Module – 3

Linear equations of second order with constant coefficients when PI = 0 or exponential – Linear equations of second order with constant coefficients when PI = sin *ax* or cos *ax* – Linear equations of second order with constant coefficients when PI = polynomial – Linear equations of second order with constant coefficients when PI = exponential with sin *ax* or cos *ax* – Linear equations of second order with constant coefficients when PI = exponential with polynomial – Linear equations of second order with constant coefficients when PI = polynomial with sinh *ax* or cosh *ax* – Linear equations of second order with variable coefficients – Homogeneous equation of Euler type – Homogeneous equation of Legendre's type – Equations reducible to homogeneous form – Variation of parameters – Simultaneous first order equations with constant co-efficients – Applications of Differential equations in Engineering.

UNIT III - DIFFERENTIAL EQUATIONS

A **differential equation** is a mathematical equation involving an unknown function and its derivatives.

Example

(i) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{-2x}$

(ii) $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+2y=\sin 3x$

(iii) $\left(\frac{d^2y}{dx^2}\right)^2+3\frac{dy}{dx}+2y=5x$

(iv) $\frac{dy}{dx}+3y=5x$

The **order** of a differential equation is the order of the highest derivative of the unknown function involved in the equation. The order of the differential equations (i), (ii) and (iii) is two, whereas the order of the differential equation (iv) is one.

The **degree** of a differential equation is the degree of the highest derivative of the unknown function involved in the equation, after it is expressed free from radicals. The degree of the differential equations (i), (ii) and (iv) is one, whereas the degree of the differential equation (iii) is two

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form $a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_ny=f(x)$, where $a_0, a_1, a_2, \ldots, a_n$ are constants, is called a linear differential equation of degree n with constant coefficients.

Let $D=\frac{d}{dx}, D^2=\frac{d^2}{dx^2},\cdots$ and so on. Then the above equation can be written as

$(a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\cdots+a_n)y=f(x)$. (i.e.) $\phi(D)\,y=f(x)$

The general or complete solution consists of two parts namely (i) Complementary Function (C.F.) and (ii) Particular Integral (P.I.).

To Find C.F.

S.No.	**Roots of A.E.**	**C.F.**
1	If the roots are real and different m_1, m_2 $(m_1 \neq m_2)$	$Ae^{m_1 x} + B\, e^{m_2 x}$
2	If two roots are equal and real $m_1 = m_2 = m$(say)	$(Ax+B)e^{mx}$
3	If three roots are equal and real $m_1 = m_2 = m_3 = m$(say)	$(Ax^2+Bx+C)e^{mx}$
4	If the roots are complex $\alpha \pm i\beta$	$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$
5	If the complex roots are repeated $\alpha \pm i\beta$ (twice)	$e^{\alpha x}((Ax+B)\cos\beta x + (Cx+D)\sin\beta x)$

To Find P.I.

S.No	f(x)	P.I.

1	$e^{\alpha x}$	$P.I = \frac{1}{f(D)}e^{\alpha x} = \frac{1}{f(\alpha)}e^{\alpha x}$, *provided* $f(\alpha) \neq 0$
		If $f(\alpha) = 0$, *then*
		$P.I = x\frac{1}{f'(\alpha)}e^{\alpha x}$, *provided* $f'(\alpha) \neq 0$
		If $f'(\alpha) = 0$, *then*
		$P.I = x^2\frac{1}{f''(\alpha)}e^{\alpha x}$, *provided* $f''(\alpha) \neq 0$

2	x^n	$P.I = \frac{1}{f(D)}x^n = [f(D)]^{-1}x^n$ Expand $[f(D)]^{-1}$ and then operate on x^n.
3	sin αx (or) cos αx	$P.I = \frac{1}{f(D)}\sin\alpha x$ (or) $\cos\alpha x$ Replace $D^2 = -\alpha^2$ After replacing $D^2 = -\alpha^2$, if the denominator = 0, then multiply x with the numerator and differentiate the denominator with respect to D.

4	$e^{ax}\ g(x)$	$P.I = \frac{1}{f(D)}\,e^{ax}\ g(x) = e^{ax}\frac{1}{f(D+a)}g(x)$
5	$x\,\mathrm{V}(x)$	$P.I = \frac{1}{f(D)}\ x\,\mathrm{V}(x)$ $= x\frac{1}{f(D)}\ \mathrm{V}(x) - \frac{f'(D)}{\left[f(D)\right]^2}\ \mathrm{V}(x)$
6	x^n sinax (or) x^n cosax	$P.I = \frac{1}{f(D)}\ x^n \sin\alpha x\ (or)\ x^n \cos\alpha x$ $= \frac{1}{f(D)}$ R.P of $(x^n e^{i\alpha x})$ (or) I.P of $(x^n e^{i\alpha x})$

Binomial Expansions:

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots+\ldots$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots+\ldots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots+\ldots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots+\ldots$

1. **Solve** $\left(D^2-2D-8\right)y = -4\cosh x\sinh 3x + \left(e^{2x}+e^x\right)^2 + 1.$

Solution:

The A.E. is $\left(m^2-2m-8\right)=0$

$\Rightarrow (m-4)(m+2)=0$

$\Rightarrow m=-2,4$

C.F.: $Ae^{-2x}+Be^{4x}$

$$\text{R.H.S} = -4\cosh x\sin 3x + \left(e^{2x}+e^x\right)^2+1$$

$$= -4\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^{3x}-e^{-3x}}{2}\right)+\left(e^{2x}+e^x\right)^2+1$$

$$= -\left(e^{4x}-e^{-2x}+e^{2x}-e^{-4x}\right)+e^{4x}+2e^{3x}+e^{2x}+1$$

$$= e^{-2x}+e^{-4x}+2e^{3x}+1e^{0x}$$

$$\text{P.I.} = \frac{1}{(D-4)(D+2)}\left(e^{-2x}\right)+\frac{1}{(D-4)(D+2)}\left(-e^{-4x}+2e^{3x}+e^{0x}\right)$$

$$= \frac{-1}{(-2-4)(D+2)}e^{-2x} - \frac{e^{-4x}}{(-8)(-2)} - \frac{2e^{3x}}{(-1)(5)} + \frac{1}{(-4)(2)}$$

$$= \frac{-xe^{-2x}}{6} - \frac{e^{-4x}}{16} - \frac{2e^{3x}}{5} - \frac{1}{8}$$

G.S is $y = Ae^{-2x}+Be^{4x} - \frac{xe^{-2x}}{6} - \frac{e^{-4x}}{16} - \frac{2e^{3x}}{5} - \frac{1}{8}.$

2.	**Solve** $\frac{d^2x}{dy^2}+10x=\cos 8y.$
	Solution: Here y is independent and x is dependent variable Let $D=\frac{d}{dy}$. The A.E is $m^2+10=0$ $\Rightarrow m^2=-10$ $\Rightarrow m=\pm\sqrt{10}i$ C.F.: $A\cos\sqrt{10}y+B\sin\sqrt{10}y$ P.I $=\frac{1}{\left(D^2+10\right)}\cos 8y=\frac{\cos 8y}{-64+10}=\frac{-\cos 8y}{54}$ G.S. is $x=A\cos\sqrt{10}y+B\sin\sqrt{10}y-\frac{\cos 8y}{54}$
3.	**Solve** $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=\sin x\cos 2x$.
	Solution: The A.E is $m^2+6m+9=0$ $\left(m+3\right)^2=0$ $m=-3,\ -3.$ C.F.: $\left(A+Bx\right)e^{-3x}$ R.H.S $=\frac{2\sin x\cos 2x}{2}=\frac{1}{2}\left[\sin 3x+\sin\left(-x\right)\right]$ $=\frac{1}{2}\left[\sin 3x-\sin x\right]$ $\left[2\sin A\cos B=\sin\left(A+B\right)+\sin\left(A-B\right)\right]\quad A=x,\ B=2x$ P.I. $=\frac{1}{2}\frac{1}{\left(D+3\right)^2}\sin 3x-\frac{1}{2}\frac{1}{\left(D+3\right)^2}\sin x$ $=\frac{1}{2}\frac{1}{D^2+6D+9}\sin 3x-\frac{1}{2}\frac{1}{D^2+6D+9}\sin x$ P.I. $=\frac{1}{2}\times\frac{1}{-9+6D+9}\sin 3x-\frac{1}{2}\times\frac{1}{-1+6D+9}\sin x$ $=\frac{1}{12}\times\frac{1}{D}\sin 3x-\frac{1}{2}\times\frac{1}{8+6D}\sin x$ $=\frac{-\cos 3x}{12\left(3\right)}-\frac{\left(4-3D\right)}{4\left(4+3D\right)\left(4-3D\right)}\sin x$

	$= \frac{-1}{36}\cos 3x - \frac{1}{4}\ \frac{(4-3D)}{16-9D^2}\sin x$ $= \frac{-1}{36}\cos 3x - \frac{1}{4}\times\frac{4\sin x - 3\cos x}{16+9}$ $= \frac{-\cos 3x}{36} - \frac{\sin x}{25} + \frac{3\cos x}{100}$ y= $(A+Bx)e^{-3x} + = \frac{-\cos 3x}{36} - \frac{\sin x}{25} + \frac{3\cos x}{100}$
4.	**Solve** $(D^2+4)y = x^4 + \cos^2 x$
	Solution: The A.E. is $m^2+4=0$ $m=\pm 2i$ C.F.: $A\cos 2x + B\sin 2x$ P.I $= \frac{1}{D^2+4}x^4 + \frac{1}{D^2+4}\left(\frac{1+\cos 2x}{2}\right)$ $= \frac{1}{4}\frac{1}{\left(1+\frac{D^2}{4}\right)}x^4 + \frac{1}{2}\frac{1}{(D^2+4)}e^{0x} + \frac{1}{2}\frac{1}{D^2+4}\cos 2x$ $= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^4 + \frac{1}{2(4)} + \frac{(x\sin 2x)}{2(2)(2)}$ $= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}\right)x^4 + \frac{1}{8} + \frac{x\sin 2x}{8}$ $= \frac{x^4}{4} - \frac{12x^2}{16} + \frac{4.3.2.1}{64} + \frac{1}{8} + \frac{x\sin 2x}{8}$ G.S. is $y = A\cos 2x + B\sin 2x + \frac{4}{8} - \frac{3x^2}{4} + \frac{x^4}{4} + \frac{x\sin 2x}{8}$
5.	**Solve** $(D^2+2D-1)y = (x+e^x)^2 + \cos 2x\cosh x$.
	Solution: The A.E is $m^2+2m-1=0$ $m = \frac{-2\pm\sqrt{4+4}}{2} = -1\pm\sqrt{2}$ C.F.: $Ae^{(-1+\sqrt{2})x} + Be^{(-1-\sqrt{2})x}$ P.I.$= \frac{1}{(D^2+2D-1)}(x^2+2xe^x+e^{2x}) + \frac{1}{(D^2+2D-1)}\cos 2x\frac{(e^x+e^{-x})}{2}$ P.I$_1$ $= \frac{1}{(D^2+2D-1)}x^2 = -\frac{1}{\left[1-(2D+D^2)\right]}x^2$

$$=-\left[1+\left(2D+D^2\right)+\left(2D+D^2\right)^2+....\right]x^2$$
$$=-\left[1+2D+D^2+4D^2\right]x^2$$
$$=-\left[1\left(x^2\right)+2D\left(x^2\right)+5D^2\left(x^2\right)\right]$$
$$=-\left[\left(x^2\right)+2\left(2x\right)+5\left(2\right)\right]$$
$$=-x^2-4x-10$$
$$\frac{1}{D^2+2D-1}x^2=-x^2-4x-10$$
$$P.I_2=\frac{2}{D^2+2D-1}xe^x=\left(\frac{2e^x}{\left(D+1\right)^2+2\left(D+1\right)-1}\right)x$$
$$=2e^x\frac{1}{D^2+2D+1+2D+2-1}x$$
$$=\frac{2e^x}{D^2+4D+2}\left(x\right)$$
$$=\frac{2e^x}{2}\frac{1}{\left[1+\left(2D+\frac{D^2}{2}\right)\right]}x$$
$$=e^x\left[1+\left(2D+\frac{D^2}{2}\right)\right]^{-1}x$$
$$=e^x\left[1+\left(2D+\frac{D^2}{2}\right)+...\right]x$$
$$=e^x\left[1+2D\right]x$$
$$P.I_2=\frac{2}{\left(D^2+2D-1\right)}xe^x=e^x\left[x+2\right]=\left(x+2\right)e^x$$
$$P.I_3=\frac{1}{D^2+2D-1}e^{2x}=\frac{1}{\left(4+4-1\right)}e^{2x}=\frac{e^{2x}}{7}$$
$$P.I_4=\frac{1}{D^2+2D-1}\frac{e^x\cos 2x}{2}$$
$$=\frac{e^x}{2}\frac{1}{\left(D+1\right)^2+2\left(D+1\right)-1}\cos 2x$$
$$=\frac{e^x}{2}\frac{1}{D^2+2D+1+2D+2-1}\cos 2x$$

$$= \frac{e^x}{2} \frac{1}{-4+4D+2} \cos 2x$$

$$= \frac{e^x}{2} \frac{1}{4D-2} \cos 2x =$$

$$= \frac{e^x}{2} \frac{1}{2(2D-1)} \cos 2x$$

$$= \frac{e^x}{2} \frac{(2D+1)}{2(2D-1)(2D+1)} \cos 2x$$

$$= \frac{e^x}{2} \frac{(2D+1)}{2(4D^2-1)} \cos 2x$$

$$= \frac{e^x}{4} \frac{(2D+1)}{(4(-4)-1)} \cos 2x$$

$$= \frac{e^x}{4} \frac{1}{(-16-1)} (-2.2\sin 2x + \cos 2x)$$

$$= \frac{e^x}{4} \frac{(-4\sin 2x + \cos 2x)}{(-16-1)}$$

$$= -\frac{e^x (\cos 2x - 4\sin 2x)}{17}$$

$$P.I_5 = \frac{1}{D^2+2D-1} \frac{e^{-x}\cos 2x}{2}$$

$$= \frac{e^{-x}}{2} \frac{1}{(D-1)^2 + 2(D-1) - 1} \cos 2x$$

$$= \frac{e^{-x}}{2} \frac{1}{D^2-2} \cos 2x$$

$$= \frac{e^{-x}}{2} \frac{1}{-4-2} \cos 2x$$

$$= -\frac{e^{-x}}{12} \cos 2x$$

The General Solution is

$$y = Ae^{(-1+\sqrt{2})x} + Be^{-(1+\sqrt{2})x} - 10 - 4x - x^2 + \frac{e^{2x}}{7} + (x+2)e^x$$

$$-\frac{e^x}{17}(\cos 2x - 4\sin 2x) - \frac{e^{-x}}{12}\cos 2x$$

6.	**Solve** $\left(D^2+4\right)y = x^2\cos 2x.$
	Solution: The A.E is $m^2+4=0$ $\Rightarrow m^2=-4$ $\Rightarrow m=\pm 2i$ C.F.: $A\cos 2x + B\sin 2x$ P.I $=\dfrac{1}{D^2+4}\left(x^2\cos 2x\right)$ $= R.P\ of\ \dfrac{1}{D^2+4}x^2 e^{i2x} = R.P.\ of\ \dfrac{e^{2ix}}{\left(D+2i\right)^2+4}x^2$ P.I. $= R.P\ of\ e^{2ix}\dfrac{1}{D^2+4iD-4+4}x^2$ $= R.P\ of\ e^{2ix}\dfrac{1}{D^2+4iD}x^2 = R.P\ of\ e^{2ix}\dfrac{1}{D\left(D+4i\right)}x^2$ $= R.P\ of\ e^{2ix}\dfrac{1}{D}\dfrac{1}{4i\left(1+\dfrac{D}{4i}\right)}x^2$ $= R.P\ of\ \dfrac{e^{2ix}}{4i}\dfrac{1}{D}\left(1+\dfrac{D}{4i}\right)^{-1}x^2$ $= R.P\ of\ \dfrac{e^{2ix}}{4i}\dfrac{1}{D}\left(1-\dfrac{D}{4i}-\dfrac{D^2}{16}\right)x^2$ $= R.P\ of\ \dfrac{e^{2ix}}{4i}\int\left(x^2-\dfrac{2x}{4i}-\dfrac{2}{16}\right)dx$ $= R.P\ of\ \left(\dfrac{-ie^{2ix}}{4}\right)\left(\dfrac{x^3}{3}-\dfrac{x^2}{4i}-\dfrac{x}{8}\right)$ $= R.P\ of\ \left(\dfrac{-ie^{2ix}}{4}\right)\left(\dfrac{x^3}{3}+\dfrac{ix^2}{4}-\dfrac{x}{8}\right)$ $= R.P\ of\ \left(\dfrac{e^{2ix}}{4}\right)\left(\dfrac{-x^3 i}{3}+\dfrac{x^2}{4}+\dfrac{ix}{8}\right)$ $= R.P\ of\ \dfrac{\left(\cos 2x + i\sin 2x\right)}{4}\left(\dfrac{-x^3 i}{3}+\dfrac{x^2}{4}+\dfrac{ix}{8}\right)$ $=\dfrac{1}{4}\left[\dfrac{x^2\cos 2x}{4}+\dfrac{x^3\sin 2x}{3}-\dfrac{x\sin 2x}{8}\right]$ P.I. $=\dfrac{1}{4}\left[\dfrac{x^2\cos 2x}{4}+\dfrac{x^3\sin 2x}{3}-\dfrac{x\sin 2x}{8}\right]$

	General Solution: $y = A\cos 2x + B\sin 2x + \frac{x^2\cos 2x}{16} + \frac{x^3\sin 2x}{12} - \frac{x\sin 2x}{32}$.
7.	**Solve** $\left(D^2+4D+3\right)y = e^{-x}\sin x + xe^{3x}$ **.**
	Solution: The A.E is $m^2+4m+3=0$ $(m+1)(m+3)=0$ $m=-1,-3$ C.F.: $Ae^{-x}+Be^{-3x}$ P.I $=\frac{1}{(D+3)(D+1)}e^{-x}\sin x+\frac{1}{(D+1)(D+3)}xe^{3x}$ $=\frac{e^{-x}}{(D-1+3)(D-1+1)}(\sin x)+\frac{e^{3x}}{(D+3+1)(D+3+3)}(x)$ $=e^{-x}\frac{1}{(D+2)D}\sin x+e^{3x}\frac{1}{(D+4)(D+6)}x$ $=e^{-x}\frac{1}{(D+2)}\int\sin x\,dx+e^{3x}\frac{1}{(D+4)(D+6)}x$ $=-e^{-x}\frac{D-2}{(D+2)(D-2)}\cos x+e^{3x}\frac{1}{D^2+10D+24}x$ $=-e^{-x}\frac{D-2}{D^2-4}\cos x+\frac{e^{3x}}{24}\frac{1}{1+\frac{10D}{24}+\frac{D^2}{24}}x$ $=-e^{-x}\frac{D-2}{-1-4}\cos x+\frac{e^{3x}}{24}\left[1+\frac{5D}{12}+\frac{D^2}{24}\right]^{-1}x$ $=-e^{-x}\frac{D(\cos x)-2\cos x}{-1-4}+\frac{e^{3x}}{24}\left[1+\frac{5D}{12}+\frac{D^2}{24}\right]^{-1}x$ $=-\frac{e^{-x}(-\sin x-2\cos x)}{(-1-4)}+\frac{e^{3x}}{24}\left[1-\frac{5D}{12}\right]x$ $=-\frac{e^{-x}}{5}(\sin x+2\cos x)+\frac{e^{3x}}{24}\left(x-\frac{5}{12}\right)$ General Solution is $y=Ae^{-x}+Be^{-3x}-\frac{e^{-x}}{5}(\sin x+2\cos x)+\frac{e^{3x}}{24}\left(x-\frac{5}{12}\right)$.

	Cauchy's homogeneous linear differential equation The general form of **Cauchy's homogeneous linear differential equation is given by** $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = f(x)$ where $a_{0,} a_1, a_2, \ldots, a_n$ are constants **Procedure:** Substitute $x = e^z$ *or* $z = \log x$. $xD = D', x^2 D^2 = D'(D'-1).D = \frac{d}{dx}, D' = \frac{d}{dz}$ After substitution, convert the given equation into ordinary linear differential equation with constant coefficients.
9.	**Solve $\left(x^2 D^2 + 3xD + 5\right) y = x \cos(\log x)$.**
	Solution: Substitute $x = e^z$ *or* $z = \log x$. $xD = D', x^2 D^2 = D'(D'-1)\ .D = \frac{d}{dx}, D' = \frac{d}{dz}$ $\left(D'(D'-1) + 3D' + 5\right) y = e^z \cos z$ $\left(D'^2 + 2D' + 5\right) y = e^z \cos z$ Auxillary equation is $m^2 + 2m + 5 = 0$ $m = \frac{-2 \pm \sqrt{4-20}}{2}$ $m = \frac{-2 \pm 4i}{2} = -1 \pm 2i$ Complementary Function is $e^{-z}(A\cos 2z + B\sin 2z)$ Particular Integral= $\frac{e^z \cos z}{D'^2 + 2D' + 5}$ $= e^z \frac{\cos z}{(D'+1)^2 + 2(D'+1) + 5}$ $= e^z \frac{\cos z}{D'^2 + 4D' + 8}$ $[substitute\ D'^2 = -1]$ $= e^z \frac{\cos z}{7 + 4D'}$ $= e^z \frac{(7-4D')\cos z}{(7-4D')(7+4D')}$ $= e^z \frac{(7\cos z + 4\sin z)}{49+16}$

	$$=\frac{e^z}{65}(7\cos z+4\sin z)$$ $$y=e^{-z}(A\cos 2z+B\sin 2z)+\frac{e^z}{65}(7\cos z+4\sin z)$$ $$y=\frac{1}{x}(A\cos(2\log x)+B\sin(2\log x))+\frac{x}{65}(7\cos(\log x)+4\sin(\log x))$$
10.	**Transform the differential equation $(x^2D^2-xD+1)y=\left(\frac{\log x}{x}\right)^2$ into linear differential equation with constant coefficients and solve.**
	Solution: $$\left(x^2D^2-xD+1\right)y=\left(\frac{\log x}{x}\right)^2$$ $$x=e^z,\ z=\log x,\ xD=D',\ x^2D^2=D'\left(D'-1\right)$$ $$\therefore\ \left(D'^2-2D'+1\right)y=z^2e^{-2z}$$ $$A.E\ \text{is}\ \ m^2-2m+1=0$$ $$m=1,1$$ $$CF=(Az+B')e^z$$ $$PI=\frac{z^2e^{-2z}}{D'^2-2D'+1}$$ $$=e^{-2z}\left(\frac{z^2}{D'^2-2D'+1}\right)$$ $$replace D' by D'-2$$ $$\therefore PI=e^{-2z}\left(\frac{z^2}{(D'-2)^2-2(D'-2)+1}\right)$$

	$=e^{-2z}\left(\dfrac{z^2}{D'^2-4D'+4-2D'+4+1}\right)$ $=e^{-2z}\left(\dfrac{z^2}{D'^2-6D'+9}\right)=e^{-2z}\left(\dfrac{z^2}{(D'-3)^2}\right)$ $=e^{-2z}\left(\dfrac{z^2}{9\left(1-\dfrac{D'}{3}\right)^2}\right)=\dfrac{e^{-2z}}{9}\left(z^2\left(1-\dfrac{D'}{3}\right)^{-2}\right)$ $=\dfrac{e^{-2z}}{9}\left(1+2\left(\dfrac{D'}{3}\right)+3\left(\dfrac{D'}{3}\right)^2+..\right)z^2$ $=\dfrac{e^{-2z}}{9}\left(z^2+\dfrac{4}{3}z+\dfrac{2}{3}\right)$ Y= CF+PI $=\ (Az+B')e^z+\dfrac{e^{-2z}}{9}\left(z^2+\dfrac{4}{3}z+\dfrac{2}{3}\right)$ $=\ (A\log x+B)x+\dfrac{1}{9x^2}\left((\log x)^2+\dfrac{4}{3}(\log x)+\dfrac{2}{3}\right)$
11.	**Solve the given equation** $\boldsymbol{\left(D^2+\frac{1}{x}D\right)y=\frac{12(\log x)}{x^2}}$
	Solution: Given $\boldsymbol{\left(D^2+\frac{1}{x}D\right)y=\frac{12(\log x)}{x^2}}$ Multiply throught by x^2 We get $\mathbf{x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}}=12(\mathbf{logx})\rightarrow(1)$ $x=e^z\ (or)\ z=\log x$ $xD=D'\rightarrow(2)$ $x^2D^2=D'(D'-1)\rightarrow(3)$ Where D' denotes $\dfrac{d}{dz}$ Sub (2) & (3) in (1) we get, $(D'(D'-1)+D')y=12z$ $(i.e)(D'^2)y=12z$ The $A.E$ is $m^2=0$ C.F.: $Az+B$ To find the P.I

	$\frac{12z}{D'^2}=12\frac{1}{D'}\left(\frac{z^2}{2}\right)$ $=12\left(\frac{z^3}{6}\right)$ P.I $=2z^3$ $\therefore y=(Az+B)+2z^3$ $=(A\log x+B)+2(\log x)^3$
12.	**Solve** $\mathbf{x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=\sin(\log x)}$
	Solution: Given equation is $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=\sin(\log x)$. $(x^2D^2+4xD+2)y=\sin(\log x)\to(1)$ Put $x=e^z$ (or) $z=\log x$ $xD=D'\to(2)$ $x^2D^2=D'(D'-1)\to(3)$ Where D' denotes $\frac{d}{dz}$ Sub (2) & (3) in (1) we get $(D'(D'-1)+4D'+2)y=\sin z$ $(i.e)(D'^2-D'+4D'+2)y=\sin z$ $(D'^2+3D'+2)y=\sin z\to(4)$ The $A.E$ is $m^2+3m+2=0$ $(m+1)(m+2)=0$ $m=-1,-2$ C.F.: $Ae^{-z}+Be^{-2z}$ P.I.$=\frac{1}{D'^2+3D'+2}\sin z$ $=\frac{1}{-1+3D'+2}\sin z$ $=\frac{1}{3D'+1}\sin z$ $=\frac{3D'-1}{9D'^2-1}\sin z$ $=\frac{(3D'-1)\sin z}{9(-1)-1}$ $\left[\text{Replace } D'^2 \text{ by } -1\right]$

	$=\dfrac{3D'(\sin z)-\sin z}{-10}$ $=\dfrac{3\cos z-\sin z}{-10}$ $\therefore$ The solution of (4) is $y=Ae^{-z}+Be^{-2z}+\dfrac{3\cos z-\sin z}{-10}$ Sub $z=\log x$ *or* $x=e^{z}$, we get $y=Ae^{-\log x}+Be^{-2\log x}-\dfrac{3\cos(\log x)-\sin(\log x)}{10}$ $y=Ax^{-1}+Bx^{-2}-\dfrac{3\cos(\log x)-\sin(\log x)}{10}$ $y=\dfrac{A}{x}+\dfrac{B}{x^{2}}-\dfrac{3\cos(\log x)-\sin(\log x)}{10}$
Legendre's linear differential equation. The general form of Legendre's linear differential equation is given by $A_0(a+bx)^n\dfrac{d^ny}{dx^n}+A_1(a+bx)^{n-1}\dfrac{d^{n-1}y}{dx^{n-1}}+A_2(a+bx)^{n-2}\dfrac{d^{n-2}y}{dx^{n-2}}+...+A_ny=f(x)$ Put $bx+a=e^{z}$, $z=\log(bx+a)$, $x=\dfrac{e^{z}-a}{b}$, $(ax+b)D=aD'$ $(ax+b)^2D^2=a^2\left(D'^2-D'\right)$	
13.	**Solve the Legendre's linear equation** $\left[(3x+2)^2D^2+3(3x+2)D-36\right]y=3x^2+4x+1.$
	Solution: Let $\left[(3x+2)^2+D^2+3(3x+2)D-36\right]y=3x^2+4x+1$ Let $3x+2=e^{z}$ or $z=\log(3x+2)\Rightarrow x=\dfrac{e^{z}-2}{3}$ $\Rightarrow\dfrac{dz}{dx}=-\dfrac{3}{3x+2}$ $3x=e^{z}-2$ $x=\dfrac{1}{3}e^{z}-\dfrac{2}{3}$ $Let\,(3x+2)D=3D'$ $(3x+2)^2D^2=9D'(D'-1)$ $\left[9D'(D'-1)+3(3D')-36\right]y=3\left[\dfrac{1}{3}e^{z}-\dfrac{2}{3}\right]+4\left[\dfrac{1}{3}e^{z}-\dfrac{2}{3}\right]+1$

	$\left[9D'^2-9D'+9D'-36\right]y=3\left[\frac{1}{9}e^{2z}+\frac{4}{9}-\frac{4}{9}e^{z}\right]+\frac{4}{3}e^{z}-\frac{8}{3}+1$ $\left[9D'^2-36\right]y=\frac{1}{3}e^{2z}+\frac{4}{3}-\frac{4}{3}e^{z}+\frac{4}{3}e^{z}-\frac{8}{3}+1$ $=\frac{1}{3}e^{2z}-\frac{1}{3}$ A.E is $9m^2-36=0$ $9m^2=36$ $m^2=4$ $m=\pm\,2$ $C.F=Ae^{2z}+Be^{-2z}$ $=A(3x+2)^2+B(3x+2)^{-2}$ $P.I_1=\frac{1}{9D'^2-36}\frac{e^{2z}}{3}$ $=\frac{1}{3}.\frac{1}{36-36}e^{2z}$ $=\frac{1}{3}z\frac{1}{18D'}e^{2z}$ $=\frac{1}{54}z\frac{e^{2z}}{2}$ $=\frac{1}{108}ze^{2z}$ $=\frac{1}{108}\left[\log 3x+2\right](3x+2)^2$ $P.I_2=\frac{1}{9D^{12}-36}\frac{e^{0z}}{3}$ $=\frac{1}{3}.\frac{1}{-36}e^{0z}=-\frac{1}{108}$ $y=C.F+P.I_1-PI_2$ $=A(3x+2)^2+B(3x+2)^{-2}+\frac{1}{108}(3x+2)^2\log(3x+2)+\frac{1}{108}$ $=A(3x+2)^2+B(3x+2)^{-2}+\frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right].$
14.	**Solve $\left[(x+1)^2D^2+(x+1)D+1\right]y=4\cos\log(x+1).$**
	Solution: Put $x+1=e^z$, $z=\log(x+1)$, $x=e^z-1$ $(x+1)^2D^2=D'\left(D'-1\right)$ $(x+1)D=D'$

	$\left[D'\left(D'-1\right)+D'+1\right]y=4\cos z$ $\left[D'^{\,2}+1\right]y=4\cos z$ To find C.F : The Auxillary equation is $m^2+1=0$ $m^2=-1$ $m=\pm i$ $\therefore y=A\cos z+B\sin z$ To find P.I : $P.I=\dfrac{4\cos z}{D'^{\,2}+1}$ **put** $D'=-1$ $P.I=\dfrac{4\cos z}{-1+1}$ $=\dfrac{4\cos z}{0}$ $=4z\dfrac{1}{2D'}\cos z$ $=2z\int\cos z\,dz$ $=2z\sin z$ $\therefore y=A\cos z+B\sin z+2z\sin z$ $=A\cos\log\left(x+1\right)+B\sin\log\left(x+1\right)+2\log\left(x+1\right)\sin\log\left(x+1\right)$
15.	**Transform the differential equation $\left[\left(2x+5\right)^2D^2-6\left(2x+5\right)D+8\right]y=6x$ into linear differential equation with constant coefficients and solve the equation.**
	Solution: Put $2x+5=e^z,\ z=\log\left(2x+5\right)$ $\left(2x+5\right)^2D^2=2^2\left(D'^2-D'\right)$ $\left(2x+5\right)D=2D'$ Hence the given equation becomes $\left[4\left(D'^2-D'\right)-12D'+8\right]y=3\left(e^z-5\right)$ $\Rightarrow\left[D'^2-4D'+2\right]y=\dfrac{3}{4}\left(e^z-5\right)$ To find C.F : Auxillary equation is

	$m^2-4m+2=0$ $\Rightarrow m=\frac{(4\pm 2\sqrt{2})}{2}$ $\therefore CF=Ae^{(2+\sqrt{2})z}+Be^{(2-\sqrt{2})z}$ $PI=\frac{\frac{3}{4}\left(e^z-5\right)}{\left[D'^2-4D'+2\right]}=\frac{3}{4}\left[\frac{e^z}{D'^2-4D'+2}\right]-\frac{3}{4}\left[\frac{5e^{0z}}{D'^2-4D'+2}\right]$ $=\frac{3}{4}\left(-e^z-\frac{5}{2}\right)$ $\therefore y=CF+PI$ $=Ae^{(2+\sqrt{2})z}+Be^{(2-\sqrt{2})z}+\frac{3}{4}\left(-e^z-\frac{5}{2}\right) where z=\log(2x+5)$
16.	**Solve $Dx+y=\sin 2t, and -x+Dy=\cos 2t$**
	Solution: Dx+y=sin2t…..(1) -x+Dy=cos2t…..(2) (1) => Dx+y=sin2t Operate D on 2 => $-Dx+D^2y=-2\sin 2t$.....(3) (1)+(3)=> $y+D^2y=-\sin 2t$ $(D^2+1)y=-\sin 2t$ Auxillary equation is $m^2+1=0$ $m^2=-1$ $m=\pm i$ $\therefore y=A\cos t+B\sin t$ Particular Integral= $\frac{-\sin 2t}{D^2+1}$ (sub $D^2=-4$) $=\frac{-\sin 2t}{-3}=\frac{\sin 2t}{3}$ $\therefore y=A\cos t+B\sin t+\frac{\sin 2t}{3}$ -x+Dy=cos2t…..(2) -x+D$\left(A\cos t+B\sin t+\frac{\sin 2t}{3}\right)$=cos2t $-x-A\sin t+B\cos t+\frac{1}{3}(2\cos 2t)=\cos 2t$ $-x=A\sin t-B\cos t+\frac{1}{3}\cos 2t$

	$\therefore x=-A\sin t+B\cos t-\dfrac{\cos 2t}{3}$ $x=-A\sin t+B\cos t-\dfrac{\cos 2t}{3}$; $y=A\cos t+B\sin t+\dfrac{\sin 2t}{3}$
17.	**Solve the simultaneous ordinary differential equation** $(D+4)x+3y=t$**,** $2x+(D+5)y=e^{2t}$**.**
	Solution: Given $(D+4)x+3y=t\rightarrow(1)$ $2x+(D+5)y=e^{2t}\rightarrow(2)$ $2\times(1)-(D+4)\times(2)$ $6y-(D+4)(D+5)y=2t-(D+4)e^{2t}$ $\left[6-D^2-9D-20\right]y=2t-2e^{2t}-4e^{zt}$ $\left(D^2+9D+14\right)y=6e^{2t}-2t$ The A.E. is $m^2+9m+14=0$ $(m+7)(m+2)=0$ $m=-2,\,-7$ C.F.: $Ae^{-2t}+Be^{-7t}$ $\text{P.I.}=\dfrac{6}{\left(D^2+9D+14\right)}e^{2t}-\dfrac{2}{\left(D^2+9D+14\right)}t$ $=\dfrac{6e^{2t}}{4+18+14}-\dfrac{2}{14}\dfrac{1}{1+\dfrac{9D}{14}+\dfrac{D^2}{14}}(t)$ $=\dfrac{6e^{2t}}{36}-\dfrac{1}{7}\left(1+\dfrac{9D}{14}+\dfrac{D^2}{14}\right)^{-1}(t)$ $=\dfrac{e^{2t}}{6}-\dfrac{1}{7}\left(1-\dfrac{9D}{14}\right)(t)=\dfrac{e^{2t}}{6}-\dfrac{1}{7}\left(t-\dfrac{9}{14}\right)$ G.S. is $y=Ae^{-2t}+Be^{-7t}+\dfrac{e^{2t}}{6}-\dfrac{t}{7}+\dfrac{9}{98}$ <u>To Calculate</u> x $Dy=-2Ae^{-2t}-7Be^{-7t}+\dfrac{2e^{2t}}{6}-\dfrac{1}{7}$ $5y=5Ae^{-2t}+5Be^{-7t}+\dfrac{5e^{2t}}{6}-\dfrac{5t}{7}+\dfrac{45}{98}$ $(D+5)y=3Ae^{-2t}-2Be^{-7t}+\dfrac{7e^{2t}}{6}-\dfrac{5t}{7}-\dfrac{1}{7}+\dfrac{45}{98}$

	$(2) \Rightarrow 2x = -(D+5)y + e^{2t}$ $= -3Ae^{-2t} + 2Be^{-7t} - \frac{7e^{2t}}{6} + \frac{5t}{7} - \frac{31}{98} + e^{2t}$ $x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{7}{72}e^{2t} + \frac{5t}{14} - \frac{31}{196}$ The General solution is $x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{e^{2t}}{12} + \frac{5t}{14} - \frac{31}{196}$ $y = Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}.$
Method of variation of Parameters: $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R,$ R is a function of x. The complementary function is C.F = $c_1f_1 + c_2f_2$ Where c_1, c_2 are constants and f_1 , f_2 are functions of x. Particular Integral is P.I = P f_1 + Q f_2 Where $P = -\int \frac{f_2R}{f_1f_2' - f_1'f_2}dx$ and $Q = \int \frac{f_1R}{f_1f_2' - f_1'f_2}dx$ Where wronskian Value =w(f_1,f_2) = $f_2'f_1 - f_1'f_2$ Hence the complete solution is y = C.F + P.I	
18.	**Solve by the method of variation of parameters** $\frac{d^2y}{dx^2} + 4y = \sec 2x$ **.**
	Solution: The A.E is $m^2 + 4 = 0$ $m = \pm 2i$ C.F = $c_1\cos 2x + c_2\sin 2x$ P.I = $Pf_1 + Qf_2$ f_1 = cos 2x;f_2 = sin 2x $f_1' = -2\sin 2x$; $f_2' = 2\cos 2x$ $f_2'f_1 - f_1'f_2 = 2$ Now, $P = -\int \frac{f_2R}{f_1f_2' - f_1'f_2}dx$ $= -\int \frac{\sin 2x}{2}\sec 2x dx$ $= -\frac{1}{2}\int \tan 2x dx = \frac{1}{4}\log(\cos 2x)$

	$Q=\int \frac{f_1 R}{f_1 f_2' - f_1' f_2} dx$ $= \frac{1}{2}\int \cos 2x \sec 2x dx = \frac{1}{2}x$ $\therefore$ y = C.F + Pf_1 + Qf_2 $= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}\log(\cos 2x)\cos 2x + \frac{1}{2}x\sin 2x$.
19.	**Solve: $\frac{d^2y}{dx^2}+y=\tan x$ by method of variation of parameters.**
	Solution: A.E is $m^2+1=0$ $m=\pm i$ C.F = $c_1\cos x + c_2 \sin x$ $P.I = PI_1 + PI_2$ $f_1=\cos x$; $f_2=\sin x$ $f_1' = -\sin x$; $f_2' = \cos x$ $f_2' f_1' - f_1' f_2 = 1$ Now, $P=-\int \frac{f_2 R}{f_1 f_2' - f_1' f_2} dx$ $= -\int \sin x \tan x dx$ $= -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{(-1+\cos^2 x)}{\cos x} dx$ $= -\int \sec x dx + \int \cos x dx$ $= -\log(\sec x + \tan x) + \sin x$ $Q=\int \frac{f_1 R}{f_1 f_2' - f_1' f_2} dx$ $= \int \cos x \tan x dx$ $= -\cos x$ $\therefore$ y = C.F + Pf1 + Qf2 $= c_1\cos x + c_2 \sin x + [-\log(\sec x + \tan x) + \sin x]\cos x - \cos x \sin x$ $= c_1 \cos x + c_2 \sin x - \log(\sec x + \tan x)\cos \cos x$.
20.	**Solve $\frac{d^2y}{dx^2}+y=\cot x$ by using Method of Variation of Parameters.** **Solution:** The Auxiliary equation (A.E) is $m^2+1=0 \Rightarrow m=\pm i = 0 \pm i = \alpha \pm i\beta$

	$C.F. = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ $= A\cos x + B\sin x$ $=A.f_1+B.f_2$ $f_1 = \cos x,\quad f_2 = \sin x$ $f_1' = -\sin x,\quad f_2' = \cos x$ $f_1 f_2' - f_2 f_1' = \cos x(\cos x) - \sin x(-\sin x) = \left(\cos^2 x + \sin^2 x\right) = 1$ $P.I.=P\, f_1 + Q\, f_2$ $P = \int \frac{-f_2 R}{f_1 f_2' - f_1' f_2}\, dx$ $= \int \frac{-\sin x \cot x}{1}\, dx$ $= \int \frac{-\sin x \cos x}{\sin x}\, dx$ $= \int -\cos x\, dx$ $P = -\sin x$ $Q = \int \frac{f_1 R}{f_1 f_2' - f_1' f_2}\, dx$ $= \int \frac{-\cos x \cot x}{1}\, dx$ $= \int \frac{-\cos x \cos x}{\sin x}\, dx$ $= \int \frac{-\cos^2 x}{\sin x}\, dx$ $= \int \frac{1-\sin^2 x}{\sin x}\, dx$ $= \int (\cos ecx - \sin x)\, dx$ Q = -log(cosecx-cotx)+cosx $P.I.=-\sin x\cos x - \left[log\left(cosecx - cotx\right) + cosx\right]\sin x$ $=-\sin x log\left(cosecx - cotx\right)$ Complete solution is y=C.F +P.I y=Acosx+Bsinx $-\sin x log\left(cosecx - cotx\right)$
21.	**Solve y'' – 2y' + y = e^x logx using method of variation of parameter.** The given differential equation can be written as (D^2 – 2D + 1)y = e^x logx

A.E is $m^2 - 2m + 1 = 0$

$m = 1, 1$

C.F = $(c_1 x + c_2)e^x$

Here $f_1 = xe^x$, $f_2 = e^x$

$f_1' = xe^x + e^x$ $f_2' = e^x$

Now $f_1 f_2' - f_1' f_2 = xe^{2x} - (xe^x + e^x)e^x \Rightarrow -e^{2x}$

Particular Integral P.I = P f_1 + Q f_2

$$P = -\int \frac{f_2 R}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{e^x e^x \log x}{-e^{2x}} dx = \int \log x \, dx = x \log x - x$$

$$Q = \int \frac{f_1 R}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{xe^x e^x \log x}{-e^{2x}} dx = -\int x \log x \, dx = -\int \log x \, d\left(\frac{x^2}{2}\right)$$

$$= -\frac{x^2}{2}\log x + \int \frac{x^2}{2} d(\log x)$$

$$= -\frac{x^2}{2}\log x + \frac{1}{2}\frac{x^2}{2}$$

P.I= $(x \log x - x)xe^x + \left(-\frac{x^2}{2}\log x + \frac{x^2}{4}\right)e^x$

$$= x^2 e^x \log x - x^2 e^x - \frac{x^2 e^x \log x}{2} + \frac{x^2 e^x}{4}$$

$$= \frac{1}{2}x^2 e^x \log x - \frac{3}{4}x^2 e^x = \frac{1}{4}x^2 e^x (2\log x - 3)$$

Complete solution is

y = C.F + P.I

y=$(c_1 x + c_2)e^x + \frac{1}{4}x^2 e^x (2\log x - 3)$