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Maths Assignment II

Q1) Find the unit normal vector to the surface $x^2+y^2+z^2=1$ at the point $(1,1,1)$

Solution: $\phi = x^2+y^2+z^2-1$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\boxed{\frac{\partial\phi}{\partial x} = 2x, \ \frac{\partial\phi}{\partial y} = 2y, \ \frac{\partial\phi}{\partial z} = 2z}$$

$$[\nabla\phi]_{1,1,1} = 2\vec{i}+2\vec{j}+2\vec{k}$$

unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}+2\vec{k}}{\sqrt{2^2+2^2+2^2}}$

$$= \frac{\cancel{2}(\vec{i}+\vec{j}+\vec{k})}{\cancel{2}\sqrt{3}}$$

$$= \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$$

Q2) Find the directional derivative of $\phi = x^2+y^2+4xyz$ at the point $(1,-2,2)$ in the direction $2\vec{i}-2\vec{j}+\vec{k}$

Solution: Directional derivation, $DD = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$\phi = x^2+y^2+4xyz$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2x+4yz) + \vec{j}(2y+4xz) + \vec{k}(4xy)$$

$$(\nabla\phi)_{1,-2,2} = \vec{i}(2-16) + \vec{j}(-4+8) + \vec{k}(4(1)(-2))$$

$$= -14\vec{i} + 4\vec{j} - 8\vec{k}$$

let $\vec{a} = 2\vec{i}-2\vec{j}+\vec{k}$

$|\vec{a}| = \sqrt{(2)^2+(-2)^2+(1)^2} = \sqrt{4+4+1} = 3_u$

$$D\phi = (-14\vec{i} + 4\vec{j} - 8\vec{k}) \cdot \frac{(2\vec{i} - 2\vec{j} + \vec{k})}{3}$$

$$= \frac{-28 - 8 - 8}{3} = \boxed{\frac{-44}{3}}$$

3) Find 'a' such that $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

Solution: $\vec{F}$ is solenoidal.

$$\boxed{\operatorname{div} \vec{F} = \nabla \vec{F} = 0}$$

$$\nabla \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}\right] = 0$$

$$\Rightarrow 3 + a + 2 = 0$$

$$\Rightarrow \boxed{a = -5}$$

4) Find the constant a, b, c so that $\vec{F} = (axy + bz^3)\vec{i} + (3x^2 - cz)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution: Condition for irrotational

$$\nabla \cdot \vec{F} = 0$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (axy + bz^3) & (3x^2 - cz) & (3xz^2 - y) \end{vmatrix} = 0$$

$$\vec{i}[-1 + c] - \vec{j}[3z^2 - 3bz^2] + \vec{k}[6x - ax] = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Comparing the coefficient of $\vec{i}, \vec{j}, \vec{k}$ respectively.

$-1 + c = 0$	$3z^2 - 3bz^2 = 0$	$6x - ax = 0$
$\boxed{c = 1}$	$3z^2(1 - b) = 0$	$\boxed{a = 6}$
	$\boxed{b = 1}$	

$\therefore$ Here, the values of $a = 6, b = 1, c = 1$.

5) Using Gauss divergence theorem, evaluate $\iiint_V \nabla\cdot\vec{F}\,dV$ where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ taken over the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

Solution:

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 4z - 2y + y \Rightarrow \boxed{4z - y}$$

$$\iiint_V \nabla\cdot\vec{F}\cdot dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 [4zx - yx]_0^1\,dy\,dz$$

$$= \int_0^1\int_0^1 [4z - y]\,dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz$$

$$= \int_0^1 \left[4z - \frac{1}{2}\right] dz$$

$$= \left[\frac{4z^2}{2} - \frac{z}{2}\right]_0^1$$

$$\Rightarrow 2 - \frac{1}{2} \Rightarrow \boxed{\frac{3}{2}}$$