Module – 4 Analytic Functions

Definition of Analytic Function – Cauchy Riemann equations – Properties of analytic functions – Determination of analytic function using Milne Thomson's method – Conformal mappings: Magnification, Rotation, Inversion, Reflection – Bilinear Transformation – Cauchy's integral theorem (without proof) – Cauchy's integral theorem applications – Application of Bilinear transformation and Cauchy's Integral in Engineering.

Analytic function (or) Holomorphic function (or) Regular function.

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire (or) an Integral function.

A function which is analytic everywhere in the finite plane except at z = ∞ is called an entire function.
Example: e^z, sinz, coshz.

Necessary conditions for f(z) to be analytic.

The necessary conditions for a complex function f(z) = u(x,y) + i v(x,y) to be analytic in a region R are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \quad \text{(i.e) C – R equations.}$$

Sufficient conditions for f (z) to be analytic.

If the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial x},\frac{\partial u}{\partial y}$ exist and continuous in D and satisfies the conditions

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$. Then the function f(z) is analytic in a domain D.

Harmonic function.

Any function which possess continuous second order partial derivatives and which satisfies Laplace equation is called a harmonic function. (i.e) If $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, then f is harmonic then

Show that the function u = 2x – x^3 + $3xy^2$ is harmonic.

Solution: Given u = 2x – x^3 + $3xy^2$

$u_x = 2 - 3x^2 + 3y^2 \qquad u_y = 6xy$

$u_{xx} = -6x \qquad u_{yy} = 6x$

$u_{xx} + u_{yy} = -6x + 6x = 0$

Hence u is harmonic

Show that the function $u=\frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find f(z).

Given $u=\frac{1}{2}\log\left(x^2+y^2\right)$

$$\frac{\partial u}{\partial x}=\frac{1}{2}.\frac{1}{x^2+y^2}(2x)=\frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{2}.\frac{1}{x^2+y^2}(2y)=\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\left(x^2+y^2\right)(1)-2y^2}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2}=0$$

Hence u is harmonic function

To find conjugate of u

$$\phi_1(x,y)=\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}$$

$$\phi_1(z,o)=\frac{1}{z}$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\phi_2(z,o)=0$$

By Milne Thomson Methods

$$f'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

$$\int f'(z)\,dz=\int\frac{1}{z}dz+0$$

$$=\log z+c$$

$$f(z)=\log re^{i\theta}$$

$$f(z)=u+iv=\log r+i\theta$$

$u=\log r$, $v=\theta$

$u=\log\sqrt{x^2+y^2}$ $\left[\therefore r^2=x^2+y^2,\ \ \theta=\tan^{-1}\left(\frac{y}{x}\right)\right]$

$v=\tan^{-1}\left(\frac{y}{x}\right)$ $\therefore$ Conjugate of u is $\tan^{-1}\left(\frac{y}{x}\right)$.

Conformal transformation.

A mapping or transformation which preserves angles in magnitude and in direction between every pair of curves through a point is said to be conformal transformation.

Isogonal transformation.

A transformation under which angles between every pair of curves through a point are preserved in magnitude but altered in sense is said to be isogonal at that point.

Bilinear transformation (or) Mobius transformation (or) linear fractional transformation.

The transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b,c,d are complex numbers is called a bilinear transformation. This is also called as Mobius or linear fractional transformation.

Cross Ratio.

The cross ratio of four points z_1, z_2, z_3, z_4 is given by $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$.

Show that f (z) = $|z|^2$ is differentiable at z = 0 but not analytic at z = 0.

Solution: Let z = x + iy and $\bar{z} = x - iy$

$$|z|^2 = z\bar{z} = x^2 + y^2$$

$$f(z) = |z|^2 = (x^2 + y^2) + i0$$

$$u = x^2 + y^2 \quad , \quad v = 0$$

$$u_x = 2x \quad , \quad v_x = 0$$

$$u_y = 2y \quad , \quad v_y = 0$$

So the C-R equations $u_x = v_y$ and $u_y = - v_x$ are not satisfied everywhere except at z = 0.

So f (z) may be differentiable only at z = 0. Now $u_x = 2x$, $v_y = 0$ and $u_y = 2y$, $v_x = 0$ are continuous everywhere and in particular at (0, 0). So f (z) is differentiable at z = 0 only and not analytic.

Obtain the invariant points of the transformation $w = \frac{z-1}{z+1}$

Solution: Given: $w = \frac{z-1}{z+1}$

The invariant points are obtained by replacing w by z.

$$\text{i.e, } z = \frac{z-1}{z+1} \Rightarrow z^2 + 1 = 0 \ \therefore z = \pm i$$

Can v = $\tan^{-1}\left(\frac{y}{x}\right)$ be the imaginary part of an analytic function? If so construct an analytic function f(z) = u + iv, taking v as the imaginary part and hence find u.

Solution:

Let $v = \tan^{-1}\left(\frac{y}{x}\right)$

$$v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2};\ v_{xx} = -\left(\frac{(x^2+y^2).0 - y(2x)}{\left(x^2+y^2\right)^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2}$$

$$v_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2};\quad v_{yy} = -\left(\frac{(x^2+y^2).0 - x(2y)}{\left(x^2+y^2\right)^2}\right) = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

$v_{xx} + v_{yy} = 0 \Rightarrow v$ *is harmonic and hence v can be the imaginary part of an analytic function.*

By Milne's method, $f(z) = \int \left\{ v_y(z,0) + i v_x(z,0) \right\} dz + c$

$$v_x = \frac{-y}{x^2+y^2};\ v_x(z,0) = 0;$$

$$v_y = \frac{x}{x^2+y^2};\ v_y(z,0) = \frac{1}{z}$$

$$f(z) = \int \frac{dz}{z} + c = \log z + c = \log r + i\theta + c_1 + ic_2 \quad (\because z = re^{i\theta})$$

$$= \underbrace{\left(\frac{1}{2}\log(x^2+y^2) + c_1\right)}_{u} + i\underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{v} \qquad \left(\because r = \sqrt{x^2+y^2}\ \&\ \theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$(c_2 = 0)$$

$$\therefore u = \frac{1}{2}\log(x^2+y^2) + c_1$$

Prove that $u = x^2 - y^2$ & $v = \dfrac{-y}{x^2+y^2}$ are harmonic functions but not harmonic conjugate.

Solution:

$u = x^2 - y^2$	$v = \dfrac{-y}{x^2+y^2}$
$u_x = 2x$	$v_x = \dfrac{2xy}{\left(x^2+y^2\right)^2}$
$u_y = -2y$	$v_y = \dfrac{y^2 - x^2}{\left(x^2+y^2\right)^2}$
$u_{xx} = 2$	$v_{xx} = \dfrac{2y(y^2 - 3x^2)}{\left(x^2+y^2\right)^3}$
$u_{yy} = -2$	$v_{yy} = \dfrac{2y(3x^2 - y^2)}{\left(x^2+y^2\right)^3}$
$\therefore u_{xx} + u_{yy} = 0$	$v_{xx} + v_{yy} = 0$

Hence u and v are harmonic.

But $u_x \neq v_y$ & $v_x \neq -u_y$

C-R equations are not satisfied. Hence u+iv are not an analytic function. So they are not harmonic conjugate.

Prove that $\mathbf{w=\dfrac{z}{z+a}}$ where $\mathbf{a \neq 0}$ is analytic whereas $\mathbf{w=\dfrac{\bar{z}}{\bar{z}+a}}$ is not analytic.

Solution:

$$w=\frac{z}{z+a}=\frac{x+iy}{x+iy+a}=\frac{x+iy}{(x+a)+iy}=\frac{x+iy}{(x+a)+iy}\left(\frac{(x+a)-iy}{(x+a)-iy}\right)$$

$$=\frac{(x+iy)((x+a)-iy)}{(x+a)^2+y^2}=\frac{x(x+a)+y^2}{(x+a)^2+y^2}+i\frac{(x+a)y-xy}{(x+a)^2+y^2}$$

$$w=\underbrace{\frac{x(x+a)+y^2}{(x+a)^2+y^2}}_{u}+i\underbrace{\frac{ay}{(x+a)^2+y^2}}_{v}$$

$$u=\frac{x(x+a)+y^2}{(x+a)^2+y^2};$$

$$u_x=\frac{((x+a)^2+y^2)(2x+a)-(x(x+a)+y^2)(2(x+a))}{((x+a)^2+y^2)^2}$$

$$=\frac{2x(x+a)+2xy^2-2x^2(x+a)-2xy^2-2ax(x+a)-2ay^2}{((x+a)^2+y^2)^2}$$

$$=\frac{(x+a)(2x^2+2ax+ax+a^2-2x^2-2ax)-ay^2}{((x+a)^2+y^2)^2}$$

$$u_x=\frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\ldots(1)$$

$$u_y=\frac{((x+a)^2+y^2)(2y)-(x(x+a)+y^2)(2y)}{((x+a)^2+y^2)^2}$$

$$=\frac{2y((x+a)^2+y^2-(x(x+a)+y^2))}{((x+a)^2+y^2)^2}$$

$$=\frac{2y(x^2+ax+a^2+y^2-x^2-ax-y^2)}{((x+a)^2+y^2)^2}$$

$$u_y=\frac{2ay(x+a)}{((x+a)^2+y^2)^2}\ldots(2)$$

$$v=\frac{ay}{(x+a)^2+y^2};$$

$$v_x=\frac{((x+a)^2+y^2)(0)-(ay)(2(x+a))}{((x+a)^2+y^2)^2}$$

$$v_x=\frac{-2ay(x+a)}{((x+a)^2+y^2)^2}\dots(3)$$

$$v_y=\frac{((x+a)^2+y^2)(a)-(ay)(2y)}{((x+a)^2+y^2)^2}$$

$$=\frac{a((x+a)^2+y^2-2y^2)}{((x+a)^2+y^2)^2}$$

$$v_y=\frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\dots(4)$$

From (1) and (4), $u_x=v_y$

From (2) and (3), $u_y=-v_x$

Also u_x,u_y,v_x,v_y *are continuous functions in* x *and* y.

Hence $w=\dfrac{z}{z+a}$ *is analytic*.

Now $$w=\frac{\bar{z}}{\bar{z}+a}=\frac{x-iy}{x-iy+a}=\frac{x-iy}{(x+a)-iy}=\frac{x-iy}{(x+a)-iy}\left(\frac{(x+a)+iy}{(x+a)+iy}\right)$$

$$=\frac{(x-iy)((x+a)+iy)}{(x+a)^2+y^2}=\frac{x(x+a)+y^2}{(x+a)^2+y^2}+i\frac{(-(x+a)y+xy)}{(x+a)^2+y^2}$$

$$w=\underbrace{\frac{x(x+a)+y^2}{(x+a)^2+y^2}}_{u}+i\underbrace{\frac{-ay}{(x+a)^2+y^2}}_{v}$$

$$u=\frac{x(x+a)+y^2}{(x+a)^2+y^2};$$

$$u_x=\frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\dots(5)$$

$$u_y=\frac{2ay(x+a)}{((x+a)^2+y^2)^2}\dots(6)$$

$$v = \frac{-ay}{(x+a)^2 + y^2};$$

$$v_x = \frac{2ay(x+a)}{((x+a)^2 + y^2)^2} \ldots (7)$$

$$v_y = \frac{-a((x+a)^2 - y^2)}{((x+a)^2 + y^2)^2} \ldots (8)$$

From (5) and (8), $u_x \neq v_y$

From (6) and (7), $u_y \neq -v_x$

Hence $w = \dfrac{\bar{z}}{\bar{z}+a}$ *is not analytic.*

Properties of Analytic function

Property : 1

The function f(z) = u + iv is analytic, show that u = constant and v = constant are orthogonal

Proof:

If $f(z) = u + iv$ is an analytic function of z, then it satisfies C-R equations

$u_x = v_y,\ u_y = -v_x$

Given $u(x, y) = C_1$.............(1)

$v(x, y) = C_2$.............(2)

By total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Differentiate equation (1) & (2) we get $du = 0$, $dv = 0$

$$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \text{and} \quad \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1(say)$$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2(say)$$

$$\therefore m_1\, m_2 = -\frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial v/\partial x}{\partial v/\partial y} \quad (\because u_x = v_y \quad u_y = -v_x)$$

$$\therefore m_1\, m_2 = -1$$

The curves $u(x, y) = C_1$ and $v(x, y) = C_2$ cut orthogonally.

Property : 2

Prove that an analytic function with constant modulus is constant.

Proof:

Let $f(z) = u + iv$ be analytic

By C.R equations satisfied
i.e., $u_x = v_y,\ u_y = -v_x$
$\therefore f(z) = u + iv$
$\Rightarrow |f(z)| = \sqrt{u^2 + v^2} = C \Rightarrow |f(z)|^2 = u^2 + v^2 = C^2$
$u^2 + v^2 = C^2$...............(1)
Diff (1) with respect to x
$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$
$uu_x + vv_x = 0$.............(2)
Diff (1) with respect to y
$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$
$-uv_x + vu_x = 0$............(3)
$(2)\times u + (3)\times v \Rightarrow (u^2 + v^2)u_x = 0$
$\Rightarrow u_x = 0$
$(2)\times v - (3)\times u \Rightarrow (u^2 + v^2)v_x = 0$
$\Rightarrow v_x = 0$
W.K.T $f'(z) = u_x + iv_x = 0$
$f'(z) = 0$ Integrate w.r.to z
$f(z) = C$

Property : 3

8. Prove that the real and imaginary parts of an analytic function are harmonic function.

Proof:

Let f (z) = u + iv be an analytic function of z. Then by C- R equations we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{.......... .(1)} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{.........(2)}$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{.......... .(3)}$$

Differentiating (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \text{..........} .(4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\therefore$ u satisfies the Laplace equation.

Similarly

Differentiating (1) partially with respect to y, we get

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \text{..........} .(5)$$

Differentiating (2) partially with respect to x, we get

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \text{..........} .(6)$$

Adding (5) and (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\therefore$ v satisfies the Laplace equation.

Hence the real and imaginary parts of an analytic function are harmonic function.

Property : 4

9. The real part of an analytic function f(z) is constant, prove that f(z) is a constant function.
Proof:

Let f(z) = u + iv
Given u = constant. $\Rightarrow u_x = 0$ and $u_y = 0$
by C-R equations, $u_x = 0 \Rightarrow v_y = 0$ and $u_y = 0 \Rightarrow v_x = 0$
$f'(z) = u_x + iv_x = 0 + i0 = 0$
Integrating, f (z) = c (where c is a constant)

10. If f(z) is an analytic function, prove that $\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\} |f(z)|^2 = 4|f'(z)|^2$

Proof:

Let f(z) = u + iv be analytic.

Then $u_x = v_y$ and $u_y = -v_x$ (1)

Also $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)

Now $|f(z)|^2 = u^2 + v^2$ and $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u.u_x + 2v.v_x$$

and $\dfrac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right]$ (3)

Similarly $\dfrac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right]$ (4)

Adding (3) and (4)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) + v_x^2 + v_y^2 + v(v_{xx} + v_{yy})\right]$$

$$= 2\left[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

11. Find the map of the circle (i) $|z| = 3$ under the transformation w = 2z

(ii) $|z| = 1$ by the transformation w = z + 2 + 4i

Solution (i) : Given w = 2z **,** $|z| = 3$

$|w| = 2|z|$

$|w| = 2(3) = 6$

Hence the image of the circle $|z| = 3$ in the z-plane maps to the circle $|w| = 6$ in the w-plane.

Solution (ii) :

Given: w = z + 2 + 4i

u + iv = x + iy + 2 + 4i = (x + 2) + i (y + 4)

u = x + 2, v = y + 4

⇨ x = u – 2, y = v – 4

⇨ $|z| = 1$

$x^2 + y^2 = 1$ Hence $(u - 2)^2 + (v - 4)^2 = 1$.

$\therefore$ The circle in the z-plane is mapped into the circle in the w -plane with centre (2, 4) and radius 1.

Find the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution:

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$z = \dfrac{1}{u+iv} = \dfrac{u-iv}{u^2+v^2} \Rightarrow x = \dfrac{u}{u^2+v^2}$ __________(1) $y = -\dfrac{v}{u^2+v^2}$ ____________________(2)

Given strip is $\dfrac{1}{4} < y < \dfrac{1}{2}$ when $y = \dfrac{1}{4}$

$\dfrac{1}{4} = -\dfrac{v}{u^2+v^2}$ (by 2)

$u^2 + (v+2)^2 = 4$..........(3)

which is a circle whose centre is at $(0,-2)$ in the w-plane and radius 2.

When $y = \dfrac{1}{2}$

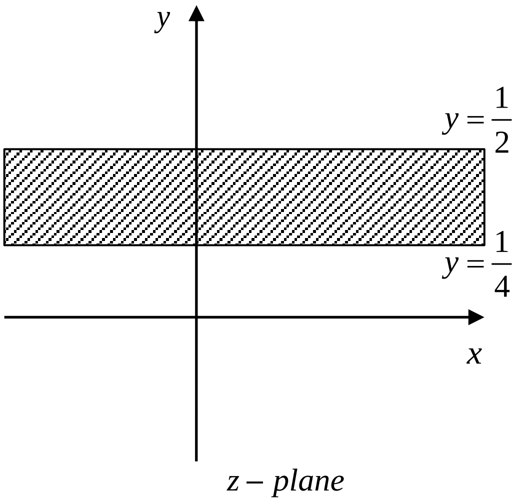
$\dfrac{1}{2} = \dfrac{-v}{u^2+v^2}$ (by 2)

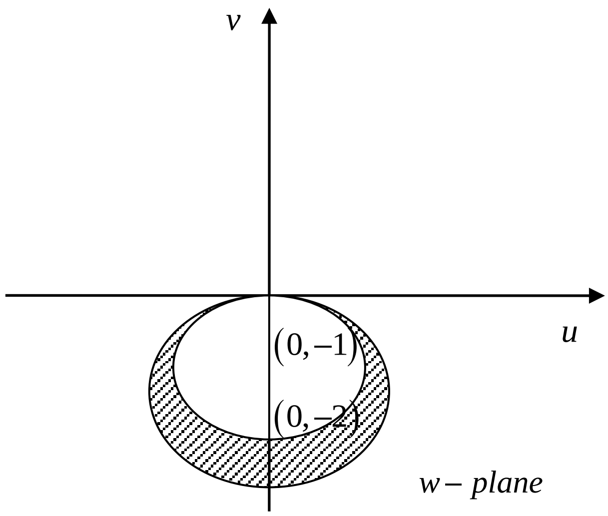
$u^2 + v^2 + 2v = 0$

$u^2 + (v+1)^2 = 1$..........(4)

which is a circle whose centre is at $(0,-1)$ and radius is 1 in the w-plane.

Hence the infinite strip $\dfrac{1}{4} < y < \dfrac{1}{2}$ is transformed into the region between circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w-plane.





Find the image of $|z-2i| = 2$ under the transformation $w = \dfrac{1}{z}$

Solution:

Given $w = \dfrac{1}{z} \Rightarrow z = \dfrac{1}{w}$

Now $w = u + iv$

$z = \dfrac{1}{w} = \dfrac{1}{u+iv} = \dfrac{u-iv}{(u+iv)(u-iv)} = \dfrac{u-iv}{u^2+v^2}$

i.e., $x + iy = \dfrac{u-iv}{u^2+v^2}$ $\therefore$ $x = \dfrac{u}{u^2+v^2}$..........(1) $y = \dfrac{-v}{u^2+v^2}$..........(2)

Given $|z-2i|=2$

$|x+iy-2i|=2 \Rightarrow |x+i(y-2)|=2$

$x^2+(y-2)^2=4 \Rightarrow x^2+y^2-4y=0$..................(3)

Sub (1) and (2) in (3)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left[\frac{-v}{u^2+v^2}\right]=0$$

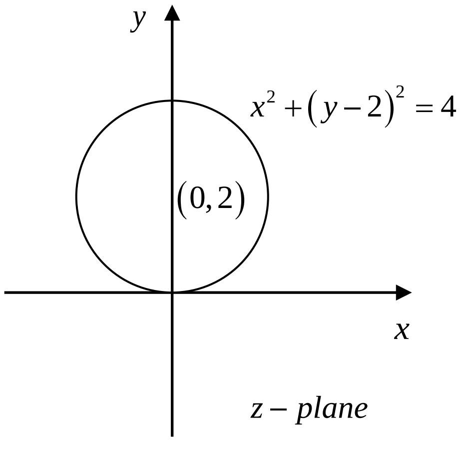
$$\frac{u^2}{\left(u^2+v^2\right)^2}+\frac{v^2}{\left(u^2+v^2\right)^2}+\left[\frac{4v}{u^2+v^2}\right]=0$$

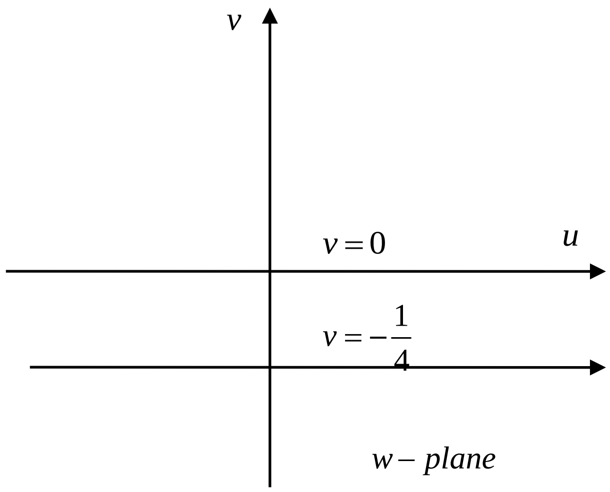
$$\frac{\left(u^2+v^2\right)+4v\left(u^2+v^2\right)}{\left(u^2+v^2\right)^2}=0$$

$$\frac{(1+4v)\left(u^2+v^2\right)}{\left(u^2+v^2\right)^2}=0$$

$1+4v=0 \Rightarrow v=-\frac{1}{4} \quad (\because u^2+v^2 \neq 0)$

which is a straight line in w-plane.





Show that the transformation $w=\frac{1}{z}$ transforms in general, circles and straight lines into circles and straight lines.

Solution:

$w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$\Rightarrow x+iy=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$

$\Rightarrow x=\frac{u}{u^2+v^2} \text{ and } y=\frac{-v}{u^2+v^2}$

Consider the equation $a\,(x^2 + y^2) + bx + cy + d = 0$ ------------(1)

This equation represents a circle if $a \neq 0$ and a straight line if $a = 0$

Under the transformation $w = \dfrac{1}{z}$ equation (1) becomes

$d\,(u^2 + v^2) + bu - cv + a = 0$ --------------(2)

This equation represents a circle if $d \neq 0$ and a straight line if $d = 0$

Value of a & d	Equation (1) and (2)	Conclusion
$a \neq 0,\ d \neq 0$	Equation (1) and (2) represents a circle, not passing through the origin, in the z–plane and w–plane	The transformation maps a circle not passing through the origin in z–plane into a circle not passing through the origin in w–plane
$a \neq 0,\ d = 0$	Equation (1) represents a circle passing through the origin in the z–plane and equation (2) represents a straight line not passing through the origin in w–plane	The transformation maps a circle passing through the origin in z–plane into a straight line not passing through the origin in w–plane
$a = 0,\ d \neq 0$	Equation (1) represents a straight line not passing through the origin in the z–plane and equation (2) represents a circle passing through the origin in w–plane	The transformation maps a straight line not passing through the origin in the z–plane into a circle passing through the origin in w–plane
$a = 0,\ d = 0$	Equation (1) and (2) represents a straight line passing through the origin in the z–plane and w–plane	The transformation maps represents a straight line passing through the origin in z–plane into a straight line passing through the origin in w–plane

Thus the transformation $w = \dfrac{1}{z}$ maps the totality of circles and straight lines as circles or straight lines.

Find the image of the circle $|z-1| = 1$ under the transformation $w = z^2$

Solution:

In polar form $z = r\,e^{i\theta}$, $w = \mathrm{R}\,e^{i\phi}$

Given

$$|z-1|=1$$

$$|re^{i\theta}-1|=1$$

$$|r\cos\theta+i\,r\sin\theta-1|=1$$

$$|(r\cos\theta-1)+i\,r\sin\theta\,|=1$$

$$(r\cos\theta-1)^2+(r\sin\theta)^2=1^2$$

$$r^2-2r\cos\theta=0$$

$$r=2\cos\theta----(1)$$

Now, we have

$$w=z^2$$

$$\mathrm{R}\,e^{i\phi}=\left(r\,e^{i\theta}\right)^2$$

$$\mathrm{R}\,e^{i\phi}=r^2\,e^{i2\theta}$$

$$\mathrm{R}=r^2,\qquad \phi=2\theta$$

$$(1)\ \Rightarrow$$

$$r^2=\left(2\cos\theta\right)^2$$

$$=4\cos^2\theta$$

$$=4\left[\frac{1+\cos 2\theta}{2}\right]$$

$$r^2=2(1+\cos 2\theta)$$

$$R=2(1+\cos\phi)$$

Find the bilinear transformation of the points –1,0,1 in z- plane onto the points 0,*i*,3*i* in w- plane.

Solution:

Given $z_1=-1,\ w_1=0$

$z_2=0,\ w_2=i$

$z_3=i,\ w_3=3i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)}=\frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)}=\frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{2w}{w-3i}=\frac{z+1}{z-1}$$

$$2wz-2w=wz+w-3iz-3i$$

$$w(2z-2-z-1)=-3i(z+1)$$

$$w(z-3)=-3i(z+1)$$

$$\therefore w=-3i\frac{(z+1)}{(z-3)}$$

Find the bilinear transformation which maps the points $z=\infty, i, 0$ into $w=0, i, \infty$ respectively.

Solution:

Given $z_1=\infty,\ w_1=0$

$z_2=i,\ w_2=i$

$z_3=0,\ w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)z_1\left(\frac{z_2}{z_1}-z_1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)}=\frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{w}{i}=\frac{i}{z},\qquad w=\frac{i^2}{z},\qquad \therefore w=-\frac{1}{z}$$

Find the bilinear transformation which maps the points z = 1, i, -1 into the points

w = 0, 1, ∞ .

Solution:

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Here, $w_3=\infty$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\,w_3\left(1-\frac{w}{w_3}\right)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{\infty}-1\right)}{(w_1-w_2)\left(1-\frac{w}{\infty}\right)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)(-1)}{(w_1-w_2)(1)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-w_1)}{(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)}{(1-0)}=\frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$w=\frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w=\frac{(z-1)(i+1)}{(z+1)(i-1)}\frac{(i+1)}{(i+1)}=i\frac{(z-1)}{(z+1)}$$

Find the bilinear transformation which maps the points $0,1,\infty$ in z-plane into itself in w-plane.

Solution:

Given $z_1=0,\ w_1=0,\quad z_2=1,\ w_2=1,\quad z_3=\infty,\ w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{(z-z_1)\,z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)}=\frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$w=z$$

Find the bilinear transformation which maps the points z=1,i,-1 into the points w= i,0,-i. Hence find the image of |z| < 1

Solution:

We know that

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-i)(0+i)}{(i-0)(-i-w)}=\frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow \frac{(w-i)(i)}{-(i)(w+i)}=\frac{(z-1)(1+i)}{-(1-i)(z+1)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{(z-1)(1+i)}{(z+1)(1-i)}\times\frac{(1+i)}{(1+i)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{(z-1)}{(z+1)}\times\frac{(1-1+2i)}{(1+1)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{(z-1)}{(z+1)}\times\frac{(2i)}{(2)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{i(z-1)}{(z+1)}\text{........(1)}$$

$$\Rightarrow \frac{(w-i)}{(w+i)}=\frac{iz-i}{z+1}$$

Applying componendo and dividendo rule, we get

$$\Rightarrow \frac{w-i+w+i}{w-i-w-i}=\frac{iz-i+z+1}{iz-i-z-1}\Rightarrow \qquad \frac{2w}{-2i}=\frac{iz-i+z+1}{iz-i-z-1}$$

$$\Rightarrow \boxed{w=-i\left[\frac{(1+i)z+1-i}{(i-1)z-1-i}\right]=\frac{(1-i)z-1-i}{(i-1)z-1-i}}$$

To find the image of $|z|<1$

From (1),

$$\frac{(z-1)}{(z+1)}=-i\frac{(w-i)}{(w+i)}=\frac{-iw-1}{w+i}$$

Applying Componendo and dividendo rule, we get

$$\frac{z-1+z+1}{z-1-z-1}=\frac{-iw-1+w+i}{-iw-1-w-i}$$

$$\Rightarrow \frac{2z}{-2} = \frac{(1-i)w+i-1}{(-1-i)w-1-i}$$

$$\Rightarrow \boxed{z = \frac{(1-i)w+i-1}{(1+i)w+1+i}}$$

Now $|z|<1 \Rightarrow \left|\frac{(1-i)w+i-1}{(1+i)w+1+i}\right|<1$

$$\Rightarrow |(1-i)w+i-1| < |(1+i)w+1+i|$$

$$\Rightarrow |(1-i)(u+iv)+i-1| < |(1+i)(u+iv)+1+i|$$

$$\Rightarrow |u+iv-iu+v+i-1| < |u+iv+iu-v+1+i|$$

$$\Rightarrow |u+v-1+i(1-u+v)| < |u-v+1+i(1+u+v)|$$

$$\Rightarrow \sqrt{(u+v-1)^2+(1-u+v)^2} < \sqrt{(u-v+1)^2+(1+u+v)^2}$$

$$\Rightarrow u^2+v^2+1-2u-2v+2uv+1+u^2+v^2-2u+2v-2uv <$$
$$u^2+v^2+1+2u-2v-2uv+1+u^2+v^2+2u+2v+2uv$$

$$\Rightarrow -2u-2u < 2u+2u$$

$$\Rightarrow -4u < 4u$$

$$\Rightarrow -8u < 0$$

$$\Rightarrow \boxed{u>0}$$

$\therefore$ the image of $|z|<1$ in z-plane is right half of w-plane u > 0.

21. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z-plane into the upper half of the w-plane.

What is the image of the circle $|z|=1$ under this transformation?

Solution:

$$w = \frac{z}{1-z} \Rightarrow w(1-z) = z$$

$$w - wz = z$$

$$w = (1+w)z$$

$$z = \frac{w}{1+w} \text{.............(1)}$$

put $z = x+iy, \quad w = u+iv$

$$x+iy=\frac{u+iv}{1+u+iv}=\frac{(u+iv)}{(1+u)+iv}\;\frac{(1+u)-iv}{(1+u)-iv}$$

$$=\frac{u(1+u)-iuv+iv(1+u)+v^2}{(1+u)^2+v^2}$$

$$=\frac{(u+u^2+v^2)+iv}{(1+u)^2+v^2}$$

Equating real and imaginary parts

$$x=\frac{(u+u^2+v^2)}{(1+u)^2+v^2},\; y=\frac{v}{(1+u)^2+v^2}$$

$$y=0\Rightarrow\frac{v}{(1+u)^2+v^2}=0$$

$$y>0\Rightarrow\frac{v}{(1+u)^2+v^2}>0$$

$$\Rightarrow v>0$$

Thus the upper half of the z- plane is mapped onto the upper half of the w- plane.

Image of $|\boldsymbol{z}|=\mathbf{1}$ **:**

by (1)

$$|z|=1\Rightarrow\left|\frac{w}{1+w}\right|=1$$

$$\Rightarrow\frac{|w|}{|1+w|}=1$$

$$\Rightarrow|w|=|1+w|$$

$$\Rightarrow|u+iv|=|1+u+iv|$$

$$\Rightarrow\sqrt{u^2+v^2}=\sqrt{(1+u)^2+v^2}$$

$$\Rightarrow u^2+v^2=(1+u)^2+v^2$$

$$\Rightarrow u^2=1+u^2+2u$$

$$\Rightarrow 2u+1=0$$

$$\Rightarrow u=\frac{-1}{2}$$

22. Determine the analytic function whose real part is $\dfrac{\mathbf{sin}\,2x}{\mathbf{cosh}\,2y-\mathbf{cos}\,2x}$

Solution:

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = -\frac{1}{\sin^2 z} = -\cos ec^2 z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = 0$$

By Milne's Thomson method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$
$$= \int -\cos ec^2 z\,dz - i0$$
$$= \cot z + c$$

23. If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$ find $f(z)$ interms of z

Solution:

$$f(z) = u + iv \qquad (1)$$

$$if(z) = iu - v \qquad (2)$$

$$\therefore (1+i)f(z) = (u-v) + i(u+v)$$

$F(z) = U + iV$, where $F(z) = (1+i)f(z)$, $U = u - v$, $V = u + v$

$$\therefore U = u - v = e^x(\cos y - \sin y)$$

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x[\cos y - \sin y]$$

$$\phi_1(z,0) = e^z$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x[-\sin y - \cos y]$$

$\phi_2(z,0) = e^z(-1) = -e^z$

By Milne's Thomson Method

$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$= \int e^z\,dz - i\int -e^z\,dz = e^z + ie^z$

$= (1+i)e^z$

$(1+i)f(z) = (1+i)e^z + C_1$

$f(z) = e^z + C$

24. Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$

Solution:

$v = e^{-x}(x\cos y + y\sin y)$

$\phi_2(x,y) = \dfrac{\partial v}{\partial x} = e^{-x}[\cos y] + (x\cos y + y\sin y)(-e^{-x})$

$\phi_2(z,0) = e^{-z} + (z)(-e^{-z}) = e^{-z} - ze^{-z} = e^{-z}(1-z)$

$\phi_1(x,y) = \dfrac{\partial u}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y(1)]$

$\phi_1(z,0) = e^{-z}[0+0+0] = 0$

By Milne's Thomson Method

$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$

$= \int 0\,dz + i\int (1-z)e^{-z}\,dz$

$= i\left[(1-z)\left[\dfrac{e^{-z}}{-1}\right] - (-1)\left[\dfrac{e^{-z}}{(-1)^2}\right]\right] + C$

$= i\left[-(1-z)e^{-z} + e^{-z}\right] + C$

$= i\left[-e^{-z} + ze^{-z} + e^{-z}\right] + C = i\left[ze^{-z}\right] + C$

25. Determine the analytic function w = u + iv if u = e^{2x} (x cos2 y – y sin2y).
Solution:

Given $u = e^{2x}[x\cos 2y - y\sin 2y]$

$\phi_1(x,y) = \dfrac{\partial u}{\partial x} = e^{2x}\cos 2y + [x\cos 2y - y\sin 2y]2e^{2x}$

$\therefore \phi_1(z,0) = e^{2z} + 2ze^{2z}$ ....(1)

$\phi_2(x,y) = \dfrac{\partial u}{\partial y} = e^{2x}[-x2\sin 2y - 2y\cos 2y - \sin 2y]$

$\therefore \phi_2(z,0) = 0$..................................(2)

By Milne Thomson method

$$F'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

From (1) & (2)

$$\int F'(z)\,dz=\int (e^{2z}+2ze^{2z})dz$$

$$=\frac{e^{2z}}{2}+2\left[z\frac{e^{2z}}{2}-(1)\frac{e^{2z}}{4}\right]$$

$$=\frac{e^{2z}}{2}+ze^{2z}-\frac{e^{2z}}{2}, \qquad \therefore F(z)=ze^{2z}$$

26. Construct the analytic function f (z) = u+iv given that 2u+3v = e^x (cos y – sin y).

Solution:

$$2u+3v=e^x\left[\cos y-\sin y\right]$$

$$f(z)=u+iv...............(1)$$

$$3if(z)=3iu-3v...............(2)$$

$$(1)\times 2\Rightarrow 2f(z)=2u+i2v...........(3)$$

$$(3)-(2)\Rightarrow (2-3i)f(z)=(2u+3v)+i(2v-3u)............(4)$$

$$F(z)=U+iV$$

$$\therefore 2u+3v=U=e^x\left[\cos y-\sin y\right]$$

$$\phi_1(x,y)=\frac{\partial U}{\partial x}=e^x\cos y-e^x\sin y$$

$$\phi_1(z,o)=e^z$$

$$\phi_2(x,y)=\frac{\partial U}{\partial y}=-e^x\sin y-e^x\cos y$$

$$\phi_2(z,o)=-e^z$$

By Milne Thomson method

$$F'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

$$\int F'(z)\,dz=\int e^z dz-i\int -e^z dz$$

$$F(z)=(1+i)e^z+C-----(5)$$

From (4) & (5)

$$(1+i)e^{z}+C=(2-3i)f(z)$$

$$f(z)=\frac{1+i}{2-3i}e^{z}+\frac{C}{2-3i}$$

$$f(z)=\frac{-1+5i}{13}e^{z}+\frac{C}{2-3i}$$