Differential Calculus

The rate of bending of a curve in any interval is called called a curvature of curve.

* The curvature of a straight line is '0'
* The curvature of a circle is constant.

→ Curvature is denoted by ρ

where, Radius of curvature $(\rho) = \frac{1}{R}$

R = Radius of a circle.

To Find Radius of curvature (R.O.C)

1) cartesian form (x, y)
2) polar form (r, θ)
3) Implicit form (x, y)
4) parametric form

cartesian co-ordinates

1) Find the R.O.C of the curve $y = e^x$ at $(0, 1)$

$$\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$$

$$y_1 = \frac{d}{dx}(e^x)$$

$$y_1 = e^x \text{ at } (0,1) = 1$$

$$y_2 = \frac{d}{dx}(y_1) = e^x \text{ at } (0,1) = 1$$

$$\rho = \frac{[1+1^2]^{3/2}}{1} = 2^{3/2} = 2\sqrt{2}$$

2) Find the R.O.C at $x = \frac{\pi}{2}$ on $y = 4\sin x$

$y = 4\sin x$ at $\pi/2$

$y_1 = 4\cos x \quad y_1 = 4(0) = 0$

$y_2 = -4\sin x \quad y_2 = -4(1) = -4$

$$\frac{(1+0)^{3/2}}{-4} = \frac{-1}{4}$$

$$|\rho| = 1/4$$

$\therefore$ Radius of curvature ~~is~~ is always +ve

3) Find R.O.C at $(1/4, 1/4)$ on $\sqrt{x}+\sqrt{y}=1$

$$\sqrt{y} = 1-\sqrt{x}$$

$$y = (1-\sqrt{x})^2$$

$$y = 1+x-2\sqrt{x}$$

$$y_1 = 1-\frac{\cancel{2}}{\cancel{2}\sqrt{x}} = 1-\frac{1}{\sqrt{x}}$$

$$y_1 = 1-x^{-1/2}$$

$$y_2 = -\left(-\frac{1}{2}x^{-3/2}\right)$$

$$y_2 = \frac{1}{2\sqrt{x}\cdot x}$$

y_1 at $(1/4, 1/4)$

$$y_1 = 1-\left(\frac{1}{4}\right)^{-1/2}$$

$$= 1-\frac{1}{\sqrt{1/4}} = 1-2 = -1$$

y_2 at $(1/4, 1/4)$

$$y_2 = \frac{1}{2}\cdot\frac{1}{\frac{1}{4}\cdot\frac{1}{2}} = \frac{8}{2} = 4$$

$$\rho = \left[\frac{(1+(-1)^2)^{3/2}}{4}\right] = \frac{2^{3/2}}{4} = \cancel{\frac{2^3}{2^2}\cdot\frac{1}{2^2} \neq \frac{8}{16} \neq \frac{1}{2}}$$

$$\rho = \frac{2\cdot\sqrt{2}}{2^2} = \frac{1}{\sqrt{2}}$$

4) Show that R.O.C at any point of catenary

$y = \cancel{\cos} c.\ \cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$. Find curvature also

~~at~~ ρ at $(0,c)$?

$$y_1 = \cancel{c}\cdot \sinh\left(\frac{x}{c}\right)\cdot\frac{1}{\cancel{c}} = \sinh\frac{x}{c}$$

$$y_2 = \cosh\frac{x}{c}\cdot\frac{1}{c}$$

$$\rho = \frac{\left(1+\sinh^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{\left(\cosh^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{\cosh^3\frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{\cosh^2\frac{x}{c}}{\frac{1}{c}}$$

$$= c\cdot\cosh^2\frac{x}{c}$$

$$= c\cdot\frac{y^2}{c^2} = \frac{y^2}{c}$$ Hence proved

at $(0,c)$ $$\rho = \frac{[1+0]^{3/2}}{\frac{1}{c}\cdot\cosh\frac{(0)}{c}} = \frac{1}{1/c} = c$$

5) $xy = c^2$ (c,c) find ρ

$$y = c^2/x$$

$$y_1 = -\frac{c^2}{x^2}$$

$$y_2 = -(-2)\frac{c^2}{x^3} = \frac{2c^2}{x^3}$$

$$\rho = \frac{\left[1+\left(-\frac{c^2}{x^2}\right)^2\right]^{3/2}}{\frac{2c^2}{x^3}} = \frac{\left[1+\frac{c^4}{x^4}\right]^{3/2}}{\frac{2c^2}{x^3}}$$

$$\rho = \frac{x^3\left[1+c^4/x^4\right]^{3/2}}{2c^2}$$

$$= \frac{c^3[1+1]^{3/2}}{2c^2} = \frac{c}{2}[2]^{3/2}$$

$$\rho = \frac{c}{2}\cdot 2\cdot\sqrt{2}$$

$$\boxed{\rho = c\sqrt{2}}$$

parametric form

$(a\cos\theta, b\sin\theta)$
$x \quad y$

R.O.C

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$x=a\cos\theta \quad y=b\sin\theta$

$\frac{dx}{d\theta}=-a\sin\theta \quad \frac{dy}{d\theta}=b\cos\theta$

$y_1=\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=-\frac{b}{a}\cot\theta$

$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right)$

$y_2=+\frac{b}{a}\operatorname{cosec}^2\theta.\frac{d\theta}{dx}$

$=+\frac{b}{a}\operatorname{cosec}^2\theta.\frac{1}{-a\sin\theta}$

$=-\frac{b}{a^2}\frac{1}{\sin^3\theta}$

$e=\frac{[1+y_1^2]^{3/2}}{y_2}$

$=\frac{1+\left(\frac{b^2}{a^2}\cot^2\theta\right)}{\left[-\frac{b}{a^2}\frac{1}{\sin^3\theta}\right]}$

$=-\left(\frac{a^2+b^2\cot^2\theta}{a^2}\right)^{3/2}\times\frac{a^2\sin^3\theta}{b}$

$=-\frac{1}{ab}\left[(a^2+b^2\cot^2\theta)^{3/2}\right]\sin^3\theta$

$e=-\frac{1}{ab}\left[(a^2\sin^2\theta+b^2\cos^2\theta)\right]^{3/2}$

Asteroid equation

$x = a\cos^3\theta \quad y = a\sin^3\theta$

$x^{2/3} + y^{2/3} = a^{2/3}$

P.T $\rho^3 = 27axy$

$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \cancel{\frac{3a\cos^2\theta(-\sin\theta)}{3a\sin^2\theta(\cos\theta)}}$

$y_1 = \frac{3a\sin^2\theta(\cos\theta)}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$

$y_2 = \frac{dy_1}{dx} = -\sec^2\theta \cdot \frac{d\theta}{dx}$

$y_2 = -\sec^2\theta \cdot \frac{1}{3a\cos^2\theta(-\sin\theta)}$

$= \cancel{\frac{-1}{3a\sin^2\theta\cos^2\theta}}$

$y_2 = \frac{1}{3a\cos^4\theta\sin\theta}$

$\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$= [1+\tan^2\theta]^{3/2} \; 3a\cos^4\theta \cdot \sin\theta$

$= \sec^3\theta \cdot 3a\cos^4\theta \cdot \sin\theta$

$\rho = 3a\cos\theta\sin\theta$

$\rho^3 = 27a^3\cos^3\theta\sin^3\theta$

$= 27a\,(a\cos^3\theta)(a\sin^3\theta)$

$\rho^3 = 27axy$

Hence proved.

cycloid equation

R.O.C

$x = a(\theta - \sin\theta)$

$y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a[1-\cos\theta]$$

$$\frac{dy}{d\theta} = a[\sin\theta]$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a[\sin\theta]}{a[1-\cos\theta]}$$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\frac{1-\cos 2\theta}{2} = \sin^2\theta$

$1-\cos\theta = 2\sin^2\theta/2$

$$= \frac{a[2\sin\theta/2\cos\theta/2]}{a[2\sin^2\theta/2]}$$

$$= \frac{\cos\theta/2}{\sin\theta/2} = \cot\theta/2$$

~~$y_2 = \frac{d}{dx}(y_1) = \frac{1}{2}[\mathrm{cosec}\frac{\theta}{2} \cot\frac{\theta}{2}]\frac{d\theta}{dx}$~~

$$y_2 = \frac{d}{dx}(y_1) = \frac{1}{2}(-\mathrm{cosec}^2\frac{\theta}{2}).\frac{d\theta}{dx}$$

$$= \frac{1}{2}(-\mathrm{cosec}^2\frac{\theta}{2})(a[1-\cos\theta])$$

$$= \frac{-1}{2}\mathrm{cosec}^2\frac{\theta}{2} \cdot \frac{1}{a(2\sin^2\theta/2)}$$

$$= -\frac{1}{4a}\mathrm{cosec}^4\theta/2$$

$$e = \frac{(1+\cot^2\theta/2)^{3/2}}{-\frac{1}{4a}\mathrm{cosec}^4\theta/2} = \frac{-4a\,\mathrm{cosec}^3\theta/2}{\mathrm{cosec}^4\theta/2}$$

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$= +4a \sin\theta/2$

($\therefore$ Radius is +ve)

Find the circle of curvature of curve at the point

$\sqrt{x}+\sqrt{y}=\sqrt{a}$ ~~at~~ $\left[\frac{a}{4},\frac{a}{4}\right]$

12M

$$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$$

$$\rho=\frac{[1+y_1^2]^{3/2}}{y_2}$$

$$\bar{x}=x-\frac{y_1}{y_2}(1+y_1^2)$$

$$\bar{y}=y+\frac{(1+y_1^2)}{y_2}$$

(1)

$$\sqrt{y}=\sqrt{a}-\sqrt{x}$$

$$y=(\sqrt{a}-\sqrt{x})^2$$

$$y=(a+x)-2\sqrt{a}\sqrt{x}$$

$$y_1=1-2\sqrt{a}\frac{1}{2\sqrt{x}}=1-\sqrt{\frac{a}{x}}$$

$$\therefore x=a/4$$

$$y_1=1-\sqrt{\frac{a}{a/4}}$$

$$y_1=1-\sqrt{4}$$

$$y_1=-1 \text{ at } (a/4, a/4)$$

$$y_1=1-\sqrt{a}\cdot x^{-1/2}$$

$$y_2=0-\sqrt{a}\cdot\left[-\frac{1}{2}x^{-3/2}\right]$$

$$y_2=\frac{\sqrt{a}}{2}\cdot x^{-3/2}=\frac{\sqrt{a}}{2}\cdot\frac{1}{x^{3/2}}$$

$$y_2\left[\frac{a}{4},\frac{a}{4}\right]=\frac{\sqrt{a}}{2}\cdot\frac{1}{a/4\cdot\sqrt{a/4}}$$

$$y_2=\frac{\cancel{4}\,2}{a}$$

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$$\varrho = \frac{a}{4}(1+1)^{3/2} = \frac{a}{4}\cdot 2\sqrt{2} = \frac{a}{\sqrt{2}}$$

$$\bar{x} = \frac{a}{4} + \frac{1}{\frac{4}{a}}(1+1)^{3/2}$$

$$= \frac{a}{4} + \frac{a}{4}(2)$$

$$= \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

$$\bar{y} = \frac{a}{4} + \frac{a(1+1)}{4}$$

$$= \frac{3a}{4}$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$$

(2) $\sqrt{x} + \sqrt{y} = 1$ $\quad \left(\frac{1}{4}, \frac{1}{4}\right)$

$$\sqrt{y} = 1 - \sqrt{x}$$

$$y = (1-\sqrt{x})^2$$

$$y = 1 + x^2 - 2\sqrt{x}$$

$$y_1 = 2x - 2\cdot\left(\frac{1}{2\sqrt{x}}\right)$$

$$y_1 = 2x - \frac{1}{\sqrt{x}}$$

$$= 2\left(\frac{1}{4}\right) - \frac{1}{(1/4)^{1/2}}$$

$$y_1 = \frac{1}{2} - 2 = -\frac{3}{4}$$

$$y_2 = \frac{d}{dx}[y_1] = \frac{d}{dx}\left[2x - \frac{1}{\sqrt{x}}\right]$$

$$= 2 - \left(-\frac{1}{2}x^{-3/2}\right)$$

$$= 2 + \frac{1}{2}\left(\frac{1}{4}\right)^{-3/2}$$

$$= 2 + \frac{1}{2}[8] = 6$$

$$\sqrt{y} = 1 - \sqrt{x}$$

$$y = (1-\sqrt{x})^2$$

$$y = 1 + x - 2\sqrt{x}$$

$$y_1 = 1 - \frac{1}{\sqrt{x}}$$

$$= 1 - \left(\frac{1}{4}\right)^{-1/2}$$

$$y_1 = -1$$

$$y_2 = +\frac{1}{2}x^{-3/2} = \frac{1}{2}\left(\frac{1}{4}\right)^{-3/2}$$

$$= 4$$

Implicit form

$$\rho = \frac{\left[f_x^2 + f_y^2\right]^{3/2}}{f_y^2 f_{xx} - 2f_{xy}f_{yx} + f_x^2 f_{yy}}$$

1. $2x^2 + 2y^2 + 2x - 5y + 1 = 0$

$f_x = 4x + 2$

$f_y = 4y - 5$

$f_{xx} = 4$

$f_{yy} = 4$

$f_{xy} = 0$

$$\rho = \frac{\left[(4x+2)^2 + (4y-5)^2\right]^{3/2}}{(4y-5)^2(4) - 2(0) + 4(4x+2)^2}$$

$$\rho = \frac{\left[(4x+2)^2 + (4y-5)^2\right]\left[(4x+2) + (4y-5)\right]^{1/2}}{4\left[\cancel{(4y-5)^2 + (4x+2)^2}\right]}$$

$$\rho = \frac{\left[(4x+2)^2 + (4y-5)^2\right]^{1/2}}{4}$$

$$= \frac{\sqrt{16x^2 + 16x + 4 + 16y^2 + 25 - 40y}}{4}$$

$$= \frac{\sqrt{8(2x^2 + 2x + 2y^2 - 5y) + 29}}{4}$$

$$\rho = \frac{\sqrt{8(-1) + 29}}{4} = \frac{\sqrt{21}}{4}$$

$$R = \frac{1}{\rho} \Rightarrow R = 4/\sqrt{21}$$

2. $2y^2 + 2x^2 + 2y - 5x + 1 = 0$

$f_x = 4x - 5$

$f_y = 4y + 2$

$f_{xx} = 4$

$f_{yy} = 4$

$f_{xy} = 0$

$$\rho = \frac{\left[(4x-5)^2 + (4y+2)^2\right]^{3/2}}{4\left[(4x-5)^2 + (4y+2)^2\right] - 2(0)}$$

$$= \frac{\left[(4x-5)^2 + (4y+2)^2\right]^{1/2}}{4}$$

$$= \frac{\left[16x^2 + 25 - 40x + 16y^2 + 16y + 4\right]^{1/2}}{4}$$

$\rho = \sqrt{8(-1)}$

$\rho = \frac{\sqrt{21}}{4}$

$R = \frac{1}{\rho} =$

polar form

$\rho = \frac{(r^2 + r_1$

$r^2 + 2r_1$

cardioid r

$r^2 =$

r_1^2

$\rho =$

$=$

$a^2\cos^2\theta + a^2$

$=$

$2a^2$

$=$

$$e = \frac{\sqrt{8(-1)+29}}{4}$$

$$e = \frac{\sqrt{21}}{4}$$

$$R = \frac{1}{e} = \frac{4}{\sqrt{21}}$$

polar form

$$e = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

cardioid $r = a(1+\cos\theta)$ prove that $\frac{e^2}{r}$ is constant

$$e = \frac{[r^2+r_1^2]^{3/2}}{r^2+2r_1^2-rr_2}$$

$r = a(1+\cos\theta)$

$r_1 = a(-\sin\theta)$

$r_2 = a(-\cos\theta)$

$r^2 = a^2(1+\cos^2\theta+2\cos\theta)$

$r_1^2 = a^2\sin^2\theta$

$$e = \frac{\left[a^2[1+\cos^2\theta+2\cos\theta]+a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos^2\theta+2\cos\theta)+2(a^2\sin^2\theta)+a^2(1+\cos\theta)(\cos\theta)}$$

$$= \frac{[a^2+a^2\cos^2\theta+2a^2\cos\theta+a^2\sin^2\theta]^{3/2}}{a^2\cos^2\theta+a^2+2a^2\cos\theta+2a^2\sin^2\theta+a^2\cos\theta+a^2\cos^2\theta}$$

$$= \frac{(2a^2[1+\cos\theta])^{3/2}}{2a^2+a^2+3a^2\cos\theta}$$

$$= \frac{(2a^2[1+\cos\theta])^{3/2}}{3a^2[1+\cos\theta]} = \frac{[4a^2(\cos^2\theta/2)]^{3/2}}{3a^2(2\cos^2\theta/2)}$$

$$\rho = \frac{8a^3\cos^3\theta}{6a^2\cos^2\theta}$$

$$\rho = \frac{4}{3}a\cos\theta/2$$

$$r = a(1+\cos\theta) = 2a\cos^2\theta/2$$

$$\rho^2 = \frac{16}{9}a^2\cos^2\theta/2$$

$$\frac{\rho^2}{r} = \frac{8}{9}\cancel{a^2\cos^2\theta}$$

$\rightarrow$ $\sqrt{x}+\sqrt{y}=1$ at $(1/4, 1/4)$

$$\sqrt{y} = 1-\sqrt{x}$$

$$y = 1+x-2\sqrt{x}$$

$$y_1 = 1-2\cdot\frac{1}{2\sqrt{x}}$$

$$y_1 = 1-2\cdot\frac{1}{2(\frac{1}{2})} = 1-2 = -1$$

$$y_2 = +\frac{1}{2}x^{-3/2} = \frac{1}{2}(4)^{3/2} = \frac{8}{2} = 4$$

$$\rho = \frac{[1+(-1)^2]^{3/2}}{4} = \frac{2\cdot\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

$$\bar{x} = \frac{1}{4} - \left(\frac{-1}{\cancel{4}}\right)[1+(-1)^2] \quad / \quad \bar{y} = \frac{1}{4} + \left(\frac{1}{4}\right)[1+(-1)^2]$$

$$= \frac{1}{4} + \cancel{\sqrt{2}}\frac{(1+1)}{4} \qquad = \frac{1}{4} + \frac{1}{4}(2)$$

$$= \frac{1}{4} + \frac{1}{2} \qquad = \frac{1}{4} + \frac{1}{2}$$

$$= \frac{1+2}{4} = \frac{3}{4} \qquad = \frac{3}{4}$$

$$(x-\bar{x})^2 + (y-\bar{y}) = \rho^2$$

$$\left(x-\frac{3}{4}\right)^2 + \left(y-\frac{3}{4}\right)^2 = \frac{1}{2}$$

$\rightarrow xy = 12$

$y = \frac{12}{x}$

$y_1 = -\frac{12}{x^2} = \frac{-12}{36} = \frac{-1}{3}$

$y_2 = \frac{2(12)}{x^3} = \frac{24}{216} = \frac{1}{9}$

$$\rho = \frac{\left[1+\left(\frac{-12}{x^2}\right)^2\right]^{3/2}}{\frac{24}{x^3}}$$

$$= \frac{\left[1+\frac{144}{x^4}\right]^{3/2}}{\frac{24}{x^3}} = \frac{(x^4+144)^{3/2}}{x^{\cancel{6}3}} \cdot \frac{x^3}{24}$$

$$\rho = \frac{(x^4+144)^{3/2}}{x^3} \cdot \frac{1}{24} = \frac{(1296+144)^{3/2}}{24(216)}$$

$(6,2)$

$$\bar{x} = x - \frac{y_1}{y_2}[1+y_1^2]$$

$$= 6 - \frac{\left(\frac{-12}{x^2}\right)}{\left(\frac{24}{x^3}\right)}\left[1+\left(\frac{-12}{x^2}\right)^2\right]$$

$$= 6 + \frac{\cancel{12}}{\cancel{x^2}} \cdot \frac{x^{\cancel{3}}}{\cancel{24}_2}\left[1+\left(\frac{144}{x^4}\right)\right]$$

$$= 6 + \frac{x}{2}\left[1+\frac{144}{1296}\right]$$

$$= 6 + \frac{6}{2}\left[1+\frac{1}{9}\right]$$

$$= 6 + 3\left[1+\frac{1}{9}\right]$$

$$= 6 + \cancel{3}\left[\frac{10}{\cancel{9}_3}\right]$$

$$= \frac{18+10}{3} = \frac{28}{3}$$

$$\rho = \frac{\left[1+\left(-\frac{1}{3}\right)^2\right]^{3/2}}{1/9}$$

$$= \frac{\left[\frac{9+1}{9}\right]^{3/2}}{\frac{1}{9}} = \frac{10^{3/2}}{27}\cdot 9 = \cancel{9}\cdot\left[\frac{10}{3}\right]^{3/2}$$

$$\bar{y} = y + \frac{1}{y_2}\left[1+y_1^2\right]$$

$$= 2 + \frac{1}{\left(\frac{1}{9}\right)}\left[1+\left(-\frac{1}{3}\right)^2\right]$$

$$= 2 + 9\left[1+\frac{1}{9}\right]$$

$$= 2 + 9\left[\frac{10}{9}\right]$$

$$= 12$$

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left[x-\frac{28}{3}\right]^2 + \left[y-12\right]^2 = \frac{1000}{9}$$

C2

17-09-19

(1) Find the equation of R.O.C at $(3,6)$

$$y^2 = 12x$$

$$2y \cdot y_1 = 12$$

$$y_1 = \frac{12}{2y}$$

$$y_1 = \frac{6}{y} \Rightarrow \frac{6}{6} = 1$$

$$y_2 = -\frac{6}{y^2}\cdot y_1$$

$$= -\frac{6}{y^2}\ \frac{6}{y} = \frac{-1}{6}$$

$$\rho = \frac{[1+1]^{3/2}}{-1/6} = -12\sqrt{2}$$

$$|\rho| = 12\sqrt{2}$$

$$\bar{x} = x - \frac{y_1}{y_2}[1+y_1^2]$$

$$= 3 - \frac{(1)}{(-1/6)}[1+1]$$

$$\bar{x} = 3+6[2] = 15$$

$$\bar{y} = y + \frac{1}{y_2}[1+y_1^2]$$

$$= 6 + \frac{1}{(-1/6)}[1+1^2]$$

$$= 6 - 6[2]$$

$$\bar{y} = -6$$

$$(x-15)^2 + (y+6)^2 = 288$$

Evolute of parabola.

(2) $y^2 = 4ax$

parametric equation

$(at^2, 2at)$

~~$2y \cdot y_1 = 4a$~~

~~$y_1 = \frac{4a}{2y} = \frac{2a}{2at} = \frac{1}{t}$~~

~~$y_2 = \frac{4a}{2}\left(-\frac{1}{y^2}\right)$~~

$$y^2 = 4ax$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a\cancel{t}}{2at} = \frac{1}{t}$$

$$y_2 = \frac{-1}{t^2} \cdot \frac{1}{2at} = \frac{-1}{2at^3}$$

$$\bar{x} = x - \frac{y_1}{y_2}[1+y_1^2]$$

$$= at^2 - \frac{1/t}{-1/2at^3}\left[1+\left(\frac{1}{t}\right)^2\right]$$

$$= at^2 + \frac{1}{t}(2at^3)\left[1+\frac{1}{t^2}\right]$$

$$= at^2 + 2at^2\left[\frac{t^2+1}{t^2}\right]$$

$$= at^2 + 2a[t^2+1]$$

$$= 2at^2 + at^2 + 2a$$

$$\bar{x} = 3at^2 + 2a \qquad \frac{\bar{x} - 2a}{3a} = t^2 \rightarrow (1)$$

$$\bar{y} = y + \frac{1}{y_2}[1+y_1^2]$$

$$= 2at + \frac{1}{-1/2at^3}\left[1+\left(\frac{1}{t^2}\right)\right]$$

$$= 2at - 2at^3\left[\frac{t^2+1}{t^2}\right]$$

$$= 2at - 2at[t^2] - 2at$$

$$\bar{y} = -2at^3 \qquad t^3 = \frac{-\bar{y}}{2a} \rightarrow (2)$$

$$\therefore \quad (1)^3 = (2)^2$$

$$\left[\frac{(\bar{x}-2a)}{3a}\right]^3 = \left[\frac{-\bar{y}}{2a}\right]^2$$

$$\frac{(\bar{x}-2a)^3}{27a^{\cancel{3}}} = \frac{(\bar{y})^2}{\cancel{4a^2}}$$

$$(\bar{x}-2a)^3 = 27a(\bar{y})^2$$

(2) $x^2 = 4ay$

$(2at, at^2)$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2at}{2a} = t$$

$$y_2 = \frac{(1)}{2a}$$

$$\bar{x} = 2at - \frac{t}{1/2a}[1+t^2]$$

$$= 2at - 2at[1+t^2]$$

$$\bar{x} = \cancel{2at} - \cancel{2at} - 2at^3 \qquad t^3 = -\bar{x}/2a \rightarrow (1)$$

$$\bar{y} = at^2 + \frac{1}{1/2a}[1+t^2]$$

$$= at^2 + 2a[1+t^2]$$

$$\bar{y} = 3at^2 + 2a$$

$$t^2 = \frac{\bar{y}-2a}{3a} \rightarrow (2)$$

$\therefore (1)^2 = (2)^3$

$$\left(\frac{\bar{x}}{4a^2}\right)^2 = \left(\frac{\bar{y}-2a}{27a^3}\right)^3$$

$$4(\bar{y}-2a)^3 = 27a(\bar{x})^2$$

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\cos\theta$

$y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$$y_1 = \frac{dy/d\theta}{dx/d\theta} = -\frac{b}{a}\cot\theta$$

$$y_2 = +\frac{b}{a}\operatorname{cosec}^2\theta \cdot \left(\frac{-1}{a\sin\theta}\right)$$

$$y_2 = -\frac{b}{a^2} \cdot \frac{1}{\sin^3\theta}$$

$$\bar{x} = x - \frac{y_1}{y_2}[1+y_1^2]$$

$$= a\cos\theta - \frac{\left(\cancel{+}\frac{\cancel{b}}{\cancel{a}}\cot\theta\right)}{\left(\cancel{+}\frac{\cancel{b}}{a^2}\cdot\frac{1}{\sin^3\theta}\right)}\left[1 + \left(-\frac{b}{a}\cot\theta\right)^2\right]$$

$$= a\cos\theta - a\sin^3\theta.\cot\theta\left[1 + \frac{b^2}{a^2}\cot^2\theta\right]$$

$$= a\cos\theta - a\sin^{\cancel{3}\,2}\theta.\frac{\cos\theta}{\cancel{\sin\theta}}\left[1 + \frac{b^2}{a^2}\cot^2\theta\right]$$

$$= a\cos\theta - a\sin^2\theta.\cos\theta - \frac{b^2}{a}\cancel{\sin^2\theta}.\cos\theta.\frac{\cos^2\theta}{\cancel{\sin^2\theta}}$$

$$= a\cos\theta - a\sin^2\theta.\cos\theta - \frac{b^2}{a}.\cos^3\theta$$

$$= a\cos\theta[1-\sin^2\theta] - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = \frac{a^2-b^2}{a}[\cos^3\theta] \Rightarrow \frac{a\bar{x}}{a^2-b^2} = \cos^3\theta \rightarrow (1)$$

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$$\bar{y} = y + \frac{1}{y_2}[1 + y_1^2]$$

$$= b\sin\theta + \frac{1}{\left(\frac{-b}{a^2\sin^3\theta}\right)}\left[1 + \left(-\frac{b}{a}\cot\theta\right)^2\right]$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b}\left[1 + \frac{b^2}{a^2}\cot^2\theta\right]$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin^3\theta \cdot \cot^2\theta$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin^3\theta \cdot \frac{\cos^2\theta}{\cancel{\sin^2\theta}}$$

$$= b\sin\theta[1 - \cos^2\theta] - \frac{a^2}{b}\sin^3\theta$$

$$= \sin^3\theta\left[b - \frac{a^2}{b}\right]$$

$$\bar{y} = \sin^3\theta\left[\frac{b^2 - a^2}{b}\right]$$

$$\sin^3\theta = \frac{b\bar{y}}{b^2 - a^2} \quad \rightarrow (2)$$

$\cancel{(1)^{3/2}}$ $\quad (1)^{2/3} + (2)^{2/3} = 1$

$$\frac{(a\bar{x})^{2/3}}{(a^2 - b^2)^{2/3}} + \frac{(b\bar{y})^{2/3}}{(b^2 - a^2)^{2/3}} = 1$$

$$\frac{(a\bar{x})^{2/3} - (b\bar{y})^{2/3}}{(a^2 - b^2)^{2/3}} = 1$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}$$

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$x = a\cos^3\theta \quad y = a\sin^3\theta ; \quad x^{2/3} + y^{2/3} = a^{2/3}$

$$\frac{2}{3}x^{2/3-1} + \frac{2}{3}y^{2/3-1} \cdot \frac{dy}{dx} = 0$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\cancel{\frac{2}{3}}x^{-1/3} = -\cancel{\frac{2}{3}}y^{-1/3} \cdot \frac{dy}{dx}$$

$$\frac{-x^{-1/3}}{y^{-1/3}} = \frac{dy}{dx}$$

$$-\left(\frac{x}{y}\right)^{-1/3} = \frac{dy}{dx}$$

$$-\left[\frac{y}{x}\right]^{1/3} = \frac{dy}{dx}$$

$$-\left[\frac{\cancel{a}\sin^3\theta}{\cancel{a}\cos^3\theta}\right]^{1/3} = \frac{dy}{dx}$$

$$-\tan\theta = \cancel{\frac{dy}{d\theta}} = \frac{dy}{dx} = y_1$$

$$y_2 = -\sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \frac{1}{-3a\cos^2\theta \cdot \sin\theta}$$

$$y_2 = \frac{\sec^4\theta}{3a\sin\theta}$$

$$e = \left[\frac{(1+\tan^2\theta)^{3/2}}{\sec^4\theta / 3a\sin\theta}\right]$$

$$e = 3\,\cancel{\sec^4\theta}.a\sin\theta\cos\theta$$

$$e^3 = 27.axy$$

Radius of curvature at any point of cycloid

$y = a(1-\cos\theta)$; $x = a(\theta - \sin\theta)$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(+\sin\theta)}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

$$y_1 = \frac{\sin\theta}{2\sin^2\theta/2} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$y_1 = \cot\theta/2$$

$$y_2 = -\operatorname{cosec}^2\theta/2 \cdot \frac{1}{2} \cdot \frac{1}{a(1-\cos\theta)}$$

$$= -\operatorname{cosec}^2\theta/2 \cdot \frac{1}{2} \frac{1}{2a\sin^2\theta/2}$$

$$y_2 = -\frac{1}{4a}\operatorname{cosec}^4\theta/2$$

$$\rho = \frac{[1+\cot^2\theta/2]^{3/2}}{-\frac{1}{4}\operatorname{cosec}^4\theta/2}$$

$$= \frac{-\operatorname{cosec}^3\theta/2}{-\frac{1}{4a}\operatorname{cosec}^4\theta/2}$$

$$\rho = \frac{4a}{\operatorname{cosec}\theta/2} = 4\sin\theta/2\,(a)$$

Cycloid

$x = a(\theta - \sin\theta)$

$y = a(1-\cos\theta)$

Asteroid

$x = a\cos^3\theta$

$y = a\sin^3\theta$

$x^{2/3} + y^{2/3} = a^{2/3}$

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Evolute

$x = a\cos^3\theta \quad y = a\sin^3\theta$

$x^{2/3} + y^{2/3} = 1$

$\frac{dy}{d\theta} = 3a\sin^2\theta \cdot \cos\theta$

$\frac{dx}{d\theta} = 3a\cos^2\theta\,(-\sin\theta)$

$y_1 = \frac{dy}{dx} = -\tan\theta \quad y_2 = \frac{d^2y}{dx^2} = +\sec^2\theta \cdot \frac{1}{3a\cos^2\theta\,(+\sin\theta)}$

$y_2 = \frac{\sec^4\theta}{3a\sin\theta}$

$\bar{x} = a\cos^3\theta + \frac{\tan\theta \cdot 3a\sin\theta}{\sec^4\theta}[1+\tan^2\theta]$

$= a\cos^3\theta + \frac{\tan\theta \cdot \cancel{\cos^4\theta} \cdot 3a\sin\theta \cdot \cancel{\sec^2\theta}}{\sec^4\theta^2}$

$= a\cos^3\theta + \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \cos^{\cancel{2}}\theta \cdot 3a\sin\theta$

$= a\cos^3\theta + 3a\cos\theta \cdot \sin^2\theta$

$= a(\cos^3\theta + 3\cos\theta \cdot \sin^2\theta)$

$\bar{y} = a\sin^3\theta + [3a\sin\theta \cdot \cos^4\theta](\sec^2\theta)$

$= a\sin^3\theta + 3a \cdot \sin\theta \cdot \cos^2\theta$

$= a(\sin^3\theta + 3\sin\theta \cdot \cos^2\theta)$

$\bar{x} + \bar{y} = a[\sin^3\theta + \cos^3\theta + 3\cos\theta\sin^2\theta + 3\sin\theta\cos^2\theta]$

$(\bar{x}+\bar{y})^{2/3} = a^{2/3}[\sin\theta + \cos\theta]^{3\,2}$

$(\bar{x}-\bar{y})^{2/3} = a^{2/3}[\sin\theta - \cos\theta]^2$

$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = a^{2/3}[2]$

* Evolute of a cycloid is a cycloid

$x = a(\theta - \sin\theta)$

$y = a(1+\cos\theta)$

$\frac{dx}{d\theta} = a(1-\cos\theta)$

$\frac{dy}{d\theta} = -a(+\sin\theta)$

$$\frac{dy}{dx} = \frac{-\sin\theta}{1-\cos\theta} = \frac{-2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$y_1 = \frac{dy}{dx} = -\cot\theta/2$$

$$y_2 = \frac{+\operatorname{cosec}^2\theta/2\left(\frac{1}{2}\right)}{a(2\sin^2\theta/2)}$$

$$y_2 = \frac{1}{4a\sin^4\theta/2}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= a(\theta - \sin\theta) + \frac{\cot\theta/2(4a\sin^4\theta/2)}{[1+\cot^2\theta/2]}$$

$$= a(\theta - \sin\theta) + \frac{\cot\theta/2}{\sin\theta/2} \cdot 4.a[\sin^4\theta/2]$$

$(-\operatorname{cosec}^2\theta/2)$

$$= a\theta - a\sin\theta + \frac{\cos\frac{\theta}{2}}{\cancel{\sin\frac{\theta}{2}}}\, 4a.\sin^{\cancel{4}}\frac{\theta}{2} \cdot \frac{(+1)}{\cancel{\sin^2\frac{\theta}{2}}}$$

$$= a\theta - a\sin\theta + 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a\theta - a\sin\theta + 2a\sin\theta$$

$$\bar{x} = a\theta + a\sin\theta$$

$$\bar{x} = a(\theta + \sin\theta)$$

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$\bar{y} = a(1+\cos\theta) + \left[4a\sin^4\frac{\theta}{2}\right]\left[1+\cot^2\theta/_2\right]$

$= a + a\cos\theta + 4a\sin^4\frac{\theta}{2}\left(\operatorname{cosec}^2\frac{\theta}{2}\right)$

$= a + \cos\theta + 4a\sin^2\frac{\theta}{2}$

$= a\left(2\cos^2\frac{\theta}{2}\right) + 4a\sin^2\frac{\theta}{2}$

$= 2a\cos^2\frac{\theta}{2} + 4a\sin^2\frac{\theta}{2}$

$= 2a\left[\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]$

$= 2a\left[1+\sin^2\frac{\theta}{2}\right] = 2a\left[\frac{1-\cos\theta}{2}+1\right]$

$= 2a + a - a\cos\theta$

$= a(3-\cos\theta)$

$\cancel{y = a(1-\cos\theta)}$

$\cancel{x = a(\theta - \sin\theta)}$

Envelope

(1) Find the envelope of $y = mx + \frac{1}{m}$ where m is parameter

$$y = \frac{m^2 x + 1}{m}$$

$$m^2 x + 1 = ym$$

$$m^2 x - ym + 1 = 0$$

$$ax^2 + bx + c = 0$$

$a = x$
$b = -y$
$c = 1$

$$\Delta = b^2 - 4ac = 0$$
$$= (y^2) - 4(x)(1) = 0$$
$$\boxed{y^2 = 4x}$$

$\therefore$ Envelope given is a parabola

(2) $y = mx + \frac{a}{m}$

$$my = m^2 x + a$$

$$m^2 x - my + a = 0$$

$$\Delta = b^2 - 4ac = 0$$

$$y^2 - 4ax = 0$$

$$y^2 = 4ax$$

(3) $y = mx + \sqrt{a^2 m^2 + b^2}$

$\cancel{y = mx +}$

$$y^2 = m^2 x^2 + (a^2 m^2 + b^2) + 2mx\sqrt{a^2 m + b^2}$$

$$y - mx = \sqrt{a^2 m^2 + b^2} \quad (4)$$

$$y^2 + m^2 x^2 - 2mxy = a^2 m^2 + b^2$$

$$y^2 + \cancel{m} (x^2 - a^2) m^2 - 2mxy - b^2 = 0$$

$\cancel{a = m^2 x^2 - a^2}$ &

$a = x^2 - a^2$
$b = -2mxy$
$c = y^2 - b^2$

$$\Delta = 4m^2 x^2 y^2 - 4(x^2 - a^2)(y^2 - b^2)$$

$$= 4[m^2 x^2 y^2 - [x^2 y^2 - a^2 y^2 + - x^2 b^2 + a^2 b^2]$$

$\Rightarrow 4\left[\cancel{x^2y^2} + \cancel{a^2y} \cancel{+ a^2b^2}\right] \cancel{A} = 0$

$\cancel{x^2y^2 + a^2y}$

$A_1 = x^2 - a^2$

$B_1 = -2xy$

$C_1 = y^2 - b^2$

$(2xy)^2 - 4[x^2 - a^2][y^2 - b^2]$

$\Rightarrow + [x^2y^2 - [x^2y^2 - a^2y^2 - x^2b^2 + a^2b^2]] = 0$

$[\cancel{x^2y^2} - \cancel{x^2y^2} + a^2y^2 + x^2b^2 - a^2b^2] = 0$

$\frac{y^2}{b^2} + \frac{x^2}{a^2} - 1 = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ Ellipse

(4) $f(\alpha) = (x\cos\alpha + y\sin\alpha)^2 = (a)^2 \rightarrow (1)$

$f'(\alpha) = (-x\sin\alpha + y\cos\alpha)^2 = (0)^2 \rightarrow (2)$

$x^2\cos^2\alpha + y^2\sin^2\alpha + \cancel{2xy\sin\alpha\cos\alpha} + x^2\sin^2\alpha +$

$y^2\cos^2\alpha \cancel{- 2xy\sin\alpha\cos\alpha} = a^2$

$(y^2 + x^2)[\cos^2\alpha + \sin^2\alpha] = a^2$

$x^2 + y^2 = a^2$

$a\cos\alpha + b\sin\alpha = \sec\alpha$

$a\cos^2\alpha + b\sin\alpha\cos\alpha = 1$

$b^2\sin^2\alpha - 4(a)(1)$

$b^2\sin^2\alpha = 4a$

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Envelope of 2 parameters family of curves

(1)

Find envelope of $\frac{x}{a}+\frac{y}{b}=1$ subject to $a+b=c$

where c is constant

$\frac{x}{a}+\frac{y}{b}=1 \rightarrow (1)$ $\quad a+b=c \rightarrow (2)$

Differentiating w.r.t 'a'. $\quad a = c-b$

$$x\left(-\frac{1}{a^2}\right)+y\left(-\frac{1}{b^2}\right)\cdot\frac{db}{da}=0$$

$$-\frac{x}{a^2}-\frac{y}{b^2}\cdot\frac{db}{da}=0$$

$$-\frac{x}{a^2}=\frac{y}{b^2}\cdot\frac{db}{da}$$

$$-\frac{x}{a^2}\cdot da=\frac{y}{b^2}\cdot db \rightarrow (3)$$

Differentiating w.r.t 'a' $\quad a+b=c$

$$1+\frac{db}{da}=0$$

$$1=-\frac{db}{da}$$

$$da=-db \rightarrow (4)$$

solving (4) in (3)

$$-\frac{x}{a^2}(-\cancel{db})=\frac{y}{b^2}\cdot\cancel{db}$$

$$\frac{x}{a^2}=\frac{y}{b^2}$$

Compodendo & dividendo Rule

$$\frac{x}{a^2}=\frac{y}{b^2} \quad \frac{\frac{x}{a}}{\frac{a}{1}}=\frac{\frac{y}{b}}{\frac{b}{1}} \Rightarrow \frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{c}$$

$$\frac{x}{a^2}=\frac{y}{b^2}=\frac{1}{c}$$

$$xc=a^2 \quad yc=b^2$$

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$a = \sqrt{xc}$ $\quad b = \sqrt{yc}$

$a + b = c$

$\sqrt{xc} + \sqrt{yc} = c$

$\sqrt{x} + \sqrt{y} = \sqrt{c}$

(2) $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$ $\quad a^2 + b^2 = c^2 \rightarrow (2)$

Differentiating w.r.t a

$x\left(\frac{-1}{a^2}\right) + y\cdot\left(-\frac{1}{b^2}\right)\cdot\frac{db}{da} = 0$

$\frac{-x}{a^2} = \frac{y}{b^2}\cdot\frac{db}{da}$

$-\frac{x}{a^2}\cdot da = \frac{y}{b^2}\cdot db \rightarrow (3)$

$a^2 + b^2 = c^2$

$2a + 2b\left(\frac{db}{da}\right) = 0$

$2a = -2b\cdot\frac{db}{da}$

$2a\cdot da = -2b\cdot db \rightarrow (4)$

$\frac{-\frac{x}{a^2}\cdot \cancel{da}}{2a\cdot\cancel{da}} = \frac{\frac{y}{b^2}\cdot\cancel{db}}{-2b\cdot\cancel{db}}$

$\frac{\cancel{-}x}{\cancel{2}a^3} = \frac{\cancel{+}y}{\cancel{2}b^3}$

$\frac{x}{a^3} + \frac{y}{b^3} = \frac{\frac{x}{a}}{\frac{a^2}{1}} = \frac{\frac{y}{b}}{\frac{b^2}{1}} = \frac{1}{c^2}$

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$$\frac{x}{a^3} = \frac{1}{c^2} \qquad \frac{y}{b^3} = \frac{1}{c^2}$$

$$a^3 = xc^2 \qquad b^3 = yc^2$$

$$a = (xc^2)^{1/3} \qquad b = (yc^2)^{1/3}$$

$$a^2+b^2=c^2$$

$$[xc^2]^{2/3} + [yc^2]^{2/3} = c^2$$

$$x^{2/3} c^{4/3} + y^{2/3} c^{4/3} = c^2$$

$$c^{-2/3}\left[x^{2/3} + y^{2/3}\right] = 1$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

(3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a+b=c$
$\rightarrow (1)$ $\quad \rightarrow (2)$

$$-\frac{2x^2}{a^3} - \frac{2y^2}{b^3} \cdot \left(\frac{db}{da}\right) = 0$$

$$-\frac{\cancel{2}x^2}{a^3} = \frac{\cancel{2}y^2}{b^3}\left(\frac{db}{da}\right)$$

$$-\frac{x^2}{a^3} \cdot da = \frac{y^2}{b^3} \cdot db \rightarrow (3)$$

$$1 + \frac{db}{da} = 0$$

$$-da = db \rightarrow (4)$$

$$\frac{x^2}{a^3} = \frac{y^2}{b^3}$$

$$\frac{x^2}{a^3} + \frac{y^2}{b^3} = \frac{\frac{x^2}{a^2}}{\frac{a}{1}} = \frac{\frac{y^2}{b^2}}{\frac{b}{1}} \qquad \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a+b} = \frac{1}{c}$$

$$\frac{x^2}{a^3} = \frac{1}{c} \; ; \; \frac{y^2}{b^3} = \frac{1}{c}$$

$$\cancel{a = \sqrt{cx}} \quad \cancel{b = \sqrt{cy}}$$

$$\cancel{x + y = \sqrt{c}}$$

$$a = (cx^2)^{1/3} \quad b = (cy^2)^{1/3}$$

$$c^{1/3} \cdot x^{2/3} + c^{1/3} \cdot y^{2/3} = c$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

(4) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a^2 + b^2 = c^2$

$$-\frac{2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0$$

$$-\cancel{2}\frac{x^2}{a^3} = \cancel{2}\frac{y^2}{b^3} \cdot \frac{db}{da}$$

$$-\frac{x^2}{a^3} \cdot da = \frac{y^2}{b^3} \cdot db$$

$$2a \cdot \cancel{da} + 2b \cdot \frac{db}{da} = 0$$

$$da\,\cancel{2}a = -\cancel{2}b \cdot db$$

$$a \cdot da = -b \cdot db$$

$$da = -\frac{b}{a} \cdot db$$

$$-\frac{x^2}{a^3} \cdot \left(-\frac{b}{a}\cancel{db}\right) = \frac{y^2}{b^3} \cdot \cancel{db}$$

$$+\frac{x^2}{a^4} = \frac{y^2}{b^4}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^4} = \frac{\frac{x^2}{a^2}}{\frac{a^2}{1}} = \frac{\frac{y^2}{b^2}}{\frac{b^2}{1}} \Rightarrow \frac{1}{c^2}$$

$\frac{x^2}{a^4} = \frac{1}{c^2}$ $\frac{y^2}{b^4} = \frac{1}{c^2}$

$a^4 = \cancel{\phi}\, x^2 c^2$ $b^4 = c^2 y^2$

$a = \sqrt{xc}$ $b = \sqrt{cy}$

$a^2 + b^2 = c^2$

$x^{2} + y^{2} = c$

(5) $\frac{x}{a} + \frac{y}{b} = 1$ $ab = c^2$

$-\frac{x}{a^2} + \cancel{v} - \frac{y}{b^2} \cdot \frac{db}{da} = 0$

$da - \frac{x}{a^2} = \frac{y}{b^2} \cdot db$

$\cancel{d}$ $a.db + b.da = 0$

$a\,db = -b\,da$

$da = -\frac{a}{b} \cdot db$

$+\frac{x}{a^2} \cdot \frac{\cancel{a}}{\cancel{b}} \cdot \cancel{db} = \frac{y}{b^2} \cdot \cancel{da}\ \cancel{db}$

$\frac{x}{a} = \frac{y}{b}$

$\frac{x}{ac^2} = \frac{y}{bc^2} \Rightarrow \frac{\frac{x}{a} \mp \frac{y}{b^2}}{2c^2} = \frac{1}{2c^2}$

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Beta, Gamma Functions

(1) $\gamma(n) = \int_0^\infty e^{-x} x^{n-1} . dx$

(2) $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} . dx$

Other forms of Beta function

(3) $\beta(m,n) = \beta(n,m)$

(4) $\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} . dy$

(5) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta . d\theta$

(6) $\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

Standard Resselts

(7) $\int_0^{\pi/2} \sin^m\theta . \cos^n\theta . d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

(8) $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(9) $\gamma(n+1) = n\gamma(n)$ (10) $\gamma(n) = (n-1)!$

Evaluate

1. $\int_{-\infty}^{\infty} e^{-x^2} . dx$ and prove that $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$f(x) = e^{-x^2}$

$f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$

$f(x) = f(-x)$ odd function

For odd function

$\int_{-\infty}^{\infty} = 2\int_0^\infty$

For even function

$\int_{-\infty}^{\infty} = 0$

$$\int_{-\infty}^{\infty} e^{-x^2} . dx = 2\int_0^{\infty} e^{-x^2} . dx$$

$$t = x^2$$

$$dt = 2x . dx$$

$$\frac{dt}{dx} = 2x$$

$$\frac{dt}{2\sqrt{t}} = dx$$

$$I = 2\int_0^{\infty} e^{-t} . \frac{dt}{2\sqrt{t}}$$

$$= \frac{2}{2}\int_0^{\infty} e^{-t} . t^{-1/2} . dt$$

$$= \int_0^{\infty} e^{-t} . t^{-1/2} . dt$$

$$\therefore \gamma(n) = \int_0^{\infty} e^{-x} . x^{n-1} . dx$$

$$\gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-t} . t^{1/2 - 1} . dt$$

$$\therefore \gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

P.T

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta . \cos^{2n-1}\theta \, d\theta$$

$$\therefore \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} . dx$$

Let us consider

$$x = \sin^2\theta$$

$$\int_0^1 \sin^{2m-2}\theta . (\cos^{2n-2})\theta . 2\sin\theta.\cos\theta.d\theta$$

$$dx = 2\sin\theta.\cos\theta . d\theta$$

$$dx$$

$$\theta = \sin^{-1}(\sqrt{x})$$

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$$\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta \cdot d\theta$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$$

Hence proved

P.T

$$\gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\cancel{\gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t} t^{1/2}}$$

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\gamma(1/2)\gamma(1/2)}{\gamma(1)}$$

$$\therefore \gamma(1) = 0! = 1$$

$$2\int_0^{\pi/2} \sin^{2(1/2)-1}\theta \cdot \cos^{2(1/2)-1}\theta \cdot d\theta = \left[\gamma\left[\tfrac{1}{2}\right]\right]^2$$

$$2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta \cdot d\theta = \left[\gamma\left(\tfrac{1}{2}\right)\right]^2$$

$$2[\theta]_0^{\pi/2} = \left[\gamma\left(\tfrac{1}{2}\right)\right]^2$$

$$2\left[\frac{\pi}{2}\right] = \left[\gamma\left(\tfrac{1}{2}\right)\right]^2$$

$$\gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

① $\int_0^1 x^6 (1-x)^9 . dx$ By comparing with $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} . dx$

$$\beta(7,10) = \frac{\gamma(7)\,\gamma(10)}{\gamma(17)}$$

$$= \frac{6!\,9!}{16!}$$

② $\int_0^{\pi/2} \sin^6\theta . \cos^{10}\theta . d\theta$

$$\int_0^{\pi/2} \sin^m\theta . \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$\frac{1}{2}\beta\left(\frac{7}{2}, \frac{11}{2}\right)$$

$$\gamma\left(\frac{7}{2}\right) = \gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\gamma\left(\frac{3}{2}+1\right) = \frac{5}{2}.\frac{3}{2}\gamma\left(\frac{1}{2}+1\right) =$$

$$\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\gamma\left(\frac{1}{2}\right) = \frac{15\sqrt{\pi}}{8}$$

$$\gamma\left(\frac{11}{2}\right) = \gamma\left(\frac{9}{2}+1\right) = \frac{9}{2}\gamma\left(\frac{7}{2}+1\right) = \frac{9}{2}.\frac{7}{2}.\gamma\left(\frac{7}{2}\right)$$

$$= \frac{63}{4}.\frac{15}{8}\sqrt{\pi} = \frac{945}{32}\sqrt{\pi}$$

$$\beta\left(\frac{7}{2}, \frac{11}{2}\right) = \frac{\gamma\left(\frac{7}{2}\right).\gamma\left(\frac{11}{2}\right)}{\gamma(9)} = \frac{\frac{1}{2}\left[\frac{15}{8}\sqrt{\pi}.\frac{945}{32}\sqrt{\pi}\right]}{8\times7\times6\times5\times4\times3\times2}$$