

Module – 2 Vector Calculus

Review of vectors in 2, 3 dimensions – Gradient, divergence, curl – Solenoidal, Irrotational fields – Vector identities (without proof) – Directional derivatives – Line integrals, Surface integrals, Volume integrals – Green's theorem (without proof) – Gauss divergence theorem (without proof), Verification, Applications to Cubes, parallelopiped only – Stoke's theorem (without proof) – Verification, Applications to Cubes, parallelopiped only – Applications of Line and Volume integrals in Engineering.

Basic Formulae

$$1. \nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

$$2. \nabla \phi = \text{grad } \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$3. \text{Directional derivative} = \nabla \phi \bullet \frac{\vec{a}}{|\vec{a}|}$$

$$4. \text{Normal derivative} = |\nabla \phi|$$

$$5. \text{Unit normal vector } \hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$6. \text{Angle between the surfaces } \cos \theta = \frac{\nabla \phi_1 \bullet \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}$$

$$7. \text{Let } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

Differentiate partially w.r.t. x

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\text{Differentiate partially w.r.t. y } \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\text{Differentiate partially w.r.t. z } \frac{\partial r}{\partial z} = \frac{z}{r}$$

1. Find $\nabla \phi$ if $\phi = \log(x^2 + y^2 + z^2)$.

Solution:

$$\begin{aligned}\nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= \vec{i} \frac{\partial}{\partial x} (\log(x^2 + y^2 + z^2)) + \vec{j} \frac{\partial}{\partial y} \log(x^2 + y^2 + z^2) + \vec{k} \frac{\partial}{\partial z} \log(x^2 + y^2 + z^2) \\ &= \vec{i} \frac{2x}{(x^2 + y^2 + z^2)} + \vec{j} \frac{2y}{(x^2 + y^2 + z^2)} + \vec{k} \frac{2z}{(x^2 + y^2 + z^2)} \\ &= \frac{2}{x^2 + y^2 + z^2} (x\vec{i} + y\vec{j} + z\vec{k}) = \frac{2\vec{r}}{r^2} \quad \because (\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \text{ \& } r^2 = x^2 + y^2 + z^2)\end{aligned}$$

2. Find $\nabla \phi$ if $\phi = xyz$ at the point (1, 2, 3).

Solution:

$$\nabla \phi = \text{grad } \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \vec{i} yz + \vec{j} xz + \vec{k} xy$$

$$\nabla \phi \text{ at } (1, 2, 3) = 6\vec{i} + 3\vec{j} + 2\vec{k}$$

3. Find ∇r .

Solution:

$$\nabla r = \vec{i} \frac{\partial r}{\partial x} + \vec{j} \frac{\partial r}{\partial y} + \vec{k} \frac{\partial r}{\partial z}$$

$$\nabla r = \vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} = \frac{\vec{r}}{r}$$

4. Find the unit normal vector to the surface $x^2 + xy + z^2 = 4$ at the point (1, -1, 2).

Solution:

$$\text{Let } \phi = x^2 + xy + z^2 - 4$$

$$\nabla \phi = \text{grad } \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 2x + y, \quad \frac{\partial \phi}{\partial y} = x, \quad \frac{\partial \phi}{\partial z} = 2z$$

$$[\nabla \phi]_{(1,-1,2)} = [(2x+y)\vec{i} + x\vec{j} + 2z\vec{k}]_{(1,-1,2)} = \vec{i} + \vec{j} + 4\vec{k}$$

The unit normal vector is

$$\therefore \hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{1^2 + 1^2 + 4^2}} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}.$$

5. Find the unit normal vector to the surface $x^2 + y^2 + z^2 = 1$ at the point $(1, 1, 1)$.

Ans $\hat{n} = \frac{i + j + k}{\sqrt{3}}$

6. Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at $(1, 1, 1)$ in the direction $2\vec{i} + 2\vec{j} - \vec{k}$.

Solution: The gradient of ϕ is $\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$

$$\frac{\partial \phi}{\partial x} = 6x, \quad \frac{\partial \phi}{\partial y} = 2, \quad \frac{\partial \phi}{\partial z} = -3$$

$$\nabla \phi = 6xi + 2j - 3k$$

Directional derivative of ϕ is

$$\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$

$$\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = \left[(6xi + 2j - 3k) \cdot \left(\frac{2\vec{i} + 2\vec{j} - \vec{k}}{3} \right) \right]_{(1,1,1)} = \frac{19}{3}$$

7. Find the directional derivative of $\phi = 2xy + z^2$ at $(1, -1, 3)$ in the direction $i + 2j + 2k$.

Ans $\frac{14}{3}$

8. Find the directional derivative of $\phi = x^2 + y^2 + 4xyz$ at $(1, -2, 2)$ in the direction $2i - 2j + k$.

Ans $-\frac{44}{3}$

9. Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at P $(1, 2, 3)$ in the direction of line PQ where Q is $(5, 0, 4)$.

Solution:

$$\nabla \phi = \text{grad } \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \text{grad } \phi = \vec{i} 2x + \vec{j}(-2y) + \vec{k} 4z$$

$$\nabla \phi \text{ at } (1, 2, 3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$$

$$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$$

$$\begin{aligned} \text{Directional derivative} &= \nabla \phi \bullet \frac{\vec{a}}{|\vec{a}|} \\ &= (2\vec{i} - 4\vec{j} + 12\vec{k}) \bullet \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}} \end{aligned}$$

10. **In what direction from (3, 1, -2) is the directional derivative of $\phi = x^2y^2z^4$ a maximum? Find the magnitude of this maximum.**

Solution: Given $\phi = x^2y^2z^4$

$$\frac{\partial \phi}{\partial x} = 2xy^2z^4, \quad \frac{\partial \phi}{\partial y} = 2x^2yz^4, \quad \frac{\partial \phi}{\partial z} = 4x^2y^2z^3$$

$$\nabla \phi = (2x^2y^2z^4)\vec{i} + (2x^2yz^4)\vec{j} + (4x^2y^2z^3)\vec{k}$$

$$[\nabla \phi]_{(3, 1, -2)} = 96\vec{i} + 288\vec{j} - 288\vec{k} = 96(\vec{i} + 3\vec{j} - 3\vec{k})$$

\therefore The maximum directional derivative occurs in the direction of $\nabla \phi = 96(\vec{i} + 3\vec{j} - 3\vec{k})$

The magnitude of this maximum directional derivative is

$$|\nabla \phi| = 96\sqrt{1^2 + 3^2 + (-3)^2} = 96\sqrt{1+9+9} = 96\sqrt{19}.$$

11. **In what direction from (1, 1, -2) is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ a maximum? Find the magnitude of this maximum.**

Ans Directional derivative is maximum in the direction of $2\vec{i} - 4\vec{j} - 16\vec{k}$

$$\text{Maximum directional derivative} = \sqrt{276}$$

12. **Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1).**

Solution: Let $\phi_1 = y^2 - x \log z - 1$

$$\frac{\partial \phi}{\partial x} = -\log z, \quad \frac{\partial \phi}{\partial y} = 2y, \quad \frac{\partial \phi}{\partial z} = -\frac{x}{z}$$

$$\nabla \phi_1 = -\log z \vec{i} + 2y\vec{j} - \frac{x}{z}\vec{k}, \quad (\nabla \phi_1)_{(1,1,1)} = 2\vec{j} - \vec{k} \quad \text{and} \quad |\nabla \phi_1| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\text{Let } \phi_2 = x^2y - 2 + z$$

$$\frac{\partial \phi}{\partial x} = 2xy, \quad \frac{\partial \phi}{\partial y} = x^2, \quad \frac{\partial \phi}{\partial z} = 1$$

$$\nabla \phi_2 = (2xy)\vec{i} + x^2\vec{j} + (1)\vec{k}, \quad (\nabla \phi_2)_{(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k} \quad \text{and} \quad |\nabla \phi_2| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{(2\vec{j} - \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k})}{(\sqrt{5})(\sqrt{6})} = \frac{0 + 2 - 1}{\sqrt{30}} \Rightarrow \theta = \cos^{-1} \left(\frac{1}{\sqrt{30}} \right).$$

13. Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, 2)$.

Ans $\cos \theta = \frac{8}{3\sqrt{21}}$

14. Find 'a' and 'b' so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ cut orthogonally at $(2, -1, -3)$.

Solution:

Let $\phi_1 = ax^3 - by^2z - (a+3)x^2$

$$\frac{\partial \phi}{\partial x} = 3ax^2 - (a+3)2x, \quad \frac{\partial \phi}{\partial y} = -2byz, \quad \frac{\partial \phi}{\partial z} = -by^2$$

$$\therefore \nabla \phi_1 = [3ax^2 - (a+3)2x]\vec{i} - 2byz\vec{j} - by^2\vec{k}$$

At $(2, -1, -3)$ $\nabla \phi_1 = (8a - 12)\vec{i} - 6b\vec{j} - b\vec{k}$

Let $\phi_2 = 4x^2y - z^3 - 11$

$$\frac{\partial \phi}{\partial x} = 8xy, \quad \frac{\partial \phi}{\partial y} = 4x^2, \quad \frac{\partial \phi}{\partial z} = -3z^2$$

$$\therefore \nabla \phi_2 = 8xy\vec{i} + 4x^2\vec{j} - 3z^2\vec{k}$$

At $(2, -1, -3)$ $\nabla \phi_2 = 16\vec{i} - 16\vec{j} - 27\vec{k}$

Since the surfaces cut orthogonally at $(2, -1, -3)$,

$$\nabla \phi_1 \cdot \nabla \phi_2 = 0$$

$$\Rightarrow -16(8a - 12) - 16(6b) + 27b = 0$$

$$\Rightarrow -128a + 192 - 69b = 0$$

$$\Rightarrow 128a + 69b = 192 \quad \rightarrow (1)$$

Since the point $s(2, -1, -3)$ lies on the surface $\phi_1(x, y, z) = 0$, we have

$$8a + 3b - 4a = 12$$

$$\Rightarrow 4a + 3b = 12 \quad \rightarrow (2)$$

$$\text{Solving (1) \& (2) we get } a = -2.333 \quad b = 7.111$$

15. **Find** $\nabla(r^n)$

Solution:

$$\begin{aligned} \nabla(r^n) &= \vec{i} \frac{\partial(r^n)}{\partial x} + \vec{j} \frac{\partial(r^n)}{\partial y} + \vec{k} \frac{\partial(r^n)}{\partial z} \\ &= \vec{i} \, nr^{n-1} \frac{x}{r} + \vec{j} \, nr^{n-1} \frac{y}{r} + \vec{k} \, nr^{n-1} \frac{z}{r} \\ &= \vec{i} \, nr^{n-2} x + \vec{j} \, nr^{n-2} y + \vec{k} \, nr^{n-2} z \\ &= nr^{n-2} (x\vec{i} + y\vec{j} + z\vec{k}) \quad (\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}) \\ \therefore \nabla(r^n) &= nr^{n-2} \vec{r}. \end{aligned}$$

Divergence, Curl, Solenoidal, Irrotational

$$\text{Let } \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$$

$$1. \operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$2. \operatorname{Curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$3. \text{Solenoidal } \nabla \cdot \vec{F} = 0$$

$$4. \text{Irrotational } \nabla \times \vec{F} = \vec{0}$$

16. **Find 'a', such that** $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ **is solenoidal.**

Solution: We know that \vec{F} is Solenoidal if $\operatorname{div} \vec{F} = 0$ or $\nabla \cdot \vec{F} = 0$

$$\left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot [(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}] = 0$$

$$\frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$$\Rightarrow 3 + a + 2 = 0$$

$$\Rightarrow 5 + a = 0 \quad \therefore a = -5.$$

17. Find the constant **a, b, c** so that $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

Solution:

Given \vec{F} is irrotational i.e., $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\begin{aligned} & \vec{i} \left(\frac{\partial}{\partial y} (4x + cy + 2z) - \frac{\partial}{\partial z} (bx - 3y - z) \right) - \vec{j} \left(\frac{\partial}{\partial x} (4x + cy + 2z) - \frac{\partial}{\partial z} (x + 2y + az) \right) \\ & + \vec{k} \left(\frac{\partial}{\partial x} (bx - 3y - z) - \frac{\partial}{\partial y} (x + 2y + az) \right) = \vec{0} \end{aligned}$$

$$= \text{i.e., } \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = \vec{0}$$

$$= \therefore c+1=0, 4-a=0, \text{ and } b-2=0$$

$$\Rightarrow a=4, b=2, c=-1$$

18. Find the constant **a, b, c** so that $\vec{F} = (axy + bz^3)\vec{i} + (3x^2 - cz)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Ans $a = 6, b = 1, c = 1$

19. Prove that $r^n \vec{r}$ is an irrotational vector for any value of 'n' but is solenoidal only if $n = -3$.

Solution:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

Similarly

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2 + z^2}} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{2z}{2\sqrt{x^2 + y^2 + z^2}} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{r}$$

$$r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$r^n \mathbf{r} = r^n (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\begin{aligned}\nabla \times (r^n \mathbf{r}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix} \\ &= \vec{i} \left(\frac{\partial}{\partial y} (r^n z) - \frac{\partial}{\partial z} (r^n y) \right) - \vec{j} \left(\frac{\partial}{\partial x} (r^n z) - \frac{\partial}{\partial z} (r^n x) \right) + \vec{k} \left(\frac{\partial}{\partial x} (r^n y) - \frac{\partial}{\partial y} (r^n x) \right) \\ &= \vec{i} \left(znr^{n-1} \frac{\partial r}{\partial y} - ynr^{n-1} \frac{\partial r}{\partial z} \right) - \vec{j} \left(znr^{n-1} \frac{\partial r}{\partial x} - xnr^{n-1} \frac{\partial r}{\partial z} \right) + \vec{k} \left(ynr^{n-1} \frac{\partial r}{\partial x} - xnr^{n-1} \frac{\partial r}{\partial y} \right) \\ &= \vec{i} \left(znr^{n-1} \frac{y}{r} - ynr^{n-1} \frac{z}{r} \right) - \vec{j} \left(znr^{n-1} \frac{x}{r} - xnr^{n-1} \frac{z}{r} \right) + \vec{k} \left(ynr^{n-1} \frac{x}{r} - xnr^{n-1} \frac{y}{r} \right) \\ &= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0\end{aligned}$$

$\therefore r^n \mathbf{r}$ is irrotational for all values of n.

$$\begin{aligned}\operatorname{div} (r^n \vec{r}) &= \nabla \cdot (r^n \vec{r}) = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (r^n (x\vec{i} + y\vec{j} + z\vec{k})) \\ &= \frac{\partial}{\partial x} (r^n x) + \frac{\partial}{\partial y} (r^n y) + \frac{\partial}{\partial z} (r^n z) \\ &= r^n + xnr^{n-1} \frac{\partial r}{\partial x} + r^n + ynr^{n-1} \frac{\partial r}{\partial y} + r^n + znr^{n-1} \frac{\partial r}{\partial z} \\ &= 3r^n + nr^{n-2} (x^2 + y^2 + z^2) = 3r^n + nr^{n-2} (r^2) = 3r^n + nr^n = (3+n)r^n\end{aligned}$$

If $n = -3$ then $\nabla \cdot (r^n \vec{r}) = 0$.

$\therefore r^n \mathbf{r}$ is solenoidal only if $n = -3$.

20. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, then find $\operatorname{div} \operatorname{curl} \vec{F}$.

Solution: $\operatorname{div} \operatorname{curl} \vec{F} = \nabla \cdot (\nabla \times \vec{F})$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix} \\ &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0} \\ \nabla \times \vec{F} &= \vec{0} \\ \therefore \nabla \cdot (\nabla \times \vec{F}) &= 0\end{aligned}$$

21. If $\nabla \phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$ find ϕ .

Solution:

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \quad \text{_____} \quad (1)$$

$$\nabla \phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k} \quad \text{_____} \quad (2)$$

Comparing (1) and (2)

$$\frac{\partial \phi}{\partial x} = y^2 - 2xyz^3 \quad \text{_____} \quad (3)$$

$$\frac{\partial \phi}{\partial y} = 3 + 2xy - x^2z^3 \quad \text{_____} \quad (4)$$

$$\frac{\partial \phi}{\partial z} = 6z^3 - 3x^2yz^2 \quad \text{_____} \quad (5)$$

Integrating (3) w.r.t. x (keeping y and z as constant)

$$\phi = y^2 x - x^2 y z^3 + f_1(y, z)$$

Integrating (4) w.r.t. y (keeping x and z as constant)

$$\phi = 3y + x y^2 - x^2 y z^3 + f_2(x, z)$$

Integrating (5) w.r.t. z (keeping x and y as constant)

$$\phi = \frac{3}{2} z^4 - x^2 y z^3 + f_3(x, y)$$

Hence $\phi = y^2 x - x^2 y z^3 + 3 y + \frac{3}{2} z^4 + c$ where c is a constant, $c = f_1(y, z) + f_2(x, z) + f_3(x, y)$

22. If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x, y, z)$ given that $\phi(1, -2, 2) = 4$.

Solution:

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \rightarrow (1)$$

$$\text{Given } \nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} \rightarrow (2)$$

\therefore comparing (1) & (2)

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \rightarrow (3)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \rightarrow (4)$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 \rightarrow (5)$$

Integrating (3) w.r.t. x (keeping y and z as constant)

$$\phi = x^2 y z^3 + f_1(y, z)$$

Integrating (4) w.r.t. y (keeping x and z as constant)

$$\phi = x^2 y z^3 + f_2(x, z)$$

Integrating (5) w.r.t. z (keeping x and y as constant)

$$\phi = x^2 y z^3 + f_3(x, y)$$

Hence $\phi = x^2 y z^3 + c$ where c is a constant, $c = f_1(y, z) + f_2(x, z) + f_3(x, y)$

Given $\phi(1, -2, 2) = 4$

$$\phi(1, -2, 2) = -16 + c = 4$$

$$c = 20$$

$$\text{Hence } \phi = x^2 y z^3 + 20$$

23. Show that the vector $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find the scalar potential function.

Solution:

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \vec{0}$$

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 \end{vmatrix} \\ &= \vec{i} \left(\frac{\partial}{\partial y} (3xz^2) - \frac{\partial}{\partial z} (2y \sin x - 4) \right) - \vec{j} \left(\frac{\partial}{\partial x} (3xz^2) - \frac{\partial}{\partial z} (y^2 \cos x + z^3) \right) \\ &\quad + \vec{k} \left(\frac{\partial}{\partial x} (y^2 \cos x + z^3) - \frac{\partial}{\partial y} (2y \sin x - 4) \right) \\ &= \vec{i} (0 - 0) - \vec{j} (3z^2 - 3z^2) + \vec{k} (2y \cos x - 2y \cos x) = \vec{0} \end{aligned}$$

$\therefore \vec{F}$ is irrotational.

To find Scalar potential ϕ we assume $\vec{F} = \nabla \phi$

$$\begin{aligned} \vec{F} = \nabla \phi &= (y^2 \cos x + z^3) \vec{i} + (2y \sin x - 4) \vec{j} + 3xz^2 \vec{k} \\ \left(\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right) &= (y^2 \cos x + z^3) \vec{i} + (2y \sin x - 4) \vec{j} + 3xz^2 \vec{k} \end{aligned}$$

comparing coefficient of \vec{i}, \vec{j} & \vec{k}

$$\frac{\partial \phi}{\partial x} = y^2 \cos x + z^3 \quad \rightarrow (1)$$

$$\frac{\partial \phi}{\partial y} = 2y \sin x - 4 \quad \rightarrow (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 \quad \rightarrow (3)$$

Integrating (1) w.r.t. x (keeping y and z as constant)

$$\phi = y^2 (\sin x) + xz^3 + f_1(y, z)$$

Integrating (2) w.r.t. y (keeping x and z as constant)

$$\phi = y^2 \sin x - 4y + f_2(x, z)$$

Integrating (3) w.r.t. z (keeping x and y as constant)

$$\phi = xz^3 + f_3(x, y)$$

Hence $\phi = y^2 \sin x + xz^3 - 4y + c$ where c is a constant, $c = f_1(y, z) + f_2(x, z) + f_3(x, y)$

24. Show that the vector $\vec{F} = (6xy + z^3) \vec{i} + (3x^2 - z) \vec{j} + (3xz^2 - y) \vec{k}$ is irrotational and find the

scalar potential function.

Solution:

Given $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \vec{0}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

To find scalar potential ϕ we assume $\vec{F} = \nabla \phi$

$$\vec{F} = \nabla \phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$$\left(\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right) = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

comparing coefficient of \vec{i}, \vec{j} & \vec{k}

$$\frac{\partial \phi}{\partial x} = (6xy + z^3) \rightarrow (1)$$

$$\frac{\partial \phi}{\partial y} = (3x^2 - z) \rightarrow (2)$$

$$\frac{\partial \phi}{\partial z} = (3xz^2 - y) \rightarrow (3)$$

Integrating (1) w.r.t. x (keeping y and z as constant)

$$\phi = 3x^2 y + x z^3 + f_1(y, z)$$

Integrating (2) w.r.t. y (keeping x and z as constant)

$$\phi = 3x^2 y - y z + f_2(x, z)$$

Integrating (3) w.r.t. z (keeping x and y as constant)

$$\phi = x z^3 - y z + f_3(x, y)$$

Hence $\phi = 3x^2 y + x z^3 - y z + c$ where c is a constant, $c = f_1(y, z) + f_2(x, z) + f_3(x, y)$

25. If \vec{A} and \vec{B} are irrotational, then prove that $\vec{A} \times \vec{B}$ is solenoidal.

Solution:

\vec{A} and \vec{B} are irrotational.

$$\therefore \nabla \times \vec{A} = \vec{0} \text{ and } \nabla \times \vec{B} = \vec{0}$$

$$\text{Now } \nabla \bullet (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \bullet \vec{B} - (\nabla \times \vec{B}) \bullet \vec{A} = 0 - 0 = 0$$

$\therefore \vec{A} \times \vec{B}$ is solenoidal.

26. Prove that (i) $\text{div } \vec{r} = 3$ (ii) $\text{curl } \vec{r} = 0$.

Solution:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\begin{aligned} \text{div } \vec{r} &= \nabla \bullet \vec{r} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \bullet (x\vec{i} + y\vec{j} + z\vec{k}) \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3 \end{aligned}$$

$$\text{Curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

27. If $\phi = x^2 - y^2$, then prove that $\nabla^2 \phi = 0$.

Solution:

$$\nabla^2 \phi = \nabla \bullet \nabla \phi$$

$$\begin{aligned} &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \bullet \left(\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \right) \\ &= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(0) = 2 - 2 = 0 \end{aligned}$$

28. Prove that $\text{curl}(\text{grad } \phi) = 0$.

Solution:

$$\begin{aligned}
 \text{Curl}(\text{grad } \varphi) &= \nabla \times \nabla \varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} \\
 &= \vec{i} \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \right) - \vec{j} \left(\frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^2 \varphi}{\partial z \partial x} \right) + \vec{k} \left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right) \\
 &= 0\vec{i} + 0\vec{j} + 0\vec{k} \quad (\text{Since mixed partial derivatives are equal.})
 \end{aligned}$$

29. State Green's Theorem.

Statement: If $P(x, y)$ and $Q(x, y)$ are continuous functions with continuous partial derivatives in a region R of the xy plane bounded by a simple closed curve C , then

$$\oint_C (P dx + Q dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \quad \text{where } C \text{ is the curve described in positive direction.}$$

30. Verify Green's theorem for $\int_C [x^2(1+y)dx + (x^3 + y^3)dy]$ where C is the boundary of the region defined by the lines $x = \pm 1$ and $y = \pm 1$.

Solution:

By Green's theorem

$$\oint_C (P dx + Q dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

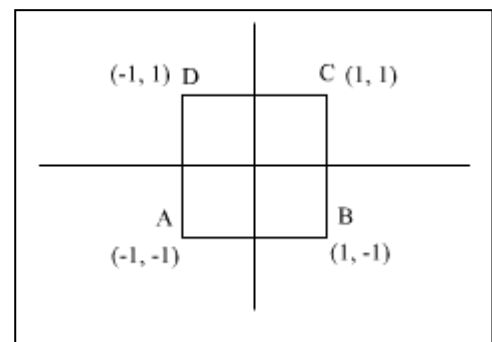
$$\text{Given } \int_C x^2(1+y)dx + (y^3 + x^3)dy$$

$$P = x^2(1+y)$$

$$Q = y^3 + x^3$$

$$\frac{\partial P}{\partial y} = x^2$$

$$\frac{\partial Q}{\partial x} = 3x^2$$



Consider

$$\begin{aligned}
 & \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
 &= \int_{-1}^1 \int_{-1}^1 (3x^2 - x^2) dy dx \\
 &= \int_{-1}^1 \int_{-1}^1 (2x^2) dy dx \\
 &= \int_{-1}^1 2 \left[\frac{x^3}{3} \right]_{-1}^1 dy \\
 &= \int_{-1}^1 \frac{2}{3} [1^3 - (-1)^3] dy = \int_{-1}^1 \frac{2}{3} 2 dy = \int_{-1}^1 \left[\frac{4}{3} \right] dy \\
 &= \left[\frac{4}{3} y \right]_{-1}^1 = \left[\frac{4}{3} \right] [1 - (-1)] = \frac{8}{3} \quad \rightarrow (1)
 \end{aligned}$$

Consider

$$\int_c P dx + Q dy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB, $y = -1$, $dy = 0$ and x varies from -1 to 1

$$\therefore \int_{AB} P dx + Q dy = \int_{-1}^1 x^2 (1 + y) dx = \int_{-1}^1 x^2 (1 - 1) dx = 0$$

Along BC, $x = 1$, $dx = 0$ and y varies from -1 to 1

$$\begin{aligned}
 \therefore \int_{BC} P dx + Q dy &= \int_{-1}^1 (x^3 + y^3) dy = \int_{-1}^1 (1 + y^3) dy \\
 &= \left[y + \frac{y^4}{4} \right]_{-1}^1 = \left[1 + \frac{1}{4} \right] - \left[-1 + \frac{1}{4} \right] \\
 &= 1 + \frac{1}{4} + 1 - \frac{1}{4} = 2
 \end{aligned}$$

Along CD, $y = 1$, $dy = 0$ and x varies from 1 to -1

$$\therefore \int_{CD} P dx + Q dy = \int_{-1}^1 x^2 (1 + y) dx = \int_{-1}^1 2x^2 dx = \left[\frac{2x^3}{3} \right]_{-1}^1 = \frac{2}{3} [(-1)^3 - (1)^3] = \frac{2}{3} [-1 - 1] = -\frac{4}{3}$$

Along DA, $x = -1$, $dx = 0$ and y varies from 1 to -1

$$\begin{aligned}
 \therefore \int_{DA} P dx + Q dy &= \int_1^{-1} (x^3 + y^3) dy = \int_1^{-1} (-1 + y^3) dy \\
 &= \left[\frac{y^4}{4} - y \right]_1^{-1} = \frac{1}{4} + 1 - \frac{1}{4} + 1 = 2
 \end{aligned}$$

$$\int_C Pdx + Qdy = 0 + 2 - \frac{4}{3} + 2 = 4 - \frac{4}{3} = \frac{8}{3} \rightarrow (2)$$

$$\therefore (1) = (2)$$

Hence the theorem is verified.

18

Using Green's theorem, Evaluate $\int_C (y - \sin x)dx + \cos x dy$ where C is the plane triangle bounded by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \left(\frac{2}{\pi}\right)x$

Solution:

Green's theorem states that

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$\text{Given } \int_C (y - \sin x)dx + \cos x dy$$

$$P = y - \sin x$$

$$Q = \cos x$$

$$\frac{\partial P}{\partial y} = 1$$

$$\frac{\partial Q}{\partial x} = -\sin x$$

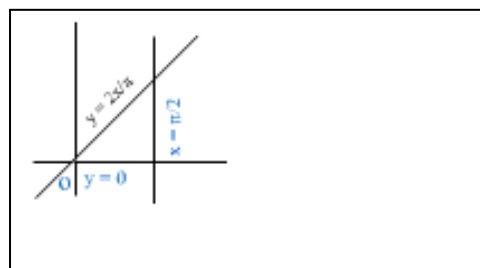
$$\int_C (y - \sin x)dx + \cos x dy = \iint_R (-\sin x - 1) dxdy$$

$$\iint_R (-\sin x - 1) dxdy = \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (-\sin x - 1) dxdy$$

$$= \int_0^1 \left[\cos x - x \right]_{\frac{\pi y}{2}}^{\frac{\pi}{2}} dy$$

$$= \int_0^1 \left[\left(\cos \frac{\pi}{2} - \frac{\pi}{2} \right) - \left(\cos \frac{\pi y}{2} - \frac{\pi y}{2} \right) \right] dy$$

$$= \left[-\frac{\pi y}{2} - \frac{\sin \frac{\pi y}{2}}{\frac{\pi}{2}} + \frac{\pi y^2}{4} \right]_0^1 = \left[-\frac{\pi}{2} - \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{4} \right] - [0] = -\left(\frac{\pi}{4} + \frac{2}{\pi} \right)$$



19

Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by the lines $x = 0$, $y = 0$ and $x + y = 1$.

Solution:

Green's theorem states that

Given $\int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$

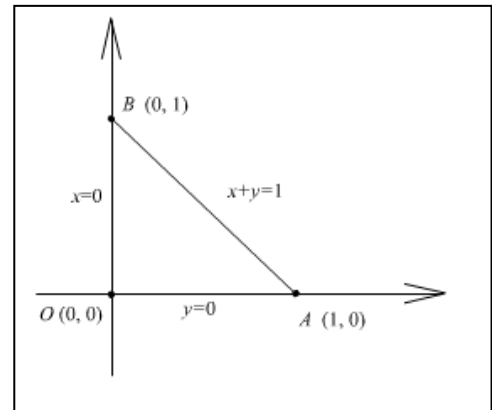
$$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$P = 3x^2 - 8y^2$$

$$\frac{\partial P}{\partial y} = -16y$$

$$Q = 4y - 6xy$$

$$\frac{\partial Q}{\partial x} = -6y$$



Evaluation of RHS:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (-6y + 16y) dx dy$$

$$= \int_0^1 \int_0^{1-y} 10y dx dy = \int_0^1 10y [x]_0^{1-y} dy$$

$$= \int_0^1 10y(1-y) dy$$

$$= 10 \int_0^1 (y - y^2) dy$$

$$= 10 \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= 10 \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{10}{6}$$

$$= \frac{5}{3}$$

Evaluation of LHS:

$$\int_C (P dx + Q dy) = \int_{OA} (P dx + Q dy) + \int_{AB} (P dx + Q dy) + \int_{BO} (P dx + Q dy)$$

Along OA : $y = 0 \Rightarrow dy = 0$

$$\int_{OA} P dx + Q dy = \int_{OA} (3x^2) dx$$

$$= \left[\frac{3x^3}{3} \right]_0^1 = 1 - 0 = 1$$

Along AB :

$$\begin{aligned} x + y = 1 &\Rightarrow y = 1 - x \\ &\Rightarrow dy = -dx \end{aligned}$$

$$\begin{aligned} \int_{AB} Pdx + Qdy &= \int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy \\ &= \int_{AB} [3x^2 - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)](-dx) \\ &= \int_1^0 (-11x^2 + 26x - 12)dx \\ &= \left[\frac{-11x^3}{3} + \frac{26x^2}{2} - 12x \right]_1^0 = (0) - \left(\frac{-11}{3} + \frac{26}{2} - 12 \right) = \frac{11}{3} - 1 = \frac{8}{3} \end{aligned}$$

Along BO : $x = 0 \Rightarrow dx = 0$

$$\begin{aligned} \int_{BO} Pdx + Qdy &= \int_{BO} 4y dy \\ &= \left[\frac{4y^2}{2} \right]_1^0 = 2[0 - (1)] \\ &= -2 \end{aligned}$$

$$\therefore \int_C Pdx + Qdy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

Hence Green's theorem is verified.

20 State Stoke's theorem.

Statement: If S is an open surface bounded by a simple closed curve C and if a vector function \vec{F} is continuous and has continuous partial derivatives in S and on C , then

$$\iint_S \text{curl} \vec{F} \cdot \hat{n} ds = \int_C \vec{F} \cdot d\vec{r} \quad \text{where } \hat{n} \text{ is the outward unit normal vector to the surface.}$$

21 Verify Stoke's theorem for the vector $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes $x = 0$, $y = 0$, $x = 1$, $y = 2$ and $z = 3$ above the XOY plane.

Solution:

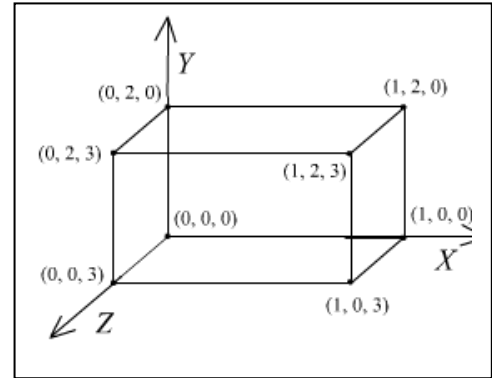
By Stoke's theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} \, ds$$

$$\int_C \vec{F} \cdot d\vec{r} = xydx - 2yzdy - xzdz$$

Evaluation of L.H.S :

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BD} \vec{F} \cdot d\vec{r} + \int_{DO} \vec{F} \cdot d\vec{r}$$



Along OA : $y = 0, z = 0, dy = 0, dz = 0$

$$\int_{OA} \vec{F} \cdot d\vec{r} = 0$$

Along AB : $x = 1, z = 0, dx = 0, dz = 0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{AB} 0 = 0$$

Along BC : $y = 2, z = 0, dy = 0, dz = 0$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_{BC} (2x) dx = \int_1^0 2x \, dx = \left[\frac{2x^2}{2} \right]_1^0 = 0 - 1 = -1$$

Along CO: $x = 0, z = 0, dx = 0, dz = 0$

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int_{CO} 0 = 0$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = 0 + 0 - 1 + 0 = -1$$

Evaluation of RHS:

$$\iint_S \nabla \times \vec{F} \cdot \vec{n} \, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

Given, $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix} = \vec{i}[0 - (-2y)] - \vec{j}[0 - (-z)] + \vec{k}[0 - x] \\ &= 2y\vec{i} + (z)\vec{j} - x\vec{k}\end{aligned}$$

Over S_1 : $x = 0$, $n = -\vec{i}$

$$\begin{aligned}\iint_{S_1} (\nabla \times \vec{F}) \cdot n \, ds &= \int_0^3 \int_0^2 [2y\vec{i}] \cdot (-\vec{i}) \, dy \, dz \\ &= \int_0^3 \int_0^2 -2y \, dy \, dz \\ &= \int_0^3 \int_0^2 -2y \, dy \, dz = \int_0^3 \left[\frac{-2y^2}{2} \right]_0^2 \, dz \\ &= -4(z)_0^3 = -12\end{aligned}$$

Over S_2 : $x = 1$, $n = \vec{i}$

$$\begin{aligned}\iint_{S_2} (\nabla \times \vec{F}) \cdot n \, ds &= \int_0^3 \int_0^2 [2y\vec{i}] \cdot (\vec{i}) \, dy \, dz \\ &= \int_0^3 \int_0^2 2y \, dy \, dz = \int_0^3 \left[\frac{2y^2}{2} \right]_0^2 \, dz = 12\end{aligned}$$

Over S_3 : $y = 0$, $n = -\vec{j}$

$$\begin{aligned}\iint_{S_3} (\nabla \times \vec{F}) \cdot n \, ds &= \int_0^3 \int_0^1 [z\vec{j}] \cdot (-\vec{j}) \, dx \, dz = - \int_0^3 \int_0^1 (z) \, dx \, dz \\ &= - \int_0^3 (xz)_0^1 \, dz = - \int_0^3 (z) \, dz = - \left(\frac{z^2}{2} \right)_0^3 = -\frac{9}{2}\end{aligned}$$

Over S_4 : $y = 1$, $n = \vec{j}$

$$\iint_{S_4} (\nabla \times \vec{F}) \cdot n \, ds = \int_0^3 \int_0^1 z \vec{j} \cdot \vec{j} \, dx \, dz$$

$$\begin{aligned}
 &= \int_0^3 \int_0^1 (z) \, dx \, dz = \int_0^3 (xz)_0^1 \, dz \\
 &= \left(\frac{z^2}{2} \right)_0^3 = \frac{9}{2}
 \end{aligned}$$

Over $S_5 : z = 1$, $n = \vec{k}$

$$\begin{aligned}
 \iint_{S_5} (\nabla \times \vec{F}) \cdot n \, ds &= \int_0^2 \int_0^1 (-x\vec{k}) \cdot \vec{k} \, dx \, dy \\
 &= \int_0^2 \int_0^1 (-x) \, dx \, dy = \int_0^2 \left(-\frac{x^2}{2} \right)_0^1 \, dy \\
 &= \int_0^2 \left(\frac{-1}{2} \right) \, dy = \left(\frac{-1}{2} \right) (y)_0^2 = -1
 \end{aligned}$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} = -12 + 12 - \frac{9}{2} + \frac{9}{2} - 1 = -1$$

\therefore L.H.S = R.H.S.

Hence Stoke's theorem is verified.

- 22** Verify the Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ in the rectangular region of $z = 0$ plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

Solution:

$$\text{Given } \vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$$

$$\text{By Stoke's theorem } \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds$$

Evaluation of LHS:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA : $y = 0 \Rightarrow dy = 0$, x varies from 0 to a

$$\begin{aligned}
 \therefore \int_{OA} \vec{F} \cdot d\vec{r} &= \int_0^a (x^2) \, dx \\
 &= \left(\frac{x^3}{3} \right)_0^a = \frac{a^3}{3}
 \end{aligned}$$

Along AB: $x = a \Rightarrow dx = 0$, y varies from 0 to b

$$\begin{aligned}\int_{AB} \vec{F} \cdot d\vec{r} &= \int_0^b -2ay \, dy \\ &= -2a \left(\frac{y^2}{2} \right)_0^b = -ab^2\end{aligned}$$

Along BC: $y = b$, $dy = 0$, x varies from a to 0

$$\begin{aligned}\int_{BC} \vec{F} \cdot d\vec{r} &= \int_a^0 (x^2 + b^2) dx \\ &= \left(\frac{x^3}{3} + b^2 x \right)_a^0 \\ &= -\frac{a^3}{3} - ab^2\end{aligned}$$

Along CO: $x = 0$, $dx = 0$, y varies from b to 0

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int_b^0 (0 + y^2) 0 + 0 = 0$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0 = -2ab^2$$

Evaluation of RHS:

$$\begin{aligned}\text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} \\ &= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y] = -4y\vec{k}\end{aligned}$$

As the region is in the xy plane we can take $\vec{n} = \vec{k}$ and $ds = dxdy$

$$\begin{aligned}\iint_S \text{curl } \vec{F} \cdot \vec{n} \, ds &= \iint -4y\vec{k} \cdot \vec{k} \, dxdy \\ &= -4 \int_0^b \int_0^a y \, dxdy \\ &= -4 \left(\frac{y^2}{2} \right)_0^b (x)_0^a \\ &= -2ab^2\end{aligned}$$

$$= -2ab^2$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} ds$$

Hence Stoke's theorem is verified.

23 State Gauss Divergence Theorem.

Statement: If V is the volume bounded by a closed surface S and if a vector function \vec{F} is continuous partial derivative in V and on S , then $\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \text{div} \vec{F} dV$

where \hat{n} is the outward unit normal vector to the surface.

24 Verify Gauss divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ taken over the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0$ and $z = a$

Solution:

By Gauss – Divergence theorem $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \text{div} \vec{F} \cdot dV$

Evaluation of LHS:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n} ds + \iint_{S_2} \vec{F} \cdot \vec{n} ds + \dots + \iint_{S_6} \vec{F} \cdot \vec{n} ds$$

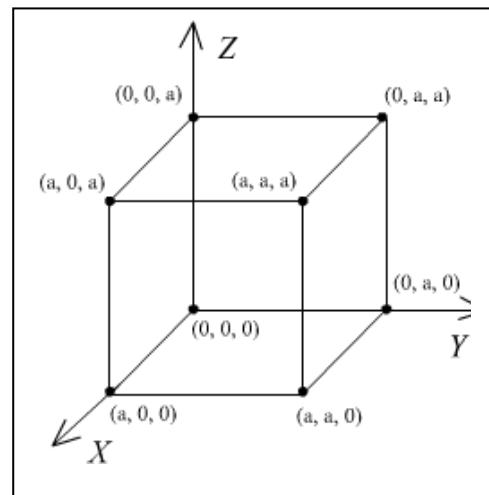
Over S_1 : $x = 0, \hat{n} = -\vec{i}$

$$\begin{aligned} \iint_{S_1} \vec{F} \cdot \hat{n} ds &= \int_0^a \int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}) \cdot (-\vec{i}) dy dz = \int_0^a \int_0^a -x^3 dy dz \\ &= 0 \end{aligned}$$

Over S_2 : $x = a, \hat{n} = \vec{i}$

$$\begin{aligned} \iint_{S_2} \vec{F} \cdot \hat{n} ds &= \int_0^a \int_0^a (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}) \cdot (\vec{i}) dy dz = \int_0^a \int_0^a x^3 dy dz \\ &= \int_0^a \int_0^a a^3 dy dz = a^3 \int_0^a [y]_0^a dz = a^3 \int_0^a a dz \\ &= a^4 [z]_0^a = a^4(a) = a^5 \end{aligned}$$

Over S_3 : $y = 0, \hat{n} = -\vec{j}$



$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^a (x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}) \cdot (-\vec{j}) \, dx \, dz = \int_0^a \int_0^a -y^3 \, dx \, dz = 0$$

Over S_4 : $y = a$, $\hat{n} = \vec{j}$

$$\begin{aligned} \iint_{S_4} \vec{F} \cdot \hat{n} \, ds &= \int_0^a \int_0^a (x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}) \cdot (\vec{j}) \, dx \, dz = \int_0^a \int_0^a y^3 \, dx \, dz \\ &= \int_0^a \int_0^a a^3 \, dx \, dz = a^3 \int_0^a [x]_0^a \, dz = a^3 \int_0^a [a - 0] \, dz = a^4 [z]_0^a = a^4(a) = a^5 \end{aligned}$$

Over S_5 : $z = 0$, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^a (x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}) \cdot (-\vec{k}) \, dx \, dy = \int_0^a \int_0^a -z^3 \, dx \, dy = 0$$

Over S_6 : $z = a$, $\hat{n} = \vec{k}$

$$\begin{aligned} \iint_{S_6} \vec{F} \cdot \hat{n} \, ds &= \int_0^a \int_0^a (x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}) \cdot (\vec{k}) \, dx \, dy = \int_0^a \int_0^a z^3 \, dx \, dy \\ &= a^3 \int_0^a [x]_0^a \, dy = a^3 \int_0^a a \, dy = a^4 [y]_0^a = a^4(a) = a^5 \end{aligned}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = 0 + a^5 + 0 + a^5 + 0 + a^5 = 3a^5$$

Evaluation of RHS:

$$\nabla \cdot \vec{F} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k})$$

$$\nabla \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\iiint_V \nabla \cdot \vec{F} \, dV = \int_0^a \int_0^a \int_0^a 3x^2 + 3y^2 + 3z^2 \, dx \, dy \, dz$$

$$= 3 \int_0^a \int_0^a \int_0^a x^2 + y^2 + z^2 \, dx \, dy \, dz$$

$$\begin{aligned}
&= 3 \int_0^a \int_0^a \left[\frac{x^3}{3} + (y^2 + z^2)x \right]_0^a dy dz \\
&= 3 \int_0^a \int_0^a \left[\frac{a^3}{3} + (y^2 + z^2)a \right] dy dz \\
&= 3 \int_0^a \left[\frac{a^3}{3} y + a \frac{y^3}{3} + az^2 y \right]_0^a dz \\
&= 3 \int_0^a \left[\frac{a^4}{3} + \frac{a^4}{3} + a^2 z^2 \right] dz \\
&= 3 \left[\frac{a^4}{3} z + \frac{a^4}{3} z + a^2 \frac{z^3}{3} \right]_0^a \\
&= 3 \left[\frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3} \right] \\
&= \frac{9a^5}{3} = 3a^5
\end{aligned}$$

25 **Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$.**

Solution:

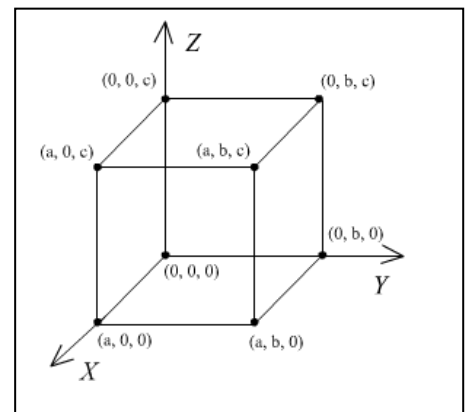
By Gauss – Divergence theorem $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \text{div} \vec{F} \cdot dV$

Evaluation of LHS:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n} ds + \iint_{S_2} \vec{F} \cdot \vec{n} ds + \dots + \iint_{S_6} \vec{F} \cdot \vec{n} ds$$

$$\text{Over } S_1: x = 0, \hat{n} = -\vec{i} \quad \iint_{S_1} \vec{F} \cdot \hat{n} ds =$$

$$= \int_0^c \int_0^b (0 - yz)(-1) dy dz = \int_0^c \int_0^b (yz) dy dz = \int_0^c \left[z \left(\frac{y^2}{2} \right)_0^b \right] dz = \frac{b^2}{2} \left(\frac{z^2}{2} \right)_0^c = \frac{b^2 c^2}{4}$$



Over S_2 : $x = a$, $\hat{n} = \vec{i}$

$$\begin{aligned}\iint_{S_2} \vec{F} \cdot \hat{n} \, ds &= \int_0^c \int_0^b (-yz + a^2) \, dy \, dz = \int_0^c \left[-y \left(\frac{z^2}{2} \right)_0^b + a^2 [z]_0^c \right] dz \\ &= -\frac{c^2}{2} \left(\frac{y^2}{2} \right)_0^b + ca^2 [y]_0^b = a^2 bc - \frac{b^2 c^2}{4}\end{aligned}$$

Over S_3 : $y = 0$, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^a (xz) \, dx \, dz = \int_0^c \left(\frac{x^2}{2} z \right)_0^a dz = \frac{a^2}{2} \left(\frac{z^2}{2} \right)_0^c = \frac{a^2 c^2}{4}$$

Over S_4 : $y = b$, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^a (-xz + b^2) \, dx \, dz = \int_0^c \left[-z \left(\frac{x^2}{2} \right)_0^a + b^2 a \right] dz = ab^2 c - \frac{a^2 c^2}{4}$$

Over S_5 : $z = 0$, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a (xy) \, dx \, dy = \int_0^b \left[y \left(\frac{x^2}{2} \right)_0^a \right] dy = \frac{a^2 b^2}{4}$$

Over S_6 : $z = c$, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a (-xy + c^2) \, dx \, dy = \int_0^b \left[-y \left(\frac{x^2}{2} \right)_0^a + c^2 a \right] dy = abc^2 - \frac{a^2 b^2}{4}$$

$$\begin{aligned}\therefore \iint_S \vec{F} \cdot \hat{n} \, ds &= \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + a b^2 c - \frac{a^2 c^2}{4} + \frac{a^2 b^2}{4} + a bc^2 - \frac{a^2 b^2}{4} \\ &= a^2 bc + ab^2 c + abc^2 = abc(a + b + c)\end{aligned}$$

Evaluation of RHS:

S	OAB
1	C
S	DEF
2	G
S	OCD
3	E
S	ABG
4	F
S	OEF
5	A
S	CDG
6	B

$$\nabla \cdot \vec{F} = 2(x + y + z)$$

$$\begin{aligned} \iiint_V \nabla \cdot \vec{F} \, dV &= \int_0^c \int_0^b \int_0^a 2(x + y + z) \, dx \, dy \, dz \\ &= 2 \int_0^c \int_0^b \left[\frac{x^2}{2} + xy + xz \right]_0^a \, dy \, dz \\ &= 2 \int_0^c \int_0^b \left[\frac{a^2}{2} + ay + az \right] \, dy \, dz \\ &= 2 \int_0^c \left[\frac{a^2}{2} y + a \frac{y^2}{2} + ayz \right]_0^b \, dz \\ &= 2 \left[\frac{a^2 bz}{2} + \frac{ab^2 z}{2} + \frac{abz^2}{2} \right]_0^c \\ &= 2 \left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right] = a^2 bc + ab^2 c + abc^2 = abc(a + b + c) \end{aligned}$$

Hence, Gauss divergence theorem is verified.

26

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ and **C** is the straight line from **A(0,0,0)** to **B(2,1,3)**

Solution:

$$\text{Given } \vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y)dy + z dz.$$

$$\text{The equation of AB is } \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)} \quad \left(\because \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \right)$$

$$\Rightarrow x = 2t \Rightarrow dx = 2dt$$

$$y = t \Rightarrow dy = dt$$

$$z = 3t \Rightarrow dz = 3dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 3x^2 dx + (2xz - y)dy + z dz$$

$$= \int_0^1 (36t^2 + 8t) dt = \left[36 \frac{t^3}{3} + 8 \left(\frac{t^2}{2} \right) \right]_0^1 = 16$$

- 27 Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle in the XY - Plane from (0, 0) to (1,1) along the parabola $y^2 = x$**

Solution:

$$\text{Given } \vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy.$$

$$\text{Given } y^2 = x$$

$$2ydy = dx$$

$$\begin{aligned} \therefore \vec{F} \cdot d\vec{r} &= (x^2 - x + x)dx - (2y^3 + y)dy \\ &= x^2dx - (2y^3 + y)dy \end{aligned}$$

$$\begin{aligned} \int_c \vec{F} \cdot d\vec{r} &= \int_0^1 x^2dx - \int_0^1 (2y^3 + y)dy \\ &= \left[\frac{x^3}{3} \right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2} \right]_0^1 \\ &= \left(\frac{1}{3} - 0 \right) - \left[\left(\frac{2}{4} + \frac{1}{2} \right) - (0 + 0) \right] = \frac{-2}{3} \end{aligned}$$

$$\therefore \text{Work done} = \frac{2}{3}$$