SRM Institute of Science and Technology
Ramapuram Campus
Department of Mathematics
18MAB101T - Calculus And Linear Algebra

Year/Sem: I/I

Branch: Common to ALL B.Tech. except B.Tech. (Business Systems)

Unit – IV

DIFFERENTIAL CALCULUS

Part – C

1. Find the radius of curvature of the curve $y^2 = 12x$ at the point (3, 6).

Solution:

$$\frac{dy}{dx} = \frac{6}{y}$$

At (3, 6), $\frac{dy}{dx} = 1$

$$\frac{d^2y}{dx^2} = 6\left(\frac{-1}{y^2}\right)\frac{dy}{dx}$$

At (3, 6), $\frac{d^2y}{dx^2} = 6\left(\frac{-1}{36}\right)1 = \frac{-1}{6}$

$$\rho = \frac{\left(1 + {y_1}^2\right)^{\frac{3}{2}}}{y_2}$$

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{-1/6} = -12\sqrt{2}$$

2. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point (a/4, a/4).

Solution:

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

At (a/4, a/4), $\frac{dy}{dx} = -1$

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}.\frac{1}{2}y^{-1/2}\frac{dy}{dx} - \sqrt{y}.\frac{1}{2}x^{-1/2}}{x}\right]$$

At (a/4, a/4), $\frac{d^2y}{dx^2} = \frac{4}{a}$

$$\rho = \frac{\left(1+{y_1}^2\right)^{\frac{3}{2}}}{y_2}$$

$$\rho = \frac{a}{\sqrt{2}}$$

3. Find the radius of curvature of the curve $xy = c^2$ at the point (c, c).

Solution:

$$x.\frac{dy}{dx} + y.1 = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

At (c, c), $\frac{dy}{dx} = -1$

$$\frac{d^2y}{dx^2} = -\left[\frac{x\frac{dy}{dx} - y.1}{x^2}\right]$$

At (c, c), $\frac{d^2y}{dx^2} = -\left(\frac{-2c}{c^2}\right) = \frac{2}{c}$

$$\rho = \frac{\left(1 + {y_1}^2\right)^{\frac{3}{2}}}{y_2}$$

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{2/c} = \sqrt{2}\,c$$

4. If $\boldsymbol{x = a\,cos}\theta$, $y = \boldsymbol{b\,sin}\,\theta$, then find $\frac{dy}{dx}$.

Solution:

$$\frac{dx}{d\theta} = -a\sin\theta,\; \frac{dy}{d\theta} = -a\cos\theta,\; \frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

5. Find the envelope of x cos θ + y sin θ = 1, θ being the parameter.

Solution:

$x\cos\theta + y\sin\theta = 1$ ____________________(1)

Differentiate partially w.r.t. θ.

$x(-\sin\theta) + y(\cos\theta) = 0$ ____________________(2)

Squaring and adding (1) and (2)

$x^2 + y^2 = 1$

6. Find the envelope of x cos α + y sin α = a sec α, α being the parameter.

Solution:

$x\cos\alpha + y\sin\alpha = a\sec\alpha$

Divide by cos α.

$x + y\tan\alpha = a\sec^2\alpha$

$x + y\tan\alpha = a(1 + \tan^2\alpha)$

$a\tan^2\alpha - y\tan\alpha + (a - x) = 0$

Here $A = a$, $B = -y$, $C = a - x$

Envelope is given by $B^2 - 4AC = 0$.

$$y^2 = 4a\,(a - x)$$

7. Find $\int_0^1 x^6 (1-x)^9\,dx$**.**

Solution:

$m = 7,\ n = 10$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 x^{7-1}(1-x)^{10-1}\,dx$$

$$= \frac{\Gamma(7)\Gamma(10)}{\Gamma(17)} = \frac{6!9!}{16!}$$

8. Prove that $\frac{B(m+1,n)}{B(m,n+1)} = \frac{m}{n}$**.**

Solution:

$$\frac{B(m+1,n)}{B(m,n+1)} = \frac{\dfrac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}}{\dfrac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}} = \frac{m\Gamma(m)\Gamma(n)}{n\Gamma(m)\Gamma(n)} = \frac{m}{n}$$

9. Find $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$**.**

Solution:

$$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\,\cos^{-1/2}\theta\,d\theta$$

$$= \frac{1}{2}B\left(\frac{3/2}{2}, \frac{1/2}{2}\right)$$

$$= \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

Formula $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$= \frac{\pi}{\sqrt{2}}$$

10. Find $\int_0^{\pi/2} \sin^6\theta \cos^6\theta\, d\theta$ **.**

Solution: m = 6, n = 6

$$\int_0^{\pi/2} \sin^6\theta \cos^6\theta\, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{7}{2}, \frac{7}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(7)}$$

$$= \frac{1}{2} \frac{\left(\frac{15}{8}\sqrt{\pi}\right)\left(\frac{15}{8}\sqrt{\pi}\right)}{6!}$$

11. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Solution:

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\, d\theta$$

Put m = ½, n = ½ .

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta$$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = 2.\frac{\pi}{2}$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$$

Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

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