SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 3 – Laplace Transforms

Part – B (Each question carries 3 Marks)

1. Find $L[2e^{-3t}]$.

Solution

$$L[e^{-at}] = \frac{1}{s+a}$$

$$L[2e^{-3t}] = 2L[e^{-3t}] = 2\left(\frac{1}{s+3}\right)$$

2. Find $L[e^{3t+5}]$.

Solution

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{3t}.e^{5}] = e^{5}L[e^{3t}] = e^{5}\left(\frac{1}{s-3}\right)$$

3. Find the Laplace transform of $f(t) = cos^2(3t)$.

Solution

$$L[\cos^2 3t] = L\left[\frac{1+\cos 6t}{2}\right] = \frac{L(1)+L(\cos 6t)}{2} \qquad \because \cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+36)} \qquad \because L(1) = \frac{1}{s}, L(\cos at) = \frac{s}{s^2+a^2}$$

$\therefore$ **L[cos² 3t]** $= \dfrac{s^2+18}{s(s^2+36)}$

4. Find $L\left(t^2 - 4\sin 2t + 2\cos 3t\right)$**.**

Solution

$$L\left(t^2 - 4\sin 2t + 2\cos 3t\right) = \frac{2}{s^3} - 4\left(\frac{2}{s^2+4}\right) + 2\left(\frac{s}{s^2+9}\right)$$

5. Find the Laplace transform of e^{-t} **sin 2t.**

Solution

$$L[e^{-t}\sin 2t] = L[e^{-at} f(t)] = F(s+a) = F(s+1)$$

$$F(s) = L[f(t)] = L(\sin 2t) = \frac{2}{s^2+4}$$

$$F(s+1) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

6. Obtain the Laplace transform of $\boldsymbol{\sin 2t - 2t\cos 2t}$ **.**

Solution

$$L[\sin 2t - 2t\cos 2t] = L[\sin 2t] - 2L[t\cos 2t] = L[\sin 2t] - 2\left(-\frac{d}{ds}L[\cos 2t]\right)$$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4} + 2\left(\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{2(s^2+4) + 2(4-s^2)}{(s^2+4)^2}$$

$$\therefore L[\sin 2t - 2t\cos 2t] = \frac{16}{(s^2+4)^2}$$

7. Find $L\left(t\,e^t\right)$**.**

Solution

$$L\left(t\,f(t)\right) = -\frac{d}{ds}L(f(t))$$

$$L\left(t\,e^{t}\right)=-\frac{d}{ds}L(e^{t})$$

$$=-\frac{d}{ds}L\left(\frac{1}{s-1}\right)=\frac{1}{(s-1)^{2}}$$

8. Find $L\left(t\sin 2t\right)$**.**

Solution

$$L\left(t\ f(t)\right)=-\frac{d}{ds}L\left(f(t)\right)$$

$$L\left(t\sin 2t\right)=-\frac{d}{ds}L\left(\sin 2t\right)$$

$$=-\frac{d}{ds}\left(\frac{2}{s^{2}+4}\right)=\frac{4s}{\left(s^{2}+4\right)^{2}}$$

9. Find the Laplace transform of $f(t)=t^{2}\cos t$.

Solution

$$L\left[t^{2}\cos t\right]=\left[\frac{d^{2}}{ds^{2}}L\left[\cos t\right]\right]=\frac{d^{2}}{ds^{2}}\left(\frac{s}{s^{2}+1}\right)$$

$$=\frac{d}{ds}\left(\frac{\left(s^{2}+1\right).1-1.2s.s}{\left(s^{2}+1\right)^{2}}\right)=\frac{d}{ds}\left(\frac{1-s^{2}}{\left(s^{2}+1\right)^{2}}\right)$$

$$=\frac{\left(s^{2}+1\right)^{2}(-2s)-\left(1-s^{2}\right)2\left(s^{2}+1\right)2s}{\left(s^{2}+1\right)^{3}}=\frac{-2s\left(3-s^{2}\right)}{\left(s^{2}+1\right)^{3}}$$

10. Find the Laplace transform of $\boldsymbol{f(t)=te^{-3t}cos2t}$

Solution

$$L[f(t)]=L[te^{-3t}cos2t]=-\frac{d}{ds}L[\cos 2t]_{s\to s+3}=-\frac{d}{ds}\left[\frac{s}{s^{2}+4}\right]_{s\to s+3}$$

$$=-\left[\frac{(s^{2}+4)(1)-s(2s)}{(s^{2}+4)^{2}}\right]_{s\to s+3}=\left[\frac{s^{2}-4}{(s^{2}+4)^{2}}\right]_{s\to s+3}$$

$$= \frac{(s+3)^2-4}{((s+3)^2+4)^2}$$
$$= \frac{s^2+6s+5}{(s^2+6s+13)^2}$$

11. Find the Laplace Transform of $f(t) = e^{-t}t\, cost.$

Solution

$$L[e^{-t}tcost] = -\frac{d}{ds}L[\cos t]_{s\to s+1} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= -\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]_{s\to s+1}$$
$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to s+1}$$
$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2} = \frac{s^2+2s}{(s^2+2s+2)^2}$$

$$= \frac{s(s+2)}{(s^2+2s+2)^2}$$

12. Find $L\left[\frac{sint}{t}\right]$.

Solution

$$L\left[\frac{sint}{t}\right]=L\left[\frac{f(t)}{t}\right]=\int_s^{\infty} F(s)ds$$

$$F(s) = L\,[sint]=\frac{1}{s^2+1^2}$$

$$\int_s^{\infty} F(s)ds=\int_s^{\infty}\frac{1}{s^2+1}ds=[tan^{-1}(s)]_s^{\infty}$$

$$=[tan^{-1}\infty - tan^{-1}s]=\left[\frac{\pi}{2} - tan^{-1}s\right] = cot^{-1}s$$

13. Find the Laplace transform of $f(t) = \dfrac{e^{-t}\sin t}{t}$.

Solution

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L\left(e^{-t}\sin t\right)ds$$

$$= \int_s^\infty L\left(\sin t\right)_{s+1} ds = \int_s^\infty \left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^\infty \frac{1}{(s+1)^2+1}ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

14. Find the Laplace Transform of $f(t) = \frac{1-cost}{t}$.

Solution

$$L[1-cost] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[1-cost]ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}[\log(s^2+1) - logs^2]_s^\infty$$

$$= -\frac{1}{2}\left[log\frac{s^2+1}{s^2}\right]_s^\infty = -\frac{1}{2}\left[log\left(1+\frac{1}{s^2}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}log\left[1+\frac{1}{s^2}\right] = \frac{1}{2}log\left(\frac{s^2+1}{s^2}\right)$$

15. Find $L\left[\frac{cos\ at - cos\ bt}{t}\right]$.

Solution

$$L\left[\frac{cosat - cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}log(s^2+a^2) - \frac{1}{2}log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[log\,\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty=\frac{1}{2}\left[log\,\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$=\frac{1}{2}\left[log1-log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]=\frac{1}{2}\;log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

16. Evaluate $\int_0^\infty t\,e^{-2t}\,sin\,t\,dt$ using Laplace transform.

Solution

$$\int_0^\infty te^{-2t}\sin t\,dt=\int_0^\infty e^{-st}f(t)dt=F(s)\text{ Here s = 2.}$$

$$F(s)\;=\;L\,[f\;(t)],F(s)\;=\;L\,[t\sin t]$$

$$=-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]=\frac{2s}{(s^2+1)^2}$$

$$\int_0^\infty \boldsymbol{t\,e^{-2t}\,sint\,dt}\;=[\boldsymbol{F(s)}]_{s\,=\,2}=\frac{\mathbf{4}}{(\mathbf{4+1})^2}=\frac{\mathbf{4}}{\mathbf{25}}$$

17. Verify initial value theorem for the function *f* (*t*) = 2 – cos t.

Solution

Initial value theorem states that $\lim_{t\to0}f\left(t\right)=\lim_{s\to\infty}sF\left(s\right)$

L. H. S. = $\lim_{t\to0}f(t)=2-\cos0=1$

R. H. S. = $\lim_{s\to\infty}sL(f(t))=\lim_{s\to\infty}sL(2-\cos t)$

$$=\lim_{s\to\infty}s\left(2-\frac{s^2}{s^2+1}\right)==\lim_{s\to\infty}s\left(2-\frac{1}{1+\frac{1}{s^2}}\right)=2-1=1$$

L.H.S=R.H.S

Initial value theorem verified.

18. Verify final value theorem for the function f(t) = 1 + e^{-t} (sin t + cos t).

Solution

$$L\left[f(t)\right]=F(s)$$

$$=\frac{1}{s}+L\left[\sin t+\cos t\right]_{s\to s+1}$$

$$=\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}=\frac{1}{s}+\frac{s+2}{(s+1)^2+1}$$

Final value theorem states that $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$

$$\text{L.H.S.}=\lim_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right]=1+0=1$$

$$\text{R. H. S.}=\lim_{s\to 0} s\left[\frac{1}{s}+\frac{s+2}{(s+1)^2+1}\right]=\lim_{s\to 0}\left[1+\frac{s^2+2s}{s^2+2s+2}\right]=1$$

L.H.S.=R.H.S

Hence final value theorem verified

19. Find $L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)$**.**

Solution

$$L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)=e^{3t}+1+\cosh 3t$$

20. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$**.**

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)=L^{-1}\left(\frac{1}{(s+2)}\right)-2L^{-1}\left(\frac{1}{(s+2)^2}\right)=e^{-2t}-2te^{-2t}$$

21. Find $L^{-1}\left(\frac{1}{s^2+2s+5}\right)$**.**

Solution

$$L^{-1}\left(\frac{1}{s^2+2s+5}\right)=L^{-1}\left(\frac{1}{(s+1)^2+4}\right)=\frac{e^{-t}\,\sin 2t}{2}$$

22. Find $\boldsymbol{L^{-1}\left(\frac{s}{s^2+4s+5}\right)}$**.**

Solution

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right)=L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right)=e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$=e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right)-2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$=e^{-2t}\left[\cos t-2\sin t\right]$$

23. Find $L^{-1}\left(\frac{s-5}{s^2-3s+2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s-5}{s^2-3s+2}\right)=L^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}\right)=L^{-1}\left(\frac{4}{s-1}\right)+L^{-1}\left(\frac{-3}{s-2}\right)=4e^{t}-3e^{2t}$$

24. Find $\boldsymbol{L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]}$**.**

Solution:

$$L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]\ =\ L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right]\ \because\ L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]$$

$$=\ L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right]+L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]\right)=\mathrm{e}^{-\mathrm{t}}\ (\cos \mathrm{t}+\sin \mathrm{t})$$

25. Find $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$.

Solution

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right]=L^{-1}\left[\frac{1}{(s+3)^2+4}\right]=L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$
$$=\frac{1}{2}L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]=\frac{1}{2}\mathrm{e}^{-3\mathrm{t}}\sin 2t.$$

26. Find $L^{-1}\left[\cot^{-1}(s+1)\right]$.

Solution:

Let $L^{-1}\left[\cot^{-1}(s+1)\right]=f(t)$

$$\therefore\ L\left[f(t)\right]=\cot^{-1}(s+1)$$

$$L\left[tf(t)\right]=-\frac{d}{ds}\left[\cot^{-1}(s+1)\right]=\frac{1}{(s+1)^2+1}$$

$$tf(t)=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]=e^{-t}\sin t$$

$$\therefore\ f(t)=\frac{e^{-t}\sin t}{t}$$

27. Find the inverse Laplace transform of $\frac{s}{(s+2)^2}$.

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$
$$=\frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)=\frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$
$$=\frac{d}{dt}\left(e^{-2t}\,t\right)=e^{-2t}+t\left(-2e^{-2t}\right)=e^{-2t}(1-2t)$$

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