UNIT 4 ANALYTIC FUNCTIONS

INTRODUCTION

This unit focuses on Complex variables and analytic functions. The theory of functions of a complex variable is the most important in solving a large number of Engineering and Science problems. Many complicated integrals of real functions are solved with the help of complex variable.

INTRODUCTORY CONCEPT

5.0 FUNCTIONS OF A COMPLEX VARIABLE

If $z = x + iy$ and $w = u + iv$ are two complex variables , and if for each value of z in a given region R of the complex plane three corresponds one or more values of w, then w is said to be a function of z and is denoted by $w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are called the real functions of the real variables x and y.

Worked Examples 5.0(A)

Example 1

Test the continuity of the function $f(x,y) = \frac{2xy^2}{x^2+y^4}$ at the origin given $f(0) = 0$.

Solution:

$$f(x,y) = \frac{2xy^2}{x^2+y^4}$$

$$\lim_{z\to 0}[f(z)] = \lim_{\substack{y\to mx \\ x\to 0}}[f(z)] = \lim_{x\to 0}\left[\frac{2m^2x}{1+m^4x^2}\right] = 0$$

$$\lim_{\substack{y^2\to x \\ x\to 0}}[f(z)] = \lim_{x\to 0}\left[\frac{2x^2}{x^2+x^2}\right] = \lim_{x\to 0}\left[\frac{2x^2}{2x^2}\right] = 1 \neq 0$$

The given function f(z) is discontinuous.

Example 2

State the basic difference between the limits of a function of a real variable and that of a complex variable**.**

Solution:

In real variables, $x \to x_0$ implies that x approaches x_0 along the number line, either from left or from right. In complex variables, $z \to z_0$ implies that z approaches z_0 along any path, straight or curved, since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

Example 3

Write the application of analytic functions to the global solvability of the Cauchy problem for equations of Naiver-Stokes**.**

Solution:

The interrelation between analytic functions and real-valued functions is formulated in the work. It is shown such an interrelation realizes nonlinear representations for real-valued functions that allow developing new methods of estimation for them. These methods of estimation are approved by solving the Cauchy problem for equations of viscous incompressible liquid.

5.1 ANALYTIC FUNCTIONS

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point. It is also called as holomorphic and regular functions.

Entire function: A function which is analytic everywhere in the finite plane is called as entire function. An entire function is analytic everywhere except at $z=\infty$.

Example: $e^z, \sin z, \cos z, \sinh z, \cosh z$.

Cauchy-Riemann Equations:

(i) The necessary conditions for a complex function $f(z)=u(x,y)+iv(x,y)$ to be analytic in a region R are $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$. (i.e) $u_x=v_y$ and $v_x=-u_y$

(ii) The sufficient condition for complex function is nothing but if the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x=v_y$ and $v_x=-u_y$, then the function $f(z)$ is analytic in a domain D.

(iii) **C-R Equations for polar form:**

In Cartesian coordinates any point z is $z=x+iy$.

In polar coordinates , $z=re^{i\theta}$ where r is the modulus and θ is the argument.

$$u_r=\frac{\partial u}{\partial r}=v_r=\frac{1}{r}v_\theta \qquad v_r=\frac{\partial u}{\partial r}=v_r=-\frac{1}{r}u_\theta \qquad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Worked Examples 5.1(A)

Example 1

Find the values of a, b, c, d so that the function $f(z)=(x^2+axy+by^2)+i(cx^2+dxy+y^2)$ may be analytic function.

Solution:

Given $f(z)=(x^2+axy+by^2)+i(cx^2+dxy+y^2)$

$u+iv=(x^2+axy+by^2)+i(cx^2+dxy+y^2)$

$u=x^2+axy+by^2 \qquad v=cx^2+dxy+y^2$

$u_x=2x+ay \qquad v_x=2cx+dy$

$u_y=ax+2by \qquad v_y=dx+2y$

Given f(z) is analytic. Therefore, by C-R equation,

$u_x=v_y$, $u_y=-v_x$

$2x+ay=dx+2y \qquad ax+2by=-2cx-dy$

From 1st and 2nd condition equating the coefficients of x and y, we get
d = 2, a= 2 and a =-2c, 2b= -d which implies c = -1, b = -1

Example 2

If $f(z)=r^2(\cos 2\theta+i\sin p\theta)$ is analytic, then find the value of p**.**

Solution:

Given

$f(z)=r^2(\cos 2\theta+i\sin p\theta)$

$u+iv=r^2\cos 2\theta+ir^2\sin p\theta$

$u = r^2 \cos 2\theta \qquad\qquad v = r^2 \sin p\theta$

$\frac{\partial u}{\partial r} = 2r\cos 2\theta \qquad \frac{\partial v}{\partial r} = 2r\sin p\theta$

$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta \qquad \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$

Given f(z) is analytic. Therefore, by C-R equation,

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad\qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$2r\cos 2\theta = \frac{1}{r} pr^2 \cos p\theta \qquad\qquad 2r\sin p\theta = \left(-\frac{1}{r}\right)(-2r^2 \sin 2\theta)$

$2\cos 2\theta = p\cos p\theta \qquad\qquad \sin p\theta = \sin 2\theta$

$\Rightarrow p = 2 \qquad\qquad \Rightarrow p = 2$

Both the cases p = 2. Therefore, p = 2

Example 3

Test the analyticity of the function $f(z) = e^x(\cos y + i\sin y)$.

Solution:

$f(z) = e^x(\cos y + i\sin y)$

$u + iv = e^x \cos y + ie^x \sin y$

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y$

Thus $u_x = v_y, \qquad u_y = -v_x$

$u_x = v_y, \quad u_y = -v_x$

C-R Equation is satisfied. Hence f (z) is analytic.

Example 4

Check whether $w = \bar{z}$ is analytic everywhere.

Solution:

Let $w = f(z) = \bar{z}$

$u + iv = x - iy$

$u = x, \ v = -y$

$u_x = 1, \qquad v_x = 0$

$u_y = 0, \qquad v_y = -1$

$u_x \neq v_y$ at any point p(x,y).

Hence CR equations are not satisfied. Therefore, the function f(z) is nowhere analytic.

Example 5

Prove that $w = \sin 2z$ is analytic functions.

Solution:

$w = \sin 2z$

$u + iv = \sin 2(x + iy)$

$= \sin 2x \cos 2iy + \cos 2x \sin 2iy$

$= \sin 2x \cosh 2y + i\cos 2x \sinh 2y$

$$u = \sin 2x \cosh 2y \qquad v = \cos 2x \sinh 2y$$

$$u_x = 2\cos 2x \cosh 2y \qquad v_x = -2\sin 2x \sinh 2y$$

$$u_y = 2\sin 2x \sinh 2y \qquad v_y = 2\cos 2x \cosh 2y$$

$$u_x = v_y \text{ and } u_y = -v_x$$

CR equations are satisfied, $w = \sin 2z$ is an analytic function.

Example 6

Give an example of a complex valued function which is differentiable at a point but not analytic at the Point.

Solution:

We know that z= x+iy

$$|z| = |x+iy|$$

$$|z| = \sqrt{x^2 + y^2}$$

$$|z|^2 = \left(\sqrt{x^2 + y^2}\right)^2$$

$$|z|^2 = x^2 + y^2$$

$$i.e\ f(z) = |z|^2 = x^2 + y^2$$

$$u + iv = x^2 + y^2$$

$$u = x^2 + y^2; \qquad v = 0$$

$$u_x = 2x\ ;\ u_y = 2y\ ;\ v_x = 0\ ;\ v_y = 0 \text{.....(1)}$$

From the equation we conclude that the CR equations is satisfied only at the point z= 0(x=0,y=0) and not everywhere.

So f (z) may not be differentiable at z=0

Now $u_x = 2x$; $u_y = 2y$; $v_x = 0$; $v_y = 0$ are continuous everywhere and in Particular (0, 0).

Hence the sufficient conditions for differentiability are satisfied by f(z) at z= 0. So f(z) is differentiable at z= 0 only and not analytic there.

Example 7

If $f(z) = (x-y)^2 + 2i(x+y)$ show that the CR equation is satisfied along the curve $x - y = 1$.

Solution:

$$\text{Given}: f(z) = (x-y)^2 + 2i(x+y)$$

$$i.e., u + iv = (x-y)^2 + 2i(x+y)$$

$$i.e.,\ u = (x-y)^2 \qquad v = 2(x+y)$$

$$u_x = 2(x-y) \qquad v_x = 2$$

$$u_y = 2(x-y)(-1) \qquad v_y = 2$$

$$= -2(x-y)$$

$$u_x = v_y \qquad u_y = -v_x$$

$$\Rightarrow 2(x-y) = 2 \qquad -2(x-y) = -2$$

$$x - y = 1 \qquad x - y = 1$$

C-R equations are satisfied only if x-y = 1.

Therefore it is analytic along the curve x-y = 1.

Example 8

Show that $|z|^2$ is not analytic at any point.

Solution:

$f(z)=|z|^2=x^2+y^2$

$u+iv=x^2+y^2$

$u=x^2+y^2; \qquad v=0$

$u_x=2x$; $u_y=2y$; $v_x=0$; $v_y=0$.....(1)

From the equation (1) we conclude that the CR equations is satisfied only at the point $z=0(x=0, y=0)$ and not everywhere.

So $f(z)$ may not be differentiable at $z=0$

Now $u_x=2x$; $u_y=2y$; $v_x=0$; $v_y=0$ are continuous everywhere and in Particular (0, 0).

Hence the sufficient conditions for differentiability are satisfied by $f(z)$ at $z=0$. So $f(z)$ is differentiable at $z=0$ only and not analytic.

Example 9

Determine whether the Cauchy Riemann condition are satisfied for the function $w=2z^2$.

Solution:

Given $w=2z^2$

$u+iv=2(x+iy)^2=2[(x^2+y^2)+i\,2xy]$

$u=2(x^2+y^2) \qquad v=4xy$

$u_x=4x \qquad v_x=4x$

$u_y=4y \qquad v_y=4y$

$u_x\neq v_y$ and $u_y\neq v_x$, CR equations are not satisfied.Therefore, it is not analytic.

Example10

Verify whether $w=(x^2-y^2-2xy)+i(x^2-y^2+2xy)$ is an analytic function of $z=x+iy$.

Solution:

Let $w=(x^2-y^2-2xy)+i(x^2-y^2+2xy)$

$u+iv=(x^2-y^2-2xy)+i(x^2-y^2+2xy)$

$u=x^2-y^2-2xy \qquad v=x^2-y^2+2xy$

$u_x=2x-2y \qquad v_x=2x+2y$

$u_y=-2y-2x \qquad v_y=-2y+2x$

$\therefore u_x=v_y$ and $u_y=-v_x$,

CR equations are satisfied. Hence f (z) is analytic.

Worked Examples 5.1(B)

Example 1

Prove that $f(z)=z^n$ is analytic for all values of n and find its derivative.

Solution:

$Let\ z=re^{i\theta}$

$z^n=r^ne^{in\theta}=r^n(\cos\theta+i\sin\theta)^n$

$=r^n(\cos n\theta+i\sin n\theta)$

$u=r^n\cos n\theta$

$v=r^n\sin n\theta$

$$\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta$$
$$\frac{\partial v}{\partial r} = nr^{n-1} \sin n\theta$$
$$\frac{\partial u}{\partial \theta} = -nr^{n} \sin n\theta$$
$$\frac{\partial u}{\partial \theta} = nr^{n} \cos n\theta$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\ ,\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$
By C-R equation it is analytic and continuous.

Example 2

Prove that $w = z / z + a$ where $a \neq 0$ is analytic whereas $w = \overline{z} / \overline{z} + a$ is not analytic**.**

Solution:

Now, $u + iv = \dfrac{\overline{z}}{\overline{z} + a} = a\ \ function\ of\ \overline{z}$

Since, a function of $\overline{z}$ cannot be analytic.

$u + iv$ is not an analytic function of z.

Let $f(z) = u - iv = conjugate\ of\ u + iv = \dfrac{z}{z + a} = f(z)\ \ say.$

f(z) is a function of z alone and $f\,'(z) = \dfrac{a}{(z+a)^2}$

That exists everywhere except z = -a.

Therefore f(z) is analytic, except at z = -a.

Hence $w = \dfrac{z}{z+a}$ where $a \neq 0$ is analytic whereas $w = \dfrac{\overline{z}}{\overline{z} + a}$ is not analytic

Example 3

Show that $f(z) = 1/z$ is analytic except at $z = 0$ & find $f'(z)$**.**

Solution:

Let $z = re^{i\theta}$

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$
$$= \frac{1}{r}\left[\cos\theta - i\sin\theta\right]$$

At $z = 0, r = 0$ and so $f(z)$ is not defined at $z = 0.$

Now, the C-R equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$
$$u = \frac{\cos\theta}{r} \qquad v = -\frac{\sin\theta}{r}$$
$$u_r = -\frac{\cos\theta}{r^2} \qquad v_r = \frac{\sin\theta}{r^2}$$
$$u_\theta = -\frac{\sin\theta}{r} \qquad v_\theta = -\frac{\cos\theta}{r}$$

$\therefore C - R$ equations are satisfied.

Hence, $f(z)$ is analytic everywhere except at $z=0$.

In polar form,

$$f'(z)=\frac{u_r+iv_r}{e^{i\theta}}$$

$$=\frac{-\cos\theta+i\sin\theta}{r^2 e^{i\theta}}$$

$$=\frac{-[\cos\theta-i\sin\theta]}{r^2 e^{i\theta}}$$

$$=-\frac{e^{-i\theta}}{r^2 e^{i\theta}} \quad =-\frac{1}{\left(r e^{i\theta}\right)^2}=-\frac{1}{z^2}$$

Example 4

Prove that $f(z)=\sinh z$ is an analytic function and find its derivative.

Solution:

$$f(z)=\sinh z=\frac{1}{i}\sin(iz)$$

$$=\frac{1}{i}\sin(i[x+iy])$$

$$=\frac{1}{i}\sin(ix-y)$$

$$=\frac{1}{i}[\sin ix\cos y-\cos ix\sin y]$$

$$=\frac{1}{i}[i\ \sinh x\cos y-\cosh x\sin y]$$

$$=\sinh x\cos y+i\cosh x\sin y$$

$i.e., u=\sinh x\cos y \quad ...(1)$

$v=\cosh x\sin y \quad ...(2)$

$u_x=\cosh x\cos y \quad u_y=-\sinh x\sin y$

$v_x=\sinh x\sin y \quad v_y=\cosh x\cos y$

$\therefore u_x, u_y, v_x$ and v_y exist, and are continuous.

$u_x=v_y$ and $u_y=-v_x$

C-R equations are satisfied.

Therefore f(z) is analytic everywhere.

Now $f'(z)=u_x+iv_x$

$$=\cosh x\cos y+i\sinh x\sin y$$

$$=\cos ix\cos y+\sin ix\sin y$$

$$=\cos(ix-y)$$

$$=\cos i(x+iy)$$

$$f'(z)=\cosh(x+iy)=\cosh z$$

Example 5

If f (z) = u+iv is a regular function of z in a domain D then the following relations holds $\nabla^2[|u|^2]=2|f'(z)|^2$.

Solution:

$$f(z)=u+iv$$
$$\operatorname{Re} f(z)=u$$
$$|\operatorname{Re} f(z)|^2=u^2$$
$$\nabla^2|\operatorname{Re} f(z)|^2=\nabla^2(u^2)$$
$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2)$$
$$=\frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial y^2}(u^2)\text{..........(1)}$$
$$\frac{\partial^2}{\partial x^2}(u^2)=\frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right]$$
$$=2\left[u\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right]$$
$$\frac{\partial^2}{\partial x^2}(u^2)=2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial x}\right)^2\text{........(2)}$$
Similarly $\frac{\partial^2}{\partial y^2}(u^2)=2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2$ ....(3)

Substituting (2) and (3) in (1),
$$=2\left[u_x{}^2+u_y^2\right]\qquad(\because u_{xx}+u_{yy}=0)$$
$$\nabla^2(|u|^2)=2|f'(z)|^2$$

Example 6

If f (z) = u+iv is an analytic function, then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^p=p(p-1)u^{p-2}|f'(z)|^2$.

Solution:
$$\frac{\partial}{\partial x}(u^p)=pu^{p-1}\frac{\partial u}{\partial x}$$
$$\frac{\partial^2}{\partial x^2}(u^p)=p\left[u^{p-1}\frac{\partial^2 u}{\partial x^2}+(p-1)u^{p-2}\left(\frac{\partial u}{\partial x}\right)^2\right]\quad\text{....(1)}$$
$$\frac{\partial^2}{\partial y^2}(u^p)=p\left[u^{p-1}\frac{\partial^2 u}{\partial y^2}+(p-1)u^{p-2}\left(\frac{\partial u}{\partial y}\right)^2\right]\quad\text{....(2)}$$
Adding (1) and (2), we get
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^p)=pu^{p-1}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+(p-1)pu^{p-2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$
As f(z) = u+ iv is analytic, then we have
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\ .$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^p)=p(p-1)u^{p-2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$=p(p-1)u^{p-2}\left|f'(z)\right|^2 \qquad \text{as} \quad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Example 7

Prove that both real and imaginary parts of an analytic function in z plane satisfy the Laplace equation in two variables.

Solution:

Since w= u+iv is analytic in some region of the z –plane, u and v satisfy the C-R equations,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad ....(1)$$

Differentiating these two equations partially with respect to x and y respectively, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}; \quad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x} \qquad .....(2)$$

Adding the above two equations(1) &(2),

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=0 \quad ....(3)$$

Differentiating equations (1) partially with respect to y and x, respectively we get,

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 v}{\partial y^2}; \qquad \frac{\partial^2 u}{\partial x\partial y}=-\frac{\partial^2 v}{\partial x^2}$$

Adding these two equations, we get

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial y\partial x}-\frac{\partial^2 u}{\partial x\partial y}=0$$

$$(i.e)., \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad ....(4)$$

From (3) and (4), we see that u and v satisfy their Laplace’s equations.

Hence the theorem.

Example 8

When the function f (z) = u+iv is analytic, show that u=constant and v=constant are orthogonal.

Solution:

Given f(z) is an analytic function.

Therefore, by C.R equation $u_x=v_y$ and $u_y=-v_x$

Given: u (x, y) = a and v (x, y) = b

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0, \quad \frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=m_1\,(say),$$

$$\frac{dy}{dx}=\frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

$$= \frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)} \quad \text{by C.R. equations}$$

$$= m_2 \ (say)$$

Product of slopes at their point of intersection = $m_1 m_2$

$$= \frac{-\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)}. = -1$$

Hence the two families of curves form an orthogonal system.

Example 9

Show that $Z\overline{Z}$ is differentiable but not analytic at z =0.

Solution:

$$\text{Given}: z\,\bar{z} = x^2 + y^2$$

$$i.e., \quad f(z) = z\,\bar{z} = (x^2 + y^2) + i\,0$$

$$u = x^2 + y^2, \qquad v = 0$$

$$u_x = 2x \quad , \qquad v_x = 0$$

$$u_y = 2y \quad , \qquad v_y = 0$$

The C-R equations are satisfied if 2x = 0, 2y = 0. That is x = 0 and y = 0.
Thus, the C-R equations are satisfied at the origin. Further, the partial derivatives of u and v are continuous. Hence $z\,\bar{z}$ has a derivative at the origin (only).

Though f(z) has a derivative at the origin, there is no neighbourhoods of the origin where it has a derivative. Therefore f(z) is differentiable but not analytic at z = 0.

Exercise 5.1

1.Check the analyticity of $\log z$.

2. If f(z)= u+iv is analytic on a domain D and $|f|$ is a constant on D. Prove that f must be a constant on D.

3. If f(z) is an analytic function z, prove that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

4. If f(z) = u+iv is a regular function of z in a domain D then the following relation holds $\nabla^2[|u|^2] = 2|f'(z)|^2$

5. If $w = f(z)$ is analytic prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$ and prove that $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$

6.Show that $f(z) = \frac{1}{z}$ is analytic everywhere except at $z = 0$ and find $f^{'}(z)$.

5.2 HARMONIC FUNCTIONS

A real function of two variables x and y possesses continuous second order partial derivatives and that satisfies the Laplace equation is called as harmonic function. It is also known as potential function.

Harmonic Conjugate

If u and v are harmonic functions such that u+iv is analytic, then each is called harmonic conjugate of the other.

Laplace equation

$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$ is known as Laplace equation in two dimension.

Worked Examples 1.2(A)

Example 1

Verify whether $u=(x-y)(x^2+4xy+y^2)$ is harmonic function.

Solution:

$u=(x-y)(x^2+4xy+y^2)$

i.e) $u=x^3+4x^2y+xy^2-yx^2-4xy^2-y^3$

$u_x=3x^2+8xy+y^2-2yx-4y^2=3x^2+6xy-3y^2$

$u_{xx}=6x+6y$

$u_y=4x^2+2xy-x^2-8yx-3y^2$

$u_{yy}=2x-8x-6y=-6x-6y$

$u_{xx}+u_{yy}=6x+6y-6x-6y=0$

Therefore u is a harmonic function.

Example 2

Find the value of m if $u=2x^2-my^2+3x$ is harmonic.

Solution:

Since u is harmonic, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\rightarrow(*)$

Given $u=2x^2-my^2+3x$ .....(1)

Differentiating (1) w.r.t 'x', $\frac{\partial u}{\partial x}=4x+3$

Again differentiating w.r.t 'x', $\frac{\partial^2 u}{\partial x^2}=4\rightarrow(2)$

Differentiating (1) w.r.t 'y', $\frac{\partial u}{\partial y}=-2my$

Again differentiating w.r.t 'y', $\frac{\partial^2 u}{\partial y^2}=-2m\rightarrow(3)$

Substituting (2) and (3) in (*),

$4-2m=0\Rightarrow 4=2m$

$\therefore m=2$

Example 3

Show that $f(x,y)=\log\sqrt{x^2+y^2}$ is harmonic.

Solution:

i.e., to prove $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

Consider, $\frac{\partial f}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}(x^2+y^2)^{-1/2}2x=\frac{x}{x^2+y^2}$

$$\frac{\partial^2 f}{\partial x^2}=\frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2}=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}=\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{x^2+y^2-y(2y)}{(x^2+y^2)^2}=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

Now consider, $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=\frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}=0$

Hence $f(x,y)=\log\sqrt{x^2+y^2}$ is harmonic.

Example 4

Give an example of a function where u and v are harmonic but $\mathrm{u}+\mathrm{iv}$ is not analytic.

Solution:

Example of harmonic but not analytic.

$f(z)=x^2-iy^2$

$u=x^2 \quad v=-y^2 \qquad u_x=2x \quad v_y=-2y$

$u_{xx}=2 \quad v_{yy}=-2 \qquad u_y=0 \quad v_x=0$

$u_{xx}+v_{yy}=0. \qquad \therefore\ u_x\neq v_y \ \ and \ \ v_x\neq -u_y$

Hence for the given above function $f(z)$ u and v are harmonic but $\mathrm{u}+\mathrm{iv}$ is not analytic.

Example 5

Show that the function $\mathrm{e}^{x}\sin \mathrm{y}$ is harmonic.

Solution:

Given $u=e^x\sin y \ \ u_x=e^x\sin y$

$u_{xx}=e^x\sin y$

$u_y=e^x\cos y$

$u_{yy}=-e^x\sin y$

$u_{xx}+u_{yy}=e^x\sin y-e^x\sin y=0$

$u_{xx}+u_{yy}=0$

Example 6

Show that the function $\mathrm{u}(\mathrm{x},\mathrm{y})=-\sin \mathrm{x}\sinh \mathrm{y}$ is harmonic.

Solution:

Given $u(x,y)=-\sin x\sinh y$

$\frac{\partial u}{\partial x}=-\cos x\sinh y \qquad \frac{\partial u}{\partial y}=-\sin x\cosh y$

$\frac{\partial^2 u}{\partial x^2}=\sin x\sinh y \qquad \frac{\partial^2 u}{\partial y^2}=-\sin x\sinh y$

$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Hence $u(x,y)=-\sin x\sinh y$ is harmonic.

Example 7

Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$

$u_{xx} = 6x + 6$

$u_y = -6xy - 6y$

$u_{yy} = -6x - 6$

$\therefore u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$

Hence the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic

Example 9

Show that $u(x, y) = (x\cos y - y\sin y)$ is harmonic.

Solution:

Given $u(x, y) = (x\cos y - y\sin y)$

$u_x = \cos y$

$u_{xx} = 0$

$u_y = -x\sin y - y\cos y - \sin y$

$u_{yy} = -x\cos y + y\sin y - \cos y - \cos y$

$u_{xx} + u_{yy} \neq 0$ which is not harmonic.

Worked Examples 5.2(B)

Example1

Prove that the real and imaginary parts of an analytic function are harmonic functions.

Solution:

Let $f(z) = w = u + iv$ be analytic

To prove: u and v satisfy the Laplace equation.

i.e., To prove:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Given: f(z) is analytic. u and v satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} ....(1) \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} ....(2)$$

Differentiate (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} ....(3)$$

Differentiate (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} ....(4)$$

The second order mixed partial derivatives are equal

Differentiate (1) partially with respect to x, we get

$$i.e., \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

$$(3)+(4) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\therefore$ u satisfies Laplace equation

Differentiate (1) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} ....(5)$$

Differentiate (2) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} ....(6)$$

$$(5)+(6) \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ v satisfies Laplace equation.

Example 2

Prove that $u = x^2 - y^2$ and $v = \frac{-y}{x^2 + y^2}$ are harmonic but not harmonic conjugate.

Solution

Given $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$

To prove u and v are harmonic

(i.e.,) to prove $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

$u_x = 2x \qquad u_{xx} = 2$

$u_y = -2y \qquad u_{yy} = -2$

$u_{xx} + u_{yy} = 0 \qquad \therefore$ u is harmonic.

$$v_x = \frac{2xy}{(x^2+y^2)^2}$$

$$v_{xx} = \frac{(x^2+y^2)^2 2y - (x^2+y^2)8x^2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\{(2yx^2+2y^3)-8x^2y\}}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\{(2yx^2+2y^3)-8x^2y}{(x^2+y^2)^4}$$

$$= \frac{(2yx^2+2y^3)-8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3-6x^2y}{(x^2+y^2)^3}$$

$$v_y = \frac{-2y^2}{(x^2+y^2)^2}$$

$$v_{yy}=\frac{(x^2+y^2)^2(-4y)+8y^3(y^2+x^2)}{(x^2+y^2)^4}$$

$$=\frac{\{(-4y)\left(x^2+y^2\right)+8y^3\}}{(x^2+y^2)^4}$$

$$=\frac{\{(-4yx^2+4y^3\}}{(x^2+y^2)^3}$$

$$v_{xx}+v_{yy}\neq 0$$

$\therefore$ v is not harmonic.

To find harmonic conjugate

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

$$=-2ydx+2xdy$$

$$v=-2xy+2xy=0$$

$\therefore$ u and v are not harmonic conjugate.

Example 3

Prove that v= $\tan^{-1}\left(\frac{y}{x}\right)$ is harmonic, and also find u

Solution

Given: $v=\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial v}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left[\frac{-y}{x^2}\right]=\frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{-y(2x)}{(x^2+y^2)^2}=\frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left[\frac{1}{x}\right]=\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 v}{\partial y^2}=\frac{x(2y)}{(x^2+y^2)^2}$$

$$=\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{2xy-2xy}{(x^2+y^2)^2}=0$$

Hence u is harmonic.

To find the conjugate:

$$v_x=\frac{-y}{x^2+y^2} \qquad v_x(z,0)=0$$

$v_y = \dfrac{x}{x^2+y^2}$ $\qquad v_y(z,0) = \dfrac{z}{z^2} = \dfrac{1}{z}$

$f'(z) = u_x + iv_x = v_y + iv_x$ (By CR equations)

$$= \frac{1}{z}$$

$w = f(z) = \log z$

To find *u*:

Substitute $z = re^{i\theta}$

$u + iv = \log(re^{i\theta}) = \log r + \log e^{i\theta}$

$u + iv = \log r + i\theta$ $\qquad (\because z = x+iy$, $r = |z| = \sqrt{x^2+y^2})$

$u = \log r = \log\sqrt{x^2+y^2}$

Example 4

Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Determine its analytic function. Find also its conjugate.

Solution

Given: $u = \dfrac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{(x^2+y^2-2x^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{(x^2+y^2-2y^2)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

Hence u satisfies Laplace equation.

$\therefore$ u is harmonic.

To find the conjugate of u i.e., to find v:

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \left[\textit{Exact differential method}\right]$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \text{ by } C-R$$

$$\int dv = \int -u_y \; dx \;(\text{treating y as a constant}) + \int [\text{terms } (u_x) \text{ not contributing } x] dy$$

$$v = \int -\frac{y}{x^2+y^2} dx + 0 = \int \frac{1}{1+\left(\frac{y}{x}\right)^2} d\left[\frac{y}{x}\right]$$

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

To find analytic function:

$$f(z) = u + iv$$

$$= \frac{1}{2}\log\left(x^2+y^2\right) + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \log(x+iy)$$

$$= \log z$$

Example 5

Show that the harmonic function 'u' satisfies the formal differential equation $\frac{\partial^2 u}{\partial z \partial \overline{z}} = 0$ and hence prove that $\log|f'(z)|$ is harmonic, where $f(z)$ is a regular function.

Solution

We know that $|f(z)|^2 = u^2 + v^2$

$$|f(z)| = \left(u^2+v^2\right)^{\frac{1}{2}}$$

$$\log|f(z)| = \frac{1}{2}\log\left(u^2+v^2\right)$$

$$\frac{\partial}{\partial x}\log|f(z)| = \frac{1}{2}\frac{\partial}{\partial x}\log\left(u^2+v^2\right)$$

$$= \frac{1}{2}\left[\frac{1}{u^2+v^2}\right][2\,uu_x + 2\,vv_x]$$

$$\frac{\partial}{\partial x}\log|f(z)| = \frac{uu_x + vv_x}{u^2+v^2} = \frac{(u^2+v^2)[uu_{xx} + v v_{xx} + u_x^2 + v_x^2] - 2[uu_x + vv_x]^2}{(u^2+v^2)^2}$$

$$|||^{ly}\; \frac{\partial^2}{\partial^2 y}\log|f(z)| = \frac{(u^2+v^2)[uu_{yy} + v v_{yy} + u_y^2 + v_y^2] - 2[uu_y + vv_y]^2}{(u^2+v^2)^2}$$

$$\left[\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right]\log|f(z)| = \frac{\left(\begin{array}{c}(u^2+v^2)[uu_{xx} + v v_{xx} + u_x^2 + v_x^2 + uu_{yy} + v v_{yy} + u_y^2 + v_y^2] \\ -2[uu_x + vv_x]^2 - 2[uu_y + vv_y]^2\end{array}\right)}{(u^2+v^2)^2}$$

$$= \frac{\left(\begin{array}{l}(u^2+v^2)[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2] \\ -2[u^2u_x^2 + v^2v_x^2 + 2\,uvu_x\,v_x + u^2u_y^2 + v^2v_y^2 + 2\,uvu_y\,v_y]\end{array}\right)}{(u^2+v^2)^2}$$

$$= \frac{\left(\begin{array}{l}(u^2+v^2)[u(0) + v(0) + u_x^2 + v_x^2 + u_y^2 + v_y^2] - 2[u^2(u_x^2 + u_y^2) + v^2(v_x^2 + v_y^2) \\ +2\,uvu_x\,v_x + 2\,uvu_y\,v_y]\end{array}\right)}{(u^2+v^2)^2}$$

Since f(z)=u+iv is a analytical function

$$\Rightarrow u_x = v_y \qquad \text{[by CR equation]}$$
$$u_y = -v_x$$

$$\Rightarrow u_{xx} + u_{yy} = 0 \, and \, v_{xx} + v_{yy} = 0$$

Since u and v are harmonic functions

$$\Rightarrow u_x v_x + u_y v_y = 0$$

By CR equation

$$\left[\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right] \log|f(z)|$$

$$= \frac{(u^2+v^2)[u_x^2+v_x^2+(-v_x)^2+u_x^2] - 2[u^2(u_x^2+v_x^2)+v^2(v_x^2+u_x^2)+2uv(u_x v_x + u_y v_y)]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[2(u_x^2+v_x^2)] - 2[(u_x^2+v_x^2)(u^2+v^2)]}{(u^2+v^2)^2}$$

$$= \frac{2[(u^2+v^2)(u_x^2+v_x^2)] - 2[(u_x^2+v_x^2)(u^2+v^2)]}{(u^2+v^2)^2}$$

$$\left[\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right] \log|f(z)| = 0$$

Example 6

If $f(z) = u + iv$ is a regular function of z in a domain D the following relations hold in D

$$\nabla^2 \log(|f(z)|) = 0 \;\; if \;\; f(z) f'(z) \neq 0 \; in \; D.$$

Solution:

We know that $|f(z)|^2 = u^2 + v^2$

$$|f(z)| = \left(u^2+v^2\right)^{\frac{1}{2}}$$

$$\log|f(z)| = \frac{1}{2}\log\left(u^2+v^2\right)$$

$$\frac{\partial}{\partial x}\log|f(z)| = \frac{1}{2}\frac{\partial}{\partial x}\log\left(u^2+v^2\right)$$

$$= \frac{1}{2}\left[\frac{1}{u^2+v^2}\right][2uu_x + 2vv_x]$$

$$\frac{\partial}{\partial x}\log|f(z)| = \frac{uu_x + vv_x}{u^2+v^2} = \frac{(u^2+v^2)[uu_{xx} + vv_{xx} + u_x^2 + v_x^2] - 2[uu_x + vv_x]^2}{(u^2+v^2)^2}$$

$$|||^{ly} \; \frac{\partial^2}{\partial^2 y}\log|f(z)| = \frac{(u^2+v^2)[uu_{yy} + vv_{yy} + u_y^2 + v_y^2] - 2[uu_y + vv_y]^2}{(u^2+v^2)^2}$$

$$\left[\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right]\log|f(z)| = \frac{\begin{pmatrix}(u^2+v^2)[uu_{xx} + vv_{xx} + u_x^2 + v_x^2 + uu_{yy} + vv_{yy} + u_y^2 + v_y^2] \\ -2[uu_x + vv_x]^2 - 2[uu_y + vv_y]^2\end{pmatrix}}{(u^2+v^2)^2}$$

$$= \frac{\begin{pmatrix}(u^2+v^2)[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2] \\ -2[u^2u_x^2 + v^2v_x^2 + 2uvu_x v_x + u^2u_y^2 + v^2v_y^2 + 2uvu_y v_y]\end{pmatrix}}{(u^2+v^2)^2}$$

$$= \frac{\begin{pmatrix}(u^2+v^2)[u(0)+v(0)+u_x^2+v_x^2+u_y^2+v_y^2]-2[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)\\ +2\,uvu_x\,v_x+2\,uvu_y\,v_y]\end{pmatrix}}{(u^2+v^2)^2}$$

Since f(z)=u+iv is a analytical function

$\Rightarrow u_x = v_y$ [by CR equation]

$u_y = -v_x$

$\Rightarrow u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

Since u and v are harmonic functions

$\Rightarrow u_x v_x + u_y v_y = 0$

By CR equation

$$\left[\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right]\log|f(z)|$$

$$= \frac{(u^2+v^2)[u_x^2+v_x^2+(-v_x)^2+u_x^2]-2[u^2(u_x^2+v_x^2)+v^2(v_x^2+u_x^2)+2\,uv(u_x\,v_x+u_y\,v_y)]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[2(u_x^2+v_x^2)]-2[(u_x^2+v_x^2)(u^2+v^2)]}{(u^2+v^2)^2}$$

$$= \frac{2[(u^2+v^2)(u_x^2+v_x^2)]-2[(u_x^2+v_x^2)(u^2+v^2)]}{(u^2+v^2)^2}$$

$$\left[\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right]\log|f(z)| = 0$$

Example 7

Prove that $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the corresponding analytic function and imaginary parts.

Solution

Given $u = e^{-2xy}\sin(x^2 - y^2)$

$$u_x = e^{-2xy}\cos(x^2-y^2)2x + \sin(x^2-y^2)e^{-2xy}(-2y)$$

$$=2e^{-2xy}[x\cos(x^2-y^2) - y\sin(x^2-y^2)]$$

$$u_{xx} = 2e^{-2xy}[-x\sin(x^2-y^2)2x + \cos(x^2-y^2) - y\cos(x^2-y^2)2x]$$

$$+[x\cos(x^2-y^2) - y\sin(x^2-y^2)]2e^{-2xy}(-2y)$$

$$u_{xx} = 2e^{-2xy}[-2x^2\sin(x^2-y^2) + \cos(x^2-y^2) - 2xy\cos(x^2-y^2)$$

$$-2xy\cos(x^2-y^2) + 2y^2\sin(x^2-y^2)] \to (1)$$

$$u_y = \frac{e^{-2xy}}{-2x}\sin(x^2-y^2) + e^{-2xy}\cos(x^2-y^2)(-2y)$$

$$\Rightarrow u_y = -2e^{-2xy}[x\sin(x^2-y^2) + y\cos(x^2-y^2)]$$

$$u_{yy} = 4xe^{-2xy}(x\sin(x^2-y^2) + y\cos(x^2-y^2)]$$

$$-2e^{-2xy}[-2xy\cos(x^2-y^2) + \cos(x^2-y^2) + 2y^2\sin(x^2-y^2)] \to (2)$$

From (1) and (2), $u_{xx} + u_{yy} = 0$

Hence *u* is harmonic.

By Milne Thomson method

$$f'(z) = u_x(z,0) + iu_y(z,0)$$
$$= 2z\cos z^2 + 2iz\sin z^2$$
$$= 2z(\cos z^2 + i\sin z^2) = 2ze^{iz^2}$$
Integrating with respect to z we get
$$f(z) = \int 2ze^{iz^2}\,dz + c$$
$$put\ t = z^2 \quad dt = 2zdz$$
$$= \int e^{it}dt + c$$
$$= \frac{e^{it}}{i} + c = \frac{1}{i}e^{iz^2} + c = -ie^{iz^2} + c$$
$$u + iv = -ie^{i(x+iy)^2} + c_1 + ic_2$$
$$= -ie^{i(x^2-y^2+2ixy)} + c_1 + ic_2$$
$$= -ie^{i(x^2-y^2)}e^{-2xy} + c_1 + ic_2$$
$$= -ie^{-2xy}\cos(x^2-y^2) + e^{-2xy}\sin(x^2-y^2)$$
Imaginary part we get $v = -e^{-2xy}\cos(x^2-y^2)$

Example 8

If f (z) = u+iv is a regular function of z in a domain D then the following relations hold in $\nabla^2[|f(z)|^2] = 4|f'(z)|^2$.

Solution:

Given $f(z) = u + iv$ be an analytic function

Therefore, by C.R equations

$$u_x = v_y, u_y = -v_x$$
$$|f(z)| = |u + iv| \quad f'(z) = u_x + iv_x, \quad |f'(z)|^2 = u_x^2 + v_x^2$$
$$\text{ie}\,|f(z)|^2 = |u+iv|^2 = u^2 + v^2$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$
$$= \frac{\partial^2}{\partial x^2}(u^2+v^2) + \frac{\partial^2}{\partial y^2}(u^2+v^2)$$
$$= \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 v^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + \frac{\partial^2 v^2}{\partial y^2} \rightarrow (1)$$
$$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$$
$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)$$
$$= 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$$
$$\text{Similarly}\ \boxed{\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]}$$

$$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right]$$

$$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+u(0)\right]\ (\because \text{u is harmonic})$$

$$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=2\left[u_x^2+u_y^2\right]$$

$$=2\left[u_x^2+(-v_x)^2\right]\ (\because u_y=-v_x)$$

$$=2\left[u_x^2+v_x^2\right]$$

$$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}=2\left|f'(z)\right|^2$$

$$\text{Similarly}\ \frac{\partial^2 v^2}{\partial x^2}+\frac{\partial^2 v^2}{\partial y^2}=2\left|f'(z)\right|^2$$

$$\therefore (1)\Rightarrow \frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 v^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+\frac{\partial^2 v^2}{\partial y^2}$$

$$=2\left|f'(z)\right|^2+2\left|f'(z)\right|^2=4\left|f'(z)\right|^2$$

$$\boxed{\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left|f(z)\right|^2=4\left|f'(z)\right|^2}$$

Exercise 5.2

1. If u(x,y) and v(x,y) are harmonic in a region R prove that $\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ **is analytic**

2. Find the analytic function f(z)=u+iv, where $v=3r^2\sin 2\theta-2r\sin\theta$**. Verify that u is a harmonic function.**

3. Find the harmonic conjugate of the function $v(x,y)=e^x\left(x\sin y+y\cos y\right)$ **if f (z) =u+iv**

5.3 CONSTRUCTION OF ANALYTIC FUNCTIONS

There are three methods to find $f(z)$. They are

1. Exact differential method 2. Substitution method 3. Milne-Thomson method

Here we use only Milne-Thomson method.

Rule of Milne-Thomson method

(i) To find $f(z)$ when u is given

Let $f(z)=u+iv$

$$f'(z)=u_x+iv_x$$

$$=u_x-iu_y \quad \text{[by C.R condition]}$$

$$\therefore f(z)=\int u_x(z,0)dz-i\int u_y(z,0)dz+C \quad \text{where C is a complex constant}$$

(ii) To find $f(z)$ when v is given

Let $f(z)=u+iv$

$$f'(z)=u_x+iv_x$$
$$=v_y+iv_x \quad \text{[by C.R condition]}$$
$$\therefore f(z)=\int v_y(z,0)dz+i\int v_x(z,0)dz+C$$ where C is a complex constant

Worked Examples 5.3(A)

Example 1

Find the real and imaginary parts of $f(z)=2z^3-3z$.

Solution:

Given

$$f(z)=2z^3-3z$$
$$f(x+iy)=2(x+iy)^3-3(x+iy)$$
$$=2(x^3-iy^3+3x^2iy-3xy^2)-3(x+iy)$$
$$=2x^3-2iy^3+6x^2iy-6xy^2-3x-3iy$$
$$=2x^3-6xy^2-3x+i(-2y^3+6x^2y-3y)$$
$$=2x^3-6xy^2-3x+i(-2y^3+6x^2y-3y)$$
$$\boxed{\text{Real}=2x^3-6xy^2-3x \ \text{Imaginary}=(-2y^3+6x^2y+3y)}$$

Example 2

Prove that $w=z^2$ is analytic and hence find $\frac{dw}{dz}$.

Solution:

Given $w=z^2$

$$w=(x+iy)^2$$
$$=x^2-y^2+2ixy$$
$$u+iv=\left(x^2-y^2\right)+i\left(2xy\right)$$
$$u=x^2-y^2 \ \& \ v=2xy$$
$$u_x=2x \quad v_x=2y$$
$$u_y=-2y \quad v_y=2x$$

From the above

$$\boxed{u_x=v_y \ \& \ u_y=-v_x}$$

Therefore $f(z)=z^2$ is analytic.

We know $\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$$=2x+i2y$$
$$=2(x+iy)$$
$$\boxed{\frac{dw}{dz}=2z}$$

Example 3

Find an analytic function f (z) for which the real part is $y+e^x\cos y$.

Solution:

Given $u=y+e^x\cos y$

$$\left(\frac{\partial u}{\partial x}\right) = e^x \cos y$$

$$\varphi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} = e^z$$

$$\left(\frac{\partial u}{\partial y}\right) = 1 - e^x \sin y$$

$$\varphi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)} = 1 - 0 = 1$$

By Milne Thomson method $f(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz + c$

$$= \int e^z dz - i\int dz + c$$

$$\boxed{f(z) = e^z - iz + c}$$

Example 4

If u+iv is analytic, show that v-iu and –v + iu are also analytic.

Solution:

Given u+iv is analytic

Then C-R equations are satisfied

That is $u_x = v_y \quad ...(\mathbf{1})$

$u_y = -v_x \quad ...(\mathbf{2})$

Since the derivatives of u and v exist and also continuous

Now to prove v-iu and –v+iu are also analytic, we should prove that

(i). $\boxed{v_x = -u_y \;\&\; v_y = u_x}$

(ii). $\boxed{-v_x = u_y \;\&\; v_y = u_x}$

(iii). $\boxed{u_x, u_y, v_x \text{ and } v_y}$ all are continuous.

Results (i) and (ii) follows from (1) and (2).

Since the derivatives of u and v exist from (1) and (2), the derivatives of u and v should be continuous.

Hence the proof.

Example 5

Find an analytic function f (z) for which the real part is $e^x \cos y$.

Solution:

Given $u = e^x \cos y$

$$\varphi_1(x,y) = \frac{\partial u}{\partial x} = e^x \cos y$$

$$\varphi_1(z,0) = e^z$$

$$\varphi_2(x,y) = \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\varphi_2(z,0) = 0$$

By Milne's Thomson method

$$f(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz$$

$$= \int e^z dz - 0$$

$$\boxed{f(z) = e^z + c}$$

Example 6

The real part of an analytic function f(z) is constant, prove that f(z) is a constant function.

Solution:

Let f(z) = u+iv be an analytic function

$$u_x = v_y \;\&\; u_y = -v_x$$

Given u = c (a constant)

$$u_x = 0, u_y = 0$$

$$v_x = 0, v_y = 0 \;\; \text{by}(1)$$

$$\text{we know that } f(z) = u + iv$$

$$f'(z) = u_x + iv_x$$

$$f'(z) = 0 + i0$$

$$\boxed{f'(z) = 0}$$

Integrating with respect to z, f(z)=c

Hence an analytic function with constant real parts is constant.

Worked Examples 5.3 (B)

Example 1

Find the analytic function for which $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ is the real part. Hence determine the analytic function u+iv for which u+v is the above function.

Solution:

$$\text{Given}: u = \frac{\sin 2x}{\cosh 2y - \cos 2x}.\text{(real part)}$$

By MilneThomson method

$$f(z) = \boxed{\int \frac{\partial u}{\partial x}(z,0)dz - i\int \frac{\partial u}{\partial y}(z,0)dz \quad ...(1)}$$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x}(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - \sin 2z(2\sin 2z)}{(1-\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$w.k.t \;\cos^2 2z + \sin^2 2z = 1$$

$$\sin^2 2z = 1 - \cos^2 2z$$

$$\frac{\partial u}{\partial x}(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}(z,0) = \frac{(1-\cos 2z)\left\{2\cos 2z - 2(1+\cos 2z)\right\}}{(1-\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}(z,0)=\frac{(1-\cos 2z)\left\{2\cos 2z-2-2\cos\ \ 2z)\right\}}{(1-\cos 2z)^2}$$

$$\frac{\partial u}{\partial x}(z,0)=\frac{\{-2\}}{(1-\cos 2z)}$$

$$w.k.t\ \sin^2 z=\frac{1-\cos 2z}{2}$$

$$2\sin^2 z=1-\cos 2z$$

$$\frac{\partial u}{\partial x}(z,0)=\frac{\{-2\}}{2\sin^2\text{z}}$$

$$\frac{\partial u}{\partial x}(z,0)=-\frac{1}{\sin^2\text{z}}$$

$$\frac{\partial u}{\partial x}(z,0)=-\cos ec^2 z$$

$$\text{Now}\ \frac{\partial u}{\partial y}=\frac{(\cosh 2y-\cos 2x)(0)-\sin 2x(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}$$

$$\frac{\partial u}{\partial y}=\frac{0-\sin 2x(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}$$

$$\frac{\partial u}{\partial y}(z,0)=0$$

$$\text{Substitute}\ \frac{\partial u}{\partial x}=\frac{0-\sin 2x(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}\ \&\ \frac{\partial u}{\partial y}(z,0)=0\ in\ (1)$$

$$f(z)=\int -\cos ec^2 z dz-0$$

$$\boxed{f(z)=\cot z+c}$$

Example 2

Determine the analytic function f (z) =u+iv given $\mathrm{u-v}=\dfrac{\cos\mathrm{x}+\sin\mathrm{x}-\mathrm{e}^{-\mathrm{y}}}{2(\cos\mathrm{x}-\cosh\mathrm{y})}$ and $\mathrm{f}\left(\dfrac{\pi}{2}\right)=0.$

Solution

Let $f(z)=u+iv$........(1)

$if(z)=iu-v$.........(2)

$(1)+(2)\Rightarrow (1+\mathrm{i})\,\mathrm{f(z)}=(\mathrm{u-v})+\mathrm{i}(u+v)$

$F(Z)=U+IV$

$$U=u-v=\frac{\cos x+\sin x-e^{-y}}{2(\cos x-\cosh y)}.$$

$$\varphi_1(\mathrm{x,y})=\frac{\partial U}{\partial x}=\frac{\begin{pmatrix}\{2(\cos x-\cosh y)\}(-\sin\mathrm{x}+\cos\mathrm{x})-\\ \{\cos x+\sin x-e^{-y}\}(-2\sin x)\end{pmatrix}}{4(\cos x-\cosh y)^2}$$

$$=\frac{\begin{pmatrix}-2\cos x\sin x+2\cos^2 x+2\cos hy\sin x-2\cosh y\cos x\\ +2\sin x\cos x+2\sin^2 x-2\sin xe^{-y}\end{pmatrix}}{4(\cos x-\cosh y)^2}$$

$$=\frac{2\cos^2 x+2\sin^2 x+2\cos hy(\sin x-\cos x)-2\sin xe^{-y}}{4(\cos x-\cosh y)^2}$$

$$= \frac{2\cos^2 x + 2\sin^2 x + 2\cos hy(\sin x - \cos x) - 2\sin xe^{-y}}{4(\cos x - \cosh y)^2}$$

$$= \frac{2 + 2\cos hy(\sin x - \cos x) - 2\sin xe^{-y}}{4(\cos x - \cosh y)^2}$$

$$\varphi_1(\mathrm{z}, 0) = \frac{2 + 2\cos h(0)(\mathrm{sinz} - \cos 0) - 2\,\mathrm{sinz}\, e^{-0}}{4(\mathrm{cosz} - \cosh 0)^2} = \frac{2 + 2(\mathrm{sinz} - 1) - 2\,\mathrm{sinz}}{4(\mathrm{cosz} - 1)^2} = 0$$

$$\varphi_2(\mathrm{x}, \mathrm{y}) = \frac{\partial U}{\partial y} = \frac{e^{-y}\{2(\cos x - \cosh y)\} - \{\cos x + \sin x - e^{-y}\}(-2\,\mathrm{sinhy})}{4(\cos x - \cosh y)^2}$$

$$\varphi_2(\mathrm{z}, 0) = \frac{\partial U}{\partial y} = \frac{e^{-0}\{2(\cos z - \cosh 0)\} - \{\cos z + \sin z - e^{-0}\}(-2\sinh 0)}{4(\cos z - \cosh 0)^2}$$

$$\varphi_2(\mathrm{z}, 0) = \frac{\{2(\cos z - 1)\}}{4(\cos z - 1)^2} = \frac{1}{2(\cos z - 1)} = -\frac{1}{2}\cos ec^2 \frac{z}{2}$$

By Milne's Thomson method,

$$F(z) = \int \phi_1(z, 0)\, dz - i\int \phi_2(z, 0)$$

$$= \int 0\, dz - i\int \frac{-1}{2}\cos ec^2 \frac{z}{2} dz$$

$$f(z) = \frac{1}{2}\left[1 - \cot\left(\frac{z}{2}\right)\right] + c$$

Example 3

Find an analytic function f (z) = u+iv, given that $2u + 3v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Solution

Let $f(z) = u + iv$

$3f(z) = 3u + 3iv \rightarrow (1)$

$2if(z) = 2iu - 2v \rightarrow (2)$

$(1) + (2) \Rightarrow (3 + 2i)f(z) = (3u - 2v) + i(2u + 3v)$

$F(z) = U + iV$

Given: $2u + 3v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\therefore 2u + 3v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\frac{\partial V}{\partial x}(z, 0) = \frac{-2}{1 - \cos 2z}$$

$$\frac{\partial V}{\partial x}(z, 0) = -\cos ec^2 z$$

$$\frac{\partial V}{\partial y}(z, 0) = 0$$

Hence, by Milne's Thomson method

$$F(z) = \int V_y(z, 0)\,\mathrm{dz} + i\int V_x(z, 0)dz$$

$$= 0 + i\int -\cos ec^2 z dz$$

$$F(z) = i\cot z + c_1$$

$$\Rightarrow (3 + 2i)f(z) = i\cot z + c_1$$

$$f(z) = \frac{1}{3+2i}\cot z + c$$

$$= \frac{i(3-2i)}{(3+2i)(3-2i)}\cot z + c$$

$$f(z) = \frac{2+3i}{13}\cot z + c$$

Example 4

Construct the analytic function $f(z) = u(r,\theta) + iv(r,\theta)$. Given that $u(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$.

Solution:

Given $u(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2.$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta - \cos\theta \quad ...(1)$$

$$\frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta \quad ...(2)$$

By C-R equation, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$$= -\frac{1}{r}[-2r^2\sin 2\theta + r\sin\theta\ ]$$

$$= 2r\sin 2\theta - \sin\theta \quad ....(3)$$

Integrating with respect to r , we get

$$v = r^2\sin 2\theta - r\sin\theta\ + \phi(\theta) \quad ....(4)$$

By C-R equation, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} = r(2r\cos 2\theta - \cos\theta\ )$$

$$\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta - r\cos\theta \quad ....(5)$$

From (4), $\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta - r\cos\theta\ + \phi'(\theta) \quad ....(6)$

Comparing (4) and (5), $\phi'(\theta) = 0 \Rightarrow \phi(\theta) = c$.

$\therefore v = r^2\sin 2\theta - r\sin\theta\ + c$

$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$= (r^2\cos 2\theta - r\cos\theta + 2) + i(r^2\sin 2\theta - r\sin\theta + c)$$

$$= r^2(\cos 2\theta + i\sin 2\theta) - r(\cos\theta + i\sin\theta) + 2 + c$$

$$= r^2 e^{i2\theta} - r\,e^{i\theta} + 2 + c$$

$$= (re^{i\theta})^2 - re^{i\theta} + 2 + c$$

$$f(z) = z^2 - z + (c+2)$$

Example 5

Find the analytic function f (z) = u+iv, given that 2u + 3v = e^x(cosx – siny).

Solution:

Given $2u + 3v = e^x(\cos x - \sin y)$

Let $f(z) = u + iv$

$2f(z) = 2u + i2v \rightarrow (1)$

$3if(z) = 3iu + 3i^2v$

i.e., $3if(z) = 3iu - 3v = -3v + 3iu \rightarrow (2)$

$(1)-(2) \Rightarrow 2f(z) - 3if(z) = 2u + i2v + 3v - 3iu$

$(2-3i)f(z) = (2u+3v) + \mathrm{i}(2\mathrm{v} - 3\mathrm{u})$

Let $F(z) = U + iV$ where $F(z) = (2-3i)f(z) \rightarrow (*)$

$U = 2u + 3v = e^x(cosx - \sin y)$

$V = 2v - 3u$

Consider $U = e^x(cosx - \sin y) \rightarrow (3)$

Differentiating (3) w.r.t x,

$U_x = e^x(cosx - \sin y) - e^x \sin x \Rightarrow U_x(z,0) = e^z \cos z - e^z \sin z \rightarrow (4)$

Differentiating (3) w.r.t y,

$U_y = -e^x \cos y \Rightarrow U_y(z,0) = -e^z \rightarrow (5)$

By Milne's Thomson method,

$F(\mathrm{z}) = \int U_x(z,0)dz - i\int U_y(z,0)dz$

$= \int e^z \cos z dz - \int e^z \sin z dz + i\int e^z dz$

$= \frac{e^z}{2}(\cos z + \sin z) - \frac{e^z}{2}(\sin z - \cos z) + ie^z + c$

$$\begin{bmatrix} \because \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] \text{and} \\ \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] \end{bmatrix}$$

$\Rightarrow F(z) = \frac{e^z}{2}\cos z + \frac{e^z}{2}\sin z - \frac{e^z}{2}\sin z + \frac{e^z}{2}\cos z + ie^z + c$

$\Rightarrow F(z) = e^z \cos z + ie^z + c$

$(2-3i)f(z) = e^z \cos z + ie^z + c$ (using*)

$f(z) = \frac{e^z(\cos z + i)}{2-3i} + c$

$= \frac{e^z(cosz + i)(2+3\mathrm{i})}{4+9} + c$

$\therefore f(z) = \frac{e^z(2cosz - 3 + i(2 + 3\cos z))}{13} + c$

Example 6

Explain the Milne Thomson's method to determine the analytic function f (z) when its real part or imaginary part is given. Apply the method to find f (z) when the real part of f (z) is $\mathrm{e}^{2x}(x\cos 2y - y\sin 2y)$.

Solution:

By Milne's Thomson method

To find f(z) when u is given

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad [\because BY \quad C-R\,Equation]$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)}$$

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)}$$

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$\int f'(z)\,dz = \int \phi_1(z,0)dz - i\int \phi_2(z,0)\,dz$$

$$\boxed{f(z) = \int \phi_1(z,0)dz - i\int \phi_2(z,0)\,dz + c}$$

If imaginary part is given then $\boxed{f(z) = \int \phi_1(z,0)dz + i\int \phi_2(z,0)\,dz + c}$

where $\phi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$, $\phi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$ and c is a complex constant.

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$\phi_1(x,y) = \left(\frac{\partial u}{\partial x}\right) = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} = 2e^{2z}\left[x cos 2(0) - 0(\sin 0)\right] + e^{2z}(\cos 0)$$

$$= 2e^{2z}(z) + e^{2z}$$

$$\boxed{\phi_1(z,0) = (2z+1)e^{2z}}$$

$$\phi_2(x,y) = \left(\frac{\partial u}{\partial y}\right) = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$$

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)} = e^{2z}(0-0-0) = 0$$

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int (2z+1)e^{2z}dz - i\int (0)\,dz$$

$$= \int (2z+1)e^{2z}dz$$

$$u = 2z+1 \quad dv = e^{2z}$$

$$u' = 2, \quad u'' = 0 \qquad and \quad v = \frac{e^{2z}}{2}, \; v_1 = \frac{e^{2z}}{4}$$

By Bernoulli's formula $\boxed{\int udv = uv - uv_1}$

$$\Rightarrow f(z) = (2z+1)\left(\frac{e^{2z}}{2}\right) - 2\left(\frac{e^{2z}}{4}\right) + c$$

$$= \left[(2z+1) - 2.\frac{1}{2}\right]\left(\frac{e^{2z}}{2}\right) + c$$

$$= [2z+1-1]\left(\frac{e^{2z}}{2}\right)+c$$

$$= 2z\left(\frac{e^{2z}}{2}\right)+c$$

$$\boxed{f(z) = ze^{2z}+c}$$

Example 7

Find the analytic function f (z) =u+iv where $u - v = e^{x}(\cos y - \sin y)$.

Solution:

Given that f(z)=u+iv is an analytic function. Then

f(z)=u+iv …(1)

if(z)=iu-v …(2)

Adding (1) and (2) f(z)+i f(z)=u+iv+iu-v = u-v +i(u+v)

f(z)(1+i)= u – v +i(u+v)

Let U = u – v

V = u + v

F(z) = (1 +i)f(z)

F(z) = U + iV

Since $U = u - v = e^{x}(\cos y - \sin y)$

$$\boxed{U_x = \frac{\partial u}{\partial x} = e^{x}(\cos y - \sin y)}$$

$$U_x(z,0) = e^{z}$$

$$\boxed{U_y = \frac{\partial u}{\partial y} = e^{x}(-\sin y - \cos y)}$$

$$U_y(z,0) = -e^{z}$$

By Milne method

$$\boxed{F(z) = \int U_x(z,0) - i\int U_y(z,0)}$$

$$F(z) = \int e^{z}dz + i\int e^{z}dz$$

$$= e^{z} + ie^{z} + c$$

$$= (1+i)e^{z} + c$$

$$But\ F(z) = (1+i)f(z)$$

$$\Rightarrow (1+i)e^{z} + c = (1+i)f(z)$$

$$\Rightarrow \boxed{f(z) = e^{z} + c}$$

Example 8

Find the analytic function, given that $u + v = \dfrac{x}{x^2 + y^2}$ subject to the condition f (1) =1

Solution:

Let $u + iv = f(z)$ …(1)

$iu - v = if(z)$ …(2)

$(1)+(2) \Rightarrow (u-v) + i(u+v) = (1+i)f(z)$

$$U + iV = F(z)$$

Where $U = u - v, \quad V = u + v, \quad F(z) = (1+i)f(z)$]

Given: $\quad V = u + v = \dfrac{x}{x^2 + y^2}$

$$\varphi_2(x,y) = \frac{\partial V}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2 + y^2 - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\varphi_2(z,0) = \frac{-z^2}{(z^2)^2} = \frac{-1}{z^2}$$

$$\varphi_1(x,y) = \frac{\partial V}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\boxed{\varphi_1(z,0) = 0}$$

By Milne's Thomson Method, we have

$$\boxed{F(z) = \int \varphi_1(z,0)\,dz + i\int \varphi_2(z,0)\,dz}$$

$$= 0 + i\int \frac{-1}{z^2}dz = -i\int \frac{1}{z^2}\,dz$$

$$F(z) = -i\int z^{-2}dz = -i\left(\frac{z^{-1}}{-1}\right) + c_1$$

$$(1+i)f(z) = \frac{i}{z} + c_1$$

$$f(z) = \frac{i}{(1+i)z} + c = \frac{i(1-i)}{z(1+i)(1-i)} + c$$

$$= \frac{i(1-i)}{z(1-i^2)} + c = \frac{i(1-i)}{z(1+1)} + c$$

$$= \frac{i - i^2}{z(1+1)} + c = \frac{i - i^2}{z(1+1)} + c = \frac{i+1}{2z} + c$$

Given: $f(1) = 1$

$$1 = \frac{i+1}{2} + c$$

$$c = 1 - \frac{i+1}{2} = \frac{2 - i - 1}{2} = \frac{1-i}{2}$$

$$\boxed{\therefore\ f(z) = \frac{1+i}{2z} + \frac{1-i}{2}}$$

<u>Exercise 5.3</u>

1. Determine the analytic function whose real part is $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

   Ans : $f(z) = \tan z + c$

2. If $f(z) = u + iv$ is an analytic function, find $f(z)$ if $v = \log(x^2 + y^2) + x - 2y$.

Ans : $f(z)=(i-2)z+2i\log z+c$

3. If w=u+iv is an analytic function and $v=x^2-y^2+\dfrac{x}{x^2+y^2}$, find u.

Ans : $\mathrm{u}=-2\mathrm{xy}+\dfrac{\mathrm{y}}{\mathrm{x}^2+\mathrm{y}^2}+\mathrm{c}$

4. Verify if the function $e^{-2x}\cos 2y$ can be the real / imaginary part of an analytic function.

5.4 APPLICATION TO FLOW PROBLEMS:

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy-plane, the velocity V of a fluid particle can be expressed as

$$V=v_xI+v_yJ \qquad \ldots(1)$$

Since the motion is irrotational, there exist a scalar function $\phi(x,y)$ such that

$$V=\nabla\phi(x,y)=\frac{\partial\phi}{\partial x}I+\frac{\partial\phi}{\partial y}J \qquad \ldots(2)$$

[The function $\phi(x,y)$ is called the velocity potential and the curves $\phi(x,y)=\mathrm{c}$ are known as equipotential lines]

Thus from (1) and (2),

$$v_x=\frac{\partial\phi}{\partial x} \text{ and } v_y=\frac{\partial\phi}{\partial y} \qquad \ldots(3)$$

Also the fluid being incompressible $divV=0$ (i.e.,) $\dfrac{\partial v_x}{\partial x}+\dfrac{\partial v_y}{\partial y}=0$

Substituting the value of v_x and v_y from (3), we get $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$

Which shows that the velocity potential ϕ is harmonic. It follows that there must exist a conjugate harmonic function $\psi(x,y)$, such that $w(z)=\phi(x,y)+iw(x,y)$ …(4) is analytic.

Also the slope at any point of the curve $\psi(x,y)=\mathrm{c}'$ is given by

$$\frac{dy}{dx}=-\frac{\partial\psi/\partial x}{\partial\psi/\partial y}=\frac{\partial\phi/\partial y}{\partial\phi/\partial x}=\frac{v_y}{v_x} \qquad \text{By(3)}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x,y)=c'$, i.e., the particle moves along this curve. Such curves are known as stream lines and $\psi(x,y)$ is called the stream function. Also the equipotential line $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From(4) $$\frac{\partial w}{\partial z} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

$$= v_x - iv_y$$

$\therefore$ The magnitude of the fluid velocity $= \sqrt{v_x^2 + v_y^2} = \left|\frac{\partial w}{\partial z}\right|$

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the complex potential. Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are equipotential lines and lines of force. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as isothermals and heat flow lines respectively. Given $\phi(x, y)$, we can find $\psi(x, y)$ and vice versa.

Worked Examples 5.4(A)

Example 1

An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2 y - y^3$, Find the stream function.

Solution:

It is readily verified that ψ satisfies the laplace's equation

$\therefore$ ϕ and ψ must satisfy the Cauchy – Riemann equations

$$\text{(i)}\ \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}\ \text{and (ii)}\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial x}\left[3x^2 y - y^3\right] = 6xy$$

Integrating w.r.t. y we get $\psi = 3xy^2$ where $\eta(y)$ is an arbitrary function of y.

(ii) gives $-3x^2 + 3y^2 + \eta'(y) = 3(y^2 - x^2)$

Where $\eta'(y) = 0$, i.e., $\eta(y) = c$, an arbitrary constant.

Example 2

An electrostatic field in the xy-plane is given by the potential function $\phi = 3xy^2 - x^3$, Find the stream function.

Solution:

It is readily verified that ψ satisfies the laplace's equation

$\therefore$ ϕ and ψ must satisfy the Cauchy – Riemann equations

$$\text{(i) } \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and (ii) } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial x}\left[3xy^2 - x^3\right] = 3y^2 - 3x^2$$

Integrating w.r.t. y we get $\psi = y^3 - 3x^2 y$ where $\eta(y)$ is an arbitrary function of y.

(ii) gives $-6xy + \eta'(y) = -6xy$

Where $\eta'(y) = 0$, i.e., $\eta(y) = c$, an arbitrary constant

Worked Examples 5.4(B)

Example 1

If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, Determine the function ϕ

Solution:

It is readily verified that ψ satisfies the laplace's equation

$\therefore$ ϕ and ψ must satisfy the Cauchy – Riemann equations

$$\text{(i) } \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and (ii) } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left[x^2 - y^2 + \frac{x}{x^2 + y^2}\right] = -2y - \frac{2xy}{\left(x^2 + y^2\right)^2}$$

Integrating w.r.t. x we get $\phi = -2xy + \dfrac{y}{x^2 + y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y.

(ii) gives $-2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2} + \eta'(y) = -2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

Where $\eta'(y) = 0$, i.e., $\eta(y) = c$, an arbitrary constant.

Example 2

In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ: (i) $\psi = \dfrac{-y}{x^2 + y^2}$ (ii) $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$

Solution:

(i) It is readily verified that ψ satisfies the laplace's equation
$\therefore$ ϕ and ψ must satisfy the Cauchy – Riemann equations

$$(1)\ \frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}\ \text{and (2)}\ \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\phi}{\partial y}=-\frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right]=\frac{2xy}{(x^2+y^2)^2}$$

Integrating w.r.t. x we get $\phi=\frac{y}{x^2+y^2}+\eta(y)$ where $\eta(y)$ is an arbitrary function of y.

(1) gives $\frac{x^2-y^2}{(x^2+y^2)^2}+\eta'(y)=\frac{y^2-x^2}{(x^2+y^2)^2}$

Where $\eta'(y)=0$, i.e., $\eta(y)=c$, an arbitrary constant.

(ii) It is readily verified that ψ satisfies the laplace's equation

$\therefore\ \phi$ and ψ must satisfy the Cauchy – Riemann equations

$$(1)\ \frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}\ \text{and (2)}\ \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\phi}{\partial y}=-\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]=-\frac{1}{\frac{x^2+y^2}{x^2}}\times\frac{-y}{x^2}=\frac{y}{(x^2+y^2)}$$

Integrating w.r.t. x we get $\phi=\frac{1}{2}\log(\mathrm{x}^2+y^2)+\eta(y)$ where $\eta(y)$ is an arbitrary function of y.

(1) gives $\frac{x}{(x^2+y^2)}+\eta'(y)=\frac{x}{(x^2+y^2)}$

Where $\eta'(y)=0$, i.e., $\eta(y)=c$, an arbitrary constant.

<u>Exercise 5.4</u>

1.In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ:

(i) $\psi=\frac{x}{x^2+y^2}$ (ii) $\psi=y+e^x\cos y$

2. An electrostatic field in the xy-plane is given by the potential function $\phi=2x^2y^2-yx^3$, Find the stream function.

5.5 CONFORMAL MAPPING

A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is said to be conformal at that point. i.e., A mapping $\mathrm{w}=\mathrm{f}(\mathrm{z})$ is said to be conformal at $\mathrm{z}=\mathrm{z}_0$, if $\mathrm{f}'(\mathrm{z}_0)\neq 0$.

If the transformation $\mathrm{w}=\mathrm{f}(\mathrm{z})$ is conformal at a point ,the inverse transformation $\mathrm{z}=\mathrm{f}^{-1}(\mathrm{w})$ is also conformal at the corresponding point.

Isogonal

A transformation under which angles between every pair of curves through the point are preserved in magnitude , but altered in sense is said to be an isogonal at that point.

Critical point

The point, at which mapping $w = f(z)$ is not conformal , i.e., $f'(z_0) = 0$ is called a critical point of the mapping.

The critical points of $z = f^{-1}(w)$ are given by $\frac{dz}{dw} = 0$. Hence the critical point of the transformation $w = f(z)$ are given by $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Invariant or Fixed point

Points of a mapping $w = f(z)$ that are mapped onto themselves are "kept fixed" under the mapping. Thus they are obtained from $w = f(z) = z$.

The identity mapping $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has infinitely many fixed points.

$w = \frac{1}{z}$ has two fixed points, a rotation has one and a translation has none in complex plane.

Some Standard transformations

1. **Translation:**

   The transformation $w = c + z$, where c is the complex constant, represents a translation.

   Let $z = x + iy$, $w = u + iv$, $c = a + ib$

   Given $w = c + z$

   $i.e)., u + iv = x + iy + a + ib$

   $u + iv = (x + a) + i(y + b)$

   Equating real and imaginary parts we get, $u = x + a,\ v = y + b$.

   These two equations are called as the transformation equations.

   The image of any point $p(x, y)$ in the z plane is mapped onto the point $p'(x + a, y + b)$ in the w-plane. Similarly every point in the z-plane is mapped onto the w-plane.

2. **Magnification:**

   The transformation $w = cz$, where c is a complex constant , represents magnification.

   $i.e)., u + iv = c(x + iy)$

   $u + iv = cx + icy$

   $i.e)., u = cx,\ v = cy$

   $\therefore$ The image of the point (x, y) is th point $(cx, c\,y)$.

   Hence the size of any figure in the z-plane is magnified c times in the w-plane.

3. **Magnification and Rotation:**

   The transformation $w = cz$, where c is a complex constant , represents both magnification and rotation.

   Let $z = re^{i\theta}$, $w = Re^{i\varphi}$ and $c = ae^{i\alpha}$

   Then $Re^{i\varphi} = (ae^{i\alpha})(re^{i\theta})$

   $Re^{i\varphi} = are^{i(\alpha+\theta)}$

   The transformation equations are $R = ar,\ \varphi = \theta + \alpha$.

   Thus the point (r, θ) in the z-plane is mapped onto the point $(ar, \theta+\alpha)$.

4. **Magnification ,Rotation and Translation:**

   The transformation $w = az + b$, where a and b are complex constant , represents magnification ,rotation and translation.

The transformation $w = az + b$ can be considered as the combination of two simple transformations $w_1 = az$ and $w = w_1 + b$

$w_1 = az$ represents magnification by $|a|$ and rotation through $amp(a)$.

$w = w_1 + b$ represents translation by vector representing b.

Thus any figure in z-plane will undergo magnification, rotation and translation by the transformation $w = az + b$.

5. Inversion and Reflection:

The transformation $w = \frac{1}{z}$ represents inversion w.r.to the unit circle $|z| = 1$, followed by reflection in real axis.

$$w = \frac{1}{z} \text{ or } z = \frac{1}{w}$$

$$x + iy = \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

$$i.e., \; x = \frac{u}{u^2 + v^2} ....(1) \text{ and } y = \frac{-v}{u^2 + v^2} ....(2)$$

W.K.T the general equation of circle in z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ---(3)$$

Substitute (1) and (2) in (3) we get

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + 2g\left(\frac{u}{u^2 + v^2}\right) + 2f\left(\frac{-v}{u^2 + v^2}\right) + c = 0$$

$i.e., c(u^2 + v)^2 + 2gu - 2fv + 1 = 0 \quad ---(4)$ which is the equation of circle in w-plane.

Hence under the transformation $w = \frac{1}{z}$ a circle in z-plane transformed to another circle in w-plane. When the circle passes through the origin we have c=0 in equation (3). When c=0 ,equation (4) gives a straight line.

<u>Worked Examples 5.5(A)</u>

Example 1

Find the points in the z-plane at which the mapping $w = z + z^{-1}$ fails to be conformal

Solution:

Given $w = z + \frac{1}{z}$

$$\frac{dw}{dz} = 1 - \frac{1}{z^2}$$

$$= \frac{z^2 - 1}{z^2}$$

$$\boxed{\frac{dw}{dz} = 0}$$

$$\Rightarrow \frac{z^2-1}{z^2}=0$$

$$\Rightarrow z^2-1=0$$

$$\Rightarrow z^2=1$$

$$z=\pm 1$$

$$\boxed{w=z+\frac{1}{z} \text{ is not conformal at } z=\pm 1}$$

Example 2

Find the image of the circle $|z|=3$ under the transformation $w=2z$.

Solution:

Given $|z|=3$

$$|w|=|2z|=2|z|$$

$$=2(3) \quad (\because |z|=3)$$

$$|w|=6$$

Hence, the image of circle $|z|=3$ in the Z-plane is transformed into circle $|w|=6$ in the w-plane under the transformation $w=2z$

Example 3

Find the image of the line x=1 under the transformation $w=z^2$

Solution:

Consider $w=z^2$

i.e., $u+iv=(x+iy)^2=x^2-y^2+2ixy$

Given $x=1$

$$u=x^2-y^2 \qquad v=2xy$$

$$\Rightarrow u=1-y^2 \qquad \Rightarrow v=2y$$

$$\Rightarrow u-1=-y^2 \qquad \Rightarrow v^2=4y^2$$

$$\Rightarrow y^2=1-u \qquad \Rightarrow v^2=4(1-u)$$

Hence, the image of a line $x=1$ in the Z-plane is transformed into a parabola $v^2=4(1-u)$ in the w-pane under the transformation $w = z^2$

Example 4

Find the image of $|z-10i|=2$ under the mapping w = z+1+i.

Solution:

Given $w^2=(z-\alpha)(z-\beta)$

Critical points occur at $\frac{dw}{dz}=0$

Differentiate with respect to z, we get

$$2w\frac{dw}{dz}=(z-\alpha)+(z-\beta)$$

$$=2z-(\alpha+\beta)----(1)$$

$$\frac{dw}{dz}=0 \Rightarrow 2z-(\alpha+\beta)=0$$

$$2z = \alpha + \beta$$

$$z = \frac{\alpha + \beta}{2}$$

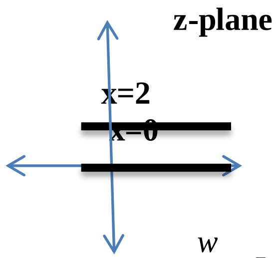
The critical points are $\boxed{z = \frac{\alpha + \beta}{2}}$

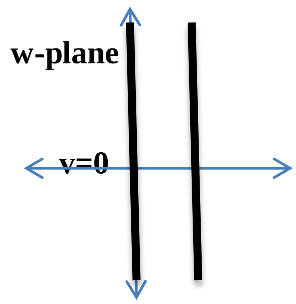
Example 5

Find the image of the $0 \le x \le 2$ under the transformation $w = iz$.

Solution:

Given $w = iz$





$$\frac{w}{i} = z$$

$$z = \frac{u + iv}{i}$$

$$z = \frac{iu - v}{-1} = \frac{v - iu}{1}$$

$$x + iy = v - iu$$

$$x = v\ ,\quad y = -u$$

$$\boxed{\begin{array}{l} given\ 0 \le x \le 2 \\ when\ \ x = 0, v = 0 \\ when\ \ x = 2, v = 2 \end{array}} \qquad \boxed{\Rightarrow 0 \le V \le 2}$$

Example 6

Find the critical points for the transformation $w^2 = (z - \alpha)(z - \beta)$

Solution:

Given $w^2 = (z - \alpha)(z - \beta)$

Critical points occur at $\frac{dw}{dz} = 0$

Differentiate with respect to z, we get

$$2w\frac{dw}{dz} = (z - \alpha) + (z - \beta)$$

$$= 2z - (\alpha + \beta) - - - - (1)$$

$$\frac{dw}{dz} = 0 \Rightarrow 2z - (\alpha + \beta) = 0$$

$$2z = \alpha + \beta$$

$$z = \frac{\alpha + \beta}{2}$$

The critical points are $\boxed{z = \frac{\alpha + \beta}{2}}$

Worked Examples 1.4 (B)

Example 1

Find the image of the circle $|z| = 2$ under the transformation $w = \sqrt{2}e^{i\pi/4}z$

.**Solution:**

Given $w = \sqrt{2}e^{i\pi/4}z$

$$u + iv = \sqrt{2}z(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

$$= \sqrt{2}z(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$$

$$= \sqrt{2}(x+iy)(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$$

$$= (x+iy)(1+i) = x + ix + iy - y$$

$$= (x-y) + i(x+y)$$

$$u = (x-y) \qquad v = x+y$$

$$x = \frac{u+v}{2} \qquad y = -\left(\frac{u-v}{2}\right) = \frac{v-u}{2}$$

Given $|z| = 2$

$i.e).,\ x^2 + y^2 = 4$

$$\left(\frac{u+v}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 = 4$$

$$\Rightarrow u^2 + v^2 + 2uv + u^2 + v^2 - 2uv = 16$$

$$\Rightarrow 2u^2 + 2v^2 = 16 \ \Rightarrow u^2 + v^2 = 8$$

Which is a circle in the w –plane.

$\therefore |z| = 2$ in the z plane maps to a circle in w-plane with centre at orgin and radius $2\sqrt{2}$

Example 2

Prove that the transformation $w = \dfrac{z}{1-z}$ maps the upper half of the z plane into the upper half of the w plane. What is the image of the circle $|z| = 1$ under this transformation?

Solution:

Given transformation is $w = \dfrac{z}{1-z}$

$$u + iv = \frac{x+iy}{1-(x+iy)} = \frac{x+iy}{1-x-iy}$$

$$= \frac{x+iy}{(1-x)-iy}\,\frac{(1-x)+iy}{(1-x)+iy}$$

$$= \frac{x(1-x) + ixy + i(1-x)y - y^2}{(1-x)^2 + y^2}$$

$$= \frac{x - x^2 - y^2 + iy}{(1-x)^2 + y^2}$$

$$\boxed{u = \frac{x - x^2 - y^2}{(1-x)^2 + y^2}, v = \frac{y}{(1-x)^2 + y^2}}$$

Upper half of the plane is y > 0,when y > 0 we have v > 0(as $(1-x)^2 + y^2$) Thus the upper half of the z-plane is mapped onto the upper half of the w-plane

To find the image of $|z| = 1$

$$|x + iy| = 1$$

$$x^2 + y^2 = 1$$

$$\therefore u = \frac{x - (x^2 + y^2)}{(1-x)^2 + y^2}$$

$$= \frac{x - (x^2 + y^2)}{1 + x^2 - 2x + y^2}$$

$$= \frac{x - (x^2 + y^2)}{1 - 2x + x^2 + y^2}$$

i.e., $$u = \frac{x - 1}{1 - 2x + 1}$$

$$u = \frac{x-1}{2-2x}$$

$$u = \frac{x-1}{-2(x-1)}$$

$$u = \frac{-1}{2} \text{ which is a straight line in the w - plane}$$

Example 3

Show that the transformation $w = 1/z$ transforms all circles and straight lines in the z-plane into circles (or) straight lines in the w-plane

Solution:

Given $w = \frac{1}{z}$ i.e., $z = \frac{1}{w}$

Now w = u + iv

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{1}{u+iv}\frac{u-iv}{u-iv} = \frac{u-iv}{u^2 - v^2}$$

$$x = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$\Rightarrow \boxed{x = \frac{u}{u^2+v^2} \quad and \quad y = \frac{-v}{u^2+v^2}}$$

The general equation is $a\left(x^2 + y^2\right) + 2gx + 2fy + c = 0$

$$a\left(\frac{u^2}{\left(u^2+v^2\right)^2} + \frac{v^2}{\left(u^2+v^2\right)^2}\right) + 2g\frac{u}{u^2+v^2} + 2f\frac{-v}{u^2+v^2} + c = 0$$

$$a\left(\frac{u^2+v^2}{\left(u^2+v^2\right)^2}\right)+2g\,\frac{u}{u^2+v^2}-2f\,\frac{v}{u^2+v^2}+c=0$$

The transformed equation is $\boxed{c\left(u^2+v^2\right)+2gu-2fv+a=0}$

i. $a \neq 0$, $c \neq 0$ $\Rightarrow$ circles not passing through the origin in z – plane map into circles not passing through the origin.
ii. $a \neq 0$, $c = 0$ $\Rightarrow$ circles through the origin in z – plane map onto straight lines not through the origin.
iii. $a = 0$, $c \neq 0$ $\Rightarrow$ straight lines not passing through the origin in z – plane map onto circles through the origin in w - plane.
iv. $a = 0$, $c = 0$ $\Rightarrow$ straight lines through the origin in z – plane map onto straight lines through the origin in w - plane.

Example 4

Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$.

Solution:

Given $w=\frac{1}{z}$

$$z=\frac{1}{w}$$

Now $w=u+iv$

$$w=\frac{1}{z}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

$$i.e.,\ \ x+iy=\frac{u-iv}{u^2+v^2}$$

$$i.e.,\ \ x=\frac{u}{u^2+v^2}\,.......(1)\ \ and\ \ y=\frac{-v}{u^2+v^2}\,.......(2)$$

$$Given: |z-2i|=2$$

$$|x+iy-2i|=2$$

$$|x+i(y-2)|=2$$

$$x^2+(y-2)^2=2$$

$$x^2+y^2-4y=0.......(3)$$

Substituting (1) and (2) in (3) we get

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left(\frac{-v}{u^2+v^2}\right)=0$$

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{v}{u^2+v^2}\right)^2+4\left(\frac{v}{u^2+v^2}\right)=0$$

$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$\frac{u^2+v^2(1+4v)}{(u^2+v^2)^2}=0$$

$$1+4v=0 \Rightarrow v=\frac{-1}{4} \text{ which is a straight line in w plane.}$$

Example 5

Find the image of the strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w=\frac{1}{z}$. Also show the regions graphically.

Solution:

$$Given \quad w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$$

$$z=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

$$x+iy=\frac{u-iv}{u^2+v^2}=\left[\frac{u}{u^2+v^2}\right]+i\left[\frac{-v}{u^2+v^2}\right]$$

$$i.e., \boxed{x=\frac{u}{u^2+v^2} \quad -----(1)}$$

$$\boxed{y=\frac{-v}{u^2+v^2} \quad -----(2)}$$

Given strip is $\frac{1}{4} \le y \le \frac{1}{2}$

$$when\ y=\frac{1}{4}, \quad \frac{1}{4}=\frac{-v}{u^2+v^2} \quad by(2)$$

$$u^2+v^2=-4v$$

$$u^2+v^2+4v=0$$

$$u^2+(v+2)^2-4=0$$

$$\boxed{u^2+(v+2)^2=4 \quad -----(3)}$$

Which is a circle whose centre is at (0,-2) in the w plane and radius is 2.

$$when\ y=\frac{1}{2}, \quad \frac{1}{2}=\frac{-v}{u^2+v^2} \quad by(2)$$

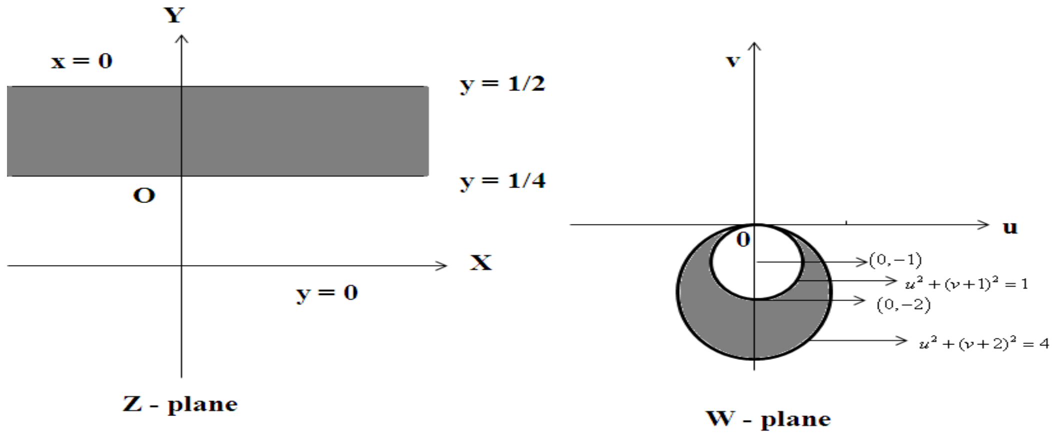
$$u^2+v^2=-2v$$

$$u^2+v^2+2v=0$$

$$u^2+(v+1)^2-1=0$$

$$\boxed{u^2+(v+1)^2=1 \quad -----(4)}$$

Which is a circle whose centre is at (0,-1) in the w plane and radius is 1.



Hence the given strip is $\frac{1}{4} \le y \le \frac{1}{2}$ transformed into the region inbetween circles. $\boxed{u^2+(v+1)^2=1}$ and $\boxed{u^2+(v+2)^2=4}$ in the w plane.

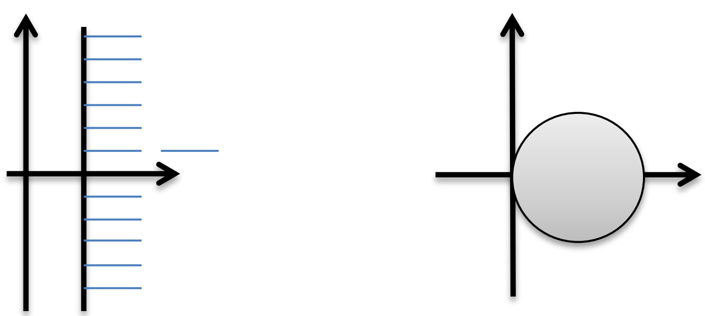
Example 6

Find the image of the half plane $x > c$ when $c > 0$ and the transform $\omega = 1/z$.

Solution:

The given transformation of

$$\omega = \frac{1}{z}, \quad z = \frac{1}{\omega}$$



z-plane

w-plane

$$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

Given that $x > c$,

$$\frac{u}{u^2+v^2} > c$$

$$u > c(u^2+v^2)$$

$$\frac{u}{c} > (u^2+v^2)$$

$$u^2+v^2-\frac{u}{c} < 0$$

$$u^2+v^2-\frac{u}{c}+\left(\frac{1}{2c}\right)^2<\left(\frac{1}{2c}\right)^2$$

$$\boxed{\therefore\left(u-\frac{1}{2c}\right)^2+v^2<\left(\frac{1}{2c}\right)^2}$$

The image of the half plane x>c is transformed to circle with centre $\left(\frac{1}{2c},0\right)$ and the radius $\frac{1}{2c}$ under the transformation $w=\frac{1}{z}$.

Exercise 5.5

1. What will be the image of a circle passing through the origin in the XY plane under the transformation $w=\frac{1}{z}$?

2. Find the image of $|z+1|=1$ under the map w=$\frac{1}{z}$.

3. Determine the image of 1<x<2 under the mapping $w=\frac{1}{z}$.

4. Find the image of the circle $|z+1|=2$ under the transformation $w=3z$

5. Draw the image of the square whose vertices are at $(0,0),(1,0),(1,1),(0,1)$ in the z-plane under the transformation $w=(1+i)z$. What has this transformation done to the original square?

5.6 BILINEAR TRANSFORMATION

The transformation $w=\frac{az+b}{cz+d}$, $ad-bc\neq 0$ where a, b, c and d are complex numbers , is called bilinear transformation.

It was first introduced by A.F.Mobius , so it is also called as Mobius transformation. And it is also known as linear fractional transformation.

Fixed points (or) Invariant points

The fixed points of the transformation $w=\frac{az+b}{cz+d}$ is obtained from $z=\frac{az+b}{cz+d}$ (or) $cz^2+(d-a)z-b=0$.

These points are two in number unless the discriminant is zero in which case the number of points is one.

Cross ratio

Let the four points be z_1,z_2,z_3,z_4 in this order, then the ratio of $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

Critical Point of Bilinear Transformation

The bilinear transformation $w=\frac{az+b}{cz+d}$ is conformal only when $\frac{dw}{dz}\neq 0$

$$i.e., \frac{ad-bc}{(cz+d)^2}\neq 0$$

$$i.e., ad-bc\neq 0$$

$i.e.,$ if $\boxed{ad-bc=0}$ then every point in the z – plane is a critical point.

Note:1 The bilinear transformation which transforms

z_1, z_2, z_3 into w_1, w_2, w_3 *is* $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Note:2 The bilinear transformation which transforms

z_1, z_2, z_3, z_4 into w_1, w_2, w_3, w_4 *is* $\dfrac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

Worked Examples 5.6(A)

Example 1

Find the fixed points of the transformation $w = \dfrac{2z+6}{z+7}$.

Solution:

The fixed points are given by replacing w = z.

$$i.e., \ w = \frac{2z+6}{z+7} \Rightarrow z = \frac{2z+6}{z+7}$$

$$z^2 + 7z = 2z + 6$$

$$z^2 + 5z - 6 = 0$$

$$(z-1)\ (z-6) = 0$$

$$\boxed{z = 1, 6}$$

The fixed points are 1, 6.

Example 2

Find the points where the function ceases to be analytic f (z) = $\dfrac{z^2-4}{z^2+1}$.

Solution:

$$f'(z) = \frac{(z^2+1)(2z)-(z^2-4)(2z)}{(z^2+1)^2}$$

$$= \frac{2z[z^2+1-z^2+4]}{(z^2+1)^2}$$

$$= \frac{2z(5)}{(z^2+1)^2} = \frac{10z}{(z^2+1)^2}$$

Ceases to be analytic $\Rightarrow f'(z) \to \infty$

$\Rightarrow$ Denominator of $f'(z) = 0$

$$\therefore (z^2+1)^2 = 0$$

$$z^2 + 1 = 0$$

$$z^2 = -1$$

$$\boxed{z = \pm\, i}$$

Example 3

Find the invariant point of transformation $w = \dfrac{1+iz}{1-iz}$.

Solution:

Given $w = \dfrac{1+iz}{1-iz}$

$$z = \frac{1+iz}{1-iz}$$

On cross multiplying we get

$$z - iz^2 = iz + 1$$

$$iz^2 + (i-1)z + 1 = 0$$

$$\Rightarrow z = \frac{(1-i) \pm \sqrt{(1-i)^2 - 4i}}{2i} = \frac{(1-i) \pm \sqrt{(-6i)}}{2i}$$

$$\Rightarrow z = \frac{(1-i) \pm \sqrt{(-6i)}}{2i}$$

The points are

$$\Rightarrow \boxed{z = \frac{(1-i) \pm \sqrt{(-6i)}}{2i}}$$

Example 4

Find the fixed points under the transformation $w = \frac{z-1}{z+1}$

Solution:

Given $w = \frac{z-1}{z+1}$

$$z = \frac{z-1}{z+1}$$

$$z(z+1) = z - 1$$

$$z^2 + z = z - 1$$

$$z^2 = -1 \Rightarrow z = \pm i$$

The fixed point is $\boxed{z = \pm i}$

Example 5

Find the invariant point of $f(z) = z^2$.

Solution:

Given $f(z) = z^2$.

$$w = z^2.$$

$$z = z^2$$

$$z^2 - z = 0$$

$$z(z-1) = 0$$

$$\boxed{z = 0,1}$$

The invariant points are 0,1.

Example 6

Prove that a bilinear transformation has at least two fixed points.

Solution:

The equation $z = \frac{az+b}{cz+d}$ can be written as

$$cz^2 + dz = az + b$$
$$cz^2 + dz - az - b = 0,$$

and this quadratic equation has, at most, two distinct solutions:

$$z = \frac{a - d + \sqrt{(d-a)^2 + 4bc}}{2c} \quad \text{and}$$

$$\boxed{z = \frac{a - d - \sqrt{(d-a)^2 + 4bc}}{2c}}$$

Example 7

Find the invariant points of the mapping $f(z) = \dfrac{z^3 + 7z}{7 - 6zi}$

Solution:

Given $f(z) = \dfrac{z^3 + 7z}{7 - 6zi}$

$$z = \frac{z^3 + 7z}{7 - 6zi}$$
$$z(7 - 6zi) = z^3 + 7z$$
$$z^3 + 6z^2 i = 0$$
$$z^2(z + 6i) = 0$$
$$z = 0, z = 0, z = -6i$$

Example 8

Find the invariant point of $w = \dfrac{z}{z^2 - 2}$

Solution:

Given $w = \dfrac{z}{z^2 - 2}$

$$z = \frac{z}{z^2 - 2}$$
$$1 = \frac{1}{z^2 - 2}$$

$$z^2 - 2 = 1$$
$$z^2 - 3 = 0$$
$$\Rightarrow z = \sqrt{3}, -\sqrt{3}$$

Worked Examples 5.6(B)

Example 1

Find the bilinear transformation that maps the points 1+i, -i, 2-i of the z-plane into the points 0, 1, i of the w-plane

Solution:

Let $z_1 = 1 + i$; $\quad z_2 = -i$; $\quad z_3 = 2 - i$.

$w_1 = 0; \qquad w_2 = 1; \qquad w_3 = i.$

The required transformation is given by

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\Rightarrow \frac{(z-1-i)(-i-2+i)}{(1+2i)(2-i-z)} = \frac{(w-0)(1-i)}{(0-1)(i-w)}$$

$$\Rightarrow \frac{(z-1-i)(-2)}{(1+2i)(2-i-z)(1-i)} = \frac{w}{w-i}$$

$$\Rightarrow \frac{(1+2i)(2-i-z)(1-i)}{(z-1-i)(-2)} = \frac{w-i}{w}$$

$$\Rightarrow \frac{(7-i-3z-zi)}{(-2z+2+2i)} = 1-\frac{i}{w}$$

$$\Rightarrow \frac{i}{w} = 1-\frac{(7-i-3z-zi)}{(-2z+2+2i)}$$

$$\Rightarrow \frac{i}{w} = \frac{z-5+3i+zi}{(-2z+2+2i)}$$

$$\Rightarrow w = i\left(\frac{-2z+2+2i}{z-5+3i+zi}\right)$$

$$\Rightarrow w = \left(\frac{-2+(2-2z)i}{(z-5)+(3+z)i}\right)$$

Example 2

Find the bilinear transformation that maps the points 0, -1, i onto the points 1, 0, ∞

Solution:

The bilinear transformation is given by

Given: $z_1 = 0, \quad z_2 = -1, \qquad z_3 = i$

$w_1 = 1, \quad w_2 = 0, \quad w_3 = \infty$

Let the required transformation be

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$

$$\frac{(w-w_1)\,w_3(\frac{w_2}{w_3}-1)}{w_3(\frac{w}{w_3}-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)}{(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)}{(0-1)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$(w-1) = \frac{-z(1+i)}{(z-i)}$$

$$(w-1)(z-i)=-z(1+i)$$
$$wz-wi-z+i=-z-zi$$
$$w(z-i)=-z-zi+z-i$$
$$\boxed{w=\frac{-i(1+z)}{z-i}}$$ is the required transformation.

Example 3

Find the bilinear transformation that maps the points 1, i, -1 in the z plane onto the points i, 0, -i in the w plane. Hence find the image of $|\mathbf{z}|<\mathbf{1}$.

Solution:

Given $z_1=1, z_2=i, z_3=-1$

$w_1=i, w_2=0, w_3=-i,$

The bilinear transformation is given by

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$
$$\frac{(w-i)(0+i)}{(w+i)(0-i)}=\frac{(z-1)(i+1)}{(z+1)(i-1)}$$
$$\frac{(w-i)i}{(w+i)(-i)}=\frac{(z-1)(i+1)}{(z+1)(i-1)}$$
$$\frac{(i-w)}{(i+w)}=\frac{(z-1)(i+1)}{(z+1)(i-1)}$$
$$\frac{i-w}{i+w}=\frac{zi+z-i-1}{iz-z+i-1}$$

Using the fact $\frac{A}{B}=\frac{C}{D}\Rightarrow\frac{A+B}{A-B}=\frac{C+D}{C-D}$

$$\Rightarrow\frac{i-w+i+w}{i-w-i-w}=\frac{zi+z-i-1+iz-z+i-1}{zi+z-i-1-iz+z-i+1}$$
$$\frac{2i}{-2w}=\frac{2iz-2}{2z-2i}$$
$$\frac{i}{-w}=\frac{iz-1}{z-i}$$
$$\frac{-w}{i}=\frac{z-i}{iz-1}$$
$$\frac{w}{i}=\frac{-z+i}{iz-1}=\frac{i-z}{iz-1}$$
$$w=\frac{-1-iz}{-1+iz}=\frac{1+iz}{1-iz}$$

$\boxed{w=\frac{1+iz}{1-iz}}$ To find the image of $|z|<1$

$$w=\frac{1+iz}{1-iz}$$
$$w-wiz=1+iz$$
$$ziw+iz=w-1$$

$z(iw+i) = w-1$

$z = \dfrac{w-1}{i(w+1)}$ i.e) $|z|<1$

$\boxed{\left|\dfrac{w-1}{i(w+1)}\right|<1} = |w-1|<|w+1|$

$|u+iv-1|<|u+iv+1|$

$|(u-1)+iv|<|(u+1)+iv|$

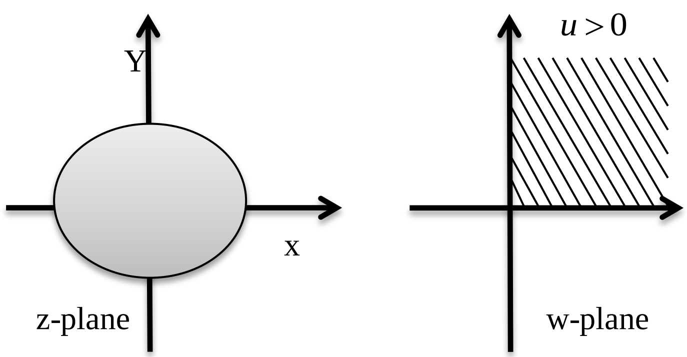
$(u-1)^2+v^2<(u+1)^2+v^2$

$1-2u+u^2+v^2<1+2u+u^2+v^2$

$-2u<2u$

$0<4u$

$\boxed{\therefore 4u>0}$



The interior of the unit circle $|z|=1$ maps into the half plane u > 0 of the w plane.

Example 4

Find the bilinear transformation which maps the points 1, i, -1 onto the points 0, 1, ∞, show that the transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane

Solution:

Given: $Z_1=1$, $Z_2 = i$, $Z_3 = -1$ and $W_1 = 0$, $W_2 = 1$, $W_3 = \infty$

Let the transformation be

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}} = \boxed{\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$

$$\frac{(w-w_1)w_3(\frac{w_2}{w_3}-1)}{w_3(\frac{w}{w_3}-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)w_3(\frac{w_2}{w_3}-1)}{(\frac{w}{w_3}-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w=\frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w=i\frac{(z-1)}{(z+1)}$$

$$w=i\frac{(z-1)}{(z+1)} \Rightarrow w(z+1)=i(z-1)$$

$$wz+w-iz+i=0$$

$$z(w-i)+(w+i)=0$$

$$z=\frac{-(w-i)}{(w+i)}$$

Thus the region |z|<1 gives onto the region $\left|\frac{w-i}{w+i}<1\right|$

Let $w=u+iv$ we get $\left|\frac{u+iv-i}{u+iv+i}\right|<1$

$$|u+iv-i|<|u+iv+i|$$

$$|u+i(v-1)|<|u+i(v+1)|$$

$$u^2+(v-1)^2<u^2+(v+1)^2$$

$$(v-1)^2<(v+1)^2$$

$$v^2-2v+1<v^2+2v+1$$

$$\boxed{\begin{array}{l}-4v<0\\ i.e., v>0\end{array}}$$

Hence the interior of unit circle is mapped onto the upper half of the w plane.

Example 5

Find the bilinear transformation which maps the points -1, 0, 1 of the z-plane onto the points -1, -i, 1 of the w-plane respectively. Show that under this transformation the upper half of the z-plane maps on to the interior of the unit circle $|W|=\mathbf{1}$

Solution:

Given $z_1=-1, z_2=0, z_3=1$ and $w_1=-1, w_2=-i, w_3=1$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)((z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w+1)(-i-1)}{(w-1)(-i+1)}=\frac{(z+1)(0-1)}{(z-1)(0+1)}$$

$$\Rightarrow \frac{(w+1)(-i-1)}{(w-1)(-i+1)}=\frac{-(z+1)}{(z-1)}$$

$$\Rightarrow \frac{iw-i-w-1}{-iw+i+w-1}=\frac{-z-1}{z-1}$$

$$\Rightarrow (z-1)(-iw-i-w-1)=(-z-1)(-iw+i+iw-1)$$

$$\Rightarrow -iwz-iz-wz-z+iw+i+w+1=iwz-iz-wz+z+iw-i-w+1$$

$$\Rightarrow -iwz-wz+iw+w-iwz+wz-iw+w=iz+z-i-1-iz+z-i+1$$

$$\Rightarrow -2iwz+2w=2z-2i$$

$$\Rightarrow 2w(1-iz)=2(z-i)$$

$\therefore w = \dfrac{z-i}{1-iz}$ which is the required bilinear transformation.

Example 6

Find the bilinear transformation which maps the points $z = 0, 1, -1$ onto the points $w = -1, 0, \infty$. Find also the invariant points of the transformation

Solution:

Given $w_1 = -1, w_2 = -0, w_3 = \infty$ $z_1 = 0, z_2 = 1, z_3 = -1$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)((z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w+1)(0-\infty)}{(w-\infty)(0+1)} = \frac{(z-0)(1+1)}{(z+1)(1-0)}$$

$$\Rightarrow \frac{(w+1)}{(w-\infty)(0+1)} = \frac{(z-0)(1+1)}{(z+1)(1-0)}$$

$$\Rightarrow \frac{w+1}{0+1} = \frac{z(2)}{(z+1)(1)}$$

$$\Rightarrow \frac{w+1}{1} = \frac{2z}{(z+1)}$$

$$\Rightarrow w+1 = \frac{2z}{z+1}$$

$$\Rightarrow w = \frac{2z}{z+1} - 1$$

$$\Rightarrow w = \frac{2z-z-z}{z+1}$$

$\therefore w = \dfrac{z-1}{z+1}$ which is the required bilinear transformation.

Example 7

Find the bilinear transformation that maps the points ∞, 2, -1 onto the points 1, ∞ and 0 respectively.

Solution:

Given : $z_1 = \infty,\ z_2 = 2,\ z_3 = -1$ and $w_1 = 1,\ w_2 = \infty,\ w_3 = 0$

Let the required transformation be

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$

$$\frac{(w-w_1)(1-\frac{w_3}{w_2})}{(w-w_3)(1-\frac{w_1}{w_2})} = \frac{(\frac{z}{z_1}-1)(z_2-z_3)}{(z-z_3)(\frac{z_2}{z_1}-1)}$$

$$\frac{(w-1)(1-\frac{0}{\infty})}{(w-0)(1-\frac{1}{\infty})} = \frac{(\frac{z}{\infty}-1)(2+1)}{(z+1)(\frac{2}{\infty}-1)}$$

$$\frac{(w-1)}{(w-0)}=\frac{-3}{(z+1)(-1)}$$
$$(w-1)(-z-1)=-3w$$
$$-wz+z-w+1=-3w$$
$$3w-w-wz=-z-1$$
$$w(3-1-z)=-(z+1)$$
$\therefore w=\dfrac{-(z+1)}{(2-z)}$ which is the required bilinear transformation.

Exercise 5.6

1. Find the bilinear transformation that maps the points 1, i, -1 onto the points 2, i, -2.

   Ans: $w=2\left(\dfrac{3z(1+i)+(1-i)}{3(1+i)+z(1-i)}\right)$
2. Determine the bilinear transformation that maps the points -1, 0, i in the z plane onto the points -1, i, 1 in the w plane. **Ans:** $w=\dfrac{z+i}{z+1}$
3. Find the bilinear transformation that maps the points $1,-i,0$ onto the points $0,i,\infty$

   Ans: $w=\dfrac{1+iz-z-i}{2z}$
4. Find the bilinear transformation which maps the points z=0, z=1 and z=∞ into the points w=i, w=1 and w=-i. **Ans:** $w=\dfrac{iz-1}{(i-z)}$
5. Find the bilinear transformation that maps the points $\infty,i,0$ onto the points $0,i,\infty$.

   Ans: $w=-\dfrac{1}{z}$