UNIT 5 COMPLEX INTEGRATION

INTRODUCTION

This unit focuses on Complex variables and Complex integration. Complex variable techniques have been used in a wide variety of areas of engineering such as electromagnetic field theory, fluid dynamics, aerodynamics and elasticity. Many complicated integrals of real functions are solved with the help of complex variable.

COMPLEX INTEGRATION

In case of real variable, the path of integration of $\int_a^b f(x)dx$ is always along the x - axis, from $x = a$ to $x = b$. But, in case of complex function $f(z)$ the path of the definite integral $\int_a^b f(z)dz$ can be along any curve from $z = a$ to $z = b$. Its value depends upon the path (curve) of integration. But, the value of integral from a to b remains the same, if the different curves from a to b are regular curves.

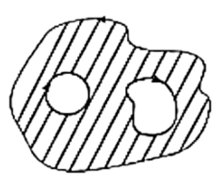
SIMPLY AND MULTIPLY CONNECTED REGIONS

A region R is called simply connected, if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply connected is called multiply connected.

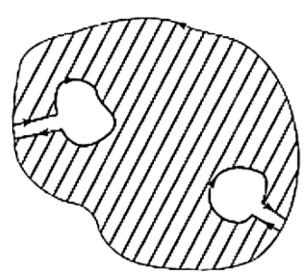
obviously, a simply connected region is one which does not have any "holes" in it, whereas a multiply connected region is one which has.

A multiply connected region can be converted into a simply connected region by introducing cross- cuts as show in the below Figs

Simply connected region



Multiply connected region



Multiply connected converted into Simply connected region by cross cuts

6.1 CAUCHY'S INTEGRAL THEOREM AND INTEGRAL FORMULA

CAUCHY'S INTEGRAL THEOREM (OR) CAUCHY'S THEOREM (OR) CAUCHY'S FUNDAMENTAL THEOREM:

Statement: If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C, then $\int_C f(z)dz = 0$.

CAUCHY'S THEOREM FOR MULTIPLY CONNECTED REGION:

If $f(z)$ is analytic in the doubly connected region R bounded by two simple closed curves C_1 and C_2, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

If there are finite number of simple closed curves $C_1, C_2, ...C_n$ inside C and $f(z)$ is analytic in the region within the regions between the curves $C_1, C_2, ...C_n$, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz$$

CAUCHY'S INTEGRAL FORMULA:

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R if 'a' is any point within C, then $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{z-a}dz,$ the integration around C being taken in the positive direction.

NOTE: $\displaystyle\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$

CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVE:

If a function $f(z)$ is analytic within and on a simple closed curve C and 'a' is any point lying in it, $\displaystyle\int_C \frac{f(z)}{(z-a)^2}dz = 2\pi i f'(\text{a})$

$$\int_C \frac{f(z)}{(z-a)^3}dz = \frac{2\pi i}{2!} f''(\text{a})$$

In General, $\displaystyle\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^n(a), n = 1, 2, 3, ...$

Worked Examples 6.1(A)

Example 1

Evaluate $\displaystyle\int_c \left(z^2 - z + 1\right)dz$ Where C is the Circle $|z| = 2$.

Solution:

Let f(z) = $\left(z^2 - z + 1\right)$ is analytic and C is closed circle.

Hence, by Cauchy's integral theorem $\displaystyle\int_c f(z)\,dz = 0$

Therefore $\displaystyle\int_c \left(z^2 - z + 1\right)dz = 0$

Example 2

What is the value for the integral $\displaystyle\int_c \left(\frac{3z^2 + 7z + 1}{z+1}\right)dz$ where C is $|z| = \dfrac{1}{2}$

Solution:

$$\int f(z)dz = \int_c \left(\frac{3z^2 + 7z + 1}{z+1}\right)dz$$

The point z = -1 lies outside the circle $|z| = \dfrac{1}{2}$

Therefore by

Cauchy's integral theorem, $\int_c f(z)\,dz = 0$ Hence $\int_c \left(\frac{3z^2+7z+1}{z+1}\right) dz = 0$

Example 3

Evaluate $\int_c \frac{z+4}{z^2+2z}\,dz$ where c is the circle $\left|z-\frac{1}{2}\right| = \frac{1}{3}$.

Solution:

$$\int_c \frac{z+4}{z^2+2z}\,dz = \int_c \frac{z+4}{z(z+2)}\,dz$$

Given $\left|z-\frac{1}{2}\right| = \frac{1}{3}$ is circle of center is (½ , 0), radius 1/3

here z = 0 is lies outside $\left|z-\frac{1}{2}\right| = \frac{1}{3}$

z = -2 is lies outside $\left|z-\frac{1}{2}\right| = \frac{1}{3}$

By Cauchy's integral theorem, we get

$$\int_c \frac{z+4}{z^2+2z}\,dz = 0$$

Example 4

Evaluate $\int_c \frac{e^z\,dz}{(z-2)}$, where c is the unit circle with center as origin.

Solution:

Given $\int_c \frac{e^z\,dz}{(z-2)}$

Here C is the unit circle with Centre as origin.

Let $f(z) = \frac{e^z}{z-2}$

Here $z = 2$ lies outside C

$\therefore f(z)$ is analytic inside and on C. $f'(z)$ is continuous inside C.

Hence by Cauchy's integral theorem $\int_c f(z)dz = 0$

$$\Rightarrow \int_c \frac{e^z\,dz}{(z-2)} = 0$$

Example 5

Use Cauchy integral formula, evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2 dz}{(z+1)(z+2)}$ where C is $|z| = \frac{\pi}{2}$

Solution:

Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)\ (z+2)}$

Here, z = -1 is lies inside c

z= -2 is lies outside c.

Therefore f(z) is analytic inside and on c.

$f'(z)$ is continuous inside c.

Hence, by Cauchy's integral formula

$$\int_c \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i f(a) & \text{if [a lies inside C]} \\ 0 & \text{if [a lies outside C]} \end{cases}$$

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2 dz}{(z+1)(z+2)} = \int_c \frac{\left(\sin \pi z^2 + \cos \pi z^2 / z+2\right)}{z+1}dz = 2\pi i f(-1)$$

Since $f(z) = \left(\sin \pi z^2 + \cos \pi z^2 / (z+2)\right)$

$$f(z) = \left(\sin \pi (-1)^2 + \cos \pi (-1)^2 / (-1+2)\right) = \sin \pi + \cos \pi = -1$$

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2 dz}{(z+1)(z+2)} = 2\pi i f(-1) = 2\pi i(-1) = -2\pi i$$

Example 6

If C is the circle $|z| = 3$ and if $g(z_0) = \int_c \frac{2z^2 - z - 2}{z - z_0} dz$ then find $g(2)$.

Solution:

We know that, Cauchy's integral formula is

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a) \quad ...(1)$$

Given: $g(z_0) = \int_c \frac{2z^2 - z - 2}{z - z_0} dz$

$$g(2) = \int_C \frac{2z^2 - z - 2}{z-2} dz = \int_C \frac{f(z)}{z-2} dz = 2\pi i f(2) ...(2)$$

Where $f(z) = 2z^2 - z - 2$

$$f(2) = 2(2)^2 - 2 - 2 = 8 - 2 - 2 = 4$$

$(2) \Rightarrow g(2) = (2\pi i)(4) = 8\pi i$

Example 7

Evaluate $\int_c \frac{e^z}{z-1} dz$ where c is $|z+3| = 1$

Solution:

Given $\int_c \frac{e^z}{z-1} dz$

$z - 1 = 0$

$z = 1$ lies outside C

$$\therefore \int_c \frac{e^z}{z-1} dz = 0 \text{ by Cauchy's Integral theorem.}$$

Example 8

Evaluate $\int_c \frac{5z^2 + 30z + 100}{(z-2)} dz$ where c is the circle $|z-2| = 4$

Solution:

Given $|z-2|=4$ is a circle whose centre is 2 and radius 4

2 is lies inside $|z-2|=4$

Cauchy's integral formula is

$$\int_c \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i f(a) & \text{if [a lies inside C]} \\ 0 & \text{if [a lies outside C]} \end{cases}$$

$$f(z)=5z^2+30z+100$$

Here $f(2)=5(2)^2+30(2)+100=180$

$$\int \frac{f(z)}{z-2}dz = 2\pi i f(2) = 2\pi i(180) = 360\pi i$$

$$\int_c \frac{5z^2+30z+100}{(z-2)}dz = 360\pi i$$

Worked Examples 6.1(B)

Example 1

Using Cauchy's integral formula, evaluate $\int_c \frac{z}{(z-1)(z-2)^2}dz$ where c is $|z-2|=\frac{1}{2}$.

Solution:

Cauchy's integral derivative formula is $\int_c \frac{f(z)}{(z-a)^2}dz = \frac{2\pi i}{1!} f'(a)$

Given circle c is $|z-2|=\frac{1}{2}$ i.e., centre 2 ; radius $\frac{1}{2}$

z = 1 lies outside $|z-2|=\frac{1}{2}$,

z = 2 lies inside $|z-2|=\frac{1}{2}$

$$\therefore \int_c \frac{z}{(z-1)(z-2)^2}dz = \int_c \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2}dz$$

Here, $f(z)=\frac{z}{z-1}$, and a = 2

$$f'(z)=\frac{(z-1)(1)-z(1)}{(z-1)^2}=\frac{z-1-z}{(z-1)^2}=\frac{-1}{(z-1)^2}$$

$$\int_c \frac{f(z)}{(z-2)^2}dz = \frac{2\pi i}{1!} f'(2) = 2\pi i\left[\frac{-1}{(2-1)^2}\right] = -2\pi i$$

Example 2

Evaluate $\int_c \frac{z^2}{(z^2+1)^2}dz$ Where C is the circle $|z-i|=1$ by using Cauchy's integral formula.

Solution:

W.K.T. Cauchy's integral formula is

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \qquad ...(1)$$

Given: $|z-i|=1$ is a circle, whose centre is (0, 1) and radius is 1.

$$\int_c \frac{z^2}{(z^2+1)^2}\,dz = \int_C \frac{z^2}{[(z-i)(z+i)]^2}dz$$

$$= \int_C \frac{z^2}{(z-i)^2\,(z+i)^2}dz$$

Here z = i lies inside C.
Here z = -i lies outside C.

$$= \int_C \frac{\left[\dfrac{z^2}{(z+i)^2}\right]}{(z-i)^2}dz$$

Here $f(z) = \dfrac{z^2}{(z+i)^2}$

$$f'(z) = \frac{(z+i)^2 2z - z^2 2(z+i)}{(z+i)^4}$$

$$i.e.,\ f'(z) = \frac{(z+i)2z - 2z^2}{(z+i)^3}$$

$$f'(i) = \frac{(2i)(2i) - 2(i)^2}{(2i)^3} = \frac{2i^2}{8i^3} = \frac{1}{4i}$$

$$(1) \Rightarrow \int_C \frac{z^2}{(z^2+1)^2}dz = \frac{2\pi i}{1!}\left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

Example 3

If $g(a) = \int_c \dfrac{3z^2+7z+1}{z-a}dz$, Where C is the circle $|z|=2$, find the values of $g(3), g'(1-i)$ and $g''(1-i)$

Solution:

C is the circle $|z|=2$

(i) To find: $g(3)$

$$g(a) = \int_c \frac{3z^2+7z+1}{z-a}dz$$

$$\therefore g(3) = \int_c \frac{3z^2+7z+1}{z-3}dz = \int_{|z|=2} \frac{3z^2+7z+1}{z-3}\,dz$$

z=3 lies outside the circle $|z|=2$

By Cauchy's integral formula $\int_c \frac{f(z)}{(z-a)} dz = \begin{cases} 2\pi i f(a) & if \ a \ is \ inside \ c \\ 0 & if \ a \ is \ outside \ c \end{cases}$

$$\therefore \int_{|z|=2} \frac{3z^2 + 7z + 1}{z-3} d = 2\pi i \, g(3)$$

$$\int_{|z|=2} \frac{3z^2 + 7z + 1}{z-3} dz = 0$$

$Hence \ g(3) = 0$

(ii) to find $g'(1-i), g''(1-i)$

Z = 1 – i lies inside the $|z| = 2$

By Cauchy's integral formula we get

$$\therefore \int_c \frac{3z^2 + 7z + 1}{z-3} d = 2\pi i \, g(a)$$

$$g(a) = 2\pi i \left(3a^2 + 7a + 1\right) \quad ..(1)$$

$$g'(a) = 2\pi i \left(6a + 7\right) \quad ..(2)$$

$$g''(a) = 2\pi i \left(6\right) = 12\pi i \quad ..(3)$$

$$g'(1-i) = 2\pi i \left(6(1-i) + 7\right) = 2\pi i \left(6 - 6i + 7\right) = 2\pi i \left(13 - 6i\right)$$

$$g''(1-i) = 2\pi i \left(6\right) = 12\pi i$$

Example 4

Evaluate $\int_c \frac{z+1}{\left(z^2 + 2z + 4\right)^2} dz$, where C is the $|z + 1 + i| = 2$, Cauchy's integral formula.

Solution:

$Solving \ z^2 + 2z + 4 = 0 \ we \ get$

$$z = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= -1 \pm \sqrt{3}i$$

$z = -1 + \sqrt{3}i,$ is outside on c

$z = -1 - \sqrt{3}i$ is inside on c

$$z^2 + 2z + 4 = \left[z - (-1 + \sqrt{3}i)\right]\left[z - (-1 - \sqrt{3}i)\right]$$

Now $\int_C \frac{z+1}{\left(z^2 + 2z + 4\right)^2} dz = \int_C \frac{z+1}{\left[z - (-1 + \sqrt{3}i)\right]^2 \left[z - (-1 - \sqrt{3}i)\right]^2}$

$$= \int_C \frac{z+1 \Big/ \left[z - (-1 + \sqrt{3}i)\right]^2}{\left[z - (-1 - \sqrt{3}i)\right]^2} dz$$

By Cauchy's integral formula, we get

$$\int_C \frac{z+1}{\left(z^2+2z+4\right)^2}dz = 2\pi i f'(-1-\sqrt{3}i)$$

Here $f(z)=\dfrac{z+1}{\left[z-(-1+\sqrt{3}i)\right]^2}$

$$f'(z)=\frac{(1)\left[z-(-1+\sqrt{3}i)\right]^2-2(z+1)\left[z-(-1+\sqrt{3}i)\right]}{\left[z-(-1+\sqrt{3}i)\right]^4}$$

$$=\frac{\left[z-(-1+\sqrt{3}i)\right]\left[\left(z-\left(1+\sqrt{3}i\right)\right)-2(z+1)\right]}{\left[z-(-1+\sqrt{3}i)\right]^4}$$

$$f'(z)=\frac{\left(z-\left(1+\sqrt{3}i\right)\right)-2(z+1)}{\left[z-(-1+\sqrt{3}i)\right]^3}$$

$$f'\left(-1-\sqrt{3}i\right)=\frac{\left(\left(-1-\sqrt{3}i\right)-\left(1+\sqrt{3}i\right)\right)-2\left(\left(-1-\sqrt{3}i\right)+1\right)}{\left[\left(-1-\sqrt{3}i\right)-(-1+\sqrt{3}i)\right]^3}$$

$$=\frac{\left(-1-\sqrt{3}i-1-\sqrt{3}i\right)-2\left(-1-\sqrt{3}i+1\right)}{\left[-1-\sqrt{3}i+1-\sqrt{3}i\right]^3}$$

$$=\frac{-2-2\sqrt{3}i+2\sqrt{3}i}{\left[-2\sqrt{3}i\right]^3}$$

$$=\frac{-2}{-8\times 3\times\sqrt{3}(-i)}=\frac{-1}{12\sqrt{3}i}$$

$$\int_C \frac{z+1}{\left(z^2+2z+4\right)^2}dz = 2\pi i\left[\frac{-1}{12\sqrt{3}i}\right]=\frac{-\pi}{6\sqrt{3}}$$

Example 5

Evaluate $\int_c \frac{e^z}{z(1-z)^3}dz$ if C is $|z|=2$, by using Cauchy's integral formula.

Solution:

Given $\int_c \frac{e^z}{z(1-z)^3}dz$ on $|z|=2$.

W.K.T. Cauchy's integral formula is

$$\int_c \frac{f(z)}{z-a}dz = 2\pi i\, f(a) \qquad ...(1)$$

Here take $f(z)=e^z$

The poles are z = 0 and z =1 both are lies inside the region $|z|=2$

$$\frac{1}{z(1-z)^3}=\frac{A}{z}+\frac{B}{(1-z)}+\frac{C}{(1-z)^2}+\frac{D}{(1-z)^3}$$

$$1=A(1-z)^3+Bz(1-z)^2+Cz(1-z)+Dz$$

Put $z=0$ *Put* $z=1$ *Compare the coefficent of* z^3

$1=A$ $1=D$ $0=-A+B\Rightarrow B=A=1$

Compare the coefficient of z^2

$0=3A-2B-C$

$0=3-2-C$

$-1=-C\Rightarrow C=1$

$$\frac{1}{z(1-z)^3}=\frac{1}{z}+\frac{1}{(1-z)}+\frac{1}{(1-z)^2}+\frac{1}{(1-z)^3}$$

Here $f(z)=e^z,\ f'(z)=e^z,\ f''(z)=e^z$

$f(0)=1,\ f(1)=e,\ f'(1)=e,\ f''(1)=e$

Hence by Cauchy's integral formula is

$$\int_c \frac{e^z}{z(1-z)^3}dz=\int_c \frac{f(z)}{z}dz+\int_c \frac{f(z)}{(1-z)}dz+\int_c \frac{f(z)}{(1-z)^2}dz+\int_c \frac{f(z)}{(1-z)^3}dz$$

$$=2\pi i f(0)+2\pi i f(1)+2\pi i f'(1)+2\pi i f''(1)$$

$$=2\pi i+2\pi ie+2\pi ie+2\pi ie$$

$$\int_c \frac{e^z}{z(1-z)^3}dz=2\pi i(1+3e)$$

Example 6

Evaluate $\int_c \frac{z+4}{z^2+2z+5}dz$, where C is the circle $|z+1+i|=2,$ using Cauchy's integral formula.

Solution:

Given: | z + 1 + i | = 2

i.e., | z – [- (1 + i)] | = 2 is the circle whose centre is (-1,-1) and radius is 2.

i.e., centre is (-1, -1) and radius is 2.

[Note: $z^2+2z+5=0$

$$z=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

z = – 1 +2i ; z = – 1 – 2i]

$$z^2+2z+5=[z-(-1+2i)][z-(-1-2i)]$$

– 1 +2i i.e., (-1, 2) lies outside *c*

– 1 – 2i i.e., (-1, -2) lies inside *c*

$$\int_c \frac{z+4}{z^2+2z+5}dz=\int_c \frac{\left[\frac{z+4}{z-(-1+2i)}\right]}{z-(-1-2i)}dz$$

Here $f(z)=\frac{z+4}{z-(-1+2i)}$ is analytic inside *c.*

By Cauchy's integral formula

$$\int_c \frac{f(z)}{z-a}\,dz = \begin{cases} 2\pi i\, f(a) & \text{if [a lies inside C]} \\ 0 & \text{if [a lies outside C]} \end{cases}$$

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\; f[-1-2i]$$

$$= 2\pi i\left[\frac{-1-2i+4}{(-1-2i)-(-1+2i)}\right]$$

$$= 2\pi i\left[\frac{3-2i}{-1-2i+1-2i}\right]$$

$$= 2\pi i\left[\frac{3-2i}{-4i}\right]$$

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = -\frac{\pi}{2}(3-2i)$$

Example 7

Using Cauchy's integral formula, evaluate $\int_c \frac{z+4}{z^2+2z+5}\,dz$, where c is $|z+1-i| = 2$.

Solution:

Given: | z + 1 - i | = 2

i.e., | z – [- (1 + i)] | = 2 is the circle whose centre is (-1,1) and radius is 2.

i.e., centre is (-1, 1) and radius is 2.

[Note: $z^2+2z+5=0$

$$z = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$$

z = – 1 +2i ; z = – 1 – 2i]

$$z^2+2z+5 = [z-(-1+2i)][z-(-1-2i)]$$

– 1 +2i i.e., (-1, 2) lies inside *c*

– 1 – 2i i.e., (-1, -2) lies outside *c*

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = \int_c \frac{\left[\dfrac{z+4}{z-(-1-2i)}\right]}{z-(-1+2i)}\,dz$$

Here $f(z) = \dfrac{z+4}{z-(-1-2i)}$ is analytic inside *c*.

by Cauchy's integral formula

$$\int_c \frac{f(z)}{z-a}\,dz = \begin{cases} 2\pi i\, f(a) & \text{if [a lies inside C]} \\ 0 & \text{if [a lies outside C]} \end{cases}$$

$$\int_c \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\; f[-1+2i]$$

$$= 2\pi i\left[\frac{-1+2i+4}{(-1+2i)-(-1-2i)}\right]$$

$$= 2\pi i\left[\frac{3+2i}{-1+2i+1+2i}\right]$$

$$= 2\pi i\left[\frac{3+2i}{4i}\right]$$

$$\int_c \frac{z+4}{z^2+2z+5}dz = \frac{\pi}{2}(3+2i)$$

Exercise 6.1

1. Evaluate $\int_c \frac{z}{z+3}dz$ where C is $|z|=2$ Ans: 0

2. Evaluate $\int_c \frac{e^{4z}}{z(z-1)}dz$ where c is $|z-3|=1$ Ans: 0

3. Using Cauchy's integral formula, evaluate $\int_c \frac{z^2+1}{z^2-1}dz$ where c is a circle of unit radius and center at $z=1$, $|z|=\frac{3}{2}$ and $|z|=\frac{1}{2}$. Ans: (i) $2\pi i$ (ii) 0 (iii) 0

4. If $f(a)=\int \frac{13z^2+27z+15}{z-a}dz$ where c is the circle $|z|=2$ then find, $f'(1-i)$ and $f(1-i)$.

Ans $f(1-i)=2\pi i[42-53i]$, $f''(1-i)=52\pi i$

6.2 TAYLOR'S AND LAURENT'S SERIES EXPANSIONS

TAYLOR'S SERIES:

1. Taylor's series about $z=a$ is

$$f(z)= f(\text{a})+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+...+\frac{f^{(n)}(a)}{\text{n}!}(z-a)^n+...\text{to}\infty$$

2. Taylor's series about $z=0$ is

$$f(z)= f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+...+\frac{f^{(n)}(0)}{\text{n}!}z^n+...\text{to}\infty$$

LAURENT'S SERIES:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n},$$

Where $a_n=\frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-a)^{n+1}}dz,\ b_n=\frac{1}{2\pi i}\int_{c_2}\frac{f(z)}{(z-a)^{-n+1}}dz$

NOTE:

1. If $f(z)$ is analytic inside c_2, then the Laurent's series residues to the Taylor's series of $f(z)$ with centre a, since in this case all the co-efficient of negative powers in Laurent's zeros.

2. The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ consisting of positive integral powers of $(z-a)$ is called the analytic part of the Laurent's series, while $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called the principal part of the Laurent's series.

Worked Examples 6.2(A)

Example 1

Expand $\frac{z-1}{z+1}$ about z=1.

Solution:

Let $f(z)=\frac{z-1}{z+1}$

Function	Value at z=1
$f(z)=\frac{z-1}{z+1}$	$f(1)=0$
$f'(z)=\frac{z+1-z+1}{(z+1)^2}=\frac{2}{(z+1)^2}$	$f'(1)=\frac{2}{4}=\frac{1}{2}$
$f''(z)=\frac{-4}{(z+1)^3}$	$f''(1)=\frac{-4}{8}=-\frac{1}{2}$
…	…

By Taylor's series, $f(z)=f(a)+(z-a)\frac{f'(a)}{1!}+(z-a)^2\frac{f''(a)}{2!}+...$

$$\Rightarrow f(z)=0+(z-1)\frac{1}{2}+(z-1)^2\frac{\left(\frac{-1}{2}\right)}{2}+...$$

$$\therefore f(z)=\frac{(z-1)}{2}-\frac{1}{4}(z-1)^3+...$$

Example 2

Expand cosz at $z=\frac{\pi}{4}$ as a Taylors series

Solution:

Function	**Value at z=π/4**
f(z)=cosz	$f'(\pi/4)=1/\sqrt{2}$
$f'(z)=-\sin z$	$f'(\pi/4)=-1/\sqrt{2}$
$f''(z)=-\cos z$	$f''(\pi/4)=-1/\sqrt{2}$
$f'''(z)=\sin z$	$f'''(\pi/4)=1/\sqrt{2}$
…	…

Taylor's series about z = π/4 is

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+...$$

$$\cos z = \frac{1}{\sqrt{2}} + \frac{z-\pi/4}{1!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^3}{3!}\left(\frac{1}{\sqrt{2}}\right) + \ldots$$

Example 3

Expand the following function in Laurent's series $f(z) = \frac{e^z}{(z-1)^2}$ about z=1

Solution:

Given $f(z) = \frac{e^z}{(z-1)^2}$

Here $z = 1$ is a singular point. To expand f(z) in Laurent's series about $z = 1$ it is enough if we expand f(z) in powers of z-1.

put $z-1 = u$ (or) $z = u+1$

$$\therefore \frac{e^z}{(z-1)^2} = \frac{e^{1+u}}{u^2} = \frac{e^1 e^u}{u^2}$$

$$= \frac{e}{u^2}\left[1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots\ldots\ldots\ldots\right]$$

$$= e\left[\frac{1}{u^2} + \frac{1}{u} + \frac{1}{2!} + \frac{u}{3!} + \ldots\ldots\ldots\ldots\right]$$

$$f(z) = e\left[\frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \ldots\ldots\ldots\right] \text{When } z \neq 1$$

Example 4

Expand $\frac{1}{z+2}$ at z = 1 as a Taylor's series

Solution:

Function	**Value at z= 1**
$f(z) = \frac{1}{z+2}$	$\frac{1}{3}$
$f'(z) = \frac{-1}{(z+2)^2}$	$\frac{-1}{9}$
$f''(z) = \frac{2}{(z+2)^3}$	$\frac{2}{27}$
$f'''(z) = \frac{-6}{(z+2)^4}$	$\frac{-6}{81} = \frac{-2}{27}$

Taylor's series about z = 1 is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

$$\frac{1}{z+2} = \frac{1}{3} - \frac{1}{9}(z-1) + \frac{2}{27}\frac{(z-1)^2}{2!} - \frac{2}{27}\frac{(z-1)^3}{3!} + \ldots$$

$$\frac{1}{z+2} = \frac{1}{3} - \frac{(z-1)}{9} + \frac{(z-1)^2}{27} - \frac{(z-1)^3}{81} + \ldots$$

Example 6

Expand $f\left(z\right)=\dfrac{1}{z^{2}}$ as a Taylor series about the point $z=2$.

Solution:

Function	Value at z= 2
$f(z)=\dfrac{1}{z^{2}}$	$\dfrac{1}{4}$
$f^{'}(z)=\dfrac{-2}{(z)^{3}}$	$\dfrac{-2}{8}=\dfrac{-1}{4}$
$f^{''}(z)=\dfrac{6}{(z)^{4}}$	$\dfrac{6}{16}=\dfrac{3}{8}$
$f^{'''}(z)=\dfrac{-24}{(z)^{5}}$	$\dfrac{-24}{32}=\dfrac{-3}{4}$

Taylor's series about z = 1 is

$$f(z)=f(a)+\frac{f^{'}(a)}{1!}(z-a)+\frac{f^{''}(a)}{2!}(z-a)^{2}+\frac{f^{'''}(a)}{3!}(z-a)^{3}+\ldots$$

$$\text{Here}\, f(z)=f(2)+\frac{f^{'}(2)}{1!}(z-2)+\frac{f^{''}(2)}{2!}(z-2)^{2}+\frac{f^{'''}(2)}{3!}(z-2)^{3}+\ldots$$

$$f\left(z\right)=\frac{1}{4}+\frac{\left(\frac{-1}{4}\right)}{1!}(z-2)+\frac{\left(\frac{3}{8}\right)}{2!}(z-2)^{2}+\frac{\left(\frac{-3}{4}\right)}{3!}(z-2)^{3}+\ldots$$

<u>Worked Examples 6.2(B)</u>

Example 1

Expand the function $f(z)=\dfrac{z^{2}-1}{(z+2)(z+3)}$ in Laurent's series for $|z|>3$ and $2<|z|<3$.

Solution:

$$Let\ \ f(z)=\frac{z^{2}-1}{(z+2)(z+3)}$$

$$\left(z+2\right)\left(z+3\right)=z^{2}+5z+6$$

$$z^{2}+5z+6\ \overline{\Big)\ z^{2}-1\ }\ \ 1$$
$$\underline{z^{2}+5z+6}$$
$$-5z-7$$

$$\frac{z^{2}-1}{(z+2)(z+3)}=1+\frac{-5z-7}{(z+2)(z+3)}$$

$$\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{(z+2)}+\frac{B}{(z+3)}$$

$$-5z-7=A\left(z+3\right)+B\left(z+2\right)$$

$$Put\ \ z=-2$$

$$10-7=A(1)+B(0)$$
$$A=3$$
$$Put\ \ z=-3$$
$$15-7=A(0)+B(-1)$$
$$B=-8$$
$$\frac{z^2-1}{(z+2)(z+3)}=1+\frac{-5z-7}{(z+2)(z+3)}=1+\frac{3}{(z+2)}+\frac{-8}{(z+3)}$$

(i) Given $|z|>3$

$$\therefore f(z)=1+\frac{3}{(z+2)}+\frac{-8}{(z+3)}$$
$$=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$
$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$
$$=1+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\ldots\right]-\frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\ldots\right]$$
$$=1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

(ii) Given $2<|z|<3$

$$2<|z|\ and\ |z|<3$$
$$\frac{2}{|z|}<1\ and\ \frac{|z|}{3}<1$$
$$\left|\frac{2}{z}\right|<1\ and\ \left|\frac{z}{3}\right|<1$$
$$\therefore f(z)=1+\frac{3}{(z+2)}+\frac{-8}{(z+3)}$$
$$=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$
$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$
$$=1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$=1+\sum_{n=0}^{\infty}(-1)^n\left[\frac{3(2^n)}{z^{n+1}}-\frac{8}{3^{n+1}}\right]z^n$$

Example 2

Expand $f\left(z\right)=\dfrac{z^2}{\left(z+1\right)\left(z+3\right)}$ in a Laurent's series valid in $\left|z\right|>3$ and $1<\left|z\right|<3$.

Solution:

Given $f(z)=\dfrac{z^2}{(z+1)(z+3)}$

$\left(z+1\right)\left(z+3\right)=z^2+4z+3$

$$z^2+4z+3\ \left|\ \begin{array}{l} \quad\ 1 \\ \hline z^2 \\ z^2+4z+3 \\ \hline \quad -4z-3 \end{array}\right.$$

$$\frac{z^2}{(z+1)\,(z+3)}=1+\frac{-4z-3}{(z+1)\,(z+3)}$$

Given $f(z)=\dfrac{z^2}{(z+1)(z+3)}$

By Partial Fractions method

$$\frac{-4z-3}{(z+1)\,(z+3)}=\frac{A}{z+1}+\frac{B}{z+3}\rightarrow(1)$$

$-4z-3=A(z+3)+B(z+1)$

put $z=-1$

$1=A(-1+3)+B(0)$

$1=2A$

$\therefore A=\dfrac{1}{2}$

put $z=-3$

$9=A(0)+B(-3+1)$

$9=-2B$

$\therefore B=-\dfrac{9}{2}$

Sub A,B *in* (1)

$$\frac{z^2}{(z+1)(z+3)}=1+\frac{\frac{1}{2}}{z+1}+\frac{\frac{-9}{2}}{z+3}$$

$$\frac{z^2}{(z+1)(z+3)}=1+\frac{1}{2}\left(\frac{1}{z+1}\right)-\frac{9}{2}\left(\frac{1}{z+3}\right)$$

Given region:

$(i)\left|z\right|>3\ ie\ 3<\left|z\right|$

$$\left|\frac{3}{z}\right|<1 \Rightarrow \left|\frac{1}{z}\right|<1$$

$$f(z)=1+\frac{1}{2z}\left(\frac{1}{1+\frac{1}{z}}\right)-\frac{9}{2z}\left(\frac{1}{1+\frac{3}{z}}\right)$$

$$=1+\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{9}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}-\frac{9}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^{n}$$

$(ii) 1<|z|<3$ *ie* $1<|z|$; $|z|<3$

$$\frac{1}{|z|}<1 ; \frac{|z|}{3}<1 \quad \left|\frac{1}{z}\right|<1 ; \left|\frac{z}{3}\right|<1$$

$$f(z)=1+\frac{1}{2z}\left(\frac{1}{1+\frac{1}{z}}\right)-\frac{9}{2(3)}\left(\frac{1}{\frac{z}{3}+1}\right)$$

$$=1+\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{3}{2}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}-\frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n}$$

Example 3

Find the Laurent's series expansion of $f(z)=\dfrac{1}{z^{2}+5z+6}$ valid in the region $1<|z+1|<2$.

Solution:

Let $f(z)=\dfrac{1}{z^{2}+5z+6}$

$$=\frac{1}{(z+2)(z+3)}$$

$$\frac{1}{(z+2)(z+3)}=\frac{A}{(z+2)}+\frac{B}{(z+3)} \rightarrow (1)$$

$$\frac{1}{(z+2)(z+3)}=\frac{A(z+3)+B(z+2)}{(z+2)(z+3)}$$

$$\Rightarrow 1=A(z+3)+B(z+2)$$

Put $z=-2$, then $1=A$

Put $z=-3$, then $1=-B \Rightarrow B=-1$

Substituting A and B in (1),

$$f(z)=\frac{1}{(z+2)(z+3)}=\frac{1}{z+2}-\frac{1}{z+3} \rightarrow (2)$$

Given $1<|z+1|<2$

Let $u = z+1$

$\Rightarrow z = u-1$

Now $1<|z+1|<2 \Rightarrow 1<|u|<2$

i.e., $1<|u|$ and $|u|<2$

i.e., $\left|\frac{1}{u}\right|<1$ and $\left|\frac{u}{2}\right|<1$

$$(2) \Rightarrow f(z) = \frac{1}{u+1} - \frac{1}{u+2}$$

$$= \frac{1}{u\left[1+\frac{1}{u}\right]} - \frac{1}{2\left[\frac{u}{2}+1\right]}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}\right]^{-1} - \frac{1}{2}\left[\frac{u}{2}+1\right]^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{u}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{u}{2}\right)^n$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z+1}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+1}{2}\right)^n$$

Example 4

Expand $f(z) = \frac{1}{(z-1)(z-2)}$ as a Laurent's series in z in the Regions $|z-1|<1$, $1<|z|<2$, $|z|<1, |z|>2$.

Solution:

Given

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

1 = A(z – 2) + B(z – 1)

Put z = 1 we get
1 = A(1 – 2) + B(0)
A = -1

Put z = 2 we get
1 = A(0) + B(2 - 1)
B = 1

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

(i) $|z-1|<1$

Let u = z – 1,

z = u + 1

$|u|<1$

$$\mathrm{f}(\mathrm{z}) = \frac{-1}{\mathrm{u}} + \frac{1}{\mathrm{u}-1} = \frac{-1}{\mathrm{u}} - \frac{1}{\mathrm{u}+1}$$

$$= \frac{-1}{u} - (1-u)^{-1}$$

$$= \frac{-1}{n} - \sum_{n=0}^{\infty} u^n$$

$$= \frac{-1}{z-1} - \sum_{n=0}^{\infty} (z-1)^n$$

ii) $1 < |z| < 2$

$1 < |z|$ and $|z| < 2$ $\qquad \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{-1}{z\left(1-\frac{1}{2}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

Valid in the region $1 < |z| < 2$

iii) $|z| < 1$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)} = \frac{1}{(1-z)} - \frac{1}{(2-z)}$$

$$= \frac{1}{(1-z)} - \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

iv) $|z| > 2$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{-1}{\left(z-\frac{1}{2}\right)} + \frac{1}{2\left(1-\frac{2}{z}\right)}$$

$$= -\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

Example 5

Expand $f(z)=\dfrac{6z+5}{(z+1)z(z-2)}$ in Laurent's Series valid for $1<|z+1|<3$.

Solution:

$$f(z)=\frac{6z+5}{(z+1)\,z(z-2)}$$

$$\frac{6z+5}{(z+1)\,z(z-2)}=\frac{A}{z}+\frac{B}{z-2}+\frac{C}{z+1}$$

$$6z+5=A(z-2)(z+1)+Bz(z+1)+Cz(z-2)$$

Put z = 2, we get

$$17=A(0)+B(2)(2+1)+C(0)$$

$$17=6B$$

$$B=17/6$$

Put z = 0, we get

$$5=A(-2)(1)+0+0$$

$$A=-5/2$$

Put z = -1, we get

$$-1=\mathrm{A}(0)+\mathrm{B}(0)+\mathrm{C}(-1)(-1-2)$$

$$-1=3\mathrm{C}\Rightarrow \mathrm{C}=-1/3$$

$$\therefore\ f(z)=\frac{-5/2}{z}+\frac{17/6}{z-2}-\frac{1/3}{z+1}=-\frac{5}{2}\left(\frac{1}{z}\right)+\frac{17}{6}\left(\frac{1}{z-2}\right)-\frac{1}{3}\left(\frac{1}{z+1}\right)\ldots(1)$$

Given region is $1<|z+1|<3$

Let u = z+1

z = u-1

i.e, 1 < |u| < 3

i.e., 1 < |u| and |u| < 3

i.e, $\left|\dfrac{1}{u}\right|<1$ and $\left|\dfrac{u}{3}\right|<1$

$$(1)\Rightarrow f(z)=-\frac{5}{2}\left(\frac{1}{u-1}\right)+\frac{17}{6}\left(\frac{1}{u-3}\right)-\frac{1}{3}\left(\frac{1}{u}\right)$$

$$=-\frac{5}{2}\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}+\left(\frac{17}{6}\right)\left(\frac{-1}{3}\right)\left(1-\frac{u}{3}\right)^{-1}-\left(\frac{1}{3}\right)\left(\frac{1}{u}\right)$$

$$=-\frac{5}{2}\frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^{n}-\frac{17}{18}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^{n}-\left(\frac{1}{3}\right)\left(\frac{1}{u}\right)$$

$$=-\frac{5}{2}\frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n}-\frac{17}{18}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^{n}-\left(\frac{1}{3}\right)\left(\frac{1}{z+1}\right)$$

Example 6

Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in a Laurent's series valid in the region $1<|z|<3$ and $|z|>3$ and $0<|z+1|<2$.

Solution:

$$Let\ \ f(z)=\frac{1}{(z+1)(z+3)}$$

$$\frac{1}{(z+1)(z+3)}=\frac{A}{(z+1)}+\frac{B}{(z+3)}$$

$$1=A(z+3)+B(z+1)$$

Put z = -3 we get

1 = A (0) + B (– 3 +1)

1 = -2B

$$B=\frac{-1}{2}$$

Put z = – 1 we get

1 = A (-1+3) + B (0)

1 = 2A

$$A=\frac{1}{2}$$

i.e., 1 < |z| ; |z| < 3

$$i.e., \left|\frac{1}{z}\right|<1;\ \left|\frac{z}{3}\right|<1$$

$$\therefore\ f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$

$$=\frac{1}{2z\left(1+\frac{1}{z}\right)}-\frac{1}{2(3)\left(1+\frac{z}{3}\right)}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n}$$

Example 7

Find the Laurent's series expansion of $f(z)=\dfrac{7z-2}{(z-2)(z+1)}$ valid in the regions $|z+1|<1$ and $|z+1|>3$

Solution:

$$\text{Given}\quad f(z)=\frac{7z-2}{(z-2)(z+1)}=\frac{A}{z-2}+\frac{B}{z+1}$$

$$7z-2=A(z+1)+B(z-2)$$

$Put\ z = 2 we\ get$ $\quad$ $Put\ z = -1 we\ get$

$14-2 = A(2+1)+B(0)$ $\quad$ $-7-2 = A(0)+B(-2-1)$

$12 = 3A$ $\quad$ $-9 = -3B$

$A = 4$ $\quad$ $B = 3$

$$f(z) = \frac{4}{z-2}+\frac{3}{z+1}$$

In $|z+1|<1$

Take u = z+1 $\Rightarrow$z=u-1 and $|u|<1$

$$f(z) = \frac{4}{z-2}+\frac{3}{z+1} = \frac{4}{u-1-2}+\frac{3}{u-1+1} = \frac{4}{u-3}+\frac{3}{u}$$

$$f(z) = \frac{4}{-3\left(1-\frac{u}{3}\right)}+\frac{3}{u}$$

$$= \frac{-4}{3}\left(1-\frac{u}{3}\right)^{-1}+\frac{3}{u}$$

$$= \frac{-4}{3}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^{n}+\frac{3}{n}$$

$$= \frac{-4}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^{n}+\frac{3}{z+1}$$

In $|z+1|>3$, Take u = z+1 $\Rightarrow$z=u-1 and $|u|>3 \Rightarrow 3<|u| \Rightarrow \frac{3}{|u|}<1$

$$f(z) = \frac{4}{z-2}+\frac{3}{z+1} = \frac{4}{u-1-2}+\frac{3}{u-1+1} = \frac{4}{u-3}+\frac{3}{u}$$

$$f(z) = \frac{4}{u\left(1-\frac{3}{u}\right)}+\frac{3}{u} = \frac{4}{u}\left(1-\frac{3}{u}\right)^{-1}+\frac{3}{u}$$

$$= \frac{4}{u}\sum_{n=0}^{\infty}\left(\frac{3}{n}\right)^{n}+\frac{3}{n}$$

$$= \frac{4}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^{n}+\frac{3}{z+1}$$

Example 8

Represent the function $f(z) = \frac{4z+3}{z(z-3)(z+2)}$ in Laurent's series (i) When $0<|z|<1$, (ii) when $1<|z|<2$ (iii) when $|z|>2$.

Solution:

$$f(z)=\frac{4z+3}{(z+2)\,z(z-3)}$$

$$\frac{4z+3}{(z+2)\,z(z-3)}=\frac{A}{z}+\frac{B}{z-3}+\frac{C}{z+2}$$

$$4z+3=A(z-3)(z+2)+Bz(z+2)+Cz(z-3)$$

Put z = 0, we get

$$3=A(-3)\,(2)+0+0$$

$$A=-1/2$$

Put z = 3, we get

$$15=B(3)\,(5)$$

$$B=1$$

Put z = -2, we get

$$-8+3=C(-2)\,(-5)$$

$$-5=10C\ \Rightarrow C=-1/2$$

$$\therefore f(z)=\frac{-1/2}{z}+\frac{1}{z-3}-\frac{1/2}{z+2}=-\frac{1}{2}\left(\frac{1}{z}\right)+\left(\frac{1}{z-3}\right)-\frac{1}{2}\left(\frac{1}{z+2}\right)$$

(i) $0<|z|<1$,

$$(1)\Rightarrow f(z)=-\frac{1}{2z}-\frac{1}{3\left(1-\frac{z}{3}\right)}-\frac{1}{4\left(1+\frac{z}{2}\right)}$$

$$=-\frac{1}{2z}-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}-\frac{1}{4}\left(1+\frac{z}{3}\right)^{-1}$$

$$=-\frac{1}{2z}-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{z}{2}\right)^{n}$$

(ii) $1<|z|<2$ $\quad (1)\Rightarrow f(z)=-\frac{1}{2z}-\frac{1}{3\left(1-\frac{z}{3}\right)}-\frac{1}{4\left(1+\frac{z}{2}\right)}$

$$=-\frac{1}{2z}-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}-\frac{1}{4}\left(1+\frac{z}{3}\right)^{-1}$$

$$=-\frac{1}{2z}-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{z}{2}\right)^{n}$$

(iii) $|z|>2$ $\quad (1)\Rightarrow f(z)=-\frac{1}{2z}+\frac{1}{z\left(1-\frac{3}{z}\right)}-\frac{1}{2z\left(1+\frac{2}{z}\right)}$

$$= -\frac{1}{2z} + \frac{1}{z}\left(1 - \frac{3}{z}\right)^{-1} - \frac{1}{2z}\left(1 + \frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^{n} - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{2}{z}\right)^{n}$$

Example 9

Find the Laurent's series of $f(z) = \frac{3z-2}{z(z^2-4)}$ valid in the region $2 < |z+2| < 4$.

Solution:

Given $f(z) = \frac{3z-2}{z(z^2-4)} = \frac{3z-2}{z(z+2)(z-2)}$

Here $\frac{3z-2}{z(z+2)(z-2)} = \frac{A}{z} + \frac{B}{z+2} + \frac{C}{z-2}$

$$3z - 2 = A(z+2)(z-2) + Bz(z-2) + Cz(z+2)$$

$Put\ z = 0$	$Put\ z = 2$	$Put\ z = -2$
$-2 = A(2)(-2)$	$4 = 2C(4)$	$-8 = -2B(-4)$
$A = \frac{1}{2}$	$C = \frac{1}{2}$	$B = -1$

$$f(z) = \frac{\left(\frac{1}{2}\right)}{z} + \frac{-1}{z+2} + \frac{\left(\frac{1}{2}\right)}{z-2} = \frac{1}{2z} - \frac{1}{z+2} + \frac{1}{2(z-2)}$$

$take\, u = z + 2 \Rightarrow z = u - 2$

$given\ 2 < |z+2| < 4 \Rightarrow\ 2 < |u| < 4$

$$2 < |u| \quad and \quad |u| < 4$$

$$\frac{2}{|u|} < 1 \quad and \quad \frac{|u|}{4} < 1$$

$$\left|\frac{2}{u}\right| < 1 \quad and \quad \left|\frac{u}{4}\right| < 1$$

$$f(z) = \frac{1}{2(u-2)} - \frac{1}{u} + \frac{1}{2(u-4)}$$

$$= \frac{1}{2u\left(1 - \frac{2}{u}\right)} - \frac{1}{u} + \frac{1}{2(-4)\left(1 - \frac{u}{4}\right)}$$

$$= \frac{1}{2u}\left(1 - \frac{2}{u}\right)^{-1} - \frac{1}{u} - \frac{1}{8}\left(1 - \frac{u}{4}\right)^{-1}$$

$$= \frac{1}{2u}\sum_{n=0}^{\infty}\left(\frac{2}{u}\right)^{n} - \frac{1}{n} - \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{u}{4}\right)^{n}$$

Example 10

Expand $f(z)=\dfrac{z}{(z-1)(z-2)}$ as a Laurent's series in z in the Regions. $|z-1|<1, 1<|z|<2$, $|z|<1$, $|z|>2$.

Solution:

Let $f(z)=\dfrac{z}{(z-1)(z-2)}$

The singular points are z = 1 and z = 2, f(z) is not analytic at these points.

$$f(z)=\frac{z}{(z-1)(z-2)}-----------(1)$$

Let $\dfrac{z}{(z-1)(z-2)}=\dfrac{A}{(z-1)}+\dfrac{B}{(z-2)}------(2)$

$z=A(z-2)+B(z-1)$

$Put\ z=1, We\ get\ A=-1$

$Put\ z=2, We\ get\ B=2$

$$(2)\Rightarrow \frac{z}{(z-1)(z-2)}=\frac{-1}{(z-1)}+\frac{2}{(z-2)}$$

$$(1)\Rightarrow f(z)=\frac{-1}{(z-1)}+\frac{2}{(z-2)}$$

$(i)|z|<1$

$$f(z)=\frac{-1}{(z-1)}+\frac{2}{2(1-z/2)}$$

$$=(-1)(1-z)^{-1}+(1-z/2)^{-1}$$

$$=-\sum_{n=0}^{\infty}z^n+\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

(ii) $1<|z|<2$

$Consider\ 1<|z|, here\ z\ is\ \max imum. So\ we\ have\ to\ take\ out\ z.$

$Consider\ |z|<2, here\ 2\ is\ \max imum. So\ we\ have\ to\ take\ out\ 2.$

$$f(z)=\frac{-1}{(z-1)}+\frac{2}{(z-2)}$$

$$=\frac{-1}{z(1-1/z)}+\frac{2}{2(1-z/2)}$$

$$=\frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n+\frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

(iii) $|z|>2$

Here z is maximum, so we can take out z.

$$f(z)=\frac{-1}{(z-1)}+\frac{2}{(z-2)}$$

$$= \frac{-1}{z(1-1/z)} + \frac{2}{z(1-2/z)}$$

$$f(z) = \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} = \frac{-1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+...\right) + \frac{2}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+...\right)$$

(iv) $|z-1|<1$

$Let\, u = z-1, u-1 = z-2$

$Given |u| < 1$

$$f(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)}$$

$$= \frac{-1}{u} + \frac{2}{u-1}$$

$$= \frac{-1}{u} - 2(1-u)^{-1}$$

$$= \frac{-1}{u} - 2(1+u+u^2+u^3+...)$$

$$f(z) = \frac{-1}{(z-1)} - 2\left(1-(z-1)+(z-1)^2-(z-1)^3+.....\right)$$

Exercise 6.2

1. Expand $\frac{e^z}{(z-1)^3}$ at z = 1 as a Taylor's series,

   Ans: $f(z) = e^z\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3}...\right]$

2. Expand $f(z) = \tanh z$ as a Taylor series about the point $z = 0$.

   Ans: $f(z) = z - \frac{z^3}{3} + ...$

3. Find the Laurent's series which represents the function $\frac{z}{(z+1)(z+2)}$ in (i) $|z|<1$, (ii) $1<|z|<2$, (iii) $|z|>2$, (iv) $|z+1|<1$

   Ans (i) $f(z) = \frac{z}{2} - \frac{3}{4}z^2 + \frac{7}{8}z^3 - .....$ (ii) $f(z) = \frac{-1}{z}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n + \sum_{n=1}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$

   (iii) $f(z) = \frac{-1}{z}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n + \frac{2}{z}\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n$

   (iv) $f(z) = \frac{-1}{(z+1)} + 2\left(1-(z+1)+(z+1)^2-(z+1)^3+.....\right)$

4. Find the Laurent's series expansion of $\frac{7z-2}{z(z-2)(z+1)}$, in $1<|z+1|<3$.

   Ans: $f(z) = -\frac{2}{z+1} + \sum_{n=2}^{\infty}\frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^n}$

6.3 RESIDUES - CAUCHY'S RESIDUE THEOREM

SINGULARITIES – CLASSIFICATION

ZEROS OF AN ANALYTIC FUNCTION:

If a function $f(z)$, analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called zero of $f(z)$

SIMPLE ZERO:

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

ZERO OF ORDER n

If $f(z_0) = f'(z_0) = ... = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$, then z_0 is called a zero of order n.

ISOLATED SINGULARITY:

A point $z = z_0$ is said to be isolated singularity of $f(z)$, if

(i) $f(z)$ is not analytic at $z = z_0$

(ii) There exists a neighbourhood of $z = z_0$ containing no other singularity

Example: $f(z) = \dfrac{1}{z}$

This function is analytic everywhere except at $z = 0$

$\therefore z = 0$ is an isolated singularity

REMOVABLE SINGULARITY:

If the principal part of $f(z)$ contains no term $i.e., b_n = 0$ for all n, then the singularity $z = z_0$ is known as the removable singularity of $f(z)$.

$$\therefore\ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

(or)

A singular point $z = z_0$ is called a removable singularity of $f(z)$, if $\lim\limits_{z \to z_0} f(z)$ exists finitely.

Example: $f(z) = \dfrac{\sin z}{z}$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1$$

$\therefore z = 0$ is a removable singularity.

POLES:

If we can find a positive integer n such that $\lim\limits_{z \to a} (z - a)^n f(z) \neq 0$, then $z = a$ is called a pole of order n for $f(z)$.

(or)

An analytic function $f(z)$ with a singularity at $z=a$ if $\lim_{z\to a} f(z)=\infty$, then $z=a$ is a pole of $f(z)$. A Pole of order one is called a simple pole.

Example: If $f(z)=\dfrac{1}{(z-4)^2(z-3)^3(z-1)}$

Here, $z=1$ is a simple pole

$z=3$ is a pole of order 3

$z=4$ is a pole of order 2

ESSENTIAL SINGULARITY:

If the principal part contains an infinite number of non-zero terms, $z=z_0$ is known as an essential singularity.

Example: $f(z)=e^{\frac{1}{z}}=1+\dfrac{\frac{1}{z}}{\lfloor 1}+\dfrac{\left(\frac{1}{z}\right)^2}{\lfloor 2}+\ldots$ has $z=0$ as an essential singularity.

Since, $f(z)$ is an infinite series of negative powers of z

ESSENTIAL SINGULARITY:

If $z=z_0$ is an isolated singular point of $f(z)$, we can find the Laurent's series of $f(z)$ about $z=z_0$

$$(i.e.,)\, f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

The co-efficient of $\dfrac{1}{z-z_0}$ in the above expansion is called the residue of $f(z)$ at $z=z_0$

(*i.e.*,) b_1 is the residue of $f(z)$ at $z=z_0$

From the definition of b_n, given in the theorem of Laurent's series

$$b_n=\frac{1}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{-n+1}}dz$$

$$b_1=\frac{1}{2\pi i}\int_c f(z)dz$$

Residue of $f(z)$ at $z=z_0$ may be denoted by Res $[f(z), z_0]$

EVALUATION OF RESIDUES:

(i) Residue at a pole of order m.

If $z=z_0$ is a pole of order m, a simple formula to determine the residue is given by

$$\text{Res}\,[f(z), z_0]=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

(ii) Residue at a simple pole

If $z=z_0$ is a simple pole,

$$\text{Res}\,[f(z), z_0]=\lim_{z\to z_0}\left[(z-z_0)^m f(z)\right]$$

(iii) If $z = z_0$ is a simple pole of $f(z)$ and if

$$f(z) = \frac{\varphi(z)}{\psi(z)} \text{ then } \operatorname{Res}[f(z), z_0] = \frac{\varphi(z_0)}{\psi'(z_0)}$$

CAUCHY'S RESIDUE THEOREM

Statement: If $f(z)$ be analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $z_1, z_2, z_3, \ldots, z_n$ inside c, then

$$\int_c f(z)dz = 2\pi i\left[\text{sum of the residues of } f(z) \text{ at } z_1, z_2, \ldots, z_n\right]$$

$$= 2\pi i \sum_{i=1}^{n} R_i, \quad \text{where } R_i \text{ is the residue of } f(z) \text{ at } z = z_i$$

Worked Examples 6.3(A)

Example 1

Classify the singularity of $f(z) = e^{\frac{1}{z^2}}$

Solution:

Let $f(z) = e^{\frac{1}{z^2}}$

Here z = 0 as an essential singularity since f(z) is an infinite series of negative power of z.

$$f(z) = e^{\frac{1}{z^2}} = 1 + \frac{1/z^2}{1!} + \frac{\left(1/z^4\right)}{2!} + \ldots$$

Example 2

Discuss the singularity of $f(z) = \frac{\sin z}{z}$

Solution:

Given $f(z) = \frac{\sin z}{z}$

The function f (z) is not defined at z = 0

But by L' Hospital's rule

$$\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \frac{\cos z}{1} = 1$$

Since the limit exists and is finite, the singularity at z=0 is a removable singularity.

Example 3

Classify the singularity of the function $f(z) = z^2 \sin\left(\frac{1}{z}\right)$

Solution:

Given $f(z) = z^2 \sin\left(\frac{1}{z}\right)$

$$f(z) = z^2\left[\frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \ldots\right]$$

$$= z^2\left[\frac{1}{z} - \frac{1}{3!\left(z^3\right)} + \frac{1}{5!\left(z^5\right)} - \frac{1}{7!\left(z^7\right)} + \ldots\right]$$

$$= z - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}\left(\frac{1}{z^3}\right) - \ldots..$$

$z = 0$ is essential singularity of f(z)

Since If the principal part contains an infinite number of non-zero terms.
removable singularity.

Example 4

Identify and classify the singularity of the function $f(z) = e^{\frac{1}{z}}$

Solution:

Given $f(z) = e^{\frac{1}{z}} = 1 + \frac{1/z}{1!} + \frac{\left(1/z\right)^2}{2!} + \ldots\ldots\ldots\ldots$

$$e^{\frac{1}{z}} = 1 + \frac{1}{1!(z)} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \ldots$$

has $z = 0$ as an essential singularity

Example 5

Identify the type of the singularity of the function $\frac{1}{\cos z - \sin z}$

Solution:

Given $f(z) = \frac{1}{\cos z - \sin z}$

Here $\cos z - \sin z = 0$

$$\cos z = \sin z$$

$$\frac{\cos z}{\sin z} = 1$$

$$\tan z = 1 = \tan\frac{\pi}{4}$$

$$\tan z = \tan\frac{\pi}{4}$$

Therefore $z = \frac{\pi}{4}$ is an isolated essential singularity.

Example 6

Identify the type of the singularity of the function $\sin\left(\frac{1}{1-z}\right)$

Solution:

Given $f(z) = \sin\left(\frac{1}{1-z}\right)$

Here z = 1 is the only singular point in the finite plane.

$$\sin\left(\frac{1}{1-z}\right) = \frac{\left(\frac{1}{1-z}\right)}{1!} - \frac{\left(\frac{1}{1-z}\right)^3}{3!} + \frac{\left(\frac{1}{1-z}\right)^5}{5!} - \ldots.$$

$$f(z)=\left(\frac{1}{1-z}\right)-\frac{1}{3!(1-z)^3}+\frac{1}{5!(1-z)^5}-....$$

Hence z = 1 is an essential singularity

Example 7

Identify the type of singularity of $f(z)=\frac{\sin z}{z^2}$

Solution:

Given $f(z)=\frac{\sin z}{z^2}=\frac{1}{z^2}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-....\right)$

$$=\frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}-....$$

Hence z = 0 is an isolated singularity

Example 8

Classify the singularity of $f(z)=e^{e^{\frac{1}{z^2}}}$.

Solution:

Given that $f(z)=e^{e^{\frac{1}{z^2}}}=e^{\left[1+\frac{1}{1!}\frac{1}{z^2}+\frac{1}{2!}\left(\frac{1}{z^2}\right)^2+.....\right]}$

$$f(z)=1+\frac{1}{1!}\left[1+\frac{1}{1!}\frac{1}{z^2}+\frac{1}{2!}\left(\frac{1}{z^2}\right)^2+.....\right]+\frac{1}{2!}\left[1+\frac{1}{1!}\frac{1}{z^2}+\frac{1}{2!}\left(\frac{1}{z^2}\right)^2+.....\right]^2+....$$

Hence z = 0 is an isolated singularity

Example 9

Classify the singularities for the function $f(z)=\frac{z-\sin z}{z}$

Solution:

Given $f(z)=\frac{z-\sin z}{z}$

The function *f(z)* is not defined at *z=0*.

By L'Hospital rule,

$$\lim_{z\to 0}\frac{z-\sin z}{z}=\lim_{z\to 0}\frac{1-\cos z}{1}=1-1=0$$

Since the limit exists and is finite, the singularity at *z=0* is a removable singularity.

Example 6.3.10

Classify the Singularity of the function $\mathrm{f}(\mathrm{z})=\mathrm{z}^2\mathrm{e}^{\frac{1}{\mathrm{z}}}$

Solution:

$$f(z)=z^2e^{\frac{1}{z}}$$

$$=z^2\left[1+\frac{1/z}{1!}+\frac{(1/z)^2}{2!}+\frac{(1/z)^3}{3!}+...\right]$$

$$= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!\,z} + \frac{1}{4!\,z^2} + \ldots$$

$$= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!} z^{-1} + \frac{1}{4!} z^{-2} + \ldots$$

Principal part of the above Laurentz's series contains infinitely many terms [$z^{-1}, z^{-2}, \ldots$]

z = 0 is an essential singularity.

Example 11

Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its poles

Solution:

$$f(z) = \frac{e^{2z}}{(z+1)^2}$$

Here z = -1 is a pole of order 2

Formula Residue of pole of order m is $\operatorname{Res}(z = z_0) = \underset{z \to z_0}{Lt} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$

Here m = 2

$$\operatorname{Res}(z = -1) = \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} \left[(z+1)^2 \frac{e^{2z}}{(z+1)^2} \right] = \lim_{z \to -1} \frac{d}{dz} \left[e^{2z} \right] = \underset{z \to -1}{Lt}\, 2e^{2z} = 2e^{-2}$$

Example 12

Evaluate the residue of $f(z) = \tan z$ at its singularities.

Solution:

Let $\mathrm{f(z) = tanz} = \frac{\sin z}{\cos z} = \frac{\phi(z)}{\psi(z)}$

$z = \frac{\pi}{2}$ is a simple pole. And also, z= - π/2 is a simple pole lies inside c

$$\phi(z) = \sin z,\ \phi\left(\frac{\pi}{2}\right) = 1,$$

$$\psi(z) = \cos z,\ \psi'(z) = -\sin z,\ \psi'\left(\frac{\pi}{2}\right) = -1$$

$$\operatorname{Res}\left[f(z), \frac{\pi}{2} \right] = \frac{\phi\left(\frac{\pi}{2}\right)}{\psi'\left(\frac{\pi}{2}\right)} = \frac{1}{-1} = -1$$

$$\operatorname{Res}\left[f(z), \frac{-\pi}{2} \right] = \frac{\phi\left(\frac{-\pi}{2}\right)}{\psi'\left(\frac{-\pi}{2}\right)} = \frac{-1}{1} = -1$$

Example 12

Find the residue of $f(z) = \frac{z}{z^2+1}$ about each singularity.

Solution:

$$Let\ f(z)=\frac{z}{z^2+1}$$

Here $z=\pm i$ are simple poles

Formula for Residue of simple pole is $\operatorname{Re}s(z=a)=\lim_{z\to a}(z-a)\,f(z)$

$$\operatorname{Res}(z=i)=\lim_{z\to i}\ (z-i)f(z)=\lim_{z\to i}\ (z-i)\frac{z}{(z+i)(z-i)} \qquad =\lim_{z\to i}\frac{z}{(z+i)}=\frac{1}{2}$$

$$\operatorname{Res}(z=-i)=\lim_{z\to -i}\ (z+i)f(z)=\lim_{z\to -i}\ (z+i)\frac{z}{(z+i)(z-i)} \qquad =\lim_{z\to -i}\ \frac{z}{(z-i)}=\frac{1}{2}$$

Example 13

Find the residue of $f(z)=\dfrac{4}{z^3(z-2)}$ at its simple poles.

Solution:

Here z = 0 is a pole of order 3

z = 2 is a simple pole

Formula for Residue of simple pole is $\operatorname{Re}s(z=a)=\lim_{z\to a}(z-a)\,f(z)$

$$\operatorname{Res}(z=2)=\underset{z\to 2}{\mathrm{Lt}}(z-2)f(z)=\underset{z\to 2}{\mathrm{Lt}}(z-2)\frac{4}{z^3(z-2)}=\underset{z\to 2}{\mathrm{Lt}}\frac{4}{z^3}=\frac{4}{8}=\frac{1}{2}$$

Example 14

Find the residue at the essential singularity of $\dfrac{\sinh z}{z^4}$.

Solution:

$$\text{Given}\ \ f(z)=\frac{\sinh z}{z^4}$$

z = 0 is a pole of order 4.

$$\operatorname{Re}s\ f(z)]_{z=a}=\frac{1}{3!}\lim_{z\to a}\frac{d^3}{dz^3}[(z-a)^4 f(z)]$$

$$\operatorname{Re}s\ f(z)]_{z=a}=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}\left[z^4\frac{\sinh z}{z^4}\right]$$

$$=\frac{1}{3!}\lim_{z\to 0}\frac{d^2}{dz^2}[\cosh z]==\frac{1}{3!}\lim_{z\to 0}\frac{d}{dz}[\operatorname{Sin}h\,z]$$

$$=\frac{1}{6}\lim_{z\to 0}[\cosh z]$$

$$=\frac{1}{6}(1)=\frac{1}{6}$$

Example 15

Find the residue of $\dfrac{1-e^{2z}}{z^4}$ at the pole.

Solution:

Given $f(z)=\frac{1-e^{2z}}{z^4}$

Here z = 0 is a pole of order 4

Formula Residue of pole of order m is $\operatorname{Res}(z=z_0)=\underset{z\to z_0}{Lt}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$

here m = 4

$$\operatorname{Res}(z=0)=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}z^4\frac{1-e^{2z}}{z^4}$$

$$=\frac{1}{6}\lim_{z\to 0}\frac{d^3}{dz^3}\left(1-e^{2z}\right)$$

$$=\frac{1}{6}\lim_{z\to 0}\frac{d^2}{dz^2}\left(-2e^{2z}\right)=\frac{1}{6}\lim_{z\to 0}\frac{d}{dz}\left(-4e^{2z}\right)=\frac{1}{6}\lim_{z\to 0}\left(-8e^{2z}\right)$$

$$\operatorname{Res}(z=0)=\frac{-8}{6}=\frac{-4}{3}$$

Example 16

Evaluate $\int_c \tan z\,dz$ where c is $|z|=2$

Solution:

Let $f(z)=\tan z=\frac{\sin z}{\cos z}=\frac{\phi(z)}{\psi(z)}$

Singular points of the function f(z) are got by equating the denominator to zero

cosz = 0

$$z=(2n+1)\frac{\pi}{2},\ n=0,\pm1,\pm2...$$

$Z=\frac{\pi}{2}$ is a simple pole. lies inside c

z= - π/2 is a simple pole lies inside c

$\phi(z)=\sin z,$

$$\phi\left(\frac{\pi}{2}\right)=1,\phi\left(\frac{-\pi}{2}\right)=-1$$

$$\psi(z)=\cos z,\ \psi'(z)=-\sin z,$$

$$\psi'\left(\frac{\pi}{2}\right)=-1,\psi'\left(\frac{-\pi}{2}\right)=1$$

$$\operatorname{Res}\left[f(z),\frac{\pi}{2}\right]=\frac{\phi\left(\frac{\pi}{2}\right)}{\psi'\left(\frac{\pi}{2}\right)}=\frac{1}{-1}=-1$$

$$\operatorname{Res}\left[f(z),\frac{-\pi}{2}\right]=\frac{\phi\left(\frac{-\pi}{2}\right)}{\psi'\left(\frac{-\pi}{2}\right)}=\frac{-1}{1}=-1$$

By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i[\text{sum of the residues}] = 2\pi i[-1-1] = -4\pi i$$

Example 17

Evaluate $\oint_C (x^2 - y^2 + 2ixy)\, dz$, where C is the contour $|z| = 1$.

Solution:

Given that $\oint_C (x^2 - y^2 + 2ixy)\, dz$ on $|z| = 1$

Here z^2 is analytic and given curve $|z| = 1$ is closed curve

$$\text{Here } \oint_C (x^2 - y^2 + 2ixy)\, dz = \int_C z^2 dz = 0$$

Worked Examples 6.3(B)

Example 1

Use calculus of residues to find $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx$ where a, b > 0.

Solution:

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Consider $\int_C f(z)dz = \frac{1}{2}\int_C \frac{dz}{(z^2+a^2)(z^2+b^2)}$ where C is the upper half of the semi-circle Γ with the bounding diameter [-R, R].

Now, $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz$ ………….(1)

When $R \to \infty, \int_\Gamma f(z)dz \to 0$

$$(1) \Rightarrow \int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx$$

$$i.e, \int f(x)dx = \int_C f(z) = 2\pi i[\text{sum of the residues}] \text{ ……..(2)}$$

Where $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{(z+ai)(z-ai)(z+bi)(z-bi)}$

The poles of f(z) are the solutions of $(z+ai)(z-ai)(z+bi)(z-bi)=0$

$z=ai$ is a simple pole lies inside Γ

$z=bi$ is a simple pole lies inside Γ

$z=-ai$ is a simple pole lies outside Γ

z=-bi is a simple pole lies outside Γ

$$\left.\begin{matrix}\operatorname{Res} f(z)\\ at\ z = ai\end{matrix}\right\} = \lim_{z\to ai}(z-ai)f(z)$$

$$= \lim_{z\to ai} (z-ai)\left[\frac{1}{(z+ai)(z-ai)(z+bi)(z-bi)}\right]$$

$$= \frac{1}{(2ai)(a+b)i(a-b)\mathrm{i}}$$

$$= \frac{-1}{2ai(a^2-b^2)}$$

$$\left.\begin{matrix}\operatorname{Res} f(z) \\ at\ z = bi\end{matrix}\right\} = \lim_{z\to bi} (z-bi) f(z)$$

$$= \lim_{z\to bi} (z-bi)\left[\frac{1}{(z+ai)(z-ai)(z+bi)(z-bi)}\right]$$

$$= \frac{1}{(2bi)(\mathrm{b}+a)i(\mathrm{b}-a)\mathrm{i}}$$

$$= \frac{-1}{2bi(\mathrm{b}^2-a^2)}$$

Hence by Cauchy's Residues theorem

$$\int_c f(z)dz = 2\pi i\left[\text{sum of residues lies within } C\right]$$

$$= 2\pi i\left[\frac{1}{2i(a^2-b^2)}\left[\frac{-1}{a}+\frac{1}{b}\right]\right]$$

$$= 2\pi i\frac{1}{2i(a-b)(a+b)}\left[\frac{a-b}{ab}\right]$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)(\mathrm{x}^2+b^2)} = \frac{1}{2}\left[\frac{\pi}{ab(a+b)}\right] = \frac{\pi}{2ab(a+b)}$$

Example 2

Using Cauchy's residue theorem, evaluate $C\int_c \frac{4-3z}{z(z-1)(z-2)}dz$ Where C is the circle $|z| = \frac{3}{2}$

Solution:

Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$, $|z| = \frac{3}{2}$

Z = 0 lies inside $|z| = \frac{3}{2}$

Z = 1 lies inside $|z| = \frac{3}{2}$

Z = 2 lies outside $|z| = \frac{3}{2}$

Formula Residue of pole of order m is

$$\operatorname{Res}(\mathrm{z}=\mathrm{z}_0) = \lim_{\mathrm{z}\to\mathrm{z}_0}\left[(\mathrm{z}-\mathrm{z}_0)\mathrm{f(z)}\right]$$

$$\operatorname{Res}[f(z),0]=\lim_{z\to 0} z\left(\frac{4-3z}{z(z-1)(z-2)}\right)$$

$$=\lim_{z\to 0}\frac{4-3z}{(z-1)(z-2)}=\frac{4}{(0-1)(0-2)}=\frac{4}{2}=2$$

$$\operatorname{Res}[f(z),1]=\lim_{z\to 1}(z-1)\left(\frac{4-3z}{z(z-1)(z-2)}\right)$$

$$=\lim_{z\to 1}\frac{4-3z}{z(z-2)}=\frac{4-3}{(1)(1-2)}=\frac{1}{-1}=-1$$

By Cauchy residue theorem,

$$\int_c f(z)\,dz=2\pi i\ \text{[sum of the residues]}$$

$$=2\pi i[2-1]=2\pi i$$

Example 3

Evaluate $\int_c \frac{zdz}{(z^2+1)^2}$, where c is the circle $|z-1|=1$, using Cauchy's residue theorem.

Solution:

Let $f(z)=\frac{z}{(z^2+1)^2}=\frac{z}{[(z+i)(z-i)]^2}$

$z=i$ is a pole of order 2, lies inside $|z-i|=1$

$z=-i$ is a pole of order 2, lies outside $|z-i|=1$

Given $|z-i|=1$

Here centre i, i.e., $(0,1)$ and radius 1.

$$\operatorname{Res}[f(z),i]=\underset{z\to i}{lt}\frac{1}{1!}\frac{d}{dz}\left[\frac{(z-i)^2z}{(z+i)^2(z-i)^2}\right]$$

$$=\underset{z\to i}{lt}\frac{d}{dz}\left[\frac{z}{(z+i)^2}\right]$$

$$=\underset{z\to i}{lt}\left[\frac{(z+i)^2(1)-2z(z+i)}{(z+i)^4}\right]$$

$$=\underset{z\to i}{lt}\left[\frac{z+i-2z}{(z+i)^3}\right]=\underset{z\to i}{lt}\left[\frac{i-z}{(z+i)^3}\right]=0$$

$\therefore$ By Cauchy's residue theorem, $\int_C f(z)dz=2\pi i[\text{sum of the residues}]$

$$=2\pi i[0]=0$$

Example 4

Evaluate $\int\frac{z^2-2z}{(z+1)^2(z^2+4)}dz$ where c is the circle $|z|=3$ using Residue Theorem.

Solution:

Let $f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}=\frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}$

Pole	order	lies

z = -1 2 inside |z|=3

z = 2i 1 inside |z|=3

z =-2i 1 inside |z|=3

Formula Residue of pole of order m is

$$\operatorname{Res}\left(z=z_0\right)=\underset{z\to z_0}{Lt}\ \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[\left(z-z_0\right)^m f(z)\right] \text{Here m = 2}$$

$$\operatorname{Res}\left[f(z)\right]_{z=-1}=\lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z^2-2z}{(z+1)^2(z^2+4)}\right]$$

$$=\lim_{z\to -1}\frac{d}{dz}\left[\frac{z^2-2z}{(z^2+4)}\right]$$

$$=\lim_{z\to -1}\left[\frac{(z^2+4)(2z-2)-\left(z^2-2z\right)(2z)}{(z^2+4)^2}\right]$$

$$=\frac{((-1)^2+4)\left(2(-1)-2\right)-\left((-1)^2-2(-1)\right)\left(2(-1)\right)}{((-1)^2+4)^2}$$

$$=\frac{(5)(-4)+2(3)}{(1+4)^2}$$

$$=\frac{-20+6}{25}$$

$$\operatorname{Res}\left[f(z)\right]_{z=-1}=\frac{-14}{25}$$

$$\operatorname{Res}\left[f(z)\right]_{z=2i}=\lim_{z\to 2i}\left[(z-2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right]$$

$$=\lim_{z\to 2i}\left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]$$

$$=\left[\frac{(2i)^2-2(2i)}{(2i+1)^2(2i+2i)}\right]$$

$$=\frac{-4-4i}{(4i)(2i+1)^2}$$

$$=\frac{-1-i}{i(2i+1)^2}$$

$$=\frac{-i(-1-i)}{(2i+1)^2}$$

$$\operatorname{Res}\left[f(z)\right]_{z=2i}=\frac{(i-1)}{(2i+1)^2}$$

$$\operatorname{Res}\left[f(z)\right]_{z=-2i}=\lim_{z\to -2i}\left[(z+2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right]$$

$$= \lim_{z\to -2i}\left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]$$

$$= \left[\frac{(-2i)^2-2(-2i)}{(-2i+1)^2(-2i+2i)}\right]$$

$$= \frac{-4+4i}{(4i)(1-2i)^2}$$

$$= \frac{-1+i}{-i(1-2i)^2}$$

$$= \frac{i(-1+i)}{(1-2i)^2}$$

$$\operatorname{Res}\left[f(z)\right]_{z=2i} = \frac{-(i+1)}{(1-2i)^2}$$

By Cauchy's Residue theorem

$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz = 2\pi i\left[\begin{matrix}\text{sum of the residues at each of its}\\ \text{poles which lies inside } C\end{matrix}\right]$$

$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz = 2\pi i\left[\frac{-14}{25}+\frac{(i-1)}{(2i+1)^2}-\frac{(i+1)}{(1-2i)^2}\right]$$

Example 5

Using Cauchy's residue theorem evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$ where c is $|z-i|=2$.

Solution:

Given $f(z)=\frac{z-1}{(z+1)^2(z-2)}$

Here C is circle $|z-i|=2$ with center (0,1), and radius 2

z=-1 is a pole of order 2 and lies inside C

z=2 is a simple pole and lie outside C

By Cauchy's residue theorem $\int_c f(z)\,dz = 2\pi i$ [sum of the residues]

To find Res{f(z)} at z=-1

Formula Residue of pole of order m is

$$\operatorname{Res}(z=z_0) = \underset{z\to z_0}{Lt}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Here m = 2

$$\operatorname{Res}(z=-1) = \lim_{z\to -1}\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}(z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}$$

$$= \lim_{z\to -1} \frac{d}{dz}\frac{z-1}{z-2} = \lim_{z\to -1} \frac{(z-2)(1)-(z-1)(1)}{(z-2)^2} \qquad \because d\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$$

$$= \lim_{z\to -1} \frac{z-2-z+1}{(z-2)^2} = \lim_{z\to -1} \frac{-1}{(z-2)^2} = \frac{-1}{(-1-2)^2} = \frac{-1}{9}$$

$$\int_c f(z)\,dz = 2\pi i \text{ [sum of the residues]} = 2\pi i\left[\frac{-1}{9}\right] = \frac{-2\pi i}{9}$$

$$\int_c \frac{z-1}{(z+1)^2(z-2)}dz = \frac{-2\pi i}{9}$$

Exercise 6.3

1. Classify the singularity of $f(z) = \dfrac{e^{\frac{1}{z}}}{(z-a)^2}$, Ans: $z=0$ is a removable singularity
2. Classify the singularity of $f(z) = \dfrac{e^z}{z^2+4}$, Ans: $z=0$ is a removable singularity
3. Find the residue of $f(z) = \dfrac{4}{z^3(z-2)}$ at its simple poles. Ans : $\dfrac{1}{2}$
4. Find the residue at the essential singularity of $\dfrac{\sinh z}{z^4}$. Ans : $\dfrac{1}{6}$
5. Using Cauchy's residue theorem, evaluate $\int_c \dfrac{\sin \pi z + \cos \pi z}{(z+2)(z+1)^2}dz$ where c is $|z|=3$.

   Ans: $2\pi i(2-\pi)$
6. Using Cauchy's residue theorem, evaluate $\int_c \dfrac{1-2z}{z(z-1)(z-2)}dz$ Where C is the circle $|z| = \dfrac{3}{2}$

   Ans: $3\pi i$

6.4 EVALUATION OF REAL DEFINITE INTEGRALS AROUND THE UNIT CIRCLE

CONTOUR INTEGRATION:

The complex integration along the scro curve used in evaluation the definite integral is called contour integration

TYPE I: Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where f is rational function in $\cos\theta$ and $\sin\theta$. Suppose the given integral is $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$

Here, we shall choose the contour as the unit circle $C:|z|=1$ or $z=e^{i\theta}, 0\le\theta\le 2\pi$

On this circle $z=\cos\theta+i\sin\theta, \frac{1}{z}=\cos\theta-i\sin\theta$

$$z+\frac{1}{z}=2\cos\theta \Rightarrow \cos\theta=\frac{z^2+1}{2z}$$

$$\cos\theta=\frac{1}{2}\left[z+\frac{1}{z}\right]=\frac{1}{2}\left[\frac{z^2+1}{z}\right]$$

$$z-\frac{1}{z}=2i\sin\theta \Rightarrow \sin\theta=\frac{z^2-1}{2iz}$$

$$\sin\theta=\frac{1}{2i}\left[z-\frac{1}{z}\right]=\frac{1}{2i}\left[\frac{z^2-1}{z}\right]$$

Also, $z=e^{i\theta}; dz=ie^{i\theta}d\theta=izd\theta \Rightarrow d\theta=\frac{1}{iz}dz$

When these results are used in the complex integral, we get

$$\int_0^{2\pi} f(\cos\theta,\sin\theta)d\theta=\int_c f\left[\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{1}{iz}dz$$

Worked Examples 6.4(B)

Example 1

Evaluate $\int_0^{2\pi}\frac{\cos 2\theta}{5-4\cos\theta}d\theta$ using contour integration.

Solution:

Let $z=e^{i\theta}$ $\cos\theta=\frac{1}{2}\left[z+\frac{1}{z}\right]=\frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$dz=i\,e^{i\theta}\,d\theta$ $\sin\theta=\frac{1}{2i}\left[z-\frac{1}{z}\right]=\frac{1}{2i}\left[\frac{z^2-1}{z}\right]$

$=i\,z\,d\theta$

$d\theta=\frac{1}{i\,z}dz$

$z^2=(e^{i\theta})^2=e^{2i\theta}=\cos 2\theta+i\sin 2\theta$

Real part of $e^{2i\theta}=\cos 2\theta$

Given: $\int_0^{2\pi}\frac{\cos 2\theta}{5-4\cos\theta}d\theta=\text{R.P.}\int_c\frac{z^2}{5-4\left[\frac{z^2+1}{2z}\right]}\frac{1}{iz}dz$

where c is | z | = 1

$=\text{R.P.}\ \frac{1}{i}\int_c\frac{z}{5-2\left[\frac{z^2+1}{z}\right]}\,dz=\text{R.P.}\ \frac{1}{i}\int_c\frac{z}{\left[\frac{5z-2z^2-2}{z}\right]}\,dz$

$$= \text{R.P. } \frac{1}{i}\int_c \frac{z^2}{5z-2z^2-2}\,dz = \text{R.P. } \frac{1}{-i}\int_c \frac{z^2}{2z^2-5z+2}\,dz$$

$$= \text{R.P. } \frac{-1}{i}\int_c \frac{z^2}{2z^2-5z+2}\,dz$$

Let $f(z) = \frac{z^2}{2z^2-5z+2} = \frac{z^2}{(z-2)(2z-1)}$

Here z = 2 is a simple pole, lies outside c

z = 1/2 is a simple pole, lies inside c

Formula for Residue of simple pole is

$$\text{Res}(z=a) = \lim_{z\to a}[z-a]f(z)$$

$$\text{Res}\left(z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left[z-\frac{1}{2}\right]f(z)$$

$$= \lim_{z\to\frac{1}{2}}\left[\frac{2z-1}{2}\right]\left[\frac{z^2}{(z-2)(2z-1)}\right]$$

$$= \lim_{z\to\frac{1}{2}}\frac{1}{2}\left[\frac{z^2}{z-2}\right] = \frac{1}{2}\frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}-2\right)}$$

$$= \frac{1}{2}\frac{\left(\frac{1}{4}\right)}{\left(\frac{-3}{2}\right)} = \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{-2}{3}\right) = \frac{-1}{12}$$

Hence by Cauchy's residue theorem

$$\int_c f(z)\,dz = 2\pi i \text{ [sum of the residues]}$$

$$\int_c f(z)\,dz = 2\pi i\left[\frac{-1}{12}\right] = \frac{-\pi i}{6}$$

Therefore $\int_0^{2\pi}\frac{\cos 2\theta}{5-4\cos\theta}\,d\theta = \text{R.P.}\frac{-1}{i}\left[\frac{-\pi i}{6}\right] = \text{R.P.}.\frac{\pi}{6} = \frac{\pi}{6}$

Example 2

Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$ using contour integration.

Solution:

Given: $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$

Change θ to Z

$Let\ \ z = e^{i\theta}$

$dz = ie^{i\theta}d\theta = izd\theta$

$d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$$

Convert the given integral into a Contour integral

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_c \frac{1}{2+\frac{1}{2}\left[\frac{z^2+1}{z}\right]}\frac{dz}{iz}$$

$$= \int_c \frac{2z}{4z+z^2+1}\frac{dz}{iz}$$

$$= \frac{2}{i}\int_c \frac{dz}{z^2+4z+1} \rightarrow (1)$$

To find the Poles of f(z)

$$z^2+4z+1=0$$

$$z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm\sqrt{12}}{2}$$

$$z = -2\pm\sqrt{3}$$

Here $\alpha = -2+\sqrt{3} = -2+1.7321 = -0.268$

is a simple pole which lies inside C

Here $\beta = -2-\sqrt{3} = -2-1.7321 = -3.7321$

is a simple pole which lies outside C

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_c \frac{dz}{(z-\alpha)(z-\beta)}$$

$$f(z) = \frac{1}{(z-\alpha)(z-\beta)}$$

To find the Residues

Formula for Residue of simple pole is $\operatorname{Re}s(z=a) = \lim_{z\to a}(z-a)f(z)$

$$\operatorname{Re}s(z=\alpha) = \lim_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)} = \lim_{z\to\alpha}\frac{1}{(z-\beta)} = \frac{1}{(\alpha-\beta)}$$

$$= \frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

Hence by Cauchy's Residues theorem

$$\int_c f(z)dz = 2\pi i\left[sum\ of\ \ residues\ \ lies\ within\ C\right]$$

$$\int_c \frac{dz}{(z-\alpha)(z-\beta)} = 2\pi i\left[\frac{1}{2\sqrt{3}}\right] = \frac{\pi i}{\sqrt{3}}$$

To find the Value of the contour integral

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = 2\pi i\left(\frac{2}{i}\right)\left(\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

Example 3

Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}(0<a<1)$ using contour integration.

Solution:

On circle $|z|=1, z=e^{i\theta}, dz=ie^{i\theta}d\theta, d\theta=\frac{dz}{iz}$ *and* $\sin\theta=\frac{z^2-1}{2iz}$

The given integral $I=\int_C \frac{dz/iz}{1-2a\left(\frac{z^2-1}{2iz}\right)+a^2}$ where C is $|z|=1$

$$I=\int_C \frac{dz}{iz-az^2+a+ia^2z}$$

$$=\frac{1}{a}\int_c \frac{dz}{\frac{iz}{a}-z^2+1+iaz}$$

$$=\frac{-1}{a}\int_c \frac{dz}{-\frac{iz}{a}+z^2-1-iaz}$$

$$=-\frac{1}{a}\int_C \frac{dz}{z^2-i\left(a+\frac{1}{a}\right)z-1}$$

$$=-\frac{1}{a}\int_C \frac{dz}{(z-ia)\left(z-\frac{i}{a}\right)} \qquad ....(1)$$

The singularities of $(z-ia)\left(z-\frac{i}{a}\right)$ are z = ia,

and z = i/a. which are simple poles

Now $|ia|=|a|<1$ *as* $0<a<1$

Since *choose* $a=\frac{1}{2}, 0<\frac{1}{2}<1: \frac{1}{a}=2$

$a=\frac{1}{2}$ *lies inside* c, $\frac{1}{a}=2$ *lies outside* C

Hence the pole z = ia lies inside C , but z = i/a lies outside C

Formula for Residue of simple pole is $\operatorname{Re}s(z=a)=\lim_{z\to a}(z-a)f(z)$

$$\operatorname{Re}s(z=ia)=\lim_{z\to ia}\left[(z-ia)\frac{1}{(z-ia)(z-i/a)}\right]$$

$$=\lim_{z\to ia}\left[\frac{1}{(z-i/a)}\right]$$

$$=\frac{1}{ia-i/a}=\frac{1}{\frac{ia^2-i}{a}}=\frac{a}{ia^2-i}=\frac{a}{i(a^2-1)}=\frac{-ai}{-(i)^2(a^2-1)}=\frac{-ai}{(a^2-1)}$$

$$\operatorname{Re}s\left(z=ia\right)=\frac{ia}{1-a^2}$$

By Cauchy's residue theorem

$$\int_c f(z)\,dz = 2\pi i \ \text{[sum of the residues]}$$

$$\mathrm{I}=-\frac{1}{\mathrm{a}}\times 2\pi\mathrm{i}\times\frac{\mathrm{ia}}{1-\mathrm{a}^2}=\frac{2\pi}{1-\mathrm{a}^2}$$

Example 4

Evaluate $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}$ where $(a>b>0)$ using contour integration.

Solution:

Given $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}$

$$z=e^{i\theta} \qquad dz = i\,z\,d\theta$$

$$dz = i\,e^{i\theta}\,d\theta \qquad d\theta=\frac{dz}{i\,z}$$

And $\cos\theta=\frac{1}{2}\left[\frac{z^2+1}{z}\right]$

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\int_c\frac{1}{a+b\frac{1}{2}\left[\frac{z^2+1}{z}\right]}\frac{dz}{i\,z}$$

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\int_c\frac{2z}{2az+bz^2+b}\ \frac{dz}{i\,z}$$

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2}{i}\int_c\frac{1}{bz^2+2az+b}dz$$

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2}{ib}\int_c\frac{1}{z^2+\frac{2a}{b}z+1}dz\text{.............(1)}$$

From equation (1)

Here f(z) = $\frac{2}{ib}\frac{1}{z^2+\frac{2a}{b}z+1}$

$$equate\,Dr\,to\,zero\ \ i.e\ \ z^2+\frac{2a}{b}z+1=0$$

$$subsitute\,in\,z=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$where\ A=1;B=\frac{2a}{b};C=1$$

$$z=\frac{-\frac{2a}{b}\pm\sqrt{\left(\frac{2a}{b}\right)^2-4}}{2}$$

$$z=\frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}}{2}$$

$$z=\frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2-4b^2}{b^2}}}{2}$$

$$z=\frac{-\frac{2a}{b}\pm\frac{2}{b}\sqrt{a^2-b^2}}{2}$$

$$z=\frac{-a\pm\sqrt{a^2-b^2}}{b}$$

$$i.e\, z=\frac{-a+\sqrt{a^2-b^2}}{b} \;;\; z=\frac{-a-\sqrt{a^2-b^2}}{b}$$

Let $\alpha=\frac{-a+\sqrt{a^2-b^2}}{b} \;;\beta=\frac{-a-\sqrt{a^2-b^2}}{b}$

$$\alpha-\beta=\frac{-a+\sqrt{a^2-b^2}}{b}-\left\{\frac{-a-\sqrt{a^2-b^2}}{b}\right\}$$

$$\alpha-\beta=\frac{2\sqrt{a^2-b^2}}{b}$$

Therefore f(z) = $\frac{2}{ib}\frac{1}{(z-\alpha)(z-\beta)}$

We have found two poles $z=\frac{-a+\sqrt{a^2-b^2}}{b} \;;z=\frac{-a-\sqrt{a^2-b^2}}{b}$ simple pole or pole of order

To find these poles lies inside or outside the contour $|z|=1$

Consider values for a and b such way that they satisfies (a> b >0)

Let a= 2, b= 1

$$z=\frac{-2+\sqrt{4-1}}{1} \;;z=\frac{-2-\sqrt{4-1}}{1}$$

$$z=-2+\sqrt{3} \;;\; z=-2-\sqrt{3}$$

$$z=-2+1.732 \;;\; z=-2-1.732$$

$$z=-0.268 \;;\; z=-3.732$$

Substitute in $|z|=1$

$$|-0.268|<1$$

$$0.268<1$$

This implies $z=\frac{-a+\sqrt{a^2-b^2}}{b}$ is a pole of order which lies inside the contour $|z|=1$

Now substitute $z=-3.732\ in|z|=1$

$|-3.732|>1$

$3.732>1$

This implies $z=\frac{-a-\sqrt{a^2-b^2}}{b}$ is a pole of order which lies outside the contour $|z|=1$

Res [f(z)]= $lt_{z\to a}(z-\alpha)\,f(z)$

$$\text{Res}\left[\text{f}\left(\text{z}\right)\right]=lt_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}\frac{2}{ib}$$

$$\text{Res}\left[\text{f}\left(\text{z}\right)\right]=lt_{z\to\alpha}\frac{1}{(z-\beta)}\frac{2}{ib}$$

$$\text{Res}\left[\text{f}\left(\text{z}\right)\right]=lt_{z\to\alpha}\frac{1}{(\alpha-\beta)}\frac{2}{ib}$$

$$\text{Res}\left[\text{f}\left(\text{z}\right)\right]=\frac{1}{\frac{2\sqrt{a^2-b^2}}{b}}\frac{2}{ib}$$

$$\text{Res}\left[\text{f}\left(\text{z}\right)\right]=\frac{1}{i\sqrt{a^2-b^2}}$$

Hence by Cauchy's residue theorem

$$\int_c f(z)\,dz=2\pi i(sum\,of\ the\,residues)$$

$$\int_c f(z)\,dz=2\pi i\ \frac{1}{i\sqrt{a^2-b^2}}$$

$$\int_c f(z)\,dz=2\pi\ \frac{1}{\sqrt{a^2-b^2}}$$

Example 5

Evaluate $\int_0^{2\pi}\frac{d\theta}{13+5\cos\theta}$ using contour integration

Solution:

Given: $\int_0^{2\pi}\frac{d\theta}{13+5\cos\theta}$

Change θ to Z

$Let\ \ z=e^{i\theta}$

$dz=ie^{i\theta}d\theta=izd\theta$

$d\theta=\frac{dz}{iz}$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] = \frac{1}{2}\left[\frac{z^2+1}{z}\right]$$

$$13 + 5\cos\theta = 13 + 5\left(\frac{z^2+1}{2z}\right) = \frac{26z + 5z^2 + 5}{2z}$$

Convert the given integral into a Contour integral

$$\int_0^{2\pi} \frac{1}{13+5\cos\theta} d\theta = \int_c \frac{2z}{5z^2+26z+5} \times \frac{1}{iz} dz$$

$$= \frac{2}{5i} \int_c \frac{z^2}{z^2 + \frac{26}{5} z + 1} dz$$

$$= \frac{2}{5i}[2\pi i \quad \text{sum ofthe residues}]...(1)$$

Now $z^2 + \frac{26}{5} z + 1 = 0$

$$a = 1, b = \frac{26}{5}, c = 1$$

$$z = \frac{-26/5 \pm \sqrt{\frac{676}{25} - 4}}{2}$$

$$= \frac{-26/5 \pm \sqrt{\frac{676-100}{25}}}{2}$$

$$= \frac{-26/5 \pm \sqrt{\frac{576}{25}}}{2}$$

$$= \frac{-\frac{26}{5} \pm \frac{24}{5}}{2} = \frac{-26 \pm 24}{10}$$

$$= \frac{-26-24}{10}, \frac{-26+24}{10}$$

$$\text{z} = -5, \frac{-1}{5}$$

$z = \frac{-1}{5}$ lies inside $|z| = 1$

$z = -5$ lies outside the contour $|z| = 1$

$$\operatorname{Res}[f(z), -\frac{1}{5}] = \lim_{z \to -\frac{1}{5}} \left(z + \frac{1}{5}\right)\left(\frac{1}{\left(z + \frac{1}{5}\right)(z+5)}\right)$$

$$= \lim_{z \to -\frac{1}{5}} \frac{1}{(z+5)}$$

$$= \frac{1}{-\frac{1}{5}+5} = \frac{1}{\frac{24}{5}} = \frac{5}{24}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{1}{13+5\cos\theta} d\theta = \frac{2}{5i} \times 2\pi i \ \times \frac{5}{24} = \frac{\pi}{6}$$

TYPE II:

Improper integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ where $P(x)$ and $Q(x)$ are polynomials in x such that the degree of Q exceeds that of P at least by two and $Q(x)$ does not vanish for any x.

Example 6

Evaluate $\int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ $a>0, b>0$ using contour integration. (or) Prove that

$\int_{-\infty}^{\infty} \frac{x^2 dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)} = \frac{\pi}{a+b}$, $a>b>0$.

Solution:

$$\int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} \qquad ....(1)$$

To find: $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$

Consider $\int_c f(z)\,dz = \int_c \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)}$ where c is the upper half of the semi-circle Γ with the bounding diameter[-R, R].

By Cauchy's residue theorem

We have $\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$

The poles of f(z) are the solutions of $(z^2+a^2)(z^2+b^2)=0$

$z^2+a^2=0$, $\qquad z^2+b^2=0$

$z = \pm ai, \ z = \pm bi$

z = ai is a simple pole lies inside Γ

z = bi is a simple pole lies inside Γ

z = -ai is a simple pole lies outside Γ

z = -bi is a simple pole lies outside Γ

Formula for Residue of simple pole is $\text{Res}(z=a) = \lim_{z \to a}[z-a]f(z)$

$\text{Res}[z=ai] = \underset{z \to ai}{\text{Lt}}(z-ai)f(z)$

$$= \underset{z \to ai}{Lt}\,(z-ai)\frac{z^2}{(z-ai)(z+ai)(z^2+b^2)}$$

$$= \underset{z \to ai}{Lt}\,\frac{z^2}{(z+ai)(z^2+b^2)} = \frac{-a^2}{(2ai)(-a^2+b^2)}$$

$$= \frac{-a}{(2i)(b^2-a^2)} = \frac{a}{(2i)(a^2-b^2)}$$

$$\mathrm{Res}[z=bi] = \underset{z \to bi}{\mathrm{Lt}}\,(z-bi)f(z)$$

$$= \underset{z \to bi}{Lt}\,(z-bi)\frac{z^2}{(z^2+a^2)(z-bi)(z+bi)}$$

$$= \underset{z \to bi}{Lt}\,\frac{z^2}{(z+bi)(z^2+a^2)} = \frac{-b^2}{(2bi)(-b^2+a^2)}$$

$$= \frac{-b^2}{(2bi)(-b^2+a^2)} = \frac{-b}{(2i)(a^2-b^2)}$$

Hence by Cauchy's residue theorem

$$\int_c f(z)\,dz = 2\pi i\ [\text{ sum of the residues}]$$

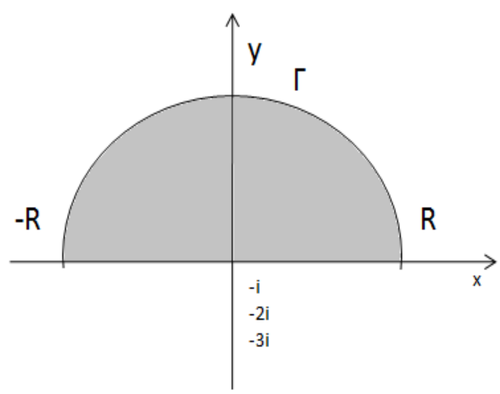
$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i(a^2-b^2)}[a-b]$$

$$= \frac{\pi}{(a-b)(a+b)}(a-b) = \frac{\pi}{(a+b)}$$

$$i.e.,\ \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{\pi}{a+b}$$

If $R \to \infty$ then $\int_{\Gamma} f(z)\,dz \to 0$



$$(1) \Rightarrow \int_0^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{(a+b)}}$$

$$\boxed{\therefore \int_0^{\infty} f(x)\,dx = \frac{1}{2}\frac{\pi}{(a+b)}}$$

Example 7

Evaluate $\int_0^{\infty} \frac{dx}{\left(a^2+x^2\right)^2}$, $(a>0)$ by using contour integration.

Solution:

Let $f(x) = \frac{1}{\left(a^2+x^2\right)^2}$

Since $f(x)$ is even, $\int_{0}^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$

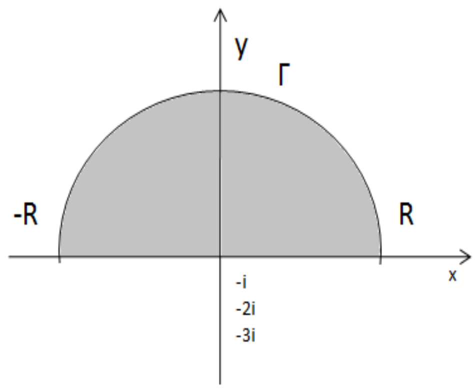
Let $I=\int_0^\infty \frac{dx}{\left(a^2+x^2\right)^2}$

The integrand is of the form $\frac{P(x)}{Q(x)}$, where degree of Q(x) is atleast 2 more than that of P(x) and Q(x) does not vanish for any real x. consider $\int_c \frac{1}{\left(z^2+a^2\right)^2}dz$

where c is the upper half of the semi circle Γ with the bounding diameter [-R,R].

By Cauchy's residue theorem

We have $\int_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz$



$$f(z) = \frac{1}{\left(z^2+a^2\right)} = \frac{1}{\left[(z+ai)(z-ai)\right]^2}$$

z = ai is a pole order 2 lies inside c

z= - ai is a pole of order 2 lies outside c

By Cauchy's Residue theorem,

$$\int_c f(z)dz = 2\pi i\left[R_1+R_2\right]$$

To find Residue of f(z) at z = ai

Formula $\operatorname{Res}(z=z_0) = \underset{z\to z_0}{Lt}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$

Here m = 2

$$\operatorname{Res}\left[f(z)\right]_{z=ai} = \lim_{z\to ai}\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}(z-ai)^2\frac{1}{\left[(z-ai)^2(z+ai)^2\right]}$$

$$=\lim_{z\to ai}\frac{d}{dz}\frac{1}{(z+ai)^2} = \lim_{z\to ai} -2\frac{1}{(z+ai)^3} = \frac{-2}{(2ai)^3} = \frac{-2}{-8ai} = \frac{1}{4ai}$$

$$\int_c f(z)dz = 2\pi i\left(\frac{1}{4ai}\right) = \frac{\pi}{2a}$$

By Cauchy's lemma $\int_\Gamma f(z)dz = 0$

Now $\int_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz$

Taking limits as R→ ∞ we get

$$\frac{\pi}{2a} = \int_{-\infty}^{\infty}\frac{1}{\left(a^2+x^2\right)^2}dx + 0$$

$$\frac{\pi}{2a} = 2\int_0^{\infty} \frac{1}{\left(a^2 + x^2\right)^2} dx$$

$$\int_0^{\infty} \frac{1}{\left(a^2 + x^2\right)^2} dx = \frac{\pi}{4a}$$

Example 8

Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$ using contour integration.

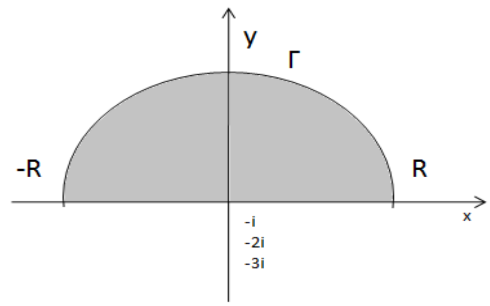
Solution:

Consider $\int_c f(z)dz = \int_c \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$

where c is the upper half of the semi circle Γ with the bounding diameter [-R,R].

By Cauchy's residue theorem

We have $\int_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz$



The poles of f(z) are the solutions of $z^4 + 10z^2 + 9 = 0$

put $t = z^2 \Rightarrow t^2 + 10t + 9 = 0$

$(t+9)(t+1) = 0$

i.e., $z^2 + 9 = 0$ $\quad\quad$ $z^2 + 1 = 0$

$z^2 = -9$ $\quad\quad$ $z^2 = -1$

$z = \pm 3i$ $\quad\quad$ $z = \pm i$

The poles i and 3i lies inside Γ

Formula for Residue of simple pole is $\operatorname{Res}(z = a) = \lim_{z \to a} (z - a) f(z)$

$$\operatorname{Res}(z = i) = \lim_{z \to i} (z - i) f(z)$$

$$= \underset{z \to i}{\text{Lt}} (z - i) \frac{z^2 - z + 2}{\left(z^2 + 9\right)(z + i)(z - i)} = \underset{z \to i}{\text{Lt}} \frac{z^2 - z + 2}{\left(z^2 + 9\right)(z + i)}$$

$$= \frac{-1 - i + 2}{(-1 + 9)(2i)} = \frac{1 - i}{(8)(2i)}$$

$$= \frac{1 - i}{16i}$$

Formula for Residue of simple pole is $\operatorname{Res}(z = a) = \lim_{z \to a} (z - a) f(z)$

$$\operatorname{Res}(z = 3i) = \underset{z \to 3i}{Lt} (z - 3i) f(z)$$

$$= \underset{z \to 3i}{Lt} (z - 3i) \frac{z^2 - z + 2}{(z + 3i)(z - 3i)\left(z^2 + 1\right)}$$

$$= \underset{z \to 3i}{Lt} \frac{z^2 - z + 2}{(z + 3i)\left(z^2 + 1\right)} = \frac{-9 - 3i + 2}{(6i)(-9 + 1)} = \frac{-7 - 3i}{(6i)(-8)} = \frac{7 + 3i}{48i}$$

$By\ Cauch's\ residue\ \ theorem$

$$\int_c f(z) = 2\pi i(sum\ of\ \ the\ residues)$$

$$= 2\pi i\left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right] = \frac{2\pi i}{16i}\left(1-i+\frac{7+3i}{3}\right)$$

$$= \frac{\pi}{24}(10) = \frac{5\pi}{12}$$

$$As\ R \to \infty, \int_c f(z) = 0 \ \therefore \int_{-\infty}^{\infty} f(x)dx = \frac{5\pi}{12}$$

$$i.e., \int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}$$

TYPE III:

Integrals of the form $\int_{-\infty}^{\infty} f(x)\cos mxdx$ (or) $\int_{-\infty}^{\infty} f(x)\sin mxdx$ where $f(x) \to 0$ as $x \to \infty$

Example 9

Evaluate $\int_0^{\infty} \frac{\cos mx}{x^2+a^2}dx$, using contour integration.

Solution:

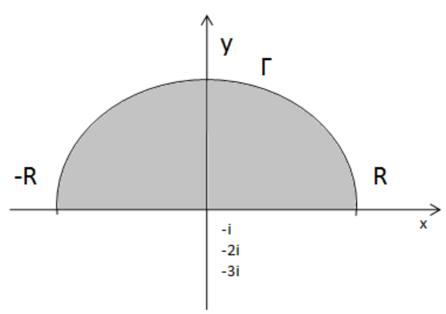
$$\int_0^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}dx$$

To find: $\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}dx$

Consider $\int_c f(z)dz = \frac{1}{2}\int_c \frac{\cos mz}{z^2+a^2}dz = R.P.\int_c \frac{e^{imz}}{z^2+a^2}dz$ where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R]

By Cauchy's residue theorem, we haveWe have

$$\int_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz$$



The poles of f(z) are the solutions of $z^2 + a^2 = 0$

$z^2 + a^2 = 0,$

$z = \pm ai,$

z = ai is a simple pole lies inside Γ

z = -ai is a simple pole lies outsideΓ

Formula for Residue of simple pole is

$$\text{Res}(z=a) = \lim_{z\to a}[z-a]f(z)$$

$$\text{Res}[z=ai] = \underset{z\to ai}{\text{Lt}}(z-ai)f(z)$$

$$= \underset{z\to ai}{Lt}(z-ai)\frac{e^{imz}}{(z-ai)(z+ai)}$$

$$= \underset{z\to ai}{Lt} \frac{e^{imz}}{(z+ai)}$$

$$= \frac{e^{im(ia)}}{(ai+ai)} = \frac{e^{-ma}}{2ai}$$

By Cauchy's Residues theorem

$$\int_c f(z)\,dz = R.P.\ 2\pi i\ [\text{ sum of the residues}]$$

$$\int_c \mathrm{f(z)\,dz} = R.P.\ 2\pi i\left[\frac{e^{-ma}}{2ai}\right]$$

$$= R.P.\ \frac{\pi e^{-ma}}{a} = \frac{\pi e^{-ma}}{a}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{\pi e^{-ma}}{a}$$

If R $\to \infty$ then $\int_{\Gamma} f(z)\,dz \to 0$

$$\int_{-\infty}^{\infty} f(x)dx = \pi e^{-ma}$$

$$i.e., \int_0^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{2a}$$

Example 10

Evaluate $\int_0^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx$, where $a>0$, $m>0$, using contour integration.

Solution:

$$\int_0^{\infty} \frac{\mathrm{x}\sin mx}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{x}\sin mx}{x^2+a^2}\,dx$$

$$\text{To find:} \int_{-\infty}^{\infty} \frac{\mathrm{x}\sin mx}{x^2+a^2}\,dx$$

Consider $\int_c f(z)\,dz = \frac{1}{2}\int_c \frac{\mathrm{z}\sin mz}{z^2+a^2}\,dz = IM.P.\frac{1}{2}\int_c \frac{ze^{imz}}{z^2+a^2}\,dz$ where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R].

By Cauchy's residue theorem, we have

We have $\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$

The poles of f(z) are the solutions of $z^2 + a^2 = 0$

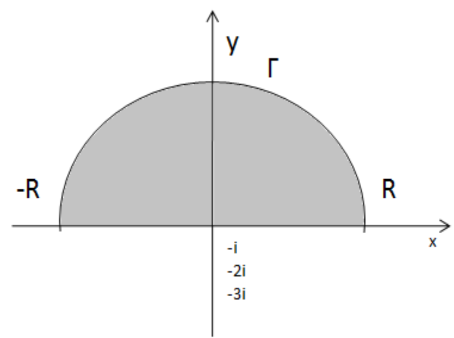
$$z^2 + a^2 = 0,$$

$z = \pm ai,$

z = ai is a simple pole lies inside Γ

z = -ai is a simple pole lies outside Γ

Formula for Residue of simple pole is



$$\operatorname{Res}(z=a)=\lim_{z\to a}[z-a]f(z)$$

$$\operatorname{Res}[z=ai]=\underset{z\to ai}{Lt}(z-ai)f(z)$$

$$=\underset{z\to ai}{Lt}(z-ai)\frac{1}{2}\frac{ze^{imz}}{(z-ai)(z+ai)}$$

$$=\underset{z\to ai}{Lt}\frac{1}{2}\frac{ze^{imz}}{(z+ai)}$$

$$=\frac{ai\,e^{im(ia)}}{2(ai+ai)}=\frac{aie^{-ma}}{4ai}=\frac{e^{-ma}}{4}$$

By Cauchy's Residues theorem

$$\int_c f(z)\,dz=IM.P.2\pi i\ \text{[sum of the residues]}$$

$$\int_c f(z)\,dz=IM.P.\ 2\pi i\left[\frac{e^{-ma}}{4}\right]=\frac{\pi e^{-ma}}{2}$$

$$\therefore\int_{-R}^{R}f(x)\,dx+\int_{\Gamma}f(z)\,dz=\frac{\pi e^{-ma}}{2}$$

If $R\to\infty$ then $\int_{\Gamma}f(z)\,dz\to 0$

$$\int_{-\infty}^{\infty}f(x)dx=\frac{\pi e^{-ma}}{2}\qquad i.e.,\int_0^{\infty}\frac{\cos mx}{x^2+a^2}dx=\frac{\pi e^{-ma}}{4}$$

Exercise 6.4

1. Evaluate $\int_0^{2\pi}\frac{1}{1+\cos 2\theta}d\theta$ using contour integration. Ans: 0
2. Evaluate $\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}d\theta$ using contour integration. Ans: $\frac{\pi}{6}$
3. Evaluate $\int_0^{2\pi}\frac{d\theta}{13+12\cos\theta}$ using contour integration. Ans : $\frac{2\pi}{5}$
4. Evaluate $\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+4)(x^2+9)}$ using contour integration. Ans : $\frac{\pi}{5}$
5. Evaluate $\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+4)(x^2+1)}$ using contour integration. Ans : $\frac{\pi}{3}$
6. Evaluate $\int_0^{\infty}\frac{\cos x}{x^2+a^2}dx$, using contour integration. Ans : $\frac{\pi e^{-a}}{2a}$
7. Evaluate $\int_0^{\infty}\frac{\sin x}{x^2+1}dx$, using contour integration. Ans : $\frac{\pi}{2e}$