18MAB102T

UNIT -2

VECTOR CALCULUS

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SYLLABUS

Gradient and Directional derivative - Divergence and curl – Vector Identities- Irrotational and Solenoidal vector fields –Line integral over a plane curve – Surface integral- Area of a curved surface- Volume integral- Green’s , Gauss divergence and Stoke’s theorems – Verification and application in evaluating line, surface and volume integrals.

YOU TUBE, NPTEL VIDEOS, WEBSITES

Websites:

http://www-math.mit.edu/~djk/18_022/chapter10/section01.html

www.math.oregonstate.edu/home/programs/undergrad/.../so_var.html

www.sosmath.com/diffeq/laplace/basic/basic.html

www.jstor.org/stable/2321164

http://mathworld.wolfram.com/CauchyIntegralTheorem.html

Videos:

- http://nptel.ac.in/video.php?subjectId=122104017
- http://nptel.ac.in/video.php?subjectId=122107037
- http://www.youtube.com/watch?v=kbp9qWS-Bsk
- http://nptel.ac.in/courses/111103070/26
- http://nptel.ac.in/courses/111103070/39

Vector calculus –Topic

1. Gradient
2. Unit normal vector
3. Directional derivative
4. Angle between the surfaces
5. Divergence and curl
6. Vector Identities
7. Irrotational and Solenoidal vector fields
8. Vector integration,Line and surface integrals
9. Green's theorem in a plane
10. Gauss divergence theorem
11. Stoke’s theorem
12. Application in evaluating line, surface and volume integrals.

APPLICATION OF VECTOR CALCULUS:

Vector calculus is used in engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

1.GRADIENT

Definition of $grad\,\phi$ and its Geometrical meaning :

Let $\phi(x,y,z)$ be a scalar point function and is continuously differentiable. Then the vector

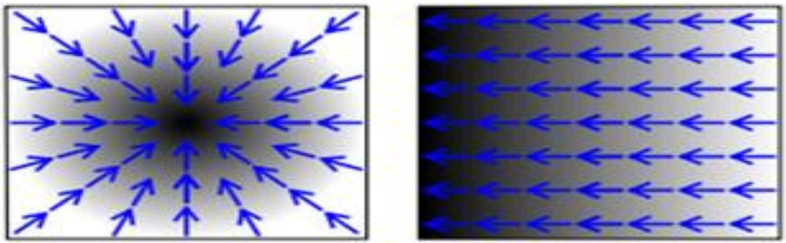
$$\nabla\phi=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

is called the gradient of the scalar function ϕ and is written as $Grad\,\phi=\nabla\phi$. Geometrical meaning of $grad\,\phi$ is the vector perpendicular to the level surface ϕ at the point.

Note:

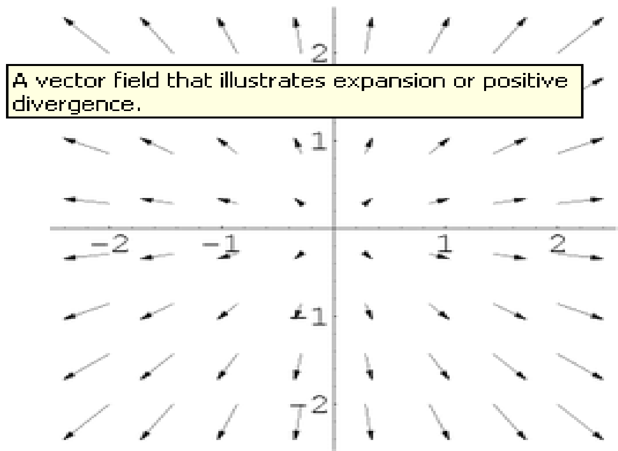
- The gradient represents the direction of greatest change
- The gradient perpendicular to lines of equal potential
- Lines of equal potential ("equipotential") are the points with the same energy (or value for F(x,y,z)). In the simplest case, a circle represents all items the same distance from the center.

DIAGRAM REPRESENTATION OF GRADIENT



In the above two images, the values of the function are represented in black and white, black representing higher values, and its corresponding gradient is represented by blue arrows.

Gradient represents velocity of some fluid flow



EXAMPLE:

1.**Find** $\nabla\phi$ **if** $\phi = x^2 + y^2 + z^2$ **at (1,-1,1).**

Solution:

Given $\phi = x^2 + y^2 + z^2$

W.K.T $\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$

$$= \vec{i}\,\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \vec{j}\,\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \vec{k}\,\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\left(\nabla\phi\right)_{(1,-1,1)} = 2(1)\vec{i} + 2(-1)\vec{j} + 2(1)\vec{k} \quad = 2\vec{i} - 2\vec{j} + 2\vec{k}$$

2. Prove that $\nabla r^n = nr^{n-2}\vec{r}$

Solution: $\nabla\left(r^n\right) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(r^n)$

$$= \sum \vec{i}\frac{\partial}{\partial x}\left(r^n\right) = \sum \vec{i}\, n\, r^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\, n\, r^{n-1}\frac{x}{r} = \sum \vec{i}\, n\, r^{n-2} x \qquad \because \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= n\, r^{n-2}\sum x\vec{i} = n\, r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= n\, r^{n-2}\vec{r} \qquad \because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\therefore\ \nabla\left(r^n\right) = nr^{n-2}\vec{r}$$

2. UNIT NORMAL VECTOR

If $\varphi(x, y, z)$ be a scalar function, then $\varphi(x, y, z) = c$ represents a surface and the unit normal vector to the surface φ is given by $\dfrac{\nabla\varphi}{|\nabla\varphi|}$.

It is denoted by Unit normal vector $= \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

EXAMPLE:

1. Find the unit normal to the surface of the sphere $x^2 + y^2 + z^2 = 1$ **.**

Solution:

Given: $\phi = x^2 + y^2 + z^2 - 1$

W.K.T $\quad \nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$

$$= \vec{i}\,\frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 1) + \vec{j}\,\frac{\partial}{\partial y}(x^2 + y^2 + z^2 - 1) + \vec{k}\,\frac{\partial}{\partial z}(x^2 + y^2 + z^2 - 1)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla \phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}$$

$$\therefore \text{ Unit normal vector } = n = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

2. Find the unit normal to the surface of the sphere $x^2 + y^2 - 2z + 3 = 0$ at (1,2,-1)

Solution:

Given $\phi = x^2 + y^2 - 2z + 3$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} - 2\vec{k}$$

$$\nabla\phi_{(1,2,-1)} = 2\vec{i} + 4\vec{j} - 2\vec{k}$$

$$|\nabla\phi| = \sqrt{4+16+4} = \sqrt{24}$$

$$\therefore \text{ Unit normal vector } = n = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2\vec{i} + 4\vec{j} - 2\vec{k}}{\sqrt{24}}$$

$$\therefore \quad \boldsymbol{\hat{n}} = \frac{\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}}{\sqrt{\boldsymbol{6}}}$$

3. DIRECTIONAL DERIVATIVE

The derivative of a point function in a particular direction is called its directional derivative along the direction.

That is directional derivative of a scalar point function $\varphi(x,y,z)$ in the direction $\vec{a}$ is given by $\nabla\varphi.\dfrac{\vec{a}}{|\vec{a}|}$

It is denoted by $\text{Directional Derivative} = \nabla\phi.\dfrac{\vec{a}}{|\vec{a}|}$

EXAMPLE:

1. Find the directional derivative of $\phi = xyz$ at the point (1,1,1) in the direction of $\vec{i} + \vec{j} + \vec{k}$.

Solution:

Given: $\phi = xyz$

and $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$$\therefore \left|\vec{a}\right| = \sqrt{1+1+1} = \sqrt{3}$$

WKT $\nabla \phi = \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z}$

$$\nabla \phi = (yz)\vec{i} + (xz)\,\vec{j} + (xy)\vec{k}$$

$$(\nabla \phi)_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$$

$$\therefore \text{Directional Derivative} = \nabla \phi . \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i} + \vec{j} + \vec{k}) . \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}(1 + 1 + 1) = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\therefore \text{D. D} = \sqrt{3}$$

ıd the directional derivative of $\varphi = x^2yz + 4x^2z^2$ at the P(1, -2, -1) in the direction of PQ, where Q is (3, -3, -2).

Solution:

Given: $\phi = \left(x^2yz + 4x^2z^2\right)$

$$\text{WKT} \quad \nabla\varphi = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\left(x^2yz + 4x^2z^2\right)$$

$$= \vec{i}\,(2xyz + 8xz^2) + \vec{j}\,(x^2z) + \vec{k}\,(x^2y + 8x^2z)$$

$$(\nabla\phi)_{(1,-2,-1)} = \vec{i}\left[2(1)(-2)(-1) + 8(1)(-1)^2\right] + \vec{j}\left[(1)^2(-1)\right] + \vec{k}\left[(1)^2(-2) + 8(1)^2(-1)\right]$$

$$\nabla\varphi = 12\vec{i} - \vec{j} - 10\vec{k}$$

Given : P(1, -2, -1) and Q (3,-3,-2)

$$\overline{PQ} = \overline{OQ} - \overline{OP}$$

$$= (3\vec{i} - 3\vec{j} - 2\vec{k}) - (\vec{i} - 2\vec{j} - \vec{k}) = 2\vec{i} - \vec{j} - \vec{k}$$

$$\overline{PQ} = 2\vec{i} - \vec{j} - \vec{k}$$

$$\left|\overline{PQ}\right| = \sqrt{4+1+1} = \sqrt{6}$$

Directional derivative of ϕ

in the direction of $\overline{PQ} = \dfrac{\nabla\varphi.\overline{PQ}}{\left|\overline{PQ}\right|}$

$$= \frac{(12\vec{i} - \vec{j} - 10\vec{k}).(2\vec{i} - \vec{j} - \vec{k})}{\sqrt{6}}$$

$$= \frac{24+1+10}{\sqrt{6}}$$

$= \dfrac{35}{\sqrt{6}}$ units

4. ANGLE BETWEEN THE SURFACES

EXAMPLE:

1. Find the angle between the normal to the surface $x^2 = yz$ **at the points** $(1,1,1)$ **and** $(2,4,1)$**.**

Solution:

Let $\varphi = yz - x^2$

$\nabla\varphi$ is normal to the surface $\varphi(x, y, z) = c$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,\frac{\partial(yz - x^2)}{\partial x} + \vec{j}\,\frac{\partial(yz - x^2)}{\partial y} + \vec{k}\,\frac{\partial(yz - x^2)}{\partial z}$$

$$\nabla\varphi = -\vec{i}\,2x + \vec{j}z + \vec{k}y$$

$$\nabla\varphi_{1(1,1,1)} = -2\vec{i} + \vec{j} + \vec{k}$$

$$\nabla\varphi_{2(2,4,1)} = -4\vec{i} + \vec{j} + 4\vec{k}$$

Let θ be the angle between the normals.

Then $\cos\theta = \dfrac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

$$= \frac{(-2\vec{i} + \vec{j} + \vec{k}).(-4\vec{i} + \vec{j} + 4\vec{k})}{\sqrt{4+1+1}\sqrt{16+1+16}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}$$

$$= \frac{13}{\sqrt{3\times 2}\sqrt{11\times 3}} = \frac{13}{3\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$$

2. Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - xz = 18$ at the point (6, 4, 3)

Solution:

Given $\phi_1 = x^2 - y^2 - z^2 - 11$

$$\nabla\varphi_1 = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial k}\right)\left(x^2 - y^2 - z^2 - 11\right)$$

$$= 2x\,\vec{i} - 2y\,\vec{j} - 2z\,\vec{k}$$

$$\left(\nabla\varphi_1\right)_{(6,4,3)} = 12\,\vec{i} - 8\,\vec{j} - 6\,\vec{k} \qquad .....(1)$$

\

$$\left|\nabla\phi_1\right| = \sqrt{144 + 64 + 36}$$

$$= \sqrt{244}$$

Given $\phi_1 = xy + yz - xz - 18$

$$\nabla\varphi_2 = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial k}\right)(xy + yz - xz - 18)$$

$$= (y - z)\vec{i} + (x + z)\vec{j} + (y - x)\vec{k}$$

$$(\nabla\varphi_2)_{(6,4,3)} = 1\vec{i} + 9\vec{j} - 2\vec{k} \qquad .....(2)$$

$$|\nabla\phi_1| = \sqrt{1 + 81 + 4}$$

$$= \sqrt{86}$$

If θ is the angle between

the surfaces at (6,4,3) then $\cos\theta = \dfrac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$\cos\theta = \frac{\left(12\vec{i} - 8\vec{j} - 6\vec{k}\right).\left(1\vec{i} + 9\vec{j} - 2\vec{k}\right)}{\sqrt{244}\sqrt{86}}$$

$$= \frac{(12 - 72 + 12)}{2\sqrt{61} \times \sqrt{86}}$$

$$= \frac{-48}{2\sqrt{61} \times \sqrt{86}} = \frac{-24}{\sqrt{61 \times 86}}$$

$$\cos\theta = \frac{-24}{\sqrt{5246}}$$

$$\therefore\ \theta = \cos^{-1}\left[\frac{-24}{\sqrt{5246}}\right]$$

5. DIVERGENCE AND CURL

Divergence :

If $\vec{F}(x,y,z)$ is a differentiable vector function. Then the divergence of $\vec{F}$ is

$$\nabla.\vec{\mathrm{F}} = \vec{i}.\frac{\partial \vec{F}}{\partial x} + \vec{j}.\frac{\partial \vec{F}}{\partial y} + \vec{k}.\frac{\partial \vec{F}}{\partial z}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \because \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Curl:

If $\vec{F}(x,y,z)$ is a differentiable vector function. Then the curl of $\vec{F}$ is

$$\text{curl } \overline{F} = \nabla X\vec{F} = \vec{i}\times\frac{\partial\vec{F}}{\partial x}+\vec{j}\times\frac{\partial\vec{F}}{\partial y}+\vec{k}\times\frac{\partial\vec{F}}{\partial z}$$

EXAMPLE:

1. **Prove that** $div\ \vec{r} = 3\ and\ \ curl\ \vec{r} = 0$ **.**

   Solution:

   (i) To Prove that $div\ \vec{r} = 3$

$$\text{W.K.T } \vec{\mathrm{r}} = \mathrm{x}\vec{\mathrm{i}} + \mathrm{y}\vec{\mathrm{j}} + \mathrm{z}\vec{\mathrm{k}}$$

$$\nabla .\vec{r} = (\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}).(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= (\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z})$$

$$\therefore \nabla . \vec{r} = (1+1+1) = 3$$

(ii) To Prove that $curl\ \vec{r} = 0$

$$curl\,\vec{r} = \nabla \text{x} \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)$$

$$= \vec{i}\,0 - \vec{j}\,0 + \vec{k}\,0$$

$$\therefore \nabla \text{x} \vec{F} = 0$$

2. **If** $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ **, find** $div(curl\vec{F})$

Solution:

$$\text{W.K.T} \quad Curl\,\overline{F} = \nabla X\,\overline{F} = \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{pmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(\text{zx}) - \frac{\partial}{\partial z}(yz)\right) - \vec{j}\left(\frac{\partial}{\partial x}(\text{zx}) - \frac{\partial}{\partial z}(\text{xy})\right) + \vec{k}\left(\frac{\partial}{\partial x}(\text{yz}) - \frac{\partial}{\partial y}(\text{xy})\right)$$

$$= \vec{i}\,(0 - y) - \vec{j}(\text{z} - 0) + \vec{k}(0 - x)$$

$$= -y\vec{i} - z\vec{j} - x\vec{k}$$

$$= -(y\vec{i} + z\vec{j} + x\overrightarrow{k)}$$

$$\therefore div(curl\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).\left[-(y\vec{i} + z\vec{j} + x\vec{k})\right]$$

$$= -\left(\frac{\partial y}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial x}{\partial z}\right) = \vec{0}$$

7. IRROTATIONAL AND SOLENOIDAL VECTOR FIELDS

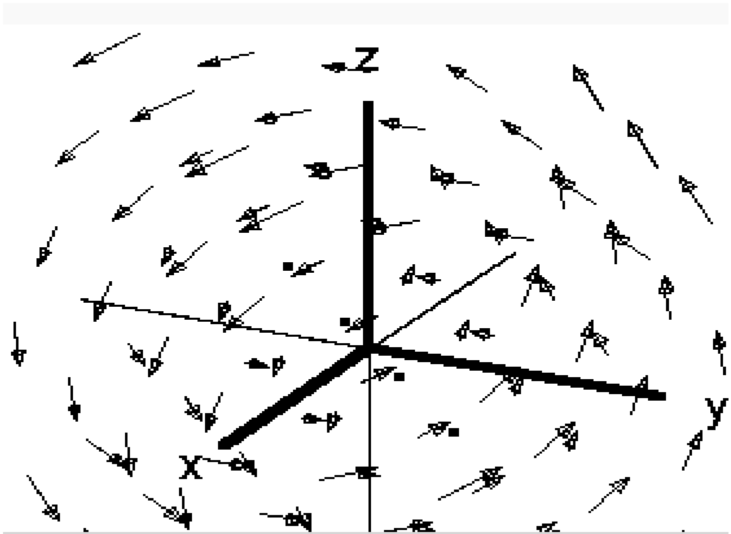
Solenoidal vectors:

If $\vec{F}$ is a vector point function such that $\nabla.\vec{F} = 0$ at all points in a given region , then it is said to be Solenoidal vector in that region.

Irrotational vectors:

If $\vec{F}$ is a vector point function such that $\nabla \times \vec{F} = 0$ at all points in a given region , then it is said to be Irrotational vector in that region.

ROTATION OF A FLUIDE FLOW



EXAMPLE:

1. Prove that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ **is irrotational.**

Solution:

Given $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

To prove that $\nabla \times \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(\mathrm{xy}) - \frac{\partial}{\partial z}(\mathrm{zx})\right) - \vec{j}\left(\frac{\partial}{\partial x}(\mathrm{xy}) - \frac{\partial}{\partial z}(\mathrm{yz})\right) + \vec{k}\left(\frac{\partial}{\partial x}(\mathrm{zx}) - \frac{\partial}{\partial y}(\mathrm{yz})\right)$$

$$= \vec{i}\,(x - x) - \vec{j}\,(y - y) + \vec{k}\,(z - z)$$

$$= \vec{0i} + 0\vec{j} + 0\vec{k} = 0$$

$$\nabla \times \vec{F} = \vec{0}$$

Hence $\vec{F}$ is irrotational.

2. Show that $\vec{F} = (x+2y)\vec{i} + (y+3z)\vec{j} + (x-2z)\vec{k}$ **is solenoidal.**

Solution:

Given $\vec{F} = (x+2y)\vec{i} + (y+3z)\vec{j} + (x-2z)\vec{k}$

To prove that $\nabla.\vec{F} = 0$

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left((x+2y)\vec{i} + (y+3z)\vec{j} + (x-2z)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x+2y) + \frac{\partial}{\partial y}(y+3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1 + 1 - 2$$

$$\nabla.\vec{F} = 0$$

Hence $\vec{F}$ is solenoidal.

3. Determine $f(r)$, so that the vector $f(r)\vec{r}$ is solenoidal

Solution:

We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$f(r)\ \vec{r} = f(r)\, x\vec{i} + f(r)\, y\vec{j} + f(r)\, z\vec{k}$$

Given: $f(r)\ \vec{r}$ is solenoidal.

$$\Rightarrow\ \nabla.[f(r)\ \vec{r}] = 0$$

$$\Rightarrow\ \nabla.[f(r)\ \vec{r}] = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).[f(r)\, x\vec{i} + f(r)\, y\vec{j} + f(r)\, z\vec{k}] = 0$$

$$\Rightarrow\ \frac{\partial}{\partial x}[xf(r)] + \frac{\partial}{\partial y}[yf(r)] + \frac{\partial}{\partial z}[zf(r)] = 0$$

$$\Rightarrow\ \sum \frac{\partial}{\partial x}[xf(r)] = 0$$

$$\Rightarrow \qquad \sum \frac{\partial}{\partial x}[xf(r)] = 0$$

$$\Rightarrow \qquad \sum\left[x\left[f'(r)\frac{\partial r}{\partial x}\right] + f(r)(1)\right] = 0$$

$$\Rightarrow \qquad \sum\left[x\left[f'(r)\frac{x}{r}\right] + f(r)\right] = 0$$

$$\Rightarrow \qquad \sum\left[f'(r)\frac{x^2}{r} + f(r)\right] = 0$$

$$\Rightarrow \qquad 3f(r) + \frac{f'(r)}{r}[x^2 + y^2 + z^2] = 0$$

$$\Rightarrow \qquad 3f(r) + \frac{f'(r)}{r}[r^2] = 0 \qquad [\because\ x^2 + y^2 + z^2 = r^2]$$

$$\therefore\ 3f(r) + f'(r)\,r = 0$$

$$f'(r)r = -3f(r)$$

$$\frac{f'(r)}{f(r)} = \frac{-3}{r}$$

Integrating w.r. to r we get

$$\int \frac{f'(r)}{f(r)}dr = -3\int \frac{1}{r}dr$$

$$\log[f(r)] = -3\log r + \log c$$

$$= \log[r^{-3}] + \log c$$

$$= \log\left[\frac{1}{r^3}\right] + \log c = \log\left[\frac{c}{r^3}\right]$$

$$\Rightarrow \log[f(r)] = \log\left[\frac{c}{r^3}\right]$$

$\therefore f(r) = \frac{c}{r^3}$, where c is a constant.

4. Find the values of a, b, c so that the vector $\vec{F} = (x+y+az)\vec{i} + (bx+2y-z)\vec{j} + (-x+cy+2z)\vec{k}$ **may be irrotational.**

Solution:

Given $\vec{F} = (x+y+az)\vec{i} + (bx+2y-z)\vec{j} + (-x+cy+2z)\vec{k}$

To Prove: $\nabla \times \overrightarrow{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y+az & bx+2y-z & -x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}\left[\frac{\partial}{\partial y}(-x+cy+2z)-\frac{\partial}{\partial z}(bx+2y-z)\right]-\vec{j}\left[\frac{\partial}{\partial x}(-x+cy+2z)-\frac{\partial}{\partial z}(x+y+az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx+2y-z)-\frac{\partial}{\partial y}(x+y+az)\right]=\vec{0}$$

$$\vec{i}(c+1)-\vec{j}(-1-a)+\vec{k}(b-1)=0\vec{i}+0\vec{j}+0\vec{k}$$

Equating the coefficients of $\vec{i}$, $\vec{j}$, $\vec{k}$ we get

$$c+1=0, \quad 1+a=0, \quad b-1=0$$

$$c=-1, \quad a=-1, \quad b=1$$

ie., $\quad a=-1, \quad b=1, \quad c=-1,$

5. Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational vector and find the scalar potential such that $\vec{F} = \nabla\varphi$.

Solution:

Let $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

A vector $\vec{F}$ is said to be irrotational if $\nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right)$$

$$= \vec{i}\,(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x)$$

$$= \vec{i}\,(0) - \vec{j}(0) + \vec{k}(0) = 0$$

$\therefore \nabla \times \vec{F} = 0$ Hence $\vec{F}$ is irrotational.

To find the scalar potential φ

Let $\vec{F} = \nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

$$\text{ie., } \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

Equating like coefficients we get

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Partially integrating (1) we get

$$\partial\phi = (6xy + z^3)\partial x$$

$$\int \partial\phi = \int (6xy + z^3)\partial x \qquad \Rightarrow \phi = 6\frac{x^2}{2}y + xz^3$$

$$\Rightarrow \phi = 3x^2y + xz^3 + f(y,z) \ldots\ldots\ldots\ldots\ldots(4)$$

Similarly from (2) and (3) weget

$$\int \partial\phi = \int (3x^2 - z)\partial y$$

$$\Rightarrow \phi = 3x^2y - zy + f(x,z) \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\int \partial\phi = \int (3xz^2 - y)\partial z$$

$$\Rightarrow \phi = xz^3 - zy + f(x, y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (4), (5) and (6) weget

$\phi = 3x^2y + xz^3 - zy + c$ where 'c' is a constant.

SOME RESULT:

1. **If** $\vec{r}$ **is the position vector of the point (x, y, z), prove that** $\nabla^2 r^n = n(n+1)r^{n-2}$ **(or)** $div(grad r^n) = n(n+1)r^{n-2}$**.**
   Hence deduce that $\nabla \cdot \left(\nabla (\frac{1}{r}) \right) = 0$

   Solution:

   W.K.T $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

   $$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

   $$r^2 = x^2 + y^2 + z^2$$

   Differentiate with respect to x we get

   $$2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r},$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{Now,} \nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) r^n$$

$$= \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(r^n)\right) = \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{\partial r}{\partial x} \right]$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{x}{r} \right] = \sum \frac{\partial}{\partial x}\left[n r^{n-2} x \right] = \sum n \left[\frac{\partial}{\partial x}\left[r^{n-2} x \right] \right]$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r},$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{Now,} \nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) r^n$$

$$= \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(r^n)\right) = \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{\partial r}{\partial x} \right]$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{x}{r} \right] = \sum \frac{\partial}{\partial x}\left[n r^{n-2} x \right] = \sum n \left[\frac{\partial}{\partial x}\left[r^{n-2} x \right] \right]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{\partial r}{\partial x} + r^{n-2}(1) \right]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{x}{r} + r^{n-2} \right]$$

$$= \sum n\left[x^2(n-2)r^{n-4} + r^{n-2} \right]$$

$$= \sum n(n-2)r^{n-4}x^2 + nr^{n-2}$$

Expanding the summation, we get

$$= n(n-2)r^{n-4}(x^2+y^2+z^2) + 3nr^{n-2}$$

$$= n(n-2)r^{n-4}r^2 + 3nr^{n-2} \qquad \because r^2 = x^2+y^2+z^2$$

$$= n(n-2)r^{n-2} + 3nr^{n-2}$$

$$= nr^{n-2}[(n-2)+3] = nr^{n-2}(n+1) = n(n+1)r^{n-2}$$

Hence $\nabla^2 r^n = n(n+1)r^{n-2}$ .....(1)

To Deduce that:

$$\nabla.\left(\nabla(\frac{1}{r})\right)=\nabla.\nabla(\frac{1}{r})=\nabla^2(\frac{1}{r})=\nabla^2 r^{-1} \qquad \because \nabla.\nabla=\nabla^2$$

Put in the above result (1), $r=-1$ we get

$$\therefore \ \nabla.\left(\nabla(\frac{1}{r})\right)=\nabla^2 r^{-1}=-1(-1+1)r^{-1-2}=0$$

1. Show that $div(\vec{u}\times\vec{v})=\vec{v}.curl\,\vec{u}-\vec{u}.curl\,\vec{v}$

Solution:

$$\nabla.(\vec{u}\times\vec{v})=(\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}).(\vec{u}\times\vec{v})=\sum\vec{i}\cdot\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$

$$=\sum\vec{i}\cdot\left[(\frac{\partial\vec{u}}{\partial x}\times\vec{v})+(\vec{u}\times\frac{\partial\vec{v}}{\partial x})\right]$$

$$= \sum \vec{i} \cdot \left(\frac{\partial \vec{u}}{\partial x} \times \vec{v}\right) - \sum \vec{i} \cdot \left(\frac{\partial \vec{v}}{\partial x} \times \vec{u}\right) \qquad \left(\because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}\right)$$

$$= \sum \vec{i} \times \left(\frac{\partial \vec{u}}{\partial x} . \vec{v}\right) - \sum \vec{i} \times \left(\frac{\partial \vec{v}}{\partial x} . \vec{u}\right)$$

($\because$ In a scalar triple product dot and cross can be interchaned)

$$= \left[\sum \vec{i} \times \frac{\partial \vec{u}}{\partial x}\right] . \vec{v} - \left[\sum \vec{i} \times \frac{\partial \vec{v}}{\partial x}\right] . \vec{u}$$

$$= (curl\, \vec{u}) . \vec{v} - (\mathrm{curl}\, \vec{v}) . \vec{u}$$

$$div(\vec{u} \times \vec{v}) = \vec{v} . curl\, \vec{u} - \vec{u} . curl\, \vec{v} \qquad \left(\because \vec{a} . \vec{b} = \vec{b} . \vec{a}\right)$$

Hence proved

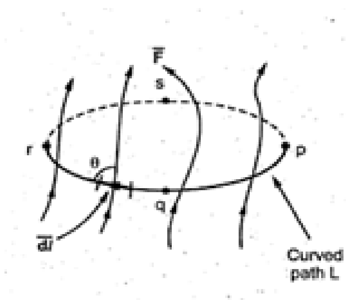
8.VECTOR INTEGRATION,LINE AND SURFACE INTEGRALS

Physically interpretation of the line integral states that,

If $\vec{F}(x,y,z)$ is a force acting on a particle which moves along a given curve C, then $\int_A^B \vec{F}.d\vec{r}$ gives the total work done by the force $\vec{F}$ in the displacing along C.

$$\therefore \text{ Work done by force } \vec{F} = \int_A^B \vec{F}.d\vec{r}$$

Line integral



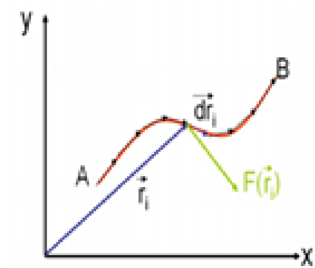


Figure 2: Line integral.

EXAMPLE:

1. **Find the Work done in moving a particle in the force given by** $\vec{F} = 3x^2\vec{i} - (2xz - y)\vec{j} + z\vec{k}$ **along the straight line from (0, 0, 0)to(2, 1, 3)**

   Solution:

   Given $\vec{F} = 3x^2\,\vec{i} - \left(2xz - y\right)\vec{j} + z\,\vec{k}$

   $$\vec{dr} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

   $$\vec{F}\,.\vec{dr} = 3x^2 dx - \left(2xz - y\right)dy + zdz$$

   The equation of the Straight line formula is

   $$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\frac{x-0}{0-2}=\frac{y-0}{0-1}=\frac{z-0}{0-3}$$

$$\frac{x}{-2}=\frac{y}{-1}=\frac{z}{-3}$$

$$\frac{x}{2}=\frac{y}{1}=\frac{z}{3}=t\,(say)$$

$$x=2t \qquad y=t \qquad z=3t$$

$$dx=2dt \qquad dy=dt \qquad dz=3dt$$

$$\int_C \vec{F}.\vec{dr}=\int_0^1\left[3(2t)^2\,2dt-\left(12t^2-t\right)dt+9tdt\right]$$

$$=\int_0^1\left[24t^2-12t^2+t+9t\right]dt$$

$$=\int_0^1\left[12t^2+10t\right]dt$$

$$= \left[\frac{12t^3}{3} + \frac{10t^2}{2} \right]_0^1$$

$$= \left[\frac{12(1)^3}{3} + \frac{10(1)^2}{2} \right]$$

$$= [4+5]$$

$$= 9$$

$$\therefore \text{Work done} = \int_C \vec{F} \cdot \vec{dr} = 9$$

2. Show that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force field. Find the scalar potential and the work done by $\vec{F}$ in moving an object in this field from (1,-2,1) to (1,1,4).

Solution:

Given: $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$

To show that $\vec{F}$ is a conservative force

$w.k.t\ \vec{F}$ is a conservative force $then\ \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$=\vec{i}\left(\frac{\partial 3xz^2}{\partial y}-\frac{\partial x^2}{\partial z}\right)-\vec{j}\left(\frac{\partial 3xz^2}{\partial x}-\frac{\partial 2xy+z^3}{\partial z}\right)+\vec{k}\left(\frac{\partial x^2}{\partial x}-\frac{\partial 2xy+z^3}{\partial y}\right)$$

$$=\vec{i}0-\vec{j}\left(3z^2-3z^2\right)+\vec{k}\left(2x-2x\right)$$

$$\nabla \times \vec{F}=0$$

Hence $\vec{F}$ is a conservative force

(ii) To find the scalar potential: $\vec{F}=\nabla\varphi$

Where $\vec{F}=(2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$ and

$$\nabla\varphi=\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}$$

$$(2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}=\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}$$

Equating we get

$$(2xy+z^3)=\frac{\partial\varphi}{\partial x};\ x^2=\frac{\partial\varphi}{\partial y};\ 3xz^2=\frac{\partial\varphi}{\partial z}$$

Integrating w.r.t to x,y and z respectively we get

Consider $(2xy+z^3)=\frac{\partial\varphi}{\partial x}$

$$\varphi=\int (2xy+z^3)\,dx$$

$$\varphi=(\frac{2x^2y}{2}+xz^3)+c \quad \ldots\ldots\ldots\text{(i)}$$

Consider $x^2=\frac{\partial\varphi}{\partial y}$

$$\frac{\partial\varphi}{\partial y}=x^2$$

$$\varphi=x^2y+c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Consider $\frac{\partial\varphi}{\partial z}=3xz^2$

$$\varphi = \frac{3xz^3}{3} + c$$

$$\varphi = xz^3 + c \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

From (i),(ii) and (iii) we conclude

$\varphi = x^2 y + xz^3 + c \quad (\text{excluding repeatation})$

(iii) The work done by $\vec{F}$ in moving an object in this field from (1,-2, 1) to (1, 1, 4).

w.k.t work done $= \int \vec{F}.d\vec{r}$, since path is not given we consider along any path

$$= \int_{1,-2,1}^{1,1,4} \vec{F}.d\vec{r}$$

$$= \int_{1,-2,1}^{1,1,4} \nabla\varphi.d\vec{r}$$

$$= \int_{1,-2,1}^{1,1,4} d\phi \quad ; w.k.t\, d\phi = \nabla \varphi . d\vec{r}$$

$$= [\phi]_{1,-2,1}^{1,1,4}$$

$$= \left[x^2 y + xz^3 + c \right]_{1,-2,1}^{1,1,4}$$

$$= [(1+64+c)-(-2+1+c)] = 65+1 = 66$$

$$\therefore \text{Work done} = 66$$

9. GREEN'S THEOREM IN A PLANE

If $u, v, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are continuous and single valued functions in the region R enclosed by the curve C, then

$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$$

THE CONCEPTS OF GREEN'S THEOREM AND EXAMPLE

Green's theorem works only for the case where C is a simple closed curve. If C is an open curve we cannot use.

For example, we can compute $\int_C (ydx + xydy)$ where C is the counter clockwise unit circle

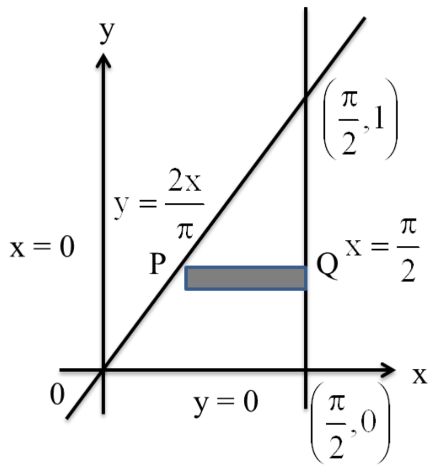
EXAMPLE:

1. Verify Green's theorem in a plane for $\int_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ **where 'c' is the boundary of the region defined by x=0, y=0 and x+y=1.**

Solution:

Let u = $3x^2-8y^2$

and v = 4y-6xy



$$Then\ \frac{\partial u}{\partial y} = -16y \quad and \quad \frac{\partial v}{\partial x} = -6y$$

$$\therefore \int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_C udx + vdy$$

By Green's Theorem in plane,

$$\int_C udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

To Prove R.H.S:

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = \iint_R \left(-6y + 16y\right) dxdy = \int_0^1 \int_0^{1-y} (10y)dydx = 10\int_0^1 y\left[x\right]_0^{1-y} dy$$

$$= 10\int_0^1 \mathrm{y}(1-y)dy = 10\int_0^1 (\mathrm{y} - y^2)dy = 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 10\left[\frac{1}{2} - \frac{1}{3}\right] = 10(\frac{1}{6}) = \frac{5}{3}$$

To Prove L.H.S:

Let us now evaluate the line integral directly.

$$\int_C udx + vdy = \int_{along\, OA} + \int_{along\, AB} + \int_{along\, BO}$$

Along OA, y=0 implies dy = 0 and x varies from 0 to 1.

$$\therefore \int_{OA} udx + vdy = \int_0^1 3x^2 dx = (x^3)_0^1 = 1$$

$Along \;\; AB, y = 1 - x \Rightarrow dy = -dx \, and \; x$ varies from 1 to 0.

$$\therefore \int_{AB} udx + vdy = \int_0^1 \left[3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x) \right] dx$$

$$= \left[\frac{3x^3}{3} + \frac{8(1-x)^3}{3} + \frac{4(1-x)^2}{2} + 3x^2 - 2x^3 \right]_0^1$$

$$= -1 + \frac{8}{3} + 2 - 3 + 2 = \frac{8}{3}$$

$Along\ \ BO, x = 0 \Rightarrow dx = 0\ \ and\ \ y$ varies $\ from\ 1\ to\ 0.$

$$\therefore \int_{BO} udx + vdy = \int_{1}^{0} 4ydy = (2y^2)_0^1 = -2$$

$$\therefore \int_{C} udx + vdy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

From L.H.S and R.H.S we get

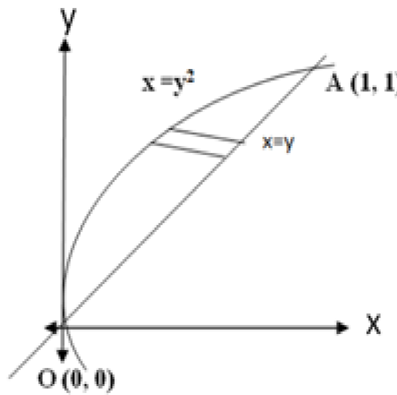
$$\therefore \int_{C} udx + vdy = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy \quad = \frac{5}{3}$$

Hence Green's theorem is verified.

Verify Green's theorem for $\int_c (xy+y^2)dx + x^2dy)$, where C is boundary of the common area between $y^2 = x \text{ and } y = x$ (M/J 2014)(N\D 2009)

Solution: Green's theorem on XY plane is given by:

$$\int_c udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy \text{........(1)}$$



Given: $\int_c (xy+y^2)dx + x^2dy$.......(2)

Comparing L.H.S of eqn. (1) and (2)

$u = xy + y^2\ ; v = x^2$

Therefore $\frac{\partial u}{\partial y} = x + 2y; \frac{\partial v}{\partial x} = 2x$

Plotting $y^2 = x \text{and } y = x$

Now consider L.H.S of (1)

$\int_c udx + vdy$, we shall take c in two different paths i.e

along $AO\,(y^2 = x)$; along $OA(y = x)$

$$\int_c udx + vdy = \int_{OA} udx + vdy + \int_{AO} udx + vdy \text{.........}(3)$$

(i) *along* $AO, x = y^2, dx = 2ydy$; Limits along AO: y = 1 to y = 0

(ii) *along* $OA, x = y, dx = dy$; Limits along OA: y = 0 to y = 1

Using (i) and (ii) in (3)

$$\int_c udx + vdy = \int_1^0 \left(y^2 y + y^2\right) 2ydy + y^4\, dy + \int_0^1 \left(y^2 + y^2\right) dy + y^2 dy$$

$$\int_c (xy + y^2)dx + x^2 dy = \int_1^0 \left(2y^4 + 2y^3\right) dy + y^4\, dy + \int_0^1 \left(3y^2\right) dy$$

$$= \left[\frac{2y^5}{5} + \frac{2y^4}{4} + \frac{y^5}{5}\right]_1^0 + \left[\frac{3y^3}{3}\right]_0^1$$

$$= -\left[\frac{2}{5}+\frac{2}{4}+\frac{1}{5}\right]+[1-0]$$

$$= -\left[\frac{8+10+4}{20}\right]+[1]$$

$$\int_c (xy+y^2)dx + x^2dy = -\frac{22}{20}+1 = -\frac{1}{10} \qquad \ldots.(4)$$

Now consider R.H.S of eqn:(1)

$$\iint_R \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy$$

To find limits of x and y:

From the above fig: $x = y^2$ to $x= y$ (left to right)

$y = 0$ to 1 (bottom to top)

$$\iint_R \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\right)dxdy = \int_0^1\int_{y^2}^{y}(2x-x-2y)\,dx\,dy$$

$$= \int_0^1\int_{y^2}^{y}(x-2y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy \right]_{y^2}^{y} dy$$

$$= \int_0^1 \left[\frac{y^2}{2} - 2y^2 - \left\{ \frac{y^4}{2} - 2y^3 \right\} \right] dy$$

$$= \int_0^1 \left[\frac{y^2}{2} - 2y^2 - \frac{y^4}{2} + 2y^3 \right] dy$$

$$= \left[\frac{y^3}{6} - \frac{2y^3}{3} - \frac{y^5}{10} + \frac{2y^4}{4} \right]_0^1$$

$$= \left[\frac{1}{6} - \frac{2}{3} - \frac{1}{10} + \frac{1}{2} \right]$$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \right) dxdy = \left[\frac{1}{6} - \frac{2}{3} - \frac{1}{10} + \frac{1}{2} \right] = \frac{10-40-6+30}{60} = \frac{-1}{10} \qquad \ldots(5)$$

From (4) & (5), we get $\int_c udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$

Hence Green's theorem is verified.

10. GAUSS DIVERGENCE THEOREM

STATEMENT:

The surface integral of the normal component of a vector function taken over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of $\vec{F}$ taken through the volume V.

$$\iint_S \vec{F}.n\,ds = \iiint_V \nabla.\vec{F}\,dv$$

THE PHYSICAL REPRESENTATION OF GAUSS DIVERGENCE THEOREM AND EXAMPLE:

If a vector field F represents the flow of a fluid, then the divergence of F represents the expansion or compression of the fluid. The divergence theorem says that the total expansion of the fluid inside some three-dimensional region W equals the total flux of the fluid out of the boundary of W. Example ,pump air into one of the car's tires

We cannot use the divergence theorem to calculate a surface integral over S if S is an open surface, like part of a cone or a paraboloid.

EXAMPLE:1

Verify Gauss's divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ **over the cube bounded by** $x = 0, x = 1, y = 0, y = 1, z = 0 \; and \; z = 1$

Solution:

Gauss divergence theorem is

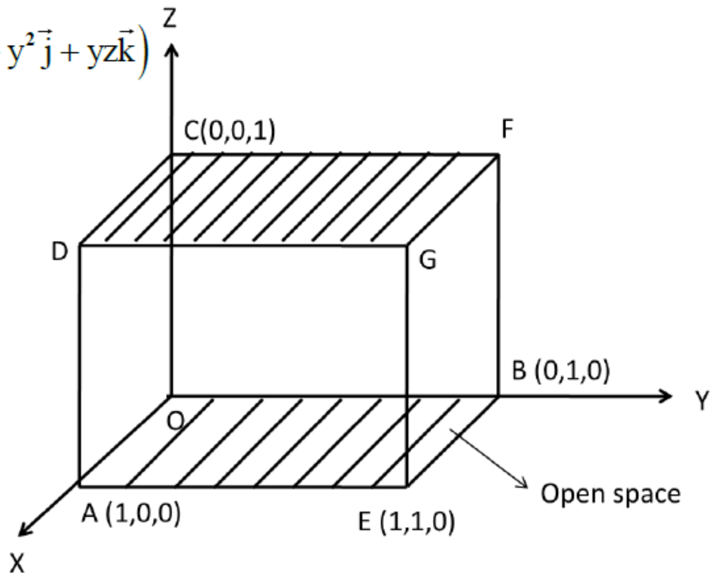
$$\iint_S \vec{F}.n \, ds = \iiint_V \nabla.\vec{F} \, dv$$

$$\iint_S \vec{F}.n\,ds = \iiint_V \nabla.\vec{F}\,dv$$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= 4z - 2y + y = 4z - y$$



$$\iiint_V \nabla.\vec{F}\ dv = \int_0^1\int_0^1\int_0^1 (\mathbf{4}z - y)dxdydz$$

$$= \int_0^1\int_0^1 \left[(\mathbf{4}zx - yx)\right]_0^1 dydz$$

$$= \int_0^1\int_0^1 \left[(\mathbf{4}z - y)\right]dydz$$

$$= \int_0^1 \left(\mathbf{4}zy - \frac{y^2}{\mathbf{2}}\right)_0^1 dz$$

$$= \int_0^1 \left(4z - \frac{1}{2} \right) dz$$

$$= \left[\frac{4z^2}{2} - \frac{z}{2} \right]_0^1$$

$$= \left(2 - \frac{1}{2} \right) - 0$$

Surface	Face	Equation	Value of ds	Unit Outward drawn normal n
S_1	AEGD	$x=1\&dx=0$	$ds=dydz$	$\vec{i}$
S_2	OBFC	$x=0\&dx=0$	$ds=dydz$	$-\vec{i}$
S_3	EBFG	$y=1\&dy=0$	$ds=dxdz$	$\vec{j}$
S_4	OADC	$y=0\&dy=0$	$ds=dxdz$	$-\vec{j}$
S_5	DGFC	$z=1\&dz=0$	$ds=dxdy$	$\vec{k}$
S_6	OAEB	$z=0\&dz=0$	$ds=dxdy$	$-\vec{k}$

$$\therefore \iiint_V \nabla . \vec{F}\ dv = \frac{\mathbf{3}}{\mathbf{2}} \qquad \text{....}(\mathbf{1})$$

To Prove L.H.S:

$$\iint_s \vec{F}.n\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

ON AEGD(S_1)

$$\iint_{S_1} \vec{F}.nds = \iint_{S_1} \left(\mathbf{4}xz\vec{i} - y^{\mathbf{2}}\vec{j} + yz\vec{k}\right).\vec{i}\ ds$$

$$= \iint_{S_1} \mathbf{4}xzdydz \qquad \because x = \mathbf{1}$$

$$= \int_{0}^{1}\int_{0}^{1} 4zdydz$$

$$= \int_{0}^{1} \left(4yz\right)_{0}^{1} dz = \int_{0}^{1} 4zdz = \left[\frac{4z^2}{2}\right]_{0}^{1} = 4/2$$

$$\iint_{S_1} \vec{F}.nds = 2$$

ON OBFC(S_2)

$$\iint_{S_2} \vec{F}.nds = \iint_{S_2} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\left(-\vec{i}\right)ds$$

$$= \iint_{S_2} -4xz\,dy\,dz \qquad \because x = 0$$

$$\iint_{S_2} \vec{F}.n\,ds = 0$$

ON EBFG(S_3)

$$\iint_{S_3} \vec{F}.n\,ds = \iint_{S_3} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\vec{j}\,ds$$

$$= \iint_{S_3} -y^2\,dx\,dz \qquad \because y = 1$$

$$= -\int_0^1 \int_0^1 dxdz$$

$$\iint_{S_3} \vec{F}.nds = -\mathbf{2}$$

ON OADC(S_4)

$$\iint_{S_4} \vec{F}.nds = \iint_{S_4} \left(\mathbf{4}xz\vec{i} - y^{\mathbf{2}}\vec{j} + yz\vec{k}\right).\left(\overrightarrow{-j}\right)ds$$

$$= \iint_{S_4} y^{\mathbf{2}}dxdz \qquad \because y = \mathbf{0}$$

$$\iint_{S_4} \vec{F}.nds = \mathbf{0}$$

ON DEFC(S_5)

$$\iint_{S_5} \vec{F}.nds = \iint_{S_5} \left(\mathbf{4}xz\vec{i} - y^{\mathbf{2}}\vec{j} + yz\vec{k}\right).\vec{k}\, ds$$

$$= \iint_{S_5} yzdxdy \qquad \because z = \mathbf{1}$$

$$= \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} ydxdy$$

$$= \int_0^1 \int_0^1 -yz\,dx\,dy = \mathbf{0}$$

$$\iint_s \vec{F}.n\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} = \mathbf{2} + \mathbf{0} - \mathbf{1} + \mathbf{0} + (\mathbf{1}/\mathbf{2}) + \mathbf{0} = \mathbf{3}/\mathbf{2} \quad ....(\mathbf{2})$$

From (1) and (2), weget

Hence Gauss divergence theorem is verified.

2. Verify Gauss-divergence theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, z=0 and z=2.

Solution:

Gauss Divergence theorem is

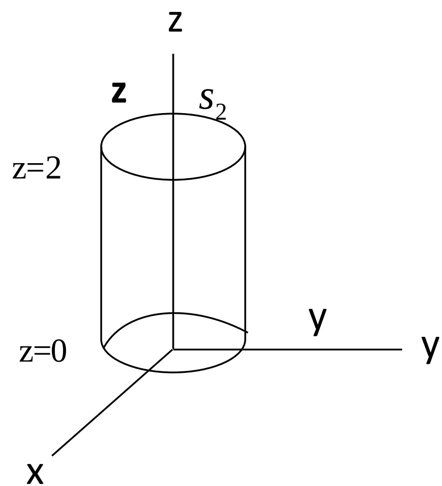
$$\iint_s \vec{F}.\hat{n}ds = \iiint_v \nabla.\vec{F}dV$$

Given $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(\mathrm{y}) + \frac{\partial}{\partial y}(\mathrm{x}) + \frac{\partial}{\partial z}(\mathrm{z}^2) = 2\,\mathrm{z}$$

The region bounded by $x^2 + y^2 = 9$

z=0 and z=2



$$\text{R.H.S} = \iiint_v \nabla.\vec{F}dv$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{2} 2z dz dy dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left[\frac{z^2}{2}\right]_0^2 dydx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[z^2\right]_0^2 dydx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-0)dydx$$

$$= 4\int_{-3}^{3} [\text{y}]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= 4\int_{-3}^{3} 2\sqrt{9-x^2}\,dx$$

$$= 8\int_{-3}^{3} \sqrt{9-x^2}\,dx$$

$$= 8\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_{-3}^{3} \quad \because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= 8\left[\left(0+\frac{9}{2}\frac{\pi}{2}\right)-\left(0-\frac{9}{2}\frac{\pi}{2}\right)\right] \qquad \because \sin\frac{\pi}{2}=0$$

$$= 8\left[\frac{9\pi}{4}+\frac{9\pi}{4}\right]$$

$$= 8\left[\frac{9\pi}{2}\right]$$

$$= 36\pi$$

$$\text{L.H.S} = \iint_{s} \vec{F}.nds = \iint_{s_1} + \iint_{s_2} + \iint_{s_3}$$

ON S_1

$$\iint_{s_1} \vec{F}.nds = \iint_{s_1} \vec{F}.(-\vec{k})\,\mathrm{dxdy} = \iint_{s_1} -\mathrm{z}^2\mathrm{dxdy} = 0 \quad \because z = 0 \text{ and } S_1$$

ON S_2

$$\iint_{s_2} \vec{F}.nds = \iint_{s_2} \vec{F}.\vec{k}\,\mathrm{dxdy} = \iint_{s_2} \mathrm{z}^2\mathrm{dxdy}$$

$$= 4\iint_{s_2} \mathrm{dxdy} \qquad \text{[Since Z = 2 on } s_2\text{]}$$

$= 4(9\pi) = 36\pi$ [Since s_2 =area of circle= $\pi r^2 = \pi(3^2) = 9\pi$]

ON S_3 To find $\iint_{s_3} \vec{F}.nds$

Given $x^2 + y^2 = 9$

$$x = 3\cos\theta, y = 3\sin\theta$$

$$ds = 3d\theta dz$$

[since in polar coordinates ds = rdrdθ]

$$n = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{9} = 6$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{6} = \frac{x\vec{i} + y\vec{j}}{3}$$

$$\therefore \iint_{s_3} \overrightarrow{F}.nds = \iint_{s_3} (y\vec{i} + x\vec{j} + z^2\vec{k}).\frac{(x\vec{i} + y\vec{j})}{3} 3dz.d\theta$$

$$= \iint_{s_3} (\mathrm{xy} + \mathrm{xy})\,\mathrm{dzd}\,\theta$$

$$= 2\iint_{s_3} \mathrm{xydzd}\,\theta$$

$$= 2\iint_{s_3} (3\cos\theta 3\sin\theta)\,\mathrm{dzd}\,\theta$$

$$= 9\iint_{s_3} (\sin 2\theta)\,\mathrm{dzd}\,\theta$$

$$= 9\int_0^{2\pi}\int_0^{2} \sin 2\theta dzd\theta$$

$$= \int_0^{2\pi} \sin 2\theta \left[z\right]_0^2 d\theta$$

$$= \int_0^{2\pi} (\sin 2\theta)(2-0)\,d\theta$$

$$= 18 \int_0^{2\pi} \sin 2\theta$$

$$= 18 \left[\frac{\cos 2\theta}{2} \right]_0^{2\pi}$$

$$= -9[-1-1]$$

$$= 0$$

$$\therefore \iint_s \;= \iint_{s_1} + \iint_{s_2} + \iint_{s_3}$$

$$= 0 + 36\pi + 0$$

$$= 36\pi$$

$\therefore$ L.H.s = R.H.S

Hence Gauss divergence theorem verified

11. STOKE'S THEOREM

STATEMENT:

The Surface integral of the normal component of the curl of a vector function F over an open surface S is equal to the line integral of the tangential component of F around the closed curve C bounding.

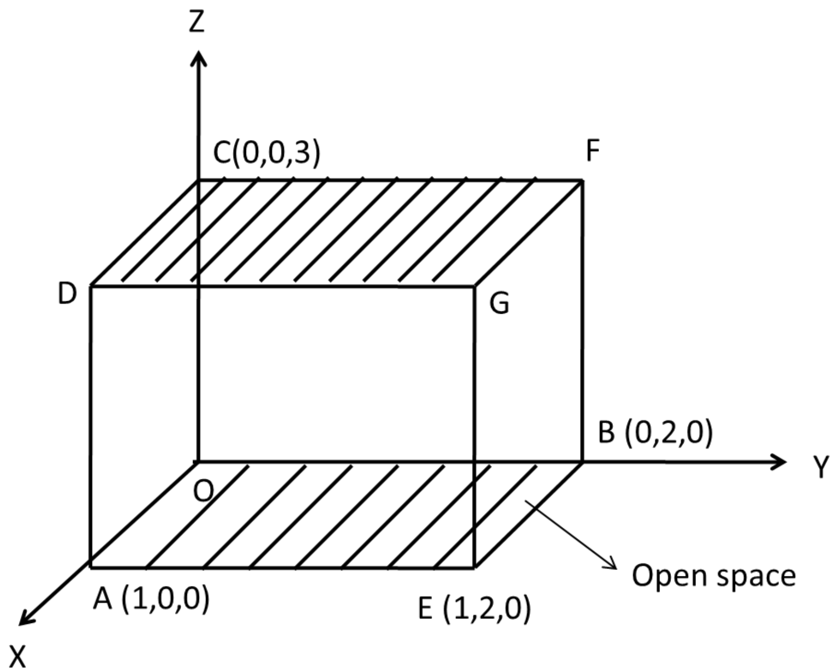
$$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\,ds$$

EXAMPLE:

Verify Stoke's theorem for $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$, **where S is the open surface of the rectangular parallelepiped formed by the planes** $x = 0, x = 1, y = 0, y = 2, and\ \ z = 3$ **above the XY plane.**

Solution:

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds$


Z
C(0,0,3)
F
D
G
B (0,2,0)
Y
O
Open space
A (1,0,0)
E (1,2,0)
X

Given $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(-2yz)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(xy)\right) + \vec{k}\left(\frac{\partial}{\partial x}(-2yz) - \frac{\partial}{\partial y}(xy)\right)$$

$$= \vec{i}(0-(-2y)) - \vec{j}(-z-0) + \vec{k}(0-x)$$

$$= 2y\vec{i} + z\vec{j} - x\vec{k}$$

To Prove R.H.S:

$$\iint_S \nabla \times \vec{F}.\hat{n}\,ds = \iint_{s_1} + \iint_{s_2} + \iint_{s_3} + \iint_{s_4} + \iint_{s_5}$$

$\iint_{s_6}$ is not applicable. Since given condition is above the XOY plane.

$$\iint_{s_1} = \iint_{ABGD} (2y\vec{i} + z\vec{j} - x\vec{k}).\vec{i}\,dy\,dz$$

$$= \iint\limits_{AEGD} 2y\,dy\,dz$$

$$= \int_0^3 \int_0^2 2y\,dy\,dz$$

$$= \int_0^3 \left[y^2\right]_0^2 dz = \int_0^3 4\,dz = 4\left[z\right]_0^3 = 4(3) = 12 \qquad ---(1)$$

$$\iint\limits_{s_2} = \iint\limits_{OBFC} (2y\vec{i} + z\vec{j} - x\vec{k}).(-\vec{i}\,)\,dy\,dz$$

$$= \iint\limits_{OBFC} -2y\,dy\,dz$$

$$= -\int_0^3 \left[y^2\right]_0^2 dz = -\int_0^3 4\,dz = -4\left[z\right]_0^3$$

$$= -4(3) = -12 \qquad ---(2)$$

$$\iint\limits_{s_3} = \iint\limits_{EBFG} (2y\vec{i} + z\vec{j} - x\vec{k}).\vec{j}\,dx\,dz$$

$$= \iint\limits_{EBFG} z\,dx\,dz$$

$$= \int_0^3 \int_0^1 z\,dx\,dz$$

$$= \int_0^3 \left[z\,x\right]_0^1 dz = \int_0^3 z\,dz = \left[\frac{z^2}{2}\right]_0^3 = \frac{9}{2} \qquad ---(3)$$

$$\iint\limits_{s_4} = \iint\limits_{OADC} (2y\vec{i} + z\vec{j} - x\vec{k}).(-\vec{j})\,dx\,dz$$

$$= \iint\limits_{OADC} -z\,dx\,dz$$

$$= \int_0^3 \int_0^1 -z\,dx\,dz$$

$$= -\int_0^3 \left[z\,x\right]_0^1 dz = -\int_0^3 z\,dz$$

$$= -\left[\frac{z^2}{2}\right]_0^3 = -\frac{9}{2} \qquad ---(4)$$

$$\iint_{S_5} = \iint_{DGFC} (2y\vec{i} + z\vec{j} - x\vec{k}).(\vec{k})\,dx\,dy$$

$$= \iint_{DGFC} -x\,dx\,dy$$

$$= \int_0^2 \int_0^1 -x\,dx\,dy$$

$$= -\int_0^2 \left[\frac{x^2}{2}\right]_0^1 dy = -\frac{1}{2}\int_0^2 dy$$

$$= -\frac{1}{2}[y]_0^2 = -1 \qquad ---(5)$$

Adding (1),(2),(3),(4) and (5) weget $R.H.S = -1$

To Prove L.H.S:

Given $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}.d\vec{r} = xydx - 2yzdy - zxdz$$

$$\text{L.H.S} \quad \int_C \vec{F}.d\vec{r} = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$\int_{OA} = \int_{OA} (xy\,dx - 2yz\,dy - xz\,dz)$$

$$= \int_{OA} 0 \qquad [y = 0, z = 0, dy = 0, dz = 0]$$

$$= 0$$

$$\int_{AE} = \int_{AE} (xy\,dx - 2yz\,dy - xz\,dz)$$

$$= \int_{AE} 0 \qquad [x = 1, z = 0, dx = 0, dz = 0]$$

$$= 0$$

$$\int_{EB} = \int_{EB} (xy\,dx - 2yz\,dy - xz\,dz)$$

$$= \int_{EB} 2x\,dx \qquad [y = 2, z = 0, dy = 0, dz = 0]$$

$$= \int_1^0 2x\,dx = \left[x^2\right]_1^0 = 0 - 1 = -1$$

$$\int_{BO} = \int_{BO} (xy\,dx - 2yz\,dy - xz\,dz)$$

$$= \int_{BO} 0 \qquad [x = 0, z = 0, dx = 0, dz = 0]$$

$$= 0$$

$$\therefore \text{L.H.S} = \int_C \vec{F}.d\vec{r} = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$= 0 + 0 - 1 + 0 = -1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Hence, Stoke's theorem is verified.

12. Application in evaluating line, surface and volume integrals.

Verify Gauss-divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ **, over the cube region bounded by** $x = 0, x = a, y = 0, y = a, z = 0$ and $z = a$. **(M/J 2014)**

Solution:

WKT Gauss's divergence theorem is

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}dv$$

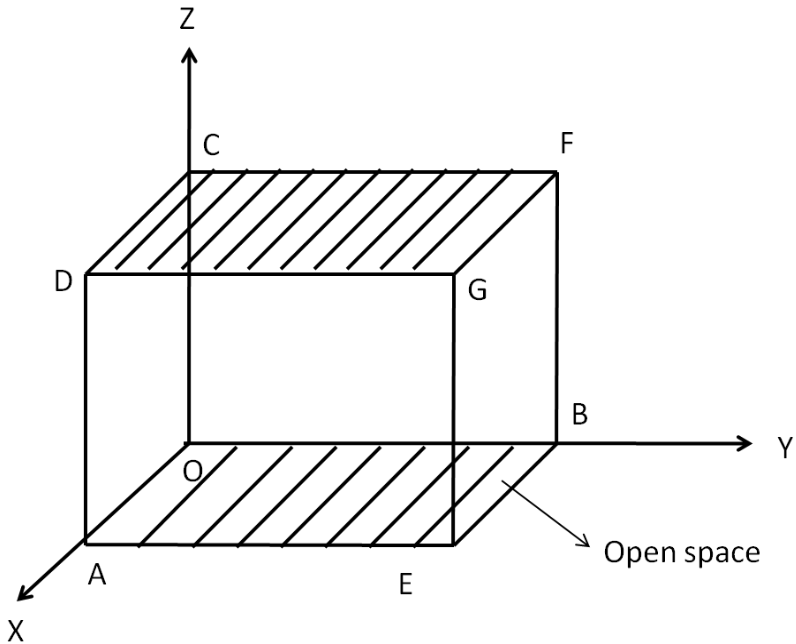
Given $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$,

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(x^3\vec{i} + y^3\vec{j} + z^3\vec{k})$$

$$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2)$$

$$\nabla.\vec{F} = 3(x^2 + y^2 + z^2)$$


Z
C
F
D
G
B
Y
O
Open space
A
E
X

To find $\iiint\limits_V \nabla.\vec{F}dv$

$$\iiint\limits_V \nabla.\vec{F}dv = \int_0^a\int_0^a\int_0^a 3(x^2+y^2+z^2)dxdydz \quad \because dv = dxdydz \qquad \text{Also x, y, z varies from 0 to a}$$

$$= 3\int_0^a\int_0^a \left(\frac{x^3}{3}+xy^2+xz^2\right)_0^a dydz$$

$$= 3\int_0^a\int_0^a \left(\frac{a^3}{3}+ay^2+az^2\right)dydz$$

$$= 3a\int_0^a\int_0^a \left(\frac{a^2}{3}+y^2+z^2\right)dydz$$

$$= 3a\int_0^a \left[\frac{a^2}{3}y+\frac{y^3}{3}+yz^2\right]_0^a dz$$

$$= 3a\int_0^a \left(\frac{a^3}{3} + \frac{a^3}{3} + az^2\right)dz$$

$$= 3a^2\int_0^a \left(\frac{2}{3}a^2 + z^2\right)dz = 3a^2\left[\frac{2}{3}a^2 z + \frac{z^3}{3}\right]_0^a$$

$$= 3a^2\left[\frac{2}{3}a^3 + \frac{a^3}{3}\right] = 3a^2\left(\frac{3a^3}{3}\right) = 3a^5$$

$$\iiint_V \nabla.\vec{F}dv = 3a^5 \qquad \ldots\ldots\ldots(a)$$

To find $\iint_S \vec{F}.\hat{n}\, ds$

Now $\iint_S \vec{F}.\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

Surface	Face	Unit outward normal vector
S_1	AEGD	$\vec{i}$
S_2	OBFC	$-\vec{i}$
S_3	EBFG	$\vec{j}$
S_4	OADC	$-\vec{j}$
S_5	DGFC	$\vec{k}$
S_6	OAEB	$-\vec{k}$

Evaluation of $\iint_{S_1} \vec{F}.\hat{n}\, ds$

$$\iint_{S_1} \vec{F}.\hat{n}\, ds = \iint_{AEGD} \left(x^3\vec{i} + y^3\vec{j} + z^3\vec{k}\right).\vec{i}\,dydz$$

($\because$ Elemental area on this face is ds=dydz)

$$= \iint_{AEGD} x^3 dydz$$

$$= \iint_{AEGD} a^3 dydz \qquad \because x = a \text{ on this face}$$

$$= a^3 \int_0^a \int_0^a dydz$$

$$= a^3 \int_0^a [y]_0^a dz$$

$$= a^3 \int_0^a [y]_0^a dz$$

$$= a^3 \int_0^a a dz = a^5 [z]_0^a = a^5 \qquad ------(1)$$

Evaluation of $\iint_{s_2} \vec{F}.\hat{n}ds$

$$\iint_{s_2} \vec{F}.\hat{n}ds = \iint_{OBFC} (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{i})dydz$$

($\because$ Elemental area on this face is ds=dydz)

$$= \iint_{OBFC} -x^3 dydz$$

$$= \iint_{OBFC} 0dydz = 0 \ldots\ldots\ldots\ldots(2) \quad \because x = 0 \text{ on this face}$$

Evaluation of $\iint_{S_3} \vec{F}.\hat{n}\,ds$

$$\iint_{S_3} \vec{F}.\hat{n}\,ds = \iint_{EBFG} \left(x^3\vec{i} + y^3\vec{j} + z^3\vec{k}\right)\cdot\vec{j}dxdz$$

($\because$ Elemental area on this face is ds=dxdz)

$$= \iint_{EBFG} y^3 dxdz \qquad \because y = a$$

$$= \int_0^a \int_0^a a^3 dxdz = a^3 \int_0^a [x]_0^a dz = a^4 \int_0^a dz = a^4 [z]_0^a = a^5 \ldots\ldots\ldots\ldots(3)$$

Evaluation of $\iint_{S_4} \vec{F}.\hat{n}\, ds$

$$\iint_{S_4} \vec{F}.\hat{n} ds = \iint_{OADC} (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{j})dxdz$$

($\because$ Elemental area on this face is ds=dxdz)

$$= \iint_{OADC} -y^3 dx\, dz$$

$$= \iint_{OADC} 0\, dy\, dz = 0 \quad \because y = 0 \qquad \text{..........(4)}$$

Evaluation of $\iint_{S_5} \vec{F}.\hat{n}\, ds$

$$\iint_{S_5} \vec{F}.\hat{n}\, ds = \iint_{DGFC} \left(x^3\vec{i} + y^3\vec{j} + z^3\vec{k}\right).\vec{k}\, dx\, dy$$

($\because$ Elemental area on this face is ds=dxdy)

$$= \iint_{DGFC} z^3 dx\, dy \quad \because z = a$$

$$= \int_0^a \int_0^a a^3 dx\, dy = a^3 \int_0^a [x]_0^a dy = a^4 \int_0^a dy = a^4 [y]_0^a = a^5 \qquad \text{...........(5)}$$

Evaluation of $\iint\limits_{S_6} \vec{F}.\hat{n}\, ds$

$$\iint\limits_{s_6} \vec{F}.\hat{n} ds = \iint\limits_{OAEB} (x^3\vec{i} + y^3\vec{j} + z^3\vec{k}).(-\vec{k}) dxdz$$

($\because$ Elemental area on this face is ds=dxdy)

$$= \iint\limits_{OAEB} -z^3 dxdy$$

$$= \iint\limits_{OAEB} 0 dxdy = 0 \ldots\ldots\ldots\ldots(6) \quad \because z = 0$$

Adding (1), (2), (3), (4), (5), (6) we get

$$\iint\limits_{S} \vec{F}.\hat{n}\, ds = a^5 + 0 + a^5 + 0 + a^5 + 0 = 3a^5 \qquad \ldots\ldots.(B)$$

From (A) and (B) we get

$$\iint\limits_{S} \vec{F}.\hat{n}\, ds = \iiint\limits_{V} \nabla.\vec{F} dv$$

Hence Gauss's divergence theorem is verified.


THANK YOU!