Engineering Mathematics

T Veerarajan

Dean, Department of Mathematics, (Retd.)
Velammal College of Engineering & Technology
Viraganoor, Madurai,
Tamil Nadu

McGraw Hill Education (India) Private Limited
NEW DELHI

McGraw Hill Education Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

Chapter 5

Multiple Integrals

5.1 INTRODUCTION

When a function $f(x)$ is integrated with respect to x between the limits a and b, we get

the difinite integral $\int_a^b f(x)\mathrm{d}x$.

If the integrand is a funtion $f(x,y)$ and if it is integrated with respect to x and y repeatedly between the limits x_0 and x_1 (for x) and between the limits y_0 and y_1 (for y),

we get a *double integral* that is denoted by the symbol $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Extending the concept of double integral one step further, we get the *triple integral*

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

5.2 EVALUATION OF DOUBLE AND TRIPLE INTEGRALS

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, we first integrate $f(x, y)$ with respect to x partially,

i.e. treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be a function of y. Then we integrate this function of y with respect to y between the limits y_0 and y_1 as usual.

The order in which the integrations are performed in the double integral is illustrated in Fig. 5.1.

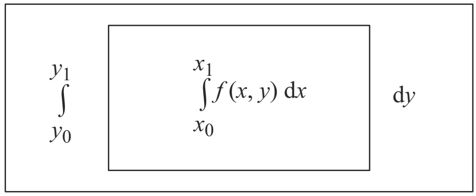

Fig. 5.1

Note ☑ Since the resulting function got after evaluating the inner integral is to be a function of y, the limits x_0 and x_1 may be either constants or functions of y.

The order in which the integrations are performed in a triple integral is illustrated in Fig. 5.2.

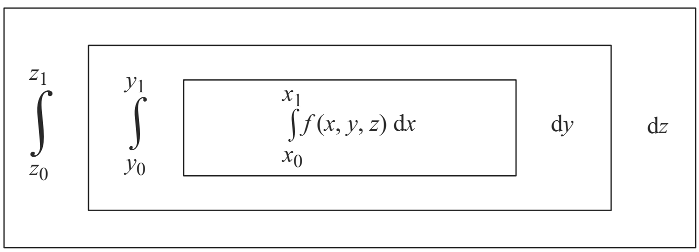

Fig. 5.2

When we first perform the innermost integration with respect to x, we treat y and z as constants temporarily. The limits x_0 and x_1 may be constants or functions of y and z, so that the resulting function got after the innermost integration may be a function of y and z. Then we perform the middle integration with respect to y, treating z as a constant temporarily. The limits y_0 and y_1 may be constants or functions of z, so that the resulting function got after the middle integration may be a function of z only. Finally we perform the outermost integration with respect to z between the constant limits z_0 and z_1.

Note ☑ Sometimes $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\mathrm{d}x\,\mathrm{d}y$ is also denoted as $\int_{y_0}^{y_1}\mathrm{d}y\int_{x_0}^{x_1} f(x, y)\mathrm{d}x$ and $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is also denoted as $\int_{z_0}^{z_1}\mathrm{d}z\int_{y_0}^{y_1}\mathrm{d}y\int_{x_0}^{x_1} f(x, y, z)\,\mathrm{d}x$. If these notations are used to denote the double and triple integrals, the integrations are performed from right to left in order.

5.3 REGION OF INTEGRATION

Consider the double integral $\int_c^d\int_{\phi_1(y)}^{\phi_2(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$. As stated above x varies from $\phi_1(y)$ to $\phi_2(y)$ and y varies from c to d.

i.e. $\phi_1(y) \le x \le \phi_2(y)$ and $c \le y \le d$.

These inequalities determine a region in the xy-plane, whose boundaries are the curves $x = \phi_1(y)$, $x = \phi_2(y)$ and the lines $y = c$, $y = d$ and which is shown in Fig. 5.3. This region $ABCD$ is known as the region of integration of the above double integral.

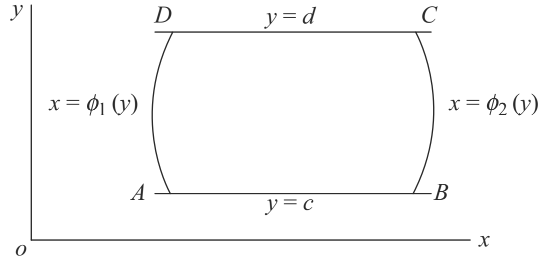

Fig. 5.3

Similarly, for the double integral $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy\,dx$, the region of integration $ABCD$, whose boundaries are the curves $y = \phi_1(x)$, $y = \phi_2(x)$ and the lines $x = a$, $x = b$, is shown in Fig. 5.4.

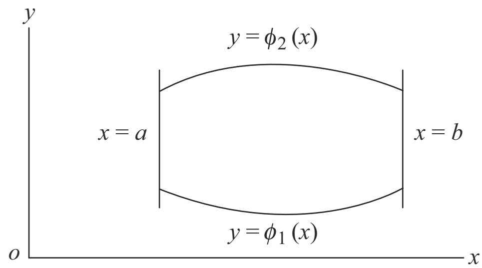

Fig. 5.4

For the triple integral $\int_{z_1}^{z_2} \int_{\psi_1(z)}^{\psi_2(z)} \int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x, y, z)\,dx\,dy\,dz$, the inequalities $\phi_1(y, z) \le x \le \phi_2(y, z)$; $\psi_1(z) \le y \le \psi_2(z)$; $z_1 \le z \le z_2$ hold good. These inequalities determine a domain in space whose boundaries are the surfaces $x = \phi_1(y, z)$, $x = \phi_2(y, z)$, $y = \psi_1(z)$, $y = \psi_2(z)$, $z = z_1$ and $z = z_2$. This domain is called the domain of integration of the above triple integral.

WORKED EXAMPLE 5(a)

Example 5.1 Verify that $\int_1^2 \int_0^1 (x^2 + y^2)\,dx\,dy = \int_0^1 \int_1^2 (x^2 + y^2)\,dy\,dx$.

$$\text{L.S.} = \int_1^2 \left[\int_0^1 (x^2 + y^2)\,dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} + y^2 x \right]_{x=0}^{x=1} \mathrm{d}y$$

Note ☑ y is treated a constant during inner integration with respect to x.

$$= \int_1^2 \left(\frac{1}{3} + y^2 \right) \mathrm{d}y = \left(\frac{y}{3} + \frac{y^3}{3} \right)_1^2 = \frac{8}{3}$$

$$\text{R.S.} = \int_0^1 \left[\int_1^2 (x^2 + y^2)\, \mathrm{d}y \right] \mathrm{d}x$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_{y=1}^{y=2} \mathrm{d}x$$

Note ☑ x is treated a constant during inner integration with respect to y.

$$= \int_0^1 \left(x^2 + \frac{7}{3} \right) \mathrm{d}x = \left(\frac{x^3}{3} + \frac{7}{3} x \right)_0^1 = \frac{8}{3}$$

Thus the two double integrals are equal.

Note ☑ From the above problem we note the following fact: If the limits of integration in a double integral are constants, then the order of integration is immaterial, provided the relevant limits are taken for the concerned variable and the integrand is continuous in the region of integration. This result holds good for a triple integral also.

Example 5.2 Evaluate $\int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 \sin\phi \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}\theta$.

$$\text{The given integral} = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi \int_0^{a} r^4 \sin\phi \, \mathrm{d}r$$

$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \left(\frac{r^5}{5} \right)_0^a \sin\phi \, \mathrm{d}\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \sin\phi \, \mathrm{d}\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} (-\cos\phi)_0^{\pi} \, \mathrm{d}\theta$$

$$= \frac{2}{5} a^5 \int_0^{2\pi} \mathrm{d}\theta$$

$$= \frac{4}{5} \pi a^5$$

Example 5.3 Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$

$$\text{The given integral} = \int_0^1 \left[\int_0^{\sqrt{1+y^2}} \frac{1}{(1+y^2)+x^2}\,dx \right] dy$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+y^2}} \tan^{-1} \frac{x}{\sqrt{1+y^2}} \right]_{x=0}^{x=\sqrt{1+y^2}} dy$$

$$= \frac{\pi}{4} \int_0^1 \frac{dy}{\sqrt{1+y^2}}$$

$$= \frac{\pi}{4} \left[\log\left(y + \sqrt{1+y^2} \right) \right]_0^1$$

$$= \frac{\pi}{4} \log\,(1+\sqrt{2})$$

Example 5.4 Evaluate $\displaystyle\int_0^1 \int_x^{\sqrt{x}} xy\,(x+y)\,dx\,dy\cdot$

Since the limits for the inner integral are functions of x, the variable of inner integration should be y. Effecting this change, the given integral I becomes

$$\mathrm{I} = \int_0^1 \left[\int_x^{\sqrt{x}} xy\,(x+y)\,dy \right] dx$$

$$= \int_0^1 \left(x^2 \frac{y^2}{2} + x\frac{y^3}{3} \right)_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(\frac{x^3}{2} + \frac{1}{3} x^{5/2} \right) - \left(\frac{x^4}{2} + \frac{x^4}{3} \right) \right] dx$$

$$= \left(\frac{x^4}{8} + \frac{2}{21} x^{7/2} - \frac{x^5}{6} \right)_0^1$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{6} = \frac{3}{56}$$

Example 5.5 Evaluate $\int_0^1\int_0^{1-z}\int_0^{1-y-z} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

$$\text{The given integral } = \int_0^1\int_0^{1-z} yz\left(\frac{x^2}{2}\right)_0^{1-y-z} \mathrm{d}y\,\mathrm{d}z$$

$$= \frac{1}{2}\int_0^1\int_0^{1-z} yz\,(1-y-z)^2\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{1}{2}\int_0^1\int_0^{1-z} yz\,\{(1-z)^2 - 2\,(1-z)y + y^2\}\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{1}{2}\int_0^1\left[z(1-z)^2\,\frac{y^2}{2} - 2z(1-z)\,\frac{y^3}{3} + z\,\frac{y^4}{4}\right]_{y=0}^{y=1-z} \mathrm{d}z$$

$$= \frac{1}{2}\int_0^1\left[\frac{1}{2}\,z(1-z)^4 - \frac{2}{3}z(1-z)^4 + \frac{1}{4}z(1-z)^4\right]\mathrm{d}z$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)\int_0^1 z(1-z)^4\,\mathrm{d}z$$

$$= \frac{1}{24}\int_0^1 \{1-(1-z)\}\,(1-z)^4\,\mathrm{d}z$$

$$= \frac{1}{24}\left[\frac{(1-z)^5}{-5} + \frac{(1-z)^6}{6}\right]_0^1$$

$$= \frac{1}{24}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{720}$$

Example 5.6 Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Since the upper limit for the innermost integration is a function of x, y, the corresponding variable of integration should be z. Since the upper limit for the middle integration is a function of x, the corresponding variable of integration should be y. The variable of integration for the outermost integration is then x. Effecting these changes, the given triple integral I becomes,

$$I = \int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{\log 2}\mathrm{d}x\int_0^{x}\mathrm{d}y\; e^{x+y}\,(e^z)_{z=0}^{z=x+y}$$

$$= \int_0^{\log 2}\mathrm{d}x\int_0^{x}(e^{2x+2y} - e^{x+y})\,\mathrm{d}y$$

$$= \int_0^{\log 2} dx \left(e^{2x} \cdot \frac{e^{2y}}{2} - e^x \cdot e^y \right)_{y=0}^{y=x}$$

$$= \int_0^{\log 2} \left(\frac{1}{2} e^{4x} - \frac{3}{2} e^{2x} + e^x \right) dx$$

$$= \left(\frac{1}{8} e^{4x} - \frac{3}{4} e^{2x} + e^x \right)_0^{\log 2}$$

$$= \frac{5}{8}$$

Example 5.7 Evaluate $\iint_R xy \, dx \, dy$, where R is the region bounded by the line $x + 2y = 2$, lying in the first quadrant.

We draw a rough sketch of the boundaries of R and identify R.

The boundaries of R are the lines $x = 0$, $y = 0$ and the segment of the line $\frac{x}{2} + \frac{y}{1} = 1$ lying in the first quadrant.

Now R is the region as shown in Fig. 5.5.

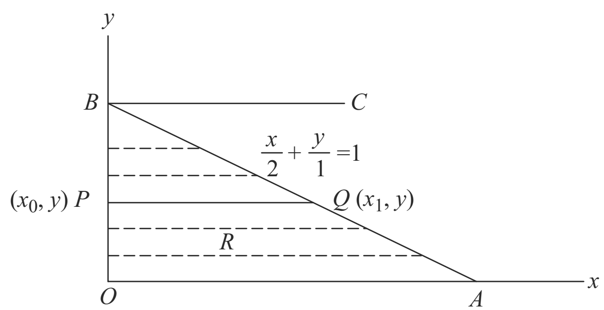

Fig. 5.5

Since the limits of the variables of integration are not given in the problem and to be fixed by us, we can choose the order of integration arbitrarily.

Let us integrate with respect to x first and then with respect to y. Then the integral I becomes

$$I = \int_R \left[\int xy \, dx \right] dy$$

When we perform the inner integration with respect to x, we have to treat y as a constant temporarily and find the limits for x.

Geometrically, treating y = constant is equivalent to drawing a line parallel to the x-axis arbitrarily lying within the region of integration R as shown in the figure.

Finding the limits for x (while y is a constant) is equivalent to finding the variation of the x co-ordinate of any point on the line PQ. We assume that the y co-ordinates of all points on PQ are y each (since y is constant on PQ) and $P \equiv (x_0, y)$ and $Q \equiv (x_1, y)$.

Thus x varies from x_0 to x_1.

Wherever the line PQ has been drawn, the left end P lies on the y-axis and hence $x_0 = 0$ and the right end Q lies on the line $x + 2y = 2$, and hence $x_1 + 2y = 2$ i.e. $x_1 = 2 - 2y$.

Thus the limits for the variable x of inner integration are 0 and $2 - 2y$. When we go to the outer integration, we have to find the limits for y.

Geometrically we have to find the variation of the line PQ, so that the region R is fully covered. To sweep the entire area of the region R, PQ has to start from the position OA where $y = 0$, move parallel to itself and go up to the position BC where $y = 1$.

Thus the limits for y are 0 and 1.

$$\therefore \qquad I = \int_0^1 \int_0^{2-2y} xy \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int_0^1 y \left(\frac{x^2}{2} \right)_0^{2-2y} \mathrm{d}y$$

$$= \int_0^1 \frac{y}{2} (2 - 2y)^2 \, \mathrm{d}y$$

$$= 2 \int_0^1 y (1 - y)^2 \, \mathrm{d}y$$

$$= 2 \left(\frac{y^2}{2} - 2\frac{y^3}{3} + \frac{y^4}{4} \right)_0^1$$

$$= \frac{1}{6}$$

5.3.1 Aliter

Let us integrate with respect to y first and then with respect to x.

Then $I = \int \left[\int_R xy \, \mathrm{d}y \right] \mathrm{d}x$

As explained above, to find the limits for y, we draw a line parallal to the y-axis (x = constant) in the region of integration and note the variation of y on this line

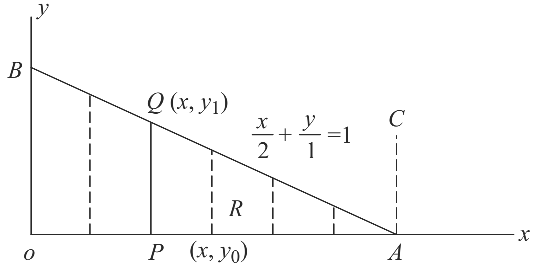

Fig. 5.6

$P(x, y_0)$ lies on the x-axis. $\therefore\ y_0 = 0$

$Q(x, y_1)$ lies on the line $x + 2y = 2$. $\therefore\ y_1 = \frac{1}{2}(2-x)$

i.e., the limits for y are 0 and $\frac{1}{2}(2-x)$.

To cover the region of integration OAB, the line PQ has to vary from OB ($x = 0$) to AC ($x = 2$)

$\therefore$ The limits for x are 0 and 2.

$$\therefore \qquad I = \int_0^2 \int_0^{\frac{1}{2}(2-x)} xy\,dy\,dx$$

$$= \int_0^2 x\left(\frac{y^2}{2}\right)_0^{\frac{1}{2}(2-x)} dx$$

$$= \frac{1}{8}\int_0^2 x(2-x)^2\,dx$$

$$= \frac{1}{8}\left(4\frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right)_0^2$$

$$= \frac{1}{6}$$

Example 5.8 Evaluate $\iint_R \frac{e^{-y}}{y}\,dx\,dy$, by choosing the order of integration suitably, given that R is the region bounded by the lines $x = 0$, $x = y$ and $y = \infty$.

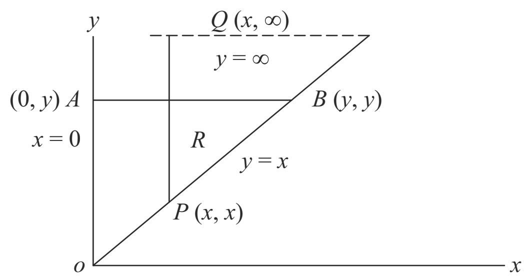

Fig. 5.7

Let $$I = \iint_R \frac{e^{-y}}{y}\,dx\,dy$$

Suppose we wish to integrate with respect to y first.

Then $$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

We note that the choice of order of integration is wrong, as the inner integration cannot be performed. Hence we try to integrate with respect to x first.

Then
$$\mathrm{I} = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^\infty \frac{e^{-y}}{y}(x)_0^y\,\mathrm{d}y$$
$$= \int_0^\infty e^{-y}\,\mathrm{d}y$$
$$= \left(e^{-y}\right)_\infty^0 = 1$$

Note ☑ From this example, we note that the choice of order of integration sometimes depends on the function to be integrated.

Example 5.9 Evaluate $\iint_R xy\,\mathrm{d}x\,\mathrm{d}y$, where R is the region bounded by the parabola $y^2 = x$ and the lines $y = 0$ and $x + y = 2$, lying in the first quadrant.

R is the region $OABCDE$ shown in Fig. 5.8.

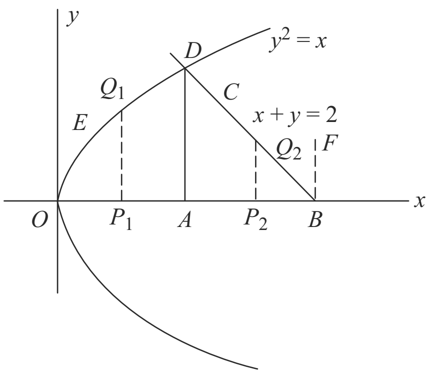

Fig. 5.8

Suppose we wish to integrate with respect to y first. Then we will draw an arbitrary line parallel to y-axis (x = constant). We note that such a line does not intersect the region of integration in the same fashion throughout.

If the line is drawn in the region $OADE$, the upper end of the line will lie on the parabola $y^2 = x$; on the other hand, if it is drawn in the region $ABCD$, the upper end of the line will lie on the line $x + y = 2$.

Hence in order to cover the entire region R, it should be divided into two, namely, $OADE$ and $ABCD$ and the line $P_1 Q_1$ should move from the y-axis to AD and the line $P_2 Q_2$ should move from AD to BF.

Accordingly, the given integral I is given by

$$\mathrm{I} = \int_0^1 \int_0^{\sqrt{x}} xy\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_0^{2-x} xy\,\mathrm{d}y\,\mathrm{d}x$$

[$\because$ the co-ordinates of D are (1, 1) and so the equation of AD is $x = 1$]

$$\mathrm{I} = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

Note ☑ This approach results in splitting the double integral into two and evaluating two double integrals. On the other hand, had we integrated with respect to x first, the problem would have been solved in a simpler way as indicated below. [Refer to Fig. 5.9]

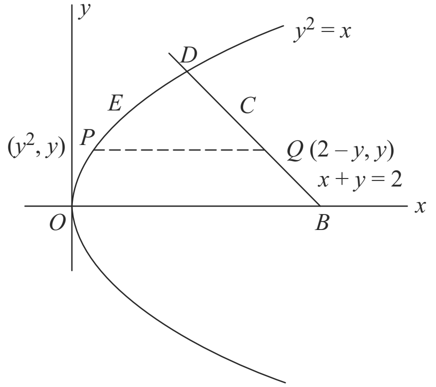

Fig. 5.9

$$\mathrm{I} = \int_0^1 \int_{y^2}^{2-y} xy\, \mathrm{d}x\, \mathrm{d}y$$

$$= \frac{1}{2}\int_0^1 y\{(2-y)^2 - y^4\}\, \mathrm{d}y$$

$$= \frac{1}{2}\int_0^1 (4y - 4y^2 + y^3 - y^5)\, \mathrm{d}y$$

$$= \frac{3}{8}$$

Note ☑ From this example, we note that the choice of order of integration is to be made by considering the region of integration so as to simplify the evaluation.

Example 5.10 Evaluate $\iiint_V (x + y + z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$, where V is the volume of the rectangular parallelopiped bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$ and $z = c$.

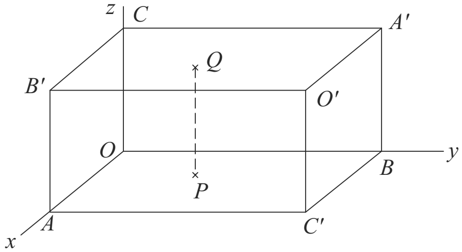

Fig. 5.10

The region of integration is the volume of the parallelopiped shown in Fig. 5.10, in which OA = a, OB = b, OC = c. Since the limits of the variables of integration are not given, we can choose the order of integration arbitrarily.

Let us take the given integral I as

$$\mathrm{I} = \iiint_V (x+y+z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

The innermost integration is to be done with respect to z, treating x and y as constants.

Geometrically, x = constant and y = constant jointly represent a line parallel to the z-axis.

Hence we draw an arbitrary line PQ in the region of integration and we note the variation of z on this line so as to cover the entire volume. In this problem, z varies from 0 to c. since $P \equiv (x, y, 0)$ and $Q \equiv (x, y, c)$

Having performed the innermost integration with respect to z between the limits 0 and c, we get a double integral.

As P take all positions inside the rectangle OAC′B in the xy-plane, the line PQ covers the entire voulme of the parallelopiped. Hence, the double integral got after the innermost integration is to be evaluated over the plane region OAC′B.

The limits for the double integral can be easily seen to be 0 and b (for y) and 0 and a (for x).

$$\therefore \qquad \mathrm{I} = \int_0^a\int_0^b\int_0^c (x+y+z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^a\int_0^b \left\{(x+y)z + \frac{z^2}{2}\right\}_{z=0}^{c} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^a\int_0^b \left\{c\,(x+y) + \frac{c^2}{2}\right\} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^a \left\{\left(cx + \frac{c^2}{2}\right)y + c\frac{y^2}{2}\right\}_0^b \mathrm{d}x$$

$$= \int_0^a \left(bcx + \frac{bc^2}{2} + \frac{b^2c}{2}\right) \mathrm{d}x$$

$$= \left[bc\frac{x^2}{2} + \frac{bc}{2}(b+c)x\right]_0^a$$

$$= \frac{abc}{2}(a+b+c)$$

Example 5.11 Evaluate $\iiint_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the finite region of space (terra- hedron) formed by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 3y + 4z = 12$.

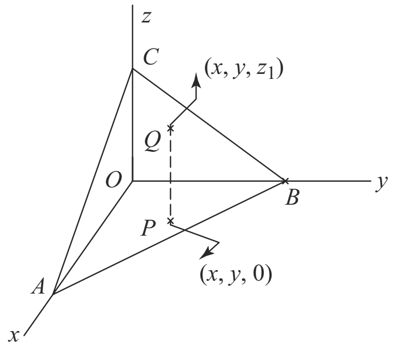

Fig. 5.11

Let I = the given integral.

Let $\mathrm{I} = \iiint\limits_V \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

The limits for z, the variable of the innermost integral, are 0 and z_1, where (x, y, z_1) lies on the plane $2x + 3y + 4z = 12$. [Refer to Fig. 5.11]

$$\therefore \qquad z_1 = \frac{1}{4}(12 - 2x - 3y)$$

After performing the innermost integration, the resulting double integral is evaluated over the orthogonal projection of the plane ABC on the xy-plane, i.e. over the triangular region OAB in the xy-plane as shown in Fig.5. 12.

In the double integral, the limits for y are 0 and $\frac{1}{3}(12 - 2x)$ and those for x are 0 and 6.

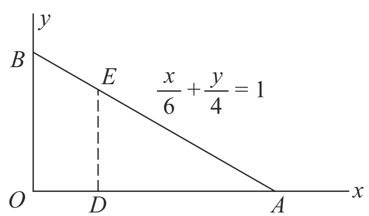

Fig. 5.12

$$\therefore \quad \mathrm{I} = \int_0^6 \mathrm{d}x \int_0^{\frac{1}{3}(12-2x)} \mathrm{d}y \int_0^{\frac{1}{4}(12-2x-3y)} \mathrm{d}z$$

$$= \frac{1}{4}\int_0^6 \mathrm{d}x \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y)\,\mathrm{d}y$$

$$= \frac{1}{4}\int_0^6 \mathrm{d}x \left[(12 - 2x)y - \frac{3y^2}{2}\right]_{y=0}^{y=\frac{1}{3}(12-2x)}$$

$$= \frac{1}{24}\int_0^6 (12 - 2x)^2\,\mathrm{d}x$$

$$= \frac{1}{6}\left\{\frac{(6 - x)^3}{-3}\right\}_0^6$$

$$= 12$$

Example 5.12 Evaluate $\iiint\limits_V \dfrac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$, where V is the region of space bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant.

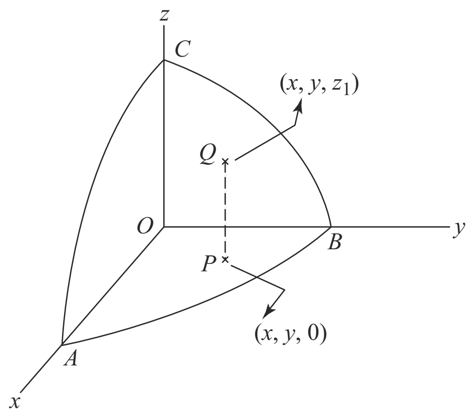

Fig. 5.13

Note ☑ In two dimensions, the x and y-axes divide the entire xy-plane into 4 quadrants. The quadrant containing the positive x and the positive y-axes is called the positive quadrant.

Similarly in three dimensions the xy, yz and zx-planes divide the entire space into 8 parts, called octants. The octant containing the positive x, y and z-axes is called the positive octant.

The region of space V given in this problem is shown in Fig. 5.13.

Let $$I = \iiint\limits_V \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

To find the limits for z, we draw a line PQ parallel to the z-axis cutting the voulme of integration.

The limits for z and 0 and z_1, where (x, y, z_1) lies on the sphere $x^2 + y^2 + z^2 = 1$

$\therefore$ $$z_1 = \sqrt{1-x^2-y^2} \quad (\because \text{ the point } Q \text{ lies in the positive octant})$$

After performing the innermost integration, the resulting double integral is evaluated over the orthogonal projection of the spherical surface on the xy-plane, i.e. over the circular region lying in the positive quadrant as shown in Fig. 5.14.

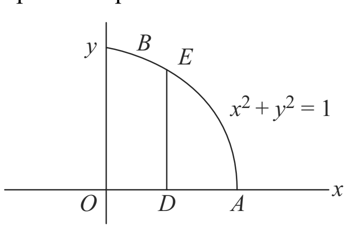

Fig. 5.14

In the double integral, the limits for y are 0 and $\sqrt{1-x^2}$ and those for x are 0 and 1.

$\therefore$

$$\mathrm{I} = \int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_0^{\sqrt{1-x^2-y^2}} \frac{\mathrm{d}z}{\sqrt{(1-x^2-y^2)-z^2}}$$

$$= \int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \left(\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right)_{z=0}^{z=\sqrt{1-x^2-y^2}}$$

$$= \frac{\pi}{2}\int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$

$$= \frac{\pi}{2}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right)_0^1$$

$$= \frac{\pi}{2}\times\frac{\pi}{4} = \frac{\pi^2}{8}$$

EXERCISE 5(a)

Part A

(Short Answer Questions)

1. Evaluate $\int_0^2\int_0^1 4xy\,\mathrm{d}x\,\mathrm{d}y$
2. Evaluate $\int_1^b\int_1^a \frac{\mathrm{d}x\,\mathrm{d}y}{xy}$
3. Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta+\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi$
4. Evaluate $\int_0^1\int_0^x \mathrm{d}x\,\mathrm{d}y$.
5. Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$
6. Evaluate $\int_0^1\int_0^2\int_0^3 xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
7. Evaluate $\int_0^1\int_0^z\int_0^{y+z} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

Sketch roughly the region of integration for the following double integrals:

8. $\int_{-b}^{b}\int_{-a}^{a} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$
9. $\int_{0}^{1}\int_{0}^{x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$
10. $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$
11. $\int_{0}^{b}\int_{0}^{\frac{a}{b}(b-y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$

Find the limits of integration in the double integral $\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where R is in the first quadrant and bounded by

12. $x = 0, y = 0, x + y = 1.$
13. $x = 0,\ y = 0,\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$
14. $x = 0, x = y, y = 1$
15. $x = 1, y = 0, y^2 = 4x$

Part B

16. Evaluate $\int_{0}^{4}\int_{y^2/4}^{y} \dfrac{y\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$ and also sketch the region of integration roughly.
17. Evaluate $\int_{0}^{a}\int_{a-x}^{\sqrt{a^2-x^2}} y\,\mathrm{d}x\,\mathrm{d}y$ and also sketch the region of integration roughly.
18. Evaluate $\int_{0}^{1}\int_{x}^{1} \dfrac{y\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$ and also sketch the region of integration roughly.
19. Evaluate $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y.$
20. Evaluate $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$
21. Evaluate $\int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x+y+z}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$

22. Evaluate $\iint x e^{-\frac{x^2}{y}} \, \mathrm{d}x \, \mathrm{d}y$, over the region bounded by $x = 0$, $x = \infty$, $y = 0$ and $y = x$.
23. Evaluate $\iint xy \, \mathrm{d}x \, \mathrm{d}y$, over the region in the positive quadrant bounded by the line $2x + 3y = 6$.
24. Evaluate $\iint x \, \mathrm{d}x \, \mathrm{d}y$, over the region in the positive quadrant bounded by the circle $x^2 - 2ax + y^2 = 0$.
25. Evaluate $\iint (x + y) \, \mathrm{d}x \, \mathrm{d}y$, over the region in the positive quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
26. Evaluate $\iint (x^2 + y^2) \, \mathrm{d}x \, \mathrm{d}y$, over the area bounded by the parabola $y^2 = 4x$ and its latus rectum.
27. Evaluate $\iint_R x^2 \, \mathrm{d}x \, \mathrm{d}y$, where R is the region bounded by the hyerbola $xy = 4$, $y = 0$, $x = 1$ and $x = 2$.
28. Evaluate $\iiint_V (xy + yz + zx) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$, where V is the region of space bounded by $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$ and $z = 3$.
29. Evaluate $\iiint_V \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{(x + y + z + 1)^3}$, where V is the region of space bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.
30. Evaluate $\iiint_V xyz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$, where V is the region of space bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant.

5.4 CHANGE OF ORDER OF INTEGRATION IN A DOUBLE INTEGRAL

In worked example (1) of the previous section, we have observed that if the limits of integration in a double integral are constants, then the order of integration can be changed, provided the relevant limits are taken for the concerned variables.

But when the limits for inner integration are functions of a variable, the change in the order of integration will result in changes in the limits of integration.

i.e. the double integral $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) \, \mathrm{d}x \, \mathrm{d}y$ will take the form $\int_a^b \int_{h_1(x)}^{h_2(x)} f(x, y) \, \mathrm{d}y \, \mathrm{d}x$, when the order of integration is changed. This process of converting a given double integral into its equivalent double integral by changing the order of integration is often called *change of order of integration*. To effect the change of order of integration, the region of integration is identified first, a rough sketch of the region is drawn and then the new limits are fixed, as illustrated in the following worked examples.

5.5 PLANE AREA AS DOUBLE INTEGRAL

Plane area enclosed by one or more curves can be expressed as a double integral both in Cartesian coordinates and in polar coordinates. The formulas for plane areas in both the systems are derived below:

(*i*) ***Cartesian System***

Let R be the plane region, the area of which is required. Let us divide the area into a large number of elemental areas like $PQRS$ (shaded) by drawing lines parallel to the y-axis at intervals of Δx and lines parallel to the x-axis at intervals of Δy (Fig. 5.l5).

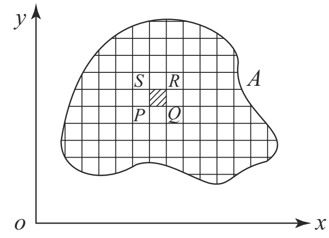

Fig. 5.15

Area of the elemental rectangle $PQRS = \Delta x.\ \Delta y$. Required area A of the region R is the sum of elemental areas like $PQRS$.

viz.,
$$A = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Sigma\Sigma\ \Delta x\ \Delta y)$$
$$= \iint_R \mathrm{d}x\,\mathrm{d}y$$

(*ii*) ***Polar System***

We divide the area A of the given region R into a large number of elemental curvilinear rectangular areas like $PQRS$ (shaded) by drawing radial lines and concentric circular arcs, where P and R have polar coordinates (r, θ) and $(r + \Delta r, \theta + \Delta\theta)$ (Fig. 5.16)

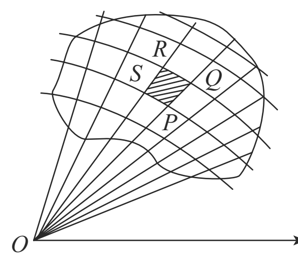

Fig. 5.16

Area of the element $PQRS = r\ \Delta r\ \Delta\theta$
($\because\ PS = r\ \Delta\theta$ and $PQ = \Delta r$)

$\therefore$ Required area
$$A = \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} (\Sigma\Sigma\ r\,\Delta r\ \Delta\theta)$$
$$= \iint_R r\,\mathrm{d}r\,\mathrm{d}\theta.$$

5.5.1 Change of Variables

(*i*) ***From Cartesian Coordinates to Plane Polar Coordinates***

If the transformations $x = x(u, v)$ and $y = y\ (u, v)$ are made in the double integral $\iint f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, then $f(x, y) \equiv g(u, v)$ and dx dy = $|J|$ du dv, where $J = \dfrac{\partial(x, y)}{\partial(u, v)}$.

[Refer to properties of Jacobians in the Chapter 4, "Functions of Several Variables" in Part I] .

When we transform from cartesian system to plane polar system,

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

In this case, $$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Hence $$\int\int_R f(x, y)\, dx\, dy = \int\int_R g(r, \theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta$$

In particular,

Area A of the plane region R is given by

$$A = \iint_R \mathrm{d}x\,\mathrm{d}y = \iint_R r\,\mathrm{d}r\,\mathrm{d}\theta$$

(*ii*) *From Three Dimensional Cartesians to Cylindrical Coordinates*

Let us first define cylindrical coordinates of a point in space and derive the relations between cartesian and cylindrical coordinates (Fig. 5.17).

Let P be the point (x, y, z) in Cartesian coordinate system. Let PM be drawn $\perp$ r to the xoy-plane and MN parallel to Oy. Let $\underline{|NOM} = \theta$ and $OM = r$. The triplet (r, θ, z) are called the cylindrical coordinates of P.

Clearly, $ON = x = r\cos\theta$; $NM = y = r\sin\theta$ and $MP = z$.

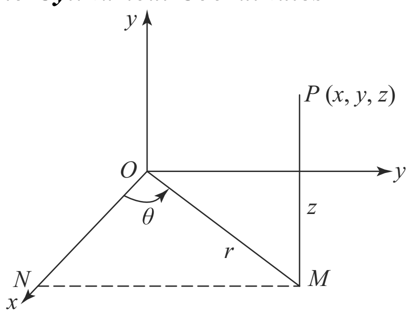

Fig. 5.17

Thus the transformations from three dimensional cartesians to cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

In this case,

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r$$

Hence $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

and $$\iiint_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_V g(r, \theta, z)\, r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

In particular, the volume of a region of space V is given by

$$\iiint_V \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_V r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z$$

Note ☑ Whenever $\iiint f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is to be evaluated throughout the volume of a right circular cylinder, it will be advantageous to evaluate the corresponding triple integral in cylindrical coordinates.

(*iii*) *From Three Dimensional Cartesians to Spherical Polar Coordinates*

Let us first define spherical polar coordinates of a point in space and derive the relations between Cartesian and spherical polar coordinates (Fig. 5.18).

Let P be the point whose Cartesian coordinates are (x, y, z). Let PM be drawn $\perp r$ to the xOy-plane. Let MN be parallel to y-axis. Let $OP = r$, the angle made by OP with the positive z-axis $= \theta$ and the angle made by OM with x-axis $= \phi$.

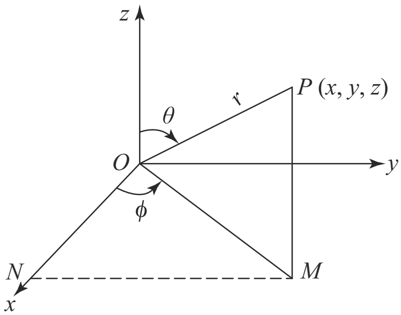

Fig. 5.18

The triplet (r, θ, ϕ) are called the spherical polar coordinates of P.

Since $\underline{|OMP} = 90°$, $MP = z = r \cos \theta$, $OM = r \sin \theta$, $ON = x = r \sin \theta \cos \phi$ and $NM = y = r \sin \theta \sin \phi$.

Thus the transformations from three dimensional cartesians to spherical polar coordinates are

$$x = r \sin \theta \cos \phi,\ y = r \sin \theta \sin \phi,\ z = r \cos \theta$$

In this case,

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$$

[Refer to example (4.8) of Worked example set 4(b) in the Chapter 4 "Functions of Several Variables." in Part I]

Hence $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = r^2 \sin\theta\, \mathrm{d}r\, d\theta\, \mathrm{d}\phi$ and $\iiint_V f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_V g(r, \theta, \phi)\, r^2 \sin \theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$.

In particular, the volume of a region of space V is given by

$$\iiint_V \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_V r^2 \sin \theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi.$$

Note ☑ Whenever $\iiint f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is to be evaluated throughout the volume of a sphere, hemisphere or octant of a sphere, it will be advantageous to use spherical polar coordinates.)

WORKED EXAMPLE 5(b)

Example 5.1 Change the order of integration in $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$ and then evaluate it.

The region of integration R is defined by $y \le x \le a$ and $0 \le y \le a$.
i.e. it is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$.
The rough sketch of the boundaries and the region R is given in Fig. 5.19.

After changing the order of integration, the given integral I becomes

$$I = \iint_R \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

The limits of inner integration are found by treating x as a constant, i.e. by drawing a line parallel to the y-axis in the region of integration as explained in the previous section.

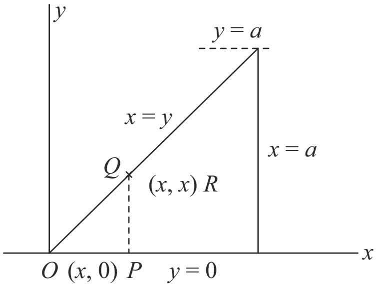

Fig. 5.19

Thus
$$I = \int_0^a \int_0^x \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

$$= \int_0^a x\left\{\log\left(y + \sqrt{y^2+x^2}\right)\right\}_{y=0}^{y=x} dx$$

$$= \int_0^a x\,[\log(x + x\sqrt{2}) - \log x]\, dx$$

$$= \log(1+\sqrt{2}) \cdot \left(\frac{x^2}{2}\right)_0^a = \frac{a^2}{2}\log(1+\sqrt{2})$$

Example 5.2 Change the order of integration in $\int_0^1 \int_x^1 \frac{x\, dx\, dy}{x^2+y^2}$ and then evaluate it.

Note ☑ Since the limits of inner integration are x and 1, the corresponding variable of integration should be y. So we rewrite the given integral I in the corrected form first.

$$I = \int_0^1 \int_x^1 \frac{x\, dy\, dx}{x^2+y^2}$$

The region of integration R is bounded by the lines $x = 0$, $x = 1$, $y = x$ and $y = 1$ and is given in Fig. 5.20.

The limits for the inner integration (after changing the order of integration) with respect to x are fixed as usual, by drawing a line parallel to x-axis (y = constant)

$\therefore$
$$I = \int_0^1 \int_0^y \frac{x\, dx\, dy}{x^2+y^2}$$

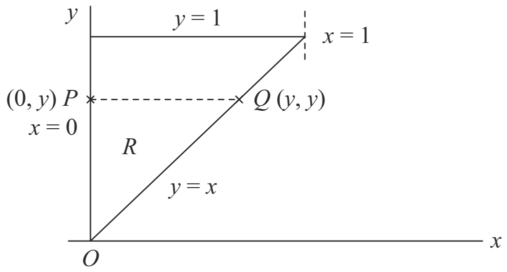

Fig. 5.20

$$= \int_0^1 \left[\frac{1}{2} \log \ (x^2 + y^2) \right]_{x=0}^{x=y} dy$$

$$= \frac{1}{2} \int_0^1 \log \left(\frac{2y^2}{y^2} \right) dy$$

$$= \frac{1}{2} \log 2.$$

Example 5.3 Change the order of integration in $\int_0^b \int_0^{\frac{a}{b}(b-y)} xy \, dx \, dy$ and then evaluate it.

The region of integration R is bounded by the lines $x = 0$, $x = \frac{a}{b}(b-y)$ or $\frac{x}{a} + \frac{y}{b} = 1$, $y = 0$ and $y = b$ and is shown in Fig. 5.21.

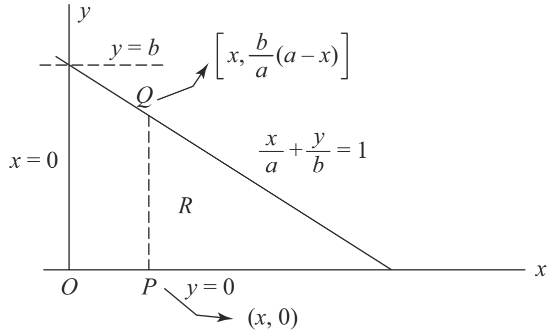

Fig. 5.21

After changing the order of integration, the integral becomes $I = \iint_R xy \, dy \, dx$. The limits are fixed as usual.

$$I = \int_0^a \int_0^{\frac{b}{a}(a-x)} xy \, dy \, dx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\frac{b}{a}(a-x)} \mathrm{d}x$$

$$= \frac{b^2}{2a^2}\int_0^a x\,(a-x)^2\,\mathrm{d}x$$

$$= \frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - 2a\frac{x^3}{3} + \frac{x^4}{4}\right]_0^a$$

$$= \frac{a^2b^2}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{a^2b^2}{24}$$

Example 5.4 Change the order of integration in $\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\,\mathrm{d}y\,\mathrm{d}x$ and then integrate it.

The region of integration R is bounded by the lines $x = 0$, $x = a$, $y = 0$ and the curve $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$ i.e. the curve $\dfrac{y^2}{b^2} = \dfrac{a^2 - x^2}{a^2}$, i.e. the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and is shown in Fig. 5.22.

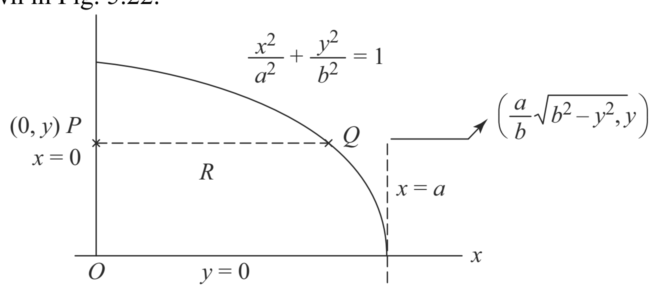

Fig. 5.22

After changing the order of integration, the integral becomes

$$\mathrm{I} = \iint_R x^2\,\mathrm{d}x\,\mathrm{d}y$$

The limits are fixed as usual.

$$\mathrm{I} = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^b \left(\frac{x^3}{3}\right)_0^{\frac{a}{b}\sqrt{b^2-y^2}} \mathrm{d}y$$

$$= \frac{a^3}{3b^3} \int_0^b (b^2 - y^2)^{\frac{3}{2}} \, dy$$

$$= \frac{a^3}{3b^3} \int_0^{\pi/2} b^4 \cos^4 \theta \, d\theta \qquad \text{(on putting } y = b \sin \theta\text{)}$$

$$= \frac{a^3 b}{3} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{16} a^3 b$$

Example 5.5 Change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y \, dx \, dy$ and then evaluate it.

The region of integration R is bounded by the line $x = a - y$, the curve $x = \sqrt{a^2 - y^2}$, the lines $y = 0$ and $y = a$.
i.e. the line $x + y = a$, the circle $x^2 + y^2 = a^2$ and the lines $y = 0$, $y = a$. R is shown in Fig. 5.23.

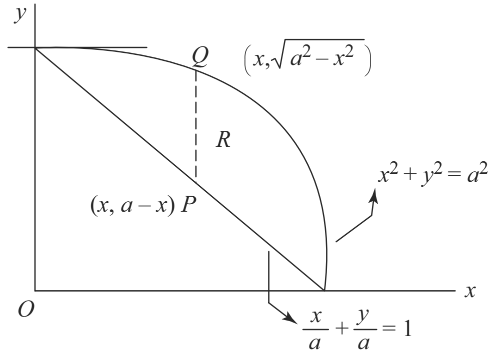

Fig. 5.23

After changing the order of integration, the integral I becomes,

$$\mathrm{I} = \iint_R y \, dy \, dx$$

$$= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y \, dy \, dx$$

$$= \int_0^a \left(\frac{y^2}{2} \right)_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a (2ax - 2x^2) \, dx$$

$$= \left(a\frac{x^2}{2} - \frac{x^3}{3} \right)_0^a$$

$$= \frac{a^3}{6}.$$

Example 5.6 Change the order of integration in $\int\limits_0^4 \int\limits_{\frac{x^2}{4}}^{2\sqrt{x}} \mathrm{d}y\, \mathrm{d}x$ and then evaluate it.

The region of integration R is bounded by the curve $y = \frac{x^2}{4}$ i.e. the parabola $x^2 = 4y$, the curve $y = 2\sqrt{x}$ i.e. the parabola $y^2 = 4x$ and the lines $x = 0$, $x = 4$. R is shown in Fig. 5.24.

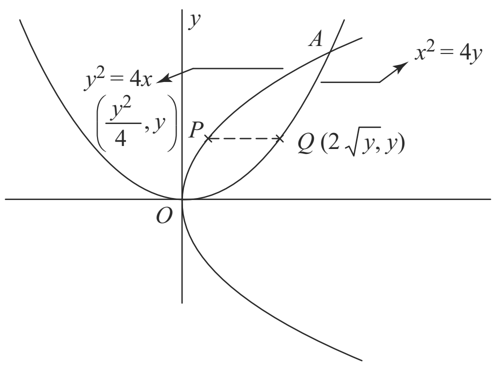

Fig. 5.24

The points of intersection of the two parabolas are obtained by solving the equations $x^2 = 4y$ and $y^2 = 4x$.

Solving them, we get $\left(\frac{x^2}{4}\right)^2 = 4x$

i.e $\quad x(x^3 - 64) = 0$

$\therefore \quad x = 0, \quad x = 4$

and $\quad y = 0, \quad y = 4$

i.e. the points of intersection are $O(0, 0)$ and $A(4, 4)$.

After changing the order of integration, the given integral

$$I = \iint_R \mathrm{d}x\, \mathrm{d}y$$

$$= \int\limits_0^4 \int\limits_{\frac{y^2}{4}}^{2\sqrt{y}} \mathrm{d}x\, \mathrm{d}y$$

$$= \int\limits_0^4 \left(2\sqrt{y} - \frac{y^2}{4} \right) \mathrm{d}y$$

$$=\left(\frac{4}{3}y^{\frac{3}{2}}-\frac{y^3}{12}\right)_0^4$$

$$=\frac{32}{3}-\frac{16}{3}$$

$$=\frac{16}{3}$$

Example 5.7 Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2+y^2}} xy\,\mathrm{d}x\,\mathrm{d}y$ and then evaluate it.

The region of integration R is bounded by the curve $x = a \mp \sqrt{a^2 - y^2}$, i.e. the circle $(x - a)^2 + y^2 = a^2$ and the lines $y = 0$ and $y = a$. The region R is shown in Fig. 5.25.

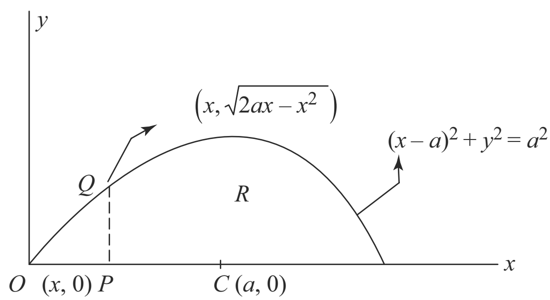

Fig. 5.25

After changing the order of integration, the integral I becomes

$$\mathrm{I} = \iint_R xy\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{2a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{2ax-x^2}} \mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2a} (2ax^2 - x^3)\mathrm{d}x$$

$$= \frac{1}{2}\left(2a\frac{x^3}{3} - \frac{x^4}{4}\right)_0^{2a}$$

$$= \frac{2}{3}a^4 \cdot$$

Example 5.8 Change the order of integration in $\int_0^1 \int_y^{2-y} xy \, dx \, dy$ and then evaluate it.

The region of integration R is bounded by the lines $x = y$, $x + y = 2$, $y = 0$ and $y = 1$. It is shown in Fig. 5.26.

After changing the order of integration, the integral I becomes

$$I = \iint_R xy \, dy \, dx$$

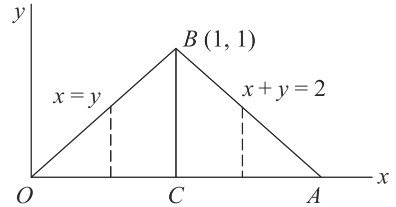

Fig. 5.26

To fix the limits for y in the inner integration, we have to draw a line parallel to y-axis (since x = constant). The line drawn parallel to the y-axis does not intersect the region R in the same fashion. If the line segment is drawn in the region OCB, its upper end lies on the line $y = x$; on the other hand, if it is drawn in the region BCA, its upper end lies on the line $x + y = 2$. In such situations, we divide the region into two sub-regions and fix the limits for each sub-region as illustrated below:

$$I = \iint_{\Delta OCB} xy \, dy \, dx + \iint_{\Delta BCA} xy \, dy \, dx$$

$$= \int_0^1 \int_0^x xy \, dy \, dx + \int_1^2 \int_0^{2-x} xy \, dy \, dx$$

$$= \int_0^1 x \left(\frac{y^2}{2} \right)_0^x dx + \int_1^2 x \left(\frac{y^2}{2} \right)_0^{2-x} dx$$

$$= \int_0^1 \frac{x^3}{2} dx + \int_1^2 \frac{x}{2} (2 - x)^2 \, dx$$

$$= \left(\frac{x^4}{8} \right)_0^1 + \frac{1}{2} \left(2x^2 - \frac{4}{3} x^3 + \frac{x^4}{4} \right)_1^2$$

$$= \frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}$$

Example 5.9 Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy \, dy \, dx$ and then evaluate it.

The region of integration R is bounded by the curve $y = \frac{x^2}{a}$, i.e. the parabola $x^2 = ay$, the line $y = 2a - x$, i.e. $x + y = 2a$ and the lines $x = 0$ and $x = a$. It is shown in Fig. 5.27.

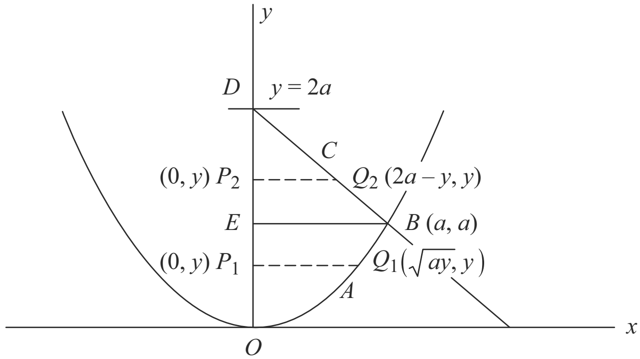

Fig. 5.27

After changing the order of integration, the integral I becomes

$$\mathrm{I} = \iint_R xy \, dx \, dy$$

When we draw a line parallel to x-axis for fixing the limits for the inner integration with respect to x, it does not intersect the region of integration in the same fashion. Hence the region R is divided into two sub-regions $OABE$ and $EBCD$ and then the limits are fixed as given below:

$$\mathrm{I} = \iint_{OABE} xy \, dx \, dy + \iint_{EBCD} xy \, dx \, dy$$

$$= \int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy + \int_a^{2a} \int_0^{2a-y} xy \, dx \, dy$$

Note ☑ The co-ordinates of the point B are obtained by solving the equations $x + y = 2a$ and $x^2 = ay$.

$B \equiv (a, a)$ and the equation of EB is $y = a$.

$$\mathrm{I} = \int_0^a y \left(\frac{x^2}{2} \right)_0^{\sqrt{ay}} dy + \int_a^{2a} y \left(\frac{x^2}{2} \right)_0^{2a-y} dy$$

$$= \frac{1}{2} \left[\int_0^a ay^2 \, dy + \int_a^{2a} y\,(2a - y)^2 \, dy \right]$$

$$= \frac{1}{2} \left[a \left(\frac{y^3}{3} \right)_0^a + \left(2a^2 \, y^2 - \frac{4a}{3} y^3 + \frac{y^4}{4} \right)_a^{2a} \right] = \frac{3}{8} a^4.$$

Example 5.10 Change the order of integration in each of the double integrals $\int_0^1 \int_1^2 \frac{dx \, dy}{x^2 + y^2}$ and $\int_1^2 \int_y^2 \frac{dx \, dy}{x^2 + y^2}$ and hence express their sum as one double integral and evaluate it.

The region of integration R_1 for the first double integral I_1 is bounded by the lines $x = 1, x = 2, y = 0$ and $y = 1$.

The region of integration R_2 for the second double integral I_2 is bounded by the lines $x = y$, $x = 2$, $y = 1$ and $y = 2$.

R_1 and R_2 are shown in Fig. 5.28.

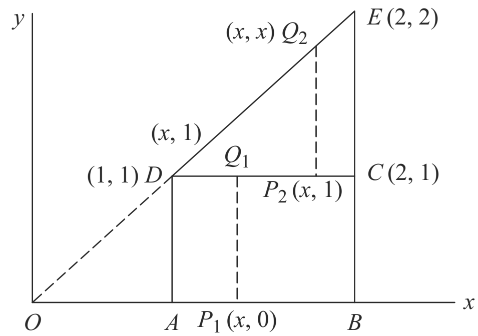

Fig. 5.28

After changing the order of integration,

$$I_1 = \int_1^2 \int_0^1 \frac{dy\,dx}{x^2 + y^2}$$

and $$I_2 = \int_1^2 \int_1^x \frac{dy\,dx}{x^2 + y^2}$$

Adding the integrals I_1 and I_2, we get

$$I = \int_1^2 dx \left(\int_0^1 \frac{dy}{x^2 + y^2} + \int_1^x \frac{dy}{x^2 + y^2} \right)$$

$$= \int_1^2 dx \int_0^x \frac{dy}{x^2 + y^2}$$

$$= \int_1^2 \left(\frac{1}{x} \tan^{-1} \frac{y}{x} \right)_{y=0}^{y=x} dx$$

$$= \int_1^2 \frac{\pi}{4} \frac{dx}{x} = \frac{\pi}{4} \log 2 .$$

Example 5.11 Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = x$ by double integration.

The region, the area of which is required is bounded by the parabolas $(y - 0)^2 = -(x - 4)$ and $y^2 = x$ and is shown in Fig. 5.29.

Required area $= \int_{OC} \int_{AB} dx\,dy$

$$= 2\int_{o}\int_{AB} \mathrm{d}x\,\mathrm{d}y, \text{ by symmetry}$$

$$= 2\int_0^{\sqrt{2}}\int_{y^2}^{4-y^2} \mathrm{d}x\,\mathrm{d}y$$

$$= 2\int_0^{\sqrt{2}} (4 - y^2 - y^2)\,\mathrm{d}y$$

$$= 2\left(4y - \frac{2}{3}y^3\right)_0^{\sqrt{2}}$$

$$= 2\left(4\sqrt{2} - \frac{4}{3}\sqrt{2}\right)$$

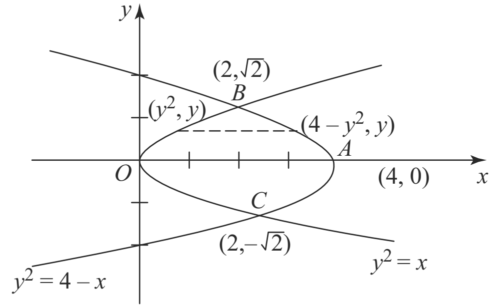

Fig. 5.29

$$= \frac{16}{3}\sqrt{2} \text{ square units}$$

Example 5.12 Find the area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ lying in the first quadrant, by double integration.

The plane region, the area of which is required, is shown in Fig. 5.30.

$$\text{Required area } = \iint_{ABC} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^a \left(\sqrt{a^2 - y^2} - a + y\right)\mathrm{d}y$$

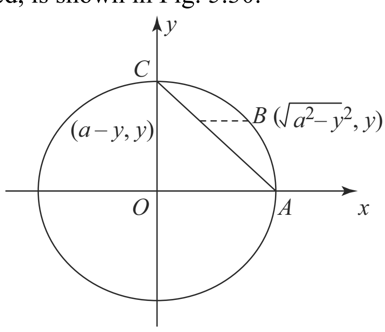

Fig. 5.30

$$= \left(\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a} - ay + \frac{y^2}{2}\right)_0^a$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} - a^2 + \frac{a^2}{2} = (\pi - 2)\frac{a^2}{4}\cdot$$

Example 5.13 Find the area enclosed by the lemniscate $r^2 = a^2 \cos 2\theta$, by double integration.

As the equation $r^2 = a^2 \cos 2\theta$ remains unaltered on changing θ to $-\theta$, the curve is symmetrical about the initial line.

The points of intersection of the curve with the initial line $\theta = 0$ are given by $r^2 = a^2$ or $r = \pm a$.

Since $r^2 = a^2 \cos 2\alpha = a^2 \cos 2\,(\pi - \alpha)$, the curve is symmetrical about the line $\theta = \dfrac{\pi}{2}$.

On putting $r = 0$, we get $\cos 2\theta = 0$. Hence $\theta = \pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}$. Hence there is a loop of the curve between $\theta = -\dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{4}$ and another loop between $\theta = -\dfrac{3\pi}{4}$ and $\theta = \dfrac{3\pi}{4}$.

Based on the observations given above the lemniscate is drawn in Fig. 5.31.

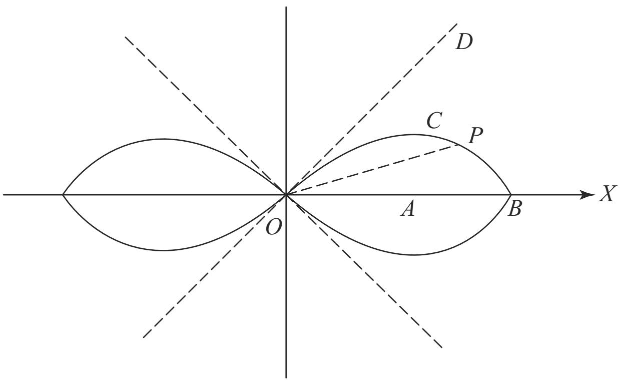

Fig. 5.31

Required area = 4 × area *OABC* (by symmetry)

$$= 4 \iint\limits_{OAB} r \, \mathrm{d}r \, \mathrm{d}\theta$$

When we perform the inner integration with respect to r, we have to treat θ as a constant temporarily and find the limits for r.

Geometrically, treating θ = constant means drawing a line *OP* arbitrarily through the pole lying within the region of integration as shown in the figure.

Finding the limits for r (while θ is a constant) is equivalent to finding the variation of the r coordinate of any point on the line *OP*. Assuming that the θ coordinates of all points on *OP* are θ each (since θ is constant on *OP*), we take $O \equiv (0, \theta)$ and $P \equiv (r_1, \theta)$; viz., r varies from 0 to r_1. Now wherever *OP* be drawn, the point $P(r_1, \theta)$ lies on the lemniscate.

Hence $r_1^2 = a^2 \cos 2\theta$ or $r_1 = a\sqrt{\cos 2\theta}$ (since r coordinate of any point is + ve)

Thus the limits for inner integration are 0 and $a\sqrt{\cos 2\theta}$.

When we perform the outer integration, we have to find the limits for θ. Geometrically, we have to find the variation of the line *OP* so that it sweeps the area of the region, namely *OABC*. To cover this area, the line *OP* has to start from the position OA ($\theta = 0$) and move in the anticlockwise direction and go up to $OD\left(\theta = \frac{\pi}{4}\right)$. Thus the limits for θ are 0 and $\frac{\pi}{4}$.

$$\therefore \quad \text{Required area} = 4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}\mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}}\cos 2\theta\,\mathrm{d}\theta$$

$$= a^2\,(\sin 2\theta)_0^{\frac{\pi}{4}} = a^2$$

Example 5.14 Find the area that lies inside the cardioid $r = a\,(1 + \cos\theta)$ and outside the circle $r = a$, by double integration.

The cardioid $r = a\,(1 + \cos\theta)$ is symmetrical about the initial line. The point of intersection of the line $\theta = 0$ with the cardioid is given by $r = 2a$, viz., the point $(2a, 0)$.

Putting $r = 0$ in the equation, we get $\cos\theta = -1$ and $\theta = \pm\pi$. Hence the cardioid lies between the lines $\theta = -\pi$ and $\theta = \pi$.

The point of intersection of the line $\theta = \frac{\pi}{2}$ is $\left(a, \frac{\pi}{2}\right)$.

Noting the above properties, the cardioid is drawn as shown in Fig. 5.32. All the points on the curve $r = a$ have the same r coordinate a, viz., they are at the same distance a from the pole. Hence the equation $r = a$ represents a circle with centre at the pole and radius equal to a.

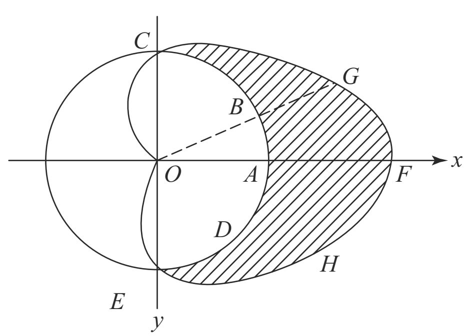

Fig. 5.32

Noting the above points, the circle $r = a$ is drawn as shown in Fig. 5.32. The area that lies outside the circle $r = a$ and inside the cardioid is shaded in the figure.

Both the curves are symmetric about the initial line. Hence the required area

$$= 2 \times AFGCB$$

$$= 2\int_{0}^{\frac{\pi}{2}}\int_{r_1}^{r_2} r\,\mathrm{d}r\,\mathrm{d}\theta$$, where (r_1, θ) lies on the circle $r = a$ and (r_2, θ) lies on the cardioid $r = a\,(1 + \cos\theta)$

$$= 2\int_{0}^{\frac{\pi}{2}}\int_{a}^{a(1+\cos\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_{0}^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_{a}^{a(1+\cos\theta)}\mathrm{d}\theta$$

$$= a^2\int_{0}^{\frac{\pi}{2}}[(1+\cos\,\theta)^2 - 1]\,\mathrm{d}\theta$$

$$= a^2\int_{0}^{\frac{\pi}{2}}\left(2\cos\theta + \frac{1+\cos\,2\theta}{2}\right)\mathrm{d}\theta$$

$$= a^2\left[2\,\sin\,\theta + \frac{\theta}{2} + \frac{1}{4}\sin\,2\theta\right]_{0}^{\frac{\pi}{2}}$$

$$= a^2\left(2 + \frac{\pi}{4}\right) = \frac{a^2}{4}\,(\pi + 8)$$

Example 5.15 Express $\displaystyle\int_{0}^{a}\int_{y}^{a}\frac{x^2\mathrm{d}x\,\mathrm{d}y}{\left(x^2+y^2\right)^{3/2}}$ in polar coordinates and then evaluate it.

The region of integration is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$, whose equations in polar system are $\theta = \dfrac{\pi}{4}$, $r = a\,\sec\,\theta$, $\theta = 0$ and $r = a$ cosec θ respectively. The region is shown in Fig. 5.33.

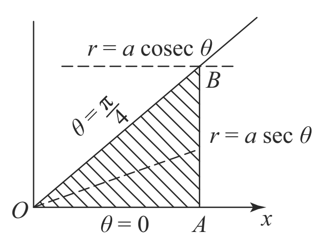

Fig. 5.33

Putting $x = r\cos\theta$, $y = r\sin\theta$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ in the given double integral I, we get

$$I = \iint_{OAB} \frac{r^3 \cos^2\theta}{r^3}\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \cos^2\theta\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_0^{\pi/4} \cos^2\,\theta \cdot [r]_0^{a\sec\theta}\, \mathrm{d}\theta$$

$$= a\int_0^{\pi/4} \cos\theta \mathrm{d}\theta = a\,[\sin\,\theta]_0^{\pi/4} = \frac{a}{\sqrt{2}}$$

Example 5.16 Transform the double integral $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{a^2-x^2-y^2}}$ in polar coordinates and then evaluate it.

The region of integration is bounded by the curves $y = \sqrt{ax - x^2}$, $y = \sqrt{a^2 - x^2}$ and the lines $x = 0$ and $x = a$.

$$y = \sqrt{ax - x^2} \text{ is the curve } x^2 + y^2 - ax = 0$$

i.e.,
$$\left(x - \frac{a}{2}\right)^2 + (y - 0)^2 = \left(\frac{a}{2}\right)^2$$

i.e. the circle with centre at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$

$$y = \sqrt{a^2 - x^2} \text{ is the curve } x^2 + y^2 = a^2$$

i.e. the circle with centre at the origin and radius a.

The polar equations of the boundaries of the region of integration are $r^2 - ar \cos\theta = 0$ or $r = a\cos\theta$, $r = a$, $r = a\sec\theta$ and $\theta = \frac{\pi}{2}$. The region of integration is shown in Fig. 5.34.

Putting $x = r\cos\theta$, $y = r\sin\theta$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ in the given double integral I, we get

$$I = \int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{a^2 - r^2}}$$

$$= \int_0^{\pi/2} \left\{-\frac{1}{2} \times 2\sqrt{a^2 - r^2}\right\}_{a\cos\theta}^{a} \mathrm{d}\theta, \text{ on putting } a^2 - r^2 = t$$

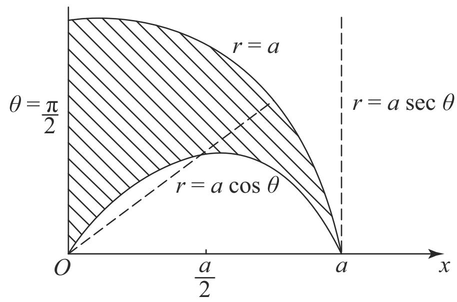

Fig. 5.34

$$= \int_0^{\pi/2} a \sin \theta \, d\theta = -a \left[\cos \theta\right]_0^{\frac{\pi}{2}} = a$$

Example 5.17 By transforming into cylindrical coordinates, evaluate the integral $\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$ taken over the region of space defined by $x^2 + y^2 \le 1$ and $0 \le z \le 1$.

The region of space is the region enclosed by the cylinder $x^2 + y^2 = 1$ whose base radius is 1 and axis is the z-axis and the planes $z = 0$ and $z = 1$. The equation of the cylinder in cylindrical coordinates is $r = 1$. The region of space is shown in Fig. 5.35.

Putting $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ and $dx\, dy\, dz = r\, dr\, d\theta\, dz$ in the given triple integral I, we get

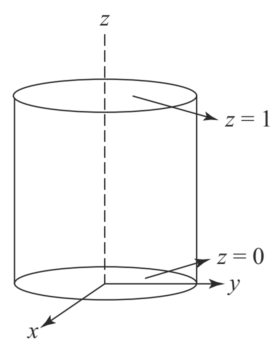

Fig. 5.35

$$I = \iiint_V (r^2 + z^2)\, r\, dr\, d\theta\, dz,$$

where V is the volume of the region of space.

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r^2 + z^2)\, r\, dr\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^4}{4} + z^2 \frac{r^2}{2} \right)_0^1 d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2} z^2 \right) d\theta\, dz$$

$$= 2\pi \int_0^1 \left(\frac{1}{4} + \frac{1}{2} z^2 \right) dz$$

$$= 2\pi\left[\frac{z}{4}+\frac{z^3}{6}\right]_0^1$$

$$= \frac{5}{6}\pi$$

Note ☑ The intersection of z = constant c and the cylinder $x^2 + y^2 = 1$ is a circle with centre at $(0, 0, c)$ and radius 1. The limits for r and θ have been fixed to cover the area of this circle and then the variation of z has been used so as to cover the entire volume.]

Example 5.18 Find the volume of the portion of the cylinder $x^2 + y^2 = 1$ intercepted between the plane $z = 0$ and the paraboloid $x^2 + y^2 = 4 - z$.

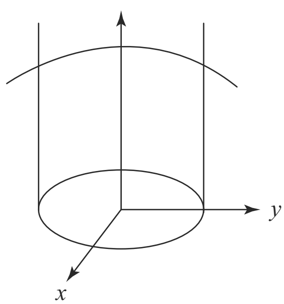

Fig. 5.36

Using cylindrical coordinates, the required volume V is given by

$$V = \iiint r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \text{ taken throughout the region of space.}$$

Since the variation of z is not between constant limits, we first integrate with respect to z and then with respect to r and θ.

Changing to cylindrical coordinates, the boundaries of the region of space are $r = 1$, $z = 0$ and $z = 4 - r^2$.

$$\therefore \qquad V = \int_0^{2\pi}\int_0^1\int_0^{4-r^2} \mathrm{d}z \, r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^1 r(4-r^2) \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^1 \mathrm{d}\theta = \frac{7}{4}\int_0^{2\pi} \mathrm{d}\theta = \frac{7}{2}\pi$$

Example 5.19 Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$, by transforming to spherical polar coordinates.

The boundaries of the region of integration are $z = 0$, $z=\sqrt{a^2-x^2-y^2}$ or $x^2 + y^2 + z^2 = a^2$, $y = 0$, $y=\sqrt{a^2-x^2}$ or $x^2 + y^2 = a^2$, $x = 0$ and $x = a$. From the boundaries, we note that the region of integration is the volume of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

By putting $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$, the given triple integral I becomes

$$I=\iiint_V r^3\sin^2\theta\cos\theta\sin\phi\cos\phi . r^2\sin\theta\,dr\,d\theta\,d\phi$$

where V is the volume of the positive octant of the sphere $r = a$, which is shown in Fig. 5.37.

To cover the volume V, r has to vary from 0 to a, θ has to vary from 0 to $\frac{\pi}{2}$ and ϕ has to vary from 0 to $\frac{\pi}{2}$.

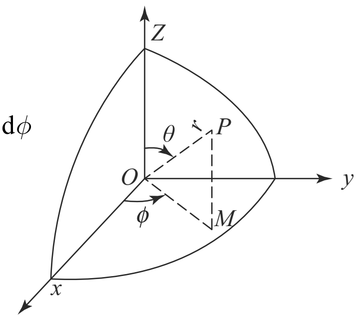

Fig. 5.37

Thus $$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^5\sin^3\theta\cos\theta\sin\phi\cos\phi\,dr\,d\theta\,d\phi$$

$$=\int_0^{\frac{\pi}{2}}\sin\phi\cos\phi\,d\phi.\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta.\int_0^{a}r^5\,dr$$

[$\because$ the limits are constants]

$$=\left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{\sin^4\theta}{4}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{r^6}{6}\right]_0^{a}$$

$$=\frac{1}{48}a^6.$$

Example 5.20 Evaluate $\iiint\sqrt{1-x^2-y^2-z^2}\,dx\,dy\,dz$, taken throughout the volume of the sphere $x^2 + y^2 + z^2 = 1$, by transforming to spherical polar coordinates.

Changing to spherical polar coordinates, the given triple integral I becomes

$$I=\iiint_V\sqrt{1-r^2}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

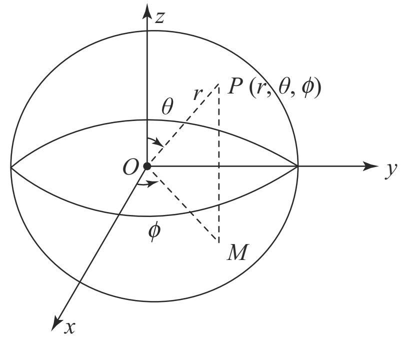

Fig. 5.38

To cover the entire volume V of the sphere, r has to vary from 0 to 1, θ has to vary from 0 to π and ϕ has to vary from 0 to 2π.

Thus $$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{1-r^2}\, r^2\, \mathrm{d}r \cdot \sin\theta\, \mathrm{d}\theta \cdot \mathrm{d}\phi$$

$$= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t\, \mathrm{d}t$$, by putting $r = \sin t$ in the innermost integral

$$= 2\pi \times (-\cos\theta)_0^{\pi} \times \left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= 4\pi \times \frac{\pi}{4} \times \frac{1}{4} = \frac{1}{4}\pi^2$$

EXERCISE 5(b)

Part A

(Short Answer Questions)

1. Change the order of integration in $\int_0^a \int_0^x f(x, y)\, \mathrm{d}y\, \mathrm{d}x$.
2. Change the order of integration in $\int_0^1 \int_y^1 f(x, y)\, \mathrm{d}x\, \mathrm{d}y$.
3. Change the order of integration in $\int_0^a \int_x^a f(x, y)\, \mathrm{d}y\, \mathrm{d}x$.
4. Change the order of integration in $\int_0^1 \int_0^y f(x, y)\, \mathrm{d}x\, \mathrm{d}y$.
5. Change the order of integration in $\int_0^1 \int_0^{1-y} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$.

6. Change the order of integration in $\int_0^a \int_0^{a-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$.

7. Change the order of integration in $\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$.

8. Change the order of integration in $\int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

9. Change the order of integration in $\int_0^1 \int_0^{2\sqrt{x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$.

10. Change the order of integration in $\int_0^\infty \int_0^{1/y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Part B

Change the order of integration in the following integrals and then evaluate them:

11. $\int_0^a \int_y^a \frac{x\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$

12. $\int_0^2 \int_x^2 (x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x$

13. $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}}\,\mathrm{d}y\,\mathrm{d}x$

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,\mathrm{d}y\,\mathrm{d}x$

15. $\int_0^1 \int_0^{1-x} e^{2x+y}\,\mathrm{d}y\,\mathrm{d}x$

16. $\int_0^2 \int_0^{\sqrt{4-y^2}} xy\,\mathrm{d}x\,\mathrm{d}y$

17. $\int_0^{2a} \int_{\frac{x^2}{4a}}^{a} (x+y)\,\mathrm{d}y\,\mathrm{d}x$

18. $\int_0^1 \int_{y^2}^{y} \frac{y\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$

19. $\int_0^3 \int_1^{\sqrt{4-x}} (x+y)\,\mathrm{d}y\,\mathrm{d}x$

20. $\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} y^2\,\mathrm{d}x\,\mathrm{d}y$

21. $\int_0^3 \int_{\frac{5}{9}x^2}^{\frac{5}{\sqrt{3}}x} \mathrm{d}x\,\mathrm{d}y$

22. $\int_1^2 \int_0^{4/x} xy\,\mathrm{d}y\,\mathrm{d}x$.

23. $\int_0^1 \int_x^{\sqrt{2-x^2}} \mathrm{d}y\,\mathrm{d}x$

24. Change the order of integration in each of the double integrals $\int_0^1 \int_0^{\sqrt{x}} xy\,\mathrm{d}y\,\mathrm{d}x$ and $\int_1^2 \int_0^{2-x} xy\,\mathrm{d}y\,\mathrm{d}x$ and hence express their sum as one double integral and evaluate it.

25. Change the order of integration in each of the double integrals $\int_{-1}^{0}\int_{-x}^{1}(x^2+y^2)\,dy\,dx$ and $\int_{0}^{1}\int_{x}^{1}(x^2+y^2)\,dy\,dx$ and hence express their sum as one double integral and evaluate it.

Find the area specified in the following problems (26–35), using double integration:

26. The area bounded by the parabola $y = x^2$ and the straight line $2x - y + 3 = 0$.
27. The area included between the parabolas $y^2 = 4a(x + a)$ and $y^2 = 4a(a - x)$.
28. The area bounded by the two parabolas $y^2 = 4ax$ and $x^2 = 4by$.
29. The area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.
30. The area bounded by the curve $y^2 = \dfrac{x^3}{2-x}$ and its asymptote.
31. The area of the cardioid $r = a(1 + \cos\theta)$.
32. The area common to the two circles $r = a$ and $r = 2a\cos\theta$.
33. The area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.
34. The area that lies inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$.
35. The area that lies outside the circle $r = a\cos\theta$ and inside the circle $r = 2a\cos\theta$.

Change the following integrals (36 − 40), into polar coordinates and then evaluate them:

36. $\displaystyle\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} e^{-\left(x^2+y^2\right)}\,dx\,dy$

37. $\displaystyle\int_{0}^{a}\int_{y}^{a}\frac{x\,dx\,dy}{\left(x^2+y^2\right)}$

38. $\displaystyle\int_{0}^{a}\int_{0}^{x}\frac{x^3\,dx\,dy}{\sqrt{x^2+y^2}}$

39. $\displaystyle\int_{0}^{2a}\int_{0}^{\sqrt{2ax-x^2}}\frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$

40. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx\,dy}{\left(a^2+x^2+y^2\right)^{3/2}}$

Evaluate the following integrals (41− 45) after transforming into cylindrical coordinates:

41. $\displaystyle\iiint_V (x+y+z)\,dx\,dy\,dz$, where V is the region of space inside the cylinder $x^2 + y^2 = a^2$, that is bounded by the planes $z = 0$ and $z = h$.

42. $\displaystyle\iiint (x^2+y^2)\,dx\,dy\,dz$, taken throughout the volume of the cylinder $x^2 + y^2 = 1$ that is bounded by the planes $z = 0$ and $z = 4$.

43. $\displaystyle\iiint dx\,dy\,dz$, taken throughout the volume of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $y + z = 3$.

44. $\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, taken throughout the volume of the cylinder $x^2 + y^2 = 4$ bounded by the plane $z = 0$ and the surface $z = x^2 + y^2 + 2$.

45. $\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, taken throughout the volume bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylindrical surface $x^2 + y^2 - 2ay = 0$.

Evaluate the following integrals (46-50) after transforming into spherical polar coordinates:

46. $\iiint \dfrac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{x^2+y^2+z^2}$, taken throughout the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

47. $\iiint \dfrac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\sqrt{1-x^2-y^2-z^2}}$, taken throughout the volume contained in the positive octant of the sphere $x^2 + y^2 + z^2 = 1$.

48. $\iiint\limits_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space bounded by the sphere $x^2 + y^2 + z^2 = a^2$ above the xOy-plane.

49. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

50. $\displaystyle\int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\left(a^2+x^2+y^2+z^2\right)^{5/2}}$

5.6 LINE INTEGRAL

The concept of a line integral is a generalisation of the concept of a definite integral $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

In the definite integral, we integrate along the x-axis from a to b and the integrand $f(x)$ is defined at each point in (a, b). In a line integral, we shall integrate along a curve C in the plane (or space) and the integrand will be defined at each point of C. The formal definition of a line integral is as follows.

Definition Let C be the segment of a continuous curve joining $A(a, b)$ and $B(c, d)$ (Fig. 5.39).

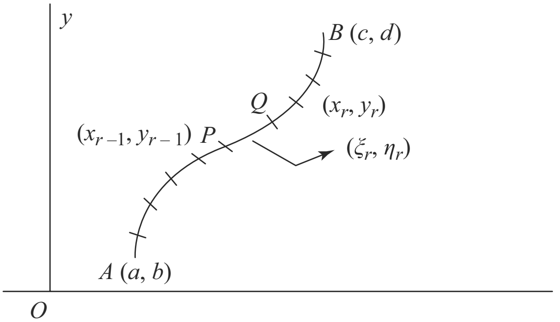

Fig. 5.39

Let $f(x, y)$, $f_1(x, y)$, $f_2(x, y)$ be single-valued and continuous functions of x and y, defined at all points of C.
Divide C into n arcs at (x_i, y_i) $[i = 1, 2, \ldots (n-1)]$
Let $x_0 = a$, $x_n = c$, $y_0 = b$, $y_n = d$.
Let $x_r - x_{r-1} = \Delta x_r$, $y_r - y_{r-1} = \Delta y_r$ and the arcual length of $PQ\,(\text{i.e.}\,\widehat{PQ}) = \Delta s_r$, where P is (x_{r-1}, y_{r-1}) and $Q(x_r, y_r)$.
Let (ξ_r, η_r) be any point on C between P and Q.

Then $$\lim_{n\to\infty}\sum_{r=1}^{n} f(\xi_r, \eta_r)\Delta s_r$$

or $$\lim_{n\to\infty}\sum_{r=1}^{n}\left[f_1(\xi_r, \eta_r)\Delta x_r + f_2(\xi_r, \eta_r)\Delta y_r\right]$$

is defined as a line integral along the curve C and denoted respectively as

$$\int_C f(x, y)\,\mathrm{d}s \quad \text{or} \quad \int_C [f_1(x, y)\,\mathrm{d}x + f_2(x, y)\,\mathrm{d}y]$$

5.6.1 Evaluation of a Line Integral

Using the equation $y = \phi(x)$ or $x = \psi(y)$ of the curve C, we express $\int_C [f_1(x, y)\,\mathrm{d}x + f_2(x, y)\,\mathrm{d}y]$ either in the form $\int_a^c g(x)\,\mathrm{d}x$ or in the form $\int_b^d h(y)\,\mathrm{d}y$ and evaluate it, which is only a definite integral.

If the line integral is in the form $\int_C f(x, y)\,\mathrm{d}s$, it is first rewritten as $\int_C f(x, y)\dfrac{\mathrm{d}s}{\mathrm{d}x}\mathrm{d}x = \int_C f(x, y)\sqrt{1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ or as $\int_C f(x, y)\dfrac{\mathrm{d}s}{\mathrm{d}y}\mathrm{d}y = \int_C f(x, y)\sqrt{1+\left(\dfrac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$ and then evaluated after expressing it as a definite integral.

5.6.2 Evaluation when C is a Curve in Space

The definition of the line integral given above can be extended when C is a curve in space. In this case, the line integral will take either the form $\int_C [f_1(x,y,z)\,\mathrm{d}x + f_2(x,y,z)\,\mathrm{d}y + f_3(x,y,z)\,\mathrm{d}z]$ or the form $\int_C f(x,y,z)\,\mathrm{d}s$. When C is a curve in space, very often the parametric equations of C will be known in the form $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$. Using the parametric equations of C, the line integral can be expressed as a definite integral. In the case of $\int_C f(x,y,z)\,\mathrm{d}s$, it is rewritten as $\int_C f(x,y,z)\dfrac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t$, where

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}.$$

5.7 SURFACE INTEGRAL

The concept of a surface integral is a generalisation of the concept of a double integral. While a double integral is evaluated over the area of a plane surface, a surface integral is evaluated over the area of a curved surface in general. The formal definition of a surface integral is given below.

Definition Let S be a portion of a regular two-sided surface. Let $f(z, y, z)$ be a function defined and continuous at all points on S. Divide S into n sub-regions Δs_1, $\Delta S_2, \ldots, \Delta S_n$. Let $P(\xi_r, \eta_r, \zeta_r)$ be any point in ΔS_r. Then $\lim\limits_{\substack{n\to\infty \\ \Delta S_r \to 0}} \sum\limits_{r=1}^{n} f(\xi_r, \eta_r, \zeta_r)\,\Delta S_r$ is called the surface integral of $f(x, y, z)$ over the surface S and denoted as $\int\limits_S f(x, y, z)\,\mathrm{d}S$ or $\iint\limits_S f(x, y, z)\,\mathrm{d}S$.

5.7.1 Evaluation of a Surface Integral

Let the surface integral be $\iint\limits_S f(x, y, z)\,\mathrm{d}S$, where S is the portion of the surface whose equation is $\phi\,(x, y, z) = c$ (Fig. 5.40).

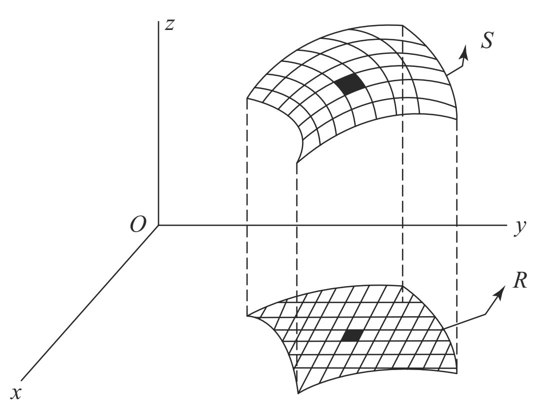

Fig. 5.40

Project the surface S orthogonally on xoy-plane (or any one of the co-ordinate planes) so that the projection is a plane region R.

The projection of the typical elemental surface ΔS (shaded in the figure) is the typical elemental plane area ΔA (shaded in the figure).

We can divide the area of the region R into elemental areas by drawing lines parallel to x and y axes at intervals of Δy and Δx respectively. Then $\Delta A = \Delta x \cdot \Delta y$.

Then $\Delta x \cdot \Delta y = \Delta S \cos\theta$, where θ is the angle between the surface S and the plane R (xoy-plane), i.e. θ is the angle between the normal to the surface S at the typical point (x, y, z) and the normal to the xoy-plane (z-axis). From Calculus, it is known that the direction ratios of the normal at the point (x, y, z) to the surface $\phi(x, y, z) = c$ are $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$. The direction cosines of the z-axis are (0, 0, 1)

$$\therefore \qquad \cos\theta = \frac{\frac{\partial\phi}{\partial z}}{\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}}$$

Thus
$$\Delta S = \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\Delta x\,\Delta y\,.$$

$$\therefore \qquad \iint_S f(x,y,z)\,\mathrm{d}S = \iint_R f(x,y,z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y$$

Thus the surface integral is converted into a double integral by using the above relation, in which the limits for the double integration on the right side are fixed so as to cover the entire region R and the integrand is converted into a function of x and y, using the equation of S.

Note ☑ Had we projected the curved surface S on the *yoz*-plane or *zox*-plane then the conversion formula would have been

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_R f(x,y,z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_x}\,\mathrm{d}y\,\mathrm{d}z$$

or
$$\iint_S f(x,y,z)\mathrm{d}S = \iint_R f(x,y,z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_y}\,\mathrm{d}z\,\mathrm{d}x, \qquad \text{respectively.}$$

5.8 VOLUME INTEGRAL

Definition Let V be a region of space, bounded by a closed surface. Let $f(x, y, z)$ be a continuous function defined at all points of V. Divide V into n sub-regions ΔV_r by drawing planes parallel to the *yoz*, *zox* and *xoy*-planes at intervals of Δx, Δy and Δz respectively. Then ΔV_r is a rectangular parallelopiped with dimensions Δx, Δy, Δz.
Let $P(\xi_r, \eta_r, \zeta_r)$ be any point in ΔV_r.

Then $\lim\limits_{\substack{n\to\infty \\ \Delta V_r \to 0}} \sum_{r=1}^{n} f(\xi_r, \eta_r, \zeta_r)\,\Delta V_r$ is called the volume integral of $f(x, y, z)$ over the region V (or throughout the volume V) and denoted as

$$\int_V f(x, y, z)\, \mathrm{d}v \quad \text{or} \quad \iiint_V f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

5.8.1 Triple Integral Versus Volume Integral

A triple integral discussed earlier is a three times repeated integral in which the limits of integration are given, whereas a volume integral is a triple integral in which the limits of integration will not be explicitly given, but the region of space in which it is to be evaluated will be specified. The limits of integration in a volume integral are fixed so as to cover the entire volume of the region of space V.

Note ☑ Though the line integral and surface integral have been defined in the scalar form in this chapter, they are also defined in the vector form. The vector form of the line and surface integrals will be discussed in Part II, Chapter 2.

WORKED EXAMPLE 5(c)

Example 5.1 Evaluate $\int_C [(3xy^2 + y^3)\, \mathrm{d}x + (x^3 + 3xy^2)\, \mathrm{d}y]$ where C is the parabola $y^2 = 4ax$ from the point (0, 0) to the point $(a, 2a)$

The given integral

$$\mathrm{I} = \int_{y^2 = 4ax} [(3xy^2 + y^3)\, \mathrm{d}x + (x^3 + 3\,xy^2)\, \mathrm{d}y]$$

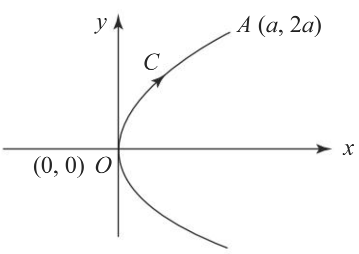

Fig. 5.41

In order to use the fact that the line integral is evaluated along the parabola $y^2 = 4ax$, we use this equation and the relation between dx and dy derived from it, namely, $2y\, \mathrm{d}y = 4a\, \mathrm{d}x$ and convert the body of the integral either to the form $f(x)\, \mathrm{d}x$ or to the form $\phi(y)\, \mathrm{d}y$. Then the resulting definite integral is evaluated between the concerned limits, got from the end points of C.

The choice of the form $f(x)\, \mathrm{d}x$ or $\phi(y)\, \mathrm{d}y$ for the body of the integral depends on convenience. In this problem, x is expressed as $\frac{1}{4a} y^2$ more easily than expressing y as $2\sqrt{ax}$.

Note ☑ From $y^2 = 4ax$, we get $y = \pm 2\sqrt{ax}$. Since the arc C lies in the first quadrant, y is positive and hence $y = 2\sqrt{ax}$.

Thus $$\mathrm{I} = \int_0^{2a} \left[\left(3 \cdot \frac{1}{4a} y^2 \cdot y^2 + y^3\right) \frac{y}{2a} \mathrm{d}y + \left(\frac{1}{64a^3} y^6 + 3 \cdot \frac{1}{4a} y^2 \cdot y^2\right) \mathrm{d}y \right]$$

(As the integration is done with respect to y, the limits for y are the y co-ordinates of the terminal points of the arc C).

$$I = \int_0^{2a} \left(\frac{5}{4a} y^4 + \frac{3}{8a^2} y^5 + \frac{1}{64a^3} y^6 \right) dy$$

$$= \left(\frac{1}{4a} y^5 + \frac{1}{16a^2} y^6 + \frac{1}{448a^3} y^7 \right)_0^{2a}$$

$$= \frac{86}{7} a^4$$

Example 5.2 Evaluate $\int_C [(2x - y)\, dx + (x + y)\, dy]$, where C is the circle $x^2 + y^2 = 9$.

In this problem the line integral is evaluated around a closed curve. In such a situation the line integral is denoted as

$\oint_C [(2x - y)\, dx + (x + y)\, dy]$, where a small circle is put across the integral symbol.

When a line integral is evaluated around a closed curve, it is assumed to be described in the anticlockwise sense, unless specified otherwise. (Fig. 5.42)

In the case of a line integral around a closed curve C, any point on C can be assumed to be the initial point, which will also be the terminal point.

Further if we take x or y as the variable of integration, the limits of integration will be the same, resulting in the value 'zero' of the line integral, which is meaningless. Hence whenever a line integral is evaluated around a closed curve, the parametric equations of the curve are used and hence the body of integral is converted to the form $f(t)\, dt$ or $f(\theta)\, d\theta$.

In this problem, the parametric equations of the circle $x^2 + y^2 = 9$ are $x = 3 \cos \theta$ and $y = 3 \sin \theta$.

$\therefore$ $dx = -3 \sin \theta \, d\theta$ and $dy = 3 \cos \theta \, d\theta$.

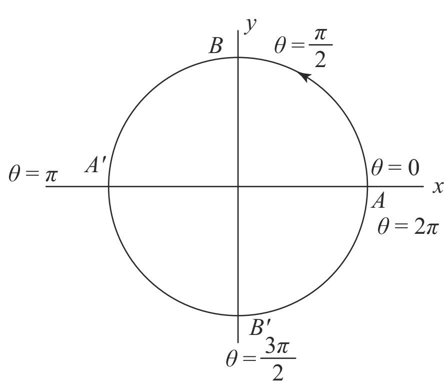

Fig. 5.42

The given integral = $\int_0^{2\pi} [(6 \cos \theta - 3 \sin \theta)\, (-3 \sin \theta \, d\theta)$

$$+ (3\cos\theta + 3\sin\theta)(3\cos\theta\, d\theta)]$$

$$= 9\int_0^{2\pi} (1 - \sin\theta\cos\theta)\, d\theta$$

$$= 9\left(\theta - \frac{\sin^2\theta}{2}\right)_0^{2\pi}$$

$$= 18\pi$$

Example 5.3 Evaluate $\int_C xy\, ds$, where C is the arc of the parabola $y^2 = 4x$ between the vertex and the positive end of the latus rectum.

Given integral $$I = \int_C xy \frac{ds}{dx} dx$$

Equation of the parabola is $y^2 = 4x$

Differentiating with respect to x, $\dfrac{dy}{dx} = \dfrac{2}{y}$

$\therefore$ $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4}{y^2}}$$

$\therefore$ $$I = \int_C xy \frac{\sqrt{y^2 + 4}}{y} dx$$

$$= \int_0^1 x\sqrt{4x + 4}\, dx$$

$$= 2\int_1^{\sqrt{2}} (t^2 - 1)\cdot t \cdot 2t\, dt, \text{ on putting } x + 1 = t^2$$

$$= 4\int_1^{\sqrt{2}} (t^4 - t^2)\, dt$$

$$= 4\left(\frac{t^5}{5} - \frac{t^3}{3}\right)_1^{\sqrt{2}}$$

$$= \frac{8}{15}\left(1 + \sqrt{2}\right)$$

Example 5.4 Evaluate $\int_C (y^2 dx - x^2 dy)$, where C is the boundary of the triangle whose vertices are $(-1, 0)$, $(1, 0)$ and $(0, 1)$ (Fig. 5.43).

C is made up of the lines BC, CA and AB.

Equations of BC, CA and AB are respectively $y = 0$, $x + y = 1$ and $-x + y = 1$.

$$\text{Given integral} = \int\limits_{\substack{BC \\ y=0 \\ dy=0}} + \int\limits_{\substack{CA \\ x+y=1 \\ dy=-dx}} + \int\limits_{\substack{AB \\ -x+y=1 \\ dy=dx}} (y^2 dx - x^2 dy)$$

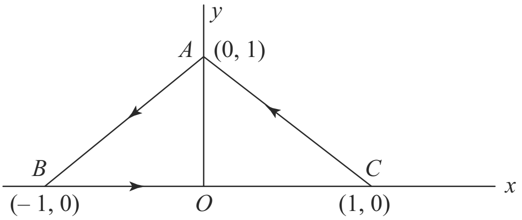

Fig. 5.43

$$= 0 + \int_1^0 [(1-x)^2 + x^2]\,dx + \int_0^{-1} [(1+x)^2 - x^2]\,dx$$

$$= \int_1^0 (1 - 2x + 2x^2)\,dx + \int_0^{-1} (1 + 2x)\,dx$$

$$= \left(x - x^2 + \frac{2x^3}{3}\right)_1^0 + (x + x^2)_0^{-1}$$

$$= -\frac{2}{3}$$

Example 5.5 Evaluate $\int_C [x^2 y\,dx + (x - z)\,dy + xyz\,dz]$, where C is the arc of the parabola $y = x^2$ in the plane $z = 2$ from (0, 0, 2) to (1, 1, 2).

$$\text{Given integral} = \int\limits_{\left(\substack{y=x^2 \\ z=2}\right)} [x^2 y\,dx + (x - z)\,dy + xyz\,dz]$$

$$= \int\limits_{y=x^2} [x^2 y\,dx + (x - 2)\,dy]$$

[$\because$ $dz = 0$, when $z = 2$]

$$= \int_0^1 [x^4 + (x-2)2x]\,dx = \left(\frac{x^5}{5} + \frac{2x^3}{3} - 2x^2\right)_0^1$$

$$= -\frac{17}{15}$$

Example 5.6 Evaluate $\int_C (x\,dx + xy\,dy + xyz\,dz)$, where C is the arc of the twisted curve $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$.

The parametric equations of C are $x = t$, $y = t^2$, $z = t^3$

$\therefore$ $dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$ on C.

Using these values in the given integral I,

$$\mathrm{I}=\int_0^1 (t+t^3\cdot 2t+t^6\cdot 3t^2)\,\mathrm{d}t$$

$$=\left(\frac{t^2}{2}+2\frac{t^5}{5}+3\frac{t^9}{9}\right)_0^1$$

$$=\frac{17}{30}$$

Example 5.7 Evaluate $\int_C (x^2+y^2+z^2)\,\mathrm{d}s$, where C is the arc of the circular helix $x = \cos t$, $y = \sin t$, $z = 3t$ from (1, 0, 0) to (1, 0, 6π)

The parametric of equations of C are

$x = \cos t$, $y \sin t$, $z = 3t$.

$$\therefore \qquad \frac{\mathrm{d}x}{\mathrm{d}t}=-\sin t, \qquad \frac{\mathrm{d}y}{\mathrm{d}x}=\cos t, \qquad \frac{\mathrm{d}z}{\mathrm{d}t}=3 \text{ on } C.$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}$$

$$=\sqrt{\sin^2 t+\cos^2 t+9}=\sqrt{10}$$

Given integral $\quad \mathrm{I}=\int_0^{2\pi} (\cos^2 t+\sin^2 t+9t^2)\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t$

Note ☑ The point (1, 0, 0) corresponds to $t = 0$ and (1, 0, 6π) corresponds to $t = 2\pi$.

$$\mathrm{I}=\left(t+3t^3\right)_0^{2\pi}\times\sqrt{10}$$

$$=2\sqrt{10}\pi(1+12\pi^2)$$

Example 5.8 Evaluate $\iint_S xyz\,\mathrm{d}S$, where S is the surface of the rectangular parallelopiped formed by $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = b$ and $z = c$ (Fig. 5.44).

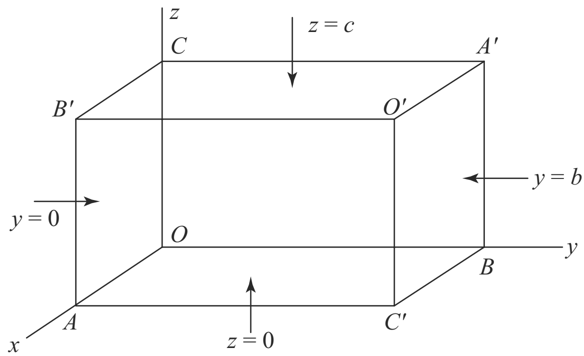

Fig. 5.44

Since S is made up of 6 plane faces, the given surface integral I is expressed as

$$\mathrm{I} = \iint_{x=0} + \iint_{x=a} + \iint_{y=0} + \iint_{y=b} + \iint_{z=0} + \iint_{z=c} (xyz\, \mathrm{d}S)$$

Since all the faces are planes, the elemental curved surface area dS becomes the elemental plane surface area dA.

On the planes $x = 0$ and $x = a$, dA = dy dz.
On the planes $y = 0$ and $y = b$, dA = dz dx.
On the planes $z = 0$ and $z = c$, dA = dx dy.

$$\therefore \quad \mathrm{I} = \iint_{x=0} + \iint_{x=a} (xyz\, \mathrm{d}y\, \mathrm{d}z) + \iint_{y=0} + \iint_{y=b} (xyz\, \mathrm{d}z\, \mathrm{d}x)$$

$$+ \iint_{z=0} + \iint_{z=c} (xyz\, \mathrm{d}x\, \mathrm{d}y)$$

Simplifying the integrands using the equations of the planes over which the surface integrals are evaluated, we get

$$\mathrm{I} = a \int_0^c \int_0^b yz\, \mathrm{d}y\, \mathrm{d}z + b \int_0^a \int_0^c zx\, \mathrm{d}z\, \mathrm{d}x + c \int_0^b \int_0^a xy\, \mathrm{d}x\, \mathrm{d}y$$

Note ☑ On the plane face $O'A'CB'$ ($z = c$), the limits for x and y are easily found to be 0, a and 0, b. Similarly the limits are found on the faces $O'B'AC'$ ($x = a$) and $O'C'BA'$ ($y = b$).]

Now
$$\mathrm{I} = a \frac{b^2}{2} \cdot \frac{c^2}{2} + b \cdot \frac{c^2}{2} \cdot \frac{a^2}{2} + c \cdot \frac{a^2}{2} \cdot \frac{b^2}{2}$$

$$= \frac{abc}{4}(ab + bc + ca)$$

Example 5.9 Evaluate $\iint_S (y + 2z - 2)\, \mathrm{d}S$, where S is the part of the plane $2x + 3y + 6z = 12$, that lies in the positive octant (Fig. 5.45).

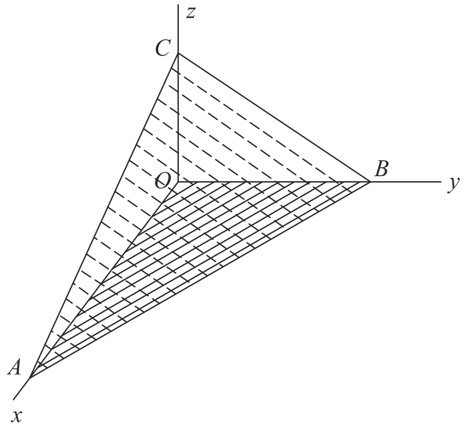

Fig. 5.45

Rewriting the equation of the (plane) surface S in the intercept form, we get

$$\frac{x}{6}+\frac{y}{4}+\frac{z}{2}=1$$

$\therefore$ S is the plane that cuts off intercepts of lengths 6, 4 and 2 on the x, y and z-axes respectively and lies in the positive octant.

We note that the projection of the given plane surface S on the xoy-plane is the triangular region OAB shown in the two-dimensional Fig. 5.46.

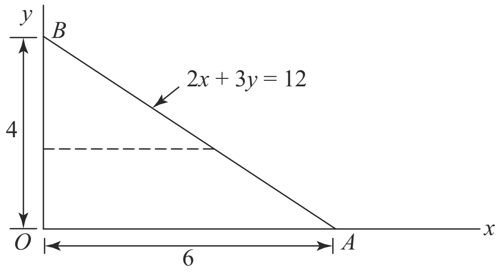

Fig. 5.46

Converting the given surface integral I as a double integral,

$$\mathrm{I}=\iint_{\Delta OAB}(y+2z-2)\frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\phi = c$ is the equation of the given surface S.

Here $\phi = 2x + 3y + 6z$

$\therefore$ $\phi_x = 2$; $\phi_y = 3$; $\phi_z = 6$.

$\therefore$
$$\mathrm{I}=\iint_{\Delta OAB}(y+2z-2)\frac{\sqrt{4+9+36}}{6}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{7}{6}\iint_{\Delta OAB}(y+2z-2)\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

Now the integrand is expressed as a function of x and y, by using the value of z (as a function of x and y) got from the equation of S, i.e. from the equation $2x + 3y + 6z = 12$

Thus
$$z=\frac{1}{6}(12-2x-3y) \tag{2}$$

Using (2) in (1), we get

$$\mathrm{I}=\frac{7}{6}\iint_{\Delta OAB}\frac{1}{3}(6-2x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{7}{18}\int_0^4 \int_0^{6-\frac{3}{2}y} (6-2x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{7}{18}\int_0^4 \left(9y - \frac{9}{4}y^2\right)\mathrm{d}y = \frac{28}{3}$$

Example 5.10 Evaluate $\iint_S z^3\, dS$, where S is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ (Fig. 5.47)

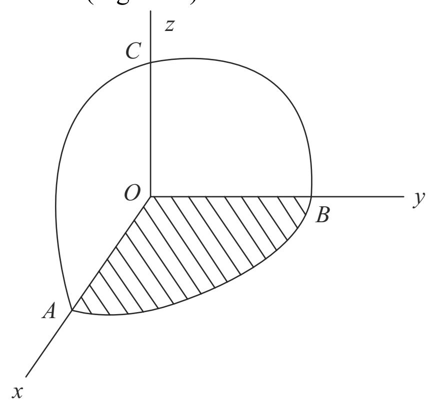

Fig. 5.47

The projection of the given surface of the sphere $x^2 + y^2 + z^2 = a^2$ (lying in the positive octant) in the *xoy* - plane is the quadrant of the circular region *OAB*, shown in the two-dimensional Fig. 5.48.

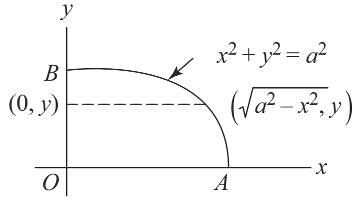

Fig. 5.48

Converting the given surface integral I as a double integral.

$$\mathrm{I} = \iint_{OAB} z^3 \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\phi \equiv x^2 + y^2 + z^2 = a^2$ is the equation of the given spherical surface.

$\phi_x = 2x;\ \phi_y = 2y;\ \phi_z = 2z.$

$\therefore$
$$\mathrm{I} = \iint_{OAB} z^3 \frac{\sqrt{4(x^2 + y^2 + z^2)}}{2z}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_{OAB} z^2 \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= a \iint_{OAB} (a^2 - x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y \qquad [\because (x, y, z) \text{ lies on } x^2 + y^2 + z^2 = a^2]$$

$$= a \int_0^a \int_0^{\sqrt{a^2 - y^2}} (a^2 - y^2 - x^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= a \int_0^a \left[(a^2 - y^2)x - \frac{x^3}{3} \right]_{x=0}^{x=\sqrt{a^2-y^2}} \mathrm{d}y$$

$$= \frac{2}{3} a \int_0^a (a^2 - y^2)^{\frac{3}{2}}\,\mathrm{d}y$$

$$= \frac{2}{3} a^5 \int_0^{\pi/2} \cos^4\theta\,\mathrm{d}\theta, \text{ on putting } x = a\sin\theta.$$

$$= \frac{2}{3} a^5 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{8} a^5 \cdot$$

Example 5.11 Evaluate $\iint_S y(z+x)\,\mathrm{d}S$, where S is the curved surface of the cylinder $x^2 + y^2 = 16$, that lies in the positive octant and that is included between the planes $z = 0$ and $z = 5$ (Fig. 5.49).

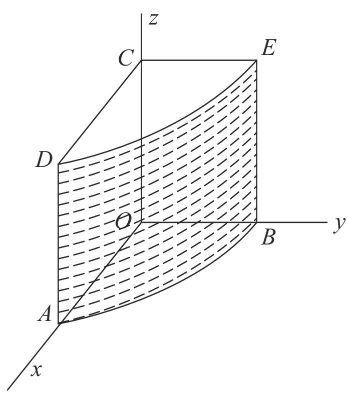

Fig. 5.49

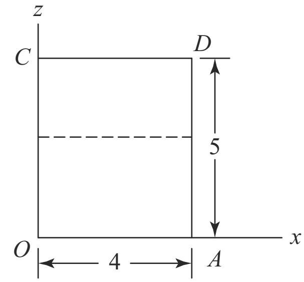

Fig. 5.50

We note that the projection of S on the xoy-plane is not a plane (region) surface, but only the arc AB of the circle whose centre is O and radius equal to 4.

For converting the given surface integral into a double integral, the projection of S must be a plane region. Hence we project S on the zox-plane (or yoz-plane). The projection of S in this case is the rectangular region $OCDA$, which is shown in Fig. 5.50.

Converting the given surface integral I as a double integral,

$$\mathrm{I} = \iint_{OADC} y(z+x)\frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{\phi y}\,\mathrm{d}z\,\mathrm{d}x,$$

where $\phi \equiv x^2 + y^2 = 16$ is the equation of the given cylindrical surface. $\phi_x = 2x$; $\phi_y = 2y$; $\phi_z = 0$.

$$\therefore \qquad \mathrm{I} = \iint_{OADC} y(z+x)\frac{\sqrt{4(x^2+y^2)}}{2y}\,\mathrm{d}z\,\mathrm{d}x$$

$$= 4\iint_{OADC}(z+x)\,\mathrm{d}z\,\mathrm{d}x \qquad [\because (x, y, z) \text{ lies on } x^2 + y^2 = 16]$$

$$= 4\int_0^5\int_0^4 (z+x)\,\mathrm{d}x\,\mathrm{d}z\,.$$

$$= 4\int_0^5 (4z+8)\,\mathrm{d}z$$

$$= 8(z^2+4z)_0^5$$

$$= 360.$$

Example 5.12 Evaluate $\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space inside the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Vide worked Example 5.11 in the section on 'Double and triple integrals' for fixing the limits of the volume integral.

$$\mathrm{I} = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} xy\left(\frac{z^2}{2}\right)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} \mathrm{d}y\,\mathrm{d}x$$

$$= \frac{c^2}{2}\int_0^a\int_0^{bt} xy\left(t-\frac{y}{b}\right)^2 \mathrm{d}y\,\mathrm{d}x\,, \text{ where } t = 1-\frac{x}{a}$$

$$= \frac{c^2}{2}\int_0^a x\left(t^2\frac{y^2}{2}-\frac{2t}{b}\frac{y^3}{3}+\frac{1}{b^2}\frac{y^4}{4}\right)_0^{bt} \mathrm{d}x$$

$$= \frac{c^2}{2}\int_0^a \left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)b^2\,xt^4\,\mathrm{d}x$$

$$= \frac{b^2 c^2}{24} \int_0^a x \left(1 - \frac{x}{a}\right)^4 \mathrm{d}x$$

$$= \frac{b^2 c^2}{24} \int_0^a a \left[1 - \left(1 - \frac{x}{a}\right)\right] \cdot \left(1 - \frac{x}{a}\right)^4 \mathrm{d}x$$

$$= \frac{ab^2 c^2}{24} \left[\frac{\left(1 - \frac{x}{a}\right)^5}{-\frac{5}{a}} + \frac{\left(1 - \frac{x}{a}\right)^6}{\frac{6}{a}} \right]_0^a$$

$$= \frac{a^2 b^2 c^2}{24} \left(\frac{1}{5} - \frac{1}{6}\right)$$

$$= \frac{1}{720} a^2 b^2 c^2 \cdot$$

Example 5.13 Express the volume of the sphere $x^2 + y^2 + z^2 = a^2$ as a volume integral and hence evaluate it. [Refer to Fig. 5.51]

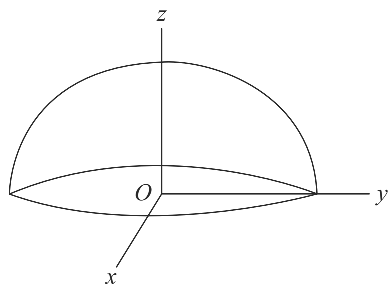

Fig. 5.51

Required volume = 2 × volume of the hemisphere above the *xoy*-plane. Vide worked example 5.12 in the section on 'Double and Triple Integrals'.

$$\text{Required volume } = 2\int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

$$= 2\int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \sqrt{(a^2 - x^2) - y^2}\, \mathrm{d}y\, \mathrm{d}x$$

Taking $a^2 - x^2 = b^2$, when integration with respect to y is performed,

$$V = 2\int_{-a}^{a} \int_{-b}^{b} \sqrt{b^2 - y^2}\, \mathrm{d}y\, \mathrm{d}x$$

$$= 4\int_{-a}^{a}\int_{0}^{b}\sqrt{b^2-y^2}\,\mathrm{d}y\,\mathrm{d}x \qquad [\because \sqrt{b^2-y^2} \text{ is an even function of } y]$$

$$= 4\int_{-a}^{a}\left(\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\frac{y}{b}\right)_0^b \mathrm{d}x$$

$$= \pi\int_{-a}^{a}(a^2-x^2)\,\mathrm{d}x$$

$$= 2\pi\left(a^2x-\frac{x^3}{3}\right)_0^a$$

$$= \frac{4}{3}\pi a^3$$

Example 5.14 Evaluate $\iiint\limits_V (x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space inside the cylinder $x^2 + y^2 = a^2$ that is bounded by the planes $z = 0$ and $z = h$ [Refer to Fig. 5.52].

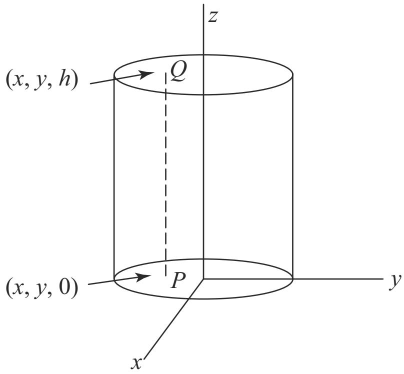

Fig. 5.52

Note ☑ The equation $x^2 + y^2 = a^2$ (in three dimensions) represents the right circular cylinder whose axis is the z-axis and base circle is the one with centre at the origin and radius equal to a.

$$\mathrm{I} = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{h}(x+y+z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[(x+y)h+\frac{h^2}{2}\right]\mathrm{d}y\,\mathrm{d}x$$

$$= 2h\cdot\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left(x+\frac{h}{2}\right)\mathrm{d}y\,\mathrm{d}x$$

[by using properties of odd and even functions]

$$= 2h\int_{-a}^{a}\left(x+\frac{h}{2}\right)\sqrt{a^2-x^2}\,\mathrm{d}x$$

$$= 2h^2\int_{0}^{a}\sqrt{a^2-x^2}\,\mathrm{d}x \qquad [\because\ x\sqrt{a^2-x^2}\text{ is odd and }\sqrt{a^2-x^2}\text{ is even}]$$

$$= 2h^2\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0^a$$

$$= \frac{\pi}{2}a^2h^2.$$

EXERCISE 5(c)

Part A

(Short Answer Questions)

1. Define a line integral.
2. What is the difference between a definite integral and a line integral?
3. Define a surface integral.
4. What is the difference between a double integral and a surface integral?
5. Define a volume integral.
6. What is the difference between a triple integral and a volume integral?
7. Write down the formula that converts a surface integral into a double integral.
8. Evaluate $\int_C (x^2\,\mathrm{d}y + y^2\,\mathrm{d}x)$ where C is the path $y = x$ from (0, 0) to (1, 1).
9. Evaluate $\int_C \sqrt{(x^2+y^2)}\,\mathrm{d}s$, where C is the path $y = -x$ from (0, 0) to (−1, 1).
10. Evaluate $\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$, where C is the circle $x^2 + y^2 = 1$ from (1, 0) to (0, 1) in the counterclockwise sense.
11. Evaluate $\iint_S \mathrm{d}S$, where S is the surface of the parallelopiped formed by $x = \pm 1, y = \pm 2, z = \pm 3$.

    [*Hint:* $\iint_S \mathrm{d}S$ gives the area of the surface S]
12. Evaluate $\iint_S \mathrm{d}S$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
13. Evaluate $\iint_S \mathrm{d}S$, where S is the curved surface of the right circular cylinder $x^2 + y^2 = a^2$, included between $z = 0$ and $z = h$.
14. Evaluate $\iiint_V \mathrm{d}V$, where V is the region of space bounded by the planes $x = 0, x = a, y = 0, y = 2b, z = 0$ and $z = 3c$.

[*Hint:* $\iiint_V \mathrm{d}V$ gives the volume of the region V]

15. Evaluate $\iiint_V \mathrm{d}V$, where V is the region of space bounded by $x^2 + y^2 + z^2 = 1$.
16. Evaluate $\iiint_V \mathrm{d}V$, where V is the region of space bounded by $x^2 + y^2 = a^2$, $z = -h$, $z = h$.

Part B

17. Evaluate $\int_{(0,0)}^{(1,3)} [x^2 y\, \mathrm{d}x + (x^2 - y^2)\,\mathrm{d}y]$ along the (i) curve $y = 3x^2$, (ii) line $y = 3x$.
18. Evaluate $\int_C [(x^2 - y^2 + x)\,\mathrm{d}x - (2x\,y + y)\,\mathrm{d}y]$ from (0, 0) to (1, 1), when C is (i) $y^2 = x$, (ii) $y = x$.
19. Evaluate $\int_{(-a,0)}^{(a,0)} (y^2\,\mathrm{d}x - x^2\mathrm{d}y)$ along the upper half of the circle $x^2 + y^2 = a^2$.
20. Evaluate $\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$, where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and described in the anticlockwise sense.
21. Evaluate $\int_C [(x^2 - y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y]$, where C is the boundary of the rectangle formed by the lines $x = 0, x = 2, y = 0, y = 1$ and described in the anticlockwise sense.
22. Evaluate $\int_C [(3x^2 - 8y^2)\,\mathrm{d}x + (4y - 6xy)\,\mathrm{d}y]$, where C is the boundary of the region enclosed by $y^2 = x$ and $x^2 = y$ and described in the anticlockwise sense.
23. Evaluate $\int_C (x - y^2)\,\mathrm{d}s$, where C is the arc of the circle $x = a\cos\theta$, $y = a \sin\theta$; $0 \le \theta \le \frac{\pi}{2}$.
24. Evaluate $\int_C x\,\mathrm{d}s$, where C is the arc of the parabola $x^2 = 2y$ from (0, 0) to $\left(1, \frac{1}{2}\right)$.
25. Evaluate $\int_C [xy\,\mathrm{d}x + (x^2 + z)\,\mathrm{d}y + (y^2 + x)\,\mathrm{d}z]$ from (0, 0, 0) to (1, 1, 1) along the curve C given by $y = x^2$ and $z = x^3$.
26. Evaluate $\int_C [(3x^2 + 6y)\,\mathrm{d}x - 14yz\,\mathrm{d}y + 20xz^2\,\mathrm{d}z]$, where C is the segment of the straight line joining (0, 0, 0) and (1, 1, 1).

27. Evaluate $\int_C [3x^2\,\mathrm{d}x + (2xy - y)\,\mathrm{d}y - z\,\mathrm{d}z]$ from $t = 0$ to $t = 1$ along the curve C given by $x = 2t^2$, $y = t$, $z = 4t^3$.
28. Evaluate $\int xy\,\mathrm{d}s$ along the arc of the curve given by the equations $x = a\tan\theta$, $y = a\cot\theta$, $z = \sqrt{2}\,a\log\tan\theta$ from the point $\theta = \frac{\pi}{4}$ to the point $\theta = \frac{\pi}{3}\cdot$
29. Evaluate $\int_C (xy + z^2)\mathrm{d}s$, where C is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1, 0, 0)$ to $(-1, 0, \pi)$.
30. Find the area of that part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ that lies in the positive octant. $\left[\textit{Hint}\text{: Area of the surface} = \iint_S \mathrm{d}s\right]$
31. Evaluate $\iint_S z\,\mathrm{d}S$, where S is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
32. Evaluate $\iint xy\,\mathrm{d}S$, where S is the curved surface of the cylinder $x^2 + y^2 = a^2$, $0 \le z \le k$, included in the positive octant.
33. Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
34. Evaluate $\iiint_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space bounded by the sphere $x^2 + y^2 + z^2 = a^2$ above the xoy-plane.
35. Evaluate $\iiint_V (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space inside the cylinder $x^2 + y^2 = a^2$ that is bounded by the planes $z = 0$ and $z = h$.

5.9 GAMMA AND BETA FUNCTIONS

Definitions The definite integral $\int_0^\infty e^{-x}\,x^{n-1}\,\mathrm{d}x$ exists only when $n > 0$ and when it exists, it is a function of n and called *Gamma function* and denoted by $\Gamma(n)$ [read as "Gamma n"].

Thus
$$\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,\mathrm{d}x$$

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$ exists only when $m > 0$ and $n > 0$ and when it exists, it is a function of m and n and called *Beta function* and denoted by β (m, n) [read as "Beta m, n"].

Thus $\quad \beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$:

30. Evaluate (i) $\int_0^\infty \frac{dx}{1+x^4}$, (ii) $\int_0^\infty \frac{x^2\,dx}{(1+x^4)^2}$ and (iii) $\int_0^\infty \frac{x^2\,dx}{(1+x^4)^3}$

[*Hint:* put $x^2 = \tan^2\theta$]

31. Find the value of $\iint x^m y^n\,dx\,dy$, taken over the area $x \geq 0, y \geq 0, x + y \leq 1$ in terms of Gamma functions, if $m, n > 0$.
32. Find the value, in terms of Gamma functions, of $\iiint x^m y^n z^p\,dx\,dy\,dz$ taken over the volume of the tetrahedron given by $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq 1$.
33. Find the area in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ and the co-ordinate axes.
34. Evaluate $\iint x^{m-1} y^{n-1} (1-x-y)^{p-1}\,dx\,dy$, taken over the area in the first quadrant enclosed by the lines $x = 0, y = 0, x + y = 1$.
35. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B and C. Find the volume of the tatrahedron $OABC$.
36. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
37. Find the volume of the region of the space bounded by the co-ordinate planes and the surface $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n+\left(\frac{z}{c}\right)^n=1$ and lying in the first octant.
38. Evaluate $\iiint \sqrt{xyz\,(1-x-y-z)}\,dx\,dy\,dz$, taken over the tetrahedral volume in the first octant enclosed by the plane $x=0, y=0, z=0$ and $x+y+z=1$.
39. Evaluate $\iiint x^2 yz\,dx\,dy\,dz$, taken throughout the volume in the first octant bounded by $x = 0, y = 0, z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.
40. Evaluate $\iiint xyz\,dx\,dy\,dz$, taken over the space defined by $x \geq 0, y \geq 0, z \geq 0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1$.

ANSWERS

Exercise 5(a)

(1) 4 (2) $\log a \log b$ (3) 2

(4) $\frac{1}{2}$ (5) $\frac{\pi}{4}$ (6) $\frac{9}{2}$

(7) $\frac{1}{2}$

(8)

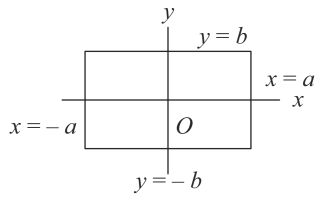

Fig. 5.59

(9)

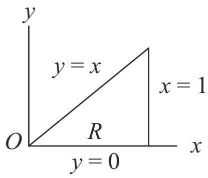

Fig. 5.60

(10)

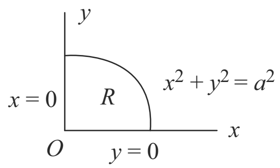

Fig. 5.61

(11)

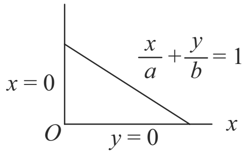

Fig. 5.62

(12) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$ (or) $\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$

(13) $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$ (or) $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} f(x,y)\,dx\,dy$

(14) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$ (or) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

(15) $\int_0^2 \int_{\frac{y^2}{4}}^1 f(x,y)\,dx\,dy$ (or) $\int_0^1 \int_0^{2\sqrt{x}} f(x,y)\,dy\,dx$

(16) 2 log 2

(17) $\frac{a^3}{6}$

(18) $\frac{\pi}{4}$

(19) $\frac{\pi a^3}{6}$

(20) $\frac{1}{720}$

(21) $\frac{8}{3}\log 2 - \frac{19}{9}$

(22) 1

(23) $\frac{3}{2}$

(24) $\frac{\pi}{2}a^3$

(25) $\frac{1}{3}ab(a+b)$

(26) $\frac{344}{105}$

(27) 6

(28) $\frac{33}{2}$

(29) $\frac{1}{16}(8\log 2 - 5)$

(30) $\frac{1}{48}$

Exercise 5(b)

(1) $\displaystyle\int_0^a\int_y^a f(x,y)\,dx\,dy$ (2) $\displaystyle\int_0^1\int_0^x f(x,y)\,dy\,dx$

(3) $\displaystyle\int_0^a\int_0^y f(x,y)\,dx\,dy$ (4) $\displaystyle\int_0^1\int_x^1 f(x,y)\,dy\,dx$

(5) $\displaystyle\int_0^1\int_0^{1-x} f(x,y)\,dy\,dx$ (6) $\displaystyle\int_0^a\int_0^{a-y} f(x,y)\,dx\,dy$

(7) $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$ (8) $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

(9) $\displaystyle\int_0^2\int_{\frac{y^2}{4}}^1 f(x,y)\,dx\,dy$ (10) $\displaystyle\int_0^\infty\int_0^{1/x} f(x,y)\,dy\,dx$

(11) $\dfrac{\pi a}{4}$ (12) $\dfrac{16}{3}$ (13) 1

(14) 1 (15) $\dfrac{1}{2}\,(e-1)^2$ (16) 2

(17) $\dfrac{9}{5}\,a^3$ (18) $\dfrac{1}{2}\,\log 2$ (19) $\dfrac{241}{60}$

(20) $\dfrac{\pi}{8}\,a^4$ (21) 3 (22) $8\log 2$

(23) $\dfrac{\pi}{4}$ (24) $\dfrac{3}{8}$ (25) $\dfrac{2}{3}$

(26) $\dfrac{32}{3}$ (27) $\dfrac{16}{3}\,a^2$ (28) $\dfrac{16}{3}\,ab$

(29) $\dfrac{\pi}{2}+\dfrac{1}{3}$ (30) 3π (31) $\dfrac{3}{2}\,\pi a^2$

(32) $a^2\left(\dfrac{2\pi}{3}-\dfrac{\sqrt{3}}{2}\right)$ (33) $\dfrac{a^2}{2}\,(3\pi-8)$ (34) πa^2

(35) $\dfrac{3}{4}\,\pi a^2$ (36) $\dfrac{\pi}{4}\left(1-e^{-a^2}\right)$ (37) $\dfrac{\pi a}{4}$

(38) $\dfrac{a^4}{4}\,\log\left(1+\sqrt{2}\right)$ (39) $\dfrac{4}{3}\,a^3$ (40) $\dfrac{2\pi}{a}$

(41) $\dfrac{\pi}{2}a^2h^2$ (42) 2π (43) 12π

(44) 16π (45) $\dfrac{16}{9}a^3(3\pi-4)$ (46) $4\pi a$

(47) $\frac{\pi^2}{8}$ (48) $\frac{\pi}{4}a^4$ (49) $\frac{\pi}{16}a^4$

(50) $\frac{\pi}{6a^2}$

Exercise 5(c)

(8) $\frac{2}{3}$ (9) 1 (10) $\frac{\pi}{2}$

(11) 88 (12) $4\pi a^2$ (13) $2\pi ah$

(14) 6 abc (15) $\frac{4}{3}\pi$ (16) $2\pi a^2 h$

(17) $-\frac{69}{10};-\frac{29}{4}$ (18) $-\frac{2}{3};-\frac{2}{3}$ (19) $\frac{4}{3}a^3$

(20) $2\pi ab$ (21) 4 (22) $\frac{3}{2}$

(23) $a^2\left(1-\frac{\pi a}{4}\right)$ (24) $\frac{1}{3}\left(2\sqrt{2}-1\right)$. (25) $\frac{163}{70}$

(26) $\frac{13}{3}$ (27) $\frac{13}{6}$ (28) $\frac{2a^3}{\sqrt{3}}$

(29) $\frac{\sqrt{2}}{3}\pi^3$ (30) $\frac{1}{2}\sqrt{a^2 b^2+b^2 c^2+c^2 a^2}$

(31) $\frac{\pi a^3}{4}$ (32) $\frac{ka^3}{2}$ (33) $\frac{abc}{6}$

(34) $\frac{\pi}{4}a^4$ (35) $\frac{\pi}{2}a^4 h$

Exercise 5(d)

(2) 0.456 (3) $\frac{8}{77}$ (4) $\frac{1}{120}$

(5) $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (7) π (10) $\frac{1}{(\log n)^{n+1}}\Gamma(n+1)$

(12) $\sqrt{\frac{\pi}{k}}$ (15) $\frac{1}{a^n b^m}\beta(m,n)$ (17) $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)$

(22) $\sqrt{\pi}$ (30) $\frac{\pi}{2\sqrt{2}};\frac{\pi}{8\sqrt{2}};\frac{5\pi\sqrt{2}}{128}$