18MAB102T - ADVANCED CALCULUS AND COMPLEX ANALYSIS (Unit I - Double and Triple Integrals)

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Outline

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Double integral

Consider a function $f(x, y)$ defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\Delta A_1, \Delta A_2, \cdots, \Delta A_n$. Let (x_k, y_k) be any point within the k-th elementary area ΔA_k. Consider the sum

$$f(x_1, y_1)\Delta A_1 + f(x_2, y_2)\Delta A_2 + \cdots + f(x_n, y_n)\Delta A_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

The limit of this sum, if exists, as the number of subdivision increases indefinitely and area of each sub-division decreases to zero, is defined as the double integral of $f(x, y)$ over the region R and it is written as

$$\iint\limits_R f(x, y)\, dA \quad \text{or} \quad \iint\limits_R f(x, y)\, dx\, dy.$$

Therefore,

$$\iint\limits_R f(x, y)\, dA = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Note: The continuity of f is a sufficient condition for the existence of the double integral, but not a necessary one. The above limit exists for many discontinuous function as well.

Properties of double integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations.

1. $$\iint\limits_R kf(x,y)\,dA = k\iint\limits_R f(x,y)\,dA, \text{ for any number } k.$$
2. $$\iint\limits_R (f(x,y)\pm g(x,y))\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA.$$
3. $$\iint\limits_R f(x,y)\,dA \geq 0, \text{ if } f(x,y)\geq 0 \text{ on } R.$$
4. $$\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA \text{ if } f(x,y)\geq g(x,y) \text{ on } R.$$
5. If R is the union of two non-overlapping regions R_1 and R_2 with boundaries that are again made of a finite number of line segments or smooth curves, then
$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA \iint\limits_{R_2} f(x,y)\,dA.$$

Evaluating double integral on rectangular domain

Theorem (First Form of Fubini's Theorem)

If $f(x,y)$ is continuous on the rectangular region $R: \ a \leq x \leq b,\ c \leq y \leq d$, then

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Example

Calculate $\iint\limits_R (1-6x^2y)\,dA$ where $R = \{(x,y) : 0 \leq x \leq 2, \ \ -1 \leq y \leq 1\}$.

Solution: By Fubini's theorem,

$$\begin{aligned}
\iint\limits_R (1-6x^2y)\,dA &= \int_{-1}^{1}\int_0^2 (1-6x^2y)\,dx\,dy \\
&= \int_{-1}^{1} \Big[x - 2x^3y\Big]_{x=0}^{x=2} dy \quad \text{(by keeping } y \text{ fixed)} \\
&= \int_{-1}^{1} (2-16y)\,dy \\
&= \Big[2y - 8y^2\Big]_{-1}^{1} = 4.
\end{aligned}$$

Evaluating double integral on rectangular domain

Note: Reversing the order of integration gives the same answer:

$$
\begin{aligned}
\iint\limits_R (1 - 6x^2y)\, dA &= \int_0^2 \int_{-1}^1 (1 - 6x^2y)\, dy\, dx \\
&= \int_0^2 \left[y - 3x^2y^2 \right]_{y=-1}^{y=1} dx \qquad \text{(by keeping } x \text{ fixed)} \\
&= \int_0^2 2\, dx \\
&= \Big[2x \Big]_0^2 \\
&= 4.
\end{aligned}
$$

Evaluating double integral on non-rectangular domain

Theorem (Stronger Form of Fubini's Theorem)

Let $f(x,y)$ be a continuous function on the region R.

1. *If R is defined by $a \le x \le b,\ g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$, then*
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$
2. *If R is defined by $c \le y \le d,\ h_1(y) \le x \le h_2(x)$ with h_1 and h_2 continuous on $[c,d]$, then*
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

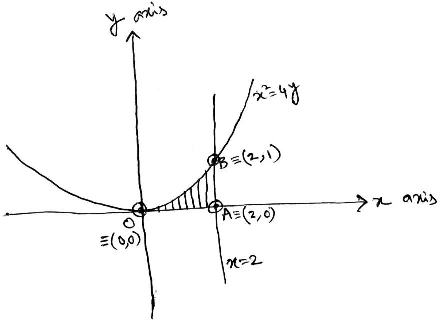
Evaluating double integral on non-rectangular domain

Example

Evaluate $\iint\limits_R xy\, dA$ where R is the domain bounded by x-axis, $x = 2$ and the parabola $x^2 = 4y$.

Solution: The parabola $x^2 = 4y$ and the line $x = 2$ intersect at $B \equiv (2, 1)$. Thus, we can write

$$R = \text{Shaded region OAB} = \{(x, y) : 0 \le x \le 2, 0 \le y \le x^2/4\}.$$



Evaluating double integral on non-rectangular domain

Therefore, by Fubini's theorem (Stronger form),

$$\iint_R xy\,dA = \int_0^2 \int_0^{x^2/4} xy\,dy\,dx \quad = \int_0^2 x\left[\frac{y^2}{2}\right]_{y=0}^{y=x^2/4} dx \quad \text{(by keeping } x \text{ fixed)}$$
$$= \frac{1}{32}\int_0^2 x^5\,dx$$
$$= \frac{1}{32}\left[\frac{x^6}{6}\right]_0^2 = \frac{1}{3}.$$

Note: The region R can be also written as

$$R = \text{Shaded region OAB} = \{(x,y) : 0 \le y \le 1,\ 2\sqrt{y} \le x \le 2\}.$$

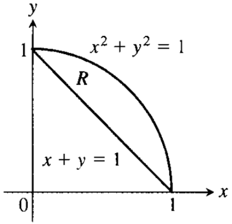
Then, by Fubini's theorem (Stronger form), we get

$$\iint_R xy\,dA = \int_0^1 \int_{2\sqrt{y}}^{2} xy\,dx\,dy \quad = \int_0^1 y\left[\frac{x^2}{2}\right]_{x=2\sqrt{y}}^{x=2} dy \quad \text{(by keeping } y \text{ fixed)}$$
$$= 2\int_0^1 (y - y^2)\,dy$$
$$= 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{3}.$$

Procedure for finding limits of integration

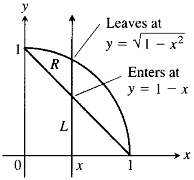
To evaluate $\iint\limits_R f(x,y)\,dA$ over a region R, integrating first with respect to y and then with respect to x, take the following steps:

1. Sketch the region of integration and label the bounding curves (here the bounding curves are $x+y=1$ and $x^2+y^2=1$).



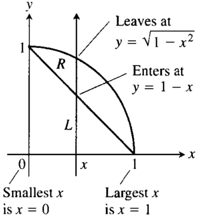
2. Imagine a vertical line L cutting through the R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration.

Procedure for finding limits of integration



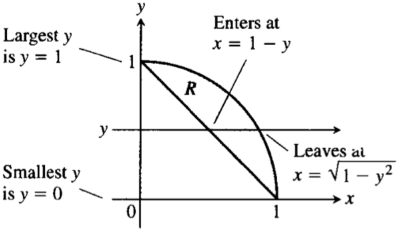
3. Choose x-limits that include all the vertical lines through R. The integral is

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$



Procedure for finding limits of integration

Note: To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines.



The integral is

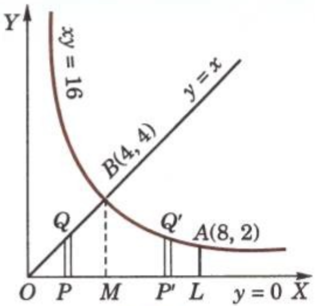
$$\iint_R f(x, y)\, dA = \int_{y=0}^{y=1} \int_{x=1-y}^{x=\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

Evaluating double integral by using decomposition of region

Example

Evaluate $\iint\limits_R x^2\,dy\,dx$, where R is the region in the first quadrant bounded by the lines $x = y$, $y = 0$, $x = 8$ and the curve $xy = 16$.

Solution: The curve $xy = 16$ and the line $x = y$ intersect at $B \equiv (4, 4)$. So, the region R = region $OLAB$ in figure. To evaluate the integral, we divide R into two parts R_1 (region OMB in figure) and R_2 (region $MLAB$ in figure) such that $R = R_1 \cup R_2$.



Evaluating double integral by using decomposition of region

Therefore,

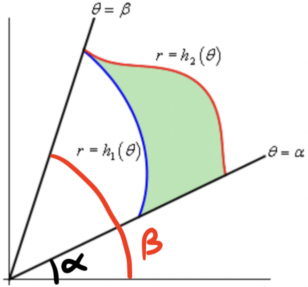
$$
\begin{aligned}
\iint_R x^2\,dy\,dx &= \iint_{R_1} x^2\,dy\,dx + \iint_{R_2} x^2\,dy\,dx \\
&= \int_0^4 \int_0^x x^2\,dy\,dx + \int_4^8 \int_0^{16/x} x^2\,dy\,dx \\
&= \int_0^4 x^2 \Big[y\Big]_0^x dx + \int_4^8 x^2 \Big[y\Big]_0^{16/x} dx \qquad \text{(by keeping } x \text{ fixed)} \\
&= \int_0^4 x^3\,dx + 16\int_4^8 x^3\,dx \\
&= \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^4}{4}\right]_4^8 \\
&= 448.
\end{aligned}
$$

Outline

Double integral in polar coordinate

Let R be a region in polar coordinate, where

$$R = \{(r, \theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}.$$



To evaluate $\int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r, \theta)\, dr\, d\theta$, we first integrate with respect to r between limits $r = h_1(\theta)$ and $r = h_2(\theta)$ by keeping θ as fixed and then, the resulting expression is integrated with respect to θ from α to β.

Double integral in polar coordinate

Note: Connection between Cartesian coordinate and Polar coordinate:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta,$$

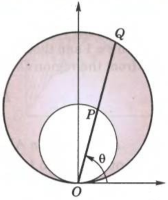
where $x = r\cos(\theta),\ y = r\sin(\theta)$ (i.e. $x^2 + y^2 = 1$).

Note: In Cartesian coordinate $dA = dx\,dy$ and in Polar coordinate $dA = r\,dr\,d\theta$.

Example

calculate $\iint_R r^2 dr\,d\theta$, where R is the region between the circles $r = 2\sin(\theta)$ and $r = 4\sin(\theta)$.

Solution: Here $R = \{(r,\theta) : 0 \le \theta \le \pi,\ 2\sin(\theta) \le r \le 4\sin(\theta)\}$.



Double integral in polar coordinate

$$\iint\limits_R r^2 dr\, d\theta = \int_0^{\pi}\int_{2\sin(\theta)}^{4\sin(\theta)} r^2 dr\, d\theta \quad = \int_0^{\pi}\left[\frac{r^3}{3}\right]_{r=2\sin(\theta)}^{r=4\sin(\theta)} d\theta \quad \text{(by keeping } \theta \text{ fixed)}$$

$$= \frac{56}{3}\int_0^{\pi} \sin^3(\theta)\, d\theta$$

$$= \frac{56}{3}\int_0^{\pi} (1-\cos^2(\theta))\sin(\theta)\, d\theta$$

$$= \frac{56}{3}\int_{-1}^{1} (1-u^2)\, du \quad \text{(let, } u=\cos(\theta))$$

$$= \frac{56}{3}\left[u - \frac{u^3}{3}\right]_{-1}^{1}$$

$$= \frac{224}{9}.$$

Outline

Change of order of integration

In double integral, the change of order of integration changes the limit of integration. The change of order of integration quite often facilitates the evaluation of a double integral.

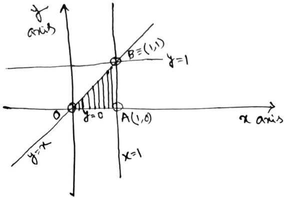
Example

Calculate $\int_0^1 \int_y^1 \frac{\sin(x)}{x}\, dx\, dy$ by changing the order of integration.

Solution: The given limits are:

$$x \text{ from } y \text{ to } 1 \; i.e. \; y \le x \le 1 \text{ (inner limit)},$$
$$y \text{ from } 0 \text{ to } 1 \; i.e. \; 0 \le y \le 1 \text{ (outer limit)}.$$

We use these to sketch the region of integration:



Change of order of integration

The given limits have inner variable x. to reverse the order of integration, we use the vertical line, The limits in this order are:

$$y \text{ from } 0 \text{ to } x \text{ (inner limit)},$$

$$x \text{ from } 0 \text{ to } 1 \text{ (outer limit)}.$$

Therefore, the integral becomes

$$\begin{aligned}
\int_0^1 \int_y^1 \frac{\sin(x)}{x}\, dx\, dy &= \int_0^1 \int_0^x \frac{\sin(x)}{x}\, dy\, dx \\
&= \int_0^1 \frac{\sin(x)}{x} \Big[y\Big]_0^x dx \\
&= \int_0^1 \sin(x)\, dx \\
&= \Big[-\cos(x)\Big]_0^1 \\
&= 1 - \cos(1).
\end{aligned}$$

Note: Observe that if we try to evaluate $\displaystyle\int_0^1 \int_y^1 \frac{\sin(x)}{x}\, dx\, dy$ without using the change of order of integration technique, then we will face the difficulty to calculate $\displaystyle\int_y^1 \frac{\sin(x)}{x}\, dx$.