then $\int\limits_{\Gamma} f(z)\,dz \to 0$

Hence $\int\limits_{-\infty}^{\infty} f(x)\,dx = \pi e^{-a}$

$$\int\limits_0^{\infty}\frac{\cos ax\;dx}{1+x^2} = \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\cos ax\;dx}{1+x^2} = \frac{\pi e^{-a}}{2}$$

Evaluate $\int\limits_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}\,dx$**, using contour integration.**

Solution:

Let $f(z)=\dfrac{z^2-z+2}{z^4+10z^2+9}$

Consider $\int\limits_c f(z)\,dz=\int\limits_c \dfrac{z^2-z+2}{z^4+10z^2+9}dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R]. By Cauchy's residue theorem, we have

$$\int\limits_c f(z)\,dz=\int\limits_{-R}^{R} f(x)\,dx+\int\limits_{\Gamma} f(z)\,dz$$

The poles f(z) are at $z^4+10z^2+9=0$

$$\left(z^2+1\right)\left(z^2+9\right)=0$$

$$z^2=-1; \qquad z^2=-9$$

$$z=\pm i; \qquad z=\pm 3i$$

The poles are at $3i,\ -3i,\ i,\ -i$

Here the poles $3i$ and i lie inside the semi-circle.

Residue at $z=3i$ is given by

$$=Lt_{z\to 3i}\left(z-3i\right)f\left(z\right)$$

$$=Lt_{z\to 3i}\left(z-3i\right)\frac{z^2-z+2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$=Lt_{z\to 3i}\left(z-3i\right)\frac{z^2-z+2}{\left(z-3i\right)\left(z+3i\right)\left(z^2+1\right)}$$

$$=Lt_{z\to 3i}\frac{z^2-z+2}{\left(z+3i\right)\left(z^2+1\right)}=\frac{7+3i}{48i}$$

Residue at $z=i$ is given by

$$=Lt_{z\to i}\left(z-i\right)f\left(z\right)$$

$$=Lt_{z\to i}\left(z-i\right)\frac{z^2-z+2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$=Lt_{z\to i}\left(z-i\right)\frac{z^2-z+2}{\left(z-i\right)\left(z+i\right)\left(z^2+9\right)}$$

$$= Lt_{z\to i}\frac{z^2-z+2}{(z+i)(z^2+9)} = \frac{1-i}{16i}$$

By Cauchy Residue theorem,

$$\int_c \frac{z^2-z+2}{z^4+10z^2+9}dz = 2\pi i\left[\frac{7+3i}{48i}+\frac{1-i}{16i}\right] = 2\pi i\left[\frac{7+3i+3-3i}{48i}\right] = 2\pi i\left[\frac{10}{48i}\right] = \frac{5\pi}{12}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{5\pi}{12}$$

If $R\to\infty$, then $\int_{\Gamma} f(z)\,dz \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \frac{5\pi}{12}$

$$\therefore \int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}$$

Evaluate $\int_0^{\infty}\frac{x\sin mx \quad dx}{(x^2+a^2)}$, where a > 0 , m > 0

Solution:

Let $f(z) = \int_0^{\infty}\frac{x\sin mxdx}{(x^2+a^2)} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin mxdx}{(x^2+a^2)}$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin mxdx}{(x^2+a^2)} = \frac{1}{2}\mathrm{IP}\int_{-\infty}^{\infty}\frac{xe^{imx}dx}{(x^2+a^2)} = \frac{1}{2}\mathrm{IP}(\mathrm{I}_1)$$

$$\mathrm{I}_1 = \int_{-\infty}^{\infty}\frac{xe^{imx}}{x^2+a^2}dx = \int_{-\infty}^{\infty}F(x)dx$$

Here $F(x) = \frac{xe^{imx}}{x^2+a^2}$ let $F(z) = \frac{ze^{imx}}{z^2+a^2}$

The poles of F(z) are given by

$\Rightarrow$ z = ±ia are poles of order 1

$\Rightarrow$ z = ia lies inside C

Consider $\int_C f(z)\ dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

$$\therefore \left[\operatorname{Res} \text{ of } f(z)\right]_{z=ai} = \underset{z\to ia}{\mathrm{Lt}} (z-ia)\,\frac{ze^{imz}}{(z+ib)(z-ib)}$$

$$= \frac{e^{-ma}(ia)}{2ia} = \frac{e^{-ma}}{2}$$

$$I_1 = 2\pi i\left(\frac{e^{-ma}}{2}\right) + \pi i(0) = i\pi e^{-ma}$$

$$I = \frac{1}{2}\mathrm{IP}(I_1) = \frac{1}{2}\mathrm{IP}(i\pi e^{-ma}) = \frac{\pi e^{-ma}}{2}$$

By Cauchy's Residue Theorem

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \qquad \because \int_C f(z)\,dz \to 0 \text{ as } R\to\infty$$

$$\Rightarrow \int_0^{\infty} f(x)\,dx = \frac{\pi e^{-ma}}{2}$$

Evaluate $\int_0^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$**, a > 0 , b > 0**

Solution:

Let $f(z) = \text{Real Part of } \frac{e^{iz}}{(z^2+a^2)\ (z^2+b^2)}$

Consider $\int_C f(z)\ dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

The poles of $f(z)$ are given by $(z^2+a^2)(z^2+b^2) = 0$

$\Rightarrow z = \pm ia, \pm ib$

$\Rightarrow z = ia, ib$ lies inside C and $z = -ia, -ib$ lies in lower half plane

$$\therefore \left[\text{Res of } f(z)\right]_{z=ai} = \underset{z\to ia}{\text{Lt}} (z-ia)\, \frac{e^{iz}}{(z+ia)(z-ia)(z^2+b^2)}$$

$$= \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$\left[\text{Res of } f(z)\right]_{z=bi} = \underset{z\to ia}{\text{Lt}} (z-ib)\, \frac{e^{iz}}{(z+ib)(z-ib)(z^2+a^2)}$$

$$= \frac{e^{-a}}{2ib(a^2-b^2)}$$

By Cauchy's Residue Theorem

$$\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = 2\pi i\left[\frac{e^{-a}}{2ia(b^2-a^2)} + \frac{e^{-b}}{2ib(a^2-b^2)}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

In (1) if $R\to\infty$, $\int_{C_1} f(z)\,dz \to 0$

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \text{Real Part of } \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$