



**SRM Institute of Science and Technology  
Ramapuram Campus**

**Department of Mathematics**

**Year / Sem: I / II**

**Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)**

**Unit 2 – Vector Calculus**

**Part – B (Each question carries 3 Marks)**

**1. Find  $\nabla\phi$  if  $\phi = \log(x^2 + y^2 + z^2)$ .**

**Solution**

$$\begin{aligned}\nabla\phi &= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \\ &= \vec{i} \frac{\partial}{\partial x} (\log(x^2 + y^2 + z^2)) + \vec{j} \frac{\partial}{\partial y} \log(x^2 + y^2 + z^2) + \vec{k} \frac{\partial}{\partial z} \log(x^2 + y^2 + z^2) \\ &= \vec{i} \frac{2x}{(x^2 + y^2 + z^2)} + \vec{j} \frac{2y}{(x^2 + y^2 + z^2)} + \vec{k} \frac{2z}{(x^2 + y^2 + z^2)} \\ &= \frac{2}{x^2 + y^2 + z^2} (x\vec{i} + y\vec{j} + z\vec{k}) = \frac{2\vec{r}}{r^2} \quad \because (\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \text{ \& } r^2 = x^2 + y^2 + z^2)\end{aligned}$$

**2. Find the unit normal vector to the surface  $x^2 + y^2 = z$  at the point  $(1, -2, 5)$ .**

**Solution**

Given

$$\begin{aligned}\phi &= x^2 + y^2 - z \\ \nabla\phi &= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} = 2x\vec{i} + 2y\vec{j} - \vec{k} \\ \nabla\phi \text{ at } (1, -2, 5) &= 2\vec{i} - 4\vec{j} - \vec{k} \\ |\nabla\phi| &= \sqrt{4 + 4 + 1} = 3\end{aligned}$$

Unit Normal vector is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$$

**3. Prove that  $\text{curl}(\text{grad}\phi) = \mathbf{0}$ .****Solution**

$$\text{grad}\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\begin{aligned} \text{Curl}(\text{grad}\phi) &= \nabla \times \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} \\ &= \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \vec{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) \\ &= 0\vec{i} + 0\vec{j} + 0\vec{k} \quad (\text{Since mixed partial derivatives are equal.}) \end{aligned}$$

**4. Find  $\text{curl}\vec{F}$  if  $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ .****Solution**

$$\text{Given } \vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$$

$$\begin{aligned} \text{curl}\vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}(0 - y) - \vec{j}(z - 0) + \vec{k}(0 - x) \\ &= -y\vec{i} - z\vec{j} - x\vec{k} \end{aligned}$$

**5. In what direction from  $(3, 1, -2)$  is the directional derivative of  $\phi = x^2y^2z^4$  maximum? Find also the magnitude of this maximum.****Solution**

$$\text{Given } \phi = x^2y^2z^4$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xy^2z^4\vec{i} + 2x^2yz^4\vec{j} + 4x^2y^2z^3\vec{k}$$

$$\nabla\phi \text{ at } (3, 1, -2) = 92\vec{i} + 144\vec{j} - 92\vec{k}$$

$$|\nabla\phi| = \sqrt{92^2 + 144^2 + 92^2} = \sqrt{37664}$$

The directional derivative is maximum in the direction  $\nabla\phi$  and the magnitude of this maximum is  $|\nabla\phi| = \sqrt{37664}$ .

**6. Find the directional derivative of  $\phi = x^2yz + 4xz^2$  at  $(1, -2, -1)$  in the direction of  $2\vec{i} - \vec{j} - 2\vec{k}$ .**

**Solution**

$$\text{Given } \phi = x^2yz + 4xz^2,$$

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}, |\vec{a}| = \sqrt{4+1+4} = 3$$

$$\nabla\phi = (2xyz + 4z^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\text{D.D.} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \frac{2\vec{i} - \vec{j} - 2\vec{k}}{3} = \frac{37}{3}$$

**7. Find the directional derivative of  $\phi = x^2 - y^2 + 2z^2$  at P  $(1, 2, 3)$  in the direction of line PQ where Q is  $(5, 0, 4)$ .**

**Solution**

$$\nabla\phi = \text{grad } \phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \text{grad } \phi = \vec{i} 2x + \vec{j} (-2y) + \vec{k} 4z$$

$$\nabla\phi \text{ at } (1, 2, 3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$$

$$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$$

$$\text{Directional derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \cdot \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

**8. Find the angle between the normals to the surfaces  $x^2 = yz$  at the points  $(1, 1, 1)$  and  $(2, 4, 1)$ .**

**Solution**

$$\text{Given } \phi = x^2 - yz$$

$$\nabla\phi = 2x\vec{i} - z\vec{j} - y\vec{k}$$

$$\nabla\phi_1 / (1,1,1) = 2\vec{i} - \vec{j} - \vec{k}$$

$$\nabla\phi_2 / (2,4,1) = 4\vec{i} - \vec{j} - 4\vec{k}$$

$$|\nabla \phi_1| = \sqrt{4+1+1} = \sqrt{6}$$

$$|\nabla \phi_2| = \sqrt{16+1+16} = \sqrt{33}$$

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{(2\vec{i} - \vec{j} - \vec{k}) \cdot (4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}.$$

**9. Find  $a$  such that  $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$  is solenoidal.**

**Solution**

$$\text{Given } \nabla \cdot \vec{F} = 0 \Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$$3 + a + 2 = 0 \Rightarrow a + 5 = 0 \Rightarrow a = -5$$

**10. Find the constant  $a$ ,  $b$ ,  $c$  so that  $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$  is irrotational.**

**Solution**

**Given**  $\vec{F}$  is irrotational i.e.,  $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\begin{aligned} & \vec{i} \left( \frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z) \right) - \vec{j} \left( \frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az) \right) \\ & + \vec{k} \left( \frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az) \right) = \vec{0} \end{aligned}$$

$$= \text{i.e., } \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = \vec{0}$$

$$= \therefore c+1=0, 4-a=0, \text{ and } b-2=0$$

$$\Rightarrow a=4, b=2, c=-1$$

11. If  $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ , then find  $\text{div curl } \vec{F}$ .

**Solution**  $\text{div curl } \vec{F} = \nabla \cdot (\nabla \times \vec{F})$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix} \\ &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0} \\ \nabla \times \vec{F} &= \vec{0} \\ \therefore \nabla \cdot (\nabla \times \vec{F}) &= 0\end{aligned}$$

12. Prove that  $\text{div } \vec{r} = 3$ .

**Solution**

$$\begin{aligned}\vec{r} &= x\vec{i} + y\vec{j} + z\vec{k} \\ \text{div } \vec{r} &= \nabla \cdot \vec{r} = \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3\end{aligned}$$

13. Show that the vector  $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$  is irrotational.

**Solution**

$$\text{Given } \vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \vec{0}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore \vec{F}$  is irrotational.

14. If  $F = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ . Evaluate  $\int_C \vec{F} \bullet d\vec{r}$  from (0,0,0) to (1,1,1) along the curve  $x = t, y = t^2, z = t^3$ .

**Solution**

The end points are (0,0,0) and (1,1,1).

These points correspond to  $t = 0$  and  $t = 1$ .

$$\therefore dx = dt, \quad dy = 2t dt, \quad dz = 3t^2 dt$$

$$\begin{aligned} \int_C \vec{F} \bullet d\vec{r} &= \int_C (3x^2 + 6y)dx - 14yzdy + 20xz^2 dz \\ &= \int_0^1 (3t^2 + 6t^2)dt - 14t^5(2tdt) + 20t^7(3t^2)dt = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = 5 \end{aligned}$$

15. If  $F = ax\vec{i} + by\vec{j} + cz\vec{k}$ ,  $a, b, c$  are constants, show that  $\iint_S \vec{F} \bullet \hat{n} ds = \frac{4\pi}{3}(a + b + c)$  where  $S$

is the surface of a unit sphere.

**Solution**

W.K.T. Gauss's divergence theorem

$$\begin{aligned} \iint_S \vec{F} \bullet \hat{n} ds &= \iiint_V \nabla \bullet \vec{F} dV = \iiint_V \left( \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right) dV \\ &= \iiint_V (a + b + c) dV = (a + b + c)V = (a + b + c) \frac{4}{3} \pi (1)^3 \\ \iint_S \vec{F} \bullet \hat{n} ds &= \frac{4}{3} \pi (a + b + c) \end{aligned}$$

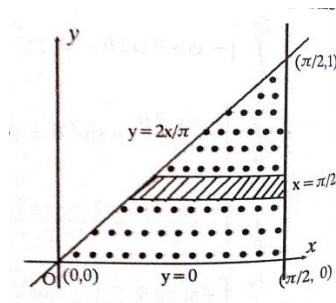
**16. Using Green's theorem, evaluate  $\int_c (y - \sin x) dx + \cos x dy$  where  $c$  is the triangle**

**formed by  $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$ .**

**Solution**

Using Green's theorem, we convert the line integral to double integral over the given region.

$$ie., \int_C u dx + v dy = \iint_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$



$$u = y - \sin x$$

$$v = \cos x$$

$$\frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = -\sin x$$

$$\text{Hence, } \int_C \{(y - \sin x) dx + \cos x dy\} = \iint_R (-\sin x - 1) dx dy$$

$$= \int_0^1 \int_{\pi/2}^{\pi/2} (-\sin x - 1) dx dy = \int_0^1 [\cos x - x]_{\pi/2}^{\pi/2} dy$$

$$= \int_0^1 \left( 0 - \frac{\pi}{2} - \cos \frac{\pi y}{2} + \frac{\pi y}{2} \right) dy$$

$$= \left[ \frac{-\pi y}{2} - \frac{\sin \frac{\pi y}{2}}{\frac{\pi}{2}} + \frac{\pi}{2} \cdot \frac{y^2}{2} \right]_0^1 = \frac{-\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= \frac{-\pi^2 - 8}{4\pi} = -\left[ \frac{\pi}{4} + \frac{2}{\pi} \right].$$

**17. Using Green's theorem, evaluate  $\int_c (3x^2 - 8y^2) dx + (4y - 6xy) dy$  where  $c$  is the boundary of the triangle formed by the lines  $x = 0, y = 0, x + y = 1$  in the  $xy$  plane.**

**Solution**

Using Green's theorem, we convert the line integral to double integral over the given region.

$$ie., \int_C u dx + v dy = \iint_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

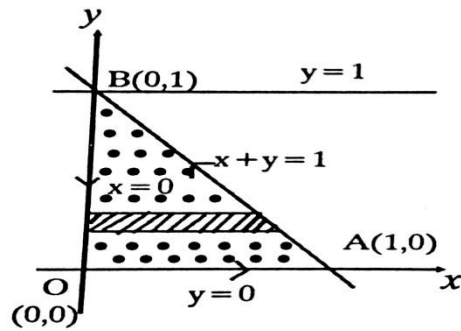
$$u = 3x - 8y^2$$

$$v = 4y - 6xy^2$$

$$\frac{\partial u}{\partial y} = -16y$$

$$\frac{\partial v}{\partial x} = -6y$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -6y + 16y = 10y$$



$$\begin{aligned} \text{Hence, } \iint_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy &= \iint_R (10y) dx dy \\ &= 10 \int_0^1 \int_0^{1-y} (y) dx dy = \int_0^1 y[x]_0^{1-y} dy \\ &= 10 \int_0^1 y(1-y) dy = 10 \int_0^1 (y - y^2) dy \\ &= 10 \left( \frac{y^2}{2} - \frac{y^3}{3} \right)_0^1 \\ &= 10 \left( \frac{1}{2} - \frac{1}{3} \right) \\ &= 10 \frac{3-2}{6} = \frac{10}{6} = \frac{5}{3} \end{aligned}$$

**18. Using Gauss divergence theorem evaluate  $\iiint_V \nabla \circ \vec{F} dv$  where  $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$  taken over the cube bounded by the planes  $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ .**

**Solution**

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 4z - 2y + y = 4z - y$$



$$\begin{aligned}\iiint_V \nabla \circ \vec{F} dv &= \int_0^1 \int_0^1 \int_0^1 (4z - y) dx dy dz = \int_0^1 \int_0^1 [4zx - yx]_0^1 dy dz = \int_0^1 \int_0^1 [4z - y] dy dz \\ &= \int_0^1 \left[ 4zy - \frac{y^2}{2} \right]_0^1 dz = \int_0^1 \left[ 4z - \frac{1}{2} \right] dz = \left[ 4 \frac{z^2}{2} - \frac{z}{2} \right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}\end{aligned}$$

**19. Using Gauss divergence theorem theorem evaluate  $\iiint_V \nabla \circ \vec{F} dv$  where**

**$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$  taken over the cube bounded by the planes**  
 **$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ .**

**Solution**

$$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

$$\begin{aligned}\iiint_V \nabla \circ \vec{F} dv &= 2 \int_0^1 \int_0^1 \int_0^1 (x + y + z) dx dy dz = 2 \int_0^1 \int_0^1 \left[ \frac{x^2}{2} + xy + xz \right]_0^1 dy dz = 2 \int_0^1 \int_0^1 \left[ \frac{1}{2} + y + z \right] dy dz \\ &= 2 \int_0^1 \left[ \frac{y}{2} + \frac{y^2}{2} + yz \right]_0^1 dz = 2 \int_0^1 \left[ \frac{1}{2} + \frac{1}{2} + z \right] dz = 2 \int_0^1 [1 + z] dz = 2 \left[ z + \frac{z^2}{2} \right]_0^1 \\ &= 2 \left( 1 + \frac{1}{2} \right) = 2 \left( \frac{3}{2} \right) = 3\end{aligned}$$

**20. Using Stokes theorem find  $\iint_S \text{curl } \vec{F} ds$  where  $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$  in the**

**rectangular region of  $x = 0, y = 0, x = a$  and  $y = a$ .**

**Solution** Stokes theorem  $\int_c \vec{F} \cdot d\vec{r} = \iint_s \text{curl } \vec{F} \cdot d\vec{s}$

$$\text{Given } \vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y \vec{k}$$

Here  $\hat{n} = \vec{k}$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} ds = \iint_S 4y dx dy = \int_0^b \int_0^a 4y dx dy = 2ab^2$$

**21. Prove that the area bounded by a simple closed curve C is given by**

$$\frac{1}{2} \int_C (x dy - y dx).$$

**Solution**

W.K.T. Green's theorem

$$\int_C (u dx + v dy) = \iint_R \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy \quad \dots 1$$

Here  $v = \frac{x}{2} \quad u = -\frac{y}{2}$

$$\frac{\partial v}{\partial x} = \frac{1}{2} \quad \frac{\partial u}{\partial y} = -\frac{1}{2}$$

$$(1) \Rightarrow \int_C \left( \frac{x}{2} dy - \frac{y}{2} dx \right) = \iint_R \left( \frac{1}{2} + \frac{1}{2} \right) dx dy$$

$$\frac{1}{2} \int_C (x dy - y dx) = \iint_R dx dy$$

**22. Find the area of the ellipse  $x = a \cos \theta$ ,  $y = b \sin \theta$  using Green's theorem.**

**Solution**

Given  $x = a \cos \theta, \quad y = b \sin \theta$

$$dx = -a \sin \theta d\theta, \quad dy = b \cos \theta d\theta$$

$\theta$  varies from 0 to  $2\pi$ .

$$\text{Area of the ellipse} = \frac{1}{2} \oint_C x dy - y dx$$

$$= \frac{1}{2} \int_0^{2\pi} (a \cos \theta)(-b \cos \theta d\theta) - (b \sin \theta)(-a \sin \theta d\theta)$$

$$= \frac{1}{2} \int_0^{2\pi} [ab \cos \theta \cos \theta + ab \sin \theta \sin \theta] d\theta$$

$$= \frac{ab}{2} \int_0^{2\pi} (\cos^2 \theta + \sin^2 \theta) d\theta = \frac{ab}{2} \int_0^{2\pi} d\theta = \frac{ab}{2} [\theta]_{\theta=0}^{\theta=2\pi}$$

$$\text{Area of the ellipse} = \frac{ab}{2} [2\pi] = \pi ab$$

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