Engineering Mathematics
[For Semesters I and II]

Third Edition

About the Author

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Chapter **4**

Complex Integration

4.1 INTRODUCTION

The concept of a real line integral, with which the reader is familiar is extended to that of a complex line integral as given below in Fig. 4.1.

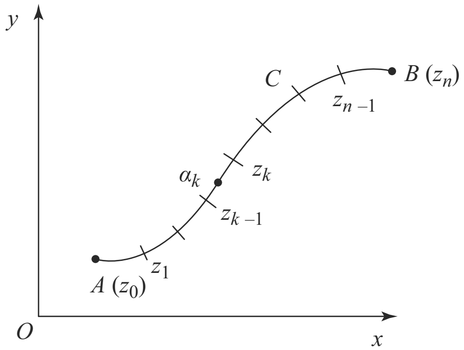


Fig. 4.1

Let $f(z)$ be a continuous function of the complex variable z (= $x + iy$) and bet C be any continuous curve connecting two points $A(z_0)$ and $B(z_n)$ on the z-plane. Let C be divided into n parts by means of the points $z_1, z_2, ...z_{n-1}$. Let $z_k - z_{k-1} = \Delta z_k$. Let α_k be an arbitrary point in the arc $z_{k-1} z_k$ ($k = 1, 2, ..., n$).

Let $$S_n = \sum_{k=1}^{n} f(\alpha_k)(z_k - z_{k-1}) \quad \text{or} \quad \sum_{k=1}^{n} f(\alpha_k)\Delta z_k.$$

If the limit of S_n exists as $n \to \infty$ in such a way that each $\Delta z_k \to 0$ and if the limit is independent of the mode of subdivision of C and the choice of the points α_k, then it is called *the complex line integral* of $f(z)$ along C from A to B and denoted as $\int_C f(z)\,dz$. Practically a complex line integral is expressed in terms of two real line integrals and evaluated.

i.e. If $f(z) = u(x, y) + iv(x, y)$, then

$$\int_C f(z)\mathrm{d}z = \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_C (u\mathrm{d}x - v\mathrm{d}y) + i \int_C (v\mathrm{d}x + u\mathrm{d}y)$$

Note ☑ When C is an open arc (of finite length) of a continuous curve, the sense of description of C is obvious, as it is traversed from $A(z_0)$ to $B(z_n)$. When C is a simple closed curve, i.e. a continuous closed curve which does not intersect itself and which encloses a finite region in the Argand plane, then C is traversed in the direction indicated by the arrows drawn on C. In this case, the complex line integral is called contour integral and denoted by the special symbol $\oint_C f(z)\mathrm{d}z$.

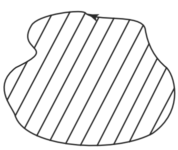
If the sense of description of C is not indicated by arrows, C is traversed in the positive sense or direction, i.e. the direction in which a person, walking along C, has the interior region of C to his or her left. Practically we shall take the anticlockwise direction of traversal of C as the positive direction.

4.1.1 Simply and Multiply Connected Regions

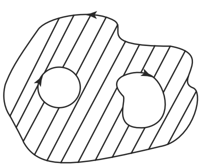
A region R is called *simply connected*, if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply connected is called *multiply connected*.

Obviously, a simply connected region is one which does not have any "holes" in it, whereas a multiply connected region is one which has.

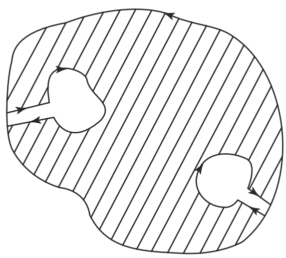
A multiply connected region can be converted into a simply connected region by introducing cross-cuts as shown in Figs 4.2 (a), (b) and (c)



(a) Simply connected region



(b) Multiply connected region



(c) Multiply connected region converted into simply connected region by cross-cuts

Fig. 4.2

4.2 CAUCHY'S INTEGRAL THEOREM OR CAUCHY'S FUNDAMENTAL THEOREM

If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points on and inside a simple closed curve C, then $\int_C f(z)\mathrm{d}z = 0$.

Proof

Let $$f(z) = u(x, y) + iv(x, y) = u + iv$$

Then $$\int_C f(z)\mathrm{d}z = \int_C (u+iv)(\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_C (u\mathrm{d}x - v\mathrm{d}y) + i\int_C (v\mathrm{d}x + u\mathrm{d}y) \tag{1}$$

Since $f'(z)$ is continuous, the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous on C and in the region R enclosed by C.

Hence Green's theorem in a plane, namely,

$$\int_C (P\mathrm{d}x + Q\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

can be applied to each of the lines integral in the R.H.S. of (1).

$$\therefore \quad \int_C f(z)\mathrm{d}z = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Since the function $f(z) = u + iv$ is analytic, u and v satisfy the Cauchy-Riemann equations in R.

i.e $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e $$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{3}$$

Using (3) in (2), we get

$$\int_C f(z)\mathrm{d}z = 0 + i0 = 0$$

Note ☑

1. The above theorem can be proved without assuming that $f'(z)$ is continuous, as was done by a French mathematician E. Goursat. We state below the modified form of the above theorem, called *Cauchy-Goursat theorem* without proof.

   "If $f(z)$ is analytic at all points on and inside a simple closed curve C, then $\int_C f(z)\mathrm{d}z = 0$"

2. We have proved Cauchy's integral theorem for a simply connected region. It can be extended to a multiply connected region as follows.

4.2.1 Extension of Cauchy's Integral Theorem

If $f(z)$ is analytic on and inside a multiply connected region whose outer boundary is C and inner boundaries are $C_1, C_2, \ldots, C_n$, then

$\int_C f(z)\mathrm{d}z = \int_{C_1} f(z)\mathrm{d}z + \int_{C_2} f(z)\mathrm{d}z + \cdots + \int_{C_n} f(z)\mathrm{d}z$ where all the integrals are taken in the same sense. We shall prove the extension for a doubly connected region for simplicity.

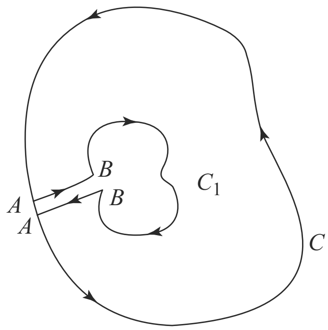


Fig. 4.3

We convert the doubly connected region into a simply connected region by introducing the cross-cut AB (Fig. 4.3).

By Cauchy's integral theorem, $\oint_{C'} f(z)\mathrm{d}z = 0$ where C' includes C described in anticlockwise sense, C_1 described in clockwise sense, AB and BA

$$\therefore \qquad \oint_{C'\circlearrowleft} f(z)\mathrm{d}z = \oint_{C\circlearrowleft} f(z)\mathrm{d}z + \oint_{C_1\circlearrowright} f(z)\mathrm{d}z + \oint_{AB} f(z)\mathrm{d}z + \oint_{BA} f(z)\mathrm{d}z = 0$$

The last two integrals in the R.H.S. are equal in value but opposite in sign and hence cancel each other.

$$\therefore \qquad \oint_{C\circlearrowleft} f(z)\mathrm{d}z + \oint_{C_1\circlearrowright} f(z)\mathrm{d}z = 0$$

i.e.

$$\oint_{C\circlearrowleft} f(z)\mathrm{d}z = -\oint_{C_1\circlearrowright} f(z)\mathrm{d}z$$

$$= \oint_{C_1\circlearrowleft} f(z)\mathrm{d}z$$

Note ☑ By introducing as many cross-cuts as the number of inner boundaries, we can give the proof in a similar manner for the extension of Cauchy's integral theorem stated above.

4.2.2 Cauchy's Integral Formula

If $f(z)$ is analytic inside and on a simple closed curve C that encloses a simply connected region R and if 'a' is any point in R, then $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\mathrm{d}z$, where C is described in the anticlockwise sense.

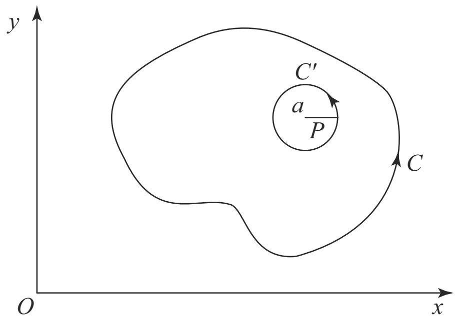


Fig. 4.4

Proof

Since $f(z)$ is analytic on and inside C, $\dfrac{f(z)}{z-a}$ is also analytic on and inside C, except at the point $z = a$. Now we draw a small circle C' with centre at $z = a$ and radius ρ, lying completely inside C. The function $\dfrac{f(z)}{z-a}$ is analytic in the doubly connected region bounded by C and C'.

$\therefore$ By Cauchy's Extended theorem, we have

$$\therefore \quad \oint_C \frac{f(z)}{z-a}\,\mathrm{d}z = \oint_{C'} \frac{f(z)}{z-a}\,\mathrm{d}z \tag{1}$$

If z is any point on C', then $|z - a| = \rho$ and hence $z - a = \rho e^{i\theta}$ or $z = a + \rho e^{i\theta}$

$$\therefore \quad \mathrm{d}z = i\rho e^{i\theta}\,\mathrm{d}\theta$$

$$\therefore \quad \oint_C \frac{f(z)}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\,\mathrm{d}\theta$$

($\because$ When z moves around C' once completely, θ varies from 0 to 2π)

$$= i\int_0^{2\pi} f(a+\rho e^{i\theta})\mathrm{d}\theta \tag{2}$$

(2) is true, however small the circle C' is and hence true when $\rho \to 0$. Taking limits of (2) as $\rho \to 0$, we get

$$\oint_{C'} \frac{f(z)}{z-a}\,\mathrm{d}z = i\int_0^{2\pi} f(a)\mathrm{d}\theta = if(a)\cdot 2\pi \tag{3}$$

Using (3) in (1), we have

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\therefore \qquad f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz.$$

4.2.3 Extension of Cauchy's Integral Formula to a Doubly Connected Region

If $f(z)$ is analytic on C_1 and C_2 (C_2 lies completely within C_1) and in the annular region R between C_1 and C_2 and if 'a' is any point in R, then

$$f(a) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,dz$$

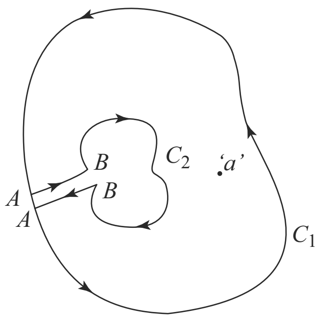


Fig. 4.5

Proof

We convert the doubly connected region to a simply connected region by introducing the cross-cut AB. 'a' lies in this region (Fig. 4.5).
By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz,$$

where C includes C_1 described in the anticlockwise sense, C_2 in the clockwise sense, AB and BA.

$$\therefore \quad f(a) = \frac{1}{2\,\pi i}\left[\oint_{C_1} \frac{f(z)}{z-a}\,dz + \oint_{C_1} \frac{f(z)}{z-a}\,dz + \int_{AB} \frac{f(z)}{z-a}\,dz + \int_{BA} \frac{f(z)}{z-a}\,dz\right]$$

$$= \frac{1}{2\,\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,dz + \frac{1}{2\,\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,dz$$ ($\because$ the last two integrals cancel each other)

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-a} \mathrm{d}z - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z-a} \mathrm{d}z$$

4.2.4 Cauchy's Integral Formulas for the Derivatives of an Analytic Function

By Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} \mathrm{d}z \tag{1}$$

Differentiating partially both sides of (1) with respect to 'a' and performing the differentiation within the integration symbol in the R.H.S., we get

$$f'(a) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} \mathrm{d}z \tag{2}$$

Proceeding further, we get

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} \mathrm{d}z \text{ etc.}$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z$$

WORKED EXAMPLE 4(a)

Example 4.1 If $f(z)$ is analytic in a simply connected region R, show that $\int_{z_0}^{z_1} f(z)\mathrm{d}z$ is independent of the path joining the points z_0 and z_1 in R and lying within R. Verify this by evaluating $\int_C (z^2 + 3z)\mathrm{d}z$ along (i) the circle $|z| = 2$ from (2, 0) to (0, 2) in the anticlockwise direction (ii) the straight line from (2, 0) to (0, 2) and (iii) the straight lines (2, 0) to (2, 2) and then from (2, 2) to (0, 2).

Let C_1 (ADB) and C_2 (AEB) be any two curves joining $A(z_0)$ and $B(z_1)$ in the region R. (Fig. 4.6)

Now $ADBEA$ may be regarded as a simple closed curve in R.

$\therefore$ By Cauchy's integral theorem,

$$\oint_{ADBEA} f(z)\mathrm{d}z = 0, \text{ i.e. } \int_{ADB} f(z)\mathrm{d}z + \int_{BEA} f(z)\mathrm{d}z = 0$$

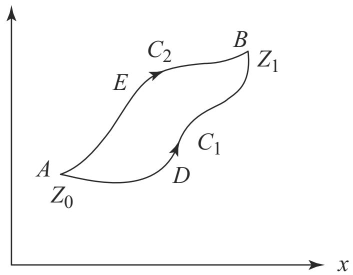


Fig. 4.6

i.e. $$\int_{ADB} f(z)\mathrm{d}z = -\int_{BEA} f(z)\mathrm{d}z = \int_{AEB} f(z)\mathrm{d}z$$

i.e. $$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z$$

$$\int_{C_1} (z^2+3z)\mathrm{d}z = \int_{|z|=2} (z^2+3z)\mathrm{d}z = \int_0^{\pi/2} \left[(2e^{i\theta})^2 + 3.2e^{i\theta}\right] 2e^{i\theta} i\mathrm{d}\theta$$

($\because$ on the circle $|z| = 2$, $z = 2e^{i\theta}$ and the end points are given by $\theta = 0$ and $\theta = \pi/2$)

$$= \left[8i\frac{e^{3i\theta}}{3i} + 12i\cdot\frac{e^{2i\theta}}{2i}\right]_0^{\pi/2}$$

$$= \frac{8}{3}(e^{i3\pi/2}-1) + 6(e^{i\pi}-1)$$

$$= \frac{8}{3}(-i-1) + 6(-2+i.0) = -\frac{44}{3} - \frac{8}{3}i$$

The equation of C_2, the line joining (2, 0) and (0, 2) is $x + y = 2$. [Fig. 4.7]

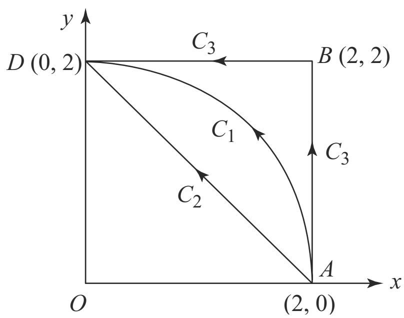


Fig. 4.7

$$\int_{C_2} (z^2+3z)\mathrm{d}z = \int_{C_2(x+y=2)} [x^2 - y^2 + i2xy + 3(x+iy)](\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_2^0 [x^2 - (2-x)^2 + i2x(2-x) + 3\{x + i(2-x)\}](\mathrm{d}x - i\mathrm{d}x)$$

$$= \int_2^0 [(-2x^2 + 8x + 2) + i(-2x^2 - 6x + 10)]\mathrm{d}x$$

$$= \left[-2\frac{x^3}{3} + 4x^2 + 2x + i\left(-\frac{2}{3}x^3 - 3x^2 + 10x \right) \right]_2^0$$

$$= -\frac{44}{3} \frac{-8}{3} i$$

$$\int_{C_3} (z^2 + 3z)\mathrm{d}z = \int_{\substack{AB \\ (x=2)}} + \int_{\substack{BD \\ (y=2)}} (z^2 + 3z)\mathrm{d}z$$

$$= \int_0^2 [(2 + iy)^2 + 3(2 + iy)]i\mathrm{d}y + \int_2^0 [(x + i2)^2 + 3(x + i2)]\mathrm{d}x$$

[$\because$ on AB, $\mathrm{d}x = 0$ and on BD, $\mathrm{d}y = 0$]

$$= \left[\frac{(2 + iy)^3}{3} + \frac{3(2 + iy)^2}{2} \right]_0^2 + \left[\frac{(x + i2)^3}{3} + \frac{3(x + i2)^2}{2} \right]_2^0$$

$$= \left(-\frac{8}{3} - 6 \right) - \frac{8}{3}i - 6 = -\frac{44}{3} - \frac{8}{3}i.$$

The values of the given integral are the same, irrespective of the curve joining the two points, since $f(z) = z^2 + 3z$ is analytic everywhere.

If the curve is not specified, the integral can be evaluated easily as follows, provided the integrand is an analytic function.

$$\int_{(2,0)}^{(0,2)} f(z)\mathrm{d}z = \int_{2+i0}^{0+2i} (z^2 + 3z)\mathrm{d}z = \left(\frac{z^3}{3} + \frac{3z^2}{2} \right)_2^{2i} = -\frac{44}{3} - \frac{8}{3}i.$$

Example 4.2 Evaluate $\int_0^{1+i} (x - y + ix^2)\mathrm{d}z$ along (i) the line joining z = 0 and $z = 1 + i$, (ii) the parabola $y = x^2$ and (iii) the curve $x = t$, $y = 2t - t^2$.

(i) The line joining the points $z = 0$ and $z = 1 + i$, i.e. the points (0, 0) and (1, 1) is $y = x$

$$\int_0^{1+i} (x - y + ix^2)(\mathrm{d}z + i\mathrm{d}y) = \int_0^1 (x - x + ix^2)(1 + i)\mathrm{d}x$$

$$= (-1 + i)\left(\frac{x^3}{3} \right)_0^1 = \frac{1}{3}(-1 + i).$$

(ii) When $y = x^2$, $\mathrm{d}y = 2x\ \mathrm{d}x$

$\therefore$ The given integral $= \int_0^1 (x - x^2 + ix^2)(\mathrm{d}x + i2x\mathrm{d}x)$

$$= \int_0^1 [(x - x^2 - 2x^3) + i(3x^2 - 2x^3)]\mathrm{d}x$$

$$= \left(\frac{1}{2} - \frac{1}{3} - \frac{2}{4}\right) + i\left(1 - \frac{2}{4}\right) = -\frac{1}{3} + \frac{i}{2}$$

(iii) (0, 0) corresponds to $t = 0$ and (1, 1) corresponds to $t = 1$.

$\therefore$ The given integral

$$= \int_0^1 (t - 2t + t^2 + i \cdot t^2)[\mathrm{d}t + i(2 - 2t)\mathrm{d}t]$$

$$= \int_0^1 [(2t^3 - t^2 - t) + i\,(-2t^3 + 5t^2 - 2t)]\,\mathrm{d}t$$

$$= \left(\frac{2}{4} - \frac{1}{3} - \frac{1}{2}\right) + i\left(-\frac{2}{4} + \frac{5}{3} - 1\right) = -\frac{1}{3} + \frac{1}{6}i$$

Note ☑ The values of the integral along three different curves are different, as the integrand is not an analytic function of z.

Example 4.3 Evaluate

(i) $\int_C \frac{\mathrm{d}z}{z - 2}$ and (ii) $\int_C (z - 2)^n \,\mathrm{d}z$

($n \neq -1$), where C is the circle whose centre is 2 and radius 4.

(i) The equation of the circle whose centre is 2 and radius 4 is $|z - 2| = 4$

$\therefore$ $z - 2 = 4e^{i\theta}$ and $\mathrm{d}z = 4e^{i\theta}\, i\mathrm{d}\theta$

To describe C once completely, θ has to vary from 0 to 2π.

$$\therefore \quad \int_C \frac{\mathrm{d}z}{z - 2} = \int_0^{2\pi} \frac{4e^{i\theta} i\mathrm{d}\theta}{4e^{i\theta}}. = i[\theta]_0^{2\pi} = 2\pi i$$

(ii) $\int_C (z - 2)^n \,\mathrm{d}z = \int_0^{2\pi} 4^n e^{in\theta}.4e^{i\theta} i\mathrm{d}\theta$

$$= 4^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \mathrm{d}\theta$$

$$= 4^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = 0, \text{since } n \neq -1$$

Aliter for (ii)

$(z-2)^n$ is analytic on and inside C.

$\therefore$ By Cauchy's integral theorem, $\int_C (z-2)^n \, dz = 0$

Example 4.4 Evaluate $\oint_C \frac{z+2}{z} dz$, where (i) C is the semicircle $|z| = 2$ in the upper half of the z-plane, (ii) C is the semicircle $|z| = 2$ in the lower half of the z-plane and (iii) C is the entire circle $|z| = 2$.

(i) On the semicircle $|z| = 2$, $z = 2e^{i\theta}$ and θ varies from 0 to π in the upper half.

$$\therefore \quad \int_C \frac{z+2}{z} dz = \int_0^{\pi} \frac{\left(2e^{i\theta}+2\right)}{2e^{i\theta}} \cdot 2e^{i\theta} i d\theta$$

$$= 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_0^{\pi} = 2\left(e^{i\pi}-1\right)+2\pi i$$

$$= -4 + 2\pi i$$

(ii) On the semicircle in the lower half, θ varies from π to 2π

$$\therefore \quad \int_C \frac{z+2}{z} dz = 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_{\pi}^{2\pi} = 4+2\pi i$$

(iii) On the entire circle, θ varies from 0 to 2π

$$\therefore \quad \int_C \frac{z+2}{z} dz = 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_0^{2\pi} = 4\pi i$$

= The sum of values of the integral along the two semi-circles.

Aliter for (iii)

$\int_{|z|=2} \frac{z+2}{z}$ is of the form $\int_C \frac{f(z)}{z-a} dz$, where $f(z)$ is analytic on and inside C that contains the point a.

Here $f(z) = z + 2$ is analytic on and inside $|z| = 2$, contains the point $z = 0$.

$\therefore$ By Cauchy's integral formula, we have

$$f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z} dz$$

$$\therefore \quad \int_C \frac{z+2}{z} dz = 2\pi i(0+2) = 4\pi i$$

Example 4.5 Evaluate $\int_C \frac{ze^z}{(z-a)^3}\,\mathrm{d}z$, where $z = a$ lies inside the closed curve C, using Cauchy's integral formula.

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^3}\,\mathrm{d}z = \frac{2\pi i}{2!} f''(a) \qquad (1)$$

where $f(z)$ is analytic on and inside C and the point $z = a$ lies inside C.

Comparing the given integral with the L.H.S. of (1), we find that $f(z) = ze^z$. ze^z is analytic everywhere and hence analytic on and inside C. Also 'a' lies inside C.

$$\therefore \qquad \int_C \frac{ze^z}{(z-a)^3}\,\mathrm{d}z = \frac{2\pi i}{2!}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(ze^z\right)\right]_{z=a}$$

$$= \pi i \,\{(z+2)e^z\}_{z=a}$$

$$= (a+2)e^a \pi i.$$

Example 4.6 Evaluate $\int_C \frac{\tan\frac{z}{2}}{(z-a)^2}\,\mathrm{d}z = (-2 < a < 2)$, where C is the boundary of the square whose sides lie along $x = \pm 2$ and $y = \pm 2$ described in the positive sense (Fig. 4.8).

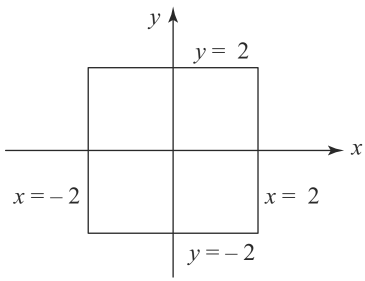


Fig. 4.8

Let $$\int_C \frac{\tan\frac{z}{2}}{(z-a)^2}\,\mathrm{d}z \equiv \int_C \frac{f(z)}{(z-a)^2}\,\mathrm{d}z$$

$f(z) = \tan\frac{z}{2}$ is analytic on and inside C, since $f'(z) = \frac{1}{2}\sec^2\frac{z}{2}$ does not exist at $z = \pm\pi, \pm 3\pi$ etc. which lie outside C.

Also, since $-2 < a < 2$, the point $z = a$ lies inside C.

Hence Cauchy's integral formula holds good.

$$\therefore \qquad \int \frac{\tan\dfrac{z}{2}}{(z-a)^2}\,dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\left(\tan\frac{z}{2}\right)\right]_{(z=a)}$$

$$= 2\pi i\cdot\left(\frac{1}{2}\sec^2\frac{z}{2}\right)_{z=a} = \pi i \sec^2\frac{a}{2}.$$

Example 4.7 Use Cauchy's integral formula to evaluate

$\displaystyle\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}\,dz$, where C is the circle $|z| = 4$.

$$\frac{1}{(z-2)(z-3)} = \frac{1}{(z-3)} - \frac{1}{(z-2)}$$

$$\therefore \qquad \text{Given integral } = \int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-3}\,dz - \int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\,dz$$

$$= \int_C \frac{f(z)}{z-3}\,dz - \int_C \frac{f(z)}{z-2}\,dz, \text{say} \qquad (1)$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic on and inside C.
The points $z = 2$ and $z = 3$ lie inside C (Fig.4.9).

$\therefore$ By Cauchy's integral formula, form (1), we get

$$\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}\,dz = 2\pi i\left(\sin\pi z^2+\cos z^2\right)_{z=3}$$

$$- 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=2}$$

$$= 2\pi i\,(\sin 9\pi + \cos 9\pi) - 2\pi i\,(\sin 4\pi + \cos 4\pi)$$

$$= - 2\pi i - 2\pi i = -4\pi i.$$

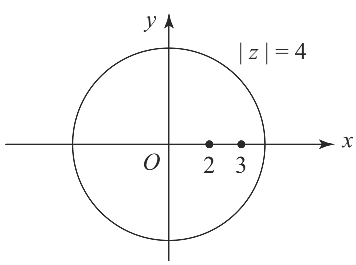


Fig. 4.9

Example 4.8 Evaluate $\int_C \frac{7z-1}{z^2-3z-4}\,dz$, where C is the ellipse $x^2 + 4y^2 = 4$, using Cauchy's integral formula. The ellipse $x^2+4y^2=4$ or $\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$ is the standard ellipse as shown in the Fig. 4.10.

$$\int_C \frac{7z-1}{z^2-3z-4}\,dz = \int_C \frac{7z-1}{(z-4)(z+1)}\,dz$$

Of the two points $z = 4$ and $z = –1$ the point $z = –1$ only lies inside C. Re-writing the given integral, we have

$$\int_C \frac{7z-1}{z^2-3z-4}\,dz = \int_C \frac{\left(\frac{7z-1}{z-4}\right)}{(z+1)}\,dz \equiv \int_C \frac{f(z)}{z+1}\,dz \qquad (1)$$

$f(z)=\frac{7z-1}{z-4}$ is analytic inside C and the point $z = –1$ lies inside C.

∴ By Cauchy's integral formula, form (1),

$$\int_C \frac{7z-1}{z^2-3z-4}\,dz = 2\pi i\, f(-1) = 2\pi i\left(\frac{-8}{-5}\right) = \frac{16}{5}\pi i.$$

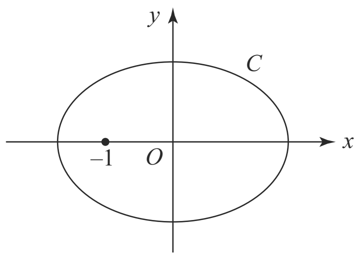


Fig. 4.10

Example 4.9 Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2|=\frac{1}{2}$, using Cauchy's integral formula.

$|z-2|=\frac{1}{2}$ is the circle with centre at $z = 2$ and radius equal to 1/2. The point $z = 2$ lies inside this circle (Fig. 4. 11).

The given integral can be re-written as

$$\int_C \frac{\left(\dfrac{z}{z-1}\right)}{(z-2)^2} \equiv \int_C \frac{f(z)}{(z-2)^2}\,\mathrm{d}z, \text{ say,}$$

$f(z)=\dfrac{z}{z-1}$ is analytic on and inside C and the point $z = 2$ lies inside C.

∴ By Cauchy's integral formula,

$$\int_C \frac{z}{(z-1)(z-2)^2}\,\mathrm{d}z = \frac{2\pi i}{1!} f'(2)$$

$$= 2\pi i \left\{ \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{z-1}\right)\right\}_{z=2}$$

$$= 2\pi i \left\{ \frac{-1}{(z-1)^2}\right\}_{z=2} = -2\pi i$$

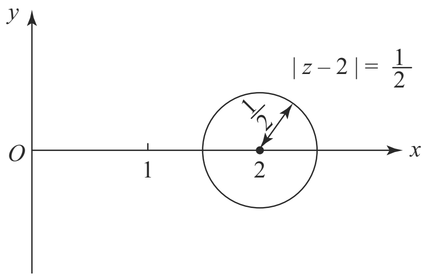


Fig. 4.11

Example 4.10 Use Cauchy's integral formula evaluate $\int_C \dfrac{z+1}{z^3-2z^2}\mathrm{d}z$, where C is the circle $|z - 2 - i| = 2$.

The circle $|z - (2 + i)| = 2$ is the circle whose centre is the point $z = 2 + i$ and radius is 2, as shown in Fig. 4.12. The point $z = 2$ lies inside this circle.

The given integral can be re-written as

$$\int_C \frac{\left(\dfrac{z+1}{z^2}\right)}{z-2}\mathrm{d}z \equiv \int_C \frac{f(z)}{z-2}\mathrm{d}z, \text{say.}$$

$f(z)=\dfrac{z+1}{z^2}$ is analytic on and inside C and the point $z = 2$ lies inside C.

∴ By Cauchy's integral formula,

$$\int_C \frac{z+1}{z^3-2z^2}\mathrm{d}z = 2\pi i\, f(2) = 2\pi i\left(\frac{z+1}{z^2}\right)_{z=2} = \frac{3}{2}\pi i.$$

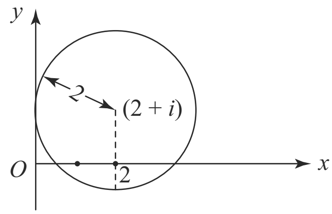


Fig. 4.12

Example 4.11 Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z + 1 + i| = 2$, using Cauchy's integral formula.

$|z + 1 + i| = 2$ is the circle whose centre is $z = -1 - i$ and radius is 2.

$$\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$$

∴ The integrand is not analytic at $z = -1 - 2i$ and $z = -1 + 2i$.

Of these, the point $z = -1 - 2i$ lies inside C. (Fig. 4.13)

Noting this, we rewrite the given integral as

$$\int_C \frac{\left(\dfrac{z+4}{z+1-2i}\right)}{z+1+2i}\,dz \equiv \int_C \frac{f(z)}{z-(-1-2i)}\,dz, \text{say.}$$

$f(z)$ is analytic on and inside C and the point $(-1 - 2i)$ lies inside C.

∴ By Cauchy's integral formula.

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\, f(-1-2i)$$

$$= 2\pi i \left\{ \frac{-1-2i+4}{-1-2i+1-2i} \right\}$$

$$= -\frac{\pi}{2}(3-2i)$$

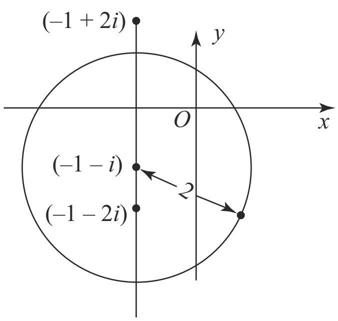


Fig. 4.13

Example 4.12 If $f(a)=\int_C \frac{3z^2+7z+1}{z-a}\,dz$, where C is the circle $|z| = 2$, find the values of $f(3)$, $f'(1-i)$ and $f''(1-i)$

Note ☑ In Cauchy's integral formula, $f(z)$ was used to denote the numerator of the integrand and $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$, but in this problem $f(a)$ is used to denote the value of the integeral.

$$f(a)=\int_C \frac{\phi(z)}{z-a}\,dz, \text{ where } \phi(z) = 3z^2+7z+1$$

$\therefore$ $$f(3)=\int_C \frac{\phi(z)}{z-3}\,dz \text{ and } \frac{\phi(z)}{z-3} \text{ is analytic on and inside } C.$$

$\therefore$ By Cauchy's integral theorem, $f(3) = 0$

Now $$f'(a)=\int_C \frac{\phi(z)}{(z-a)^2}\,dz$$

$$=\frac{2\pi i}{1!}\phi'(a), \text{ if the point '}a\text{' lies inside } C$$

(by Cauchy's integral formula)

The point $(1-i)$ lies within the circle C (Fig. 4.14).

$\therefore$ $$f'(1-i)=2\pi i\left[\frac{d}{dz}\left(3z^2+7z+1\right)\right]_{z=1-i}$$

$$= 2\pi i\,[(6(1-i)+7] = 2\pi\,(6+13i)$$

Also $$f''(a)=2\int_C \frac{\phi(z)}{(z-a)^3}\,dz = 2.\frac{2\pi i}{2!}\phi''(a)$$

$$=2\pi i\left[\frac{d^2}{dz^2}\left(3z^2+7z+1\right)\right]_{z=1-i}=12\pi i$$

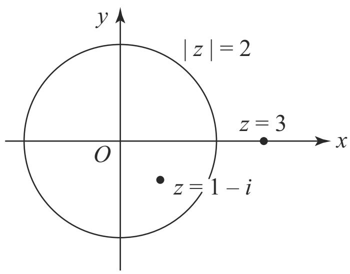


Fig. 4.14

EXERCISE 4(a)

Part A

(Short Answer Questions)

1. Define simply and multiply connected regions.
2. State Cauchy's integral theorem.

3. State Cauchy-Goursat theorem.
4. State Cauchy's extended integral theorem as applied to a multiply connected region.
5. State Cauchy's integral formula.
6. State Cauchy's extended integral formula, as applied to a doubly connected region.
7. State Cauchy's integral formula for the nth derivative of an analytic function.
8. Evaluate $\int_0^{3+i} |z|^2\, dz$, along the lines $3y = x$
9. Evaluate $\int (x^2 - iy^2)\, dz$ along the straight line from (0, 0) to (0, 1) and then from (0, 1) to (2, 1).
10. Evaluate $\int_C \bar{z}^2 dz$, where C is circle $|z - 1| = 1$.
11. Evaluate $\int_i^{2-i} \left(3xy + iy^2\right) dz$ along the line joining the points $z = i$ and $z = 2 - i$.
12. Evaluate $\int_C \frac{1}{z} dz$, where C is the semi-circular are $|z| = 1$ above the real axis.
13. Evaluate $\int_0^{1+i} \left(x^2 + iy\right) dz$ along the parabola

    (i) $y = x^2$ and (ii) $x = y^2$.
14. Evaluate $\int_C \log z\, dz$, where C is the circle $|z| = 2$.
15. Evaluate $\int_C \bar{z}\, dz$ along the curve $z = t^2 + it$ from 0 to $4 + 2i$.

Part B

16. Evaluate $\int_C \left(5z^4 - z^3 + 2\right) dz$ around (a) the circle $|z| = 1$, (b) the square with vertices at (0, 0), (1, 0) (1, 1) and (0,1) (c) the parabola $y = x^2$ from (0, 0) to (1, 1) and then the parabola $y^2 = x$ from (1, 1) to (0, 0).
17. Evaluate $\int_C \left(x^2 - iy^2\right) dz$ (i) the parabola $y = 2x^2$ from (1, 1) to (2, 8), (ii) the straight lines from (1, 1) to (1, 8) and then from (1, 8) to (2, 8), (iii) the straight line from (1, 1) to (2, 8).
18. Evaluate $\int_C \left(z^2 + 1\right)^2 dz$, along the arc of the cycloid $x = a\,(\theta - \sin \theta)$, $y = a\,(1 - \cos \theta)$ from the point $\theta = 0$ to the point $\theta = 2\pi$.

(***Hint:*** The end points are $z = 0$ and $z = 2\pi a$. The integrand is an analytic function and hence the value of the integral does not depend on the curve joining the end points.)

19. Evaluate $\int_C \frac{dz}{z-2-i}$, where C is the boundary of (i) the square bounded by the real and imaginary axes and the lines $x = 1$ and $y = 1$, (ii) the rectangle bounded by the real and imaginary axes and the lines $x = 3$ and $y = 2$, described in the counter clockwise sense.

Evaluate the following integrals using Cauchy's integral formula.

20. $\int_C \frac{\sinh 2z}{z^4} dz$ where C is the boundary of the square whose sides lie along $x = \pm 2$ and $y = \pm 2$, described in the positive sense.

21. $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$, where C is $|z| = 3$.

22. $\int_C \frac{z^2+1}{z^2-1} dz$, where C is the circle of unit radius with centre at (i) $z = 1$ and (ii) $z = i$.

23. $\int_C \frac{z^3+1}{z^2-3iz} dz$, where C is $|z| = 1$.

24. $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1} dz$, where C is $|z| = 3$.

25. $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is $|z| = 2$.

26. $\int_C \frac{dz}{(z^2+4)^2}$, where C is $|z - i| = 2$.

27. $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z + 1 - i| = 2$.

28. $\int_C \frac{z^3+z+1}{z^2-7z+6} dz$, where C is the ellipse $4x^2 + 9y^2 = 1$.

29. If $f(a) = \int_C \frac{4z^2+z+5}{z-a} dz$, where C is $|z| = 2$, find the values of $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$.

30. If $f(z)$ is analytic on and inside a simple closed curve C and z_0 is a point not lying on C, show that $\oint_C \frac{f'(z)}{z-z_0}\mathrm{d}z = \oint_C \frac{f(z)\mathrm{d}z}{(z-z_0)^2}$.

4.3 SERIES EXPANSIONS OF FUNCTIONS OF COMPLEX VARIABLE-POWER SERIES

4.3.1 Power Series

A series of the form $\sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$ is called *a power series* in powers of $(z - a)$, where z is a complex variable, the constants $c_0, c_1, c_2, \ldots$ are called the *coefficients* and the constant a is called the *centre* of the series.

Most of the definitions and theorems relating to convergence of infinite series of real terms, with which the reader is familiar, hold good for series of complex terms also.

It can be proved that there exists a positive number R such that the power series given above converges for $|z - a| < R$ and diverges for $|z - a| > R$, while it may or may not converge for $|z - a| = R$. This means that the power series converges at all points inside the circle $|z - a| = R$, diverges at all points outside the circle and may or may not converge on the circle. Due to this interpretation. R is called the *radius of convergence* of the above series and the circle $|z - a| = R$ is called the *circle of convergence*.

Power series play an important role in complex analysis, since they represent analytic functions and conversely every analytic function has a power series representation, called *Taylor series* that are similar to Taylor series in real calculus.

Analytic functions can also be represented by another type of series, called *Laurent's series*, which consist of positive and negative integral powers of the independent variable. They are useful for evaluating complex and real integrals, as will be seen later.

4.3.2 Taylor's Series (Taylor's Theorem)

If $f(z)$ is analytic inside a circle C_0 with centre at 'a' and radius r_0, then at each point z inside C_0,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots\infty$$

Proof

Let C_1 be any circle with centre at a and radius $r_1 < r_0$, containing the point z (Fig. 4.15). Let w be any point on C_1. Then, by Cauchy's integral formula,

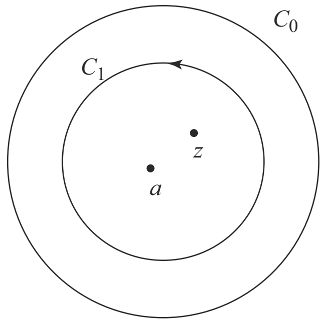


Fig. 4.15

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,\mathrm{d}w \qquad (1)$$

and $$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w \qquad (2)$$

Now $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)}\left\{1-\left(\frac{z-a}{w-a}\right)\right\}^{-1}$$

$$= \frac{1}{w-a}\left[\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}\right\} + \left(\frac{z-a}{w-a}\right)^n \cdot \frac{1}{1-\left(\frac{z-a}{w-a}\right)}\right]$$

$$\left(\because 1+\alpha+\alpha^2+\cdots\alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha} \text{ and so } 1+\alpha+\alpha^2+\cdots\alpha^{n-1}+\frac{\alpha^n}{1-\alpha} = \frac{1}{1-\alpha}\right)$$

i.e. $$\frac{1}{w-z} = \frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\cdots+\frac{(z-a)^{n-1}}{(w-a)^n}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z}$$

$$\therefore \quad \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)}\,\mathrm{d}w + (z-a)\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^2}\,\mathrm{d}w + \cdots + (z-a)^{n-1}\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^n}\,\mathrm{d}w + R_n,$$

where $$R_n = \frac{1}{2\pi i}\int_{C_1}\left(\frac{z-a}{w-a}\right)^n . \frac{f(w)}{(w-z)}\,\mathrm{d}z$$

i.e. $$f(z) = f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}+R_n, \text{by } (1) \text{ and } (2) \qquad (3)$$

Let $|z-a| = r$ and $|w-a| = r_1$.

$\therefore$ $\quad |w - z| = |(w - a) - (z - a)|$
$\geq r_1 - r$

Let M be the maximum value of $|f(w)|$ on C_1

Then $\quad |R_n| \leq \dfrac{r^n}{2\pi} \displaystyle\int_0^{2\pi} \dfrac{M}{r_1 - r} \dfrac{r_1 d\theta}{r_1^{\,n}} \left(\because w = r_1 e^{i\theta}\right)$

i.e. $\quad = \dfrac{r_1 M}{r_1 - r}\left(\dfrac{r}{r_1}\right)^n$

Since $\quad \dfrac{r}{r_1} < 1, \lim\limits_{n\to\infty}\left(\dfrac{r}{r_1}\right)^n = 0 \therefore |Rn| \to 0$ and so $Rn \to 0$ as $n \to \infty$.

Taking limits on both sides of (3) as n $\to \infty$, we get

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots\infty \tag{4}$$

Note ☑ For any point z inside C_0, we can always find C_1. So the Taylor series representation of $f(z)$ is valid for any z inside C_0.

The largest circle with centre at 'a' such that $f(z)$ is analytic at every point inside it is the circle of convergence of the Taylor's series and its radius is the radius of convergence of the Taylor's series. Clearly the radius of convergence is the distance between 'a' and the nearest singularity of $f(z)$.

2. Putting $a = 0$ in the Taylor's series, we get

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots\infty \tag{5}$$

This series is called *the Maclaurin's series* of $f(z)$.

3. The Maclaurin's series of some elementary functions, which can be derived by using (5), are given below:

(i) $e^z = 1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \ldots$, when $|z| < \infty$.

(ii) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} \ldots$, when $|z| < \infty$.

(iii) $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} \ldots$, when $|z| < \infty$.

(iv) $\sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} \ldots +$, when $|z| < \infty$.

(v) $\cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} \ldots$, when $|z| < \infty$.

(vi) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots$, when $|z| < 1$
(vii) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \ldots$, when $|z| < 1$.

4.3.3 Laurent's Series (Laurent's Theorem)

If C_1 and C_2 are two concentric circles with centre at *'a'* and radii r_1 and r_2 $(r_1 > r_2)$ and if $f(z)$ is analytic on C_1 and C_2 and throughout the annular region R between them, then at each point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where
$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w, n = 0, 1, 2, \ldots$$

and
$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, n = 1, 2, \ldots$$

Proof

By Extension of Cauchy's integral formula to a doubly connected region,

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-z)} \mathrm{d}w - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} \mathrm{d}w [\text{Fig. } 4.16] \tag{1}$$

$= I_1 + I_2$, say

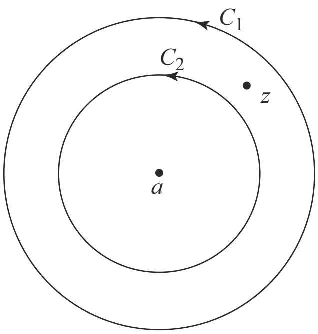


Fig. 4.16

In I_1, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{w-a}\left\{1 - \left(\frac{z-a}{w-a}\right)\right\}^{-1}$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3}$$

$$+ \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \frac{(z-a)^n}{(w-a)^n}\frac{1}{w-z} \tag{2}$$

since $|z-a| < |w-a|$ when w is on C_1

In I_2, $$-\frac{1}{w-z} = \frac{1}{(z-a)-(w-a)}$$

$$= \frac{1}{(z-a)}\left\{1 - \left(\frac{w-a}{z-a}\right)\right\}^{-1}$$

$$= \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \cdots + \frac{(w-a)^{n-1}}{(z-a)^n} + \frac{(w-a)^n}{(z-a)^n(z-w)}, \tag{3}$$

since $|w - a| < |z - a|$, when w is on C_2.

Using the expansion (2) in I_1, we have

$$I_1 = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a}\mathrm{d}w + \frac{z-a}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2}\mathrm{d}w + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^n}\mathrm{d}w + R_n$$

$$= a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + R_n,$$

where
$$R_n = \frac{(z-a)^n}{2\pi i}\oint_{C_1} \frac{f(w)\,\mathrm{d}w}{(w-a)^n(w-z)} \tag{4}$$

Using the expansion (3) in I_2, we have

$$I_2 = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{z-a}\mathrm{d}w + \frac{1}{2\pi i}\oint_{C_2} \frac{(w-a)}{(z-a)^2} f(w)\,\mathrm{d}w + \cdots$$

$$\cdots + \frac{1}{2\pi i}\oint_{C_2} \frac{(w-a)^{n-1} f(w)}{(z-a)^n}\mathrm{d}w + S_n.$$

$$= b_1(z-a)^{-1} + b_2(z-a)^{-2} + \cdots + b_n(z-a)^{-n} + S_n$$

where
$$S_n = \frac{1}{2\pi i(z-a)^n}\oint_{C_2} \frac{(w-a)^n f(w)}{z-w}\mathrm{d}w \tag{5}$$

If $|z - a| = r$, then $r_2 < r < r_1$.

$R_n \to 0$ as $n \to \infty$, as in Taylor's theorem.

Let M be the maximum value of $|f(w)|$ on C_2.

Then $\quad |S_n| \le \dfrac{1}{2\pi r^n}\displaystyle\int_0^{2\pi} \frac{r_2^n n}{r-r_2} r_2\,\mathrm{d}\theta \left(\because \text{on } C_2, w = r_2 e^{i\theta}\right)$

i.e $\quad = \dfrac{M r_2}{r-r_2}\left(\dfrac{r_2}{r}\right)^n$

$\therefore \quad |S_n| \to 0$ and so $S_n \to 0$ as $n \to \infty$, since $r_2 < r$.

Using (4) and (5) in (1) and taking limits as $n \to \infty$, we have $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$.

Note ☑

1. If $f(z)$ is analytic inside C_2, then the Laurent's series reduces to the Taylor series of $f(z)$ with centre a, since in this case all the coefficients of negative powers in Laurent's are zero.
2. As the Taylor's and Laurent's expansions in the given regions are unique, they are not usually found by the theorems given above, but by other simpler methods such as use of binomial series.
3. The part $\sum_{n=0}^{\infty} a_n (z-a)^n$, consisting of positive integral powers of $(z - a)$, is called *the analytic part* of the Laurent's series, while $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$, consisting of negative integral powers of $(z - a)$ is called the *principal part* of the Laurent's series.

4.4 CLASSIFICATION OF SINGULARITIES

We have stated, in Chapter 3, that a point at which $f(z)$ is not analytic is called a *singular point or singularity* of $f(z)$. We now consider various types of singularities.

1. Isolated Singularity

The point $z = a$ is called an *isolated singularity* of $f(z)$, if there is no other singularity in its neighbourhood. In other words, $z = a$ is called an isolated singularity of $f(z)$, if we can find a $\delta > 0$ such that the circle $|z - a| = \delta$ encloses no singularity other than a. If we cannot find any such δ, then 'a' is called a non-isolated singularity.

[If *'a'* is not a singularity and we can find $\delta > 0$ such that $|z - a| = \delta$ encloses no singularity, then *'a'* is called a *regular point* or *ordinary point* of $f(z)$.]

Note ☑

1. If a function has only a finite number of singularities in a region, those singlarities are necessarily isolated.

For example, $z = 1$ is an isolated singularity of $f(z) = \dfrac{1}{(z-1)^2}$ and $z = 0$ and $\pm i$ are isolated singularities of $f(z) = \dfrac{z+1}{z^3(z^2+1)}$.

2. If *'a'* is an isolated singularity of $f(z)$, then $f(z)$ can be expanded in Laurent's series valid throughout some neighbouhood of $z = a$ (except at $z = a$ itself), i.e., valid in $0 < |z - a| < r_1$. Here r_2 is chosen arbitrarily small.

2. Pole

If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z - a| < r_1$ has only a finite number of terms, then $z = a$ is called a *pole*.

i.e. if in
$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$
$b_m \neq 0$, $b_{m+1}=0=b_{m+2}\ldots$, then $z=a$ is called a *pole of order m*.
A pole of order 1 is called a *simple pole*.

For example, $z=0$ is a simple pole of $f(z)=\dfrac{1}{z(z-1)^2}$, as the Laurent's expansion.

of $f(z)$ valid in $0<|z|<1$ is given by $f(z)=\dfrac{1}{z}(1-z)^{-2}$

i.e.
$$f(z)=\frac{1}{z}\left(1+2z+3z^2+\cdots\infty\right)$$
$$=\left(2+3z+4z^2+\cdots\infty\right)+\frac{1}{z}$$
It has only one term i.e. $1/z$ in the principal part.
Similarly, $z=1$ is a pole of order 2 of $f(z)$, as the Laurent's expansion of $f(z)$, valid in $0<|z-1|<1$, is given by
$$f(z)=\frac{1}{(z-1)^2}\{1+(z-1)\}^{-1}$$
$$=\frac{1}{(z-1)^2}\left\{1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right\}$$
$$=\left\{1-(z-1)+(z-1)^2-\cdots\infty\right\}+\left\{-\frac{1}{z-1}+\frac{1}{(z-1)^2}\right\}$$
Here $b_2 \neq 0$ and $b_3=b_4=\cdots=0$.

3. Essential Singularity

If $z=a$ is an isolated singularity of $f(z)$ such that principal part of the Laurent's expansion of $f(z)$ at $z=a$, valid in $0<|z-a|<r_1$, has an infinite number of terms, then $z=a$ is called an *essential singularity*.

For example, $z=1$ is an essential singularity of $f(z)=e^{1/(z-1)}$, as the Laurent's

expansion is given by $f(z)=1+\dfrac{1}{1!}\dfrac{1}{z-1}+\dfrac{1}{2!}\dfrac{1}{(z-1)^2}+\dfrac{1}{3!(z-1)^3}+\cdots\infty.$

4. Removable Singularity

If a single-valued function $f(z)$ is not defined at $z=a$, but $\left[\lim_{z\to a} f(z)\right]$ exists, then z

$=a$ is called a *removable singularity*. For example, $z=0$ is a removable singularity of

$f(z) = \dfrac{\sin z}{z}$, as $f(0)$ is not defined, but $\lim\limits_{z \to 0}\left(\dfrac{\sin z}{z}\right) = 1$. The Laurent's expansion of

f (z) is given by

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \infty\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \infty$$

Note ☑

1. If $f(z)$ has a pole of order m at $z = a$, then its Laurent's expansion is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m},$$

where $b_m \neq 0$.

$$= \frac{1}{(z-a)^m}\left[\sum_{n=0}^{\infty} a_n (z-a)^{n+m} + b_m + b_{m-1}(z-a) + \cdots + b_1 (z-a)^{m-1}\right]$$

$$= \frac{1}{(z-a)^m}\phi(z), \text{say.}$$

Clearly, $\phi(z)$ is analytic everywhere that includes $z = a$ and $\phi(a) = b_m \neq 0$.

Thus for a function of the form $\dfrac{\phi(z)}{(z-a)^m}$, $z = a$ is a pole of order m, provided that $\phi(z)$ is analytic everywhere and $\phi(a) \neq 0$.

2. A function $f(z)$ which is analytic everywhere in the finite z-plane is called an *entire function* or *integral function*. An entire function can be represented by a Taylor's series whose radius of convergence is ∞ and conversely a power series whose radius of convergence is ∞ represents an entire function. The functions e^z, $\sin z$, $\cosh z$ are examples of entire functions.

3. A function $f(z)$ which is analytic everywhere in the finite plane except at a finite number of poles is called a *meromorphic function*.

For example, $f(z) = \dfrac{1}{z(z-1)^2}$ is a mermorphic function, as it has only two poles —a simple pole at $z = 0$ and a double pole at $z = 1$.

4.4.1 Residues and Evaluation of Residues

If *'a'* is an isolated singularity of any type for the function $f(z)$, then the coefficient of $\dfrac{1}{z-a}$ (viz. b_1) in the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z-a| < r_1$ is called *the residue* of $f(z)$ at $z = a$.

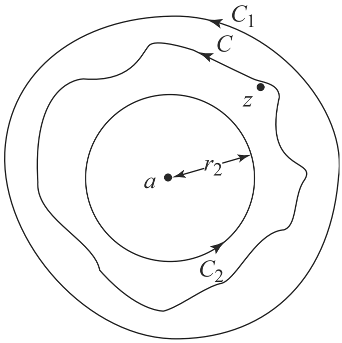


Fig. 4.17

We know, from Laurent's theorem, that $b_1 = \dfrac{1}{2\pi i}\oint_{C_2} f(w)\mathrm{d}w$, where C_2 is any circle $|z-a| = r_2 < r_1$, described in the anticlockwise sense. [Fig. 4.17]

Now if C is any closed curve around *'a'* such that $f(z)$ is analytic on and inside C except at *'a'* itself, then by Extension of Cauchy's integral theorem,

$$\oint_{C_2} f(w)\mathrm{d}w = \oint_{C} f(w)\mathrm{d}w.$$

Hence the residue of $f(z)$ *at* $z = a$ is also given by $[\text{Res.} f(z)]_{z=a} = \dfrac{1}{2\pi i}\oint_C f(z)\mathrm{d}z$, where C is any closed curve around *'a'* such that $f(z)$ is analytic on and inside it except at $z = a$ itself.

4.4.2 Formulas for the Evaluation of Residues

1. If $z = a$ is a simple pole of $f(z)$, then

$$\left[\text{Res.} f(z)\right]_{z=a} = \lim_{z\to a}\left\{(z-a)f(z)\right\}.$$

Since $z = a$ is a simple pole of $f(z)$, then the Laurent's expansion of $f(z)$ is of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$$

$$\therefore \quad (z-a)f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

$$\therefore \quad \lim_{z\to a}\left\{(z-a)f(z)\right\} = b_1 = \left[\text{Res.} f(z)\right]_{z=a}.$$

2. If $z = a$ is a simple pole of $f(z) = \dfrac{P(z)}{Q(z)}$ then [Ref. $f(z)]_{z=a}$

$$= \lim_{z \to a} \left(\frac{P(z)}{Q'(z)} \right).$$

By the previous formula,

$$[\text{Res.} f(z)]_{z=a} = \lim_{z \to a} \left[\frac{(z-a)P(z)}{Q(z)} \right] \to \frac{0}{0} \text{ form}$$

$$= \lim_{z \to a} \left[\frac{(z-a)P'(z) + P(z)}{Q'(z)} \right], \text{ by L'Hospital's rule}$$

$$= \frac{P(a)}{Q'(a)} \quad \text{or} \quad \lim_{z \to a} \left(\frac{P(z)}{Q'(z)} \right)$$

3. If $z = a$ is a pole of order m of $f(z)$, then

$$\left[\text{Res.} f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left\{ (z-a)^m f(z) \right\} \right]$$

Since $z = a$ is a pole of order m of $f(z)$, then the Laurent's expansion of $f(z)$ is of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_{m-1}}{(z-a)^{m-1}} + \frac{b_m}{(z-a)^m}$$

$$\therefore (z-a)^m f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+m} + b_1 (z-a)^{m-1} + b_2 (z-a)^{m-2} + \cdots +$$

$$b_{m-1}(z-a) + b_m.$$

$$\therefore \lim_{z \to a} \left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left\{ (z-a)^m f(z) \right\} \right] = (m-1)! b_1$$

$$\therefore b_1 = \left[\text{Res.} f(z)\right]_{z=a} = \frac{1}{(m-1)!}, \lim_{z \to a} \left[D^{m-1} \left\{ (z-a)^m f(z) \right\} \right]$$

where $\quad D \equiv \dfrac{\mathrm{d}}{\mathrm{d}z}.$

Note ☑ The residue at an essential singularity of $f(z)$ is found out using the Laurent's expansion of $f(z)$ directly.

4.4.3 Cauchy's Residue Theorem

If $f(z)$ is analytic on and inside a simple closed curve C, except for a finite number of singularities $a_1, a_2, \ldots, a_n$ lying inside C and if $R_1, R_2, \ldots, R_n$ are the residues of $f(z)$ at these singularities respectively, then

$$\oint_C f(z)\,dz = 2\pi i\left(R_1 + R_2 + \cdots + R_n\right).$$

Proof

We enclose the singularities a_i by simple closed curves C_i, which are not overlapping.
$f(z)$ is analytic in the multiply connected region bounded by the outer curve C and the inner curves $C_1, C_2, \ldots C_n$. (Fig. 4.18)
$\therefore$ By Extension of Cauchy's Integral theorem, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots \oint_{C_n} f(z)\,dz$$

$$= 2\pi i\left(R_1 + R_2 + \cdots + R_n\right), \text{ by the property of residue at a point.}$$

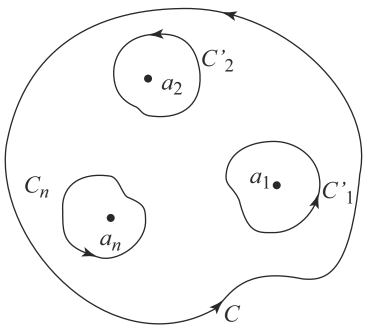


Fig. 4.18

Note ☑ Cauchy's integral formulas for $f(a)$ and $f^{(n)}(a)$ can be deduced from Cauchy's residue theorem.

Consider, $\int_C \frac{f(z)}{z-a}\,dz$, where $f(z)$ is analytic on and inside C enclosing the point *'a'*.

Now $\frac{f(z)}{z-a}$ has a simple pole at $z = a$ with residue $= \lim_{z\to a}\left\{(z-a)\frac{f(z)}{(z-a)}\right\} = f(a)$.

$\therefore$ By Cauchy's residue theorem, $\int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a)$.

$\frac{f(z)}{(z-a)^{n+1}}$ has a ploe of order $(n + 1)$ at $z = a$

with residue $= \dfrac{1}{n!}\lim_{z\to a} D^n\left[(z-a)^{n+1}\dfrac{f(z)}{(z-a)^{n+1}}\right]$

$$= \frac{1}{n!}f^{(n)}(a)$$

$\therefore$ $$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = 2\pi i\frac{f^{(n)}(a)}{n!}$$

or $$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

WORKED EXAMPLE 4(b)

Example 4.1 Expand each of the following functions in Taylor's series about the indicated point and also determine the region of convergence in each case:

(i) e^{2z} about $z = 2i$; (ii) $\cos z$ about $z = -\pi/2$;

(iii) $\dfrac{z+3}{(z-1)(z-4)}$ at $z = 2$; (iv) $\log\left(\dfrac{1+z}{1-z}\right)$ at $z = 0$.

(i) $e^{2z} = e^{2(z-2i)+4i} = e^{4i}.\, e^{2(z-2i)}$

Let $f(z) = e^{2(z-2i)}$. Then $f^{(n)}(z) = 2^n e^{2(z-2i)}$

$\therefore$ $f^{(n)}(2i) = 2^n;\ n = 1, 2, \ldots\infty.$

Also $f(2i) = 1.$

Taylor's series of $f(z)$ at $z = 2i$ is given by

$$f(z) = f(2i) + \frac{f'(2i)}{1!}(z-2i) + \frac{f''(2i)}{2!}(z-2i)^2 + \cdots$$

$\therefore$ $$e^{2z} = e^{4i}\left[1 + \frac{2(z-2i)}{1!} + \frac{2^2(z-2i)^2}{2!} + \frac{2^3(z-2i)^3}{3!} + \cdots\right]$$

The series converges in a circle whose centre is $2i$ and radius is the distance between $2i$ and the nearest singularity of $f(z)$ which is ∞.

$\therefore$ The region of convergence of the Taylor's series in this case is $|z - 2i| < \infty$.

(ii) Let $f(z) = \cos z;\ f'(z) = -\sin z;\ f''(z) = -\cos z;$

$$f'''(z) = \sin z;\ f^{\text{iv}}(z) = \cos z,\ \text{etc.}$$

$\therefore$ $$f(-\pi/2) = 0;\ f'(-\pi/2) = 1;\ f''(-\pi/2) = 0;$$

$$f'''(-\pi/2) = -1;\ f^{\text{iv}}(-\pi/2) = 0, \text{etc.}$$

Taylor's series of $f(z)$ at $z = -\pi/2$ is given by

$$f(z) = f(-\pi/2) + \frac{f'(-\pi/2)}{1!}(z+\pi/2) + \frac{f''(-\pi/2)}{2!}(z+\pi/2)^2 + \cdots$$

$$\therefore \qquad \cos z = \frac{(z+\pi/2)}{1!} - \frac{(z+\pi/2)^3}{2!} + \frac{(z+\pi/2)^5}{5!} - \frac{(z+\pi/2)^7}{7!} + \cdots$$

The region of convergence is $|z + \pi/2| < \infty$.

(iii) $$f(z) = \frac{z+3}{(z-1)(z-4)}$$

The Taylor's series of $f(z)$ at $z = 2$ is a series of powers of $(z - 2)$.
Putting $z - 2 = u$ or $z = u + 2$, we have

$$f(z) = \frac{u+5}{(u+1)(u-2)} = \frac{-4/3}{u+1} + \frac{7/3}{u-2},$$

on resolving into partial fractions

$$= -\frac{4}{3}(1+u)^{-1} - \frac{7}{6}(1-u/2)^{-1}$$

$$= -\frac{4}{3}\left\{1-u+u^2-u^3+\cdots\right\} - \frac{7}{6}\left\{1+\frac{u}{2}+\frac{u^2}{2^2}+\cdots\right\}$$

$$= -\frac{4}{3}\sum_{n=0}^{\infty}(-1)^n u^n - \frac{7}{6}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\left\{\frac{4}{3}(-1)^{n+1} - \frac{7}{6}\cdot\frac{1}{2^n}\right\}(z-2)^n$$

The Taylor's series expansion was obtained using binomial series, which are convergent in $|u| < 1$ and $|u/2| < 1$, i.e., in $|u| < 1$.

$\therefore$ The region of convergence of the Taylor's series of $f(z)$ is $|z - 2| < 1$.

Note ☑ The singularities of $f(z)$ are at $z = 1$ and $z = 4$ which lie outside the region of convergence.

(iv) $\log\left(\dfrac{1+z}{1-z}\right) = \log(1+z) - \log(1-z)$

Log $(1 \pm z)$ are many valued functions. We consider only those branches (values) which take the value zero when $z = 0$.

Now consider $f(z) = \log(1+z)$

$$f'(z) = (1+z)^{-1};\ f''(z) = -(1+z)^{-2};\ f'''(z) = (-1)(-2)(1+z)^{-3} \text{ etc.}$$

$$\therefore \qquad f(0) = 0;\ f'(0) = 1;\ f''(0) = -1;\ f'''(0) = 2!, \text{ etc.}$$

Taylor's series of $f(z)$ at $z = 0$ [or Maclaurin's series of $f(z)$] is

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots$$

$$\therefore \qquad \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \tag{1}$$

The only singularity of log $(1 + z)$ is -1 and its distance from the origin, i.e. the radius of convergence of series (1) is 1.

$\therefore$ The region of convergence of series (1) is given by $|z| < 1$.

Changing z into $-z$ in (1), the Taylor series of log $(1 - z)$ at $z = 0$ is given by

$$\log(1 - z) = -z - z^2/2 - z^3/3 - z^4/4 - \cdots$$

The region of convergence being $|z| < 1$

Using (1) and (2), we get

$$\log\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right), \text{ which converges for } |z| < 1.$$

Example 4.2 Expand each of the following functions in Laurent's series about $z = 0$. Identify the type of singularity also.

(i) ze^{-z^2} (ii) $(1 - \cos z)/z$ (iii) $z^{-1} e^{-2z}$ (iv) $\frac{1}{z^3} e^{z^2}$

(v) $(z - 1) \sin \frac{1}{z}$

(i)
$$ze^{-z^2} = z\left\{1 - \frac{z^2}{1!} + \frac{z^4}{2!} - \frac{z^6}{3!} + \cdots\right\}$$
$$= z - \frac{z^3}{1!} + \frac{z^5}{2!} - \frac{z^7}{3!} + \cdots \quad (1)$$

The Laurent's series (1) does not contain negative powers of z and the circle of convergence $|z| = \infty$ does not include any singularity $\therefore$ $z = 0$ is an ordinary point of ze^{-z^2}

(ii)
$$\frac{1 - \cos z}{z} = \frac{1}{z}\left[1 - \left\{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right\}\right]$$
$$= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots \quad (2)$$

Though $z = 0$ appears to be a singularity of $\frac{1 - \cos z}{z}$, the Laurent's series of $\frac{1 - \cos z}{z}$ at $z = 0$ does not contain negative powers of z.

$\therefore$ $z = 0$ is a removable singularity of $\frac{1 - \cos z}{z}$.

Note ☑ The value of $\left(\frac{1 - \cos z}{z}\right)$ at $z = 0$ is not defined; but $\lim_{z \to 0}\left(\frac{1 - \cos z}{z}\right) = 0$.

(iii)
$$z^{-1}e^{-2z} = \frac{1}{z}\left\{1 - \frac{2z}{1!} + \frac{2^2 z^2}{2!} - \frac{2^3 z^3}{3!} + \cdots\right\}$$
$$= \frac{1}{z} - 2 + \frac{2^2 z}{2!} - \frac{2^3 z^2}{3!} + \cdots \quad (3)$$

The principal part in the Laurent's series (3) contains the only term $\frac{1}{z}$.

$\therefore z = 0$ is a simple pole of $z^{-1} e^{-2z}$.

(iv)
$$\frac{1}{z^3} e^{z^2} = \frac{1}{z^3}\left\{1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \frac{z^6}{3!} + \cdots\right\}$$
$$= \frac{1}{z^3} + \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{3!} + \cdots \tag{4}$$

The last non-vanishing term with negative powers of z in the principal part of (4) is $\frac{1}{z^3}$.

$\therefore z = 0$ is pole of order 3 for the given function.

(v)
$$(z-1)\sin\left(\frac{1}{z}\right) = (z-1)\left[\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots\right]$$
$$= 1 - \frac{1}{z} - \frac{1}{3!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^4} - \cdots \tag{5}$$

The principal part consists of an infinite number of terms in the Laurent's series (5).

$\therefore z = 0$ is an essential singularity.

Example 4.3 Find the Laurent's series of $f(z) = \frac{1}{z(1-z)}$ valid in the region (i) $|z + 1| < 1$, (ii) $1 < |z + 1| < 2$ and (iii) $|z + 1| > 2$.

$f(z) = \frac{1}{z(1-z)}$; Since Laurent's series in powers of $(z + 1)$ are required,
put $z + 1 = u$ or $z = u - 1$.

$\therefore$
$$f(z) = \frac{1}{(u-1)(2-u)} = \frac{1}{u-1} + \frac{1}{2-u} \tag{1}$$

(i) Since the region of convergence of required Laurent's series is $|u| < 1$, the two terms in the R.H.S. of (1) should be rewritten as standard binomials whose first term is 1 and the numerator of the second term is u. Accordingly.

$$f(z) = -\frac{1}{1-u} + \frac{1}{2\left(1-\frac{u}{2}\right)} = -(1-u)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-}$$
$$= -\sum_{n=0}^{\infty} u^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$
$$= \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^{n+1}}\right)(z+1)^n \tag{2}$$

The Laurent's expansion (2) is valid, if $|u| < 1$ and $|u| < 2$, i.e. $|z + 1| < 1$.

(ii) Since the region of convergence of the required Laurent's series is given by $|u| > 1$ and $|u| < 2$, the first term in the R.H.S. (1) should be re-written as a standard binomial in such a way that u occurs in the denominator of the second member and the second term as a standard binomial in such a way that u occurs in the numerator of the second member.
Accordingly,

$$f(z) = \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{1}{2\left(1-\frac{u}{2}\right)}$$

$$= \frac{1}{u}(1-1/u)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z+1)^n \quad (3)$$

The Laurent's expansion (3) is valid, if

$$\left|\frac{1}{u}\right| < 1 \text{ and } \left|\frac{u}{2}\right| < 1 \text{ i.e., } |u| > 1 \text{ and } |u| < 2,$$

i.e. $1 < |z + 1| < 2.$

(iii) Since the region of convergence of the required Laurent's series is $|u| > 2$, both the terms in the R.H.S. of (1) should be rewritten as standard binomials in which u occurs in the denominator of the second member. Accordingly,

$$f(z) = \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{1}{u\left(1-\frac{2}{u}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1-\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} - \frac{1}{u}\sum_{n=0}^{\infty}\frac{2^n}{u^n}$$

$$= \sum_{n=0}^{\infty}(1-2^n)\frac{1}{(z+1)^{n+1}} \quad (4)$$

The Laurent's expansion (4) is valid, if

$$\left|\frac{1}{u}\right| < 1 \text{ and } \left|\frac{2}{u}\right| < 1, \text{i.e., } |u| > 1 \text{ and } |u| > 2 \text{ i.e., } |z+1| > 2$$

Example 4.4 Find all possible Laurent's expansions of the function $f(z) = \dfrac{4-3z}{z(1-z)(2-z)}$ about $z=0$. Indicate the region of convergence in each case. Find also the residue of $f(z)$ at $z = 0$, using the appropriate Laurent's series.

$$f(z) = \frac{4-3z}{z(1-z)(2-z)} = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z},$$

on resolving into partial fractions. Various Laurent's series about $z=0$ will be series of ascending and descending powers of z. They can be obtained by keeping $\dfrac{2}{z}$ unaltered and rewriting the other two terms as standard binomials in three different ways and expanding in powers of z.

Case (i)

$$f(z) = \frac{2}{z} + (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{z} + \sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} \tag{1}$$

$$= \frac{2}{z} + \sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right) z^n \text{ if } |z| < 1 \text{ and } |z| < 2, \text{ i.e. } |z| < 1.$$

But $|z| < 1$ includes $z = 0$, which is a singularity of $f(z)$. Laurent's series expansion is valid in an annular region which does not contain any singularity of the function expanded.

Hence the region of convergence of the Laurent's series, in this case, is $0 < |z| < 1$.

Case (ii)

$$f(z) = \frac{2}{z} - \frac{1}{z}(1-1/z)^{-1} + \frac{1}{2}(1-z/2)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= \frac{2}{z} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \tag{2}$$

if $\left|\dfrac{1}{z}\right| < 1$ and $\left|\dfrac{z}{2}\right| < 1$, i.e. $|z| > 1$ and $|z| < 2$

i.e. $1 < |z| < 2$, which represents the region of convergence of series (2).

Case (iii)

$$f(z) = \frac{2}{z} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

$$= \frac{2}{z} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$$= \frac{2}{z} - \sum_{n=0}^{\infty} (1 - 2^n) \frac{1}{z^{n+1}} \qquad (3)$$

if $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$, i.e. $|z| > 1$ and $|z| > 2$

i.e. $|z| > 2$ or $2 < |z| < \infty$, which represents the region of convergence of series (3).

Since $z = 0$ is an isolated singularity of $f(z)$, the residue at $z = 0$ is the coefficient of $\frac{1}{z}$ in the Laurent's expansion of $f(z)$, valid in the region $0 < |z| < 1$, i.e. in the series (1) above.

Thus $\quad [\text{Res.} f(z)]_{z=0} = 2.$

Example 4.5 Find the residues of $f(z) = \dfrac{z^2}{(z-1)(z+2)^2}$ at its isolated singularities using Laurent's series expansions.

$$f(z) = \frac{z^2}{(z-1)(z+2)^2} = \frac{1/9}{z-1} + \frac{8/9}{z+2} - \frac{4/3}{(z+2)^2},$$ on resolving into partial fractions.

Both $z = 1$ and $z = -2$ are isolated singularities of $f(z)$.

To find the residue of $f(z)$ at $z = 1$, we have to expand $f(z)$ in series of powers of $(z - 1)$, valid in $0 < |z - 1| < r$ and find the coefficient of $\frac{1}{z-1}$ in it.

Thus $$f(z) = \frac{1/9}{z-1} + \frac{8/9}{3+(z-1)} - \frac{4/3}{\{3+(z-1)\}^2}$$

$$= \frac{1/9}{z-1} + \frac{8}{27}\left\{1 + \frac{z-1}{3}\right\}^{-1} - \frac{4}{27}\left\{1 + \frac{z-1}{3}\right\}^{-2}$$

$$= \frac{1/9}{z-1} + \frac{8}{27}\sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^n}{3^n} - \frac{4}{27}\sum_{n=0}^{\infty}(-1)^n (n+1)\frac{(z-1)^n}{3^n} \qquad (1)$$

The expansion (1) is valid in the region $\left|\frac{z-1}{3}\right| < 1$, i.e. $0 < |z-1| < 3$.

[Residue of $f(z)$ at $z = 1$] = coefficient of $\frac{1}{(z-1)}$ in (1)

$$= \frac{1}{9}.$$

To find the residue of $f(z)$ at $z = -2$, we have to expand $f(z)$ in series of powers of $(z + 2)$, valid in $0 < |z + 2| < r$ and find the coefficient of $\frac{1}{z+2}$ in it.

Thus
$$f(z)=\frac{1/9}{(z+2)-3}+\frac{8/9}{z+2}-\frac{4/3}{(z+2)^2}$$
$$=-\frac{1}{27}\left\{1-\frac{z+2}{3}\right\}^{-1}+\frac{8/9}{z+2}-\frac{4/3}{(z+2)^2}$$
$$=-\frac{1}{27}\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^n}+\frac{8/9}{z+2}-\frac{4/3}{(z+2)^2} \tag{2}$$

The expansion (2) is valid in the region
$$\left|\frac{z+2}{3}\right|<1, \text{i.e.} 0<|z+2|<3.$$

[Residue of $f(z)$ at $z = -2$] = Coefficient of $\frac{1}{z+2}$ in (2)
$$=\frac{8}{9}.$$

Example 4.6 Find the singularities of $f(z)=\dfrac{z^2+4}{z^3+2z^2+2z}$ and the corresponding residues.

The singularities of $f(z)$ are given by
$$z^3+2z^2+2z=0,$$
i.e
$$z\left(z^2+2z+2\right)=0 \text{ i.e, } z=0,-1\pm i$$

For these values of z, the numerator $(z^2 + 4)$ does not vanish. Also it is analytic everywhere.

$\therefore z=0,-1\pm i$ are simple poles of $f(z)$.

$$\left[\text{Res.} f(z)\right]_{z=0}=\left[z\cdot f(z)\right]_{z=0}=\left(\frac{z^2+4}{z^2+2z+2}\right)=2.$$
$$\left[\text{Res.} f(z)\right]_{z=-1+i}=\left[(z+1-i)f(z)\right]_{z=-1+i}=\left\{\frac{z^2+4}{z(z+1+i)}\right\}_{z=-1+i}$$
$$=\frac{2-i}{-1-i}=\frac{1}{2}(-1+3i)$$

Since $z = -1 - i$ is the conjugate of $z = -1 + i$,
$$\left[\text{Res.} f(z)\right]_{z=-1-i}=\frac{-1}{2}(1+3i)$$

Example 4.7 Identify the singularities of $f(z)=\dfrac{z^2}{(z-2)^2\left(z^2+9\right)}$ and also find the residue at each singularity. The singularities of $f(z)$ are given by

$$(z-2)^2\left(z^2+9\right)=0, \text{ i.e. } z=2, \pm i3.$$

Of these $z = 2$ is a double pole and $z = \pm\, i3$ are simple poles.

$$\left[\text{Res.} f(z)\right]_{z=2}=\frac{1}{1!}\left[\frac{\mathrm{d}}{\mathrm{d}z}(z-2)^2 f(z)\right]$$

$$=\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^2}{z^2+9}\right)\right]_{z=2}=\left[\frac{18z}{\left(z^2+9\right)^2}\right]_{z=2}=\frac{36}{169}$$

$$\left[\text{Res.} f(z)\right]_{z=i3}=\left[(z-i3)f(z)\right]_{z=i3}=\left[\frac{z^2}{(z-2)^2(z+i3)}\right]_{z=i3}$$

$$=\frac{-9}{i6(i3-2)^2}=-\frac{3i}{2(5+12i)}$$

$$=-\frac{3}{338}(12+5i)$$

$$\left[\text{Res.} f(z)\right]_{z=i3}=\text{conjugate of }-\frac{3}{338}(12+5i)$$

$$=\frac{-3}{338}(12-5i)$$

Example 4.8 Find the singularities of $f(z)=\dfrac{z^2}{z^4+a^4}$ and also find the residue at each singularity. The singularities of $f(z)$ are given by $z^4 + a^4 = 0$, i.e., $z^4 = (-1)\, a^4 = e^{\,i(2r+1)\pi}.\, a^4$

i.e. $z = e^{i(2r+1)\pi/4}\, a$, where $r = 0, 1, 2, 3.$

$\therefore$ The singularities are $z=a\cdot e^{i\pi/4}, a\cdot e^{3\pi i/4}, a\cdot e^{5\pi i/4}$ and $a\cdot e^{7\pi i/4}$, all of which are simple poles.

$$\left[\text{Res.} f(z)\right]_{z=a\cdot e^{i\pi/4}}=\lim_{z\to e^{i\pi/4}}\left\{\frac{P(z)}{Q'(z)}\right\}, \text{as } f(z) \text{ is of the form } \frac{P(z)}{Q(z)}.$$

$$=\left(\frac{1}{4z}\right)_{z=ae^{i\pi/4}}=\frac{1}{4a}e^{-i\pi/4}=\frac{1}{4\sqrt{2}a}(1-i)$$

$$\left[\text{Res.} f(z)\right]_{z=ae^{i3\pi/4}}=\frac{1}{4a}e^{-i3\pi/4}=-\frac{1}{4\sqrt{2}a}(1+i)$$

$$\left[\text{Res.} f(z)\right]_{z=ae^{i5\pi/4}}=\frac{1}{4a}e^{-i5\pi/4}=\frac{1}{4\sqrt{2}a}(-1+i)$$

$$\left[\text{Res.} f(z)\right]_{z=ae^{i7\pi/4}}=\frac{1}{4a}e^{-i7\pi/4}=\frac{1}{4\sqrt{2}a}(1+i)$$

Example 4.9 Evaluate the following integrals, using Cauchy's residue theorem.

(i) $\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz$, where C is $|z| = 3$.

(ii) $\int \frac{e^z dz}{(z^2+\pi^2)^2}$, where C is $|z| = 4$.

(iii) $\int_C \frac{dz}{z \sin z}$, where C is $|z| = 1$

(i) $$\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz = \int_C f(z) dz, \text{ say.}$$

The singularities of $f(z)$ are $z = -1$ and $z = -2$, which are simple poles lying within the circle $|z| = 3$.

$$\left[\text{Re}s.f(z)\right]_{z=-1} = \left\{\frac{\cos \pi z^2 + \sin \pi z^2}{z+2}\right\}_{z=-1} = -1$$

$$\left[\text{Re}s.f(z)\right]_{z=-2} = \left\{\frac{\cos \pi z^2 + \sin \pi z^2}{z+1}\right\}_{z=-2} = -1$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \times \text{sum of the residues}$$
$$= -4\pi i.$$

(ii) Let $$\int_C f(z) dz = \int_C \frac{e^z dz}{(z^2+\pi^2)^2} = \int_C \frac{e^z dz}{(z-i\pi)^2 (z+i\pi)^2}$$

The singularities of $f(z)$ are $z = i\pi$ and $-i\pi$ each of which is a double pole of $f(z)$ and lies within the circle $|z| = 4$.

$$\left[\text{Res.} f(z)\right]_{z=i\pi} = R_1 = \frac{1}{1!}\left[\frac{d}{dz}(z-i\pi)^2 f(z)\right]_{z=i\pi}$$

$$= \left[\frac{d}{dz}\left\{\frac{e^z}{(z+i\pi)^2}\right\}\right]_{z=i\pi}$$

$$= \left[\frac{(z+i\pi)^2 e^z - 2e^z (z+i\pi)}{(z+i\pi)^4}\right]_{z=i\pi}$$

$$= \frac{e^{i\pi}(2i\pi-2)}{8i^3\pi^3} = \frac{(-1)(i\pi-1)}{-4i\pi^3} = \frac{1}{4\pi^3}(\pi+i)$$

$$\left[\text{Res. } f(z)\right]_{z=-i\pi} = R_2 = \frac{1}{4\pi^3}(\pi - i)$$

By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\left(R_1 + R_2\right)$$

$$= \frac{2\pi i}{4\pi^3} \times 2\pi = \frac{i}{\pi}$$

(iii) Let $\displaystyle\int_C \frac{\mathrm{d}z}{z\sin z} = \int_C f(z)\,\mathrm{d}z$

The singularity of $f(z)$ is given by

$$z\sin z = 0, \text{i.e.} z\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right\} = 0$$

i.e. $$z^2\left\{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right\} = 0$$

i.e. $z = 0$, which is a double pole of $f(z)$ lying within the circle $|z| = 1$.

$$\left[\text{Res. } f(z)\right]_{z=0} = \frac{1}{1!}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^2}{z\sin z}\right)\right\}_{z=0}$$

$$= \left(\frac{\sin z - z\cos z}{\sin^2 z}\right)_{z=0} \to \left(\frac{0}{0}\text{ form}\right)$$

$$= \left(\frac{z}{2\cos z}\right)_{z=0}, \text{ by L'Hospital's rule}$$

$$= 0$$

By Cauchy's residue theorem.

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \times 0 = 0.$$

Example 4.10 Evaluate the following integrals, using Cauchy's residue theorem.

(i) $\displaystyle\int_C \frac{(z+1)\,\mathrm{d}z}{z^2 + 2z + 4}$, where C is $|z + 1 + i| = 2$

(ii) $\displaystyle\int_C \frac{(12z - 7)\,\mathrm{d}z}{(z-1)^2(2z+3)}$, where C is $|z + i| = \sqrt{3}$

(iii) $\displaystyle\int_C \frac{\mathrm{d}z}{\left(z^2 + 9\right)^3}$, where C is $|z - i| = 3$

(i) Let $\int_C f(z)\,dz = \int_C \frac{(z+1)\,dz}{z^2+2z+4}$

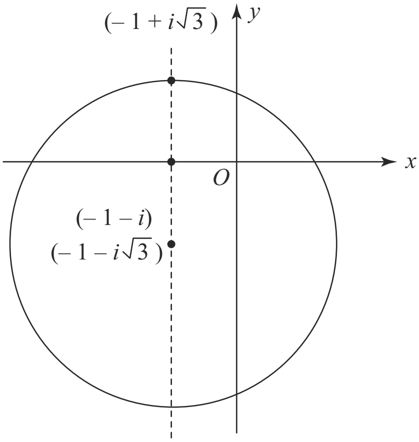


Fig. 4.19

The singularities of $f(z)$ are given by

$z^2 + 2z + 4 = 0$, i.e. $(z+1)^2 = -3$, i.e. $z = -1 \pm i\sqrt{3}$, each of which is a simple pole. Of the two poles $z = -1 - i\sqrt{3}$ alone lies inside the circle $|z-(-1-i)| = 2$, as shown in the Fig.4.19.

$$\{\text{Res.} f(z)\}_{z=-1-i\sqrt{3}} = \left\{\frac{z+1}{z+1-i\sqrt{3}}\right\}_{z=-1-i\sqrt{3}} = \frac{1}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \frac{1}{2} = \pi i$$

(ii) Let $\int_C \frac{(12z-7)}{(z-1)^2(2z+3)}\,dz = \int_C f(z)\,dz.$

The singularities of $f(z)$ are $z = 1$ and $z = -3/2$. $z = 1$ is a double pole and $z = -3/2$ is a simple pole. Only the pole $z = 1$ lies inside the circle $|z-(-i)| = \sqrt{3}$, as shown in the Fig. 4.20.

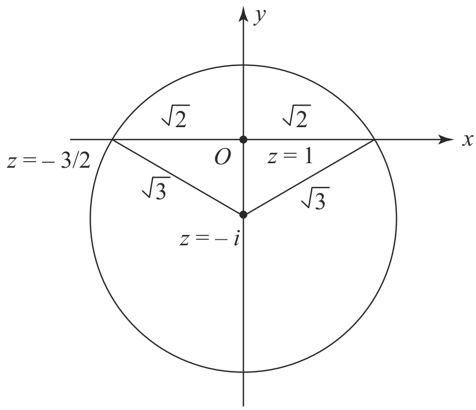


Fig. 4.20

$$[\text{Res.} f(z)]_{z=1} = \frac{1}{1!}\left\{\frac{d}{dz}\left(\frac{12z-7}{2z+3}\right)\right\}_{z=1}$$

$$= \left[\frac{50}{(2z+3)^2}\right]_{z=1} = 2.$$

By Cauchy's residue theorem,

$$\int_C f(z)\mathrm{d}z = 2\pi i \times 2 = 4\pi i.$$

(iii) Let $\displaystyle\int_C f(z)\mathrm{d}z = \int_C \frac{\mathrm{d}z}{(z^2+9)^3}$

The singularities of $f(z)$ are $z = \pm 3i$, of which $z = 3i$ lies inside the circle $|z - i| = 3$, as shown in Fig. 4.21.

$z = 3i$ is a triple pole of $f(z)$.

$$\therefore \quad [\text{Res.} f(z)]_{z=3i} = \frac{1}{2!}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{(z+3i)^3}\right]_{z=3i}$$

$$= \frac{1}{2!}\left[\frac{12}{(z+3i)^5}\right]_{z=3i}$$

$$= \frac{6}{6^5 i^5} = \frac{1}{1296i}$$

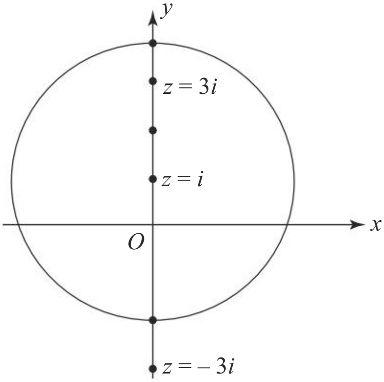


Fig. 4.21

By Cauchy's residue theorem,

$$\int_C f(z)\mathrm{d}z = 2\pi i \times \frac{1}{1296i} = \frac{\pi}{648}$$

EXERCISE 4(b)

Part A

(Short Answer Questions)

1. Define radius and circle of convergence of a power series.
2. State Taylor's theorem.
3. State Laurent's theorem.
4. What do you mean by analytic part and principal part of Laurent's series of a function of z?
5. Define regular point and isolated singularity of $f(z)$. Give one example for each.
6. Define simple pole and multiple pole of a function $f(z)$. Give one example for each.
7. Define essential singularity with an example.
8. Define removable singularity with an example.
9. Define entire function with an example.
10. Define meromorphic function with an example.
11. Define the residue of a function at an isolated singularity.
12. Express the residue of a function at an isolated singularity as a contour integral.

13. State two different formulas for finding the residue of a function at a simple pole.
14. State the formula for finding the residue of a function at a multiple pole.
15. State Cauchy's residue theorem.
16. Derive Cauchy's integral formula as a particular case of Cauchy's residue theorem.

Find the Taylor series for each of the following functions about the indicated point:

17. $\sin z$ about $z = \pi/4$.
18. $\cos z$ about $z = \pi/3$.
19. e^z about $z = -i$.
20. e^{-z} about $z = 1$.
21. $\dfrac{z-1}{z^2}$ about $z = 1$.

If each of the following functions is expanded as a Taylor's series about the indicated point, find the region of convergence in each case, without actually expanding.

22. $\dfrac{z-1}{z+1}$ about $z = 0$
23. $\dfrac{\sin z}{z^2+4}$ about $z = 0$
24. $\sec \pi z$ about $z = 1$
25. $\dfrac{z+3}{(z-1)(z-4)}$ about $z = 2$
26. $\dfrac{e^z}{z(z-1)}$ about $z = 4i$.

(***Hint:*** The radius of convergence is the distance between the centre of the Taylor's series and the nearest singularity of the concerned function.)

Find the Laurent's series of each of the following functions valid in the indicated regions:

27. $\dfrac{1}{z^2(z-2)}$, valid in $0 < |z| < 2$.
28. $\dfrac{1}{z(z-1)}$, valid in $0 < |z-1| < 1$.
29. $\dfrac{1}{z^3(1-z)}$, valid in $|z| > 1$.
30. $\dfrac{z-1}{z^2}$, valid in $|z-1| > 1$.

Find the residue at the essential singularity of each of the following functions, using Laurent's expansions:

31. $e^{1/z}$
32 $\dfrac{1-e^z}{z^2}$
33. $\dfrac{\cos z}{z}$
34. $\dfrac{\sinh z}{z^4}$
35. $\dfrac{1-\cosh z}{z^3}$

Find the residues at the isolated singularities of each of the following functions:

36. $\dfrac{z}{(z+1)(z-2)}$

37. $\dfrac{z^{2}}{z^2+a^2}$

38. $\cot z$ (at $z = 0$)

39. $\dfrac{ze^z}{(z-1)^2}$

40. $\dfrac{z\sin z}{(z-\pi)^3}$.

Evaluate the following integrals using Cauchy's residue theorem.

41. $\int_C \dfrac{z+1}{z(z-1)}\,dz$, where C is $|z| = 2$

42. $\int_C \dfrac{z^2}{z^2+1}\,dz$, where C is $|z| = 2$

43. $\int_C \dfrac{dz}{\sin z}\,dz$, where C is $|z| = 1$

44. $\int_C \tan z\,dz$, where C is $|z| = 2$

45. $\int_C \dfrac{e^{-z}}{z^2}\,dz$, where C is $|z| = 1$

Part B

46. Find the Taylor's series expansion of

$f(z) = \dfrac{1}{z(1-z)}$ about $z = -1$. States also the region of convergence of the series.

47. Find the Taylor's series expansion of

$f(z) = \dfrac{z}{z(z+1)(z+2)}$ about $z = i$.

State also the region of convergence of the series.

48. Find the Laurent's series of $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ valid in the region (i) $|z| < 2$, (ii) $2 < |z| < 3$ and (iii) $|z| > 3$.

49. Find the Laurent's series of $f(z) = \dfrac{z}{(z-1)(z-2)}$, valid in the region (i) $|z + 2| < 3$, (ii) $3 < |z + 2| < 4$; and (iii) $|z + 2| > 4$.

50. Find all possible Laurent's expansions of the function $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ about $z = -1$. Indicate the region of convergence in each case. Find also the residue of $f(z)$ at $z = -1$.

51. Find all possible Laurent's expansion of the function $f(z) = \dfrac{z^3 - 6z - 1}{(z-1)(z-3)(z+2)}$ about $z = 3$. Indicate the region of convergence in each case. Find also the residue of $f(z)$ at $z = 3$.

52. Find the residues of $f(z) = \dfrac{z^2}{(z-1)^2(z-2)}$ at its isolated singularities, using Laurent's series expansions.

53. By finding appropriate Laurent's expansions for $f(z) = \dfrac{1}{z^2(z^2+1)}$, find the residue at the poles of $f(z)$.

54. Indentify the singularities of $f(z)$ and find the corresponding residues, when

   (i) $f(z) = \dfrac{z}{z^2 + 2z + 5}$ (ii) $f(z) = \dfrac{1}{z^2(z^2 + 2z + 2)}$

55. Find the singularities of the following: (i) $f(z) = \dfrac{z}{(z+1)(z^2-1)}$

   (ii) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$; also find the corresponding residues.

56. Indentify the singularities of $f(z)$ and the corresponding residues, when

   (i) $f(z) = \dfrac{1}{z^4 + 16}$ (ii) $f(z) = \dfrac{z}{z^4 + a^4}$

57. Evaluate (i) $\int_C z e^{1/z} \mathrm{d}z$; and (ii) $\int_C \dfrac{\mathrm{d}z}{z^2 \sin z}$,

   using Cauchy's residue theorem, where C is the circle $|z| = 1$ in both cases.

58. Use Cauchy's residue theorem to evaluate the following

   (i) $\int_C \dfrac{\mathrm{d}z}{\sinh z}$, where C is the circle $|z| = 4$

   (ii) $\int_C \dfrac{z}{\cos z} \mathrm{d}z$, where C is the circle $\left| z - \dfrac{\pi}{2} \right| = \dfrac{\pi}{2}$.

59. Use Cauchy's residue theorem to evaluate the following:

   (i) $\int_C \dfrac{z\mathrm{d}z}{(z-1)(z-2)^2}$, where C is circle $|z - 2| = 1/2$.

   (ii) $\int_C \dfrac{(3z^2 + z)\mathrm{d}z}{(z-1)(z^2+9)}$, where C is the circle $|z - 2| = 2$.

   (iii) $\int_C \dfrac{(3z^2 + z + 1)\mathrm{d}z}{(z^2-1)(z+3)}$, where C is the circle $|z - i| = 2$.

60. Use Cauchy's residue theorem to evaluate the following:

(i) $\int_C \frac{(z-3)dz}{z^2+2z+5}$, where C is the circle $|z+1-i|=2$

(ii) $\int_C \frac{dz}{(z^2+4)^3}$, where C is the circle $|z-i|=2$

(iii) $\int_C \frac{(z-1)dz}{(z+1)^2(z-2)}$, where C is the circle $|z+i|=2$.

4.5 CONTOUR INTEGRATION—EVALUATION OF REAL INTEGRALS

The evaluation of certain types of real definite integrals can be done by expressing them in terms of integrals of complex functions over suitable closed paths or contours and applying Cauchy's residue theorem. This process of evaluation of definite integrals is known as contour integration.

We shall consider only three types of definite integrals which are commonly used in applications.

4.5.1 Type 1

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials in x. This integral converges (exists), if

(i) The degree of $Q(x)$ is at least 2 greater than the degree of $P(x)$.
(ii) $Q(x)$ does not vanish for any real value of x.

To evaluate this integral, we evaluate $\int_C \frac{P(z)}{Q(z)} dz$,

where C is the closed contour consisting of the real axis from $-R$ to $+R$ and the semicircle S above the real axis having this line as diameter, by using Cauchy's residue theorem and then letting $R \to \infty$. The contour C is shown in Fig. 4.22.

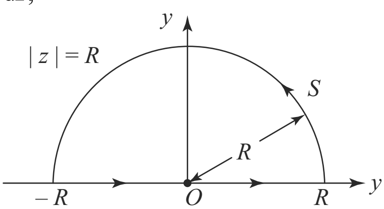


Fig. 4.22

Now we consider a result, known as Cauchy's Lemma, which will be used in evaluation of $\int_S f(z)dz$, where S is the semicircle $|z| = R$ above the real axis.

Cauchy's Lemma

If $f(z)$ is a continuous function such that

$|zf(z)| \to 0$ uniformly as $|z| \to \infty$ on S, then $\int_S f(z)dz \to 0$, as $R \to \infty$, where S is the semicircle $|z| = R$ above the real axis.

Proof

$$\left|\int_S f(z)\mathrm{d}z\right| = \left|\int_S zf(z)\cdot\frac{1}{z}\mathrm{d}z\right|$$

$$\leq \int_S |zf(z)|\cdot\frac{1}{|z|}\cdot|\mathrm{d}z|$$

i. e.
$$\int_0^{\pi} |zf(z)|\cdot\frac{1}{R}\cdot R\mathrm{d}\theta$$

[$\because$ When $|z| = R$, $z = Re^{i\theta}$ and $\mathrm{d}z = Rie^{i\theta}\mathrm{d}\theta$]

$\therefore$
$$\lim_{R\to\infty}\left|\int_S f(z)\mathrm{d}z\right| \leq \pi\times\lim_{R\to\infty}|zf(z)|$$

i. e. ≤ 0, by the given condition.

$\therefore$
$$\left|\int_S f(z)\mathrm{d}z\right| = 0 \text{ and hence } \int_S f(z)\mathrm{d}z = 0, \text{as } R\to\infty.$$

Note ☑ Similarly, we can shown that $\int_{S_1} f(z)\mathrm{d}z \to 0$ as $r \to 0$, where S_1 is the semicircle $|z - a| = r$ above the real axis, provided that $f(z)$ is a continuous function such that $|(z - a)f(z)| \to 0$ as $r \to 0$ uniformly.

4.5.2 Type 2

Integrals of the form $\int_{-\infty}^{\infty}\frac{P(x)\sin mx}{Q(x)}\mathrm{d}x$ or $\int_{-\infty}^{\infty}\frac{P(x)\cos mx}{Q(x)}\mathrm{d}x$, where $P(x)$ and $Q(x)$ are polynomials in x.

This integral converges (exists), if

(i) $m > 0$
(ii) the degree of $Q(x)$ is greater than the degree of $P(x)$
(iii) $Q(x)$ does not vanish for any real value of x. To evaluate this integral, we evaluate $\int_C \frac{P(z)}{Q(z)}e^{imz}\mathrm{d}z$, where C is the same contour as in Type 1, by using Cauchy's residue theorem and letting $R\to\infty$. After getting $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}e^{imx}\mathrm{d}x$, we find the value of the real part or the imaginary part of this integral as per the requirement.

A result, known as Jordan's Lemma, which will be used in the evaluation of $\int_S f(z)e^{imz}\,dz$, where S is the semicircle $|z| = R$ above the real axis, is given as follows.

Jordan's Lemma

If $f(z)$ is a continuous function such that $|f(z)| \to 0$ uniformly as $|z| \to \infty$ on S, then $\int_S e^{imz} f(z)dz \to 0$ as $R \to \infty$, where S is he semicircle $|z| = R$ above the real axis and $m > 0$.

Proof

On the circle $|z|\ R$, $z = Re^{i\theta}$.

$$\therefore \quad \left|\int_S e^{imz} f(z)dz\right| = \left|\int_S e^{imR(\cos\theta + i\sin\theta)}.f(z).iRe^{i\theta}d\theta\right|$$

$$\leq \int_0^{\pi} \left|e^{imR(\cos\theta+i\sin\theta)}\right| |\,f(z)\,|\,Rd\theta$$

i.e.
$$\leq \int_0^{\pi} e^{-mR\sin\theta}\ |\,f(z)|\,Rd\theta$$

i.e.
$$\leq 2\int_0^{\pi/2} e^{-mR\sin\theta}\ |\,f(z)\,|\,Rd\theta \qquad (1)$$

Now
$$\sin\theta \geq \frac{2\theta}{\pi}, \text{for } 0 \leq \theta \leq \pi/2$$

$$\therefore \quad -mR\sin\theta \leq -\frac{2mR\theta}{\pi}, \text{for } 0 \leq \theta \leq \pi/2$$

$$\therefore \quad e^{-mR\sin\theta} \leq e^{-2mR\theta/\pi}, \text{ for } 0 \leq \theta \leq \pi/2 \qquad (2)$$

Using (2) in (1), we have

$$\left|\int_S e^{imz} f(z)dz\right| \leq 2\int_0^{\pi/2} e^{-2mR\theta/\pi}\ |\,f(z)|\,Rd\theta$$

$$\therefore \quad \lim_{R\to\infty}\left|\int_S e^{imz} f(z)dz\right| \leq \int_0^{\pi/2} \lim_{R\to\infty}(2Re^{-2mR\theta/\pi}) \lim_{R\to\infty}[f(z)]d\theta$$

i.e.
$$\leq \lim_{R\to\infty}[f(z)] \times \lim_{R\to\infty}\int_0^{\pi/2} 2Re^{-2mR\theta/\pi}d\theta$$

i.e.
$$\leq \lim_{R\to\infty}[f(z)] \times \lim_{R\to\infty}\left[\frac{\pi}{m}\left(1-e^{-mR}\right)\right]$$

i.e. $\leq \frac{\pi}{m} \times 0$, by the given condition.

$\therefore$ $\left| \int_S e^{imz} f(z) \mathrm{d}z \right| = 0$ and hence $\int_S e^{imz} f(z) \mathrm{d}z = 0$

as $R \to \infty$.

4.5.3 Type 3

Integrals of the form $\int_0^{2\pi} \frac{P(\sin\theta, \cos\theta)}{Q(\sin\theta, \cos\theta)} \mathrm{d}\theta$, where P and Q are polynomials in sin θ and cos θ.

To evaluate this integral, we take the unit circle $|z| = 1$ as the contour.

When $|z| = 1$, $z = e^{i\theta}$ and so $\sin\theta = \frac{z - z^{-1}}{2i}$ and $\cos\theta = \frac{z + z^{-1}}{2}$. Also $\mathrm{d}z = e^{i\theta} i\mathrm{d}\theta$ or $\mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$. When θ varies from 0 to 2π, the point z moves once around the unit circle $|z| = 1$.

Thus $\int_0^{2\pi} \frac{P(\sin\theta, \cos\theta)}{Q(\sin\theta, \cos\theta)} \mathrm{d}\theta = \int_C f(z)\, \mathrm{d}z$,

where $f(z)$ is a rational function of z and C is $|z| = 1$.

Now applying Cauchy's residue theorem, we can evaluate the integral on the R.H.S.

WORKED EXAMPLE 4(c)

Example 4.1 Evaluate $\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{(x^2 + a^2)(x^2 + b^2)}$, using contour integration, where $a > b > 0$.

Consider $\int_C \frac{z^2 \mathrm{d}z}{(z^2 + a^2)(z^2 + b^2)}$, where C is the contour consisting of the segment of the real axis from $-R$ to $+R$ and the semicircle S above the real axis having this segment as diameter. The singularities of the integrand are given by

$$(z^2 + a^2)(z^2 + b^2) = 0$$

i.e. $z = \pm ia$ and $z = \pm ib$, which are simple poles.

Of these poles, only $z = ia$ and $z = ib$ lie inside C. (Fig. 4.23)

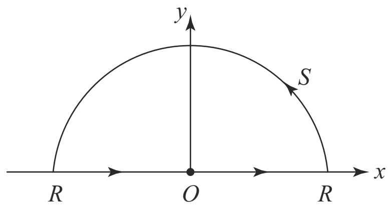


Fig. 4.23

Now $$R_1 = (\text{Res. of the integrand})_{z=ia} = \left[\frac{z^2}{(z+ia)(z^2+b^2)}\right]_{z=ia}$$

$$= \frac{-a^2}{2ia(b^2-a^2)} = \frac{a}{2i(a^2-b^2)}$$

Similarly $$R_2 = \text{Res. at } (z = ib) = -\frac{b}{2i(a^2-b^2)}$$

By Cauchy's residue theorem,

$$\int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}\mathrm{d}z = 2\pi i(R_1+R_2)$$

$$= \frac{\pi(a-b)}{a^2-b^2} = \frac{\pi}{a+b}$$

i.e., $$\int_{-R}^{R} \frac{x^2\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} + \int_S \frac{z^2\mathrm{d}z}{(z^2+a^2)(z^2+b^2)} = \frac{\pi}{a+b} \quad (1)$$

($\because$ on the real axis, $z = x$ and so $\mathrm{d}z = \mathrm{d}x$)

Now $|z^2 + a^2| \geq |z|^2 - a^2 = R^2 - a^2$ on $|z| = R$.

Similarly $|z^2 + b^2| \geq R^2 - b^2$ on $|z| = R$.

$\therefore$ $$\left|z.\frac{z^2}{(z^2+a^2)(z^2+b^2)}\right| \leq \frac{R^3}{(R^2-a^2)(R^2-b^2)}$$

Since the limit of the R.H.S. is zero as $R \to \infty$,

$$\lim_{R\to\infty}\left|z.\frac{z^2}{(z^2+a^2)(z^2+b^2)}\right| = 0, \text{on } |z| = R.$$

$\therefore$ By Cauchy's lemma, $$\int_S \frac{z^2\mathrm{d}z}{(z^2+a^2)(z^2+b^2)} = 0 \quad (2)$$

Using (2) in (1) and letting $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{x^2\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Example 4.2 Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1^2)(x^2+2x+2)}$, using contour integration.

Consider $\int_C \frac{z^2 dz}{(z^2+1^2)(z^2+2z+2)}$, where C is the same contour as in the previous example. The singularities of $f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$ are given by $(z^2 + 1)^2$ $(z^2 + 2z + 2) = 0$

i.e. $z = \pm\, i$ and $z = -1 \pm i$, of which $z = i$ and $z = -1 + i$ lie inside C.

$z = i$ is double pole and $z = -1 + i$ is a simple pole.

$$R_1 = [\text{Res.} f(z)_{z=i} = \frac{1}{1!}\left[\frac{d}{dz}\left\{\frac{(z-i)^2 \cdot z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\}\right]_{z=i}$$

$$= \left[\frac{(z+i)^2(z^2+2z+2).2z - z^2\{2(z+i)(z^2+2z+2)+(z+i)^2(2z+2)\}}{(z+i)^4(z^2+2z+2)^2}\right]_{z=i}$$

$$= \frac{9i-12}{100}$$

$$R_2 = [\text{Res.} f(z)]_{z=-1+i} = \left[\frac{z^2}{(z^2+1)^2(z+1+i)}\right]_{z=-1+i} = \frac{3-4i}{25}$$

By Cauchy's residue theorem,

$$\int_C \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i(R_1 + R_2)$$

$$= 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\}$$

$$= \frac{7\pi}{50}$$

i.e.
$$\int_{-R}^{R} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} + \int_S \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50} \qquad (1)$$

$$\lim_{R\to\infty}\left|\frac{z.z^2}{(z^2+1)^2(z^2+2z+2)}\right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Cauchy's Lemma, when $R \to \infty$

$$\int_S \frac{z^2 \mathrm{d}z}{(z^2+1)^2 (z+2z+2)} = 0 \tag{2}$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{(x^2+1)^2 (x^2+2x+2)} = \frac{7\pi}{50}$$

Example 4.3 Evaluate $\int_0^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^3}$, using contour integration, where $a > 0$.

Consider $\int_C \frac{\mathrm{d}z}{(z^2+a^2)^3}$, where C is the same contour as in example 4.1.

The singularities of $f(z) = \frac{1}{(z^2+a^2)^3}$ are $z = \pm\, ia$,which are poles of order 3. Of the two poles, $z = ia$ alone lies inside C.

$$R = [\mathrm{Res}.f(z)]_{z=ia} = \frac{1}{2!}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{\frac{1}{(z+ia)^3}\right\}\right]_{z=ia}$$

$$= \left[\frac{6}{(z+ia)^5}\right]_{z=ia} = \frac{3}{16a^5 i}$$

$\therefore$ By Cauchy's residue theorem,

$$\int_C \frac{\mathrm{d}z}{(z^2+a^2)^3} = 2\pi i R = 2\pi i \,.\, \frac{3}{16a^5 i} = \frac{3\pi}{8a^5}$$

i.e.
$$\int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+a^2)^3} + \int_S \frac{\mathrm{d}z}{(z^2+a^2)^3} = \frac{3\pi}{8a^5} \tag{1}$$

Now
$$\left| z.\frac{1}{(z^2+a^2)^3} \right| \le \frac{R}{(R^2-a^2)^3} \text{ and so}$$

$$\lim_{R\to\infty} \left| z.\frac{1}{(z^2+a^2)^3} \right| = 0 \text{ on } |z| = R$$

$\therefore$ By Cauchy's Lemma, when $R \to \infty$,

$$\int_S \frac{\mathrm{d}z}{(z^2+a^2)^3} = 0 \tag{2}$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{8a^5}.$$

Since the integrand $\frac{1}{(x^2+a^2)^3}$ is an even function of x, we have $\int_0^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{16a^5}$

Example 4.4 Use contour integration to prove that

$$\int_0^{\infty} \frac{x^2 dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a}, \text{ where } a > 0.$$

Consider $\int_C \frac{z^2 dz}{z^4+a^4}$, where C is the same contour as in Example 4.1.

The singularities of $f(z) = \frac{z^2}{z^4+a^4}$ are given by $z^4 = -a^4 = e^{i(2r+1)\pi}.\ a^4$

i.e. $z = e^{i(2r+1)\pi/4}.\ a$, where $r = 0, 1, 2, 3$.

i.e. $z = ae^{i\pi/4}, ae^{i3\pi|4}, ae^{i5\pi/4}, ae^{i7\pi/4}$, all of which are simple poles.

Of these poles, only $z = ae^{i\pi/4}$ and $z = ae^{i3\pi/4}$ lie inside C.

$$R_1 = [\text{Res.} f(z)]_{z=aei\pi/4} = \lim_{z \to ae^{i\pi/4}} \left[\frac{z^2}{4z^3}\right] \left[= \lim_{z \to a} \frac{P(z)}{Q^1(z)}\right]$$

$$= \frac{1}{4a} e^{-i\pi/4} = \frac{1}{4\sqrt{2}a}(1-i)$$

$$R_2 = [\text{Res.} f(z)_{z=aei3\pi/4} = \frac{1}{4a} e^{-i3\pi/4} = \frac{1}{4\sqrt{2}a}(-1-i)$$

∴ By Cauchy's residue theorem,

$$\int_C \frac{z^2 dz}{z^4+a^4} = 2\pi i \cdot \frac{1}{4\sqrt{2}a}(1-i-1-i) = \frac{\pi}{\sqrt{2}a}$$

i.e. $$\int_{-R}^{R} \frac{x^2 dx}{x^4+a^4} + \int_S \frac{z^2 dz}{z^4+a^4} = \frac{\pi}{\sqrt{2}a} \qquad (1)$$

Now $$\lim_{r \to \infty} \left| z.\frac{z^2}{z^4+a^4} \right| = 0 \text{ on } |z| = R.$$

∴ By Cauchy's Lemma, when $R \to \infty$,

$$\int_s \frac{z^2 dz}{z^4+a^4} = 0 \qquad (2)$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{x^4 + a^4} = \frac{\pi}{\sqrt{2}a}$$

Since the integrand is even,

$$\int_{0}^{\infty} \frac{x^2 \mathrm{d}x}{x^4 + a^4} = \frac{\pi}{2\sqrt{2}a}$$

Example 4.5 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{x^6 - a^6}\,\mathrm{d}x$, using contour integration where $a > 0$.

Consider $\displaystyle\int_C \frac{z^4}{z^6 - a^6}\mathrm{d}z$, where C is the same contour as in example 4.1, with some modifications explained below.

The singularities of $f(z) = \dfrac{z^4}{z^6 - a^6}$ are given by

$$z^6 = a^6 = e^{i2r\pi} a^6, \text{ i.e. } z = ae^{\frac{r\pi}{3}i}, \; r = 0, 1, 2, 3, 4, 5.$$

i.e. $z = a,\ ae^{\pi i/3},\ ae^{2\pi i/3},\ -a,\ ae^{4\pi i/3},\ ae^{5\pi i/3}$.

Of these singularities (simple poles),

$z = ae^{\pi i/3}$ and $z = ae^{2\pi i/3}$ lie inside C, but $z = a$ and $z = -a$ lie on the real axis. But for the evaluation of the integrals of type 1, no singularity of $f(z)$ should lie on the real axis. To avoid them, we modify C by introducing small indents, i.e., semi-circle of small radius at $z = \pm a$, as shown in Fig. 4.24.

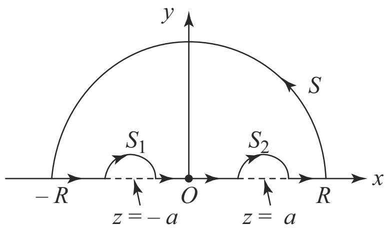


Fig. 4.24

Now the modified contour C contains only the simple poles

$$z = ae^{\pi i/3} \quad \text{and} \quad ae^{2\pi i/3}$$

$$R_1 = [\text{Res. } f(z)]_{z = ae^{\pi/3i}}$$

$$= \left(\frac{z^4}{6z^5}\right) = \frac{1}{6a} e^{-\pi i/3}$$

$$= \frac{1}{6a}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

Similarly, $R_2 = \frac{1}{6a} e^{-2\pi i/3} = \frac{1}{6a}\left(-1/2 - i\frac{\sqrt{3}}{2}\right)$

By Cauchy's residue theorem,

$$\int_C \frac{z^4}{z^6 - a^6}\,\mathrm{d}z = \frac{2\pi i}{6a}\left(1/2 - i\frac{\sqrt{3}}{2} - 1/2 - i\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{\sqrt{3}a}$$

i.e. $$\int_{-R}^{-a-r} \frac{x^4}{x^6 - a^6}\,\mathrm{d}x + \oint_{S_1} \frac{z^4}{z^6 - a^6}\,\mathrm{d}z + \int_{-a+r}^{a-r'} \frac{x^4}{x^6 - a^6}\,\mathrm{d}x + \oint_{S_2} \frac{z^4}{z^6 - a^6}\,\mathrm{d}z$$

$$+ \int_{a+r'}^{R} \frac{x^4}{x^6 - a^6}\,\mathrm{d}x + \oint_S \frac{z^4}{z^6 - a^6}\,\mathrm{d}z = \frac{\pi}{\sqrt{3}a} \quad (1)$$

where r and r' are the radii of the semicircles S_1 and S_2 whose equations are $|z + a| = r$ and $|z - a| = r'$. These two integrals taken along S_1 and S_2 vanish as $r \to 0$ and $r' \to 0$, by the note under Cauchy's Lemma.

$$\int_S \frac{z^4}{z^6 - a^6} = 0 \text{, as } R \to \infty \text{, by Cauchy's lemma.}$$

Now, letting $r \to 0$, $r' \to 0$ and $R \to \infty$ in (1),

We get $$\int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty} \frac{x^4\,\mathrm{d}x}{x^6 - a^6} = \frac{\pi}{\sqrt{3}a}$$

i.e. $$\int_{-\infty}^{\infty} \frac{x^4}{x^6 - a^6}\,\mathrm{d}x = \frac{\pi}{\sqrt{3}a}$$

Example 4.6 Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2}\,\mathrm{d}x$, by contour integration.

Consider $\int_C \frac{ze^{iz}}{z^2 + a^2}\,\mathrm{d}z$, where C is the same contour as in Example 4.1.

The singularities of $f(z) = \frac{ze^{iz}}{z^2 + a^2}$. are $z = \pm ia$, which are simple poles. Of these poles, only $z = ia$ lies inside C.

$$[\text{Res.} f(z)]_{z = ia} = \left(\frac{ze^{iz}}{z + ia}\right)_{z = ia} = \frac{1}{2} e^{-a}.$$

By Cauchy's residue theorem,

$$\int_C \frac{ze^{iz}}{z^2+a^2}\,\mathrm{d}z = 2\pi i\,.\,\frac{1}{2}e^{-a} = \pi i e^{-a}$$

i.e. $$\int_{-R}^{R} \frac{xe^{ix}}{x^2+a^2}\,\mathrm{d}x + \int_S \frac{ze^{iz}}{z+a^2}\,\mathrm{d}z = \pi i e^{-a} \qquad (1)$$

Now $$\left|\frac{z}{z^2+a^2}\right| \le \frac{R}{R^2-a^2}$$

Since the limit of the R.H.S. is zero as $R \to \infty$,

$$\lim_{R\to\infty}\left|\frac{z}{z^2+a^2}\right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Jordan's Lemma, $\displaystyle\int_S \frac{ze^{iz}}{z^2+a^2}\,\mathrm{d}z \to 0$ as $R \to \infty$, (2)

Letting $R\to\infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,\mathrm{d}x = i\pi e^{-a}$$

i.e. $$\int_{-\infty}^{\infty} \frac{x}{x^2+a^2}(\cos x + i\sin x)\mathrm{d}x = i\pi e^{-a}.$$

Equating the imaginary parts on both sides,

we get $$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x = \pi e^{-a}.$$

Since the integrand is an even function of x,

$$\int_0^{\infty} \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{2}e^{-a}.$$

Example 4.7 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$., using contour integration, where a > b > 0.

Consider $\displaystyle\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,\mathrm{d}z$, where C is the same contour as in Example 4.1.

The singularities of the integrand $f(z)$ are given by $z = \pm\, ia$ and $z = \pm\, ib$, which are simple poles. Of these poles, $z = ia$ and $z = ib$ only lie inside C.

$$R_1 = [\text{Res.} f(z)]_{z=ia} = \left[\frac{e^{iz}}{(z+ia)\left(z^2+b^2\right)} \right]_{z=ia}$$

$$= \frac{e^{-a}}{-2ia\left(a^2-b^2\right)}$$

$$R_2 = [\text{Res.} f(z)]_{z=ib} = \frac{e^{-b}}{2ib(a^2-b^2)}$$

By Cauchy's residue theorem,

$$\int_C \frac{e^{iz}}{\left(z^2+a^2\right)\left(z^2+b^2\right)} \mathrm{d}z = 2\pi i\left(R_1+R_2\right)$$

$$= \frac{\pi}{\left(a^2-b^2\right)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

i.e.
$$\int_{-R}^{R} \frac{e^{ix}}{\left(x^2+a^2\right)\left(x^2+b^2\right)} \mathrm{d}x + \int_S \frac{e^{iz}}{\left(z^2+a^2\right)\left(z^2+b^2\right)} \mathrm{d}z$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \tag{1}$$

Now
$$\left| \frac{1}{\left(z^2+a^2\right)\left(z^2+b^2\right)} \right| \le \frac{1}{(R^2-a^2)\left(R^2-b^2\right)}$$

R.H.S. $\to 0$ as $R \to \infty$. Hence

$$\lim_{R\to\infty} \left| \frac{1}{\left(z^2+a^2\right)\left(z^2+b^2\right)} \right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Jordan's Lemma, $\displaystyle\int_S \frac{e^{iz}}{\left(z^2+a^2\right)\left(z^2+b^2\right)} \mathrm{d}z \to 0$, as $R \to \infty$ (2)

Letting $R\to\infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{\left(x^2+a^2\right)\left(x^2+b^2\right)} \mathrm{d}x = \frac{\pi}{\left(a^2-b^2\right)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Equating the real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{\left(x^2+a^2\right)\left(x^2+b^2\right)} \mathrm{d}x = \frac{\pi}{\left(a^2-b^2\right)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Example 4.8 Evaluate $\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)} \mathrm{d}x = \frac{\pi}{4b^3}(1+ab)e^{-ab}$, where $a > 0$ and $b > 0$.

Consider $\int_C \frac{e^{iaz}}{(z^2+b^2)^2} \mathrm{d}z$, where C is the same contour as in Example 4.1.

The singularities of $f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$ are $z = \pm\, ib$, which are double poles. Of these poles, only $z = ib$ lies inside C.

$$[\text{Res. } f(z)]_{z=ib} = \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\frac{e^{iaz}}{(z+ib)^2}\right)\right]_{z=ib}$$

$$= \left[\frac{(z+ib)ia\,e^{iaz} - 2e^{iaz}}{(z+ib)^3}\right]_{z=ib}$$

$$= \frac{1}{4ib^3}(ab+1)e^{-ab}.$$

By Cauchy's residue theorem,

$$\int_{-R}^{R} \frac{e^{iax}}{(x^2+b^2)^2}\mathrm{d}x + \int_S \frac{e^{iaz}}{(z^2+b^2)^2}\mathrm{d}z = 2\pi i \times \frac{1}{4ib^3}(ab+1)e^{-ab}$$

$$= \frac{\pi}{2b^3}(ab+1)e^{-ab} \tag{1}$$

Now
$$\left|\frac{1}{(z^2+b^2)^2}\right| \le \frac{1}{(R^2-b^2)^2}$$

Since the R.H.S.$\to 0$ as $R \to \infty$, L.H.S. also $\to 0$ as $R \to \infty$ on $|z| = R$.

$\therefore$ By Jordan's Lemma,

$$\int_S \frac{e^{iaz}}{(z^2+b^2)^2}\mathrm{d}z \to 0 \text{ as } R \to \infty, \text{ since } a > 0 \tag{2}$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2}\mathrm{d}x = \frac{\pi}{2b^3}(ab+1)e^{-ab}.$$

Equating the real parts on both sides and noting that $\dfrac{\cos ax}{(x^2+b^2)^2}$ is an even function of x, we get

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(ab+1)e^{-ab}.$$

Example 4.9 Use contour integration to prove that

$$\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2},\ \text{when } m > 0.$$

Consider $\int_C \dfrac{e^{imz}}{z}\,dz$, where C is the usual semicircular contour, but with an indent i.e. a small semicircle at the origin, which is introduced to avoid the singularity $z = 0$, which lies on the real axis. The modified contour is shown in Fig. 4.25.

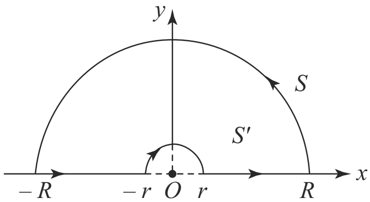


Fig. 4.25

The modified contour C does not include any singularity of $f(z) = \dfrac{e^{imz}}{z}$.
∴ By Cauchy's residue theorem,

$$\int_{-R}^{-r} \frac{e^{imx}}{x}\,dx + \oint_{S'} \frac{e^{imz}}{z}\,dz + \int_r^R \frac{e^{imx}}{x}\,dx + \oint_S \frac{e^{imz}}{z}\,dz = 0 \qquad (1)$$

The Eqn. of S' is $|z| = r$. ∴ $z = re^{i\theta}$ and $dz = re^{i\theta}\, i\,d\theta$.

As S' is described in the clockwise sense, θ varies from π to 0.

Thus
$$\int_{S'} \frac{e^{imz}}{z}\,dz = \int_\pi^0 \frac{e^{imre^{i\theta}}}{re^{i\theta}} re^{i\theta} i\,d\theta$$

∴
$$\lim_{r\to 0}\int_{S'} \frac{e^{imz}}{z}\,dz = \int_\pi^0 \left[\lim_{r\to 0}\left(e^{imre^{i\theta}}\right)\right] i\,d\theta = -i\pi \qquad (2)$$

$$\left|\frac{1}{z}\right| = \frac{1}{R} \to 0 \text{ as } R \to \infty \text{ on } |z| = R.$$

∴ By Jordan's Lemma, $\int_S \dfrac{e^{imz}}{z}\,dz \to 0$ as $R \to \infty$, since $m > 0$ (3)

Letting $r \to 0$ and $R \to \infty$ in (1) and using (2) and (3),

we get $$\int_{-\infty}^{0} \frac{e^{imx}}{x} \mathrm{d}x - i\pi + \int_{0}^{\infty} \frac{e^{imx}}{x} \mathrm{d}x = 0$$

i.e. $$\int_{-\infty}^{\infty} \frac{e^{imx}}{x} \mathrm{d}x = i\pi$$

Equating the imaginary parts on both sides and noting that $\frac{\sin mx}{x}$ is an even function of x, we get $\int_{0}^{\infty} \frac{\sin mx}{x} \mathrm{d}x = \frac{\pi}{2}$.

Example 4.10 Evaluate $\int_{0}^{\infty} \frac{\sin x \, \mathrm{d}x}{x\left(x^2 + a^2\right)}$, using contour integrations where $a > 0$.

Consider $\int_{C} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z$, where C is the same modified contour as in Example 4.9, which includes the only pole $z = ia$ of $f(z) = \frac{e^{iz}}{z\left(z^2 + a^2\right)}$

$$[\text{Res.} f(z)]_{z=ia} = \left[\frac{e^{iz}}{z(z + ia)}\right]_{z=ia} = \frac{e^{-a}}{-2a^2}$$

By Cauchy's residue theorem,

$$\int_{-R}^{-r} \frac{e^{ix}}{x\left(x^2 + a^2\right)} \mathrm{d}x + \oint_{S'} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z + \int_{r}^{R} \frac{e^{ix}}{x\left(x^2 + a^2\right)} \mathrm{d}x + \oint_{S} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z$$

$$= -\frac{\pi i}{a^2} e^{-a} \qquad (1)$$

$$\oint_{S'} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z = \int_{\pi}^{0} \frac{e^{ir e^{i\theta}} . re^{i\theta} i\mathrm{d}\theta}{re^{i\theta}\left(r^2 e^{i2\theta} + a^2\right)}$$

$\therefore$ $$\lim_{r \to 0} \int_{S'} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z = \int_{\pi}^{0} \lim_{r \to 0} \left[\frac{e^{ir e^{i\theta}}}{r^2 e^{i2\theta} + a^2}\right] i\mathrm{d}\theta$$

$$= -\frac{\pi i}{a^2} \qquad (2)$$

$$\left|\frac{1}{z\left(z^2 + a^2\right)}\right| \le \frac{1}{R\left(R^2 - a^2\right)}$$

Since R.H.S. $\to 0$ as $R \to \infty$ on $|z| = R$, L.H.S. also tends to 0 as $R \to \infty$.

$\therefore$ By Jordan's Lemma,

$$\oint_S \frac{e^{iz}}{z\left(z^2+a^2\right)} \mathrm{d}z \to 0, \text{ as } R \to \infty \tag{3}$$

Letting $r \to 0$ and $R \to \infty$ in (1) and using (2) and (3), we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x\left(x^2+a^2\right)} \mathrm{d}x = \frac{\pi i}{a^2} - \frac{\pi i}{a^2} e^{-a}.$$

Equating imaginary parts on both sides and noting that $\dfrac{\sin x}{x\left(x^2+a^2\right)}$ is an even function of x, we get,

$$\int_0^{\infty} \frac{\sin x}{x\left(x^2+a^2\right)} \mathrm{d}x = \frac{\pi}{2a^2}\left(1-e^{-a}\right).$$

Example 4.11 Evaluate $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1-2x\sin\theta+x^2} (0 < x < 1)$, using contour integration.

On the circle $|z| = 1$, $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$ or $\mathrm{d}\theta = \dfrac{\mathrm{d}z}{iz}$ and $\sin\theta = \dfrac{z-\frac{1}{z}}{2i} = \dfrac{z^2-1}{2iz}$.

$\therefore$ The given integral $I = \displaystyle\int_C \frac{\mathrm{d}z/iz}{1-2x\left(\dfrac{z^2-1}{2iz}\right)+x^2}$, where C is $|z| = 1$

i.e. $$I = \int_C \frac{\mathrm{d}z}{iz - xz^2 + x + ix^2 z}$$

$$= -\frac{1}{x}\int_C \frac{\mathrm{d}z}{z^2 - i\left(x+\frac{1}{x}\right)z - 1} = -\frac{1}{x}\int_C \frac{\mathrm{d}z}{(z-ix)\left(z-\frac{i}{x}\right)} \tag{1}$$

The singularities of $\dfrac{1}{(z-ix)\left(z-\frac{i}{x}\right)}$ are $z = ix$ and $z = \dfrac{i}{x}$, which are simple poles

Now $|ix| = |x| < 1$, as $0 < x < 1$

$\therefore$ The pole $z = ix$ lies inside C, but $z = \dfrac{i}{x}$ lies outside C.

$$\left[\text{Res.}\left\{\frac{1}{(z-ix)(z-i/x)}\right\}\right]_{z=ix} = \frac{1}{i\left(x-\frac{1}{x}\right)} = \frac{ix}{1-x^2}$$

Using Cauchy's residue theorem, from (1) we get

$$I = -\frac{1}{x} \times 2\pi i \times \frac{ix}{1-x^2} = \frac{2\pi}{1-x^2}.$$

Note ☑ We have integrated w.r.t. θ and x is a parameter.

Example 4.12 Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} (a > b > 0)$, using contour integration.

Deduce the value of $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$. On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z^2+1}{2z}$.

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{dz/iz}{a+b\frac{z^2+1}{2z}} = -\frac{2i}{b}\int_C \frac{dz}{\left(z^2+\frac{2a}{b}z+1\right)} \qquad (1)$$

where C is $|z| = 1$.

The singularities of the integrand are given by $z^2 + \frac{2a}{b}z + 1 = 0$,

i.e. $z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$, which are simple poles.

Since $a > b, \frac{a}{b} > 1$ and hence $\left|-\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}\right| > 1$ and so $z = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}$ lies outside $|z| = 1$.

$z = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}$ lies inside $|z| = 1$.

$$\left[\text{Res. of}\left(\frac{1}{z^2+\frac{2a}{b}z+1}\right)\right]_{z=-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}}$$

$$= \left(\frac{1}{2\left(z+\frac{a}{b}\right)}\right)_{z=-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}} \qquad (2)$$

$$= \frac{b}{2\sqrt{a^2-b^2}}$$

By Cauchy's residue theorem and using (2) in (1), we get

$$I = -\frac{2i}{b} \times 2\pi i \frac{b}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

i.e. $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \tag{3}$$

Differentiating both sides of (3) partially with respect to 'a', we get

$$\int_0^{2\pi} -\frac{d\theta}{(a+b\cos\theta)^2} = -\frac{2\pi a}{(a^2-b^2)^{3/2}}$$

$\therefore$ $$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Example 4.13 Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta} d\theta$, using contour integration.

On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z^2-1}{2iz}$ and $\cos\theta = \frac{z^2+1}{2z}$.

$\therefore$ $$I = \int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta} d\theta = \int_C \frac{\frac{(z^2-1)^2}{-4z^2} \cdot \frac{dz}{iz}}{5 - \frac{3(z^2+1)}{2z}}, \text{ where } C \text{ is } |z| = 1$$

$$= -\frac{i}{6} \int_C \frac{(z^2-1)^2 \, dz}{z^2(z-3)(z-1/3)} \tag{1}$$

The singularities of the integrand which lie within C are $z = 0$ and $z = 1/3$.

$z = 0$ is a double pole and $z = \frac{1}{3}$ is a simple pole.

$$R_1 = [\text{Res. of the integrand } f(z)]_{z=0}$$

$$= \frac{1}{1!}\left[\frac{d}{dz}\left\{\frac{(z^2-1)^2}{z^2 - \frac{10}{3}z + 1}\right\}\right]_{z=0} = \frac{10}{3}$$

$$R_2 = [\text{Res.} f(z)]_{z=1/3} = \left[\frac{(z^2-1)^2}{z^2(z-3)}\right]_{z=1/3} = -\frac{8}{3}$$

By Cauchy's residue theorem, from (1), we get

$$I = -\frac{i}{6} \times 2\pi i\left(R_1 + R_2\right) = \frac{\pi}{3}\left(\frac{10}{3} - \frac{8}{3}\right) = \frac{2\pi}{9}.$$

Example 4.14 Evaluate $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2}$, using contour integration, where $a^2 < 1$.

On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$ and $\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{z^4+1}{2z^2}$.

$\therefore$ The given integral $I = \int_C \frac{\frac{z^4+1}{2z^2} \frac{dz}{iz}}{1 - a\left(\frac{z^2+1}{z}\right) + a^2}$, where C is $|z| = 1$

$$= \frac{i}{2a}\int_C \frac{\left(z^4+1\right)dz}{z^2\left\{z^2 - \left(a + \frac{1}{a}\right)z + 1\right\}}$$

$$= \frac{i}{2a}\int_C \frac{\left(z^4+1\right)dz}{z^2\,(z-a)\left(z - 1/a\right)} \qquad (1)$$

The singularities of the integrand which lie inside C are $z = 0$ and $z = a$ ($\because a^2 < 1$)
$z = 0$ is a double pole and $z = a$ is a simple pole.
$R_1 = [\text{Res. of the integrand}]_{z=0}$

$$= \frac{1}{1!}\left[\frac{d}{dz}\left\{\frac{z^4+1}{z^2 - (a + 1/a)z + 1}\right\}\right]_{z=0}$$

$$= a + \frac{1}{a}$$

$$R_2 = (\text{Res. of the integrand})_{z=a} = \frac{a^4+1}{a^2\,(a - 1/a)}$$

$$\therefore \qquad R_1 + R_2 = a + \frac{1}{a} + \frac{a^4+1}{a^2\,(a-1/a)}$$

$$= \frac{2a^3}{a^2-1}$$

By Cauchy's residue theorem and from (1), we get

$$I = \frac{i}{2a} \times 2\pi i \times \frac{2a^3}{a^2-1} = \frac{2\pi a^2}{1-a^2}$$

Example 4.15 Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta$, using contour integration

On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$ and $\cos 3\theta = \frac{e^{i3\theta}+e^{-i3\theta}}{2} = \frac{z^6+1}{2z^3}$.

$\therefore$ The given integral $I = \int_C \frac{\frac{z^6+1}{2z^3} \cdot \frac{dz}{iz}}{5+4\left(\frac{z^2+1}{2z}\right)}$, where C is $|z| = 1$.

i.e. $$I = -\frac{i}{4}\int_C \frac{(z^6+1)\,dz}{z^3(z+2)(z+1/2)}$$

The singularities of the integrand $f(z)$ which lie inside C are the simple pole $z = -1/2$ and the triple pole $z = 0$.

R_1 = (Res. of the integrand)$_{z=-1/2}$ $= \left\{\frac{z^6+1}{z^3(z+2)}\right\}_{z=-1/2} = -\frac{65}{12}$

R_2, the residue at $z = 0$ is found out as the coefficient of $\frac{1}{z}$ in the Laurent's expansion of the integrand.

$$f(z) = \left(z^3 + \frac{1}{z^3}\right)\left(1+\frac{z}{2}\right)(1+2z)$$

$$= \left(z^3 + \frac{1}{z^3}\right)\left\{1 - \frac{z}{2} + \frac{z^2}{4} - \cdots\right\}\{1 - 2z + 4z^2 - \cdots\}$$

$$= \left(z^3 + \frac{1}{z^3}\right)\left\{1 - \frac{5}{2}z + \frac{21}{4}z^2 - \cdots\right\}$$

$\therefore R_2$ = Coefficient of $\frac{1}{z}$ in this expansion $= \frac{21}{4}$.

By Cauchy's residue theorem.

$$I = -i/4 \times 2\pi i\left(-\frac{65}{12} + \frac{21}{4}\right) = -\frac{\pi}{12}$$

EXERCISE 4(c)

Part A

(Short Answer Questions)

1. Give the forms of the definite integrals which can be evaluated using the infinite semi-circular contour above the real axis.

2. Explain how to convert $\int_0^{2\pi} f(\sin\theta, \cos\theta)\mathrm{d}\theta$ into a contour integral, where f is a rational function.

3. Sketch the contour to be used for the evaluation of $\int_0^{\infty} \frac{\sin mx}{x}\mathrm{d}x$.

Part B

Evaluate the following integrals by contour integration technique.

4. $\int_0^{\infty} \frac{x^2\mathrm{d}x}{(x^2+1)(x^2+4)}$

5. $\int_0^{\infty} \frac{x^2\mathrm{d}x}{(x^2+4)^2(x^2+9)}$

6. $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\mathrm{d}x$

7. $\int_0^{\infty} \frac{\mathrm{d}x}{x^4+a^4}\mathrm{d}x$

8. $\int_0^{\infty} \frac{x^2\mathrm{d}x}{(x^2+1)^3}$

9. $\int_0^{\infty} \frac{x^2}{x^6+1}\mathrm{d}x$

10. $\int_0^{\infty} \frac{x^4\mathrm{d}x}{x^6+1}$

11. $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\mathrm{d}x\,(a>0; b>0)$

12. $\int_{-\infty}^{\infty} \frac{\cos x\,\mathrm{d}x}{(x^2+4)(x^2+9)}$

13. $\int_0^{\infty} \frac{x\sin x}{x^4+1}\mathrm{d}x$

14. $\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}\mathrm{d}x$

15. $\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)^2}\mathrm{d}x$

16. $\int_{-\infty}^{\infty} \frac{\sin x\,\mathrm{d}x}{x^2+4x+5}$

17. $\int_{-\infty}^{\infty} \frac{x\cos \pi x}{x^2+2x+5}\mathrm{d}x$

18. $\int_0^{\infty} \frac{\sin x}{x(x^2+1)^2}\mathrm{d}x$

19. $\int_0^{2\pi} \frac{\mathrm{d}\theta}{1-2p\cos\theta+p^2}, 0<p<1$

20. $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} (a>b>0)$

21. $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta (a>b>0)$

22. $\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} (a>0)$

23. $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta}$

24. $\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$

25. $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta} d\theta$

ANSWERS

Exercise 4(a)

8. $\frac{10}{3}(3+i)$

9. $3 - i2$

10. $4\pi i$

11. $-\frac{4}{3}+\frac{8}{3}i$

12. πi

13. (i) $-1/6 + i5/6$; (ii) $-1/6+i\frac{13}{15}$.

14. $4\pi i$

15. $10-i\frac{8}{3}$

16. 0 in all cases.

17. (i) $\frac{511}{3}-\frac{49}{5}i$; (ii) $\frac{518}{3}-57i$; (iii) $\frac{518}{3}-8i$.

18. $\frac{1}{15}\left(96\pi^5 a^5+80\pi^3 a^3+30\pi a\right)$

19. (i) 0; (ii) $2\pi i$.

20. $\frac{8\pi i}{3}$

21. $-4\pi i$

22. (i) $2\pi i$; (ii) 0

23. $-\frac{2\pi}{3}$

24. $\sin t$

25. $\frac{8\pi i}{3e^2}$

26. $\frac{\pi}{16}$

27. $\frac{\pi}{2}(3+2i)$

28. 0

29. $20\pi i$; $2\pi(i-1)$; $-14\pi i$; $16\pi i$

Exercise 4(b)

17. $\sin z = \frac{1}{\sqrt{2}}\left\{1+(z-\pi/4)-\frac{(z-\pi/4)^2}{2!}-\frac{(z-\pi/4)^3}{3!}+\cdots\right\}$

18. $\cos z = \frac{1}{2}\left[1 - \sqrt{3}\left(z - \pi/3\right) - \frac{1}{2!}\left(z - \pi/3\right)^2 + \frac{\sqrt{3}}{3!}\left(z - \pi/3\right)^3 + \frac{1}{4!}\left(z - \pi/3\right)^4 - \cdots\right]$

19. $e^{-i}\left\{1 + \frac{(z+i)}{1!} + \frac{(z+i)^2}{2!} + \cdots\right\}$

20. $e^{-1}\left\{1 - \frac{(z-1)}{1!} + \frac{(z-1)^2}{2!} - \frac{(z-1)^3}{3!} + \cdots\right\}$

21. $(z-1) - 2(z-1)^2 + 3(z-1)^3 - 4(z-1)^4 + \cdots$

22. $|z| < 1.$

23. $|z| < 2.$

24. $|z-1| < \frac{1}{2}.$

25. $|z-2| < 1$

26. $|z-4i| < 4.$

27. $-\frac{1}{2z^2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right)$

28. $\frac{1}{z-1}\left\{1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right\}$

29. $-\frac{1}{z^4}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$

30. $\frac{1}{z-1}\left[1 - \frac{2}{z-1} + \frac{2}{(z-1)^2} - \cdots\right]$

31. 1

32. –1

33. 1

34. $\frac{1}{6}.$

35. $-\frac{1}{2}$

36. $R_{(z=-1)} = \frac{1}{3}; R_{(z=2)} = \frac{2}{3}.$

37. $\pm\frac{ia}{2}$

38. 1.

39. $2e.$

40. –1

41. $2\pi i.$

42. 0

43. $2\pi i.$

44. $-4\pi i.$

45. $-2\pi i.$

46. $\sum\left(\frac{1}{2^{n+1}} - 1\right)(z+1)^n; |z+1| < 1$

47. $\sum(-1)^n\left\{\frac{2}{(2+i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n.$

48. (i) $1+\sum(-1)^n\left\{\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right\}z^n.$

(ii) $1+3\sum(-1)^n\frac{2^n}{z^n+1}-8\sum(-1)^n\frac{z^n}{3^{n+1}}$

(iii) $1+\sum(-1)^n\left\{3.2^n-8.3^n\right\}\frac{1}{z^{n+1}}$

49. (i) $\sum\left(-\frac{1}{2.4^n}+\frac{1}{3^{n+1}}\right)(z+2)^n;$

(ii) $-\frac{1}{2}\sum\frac{(z+2)^n}{4^n}-\sum\frac{3^n}{(z+2)^{n+1}}$ (iii) $\sum\left(2.4^n-3^n\right)\frac{1}{(z+2)z^{n+1}}$

50. (i) $-\frac{3}{z+1}-\sum\left(1+\frac{2}{3^{n+1}}\right)(z+1)^n;0<|z+1|<1$

(ii) $-\frac{3}{z+1}+\sum\frac{1}{(z+1)^{n+1}}-2\sum\frac{(z+1)^n}{3^{n+1}};1<|z+1|<3.$

(iii) $-\frac{3}{z+1}+\sum\left(1+2.3^n\right)/z^{n+1},|z+1|>3.$ Res. at $(z=-1)=-3.$

51. (i) $1+\frac{4/5}{z-3}+\sum(-1)^n\left\{\frac{1}{2^{n+1}}+\frac{1}{5^{n+2}}\right\}(z-3)^n$, in $0<|z-3|<2.$

(ii) $1+\frac{4/5}{z-3}+\sum(-1)^n\left\{\frac{2^n}{(z-3)^{n+1}}+\frac{1}{5^{n+2}}.(z-3)^n\right\}$, in $2<|z-3|<5$

(iii) $1+\frac{4/5}{z-3}+\sum(-1)^n\left\{2^n+5^{n-1}\right\}\frac{1}{(z-3)^n}$, in $|z-3|>5.$ Res. at $(z=3)=\frac{4}{5}$

52. (i) $-\frac{3}{z-1}-\frac{1}{(z-2)^2}-4\sum(z-1)^n$; Res. at $(z=1)=-3$; $0<|z-1|<1$

(ii) $\frac{4}{z-2}-\sum(-1)^n(n+4)(z-2)^n$; Res. at $(z=2)=4$; $0<|z-2|<1$

53. (i) $\frac{1}{z^2}-\frac{1}{2}\sum\left\{(-1)^n+1\right\}i^nz^n$, ; Res. at $(z=0)=0.$

(ii) $\frac{i/2}{z-i}-\sum\left\{(n+1)+\frac{1}{2^{n+2}}\right\}i^n\,(z-i)^n$; Res. at $(z=i)=\frac{i}{2}.$

(iii) $\frac{-i/2}{z+i}-\sum\left\{(n+1)+\frac{1}{2^{n+2}}\right\}(-1)^n(z+i)^n$; Res. at $(z=-i)=-\frac{i}{2}.$

54. (i) Res. at the simple pole $(z=-1+2i)=\frac{1}{4}(2+i)$

and that at $(z=-1-2i)=\frac{1}{4}(2-i)$

(ii) $z = 0$ is a double pole with Res. = –1/2; $z = -1 + i$ is a simple pole with Res. = 1/4; $z = -1 - i$ is a simple pole with Res. = 1/4.

55. (i) $z = 1$ is a simple pole; Res = 1/4; $z = -1$ is a double pole; Res. = –1/4

(ii) $z = -1$ is a double pole; Res $= -\frac{14}{25}$; $z = 2i$ is a simple pole with Res. $= \frac{1}{25}(7+i)$; $z = -2i$ is a simple pole with Res. $= \frac{1}{25}(7-i)$

56. (i) $z = 2e^{i\pi/4}, 2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4}$ are simple poles. Cor. residues are,

$\frac{1}{32\sqrt{2}}(-1-i),\ \frac{1}{32\sqrt{2}}(1-i),\ \frac{1}{32\sqrt{2}}(1+i),\ \frac{1}{32\sqrt{2}}(-1+i).$

(ii) $z = ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$ are simple poles. Cor. residues are

$\frac{-i}{4a^2}, \frac{i}{4a^2}, \frac{-i}{4a^2}, \frac{i}{4a^2}.$

57. (i) πi; (ii) $\frac{\pi i}{3}$;

58. $-2\pi i$; (ii) $-\frac{\pi^2}{2}i$

59. (i) $-2\pi i$; (ii) πi; (iii) $-\frac{\pi i}{4}$

60. (i) $\pi(i-2)$; (ii) $\frac{3\pi}{256}$; (iii) $\frac{-2\pi i}{9}$.

Exercise 4(c)

4. $\frac{\pi}{6}$
5. $\frac{\pi}{200}$
6. $\frac{5\pi}{12}$
7. $\frac{\pi}{2\sqrt{2}a^3}$
8. $\frac{\pi}{16}$
9. $\frac{\pi}{6}$
10. $\frac{\pi}{3}$
11. $\frac{\pi}{b}e^{-ab}$
12. $\frac{\pi}{30}\left(\frac{3}{e^2} - \frac{2}{e^3}\right)$
13. $\frac{\pi}{2}e^{-1/\sqrt{2}} \sin\left(\frac{1}{\sqrt{2}}\right)$
14. $\frac{\pi}{3e^2}(e-1)$
15. $\frac{\pi}{2e}$
16. $-\frac{\pi}{e}\sin 2$
17. $\frac{\pi}{2}e^{-2\pi}$
18. $\frac{\pi}{2} - \frac{3\pi}{4e}$
19. $\frac{2\pi}{1-p^2}$
20. $\frac{2\pi}{\sqrt{a^2-b^2}}$
21. $\frac{2\pi}{b^2}\left\{a - \sqrt{a^2-b^2}\right\}$
22. $\frac{2\pi}{\sqrt{1+a^2}}$
23. $\frac{\pi}{6}$
24. $\frac{\pi}{12}$
25. $\frac{3\pi}{8}$