SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 4 – Analytic Functions

Part – B (Each question carries 3 Marks)

1. Test the analyticity of the function w = sin z.

Solution

$w = f(z) = \sin z$

$u + iv = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$	$v_x = -\sin x \sinh y$
$u_y = \sin x \sinh y$	$v_y = \cos x \cosh y$

$u_x = v_y$ and $u_y = -v_x$

∴ C-R equations are satisfied.

∴ The function is analytic.

2. Verify whether the function 2xy + i $(x^2 - y^2)$ is analytic or not.

Solution

$u = 2xy$	$v = x^2 - y^2$
$u_x = 2y$	$v_x = 2x$
$u_y = 2x$	$v_y = -2y$

∴ $u_x \neq v_y$ and $u_y \neq -v_x$

∴ C-R equations are not satisfied.

∴ The function is not analytic.

3. Test the analyticity of the function f (z) = e^z.

Solution

$f(z) = e^z$
$u + iv = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$u = e^x \cos y$ $v = e^x \sin y$
$u_x = e^x \cos y$ $v_x = e^x \sin y$
$u_y = -e^x \sin y$ $v_y = e^x \cos y$
$u_x = v_y$ and $u_y = -v_x$
∴ The function is analytic.

4. Verify whether w = z^3 is analytic or not.

Solution

Given $w = z^3 = (x+iy)^3 = x^3 + 3x^2 i y + 3xi^2y^2 + i^3y^3$
$= x^3 - 3xy^2 + i(3x^2y - y^3)$
$u = x^3 - 3xy^2$ $v = 3x^2y - y^3$
$u_x = 3x^2 - 3y^2$; $v_x = 6xy$
$u_y = -6xy$; $v_y = 3x^2 - 3y^2$
Now $u_x = v_y$ and $u_y = -v_x$
∴ w = z^3 is analytic.

5. Is the function f (z) = $\bar{z}$ analytic?

Solution

Given $u + iv = x - iy$
$u = x$ $v = -y$
$u_x = 1$ $v_x = -1$
$u_y = 0$ $v_y = -1$
$u_x \neq v_y$

∴ C-R equations are not satisfied.
∴ $f(z) = \bar{z}$ is not analytic.

6. Find the invariant points of the transformation $f(z) = z^2$.

Solution

Put w = f(z) = z to find the invariant points.

$$z = z^2$$
$$z - z^2 = 0$$
$$z(1 - z) = 0$$
$$z = 0,1$$

7. Find the invariant points of the transformation $w = \frac{z-1}{z+1}$**.**

Solution

The fixed points of the transformation are obtained by replacing w by z.

$$z = \frac{z-1}{z+1}$$
$$z^2 + z - z + 1 = 0$$
$$z^2 + 1 = 0$$

$z = \pm i$ are called fixed points of the transformation.

8. Find the invariant points of the transformation $w = \frac{3z-5}{z+1}$**.**

Solution

To get the invariant points, put w = z

$$\therefore z = \frac{3z-5}{1+z}$$
$$z^2 - 2z + 5 = 0$$

Solving for z,

$$Z = \frac{2\pm\sqrt{4-20}}{2} =$$
$$= \frac{2\pm 4i}{2} = 1\pm 2i$$

$\therefore$ The invariant points are $z = 1\pm 2i$

9. Find the critical point of the transformation $w = z^2$**.**

Solution

Put $\frac{dw}{dz} = 0$

$2z = 0$

The critical point is z = 0.

10. Find the critical points of the transformation $w = z + \frac{1}{z}$.

Solution

Put $\frac{dw}{dz} = 0$

$1 - \frac{1}{z^2} = 0 \Rightarrow \frac{1}{z^2} = 1 \Rightarrow z^2 = 1$

The critical points are z = 1 or z = – 1.

11. Show that the function u = 2x – x³ + 3xy² is harmonic.

Solution Given $u = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2 \qquad u_y = 6xy$

$u_{xx} = -6x \qquad u_{yy} = 6x$

$u_{xx} + u_{yy} = -6x + 6x = 0$

Hence u is harmonic

12. Prove that the function u = $e^x(x\cos y - y\sin y)$ satisfies Laplace's equation.

Solution

Given u $= e^x(x\cos y - y\sin y)$

$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$

$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$

$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x(\cos y) + e^x(\cos y)$

$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + ysin\ y)$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x(x\cos y - y\sin y + \cos y + \cos y - x\cos y - \cos y - \cos y + ysin\ y) \therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

∴u satisfies Laplace equation.

13. Prove that the function u = $x^3 - 3xy^2 + 3x^2 - 3y^2$ + 1 satisfies Laplace's equation.

Solution

Given u $= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\frac{\partial u}{\partial y} = 6xy - 6y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

∴u satisfies Laplace equation.

14. Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic.

Solution

Given $u = \frac{1}{2}\log\left(x^2 + y^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}.\frac{1}{x^2+y^2}\left(2x\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}.\frac{1}{x^2+y^2}\left(2y\right) = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2+y^2\right) - x\left(2x\right)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\left(x^2+y^2\right)\left(1\right) - 2y^2}{\left(x^2+y^2\right)^2} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2} = 0$$

Hence u is harmonic function.

15. Show that the function $u = e^x \cos y$ is harmonic.

Solution

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = e^x\,(-\sin\, y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = e^x\,(-\cos y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

∴u is harmonic.

16. Find the analytic function $f(z) = u + iv$ where $u = 3x^2\, y - y^3$.

Solution

$u = 3x^2\, y - y^3$

$u_x = 6xy$

$u_x(z,0) = 0$

$u_y = 3x^2 - 3y^2$

$u_y(z,0) = 3z^2$

Milne Thomson Method

$f(z) = \int [u_x(z,0) - iu_y(z,0)]dz + C$

$f(z) = \int -i3z^2\, dz + C$

$f(z) = -iz^3 + C$

17. Find the image of the circle $|z| = 3$ under the transformation w = 2z.

Solution

Method 1

Given w = 2z

$u + i\,v = 2\,(x + i\,y)$

$x = \frac{u}{2}, y = \frac{v}{2}$

Given $|z| = 3$

$|x + i\,y| = 3$

$\sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9 \Rightarrow \left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 9$

$u^2 + v^2 = 36$

which represents a circle with centre (0, 0) and radius 6.

(or) Method 2

w = 2 z

$|w| = 2|z|$

$|w| = 2(3) = 6$

Hence the image of the circle $|z| = 3$ in the z-plane maps to the circle $|w| = 6$ in the w-plane.

18. Find the image of the circle $|z| = 1$ by the transformation w = z + 2 + 4i.

Solution

Given: w = z + 2 + 4i

u + iv = x + iy + 2 + 4i = (x + 2) + i (y + 4)

u = x + 2, v = y + 4

⇨ x = u – 2, y = v – 4

⇨ $|z| = 1$

$x^2 + y^2 = 1$ Hence $(u – 2)^2 + (v – 4)^2 = 1$.

∴ The circle in the z-plane is mapped into the circle in the w -plane with centre (2, 4) and radius 1.

19. Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$.

Solution

Given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

Now $w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

i.e., $x+iy=\frac{u-iv}{u^2+v^2}$

$\therefore\ x=\frac{u}{u^2+v^2}$ ....(1) $y=\frac{-v}{u^2+v^2}$ ....(2)

Given $|z-2i|=2$

$|x+iy-2i|=2 \Rightarrow |x+i(y-2)|=2$

$x^2+(y-2)^2=4 \Rightarrow x^2+y^2-4y=0$..................(3)

Sub (1) and (2) in (3)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left[\frac{-v}{u^2+v^2}\right]=0$$

$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$\frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2}=0$$

$1+4v=0 \Rightarrow v=-\frac{1}{4} \quad (\therefore u^2+v^2\neq 0)$

which is a straight line in w - plane.

20. Find the bilinear transformation of the points –1, 0, 1 in z - plane onto the points 0, *i*, 3*i* in w- plane.

Solution

Given $z_1=-1,\ w_1=0\ z_2=0,\ w_2=i\ z_3=i,\ w_3=3i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)}=\frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)}=\frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{2w}{w-3i}=\frac{z+1}{z-1}$$
$$2wz-2w=wz+w-3iz-3i$$
$$w(2z-2-z-1)=-3i(z+1)$$
$$w(z-3)=-3i(z+1)$$
$$\therefore w=-3i\,\frac{(z+1)}{(z-3)}$$

21. Find the bilinear transformation which maps the points $z=\infty, i, 0$ into $w=0, i, \infty$ respectively.

Solution

Given $z_1=\infty,\ w_1=0\ z_2=i,\ w_2=i\ z_3=0,\ w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\,z_1\left(\frac{z_2}{z_1}-z_1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)}=\frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{w}{i}=\frac{i}{z},\qquad w=\frac{i^2}{z},\qquad \therefore w=-\frac{1}{z}$$

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