18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS; Unit II (Part-3) - Green's, Stoke's and Gauss Divergence theorem

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Outline

Statement (Green's theorem):

Let C be a positively oriented, piecewise smooth, simple, closed curve and let R be the region enclosed by the curve C in the xy-plane. If $P(x, y)$ and $Q(x, y)$ have continuous first order partial derivatives on R, then

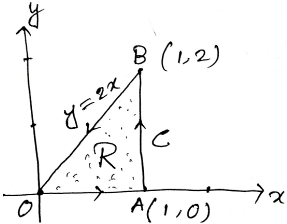
$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Applications of Green's theorem

Example 1:

Use Green's theorem to evaluate $\oint_C xydx + x^2y^3dy$, where C is the triangle with vertices $(0,0), (1,0), (1,2)$ with positive orientation.

Solution: Let $P = xy, Q = x^2y^3$ and the positive orientation curve C is as shown in the figure.



Applications of Green's theorem

Using Green's theorem,

$$\oint_C xydx + x^2y^3dy = \oint_C Pdx + Qdy$$

$$= \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint\limits_R (2xy^3 - x)dxdy$$

$$= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \int_0^1 \left[\frac{xy^4}{2} - xy\right]_0^{2x} dx$$

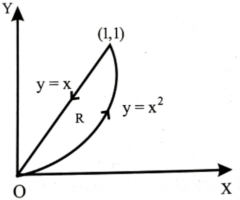
$$= \int_0^1 (8x^5 - 2x^2)dx = \left[\frac{4x^6}{3} - \frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

Applications of Green's theorem

Example 2:

Verify Green's theorem in the plane for $\oint_C [(xy + y^2)dx + x^2 dy]$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution: Let $P = xy + y^2, Q = x^2$ and the positive orientation curve C is as shown in the figure. The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$.



Applications of Green's theorem

Using Green's theorem,

$$\oint_C [(xy + y^2)dx + x^2 dy] = \oint_C Pdx + Qdy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \iint_R (2x - x - 2y)dxdy$$

$$= \iint_R (x - 2y)dxdy = \int_0^1 \int_{y=x^2}^{x} (x - 2y)dydx$$

$$= \int_0^1 \left[xy - y^2 \right]_{y=x^2}^{x} dx = \int_0^1 (x^4 - x^3)dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = -\frac{1}{20}.$$

Applications of Green's theorem

Now let us evaluate the line integral along C. Along $y = x^2, dy = 2xdx$ and the line integral equals

$$\int_0^1 [(x(x^2) + x^4)dx + x^2(2x)dx] = \int_0^1 (3x^3 + x^4)dx$$
$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}.$$

Along $y = x, dy = dx$ and the line integral equals

$$\int_1^0 [(x(x) + x^2)dx + x^2dx] = \int_1^0 (3x^2)dx = \left[\frac{3x^3}{3}\right]_1^0 = -1.$$

Therefore, the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$. Hence the theorem is verified.

Statement (Stoke's theorem):

Let S be a smooth surface that is bounded by a simple closed, smooth boundary curve C with positive orientation and $\overrightarrow{F} = F_1\overrightarrow{i} + F_2\overrightarrow{j} + F_3\overrightarrow{k}$ be any vector function having continuous first order partial derivatives, then

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint_S curl\overrightarrow{F}.\hat{n}ds,$$

where $\hat{n}$ is the outward normal unit vector at any point of S.

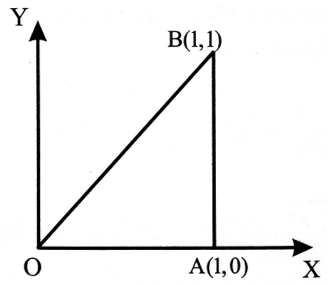
Applications of Stoke's theorem

Example 1:

Use Stoke's theorem to evaluate $\oint_C \overrightarrow{F}.\overrightarrow{dr}$, where $\overrightarrow{F} = y^2\overrightarrow{i} + x^2\overrightarrow{j} - (x+z)\overrightarrow{k}$ and C is the boundary of the triangle with vertices $(0,0,0),(1,0,0),(1,1,0)$ with positive orientation.

Solution: We note that z-coordinate of each vertex of the triangle is 0. Therefore, the triangle lies in the xy-plane. So $\widehat{n} = \overrightarrow{k}$ and the positive orientation curve C is as shown in the figure.

Applications of Stoke's theorem



Let $F_1 = y^2, F_2 = x^2, F_3 = -(x+z)$ and we have

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\,\overrightarrow{i} + \overrightarrow{j} + 2(x-y)\,\overrightarrow{k}$$

Applications of Stoke's theorem

and $curl\,\overrightarrow{F}.\hat{n} = [\overrightarrow{j} + 2(x-y)\overrightarrow{k}].\overrightarrow{k} = 2(x-y)$.
The equation of the line OB is $y = x$. Using Stoke's theorem,

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint\limits_S curl\,\overrightarrow{F}.\hat{n}ds = \int_0^1 \int_{y=0}^{x} 2(x-y)dxdy$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \frac{x^2}{2}dx = \frac{1}{3}.$$

Applications of Stoke's theorem

Example 2:

Verify Stoke's theorem for $\overrightarrow{F} = (2x - y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2z\overrightarrow{k}$ over the upper half surface S of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane and C is its boundary.

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre at origin. Let $x = \cos t, y = \sin t$, $z = 0, 0 \leq t \leq 2\pi$ are parametric equations of C.

Applications of Stoke's theorem

Now

$$\oint_C \overrightarrow{F}.\overrightarrow{dr}$$

$$= \oint_C [(2x-y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2z\overrightarrow{k}].[dx\overrightarrow{i} + dy\overrightarrow{j} + dz\overrightarrow{k}]$$

$$= \oint_C (2x-y)dx - yz^2dy - y^2zdz = \oint_C (2x-y)dx$$

$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t dt = \pi. \quad (1)$$

Applications of Stoke's theorem

Also $\widehat{n} = \overrightarrow{k}$, $ds = dxdy$,

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \overrightarrow{k}$$

and $curl\,\overrightarrow{F}.\widehat{n} = \overrightarrow{k}.\overrightarrow{k} = 1$.
Using Stoke's theorem,

$$\iint_S curl\,\overrightarrow{F}.\widehat{n}ds = \iint_S dxdy = \pi, \tag{2}$$

where $\pi(1)^2$ is the area of the circle C.
Hence from (1) and (2), the theorem is verified.

Statement (Gauss divergence theorem):

If V is the volume bounded by a closed surface S and $\overrightarrow{F}$ is a vector point function with continuous derivatives in V, then

$$\iint\limits_S curl\overrightarrow{F}.\hat{n}ds = \iiint\limits_V div\overrightarrow{F}dV,$$

where $\hat{n}$ is the outward normal unit vector at any point of S.

Applications of Gauss divergence theorem

Example 1:

Use Gauss divergence theorem to evaluate $\iint\limits_S [(x^3 - yz)dydz - 2x^2ydzdx + zdxdy]$ over the surface S of a cube bounded by the coordinate planes and the plane $x = y = z = a$.

Solution: Let $F_1 = x^3 - yz, F_2 = -2x^2y, F_3 = z$. Using Gauss divergence theorem,

$$\iint\limits_S curl\,\overrightarrow{F}.\widehat{n}ds = \iiint\limits_V div\,\overrightarrow{F}\,dV$$

$$= \iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

Applications of Gauss divergence theorem

$$= \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a}(x^2+1)dxdydz = \int_{z=0}^{a}\int_{y=0}^{a}\left[\frac{x^3}{3}+x\right]_{x=0}^{a} dydz$$

$$= \left[\frac{a^3}{3}+a\right]\int_{z=0}^{a}\int_{y=0}^{a} dydz = a\left[\frac{a^3}{3}+a\right]\int_{z=0}^{a} dz = a^2\left[\frac{a^3}{3}+a\right].$$

Applications of Gauss divergence theorem

Example 2:

Use Gauss divergence theorem to evaluate $\iint\limits_S [(x+z)dydz + (y+z)dzdx + (x+y)dxdy]$ over the surface S of the sphere $x^2+y^2+z^2=4$.

Solution: Let $F_1 = x+z, F_2 = y+z, F_3 = x+y$. Using Gauss divergence theorem,

$$\iint\limits_S curl\,\vec{F}.\hat{n}ds = \iiint\limits_V div\,\vec{F}\,dV$$

$$= \iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$$

Applications of Gauss divergence theorem

$$= \iiint\limits_V 2dV = 2\iiint\limits_V dV = 2V,$$

where V is the volume of the sphere $x^2+y^2+z^2=2^2$ ($\because$ the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$).

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64}{3}\pi.$$