Module – 1 Multiple Integrals

Evaluation of double integration Cartesian and plane polar coordinates – Evaluation of double integration by changing order of integration – Area as a double integral (Cartesian) – Area as a double integral (Polar) – Triple integration in Cartesian coordinates – Conversion from Cartesian to polar in double integrals – Volume using triple integral – Application of Multiple integral in Engineering.

Evaluation of double integration – Cartesian and Polar coordinates

Type – 1 Limits are constants

1. **Evaluate** $\int_0^1\int_1^2 \left(x^2+y^2\right)dx\,dy$ **.**

Solution:

$$\int_0^1\int_1^2 \left(x^2+y^2\right)dx\,dy = \int_0^1 \left(\frac{x^3}{3}+x\,y^2\right)_1^2 dy$$

$$= \int_0^1 \left[\left(\frac{8}{3}+2\,y^2\right)-\left(\frac{1}{3}+y^2\right)\right]dy$$

$$= \int_0^1 \left(\frac{7}{3}+y^2\right)dy$$

$$= \left(\frac{7}{3}y+\frac{y^3}{3}\right)_0^1 = \frac{8}{3}$$

Note: $\int_1^2\int_0^1 \left(x^2+y^2\right)dy\,dx = \frac{8}{3}$

If the limits of integration are constants, then the order of integration is insignificant.

2. **Evaluate** $\int_0^3\int_0^2 x\,y\left(x+y\right)dy\,dx$ **.**

Solution:

$$\int_0^3\int_0^2 x\,y\left(x+y\right)dy\,dx = \int_0^3\int_0^2 \left(x^2\,y+x\,y^2\right)dy\,dx$$

$$= \int_0^3 \left(\frac{x^2\,y^2}{2}+x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_0^3 \left(2\,x^2+\frac{8}{3}x\right)dx$$

$$= \left(2\frac{x^3}{3}+\frac{8}{3}\frac{x^2}{2}\right)_0^3 = 30$$

3. **Evaluate** $\int_2^a \int_2^b \frac{dx\,dy}{x\,y}$ **.**

Solution:

$$\int_2^a \int_2^b \frac{dx\,dy}{x\,y} = \int_2^a \left(\int_2^b \frac{dx}{x} \right) \frac{dy}{y}$$

$$= \int_2^a (\log x)_2^b \frac{dy}{y}$$

$$= (\log x)_2^b \ (\log x)_2^a$$

$$= \log\left(\frac{b}{2}\right) \log\left(\frac{a}{2}\right)$$

4. **Evaluate** $\int_0^3 \int_0^2 r\,dr\,d\theta$ **.**

Solution:

$$\int_0^3 \int_0^2 r\,dr\,d\theta = \int_0^3 \left(\frac{r^2}{2}\right)_0^2 d\theta = \int_0^3 2\,d\theta = 2(\theta)_0^3 = 6$$

Type – 2 Limits are variables

5. **Evaluate** $\int_0^1 \int_x^{\sqrt{x}} x\,y\,(x+y)\,dy\,dx$ **.**

Solution:

$$\int_0^1 \int_x^{\sqrt{x}} x\,y\,(x+y)\,dy\,dx = \int_0^1 \int_x^{\sqrt{x}} \left(x^2\,y + x\,y^2\right) dy\,dx$$

$$= \int_0^1 \left(\frac{x^2\,y^2}{2} + x\frac{y^3}{3} \right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^4}{2} - \frac{x^4}{3} \right) dx$$

$$= \left(\frac{x^4}{8} + \frac{x^{7/2}}{3\times\frac{7}{2}} - \frac{x^5}{10} - \frac{x^5}{15} \right)_0^1 = \frac{3}{56}$$

6. **Evaluate** $\int_0^a \int_0^{\sqrt{a^2-x^2}} y \; dy\, dx$ **.**

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y \; dy\, dx = \int_0^a \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\frac{a^2-x^2}{2}\right) dx = \frac{a^3}{3}$$

7. **Evaluate** $\int_0^a \int_0^{\sqrt{ay}} x\, y \; dx\, dy$ **.**

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y \; dy\, dx = \int_0^a y \left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^a y\, a\, y \; dy = \frac{a^4}{6}$$

CHANGE THE ORDER OF INTEGRATION

For changing the order of integration in a given double integral

Step 1: Draw the region of integration by using the given limits.

Step 2: After changing the order, consider

- **dxdy as horizontal strip**
- **dydx as vertical strip**

Step 3: Find the new limits.

Step 4: Evaluate the double integral.

8. **Change the order of integration in** $\int_0^a \int_y^a \frac{\mathbf{x\,dydx}}{\mathbf{x^2+y^2}}$ **and hence evaluate it.**

Solution:

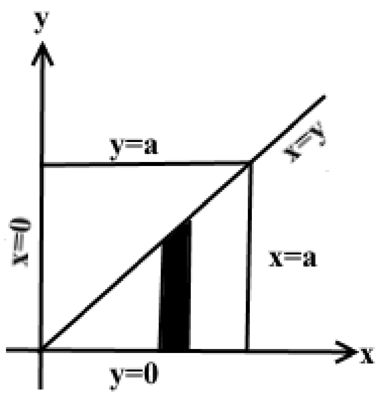
$\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$ **(Correct Form)**

Given limits $x: y \to a \;\; ; \;\; y: 0 \to a$

After changing the order,

$dy\, dx \to vertical\ strip$

Now, limit $x: 0 \to a \;\; ; \;\; y: 0 \to x$



$$\therefore \int_0^a \int_0^x \frac{\mathbf{x}}{\mathbf{x^2+y^2}} \; \mathbf{dydx} = \int_0^a \int_0^x x \left(\frac{1}{\mathbf{x^2+y^2}}\right) \mathbf{dy\, dx}$$

$$= \int_0^a x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)_0^x \mathbf{dx}$$

$$= \int_0^a x\left(\tan^{-1}(1) - \tan^{-1}(0)\right) \mathbf{dx} \quad \because \tan^{-1}(1) = \frac{\pi}{4},\ \tan^{-1}(0) = 0$$

$$= \int_0^a \left(\frac{\pi}{4}\right) \mathbf{dx}$$

$$= \left(\frac{\pi}{4}\right)(x)_0^a$$

$$= \frac{\pi a}{4}$$

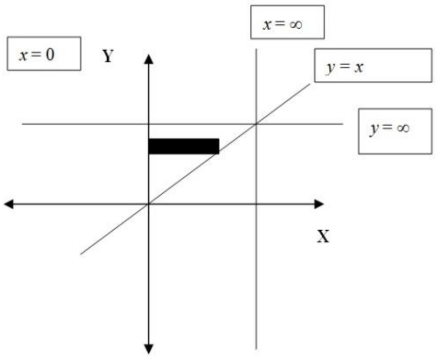
9. **Evaluate** $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ **by changing the order of integration.**

Solution:

Given limits:

***x* : 0 → ∞**

***y* : *x* → ∞**



After changing the order,

dxdy $\rightarrow$ *horizontal strip*

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \frac{e^{-y}}{y}(x)_0^y\ dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, y\, dy$$

$$= \int_0^\infty e^{-y} dy$$

$$= \left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$= \left(\frac{e^{-\infty}}{-1} - \left(\frac{e^{-0}}{-1}\right)\right)$$

$$= -e^{-\infty} + e^{-0}$$

$$= 1 \qquad \because e^{-\infty} = 0; \quad e^{-0} = e^{0} = 1$$

10. **Change the order of integration** $\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} dx\, dy$ **and hence evaluate it.**

Solution

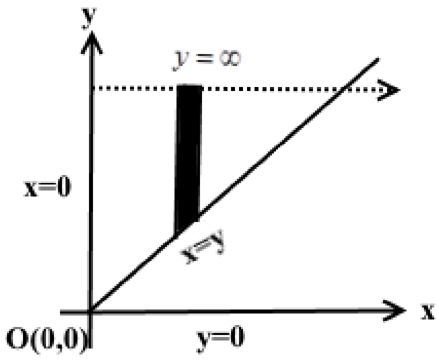
Given limits:

$x = 0, \quad x = y$

$y = 0, \quad y = \infty$

After changing the order,

$dy\, dx \to vertical\ strip$



$$\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} dx\, dy = \frac{1}{2}\int_0^\infty \int_x^\infty 2y e^{-\frac{y^2}{x}} dy\, dx$$

$$= \frac{1}{2}\int_0^\infty \left(\int_x^\infty 2y e^{-\frac{y^2}{x}} dy \right) dx$$

$$= \frac{1}{2}\int_0^\infty \left(\int_x^\infty e^{-\frac{y^2}{x}} d(y^2) \right) dx$$

$$= \frac{1}{2}\int_0^\infty \left[\int_x^\infty -x e^{-\frac{y^2}{x}} \right]_x^\infty dx$$

(Or) Use Substitution *$y^2 = t$, $2ydy = dt$, Limits: $t : x^2 \to \infty$*

$$= \frac{1}{2}\int_0^\infty \left[0 - \left(-x e^{-\frac{x^2}{x}} \right) \right] dx$$

$$= \frac{1}{2}\int_0^\infty x e^{-x} dx$$

$$= \frac{1}{2}\left[\frac{x e^{-x}}{-1} - (1)\left(\frac{e^{-x}}{(-1)(-1)} \right) \right]_0^\infty$$

$$= \frac{1}{2}\left[-x e^{-x} - e^{-x} \right]_0^\infty$$

$$= \frac{1}{2}\left[(0+0) - (0+1) \right] \qquad \because e^{-\infty} = 0,\ e^{0} = 1$$

$$= \frac{1}{2}$$

11. **Evaluate** $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$ **by changing the order of integration.**

Solution:

Given $y=0,\ y=\sqrt{a^2-x^2}$

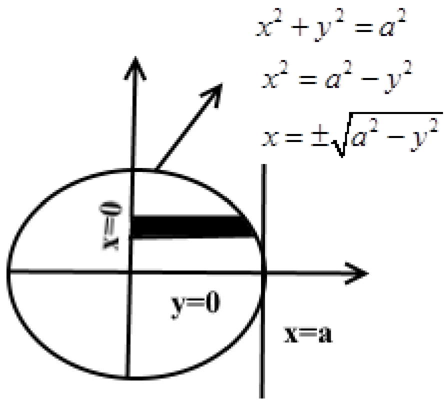
$y^2=a^2-x^2$

$x^2+y^2=a^2$

$x=0,\ x=a$

After changing the order,

dxdy $\rightarrow$ *horizontal strip*



$$\int_0^a \int_0^{\sqrt{a^2-y^2}} x\,y\ dx\,dy = \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$=\int_0^a \frac{y}{2}\left(a^2-y^2\right)dy$$

$$=\frac{a^2}{2}\int_0^a y\,dy-\frac{1}{2}\int_0^a y^3dy$$

$$=\frac{a^2}{2}\left(\frac{y^2}{2}\right)_0^a-\frac{1}{2}\left(\frac{y^4}{4}\right)_0^a$$

$$=\frac{a^4}{4}-\frac{a^4}{8}$$

$$=\frac{a^4}{4}-\frac{a^4}{8}$$

$$=\frac{a^4}{8}$$

12. **Changing the order of integration and hence evaluate** $\int_{0}^{1}\int_{x^2}^{2-x} xy\,dy\,dx$.

Solution:

Given limits:

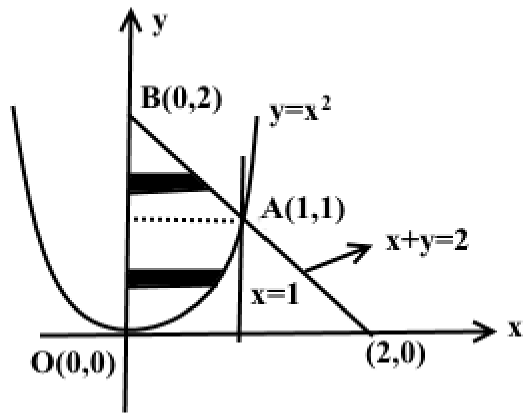
$y=x^2$

$y=2-x \Rightarrow x+y=2$

$x=0,\ \ y=1$

After changing the order,

dxdy $\rightarrow$ *horizontal strip*



$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx=\int_0^1\int_0^{\sqrt{y}} xy\,dxdy+\int_1^2\int_0^{2-y} xy\,dxdy$$

$$=I_1+I_2\quad (say)------(1)$$

To find I_1:

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\,dxdy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy$$

$$= \int_0^1 y\left(\frac{y}{2} - 0\right)dy$$

$$= \int_0^1 \frac{y^2}{2}dy$$

$$= \left(\frac{y^3}{6}\right)_0^1$$

$$= \left(\frac{1}{6} - 0\right)$$

$$I_1 = \frac{1}{6}$$

To find I_2:

$$I_2 = \int_1^2 \int_0^{2-y} xy\,dxdy$$

$$= \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_1^2 y\left(\frac{(2-y)^2}{2}\right)dy$$

$$= \int_1^2 \frac{y}{2}\left(4 - 4y + y^2\right)dy$$

$$= \frac{1}{2}\int_1^2 \left(4y - 4y^2 + y^3\right)dy$$

$$= \frac{1}{2}\left(\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right)_1^2$$

$$= \frac{1}{2}\left(\frac{4(2)^2}{2} - \frac{4(2)^3}{3} + \frac{2^4}{4} - \left(\frac{4(1)^2}{2} - \frac{4(1)^3}{3} + \frac{1^4}{4}\right)\right)$$

$$= \frac{1}{2}\left(8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(10 - \frac{28}{3} - \frac{1}{4}\right)$$

$$=\frac{1}{2}\left(\frac{10(12)-28(4)-1(3)}{12}\right)$$

$$=\frac{1}{2}\left(\frac{5}{12}\right)$$

$$I_2=\frac{5}{24}$$

$$(1)\Rightarrow I=\frac{1}{6}+\frac{5}{24}$$

$$=\frac{9}{24}$$

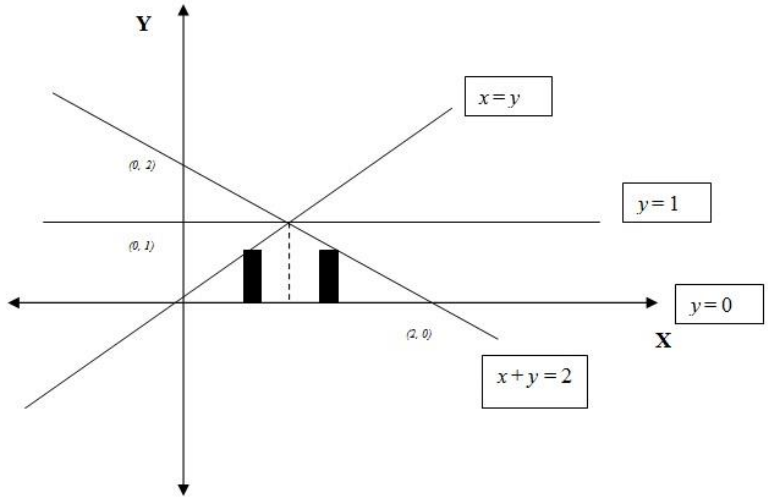
$$I=\frac{3}{8}$$

13. **Change the order of integration and hence evaluate** $\int_0^1 \int_y^{2-y} x\,y\,dx\,dy$ **.**

Solution:

Given limits: $x = y,\ x = 2 - y$

$y = 0\ ,\ y = 1$



After changing the order, $dy\,dx \rightarrow vertical\ strip$

$$\int_0^1 \int_y^{2-y} x\,y\,dx\,dy=\int_0^1 \int_0^x x\,y\,dy\,dx+\int_1^2 \int_0^{2-x} x\,y\,dy\,dx$$

$$=\int_0^1 x\left(\frac{y^2}{2}\right)_0^x dx+\int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$=\int_0^1 \frac{x^3}{2}dx+\frac{1}{2}\int_1^2 x\left(4+x^2-4x\right)dx=\frac{1}{8}+\frac{5}{24}=\frac{1}{3}$$

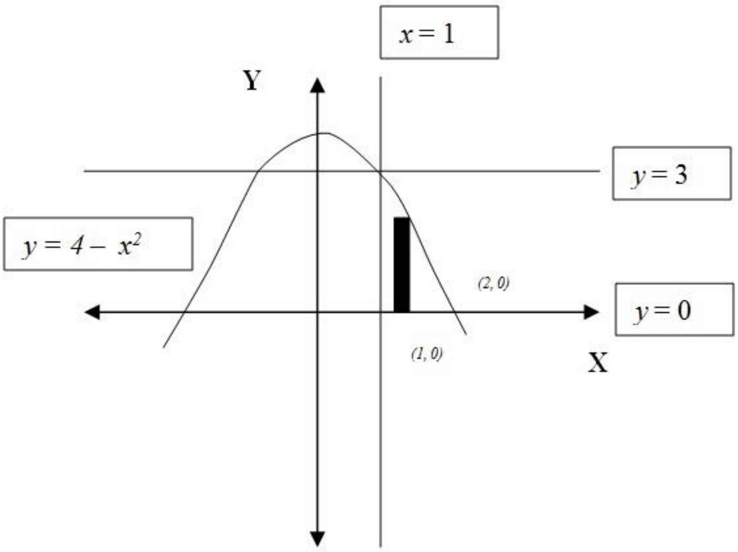
14. Change the order of integration and hence evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ **.**

Solution:

Given limits: $x = 1,\ x = \sqrt{4-y}$

$y = 0\ ,\ y = 3$

x	-2	-1	0	1	2
$y = 4 - x^2$	0	3	4	3	0



After changing the order, *dy dx* $\rightarrow$ *vertical strip*

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_1^2 \int_0^{4-x^2} (x+y)\,dy\,dx$$

$$= \int_1^2 \left(x\,y + \frac{y^2}{2} \right)_0^{4-x^2} dx$$

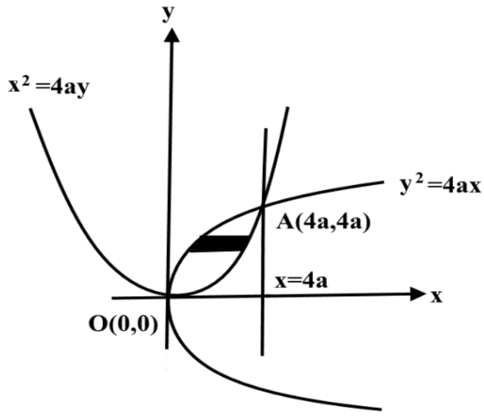
$$= \int_1^2 \left(4x - x^3 + 8 + \frac{x^4}{2} - 4x^2 \right) dx$$

$$= \frac{241}{60}$$

15. **Change the order of integration and hence evaluate** $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ **.**

Solution:

Given limits:



$$y=\frac{x^2}{4a}\Rightarrow x^2=4ay-------(1)$$

$$y=2\sqrt{ax}\Rightarrow y^2=4ax------(2)$$

$$x=0\quad,\qquad x=4a$$

Sub (1) in (2),

$$\left(\frac{y^2}{4a}\right)^2=4ay$$

$$\frac{y^4}{16a^2}=4ay$$

$$y^4=64a^3y$$

$$\left(y^4-64a^3y\right)=0$$

$$y\left(y^3-64a^3\right)=0$$

$$y=0 \;\; and \;\; y^3-64a^3=0$$

$$y=0 \quad and \quad y^3=64a^3$$

$$y=0 \quad and \quad y=4a$$

$$when \;\; y=0\Rightarrow x=0$$

$$when \;\; y=4a\Rightarrow x=\frac{16a^2}{4a}=4a$$

After changing the order, *dx dy* → *horizontal strip*

dy dx → *vertical strip*

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx=\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dxdy$$

$$=\int_0^{4a}\left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$=\int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$$

$$=\int_0^{4a}\left(2\sqrt{a}(y)^{1/2}-\frac{y^2}{4a}\right)dy$$

$$=\left(2\sqrt{a}\,\frac{(y)^{3/2}}{3/2}-y^3/12a\right)_0^{4a}$$

$$=\left(\frac{4}{3}\sqrt{a}(4a)^{3/2}-\frac{(4a)^3}{12a}\right)$$

$$=\left(\frac{32a^2}{3}-\frac{(4a)^3}{12a}\right) \quad \because (4)^{3/2}=4\sqrt{4}=8$$

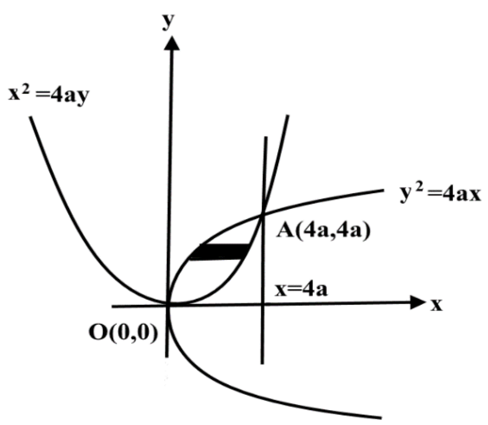
$$=\left(\frac{32a^2}{3}-\frac{64a^3}{12a}\right)$$

$$=\left(\frac{32a^2}{3}-\frac{16a^2}{3}\right)=\frac{16a^2}{3}$$

16. **Change the order of integration and hence evaluate** $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} x\,y\,dy\,dx$**.**

Solution:

Given limits:



$$y=\frac{x^2}{4a}\Rightarrow x^2=4ay-------(1)$$

$$y=2\sqrt{ax}\Rightarrow y^2=4ax------(2)$$

$$x=0 \quad , \qquad x=4a$$

Sub (1) in (2),

$$\left(\frac{y^2}{4a}\right)^2=4ay$$

$$\frac{y^4}{16a^2}=4ay$$

$$y^4=64a^3y$$

$$\left(y^4-64a^3y\right)=0$$

$$y\left(y^3-64a^3\right)=0$$

$$y=0 \ \ and \ \ y^3-64a^3=0$$

$$y=0 \quad and \quad y^3=64a^3$$

$$y=0 \quad and \quad y=4a$$

$$when\ \ y=0\Rightarrow x=0$$

$$when\ \ y=4a\Rightarrow x=\frac{16a^2}{4a}=4a$$

After changing the order,

dx dy $\rightarrow$ *horizontal strip*

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} x\,y\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} x\,y\,dx\,dy$$

$$= \int_0^{4a} y\left(\frac{x^2}{2}\right)_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left(2a\,y^2 - \frac{y^5}{32a^2}\right)dy = \frac{64}{3}a^4$$

Area as a double integral (Cartesian Coordinates) $\iint_R dx\,dy$ **or** $\iint_R dy\,dx$

17. Find the area of the circle $x^2 + y^2 = a^2$.

Solution:

Area of circle = 4 × Area in first quadrant

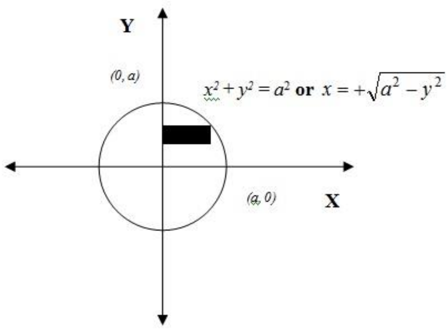
$$= 4\int_0^a \int_0^{\sqrt{a^2-y^2}} dx\,dy$$

$$= 4\int_0^a (x)_0^{\sqrt{a^2-y^2}}\,dy$$

$$= 4\int_0^a \sqrt{a^2-y^2}\,dy$$

$$= 4\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a^2}{2}\frac{\pi}{2}\right] = \pi a^2$$



18. **Find the area bounded by the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **using double integration.**

Solution: By the symmetry of the curve the area of the ellipse is

Area = 4 Area in the first quardrant

$$= 4\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx$$

$$= 4\int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\ dx$$

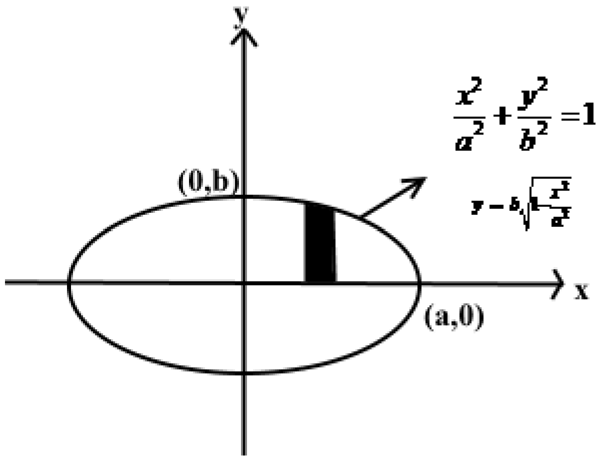
$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{\cancel{4}b}{a}\left[\frac{a^2}{\cancel{2}}\sin^{-1}\right]$$

$$= \cancel{2}ab\left(\frac{\pi}{\cancel{2}}\right)$$

$$= \pi ab.$$



19. **Find the double integration the area by the curves** $y^2 = 4ax$ **and** $x^2 = 4ay$ **.**

Solution:

The area is closed by the parabola

$y^2 = 4ax$ --------(1) **and** $x^2 = 4ay$ --------(2)

To find the limits solve (1) and (2)

$(2) \Rightarrow \quad y = \frac{x^2}{4a}$

sub in (1)

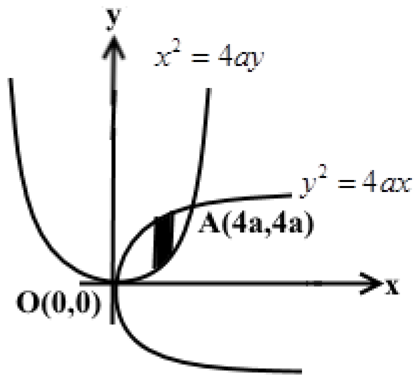
$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$x^4 = 64a^3$$

$$(x^4 - 64a^3) = 0$$

$x = 0$ **or** $(x^3 - 64a^3) = 0$

$x = 0$ **or** $x^3 = 64a^3$ Þ $x = 4a$



$$\therefore \textbf{ Area } = \int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dydx = \int_0^{4a} [y]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx = \int_0^{4a}\left[\sqrt{4ax} - \frac{x^2}{4a}\right]dx$$

$$= \int_0^{4a}\left[2\sqrt{a}\,x^{1/2} - \frac{1}{4a}x^2\right]dx = \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

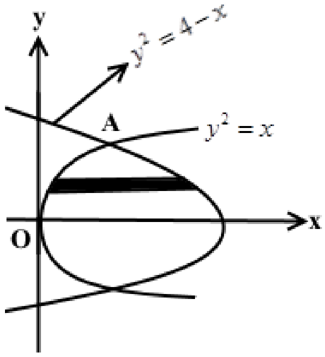
$$= \frac{4\sqrt{a}}{3}(4)^{3/2}(a)^{3/2} - \frac{1}{12a}64a^3 = \frac{4^{5/2}}{3}a^{4/2} - \frac{1}{12a}64a^3$$

$$= \frac{(2^2)^{5/2}}{3}a^2 - \frac{16}{3}a^2 = \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

20. **Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = x$ by double integration.**

Solution:



The area is bounded by

$y^2 = 4 - x$ ————————(1)

$y^2 = x$ —————————(2)

$y^2 = -(x-4)$ is a parabola with vertex $(4,0)$ and in the direction of negative x-axis both the curves are symmetric about x-axis.

To find the limits solve (1) and (2)

$4 - x = x$

$2x = 4 \Rightarrow x = 2$

$y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

$$\textbf{Area } = 2\int_0^{\sqrt{2}}\int_{y^2}^{4-y^2} dxdy$$

$$= 2\int_0^{\sqrt{2}} [x]_{y^2}^{4-y^2} dy$$

$$= 2\int_0^{\sqrt{2}} (4 - y^2 - y^2)\, dy$$

$$= 2\int_0^{\sqrt{2}} (4 - 2y^2)\, dy$$

$$= 2\left[4y - \frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

$$= 2\left[4\sqrt{2} - \frac{2\left(\sqrt{2}\right)^3}{3} - 0\right]$$

$$= 2\left[4\sqrt{2} - \frac{2(2)^{3/2}}{3}\right]$$

$$= 2\left[4\sqrt{2} - \frac{2(2)(2)^{1/2}}{3}\right]$$

$$= 2\left[4\sqrt{2} - \frac{4\sqrt{2}}{3}\right]$$

$$= 2\left(4\sqrt{2}\right)\left[1 - \frac{1}{3}\right]$$

$$= 8\sqrt{2}\left[\frac{2}{3}\right]$$

$$= \frac{16}{3}\sqrt{2}$$

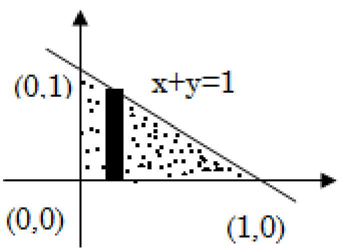
21. **Evaluate** $\iint_R (x^2 + y^2)\,dy\,dx$ **over the region R for which** $x, y \ge 0,\ x + y \le 1.$

Solution:

The region of integration is the triangle bounded by the lines

$x = 0,\ y = 0,\ x + y = 1$

Limits of y : 0 to $1 - x$; Limits of x : 0 to 1



$$\iint_R (x^2 + y^2)\,dy\,dx = \int_0^1 \int_0^{1-x} \left(x^2 + y^2\right) dy\,dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{1}{6}$$

Area as a double integral (Polar Coordinates) $\iint\limits_R r\,dr\,d\theta$

22. Find the area of the cardioid $r = a(1+\cos\theta)$ by using double integration.

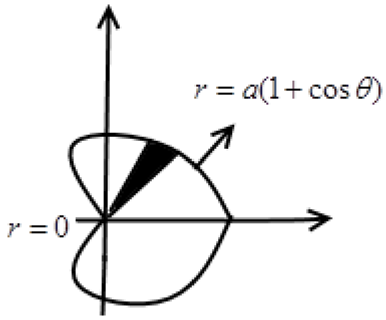
Solution:

Given the curve in polar co ordinates $r = a(1+\cos\theta)$

$\therefore$ Area of the cardioid = 2(Area above the initial line)

θ varies from 0 to π

r varies from 0 to $r = a(1+\cos\theta)$



$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\left(\frac{1+\cos\theta}{2}\right)\right]d\theta$$

$$= a^2\int_0^{\pi}\left[\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right]d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\pi} \qquad \because \sin n\pi = 0,\ \forall n$$

$$= a^2\left[\frac{3}{2}\pi\right]$$

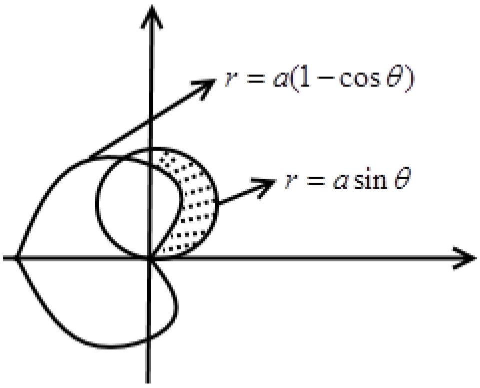
$$= \frac{3\pi a^2}{2}$$

23. **Find the area inside the circle $r = a\sin\theta$ but lying outside the cardioid $r = a(1-\cos\theta)$.**

Solution:

Given $r = a\sin\theta$ -------- (1)

and $r = a(1-\cos\theta)$ ---- (2)



Eliminating r from (1) and (2)

$$a\sin\theta = a(1-\cos\theta)$$

$$\sin\theta + \cos\theta = 1 \text{------}(3)$$

$$(3)^2 \Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$$

$$1 + 2\sin 2\theta = 1$$

$$\sin 2\theta = 0$$

$$2\theta = 0,\ \pi$$

$$\theta = 0,\ \frac{\pi}{2}$$

$$\text{Area} = \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta$$

$$\text{Area} = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta}$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \left[\sin^2\theta - \left(1 - 2\cos\theta + \cos^2\theta\right)\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \left[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \left[-1 + 2\cos\theta\right] d\theta \qquad \because \int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{a^2}{2}\left[-\theta + 2\sin\theta\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(-\frac{\pi}{2} + 2\sin\frac{\pi}{2}\right) - 0\right]$$

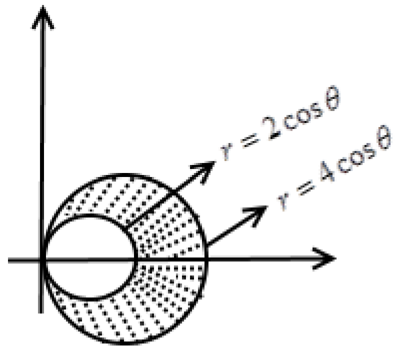
$$= \frac{a^2}{2}\left(-\frac{\pi}{2} + 2\right) \qquad = \frac{a^2}{4}(4 - \pi)$$

Find the area bounded between $r = 2\cos\theta$ and $r = 4\cos\theta$.

24. **Solution:**

$$\text{Area} = \iint_R r\,dr\,d\theta$$

Where the region R is the area between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$



$\therefore$ **r varies from $r = 2\cos\theta$ to $r = 4\cos\theta$**

θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\text{Area} = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[16\cos^2\theta - 4\cos^2\theta\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 12\cos^2\theta\,d\theta$$

$$= 6\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta$$

$$= 6(2)\int_{0}^{\pi/2} \cos^2\theta\,d\theta$$

$$= 6(2)\frac{1}{2}\frac{\pi}{2} \qquad \because \int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\frac{\pi}{2}$$

$$= 3\pi$$

Conversion from Cartesian to Polar in double integrals

Evaluation of double integrals by changing Cartesian coordinates to polar coordinates:

Changing from (x, y) *to* (r, θ), the variables are related by $x = r\cos\theta, y = r\sin\theta$

$and\ \ dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$

$\therefore \iint f(x, y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

Formula

$$\int_0^{\pi/2} \sin^n \theta\, d\theta = \int_0^{\pi/2} \cos^n \theta\, d\theta = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}\times 1 & if\ n\, is\, odd \\ \\ \dfrac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2}\times \dfrac{\pi}{2} & if\ n\, is\, even \end{cases}$$

25. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar coordinates. And hence find $\int_0^\infty e^{-x^2}\, dx$

Solution:

$x = r\cos\theta, y = r\sin\theta \quad and \quad dx\,dy = r\,dr\,d\theta$

r varies from 0 to ∞, θ varies from 0 to $\dfrac{\pi}{2}$

$$I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

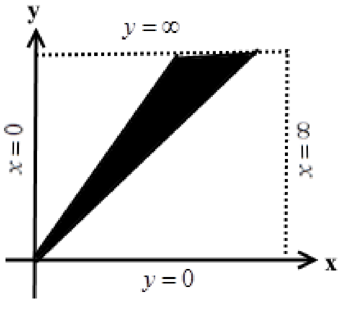
$$= \int_0^{\pi/2}\left(\int_0^\infty e^{-r^2}\ r\,dr\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\int_0^\infty e^{-t}\ dt\right)d\theta \quad \because let\, r^2 = t \Rightarrow 2r\,dr = dt \text{ and r: 0 to } \infty \Rightarrow \text{ t: 0 to } \infty$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[-e^{-t}\right]_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[-e^{-\infty}+e^0\right]d\theta \quad \because e^{-\infty}=0, e^0 = 1$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}[\theta]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4}$$



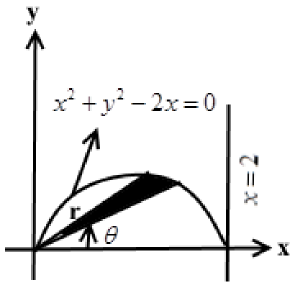
Since $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}\ dy\ = \dfrac{\pi}{4}$

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-x^2}dx\ = \frac{\pi}{4}$$

$$\Rightarrow\left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4} \Rightarrow \int_0^\infty e^{-x^2}dx\ = \frac{\sqrt{\pi}}{2}$$

26. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ by changing to polar coordinates.

Solution:



$x = r\cos\theta, y = r\sin\theta$ *and* $dx\,dy = r\,dr\,d\theta$

The limits of x are x=0 to x=2,

The limits of y are y=0 to y= $\sqrt{2x-x^2}$

$y = 0 \Rightarrow r\cos\theta = 0$

$\Rightarrow r = 0$ *and* $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2$

$x^2 + y^2 - 2x = 0$

$x = 0 \Rightarrow r\sin\theta = 0$

$\sin\theta = 0$

$\Rightarrow \theta = 0$

$\Rightarrow r^2 - 2r\cos\theta = 0$

$\Rightarrow r = 2\cos\theta$

r varies from 0 to $2\cos\theta$, θ varies from 0 to $\frac{\pi}{2}$

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4\cos^3\theta\, d\theta = 2\int_0^{\pi/2} \cos^3\theta\, d\theta = 2\left[\frac{2}{3}.1\right] = \frac{4}{3}$$

27. Evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$ by changing to polar coordinates.

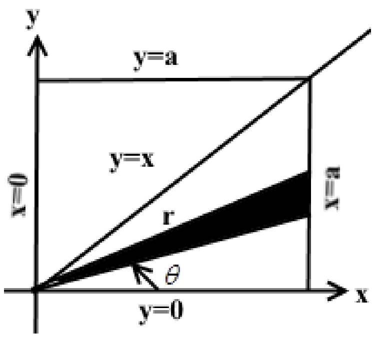
Solution:

$x = r\cos\theta, y = r\sin\theta$ *and* $dx\,dy = r\,dr\,d\theta$

The limits of x are x=y to x=a, The limits of y are y=0 to y=a

$x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$,

$x = a \Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$



$y=0 \Rightarrow r\sin\theta = 0$

$\Rightarrow r=0 \ \ and \sin\theta = 0 \Rightarrow \theta = 0$

r varies from 0 to $\dfrac{a}{\cos\theta}$, θ varies from 0 to $\dfrac{\pi}{4}$

$$I=\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$=\int_0^{\pi/4}\int_0^{a/\cos\theta}\frac{r^2\cos^2\theta}{r}\,r\,dr\,d\theta$$

$$=\int_0^{\pi/4}\int_0^{a/\cos\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$=\int_0^{\pi/4}\cos^2\theta\left[\frac{r^3}{3}\right]_0^{a/\cos\theta}d\theta$$

$$=\frac{1}{3}\int_0^{\pi/4}\cos^2\theta\left[\frac{a^3}{\cos^3\theta}-0\right]d\theta$$

$$=\frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta$$

$$=\frac{a^3}{3}\left[\log(\sec\theta+\tan\theta)\right]_0^{\pi/4}=\frac{a^3}{3}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\log\left(\sec 0+\tan 0\right)\right]=\frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right)\right]$$

28. Evaluate $\int_0^{2a}\int_0^{\sqrt{2x-x^2}}\left(x^2+y^2\right)dy\,dx$ by changing to polar coordinates

Solution:

$x=r\cos\theta, y=r\sin\theta \ \ and \ \ dx\,dy=r\,dr\,d\theta$

The limits of x are x=0 to x=2a, The limits of y are y=0 to y = $\sqrt{2ax-x^2}$

$y=0 \Rightarrow r\sin\theta=0$

$\Rightarrow r=0 \ \ and \ \ \sin\theta=0 \Rightarrow \theta=0$

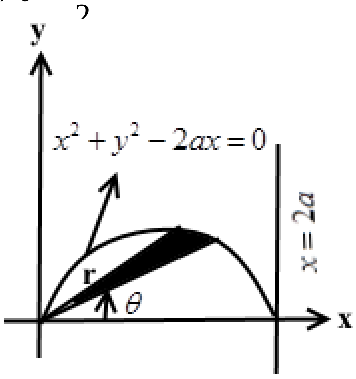
$y=\sqrt{2ax-x^2} \Rightarrow y^2=2ax-x^2$ $\qquad$ $x=0 \Rightarrow r\cos\theta=0$

$x^2+y^2-2ax=0$ $\qquad$ $\cos\theta=0 \Rightarrow \theta=\dfrac{\pi}{2}$

$\Rightarrow r^2-2ar\cos\theta=0$

$\Rightarrow r=2a\cos\theta$

r *varies from* 0 *to* $2a\cos\theta$, θ *varies from* 0 *to* $\dfrac{\pi}{2}$



$$I = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \left(x^2+y^2\right) dy\, dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \left(r^2\right) r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \left(r^3\right) dr\, d\theta$$

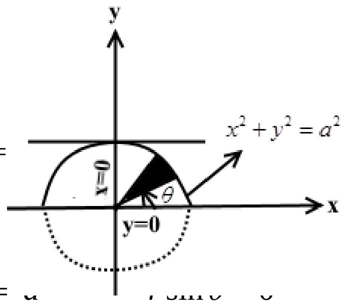
$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\left(2a\cos\theta\right)^4}{4} - 0\right) d\theta$$

$$= \frac{16a^4}{4} \int_0^{\frac{\pi}{2}} \left(\cos^4\theta\right) d\theta = 4a^4 . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} = \frac{3a^4\pi}{4}$$

29. Evaluate $\int_0^{a} \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dy\, dx$ by changing to polar coordinates.

Solution:



$x = r\cos\theta, y = r\sin\theta$ *and* $dx\, dy = r\, dr\, d\theta$

The limits of x are x=0 , $x = \sqrt{a^2-y^2}$, and limits of y are y=0, y=

$x = 0 \Rightarrow r\cos\theta = 0$ $\qquad x = \sqrt{a^2-y^2} \Rightarrow x^2+y^2$

$r = 0 \text{ and } \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ $\qquad r^2 =$

$\Rightarrow r = a$ $\qquad \sin\theta = 0 \Rightarrow \theta = 0$

r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$

$$I = \int_0^{a} \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dy\, dx$$

$$= \int_0^{\pi/2} \int_0^{a} \left(r^2\right) r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{a} \left(r^3\right) dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4} \right)_0^a d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{a^4}{4} \right) d\theta = \frac{a^4}{4} (\theta)_0^{\pi/2} = \frac{a^4}{4} \frac{\pi}{2} = \frac{a^4 \pi}{8}$$

Triple Integration (Cartesian Coordinates)

$$I = \int_{z=z_1}^{z_2} \int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} f(x,y,z)\ dx\ dy\ dz$$

Also

$$I = \int_{x=x_1}^{x_2} \int_{y=y_1}^{y_2} \int_{z=z_1}^{z_2} f(x,y,z)\ dz\ dy\ dx$$

30. Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz\ dz\ dy\ dx$

Solution:

$$I = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} xyz\ dz\ dy\ dx = \int_0^1 \int_0^2 \left(\frac{z^2}{2} \right)_0^3 dy\, dx$$

$$= \int_0^1 \int_0^2 \left(\frac{9}{2} - 0 \right)_0^3 dy\, dx = \frac{9}{2} \int_0^1 x \left(\frac{y^2}{2} \right)_0^2 dx$$

$$= \frac{9}{2} \int_0^1 x \left(\frac{4}{2} - 0 \right) dx = \frac{9}{2} \int_0^1 2x\ dx = 9 \int_0^1 x\ dx = 9 \left(\frac{x^2}{2} \right)_0^1 = 9 \left(\frac{1}{2} - 0 \right) = \frac{9}{2}$$

31. Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dxdydz$

Solution:

$$I = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} e^{x+y+z}\ dx\ dy\ dz = \int_0^1 \int_0^1 \left[e^{1+y+z} - e^{y+z} \right]\ dy\ dz$$

$$= \int_0^1 \left(e^{z+2} - 2e^{z+1} + e^z \right) dz$$

$$= e^3 - 3e^2 + 3e - 1$$

$$= (e-1)^3$$

32. **Evaluate** $\int_0^c \int_0^b \int_0^a (x+y+z)\,dx\,dy\,dz$ **.**

Solution:

$$\int_0^c \int_0^b \int_0^a (x+y+z)\,dx\,dy\,dz = \int_0^c \int_0^b \left(\frac{x^2}{2}+x\,y+x\,z\right)_0^a dy\,dz$$

$$= \int_0^c \int_0^b \left(\frac{a^2}{2}+a\,y+a\,z\right) dy\,dz$$

$$= \int_0^c \left(\frac{a^2}{2}y+a\frac{y^2}{2}+a\,z\,y\right)_0^b dz$$

$$= \int_0^c \left(\frac{a^2}{2}b+a\frac{b^2}{2}+a\,z\,b\right) dz$$

$$= \left(\frac{a^2}{2}b\,z+a\frac{b^2}{2}z+a\,b\frac{z^2}{2}\right)_0^c$$

$$= \frac{a\,b\,c\,(a+b+c)}{2}$$

33.

Evaluate $\int_0^4 \int_0^x \int_0^{\sqrt{x+y}} z\;dx\,dy\,dz$ **.**

Solution:

$$I = \int_{x=0}^{4} \int_{y=0}^{x} \int_{z=0}^{\sqrt{x+y}} z\;dz\,dy\,dx$$

$$= \int_0^4 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dydx$$

$$= \frac{1}{2}\int_0^4 \int_0^x (x+y)dydx$$

$$= \frac{1}{2}\int_0^4 \left(xy+\frac{y^2}{2}\right)_0^x dx \quad = \frac{1}{2}\int_0^4 \left(x^2+\frac{x^2}{2}\right) dx \quad = \frac{3}{4}\int_0^4 x^2 dx = \frac{3}{4}\left(\frac{x^3}{3}\right)_0^4 = 16$$

34. **Evaluate** $\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$ **.**

Solution:

$$\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_0^{\log 2} \int_0^x \left(e^z\right)_0^{x+y} e^y\;e^x\;\;dy\,dx$$

$$= \int_0^{\log 2} \int_0^{x} \left(e^{2x}\, e^{2y} - e^{x}\, e^{y}\right) dy\, dx$$

$$= \int_0^{\log 2} \left(e^{2x}\, \frac{e^{2y}}{2} - e^{x}\, e^{y}\right)_0^{x} dx$$

$$= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x}\right) dx = \frac{5}{8}$$

35. **Evaluate** $\displaystyle\int_0^{a} \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{a^2-x^2-y^2-z^2}}$

Solution:

Let $\displaystyle I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{a^2-x^2-y^2-z^2}}$

$$= \int_0^{a} \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \int_0^{a} \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\, dx$$

$$= \int_0^{a} \int_0^{\sqrt{a^2-x^2}} \left[\frac{\pi}{2} - 0\right] dy\, dx \quad = \frac{\pi}{2} \int_0^{a} [y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \frac{\pi}{2} \int_0^{a} \sqrt{a^2-x^2}\, dx \quad = \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^{a}$$

$$= \frac{\pi}{2}\left[\left(0 + \frac{a^2}{2}\frac{\pi}{2}\right) - (0+0)\right] = \frac{\pi^2 a^2}{8}$$

36. **Evaluate** $\displaystyle\int\int\int \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$ **for all positive values of x,y,z for which the integral is real.**

Solution:

Let $\displaystyle I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$

$$= \int_0^{1} \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^{1} \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{\pi}{2} - 0\right] dy\,dx \quad = \frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi^2}{8}$$

37. Evaluate $\iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes $\mathbf{x=0, y=0}$, $\mathbf{z=0, x+y+z=1}$

Solution:

Here z varies from $z=0$ to $z=1-x-y$

y varies from $y=0$ to $y=1-x$

x varies from $x=0$ to $x=1$

$$\therefore \iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x} \left[2^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x} \left[\frac{1}{4} - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}y + (x+y+1)^{-1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\left(\frac{1}{4}(1-x) + 2^{-1}\right) - \left(0 + (x+1)^{-1}\right)\right] dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{1+x}\right] dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{3}{4} - \frac{x}{4} - \frac{1}{1+x}\right] dx$$

$$=-\frac{1}{2}\left[\frac{3}{4}x-\frac{x^2}{8}-\log(1+x)\right]_0^1$$

$$=-\frac{1}{2}\left[\left(\frac{3}{4}-\frac{1}{8}-\log 2\right)-(0-0-0)\right]$$

$$=\frac{1}{2}\log 2-\frac{5}{16}$$

Volume using Triple Integral

38. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:

Since the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is symmetric about the coordinate planes.

Volume of ellipsoid = $8\times$ volume in the first octant.

In the first octant,

z varies from 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

x varies from 0 to a

$$\text{volume } =8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ dy\ dx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}[z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\ dy\ dx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\ dx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\frac{c}{b}\left(\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\right)dy\ dx$$

$$=\frac{8c}{b}\int_0^a\left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}+\frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right)\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\ dx$$

$$=\frac{4c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)\left(\sin^{-1}1-\sin^{-1}0\right)dx$$

$$=\frac{4c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}dx$$

$$=2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$=2\pi bc\left[x-\frac{1}{a^2}\frac{x^3}{3}\right]_0^a$$

$$=2\pi bc\left[a-\frac{a^3}{3a^2}-0\right]=2\pi bc\left(a-\frac{a}{3}\right)=2\pi bc\left(\frac{2a}{3}\right)=\frac{4}{3}\pi abc$$

39. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate's planes.

Solution:

The region of integration is the region bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,\ x=0,\ y=0, z=0$

z varies from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$

x varies from 0 to a

$$\text{volume }=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\ dy\ dx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\left(z\right)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=c\int_0^a\left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)b\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= c\int_0^a \left[\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= \frac{bc}{2}\int_0^a \left[\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{-3/a}\right]_0^a = \frac{-abc}{6}[0-1] = \frac{abc}{6}$$

40. Find the volume of sphere $x^2+y^2+z^2=a^2$ using triple integrals.

Solution:

Since the sphere $x^2+y^2+z^2=a^2$ is symmetric about the coordinate plane

Volume of sphere = $8 \times$ volume in the first octant.

Int the first octant,

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a

Volume of sphere = $= 8\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8\int_0^a\left[\frac{y\sqrt{a^2-x^2-y^2}}{2}+\frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a\left[0-\frac{a^2-x^2}{2}\sin^{-1}(1)\right]-\left[0-\frac{a^2-x^2}{2}\sin^{-1}(0)\right]dx$$

$$= 8\int_0^a\left(\frac{a^2-x^2}{2}\right)\left[\sin^{-1}(1)-\sin^{-1}(0)\right]dx$$

$$= 4\int_0^a \left(a^2 - x^2\right)\left[\frac{\pi}{2} - 0\right]dx$$

$$= 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\pi\left(a^3 - \frac{a^3}{3}\right) = 2\pi\left(\frac{2a^3}{3}\right) = \frac{4\pi a^3}{3}$$

* * * * *