18MAB102T

UNIT-1

MULTIPLE INTEGRALS

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Syllabus

- Double integration
- Cartesian and polar coordinates
- Change of order of integration
- Change of variables between Cartesian and polar coordinates
- Triple integration in Cartesian coordinates
- Area as double integral
- Volume as triple integral

NPTEL

Websites and videos

1) http://nptel.iitm.ac.in/video.php?subjectId=122104017
2) http://nptel.ac.in/courses/111105035/
3) http://nptel.ac.in/courses/111104025/
4) http://www.youtube.com/watch?v=If8pYknlxnM
5) http://www.youtube.com/watch?v=NETmfwOAKpQ
6) http://www.youtube.com/watch?v=lRlJ074iWH8&list=PLFD48762760B4E8D4

Application of multiple integrals

- *Multiple integrals are used in many applications in physics and engineering*
- a measure of a body's resistance to a change in its angular rotation velocity
- In electromagnetism, Maxwell's equations can be written using multiple integrals to calculate the total magnetic and electric fields.
- Force by Liquid Pressure varies depending on the shape of the object and its depth. We use integration to find the force

Double integration in Cartesian coordinates

Definition :We evaluate the function with respect to y and then we prefer to integrate with respect to x.

$$Double\ \text{int}egral\ \int_{x=a}^{x=b} \left\{ \int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\ dy \right\} dx$$

Properties of Double Integral

1. $$\iint_R kf(x,y)\,dx\,dy = k\iint_R f(x,y)\,dx\,dy$$

2. $$\iint_R \{f(x,y) \pm g(x,y)\}\,dx\,dy = \iint_R f(x,y)\,dx\,dy \pm \iint_R g(x,y)\,dx\,dy$$

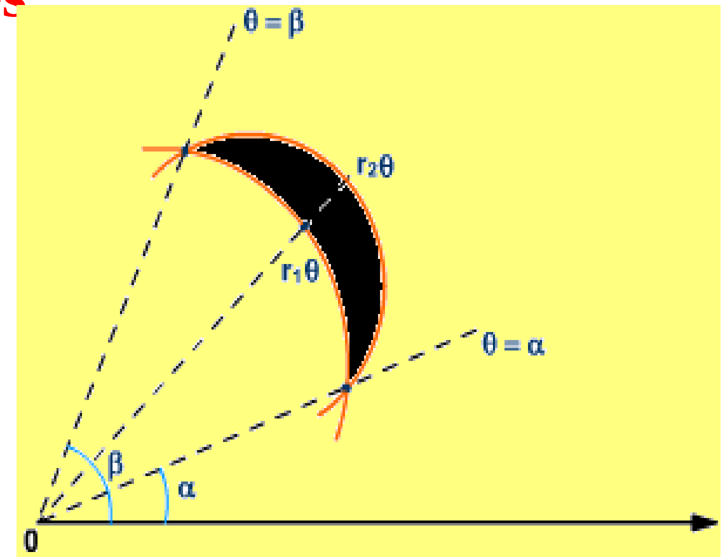
3. If $R = R_1 + R_2$ (R is the union of two non overlapping regions R_1 and R_2)

$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

Double Integral in Polar Coordinates

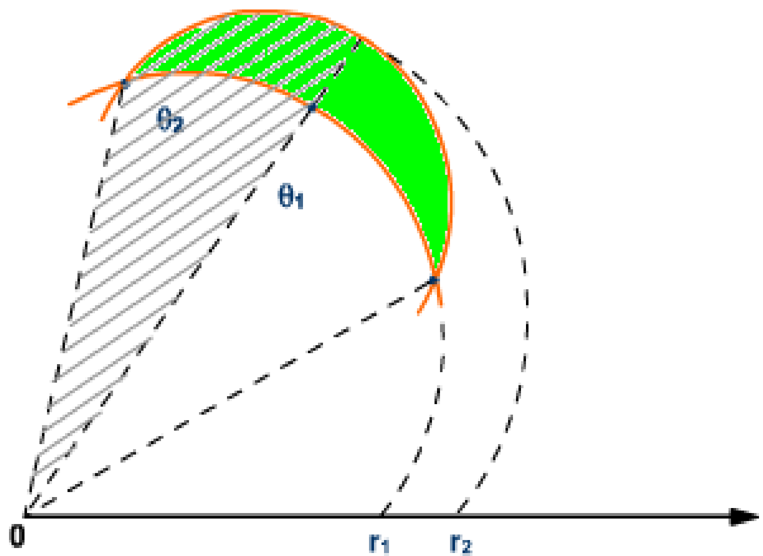
In case of polar coordinates

$$\iint_R f(r,\theta)\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \left\{ \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r_1,\theta)\, dr \right\} d\theta$$



By changing the order of integration

$$\iint_R f(r,\theta)\, dr\, d\theta = \int_{r=r_1}^{r=r_2} \left\{ \int_{\theta=\theta_1(r_1)}^{\theta=\theta_2(r_2)} f(r,\theta)\, d\theta \right\} dr$$



Double integral problems (polar & Cartesian)

Problem 01

Evaluate $\int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx.$

Solution

$$\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx = \int_0^1 \left\{ \int_{y=x}^{y=\sqrt{x}} xy\,dy \right\} dx = \int_0^1 \left. \frac{xy^2}{2} \right]_x^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{24}$$

Problem 02

$$\text{Evaluate} \int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx.$$

Solution

$$\int_0^1 \left\{ \int_x^{\sqrt{x}} (x^2 + y^2)\, dy \right\} dx \qquad = \int_0^1 \left\{ x^2 y + \frac{y^3}{3} \right\}_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4}{3} x^3 \right) dx \qquad = \left[\frac{x^{7/2}}{7/2} + \frac{1}{3} \cdot \frac{x^{5/2}}{5/2} - \frac{4}{3} \cdot \frac{x^4}{4} \right]_0^1 \qquad = \frac{3}{35}$$

Problem 03

Evaluate $\int_0^1 \int_0^y xy \, dx \, dy.$

Solution

$$\int_0^1 \int_0^y xy \, dx \; dy \quad = \int_{y=0}^{y} \left\{ \int_{x=0}^{x=y} dy \; dx \right\} dy \quad = \int_0^1 y . \frac{x^2}{2} \Bigg]_0^y \; dy$$

$$= \int_0^1 \frac{y^3}{2} dy \quad = \frac{1}{2} . \frac{y^4}{4} \Bigg]_0^1 \quad = \frac{1}{8}$$

Problem 04

Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{(1+x^2)\,(1+y^2)}$.

Solution

$$\int_0^1\int_0^1 \frac{dx\,dy}{(1+x^2)\,(1+y^2)}$$

Here the limits of both the integrals are constants and the variables can be separated.

$$\therefore \quad \int_0^1\int_0^1 \frac{dx\,dy}{(1+x^2)\,(1+y^2)} = \left(\int_0^1 \frac{dx}{1+x^2}\right)\left[\int_0^1 \frac{dy}{1+y^2}\right]$$

$$= \tan^{-1} x\Big]_0^1 \,.\, \tan^{-1} y\Big]_0^1 \quad = \frac{\pi}{4}.\frac{\pi}{4} \quad = \frac{\pi^2}{16}$$

Problem 05

Show that $\int_0^{\infty}\int_0^{\infty} e^{-x^2(1+y^2)}\, x\, dx\, dy = \dfrac{\pi}{4}.$

Solution

LHS: $\int_0^{\infty}\int_0^{\infty} e^{-x^2(1+y^2)}\, x.dx\, dy$

$$= \int_0^{\infty}\left\{\int_0^{\infty} e^{-x^2(1+y^2)}\, x\, dx\right\} dy$$

put $x^2 = t$

$2x\, dx = dt$

$x dx = \dfrac{1}{2} dt$

$$=\frac{1}{2}\int_0^\infty\left\{\int_0^\infty e^{-t(1+y^2)}\,dt\right\}dy \qquad =\frac{1}{2}\int_0^\infty \left.\frac{e^{-t(1+y^2)}}{-(1+y^2)}\right]_0^{-\infty} dy$$

$$=-\frac{1}{2}\int_0^\infty\left[0-\frac{1}{1+y^2}\right]dy \qquad =\frac{1}{2}\int_0^\infty\frac{1}{1+y^2}\,dy$$

$$=\frac{1}{2}\tan^{-1}y\Big]_0^\infty \qquad =\frac{1}{2}\left\{\frac{\pi}{2}-0\right\} \qquad =\frac{\pi}{4}$$

Problem 06

Show that $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \frac{\pi}{4}\log(1+\sqrt{2})$.

Solution

$$\int_0^1 \left\{ \int_{y=0}^{y=\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2} \right\} dx \qquad = \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \Bigg]_0^{\sqrt{1+x^2}} .\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1} 0 \right] dx \qquad = \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\frac{\pi}{4} \right] dx$$

$$= \frac{\pi}{4} \sinh^{-1} x \Big]_0^1$$

$$= \frac{\pi}{4} \log\left\{x + \sqrt{x^2 + 1}\right\}\Big]_0^1$$

$$= \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log 0\right]$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Problem 07

Show that $\int_0^{\infty}\int_y^{\infty} xe^{-x^2/y}\,dx\,dy = \frac{1}{2}$.

Solution

$$\int_0^{\infty}\int_y^{\infty} xe^{-x^2/y}\,dx\,dy \qquad = \int_0^{\infty}\left\{\int_{x=y}^{\infty} xe^{-x^2/y}\,dx\right\}dy$$

$$= \int_{y=0}^{\infty}\left\{\int_{x=y}^{\infty} -\frac{y}{2}.e^{-x^2/2}\left(-\frac{2x}{y}\right)dx\right\}dx \qquad = \int_0^{\infty} -\frac{y}{2}\left[e^{-x^2/y}\right]_{x=y}^{\infty} dy$$

$$= \int_0^{\infty} \frac{y}{2}e^{-y}\,dy \qquad = \frac{1}{2}$$

Problem 08

Evaluate: $$\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$$

.

Solution: Let I = $$\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

$$= \int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{\sin^2\theta}{2} - 0 \right] d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{8}$$

Problem 08

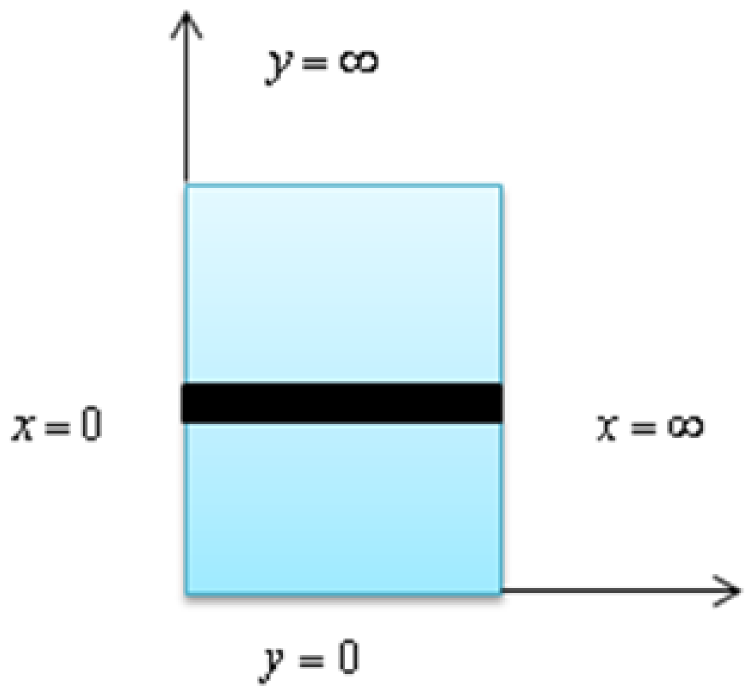
Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)}\,dx\,dy$ **using polar coordinates**

Solution:

Given: $\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)}\,dx\,dy$

Cartesian form:

$$\left.\begin{array}{l} x \text{ varies } \mathit{from}\ x = 0\ \mathit{to}\ x = \infty \\ y \text{ varies } \mathit{from}\ y = 0\ \mathit{to}\ y = \infty \end{array}\right\} \cdots\cdots\cdots\cdots (1)$$



To transform the above integral into polar

$put\ x = r\cos\theta \quad y = r\sin\theta$............(2).

$dxdy = r\,dr\,d\theta$

Substitute (1) in (2)

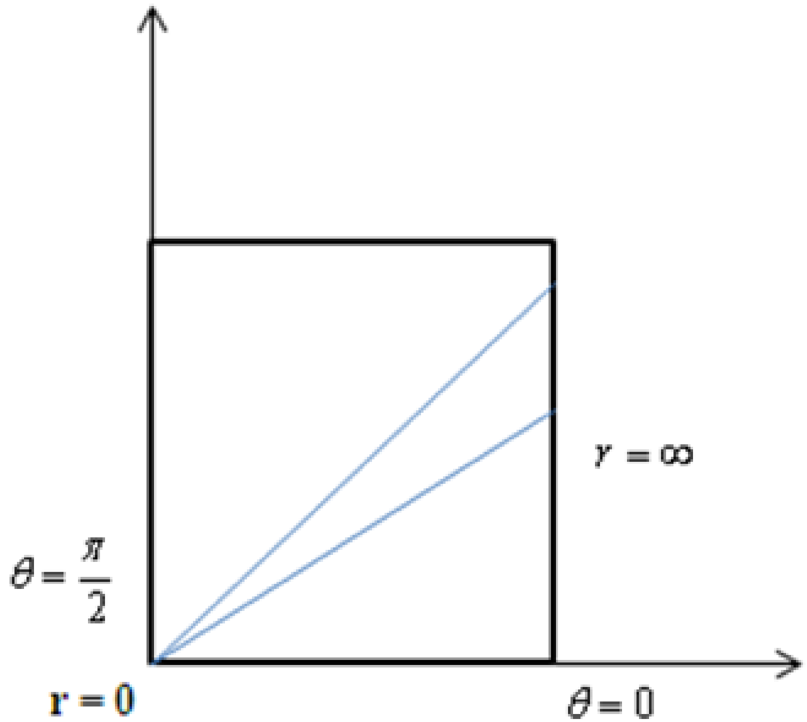
$$when\ x = 0 \quad 0 = r\cos\theta \ \Rightarrow r = 0$$

$$when\ x = \infty \quad \infty = r\cos\theta \ \Rightarrow r = \infty$$

The integrant

$$e^{-\left(x^2+y^2\right)} = e^{-\left(r^2\cos^2\theta + r^2\sin^2\theta\right)}$$

$$e^{-\left(x^2+y^2\right)} = e^{-\left(r^2\right)}$$



$$I = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \, r \; dr \; d\theta$$

$$t = r^2 \text{ ....} \mathbf{(1)}$$

Differentiating both sides

$$dt = 2r\,dr$$

$$\frac{dt}{2} = r\,dr$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-t}\,dt\;d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[e^{-t}\right]_0^{\infty}\;d\theta$$

$$= -\frac{1}{2}\int_{0}^{\frac{\pi}{2}}\left[0-1\right]\ d\theta$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}}\ d\theta$$

$$= \frac{1}{2}\left[\theta\right]_{0}^{\pi/2}$$

$$I = \frac{\pi}{4}$$

Change of order of Integration

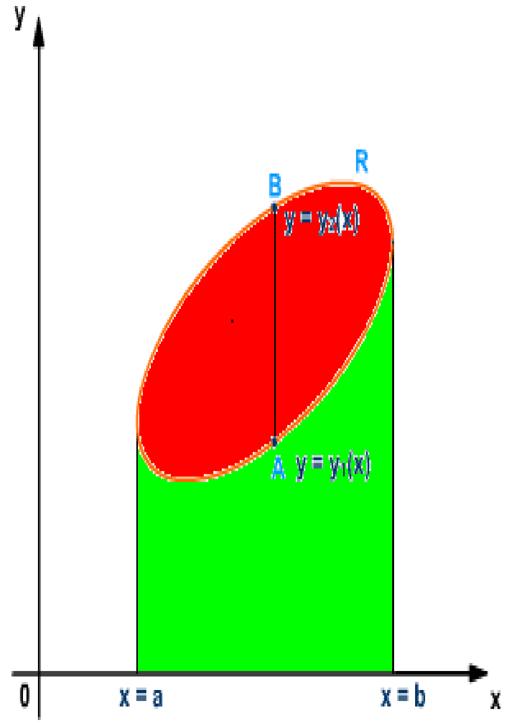
Incase the double integral $\int_{x=a}^{x=b} \left\{ \int_{y=y_1(x)}^{y=y_2(x)} f(x, y)\, dy \right\} dx$

is difficult to evaluate by integrating w.r.t y or we prefer to integrate w.r.t x first we may change the order of integration by changing the limits of integration of the variables.

Change of Order of integration

Step I: Given integral

$$\int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} f(x, y)\, dy\, dx$$



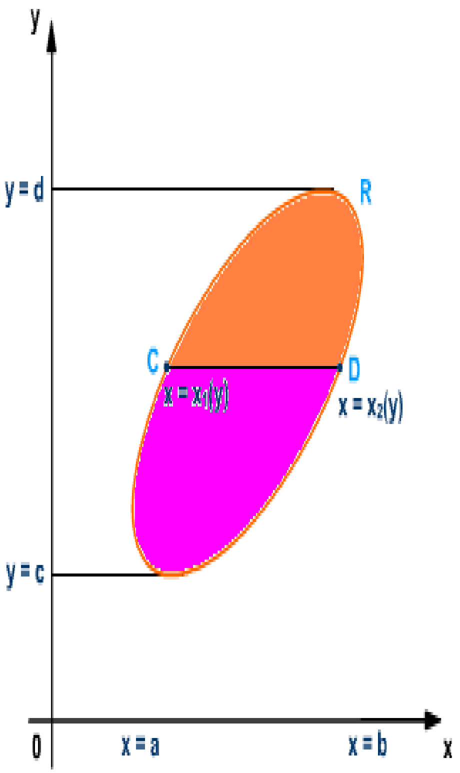
Observe that Integral is to be evaluated integrating with respect to y.

Step II: Determine the points of intersection by drawing a vertical step.

Step III:

To reverse the order of integration draw a horizontal strip in the closed region R.

Write the limits of x from $x = x_1(y)$ to $x = x_2(y)$ in the inner integral and limits of y from y = c to y = d in the outer integral.



Note:

Similar method can be used for changing order of integration for

$$\int_{y=c}^{y=d} \left\{ \int_{x=x_1(y)}^{x=x_2(y)} f(x, y)\, dx \right\} dy$$

Problem 01

Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$ by changing the order of integration.

Solution

Given $\int_0^1 \left\{ \int_x^{\sqrt{x}} xy\,dy \right\} dx$

The region is bounded by $y = x$, $y = \sqrt{x}$ and $x = 0$, $x = 1$

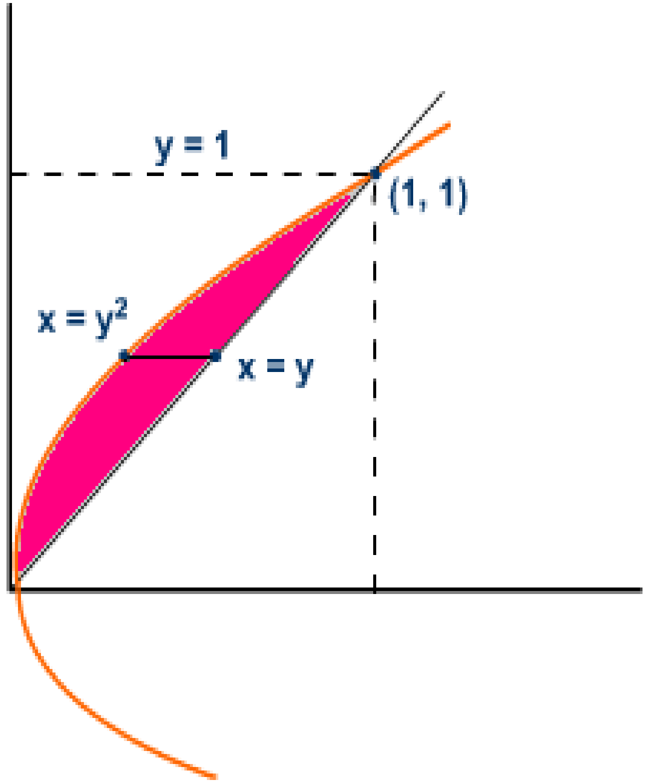
since y is expressed as a function of x.

By changing the order of integration we integrate w.r.t.x first and then evaluate the integral by integrating with respect to y.

Consider a horizontal strip in the region R.

In this region x varies from $x = y^2$ to $x = y$ and y varies from $y = 0$ to $y = 1$.

Now the double integral



$$\int_0^1 \left\{ \int_x^{\sqrt{x}} xy\, dy \right\} dx \quad = \int_{y=0}^{y=1} \left\{ \int_{x=y^2}^{x=y} xy\, dx \right\} dy$$

$$= \int_{y=0}^{y=1} \left[\frac{x^2}{2} y \right]_{y^2}^{y} dy \quad = \int_0^1 \left[\frac{y^3}{2} - \frac{y^5}{2} \right] dy$$

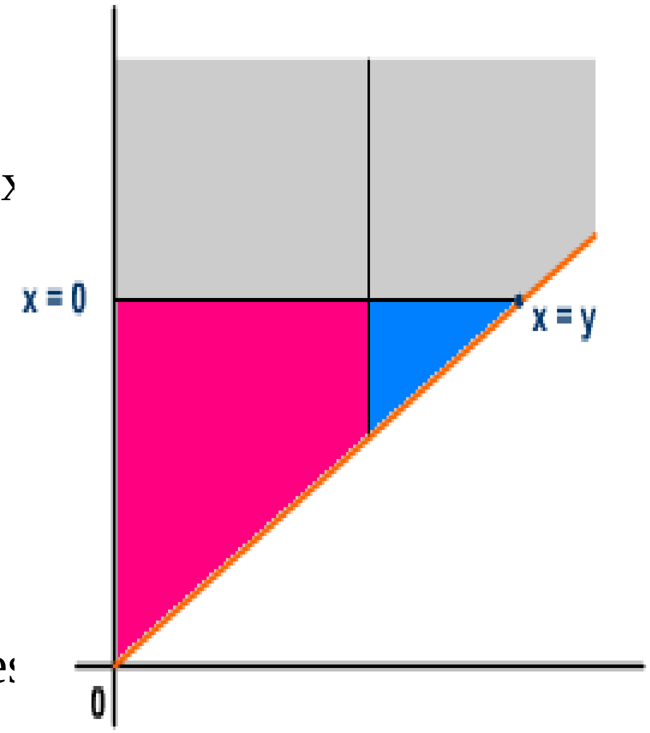
$$= \left. \frac{y^4}{8} - \frac{y^6}{12} \right]_0^1 \quad = \frac{1}{8} - \frac{1}{12} \quad = \frac{1}{24}$$

Problem 02

Evaluate $\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration.

Solution

Given double integral $\int_{x=0}^{\infty}\left\{\int_{y=x}^{\infty}\frac{e^{-y}}{y}dy\right\}dx$



In this integral, the inner integral is w.r.t.x

To change this inner integral w.r.t.y a vertical

is considered in the region R.

Now consider a horizontal strip where n varies from x = 0 to x = y and y varies from x = 0 to x = y and y varies from y = 0 to ∞

$$\therefore \quad \int_{x=0}^{\infty} \left\{ \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy \right\} dx \quad = \int_{y=0}^{\infty} \left\{ \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx \right\} dy$$

$$= \int_{0}^{\infty} \frac{e^{-y}}{+y}\, x \Big]_{0}^{y} \; dy \qquad = \int_{0}^{\infty} + e^{-y}\; dy$$

$$= e^{-y} \Big]_{0}^{\infty}$$

= -(0 - 1)

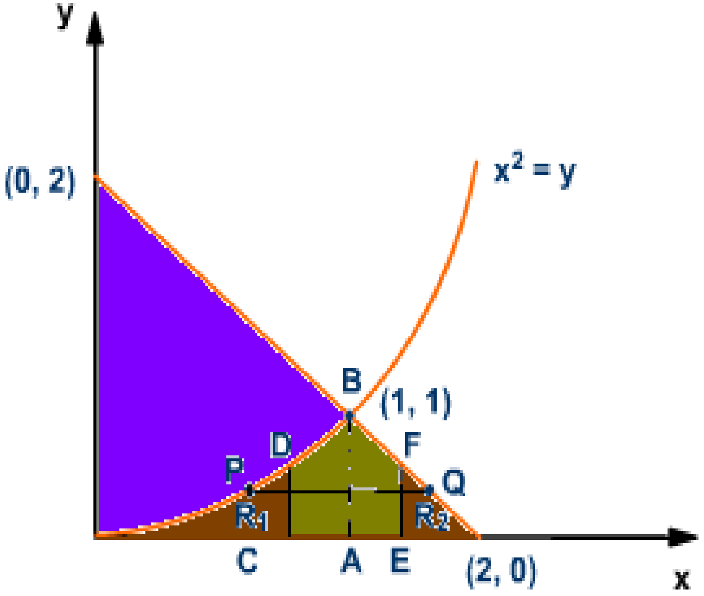
= 1

Problem 03

Change the order of integration and evaluate $\int_0^1 \int_{\sqrt{y}}^{2-y} xy \, dx \, dy.$

Solution

Given double integral $\int_0^1 \left\{ \int_{x=\sqrt{y}}^{x=2-y} xy \, dx \right\} dy$



Region is determined by $x^2 = y$ and

$x + y = 2$ on $y = 0$ to $y = 1$.

The region R be union of two non over lapping Regions R_1 and R_2.

In Region R_1, y varies from $y = 0$ to $y = x^2$ and x varies from $x = 0$ to $x = 1$.

$$\therefore \quad \iint_{R_1} xy\,dx\,dy = \int_{x=0}^{1} \left\{ \int_{y=0}^{y=x^2} xy\,dy \right\} dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} \right]_0^{x^2} dx \qquad = \int_0^1 \left[\frac{x^5}{2} - 0 \right] dx$$

$$= \frac{1}{2} \int_0^1 x^5\,dx \qquad = \frac{1}{2} \cdot \frac{x^6}{6} \Bigg]_0^1$$

$$= \frac{1}{12}$$

In Region R_2, y varies from y = 0 to y = 2 - x and x varies from x = 1 to x = 2

$$\therefore \quad \iint_{R_2} xy\ dx\ dy = \int_{x=1}^{x=2} \left\{ \int_{y=0}^{y=2-x} xy\ dy \right\} dx$$

$$= \int_{x=1}^{2} \left\{ \frac{xy^2}{2} \right\}_0^{2-x} dx = \int_1^2 \frac{x(2-x)^2}{2}\ dx$$

$$\therefore \quad \iint_{R} xy\, dx\ dy = \iint_{R_1} xy\, dx\, dy + \iint_{R_2} xy\, dx\ dy$$

$$= \frac{1}{12} + \frac{5}{24} \qquad = \frac{7}{24}$$

Problem 04

Change the order of integration in the integral $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$

Solution:

Given: $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$

The given integral is not of the form $\int_{y=c_2}^{y=c_1} \int_{x=f_2(y)}^{x=f_1(y)} \phi(x,y)\,dx\,dy$

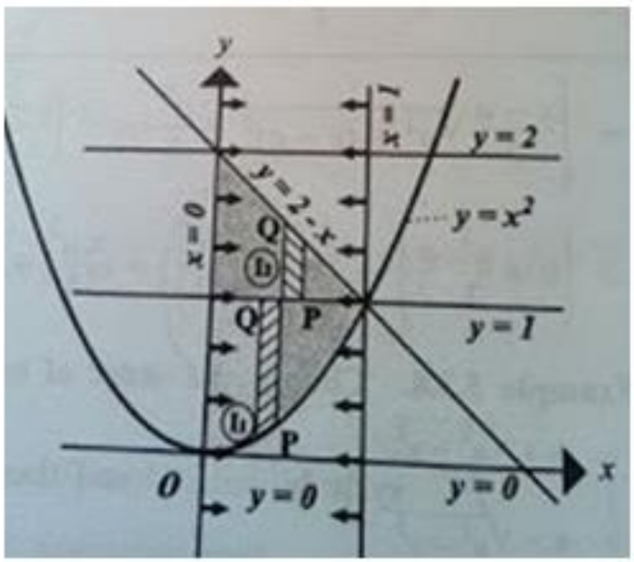
Therefore the given integral can be expressed as $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

The region of integration is bounded by $y = x^2$, $y = 2 - x$ (i.e $x + y = 2$), $x = 0$ *and* $x = 1$

Ie x varies from x= 0 to 1

y varies from $y=x^2$ to 2-x

Fig: 1 (before changing the order of integration)



For change of order of integration divide the area into two parts.

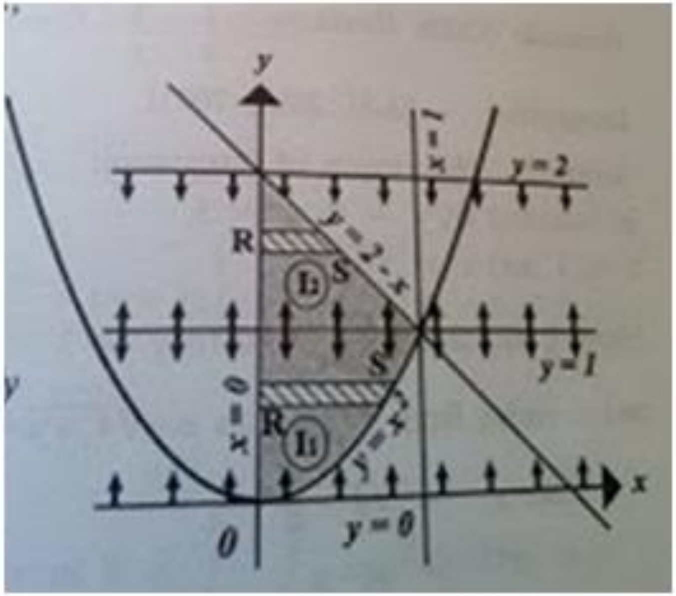
$$I = \mathrm{I}_1 + \mathrm{I}_2$$

$$I = \int_0^1 \int_{x^2}^1 xy\, dy\, dx + \int_0^1 \int_1^{2-x} xy\, dy\, dx$$

Now changing the order of integration *i.e* $\int\int xy\, dy\, dx = \int\int xy\, dx\, dy$

Changing the order of integration is nothing but to change the vertical path to horizontal path and to change the vertical strip PQ into horizontal strip RS

Now the limits of y and x can be obtained from the following fig



limits for I_1:

$x{:}\,0 \text{ to } \sqrt{y}$

$y{:}\,0 \text{ to } 1$

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy$$

$$= \int_0^1 \left[y \, \frac{x^2}{2} \right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{y^2}{2} \, dy$$

$$= \left[\frac{y^3}{6} \right]_0^1$$

$$I_1 = \frac{\mathbf{1}}{6}$$

Limits for I_2 :

x:0 to1

y:1 to 2-x

$$I_2 = \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

$$= \int_1^2 \left(\frac{yx^2}{2} \right)_0^{2-y} dy$$

$$= \int_1^2 y \frac{(2-y)^2}{2} dy$$

$$= \int_{1}^{2} y \, \frac{\left(4 + y^2 - 4y\right)}{2} \, dy$$

$$= \frac{1}{2} \int_{1}^{2} \left(4y + y^3 - 4y^2\right) dy$$

$$= \frac{1}{2} \left[\frac{4y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3} \right]_1^2$$

$$= \frac{1}{2} \left[\left(\frac{16}{2} + \frac{16}{4} - \frac{32}{3} \right) - \left(\frac{4}{2} + \frac{1}{4} - \frac{4}{3} \right) \right]$$

$$= \frac{1}{2}\left[\left(\frac{96+48-128}{12}-\frac{24+3-16}{12}\right)\right]$$

$$= \left[\frac{16}{24}-\frac{11}{24}\right] = \frac{5}{24}$$

$$I_2 = \frac{5}{24}$$

$$I = \frac{1}{6}+\frac{5}{24}$$

$$I = \frac{9}{24} = \frac{3}{8}.$$

Change of Variables between Cartesian and polar coordinates

The evaluation of a single integral is simplified by making a proper substitution (or change the variable). Similarly to evaluation of double integrals can also be evaluated by changing the variables from x,y to u,v

, Let $x = x(u,v)$ and $y = y(u,v)$ Now $f(x,y) = f(u,v)$

{expressed in terms of u and v}.

and the region R in xy plane is transformed to R' in the uv plane. Obviously the double integral will be transformed as

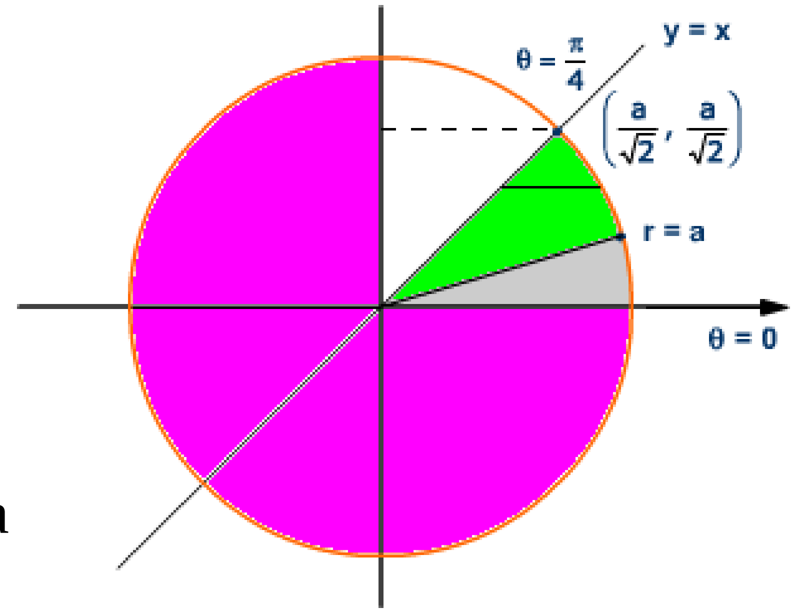
$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} F(u,v)\,J\,du\,dv \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

Problem 01

Evaluate $\int_{0}^{a/\sqrt{2}} \int_{y}^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dx\,dy$ changing to polar form.

Solution

Region of integration is



$$x = y \Rightarrow \cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

$$x = \sqrt{a^2 - y^2} \Rightarrow x^2 + y^2 = a^2 \Rightarrow r = a$$

y varies from 0 to $\frac{a}{\sqrt{2}} \Rightarrow \theta$ varies from 0 to $\frac{\pi}{4}$

$$\therefore \int_{y=0}^{y=a/\sqrt{2}} \left\{ \int_{x=y}^{x=\sqrt{a^2-y^2}} \log (x^2+y^2)\,dx \right\} dy$$

$$= \int_{\theta=0}^{\theta=\pi/4} \left\{ \int_{r=0}^{a} \log r^2\, r.dr\, d\theta \right\} \quad = \int_{\theta=0}^{\pi/4} \left\{ \int_{r=0}^{a} 2r \log r\, dr \right\} d\theta$$

$$= \int_{\theta=0}^{\pi/4} \left[a^2 \log a - \frac{a^2}{2} \right] d\theta \qquad = \left(a^2 \log a - \frac{a^2}{2} \right) \int_0^{\pi/4} d\theta$$

$$= \frac{\pi}{4} \left(a^2 \log a - \frac{a^2}{2} \right)$$

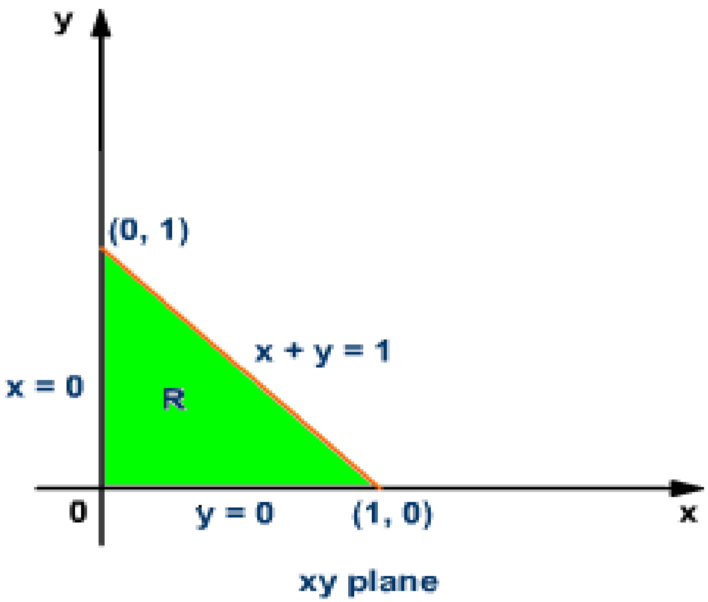
Problem 02

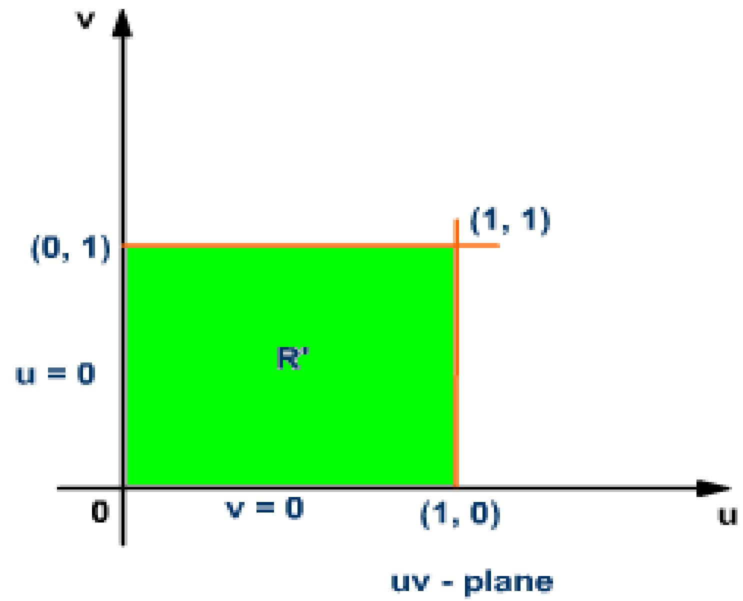
Using the transformation x + y = u and y = uv, evaluate $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dx\, dy$.

Solution

In $\iint_R e^{y/(x+y)}\, dy$

Region R is bounded by x-axis, x+y=1 and y varies from y = 0 to y = 1.





Solving form x and y we get x = u(1-v) and y = uv

$$\therefore \qquad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

To determine the region R^1.

(i) The x-axis (y = 0) gives y = 0, v = 0

since y > 0 $\Rightarrow$ u > 0 and v > 0

(ii) The y-axis, (x = 0) give u = 0, v = 1, since x > 0

$\Rightarrow$U (1 - v) > 0

$\Rightarrow$ u > 0 and v < 1

(iii) The line $x + y = 1$ gives $u = 1$.

$\therefore$ The region R^1 is bounded by $u = 0$, $v = 0$, $v = 1$ and $u = 1$

$$\therefore \quad \iint_R e^{y/x+y}\, dx\, dy$$

$$= \int_{v=0}^{v=1} \left\{ \int_{u=0}^{u=1} u\, e^{uv/u}\, du \right\} dv = \int_{v=0}^{v=1} \left. \frac{u^2}{2} e^v \right]_0^1 dv = \int_0^1 e^v \left[\frac{1}{2} \right] dv$$

$$= \frac{1}{2} e^v \Big]_0^1 = \frac{(e-1)}{2}$$

Triple Integration in Cartesian coordinates

On the same lines as we defined the double integral in two dimensions we can define a triple integral over a region R in 3-D and evaluate as follows.

$$\iiint_R f(x,y,z)\,dv = \iiint_R (x,y,z)\,dx\,dy\,dz$$

$$= \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} \int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Problem 01

Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx.$

Solution

Since all the limits are constants we can integrate in the given order.

$$\int_{-c}^{c}\int_{-b}^{b}\left(x^2x+y^2z+\frac{z^3}{3}\right)_{-a}^{a} dy\ dx \quad = \int_{-c}^{c}\int_{-b}^{b}\left[x^2(2a)+y^2(2a)+\frac{2a^3}{3}\right] dy\ dx$$

$$= \int_{-c}^{c} 2ax^2y+2a\frac{y^3}{3}+\frac{2a^3}{3}y\Bigg]_{-b}^{b} dx$$

$$= \int_{-c}^{c} \left[4ab\,x^2 + 4a\frac{b^3}{3} + \frac{4a^3b}{3} \right] dx$$

$$= 4ab\,\frac{x^3}{3} + \frac{4ab^3}{3}x + \frac{4a^3bx}{3} \Bigg]_{-c}^{c}$$

$$= \frac{8abc^3}{3} + \frac{8ab^3c}{3} + \frac{8a^3bc}{3}$$

$$= \frac{8abc}{3}(a^2 + b^2 + c^2)$$

Problem 02

Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz.$

Solution

$$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz \quad = \int_{-1}^{1}\int_{0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z}dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}(4xz+2z^2)\,dx\,dz \quad = \int_{-1}^{1}2x^2z+2z^2x\Big]_{0}^{z}\,dz$$

$$= \int_{-1}^{1}(2z^3+2z^3)dz = \int_{-1}^{1}4z^3\,dz = z^4\Big]_{-1}^{1} = 0$$

Problem 03

Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-z^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$.

Solution

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(1-x^2-y^2)^2-z^2}}\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\Bigg]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx$$

$$= \int_0^1 \frac{\pi}{2}\, y \Big]_0^{\sqrt{1-x^2}} dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\; dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] \quad = \frac{\pi^2}{8}$$

Area as double integral

(i) We $\iint\limits_{A} f(x,y)\,dx\ dy = \iint\limits_{A} f(x,y)\,d\,A$ for $f(x,y) = 1$

We have $\iint dx\,dy = \int\limits_{A} dA = A$ (total over A) of the region R.

(ii) Put $x = r\cos\theta$, $y = \sin\theta$ then $J = r$

$$\therefore \qquad \iint\limits_{R} dx\,dy = \iint\limits_{R'} r\,dr\ d\theta = A \text{ (total area of the region R)}$$

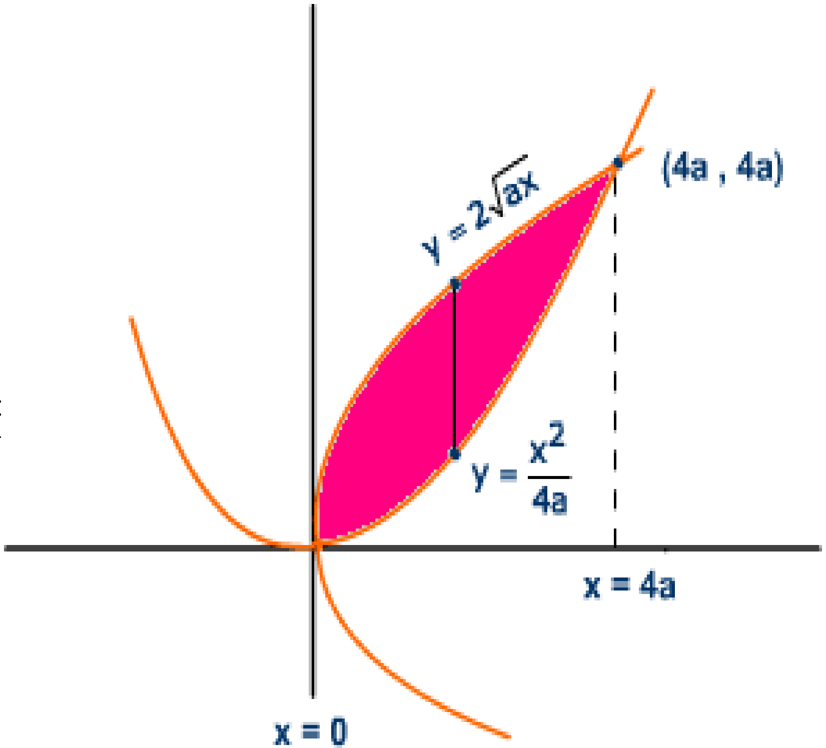
Problem 01

Find the area enclosed by the parabolas $x^2 + 4ay$ and $y^2 = 4ax$ by double integration.

Solution

Let $\quad x^2 = 4ay \quad \ldots(1)$

$\quad y^2 = 4ax \quad \ldots(2)$



Solving (1) and (2) we get the points of intersection (0, 0) and (4a, 4a)

$$\text{Required Area} = \iint_A dx\, dy$$

$$= \int_{x=0}^{4a} \left\{ \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dy \right\} dx \qquad = \int_{x=0}^{4a} y \Big]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_{0}^{4a} \left[2\sqrt{ax}\ 0\ \frac{x^2}{4a} \right] dx \qquad = 2(ax)^{3/2}\,\frac{2}{3}.\frac{1}{a} - \frac{x^3}{12a} \Big]_0^{4a}$$

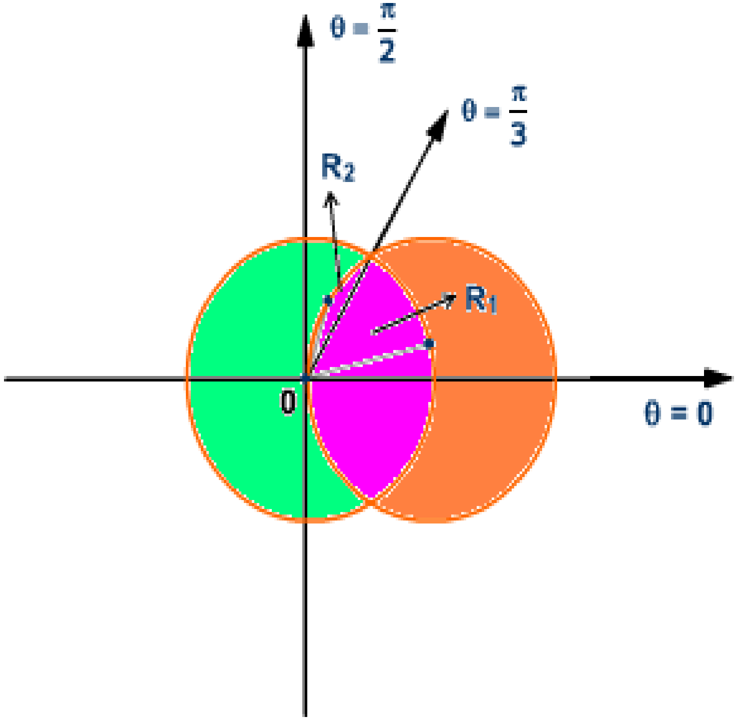
$$= \frac{4\sqrt{a}}{3}.(4a)^{3/2} - \frac{(4a)^3}{12a} \qquad = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Problem 02

Find the area common to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 2ax$.

Solution



Required area is shown in figure.

We have

$$x^2 + y^2 = a^2 \quad \Rightarrow \quad r = a \quad ...(1)$$

$$x^2 + y^2 = 2ax \quad \Rightarrow \quad r = 2a\cos\theta \quad ...($$

Solving (1) and (2) we get $\theta = \pi/3$

In R_1, r varies from 0 to r = a and θ varies from 0 to $\pi/3$

In R_2, r varies from 0 to $r = 2a\cos\theta$ and θ varies from $\pi/3$ to $\pi/2$

Required Area = $2\{R_1+R_2\}$

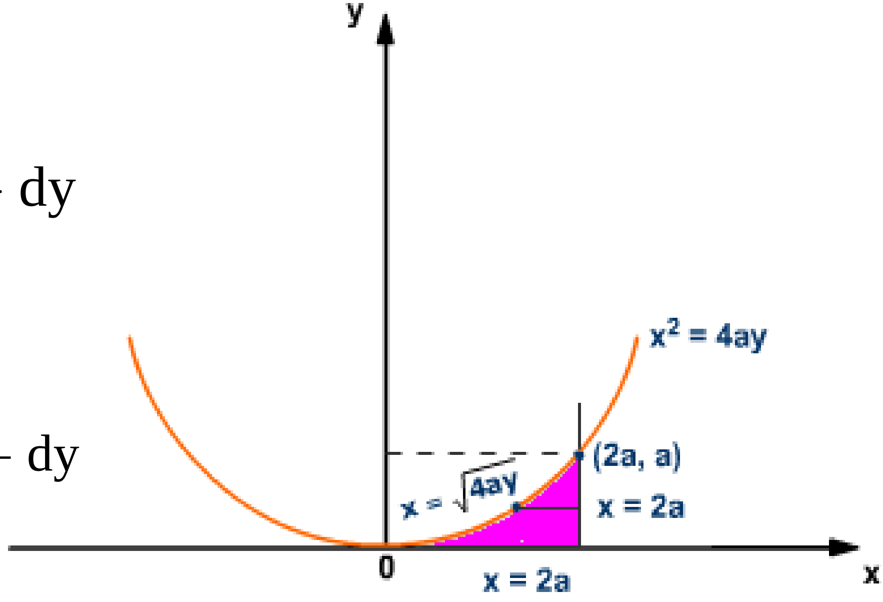
$$=2\int_{\theta=0}^{\pi/3}\left\{\int_{r=0}^{a} r\,dr\right\}d\theta + 2\int_{\theta=\pi/3}^{\pi/2}\left\{\int_{r=0}^{2a\cos\theta} r\,dr\right\}d\theta$$

$$=2.\int_{0}^{\pi/3} a^2\,d\theta + 2\frac{1}{2}\int_{\pi/3}^{\pi/2} 4a^2\cos^2\theta\,d\theta$$

$$=\frac{2}{3}\pi a^2-\frac{\sqrt{3}a^2}{2}$$

Problem 03

Evaluate $\iint\limits_A xy = dx\,dy$, where A is the region bounded by x-axis, ordinate x = 2a and the curve $x^2 = 4ay$.



Solution

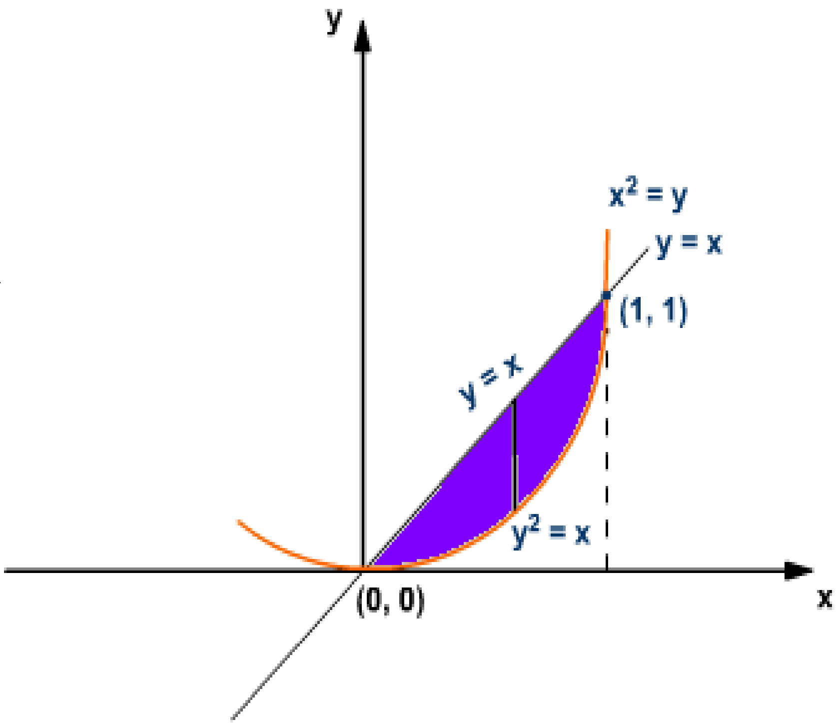
$$\iint\limits_A xy\,dx\,dy = \int_{y=0}^{a}\left\{\int_{x=\sqrt{4ay}}^{x=2a} xy\,dx\right\} dy$$

$$= \int_0^a \left.\frac{x^2 y}{2}\right]_{2\sqrt{ay}}^{2a} dy \quad = \int_0^a \frac{4a^2 y - 4ay^2}{2}\,dy$$

$$= 2a\int_0^a (ay - y^2)\,dy = 2a\left\{\frac{ay^2}{2} - \frac{y^3}{3}\right\}_0^a = \frac{a^4}{3}$$

Problem 04

Evaluate $\iint\limits_{A} xy(x+y)\,dx\,dy$ where A is the region bounded by $y = x^2$ and $y = x$.



Solution

$$\iint\limits_{A} xy\,(x+y)\,dx\,dy \quad = \int_0^1 \left\{ \int_{y=x^2}^{y=x} (x^2 y \right.$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{5}{6}x^4 - \frac{1}{2}x^6 - \frac{1}{3}x^7 \right) dx \qquad = \frac{3}{56}$$

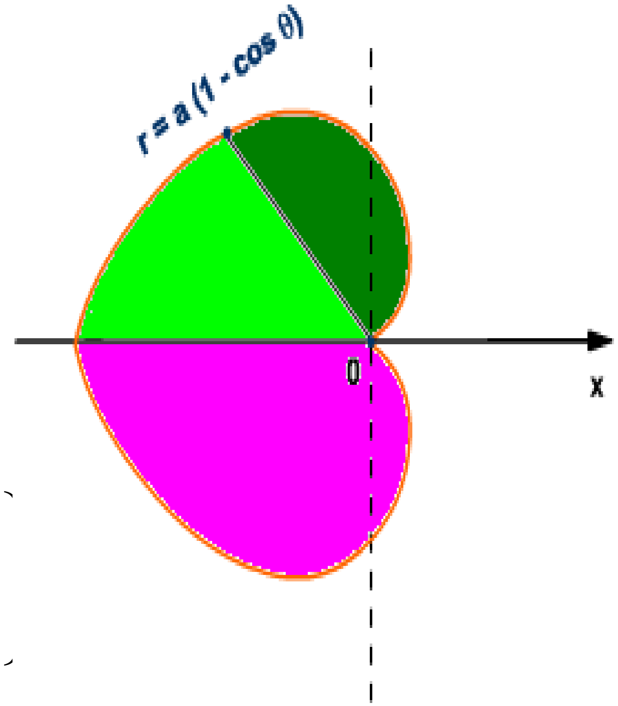
Problem 05

Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardiode $r = a(1-\cos\theta)$

above the initial line.

Solution

$$\iint r\sin\theta\, dr d\theta = \int_{\theta=0}^{\pi}\left\{\int_{r=0}^{r=a(1-\cos\theta)} r\sin\theta\, dr\right\} d\theta$$



$$= \int_0^{\pi} \sin\theta \left.\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{a^2}{2}\left\{\int_0^{\pi}\sin\theta\, d\theta - \int_0^{\pi}\cos^2\theta\,(-\sin\theta)\, d\theta - \int_0^{\pi}\sin 2\theta\, d\theta\right\}$$

$$= \frac{a^2}{2}\left\{-\cos\theta - \frac{\cos^3\theta}{3} + \frac{\cos 2\theta}{2}\right\}_0^{\pi} = \frac{4a^2}{3}$$

Problem 06

If R is the region bounded by the circle $r = 2a\cos\theta$ above the initial line show

that $$\iint_R r^2 \sin\theta \, dr\, d\theta = \frac{2}{3}a^3.$$

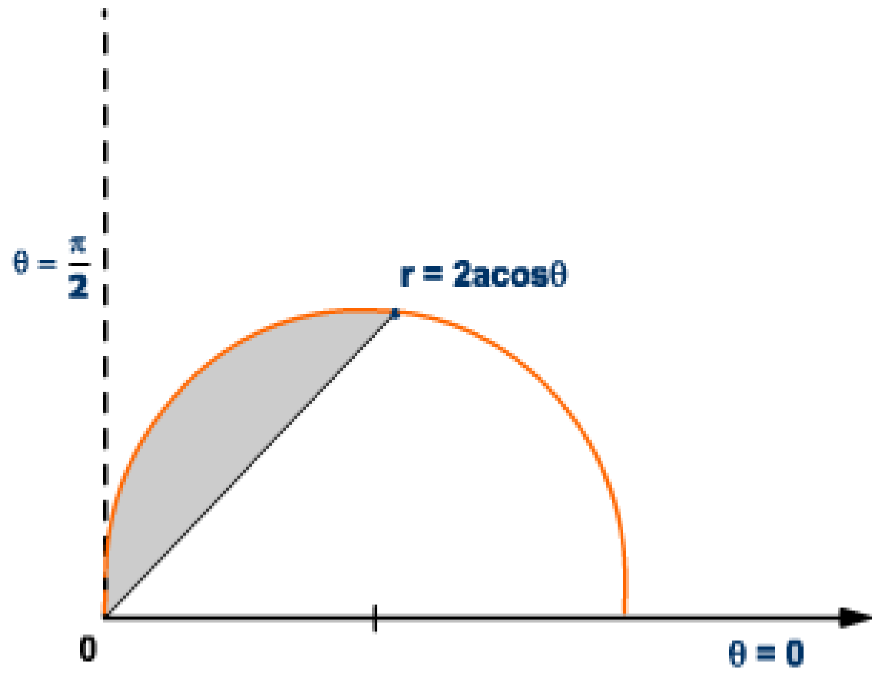
Solution

The circle $r = 2a\cos\theta$

is $r^2 = 2ar\cos\theta$

in Cartesian form it is equivalent to $x^2 + y^2 - 2ax = 0$

i.e., a circle with centre at (a,0) and passing through the origin.



$$\iint_R r^2 \sin\theta \, dr \, d\theta \qquad = \int_{\theta=0}^{\pi/2} \left\{ \int_{r=0}^{2a\cos\theta} \sin\theta . r^2 \, dr \right\} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta . \frac{r^3}{3} \Bigg]_0^{2a\cos\theta} d\theta \qquad = \int_0^{\pi/2} \frac{\sin\theta}{3} \left[8a^3 \cos^3\theta \right] d\theta$$

$$= \frac{-8a^3}{3} \int_0^{\pi/2} \cos^3\theta . (-\sin\theta) \, d\theta \qquad = -\frac{8a^3}{3} \frac{\cos^4\theta}{4} \Bigg]_0^{\pi/2}$$

$$= -\frac{2a^3}{3}[0-1] \qquad = \frac{2a^3}{3}$$

Problem 07

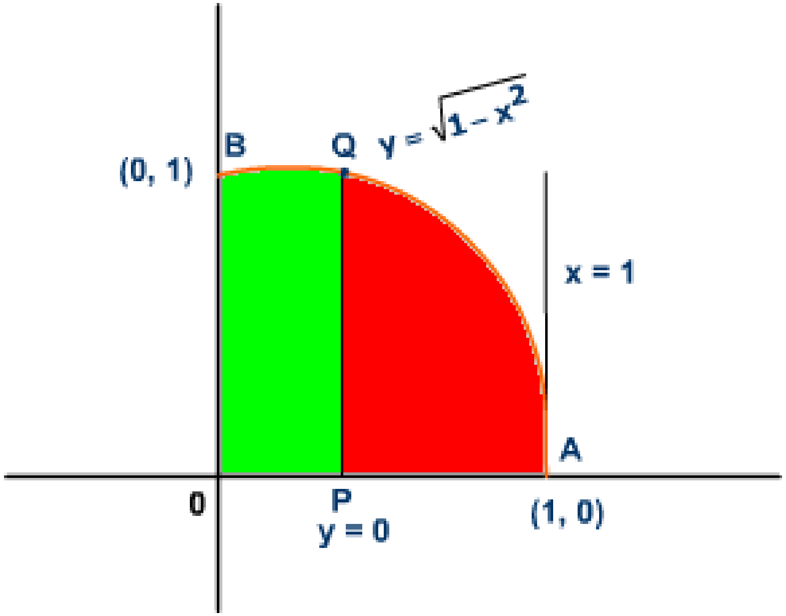
If R is the region bounded by the circle $x^2+y^2 = 1$ in the first quadrant,

evaluate $\iint_R \frac{xy}{\sqrt{1-y^2}}\, dx\, dy.$

Solution

$$\iint_R \frac{xy}{\sqrt{1-y^2}}\, dx\, dy$$

$$= \int_{x=0}^{a} \left\{ \int_{y=0}^{y=\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}}\, dy \right\} dx$$



$$= \int_{x=0}^{1} x\left[-\sqrt{1-y^2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 x\left\{-\sqrt{1-(1-x^2)} + \sqrt{1-0}\right\} dx$$

$$= \int_0^1 x\{-x+1\}\, dx \qquad = \int_0^1 (x - x^2)\, dx$$

$$= \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \qquad = \frac{1}{2} - \frac{1}{3} \qquad = \frac{1}{6}$$

Volume by using Triple Integrals

$$\iiint_R dV = V = \text{Volume of the region } R = \iiint_R dx\,dy\,dz$$

Note:

1. $V = \iiint_R dV = \iiint_R R\,dR\,d\phi\,dz$ in cylindrical polar coordinates

2. $V = \iiint_R dV = \iiint_R r^2 \sin\theta\,dr\,d\theta\,d\phi$ in spherical polar coordinates

Problem 01

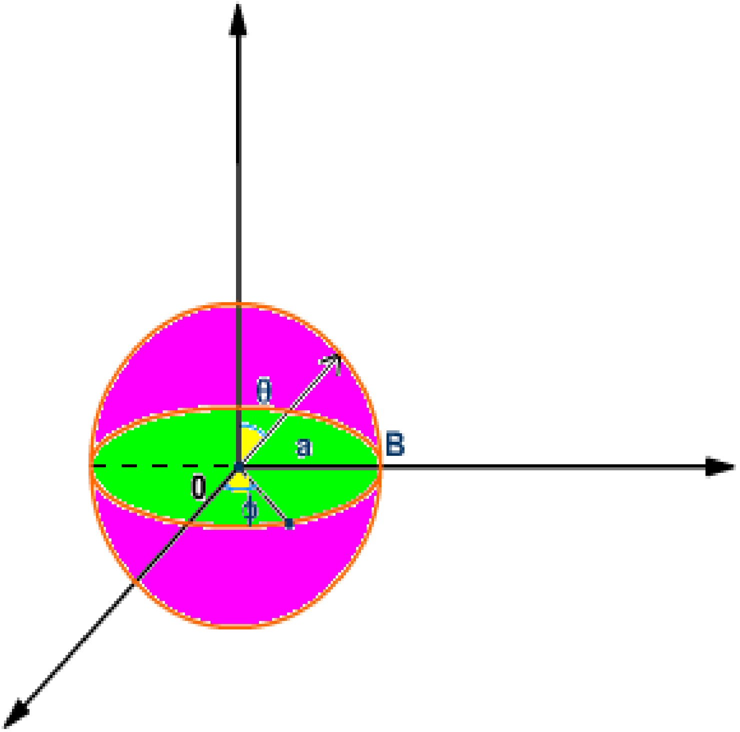
Find by triple integration the volume of the sphere of radius a.

Solution

$$\iiint_V dx\,dy\,dz$$

$$a \quad \sqrt{a^2 - x^2} \quad \sqrt{a^2 - x^2 - y^2}$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$



Changing to spherical coordinate system

dr dy dz = $r^2 . \sin\theta \; dr\, d\phi\, d\theta$

$$V = 8 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{r=0}^{a} r^2 \sin\theta dr d\phi d\theta$$

$$= 8 \int_{\theta=0}^{\pi/2} \sin\theta d\theta \int_{\phi=0}^{\pi/2} d\phi \int_{r=0}^{a} r^2 dr$$

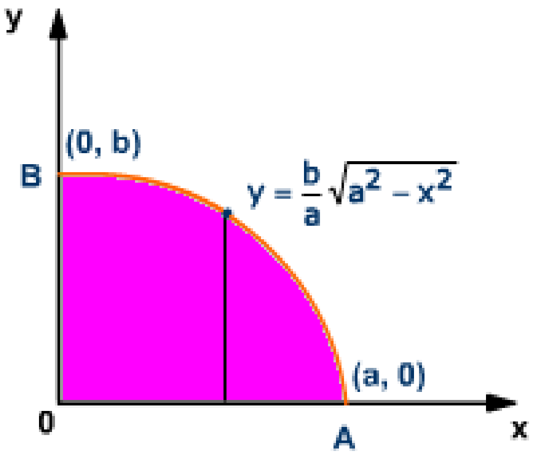
$$= \frac{4\pi a^3}{3}(1)$$

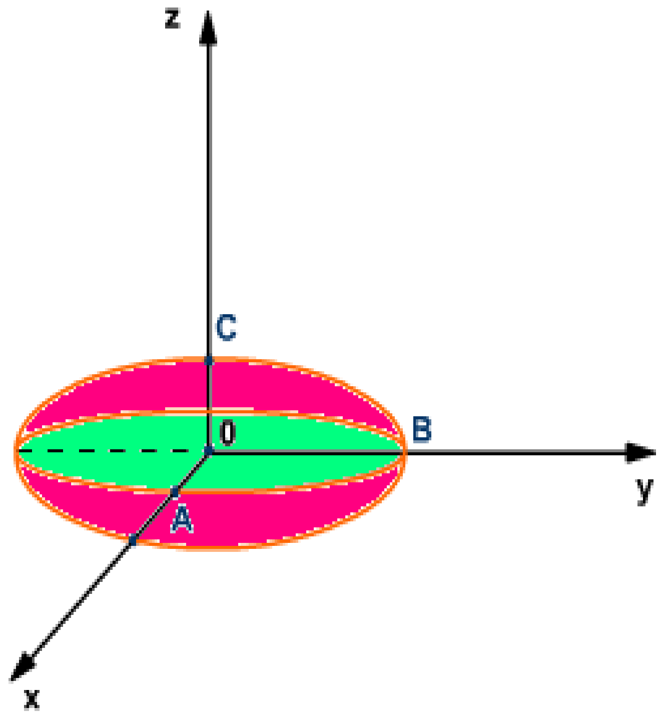
Problem 02

Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using triple integration

Solution

$$\text{Volume } V = 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$





$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-x^2/a^2}} z \Big]_{0}^{c\sqrt{1-\frac{x^2}{a^2}\frac{y^2}{b^2}}} \; dy \; dx$$

$$= \int_{x=0}^{a} \left\{ \int_{y=0}^{\frac{b}{a}\sqrt{1^2-x^2/b^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \; dy \right\} dx$$

$$=8 \int_{x=0}^{a} \left\{ \int_{y=0}^{b\sqrt{1-x^2/a^2}} c\sqrt{\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}} \; dy \right\} dx$$

$$=8\int_{x=0}^{a}\left\{\int_{y=0}^{b\sqrt{1-x^2/a^2}}\frac{c}{b}\sqrt{b^2\left[1-\frac{x^2}{a^2}\right]-y^2}\;dy\right\}dx$$

$$=\frac{8c}{b}\int_{x=0}^{a}\left[\frac{b^2(1-\frac{x^2}{a^2})}{2}\sin^{-1}\left[\frac{y}{b\left[\sqrt{1-x^2/a^2}\right]}\right]+\frac{y}{2}\right.$$

$$\left.+\frac{y}{2}\sqrt{b^2\left(\frac{1-x^2}{a^2}\right)-y^2}\right]_0^{b\sqrt{1-x^2/a^2}}$$

$$=\frac{8bc}{2}\int_{x=0}^{a}\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}\,dx \qquad =\frac{8\pi ab}{4a^2}\int_0^a (a^2-x^2)\,dx$$

$$=\frac{8\pi abc}{4a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a \qquad =\frac{8\pi bc}{4a^2}\left[a^3-\frac{a^3}{3}\right]$$

$$=\frac{8\pi bc}{4a^2}\cdot\frac{2a^3}{3}$$

$$=\frac{4\pi abc}{3}$$

- **Problem 03**

Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$

Solution:

$$Given: \int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$$

$$= \int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x} e^{y} e^{z}\,dx\,dy\,dz$$

Correct form:

$$I = \int_{z_1=c_1}^{z_2=c_2}\int_{y_1=g_1(z)}^{y_2=g_2(z)}\int_{x_1=f_1(y,z)}^{x_2=f_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

Rearrange the order

$$= \int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y+z} \right]_0^{x+y} dy\, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y+x+y} - e^{x+y} \right] dy\, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{2x+2y} - e^{x+y} \right] dy\, dx$$

$$= \int_0^{\log 2} \left[\frac{e^{2x+2y}}{2} - e^{x+y} \right]_0^x dx$$

$$= \int_0^{\log 2} \left[\frac{e^{2x+2x}}{2} - e^{x+x} - \left\{ \frac{e^{2x}}{2} - e^{x} \right\} \right] dx$$

$$= \int_0^{\log 2} \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \right] dx$$

$$= \int_0^{\log 2} \left[\frac{e^{4x}}{2} - 3\frac{e^{2x}}{2} + e^{x} \right] dx$$

$$= \left[\frac{e^{4x}}{8} - 3\frac{e^{2x}}{4} + e^{x} \right]_0^{\log 2}$$

$$= \left[\frac{e^{4x}}{8} - 3\frac{e^{2x}}{4} + e^{x} \right]_0^{\log 2}$$

$$= \left[\frac{e^{4\log 2}}{8} - 3\frac{e^{2\log 2}}{4} + e^{\log 2} - \left\{ \frac{1}{8} - \frac{3}{4} + 1 \right\} \right]$$

$$= \left[\frac{e^{\log 2^{4}}}{8} - 3\frac{e^{\log 2^{2}}}{4} + e^{\log 2} - \left\{ \frac{3}{8} \right\} \right]$$

$$= \left[\frac{2^4}{8} - (3)\frac{2^2}{4} + 2 - \left\{ \frac{3}{8} \right\} \right]$$

$$= \left[\frac{16}{8} - 3 + 2 - \left\{ \frac{3}{8} \right\} \right]$$

$$= \left[\frac{13}{8} - 1 \right]$$

$$= \frac{5}{8}$$

- **Problem 04**

Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ **and the coordinate plane x= 0,y=0,z=0**

Solution:

$$\text{volume} = \int\int\int dz\,dy\,dx$$

Here z varies from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$

x varies from o to a

$$V=\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\right] dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} (z)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dydx$$

$$= c\int_0^a \left\{ \int_0^{b\left(1-\frac{x}{a}\right)} \left[\left(1-\frac{x}{a}\right)-\frac{y}{b}\right] dy \right\} dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right) b \left(1-\frac{x}{a}\right) - \frac{\left\{ b\left(1-\frac{x}{a}\right) \right\}^2}{2b} \right] dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)^2 \left(1-\frac{1}{2}\right) \right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= \frac{bc}{2}\left[\frac{-a}{3}\right]\left[0-1\right]$$

$$= \frac{abc}{6} \text{ cubic units}$$

PROBLEM : 5

Evaluate $\iiint x^2yz\,dx\,dy\,dz$ **taken over the tetrahedron bounded by the planes x=0,y=0,z=0 and** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Solution:

Limits are

z varies from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$

x varies from 0 to a

$$\iiint x^2yz\,dx\,dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2yz\,dx\,dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} x^2y\left(\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} z\,dz\right)dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} x^2y\left(\frac{x^2}{2}\right)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx$$

$$=\frac{1}{2}\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} x^2yc^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2 dydx$$

$$=\frac{c^2}{2}\int_0^a x^2\left(\int_0^{bk} y\left(k-\frac{y}{b}\right)^2 dy\right)dx, \qquad k=1-\frac{x}{a}$$

$$=\frac{c^2}{2}\int_0^a x^2\left(\int_0^{bk} y\left(k^2+\frac{y^2}{b^2}-\frac{2ky}{b}\right)dy\right)dx$$

$$=\frac{c^2}{2}\int_0^a x^2\left(\int_0^{bk}\left(k^2y+\frac{y^2}{b^2}y-\frac{2ky^2}{b}\right)dy\right)dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\int_0^{bk}\left(k^2y+\frac{y^2}{b^2}y-\frac{2ky^2}{b}\right)dy\right)dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(k^2\frac{y^2}{2}+\frac{y^4}{4b^2}-\frac{2ky^3}{3b}\right)_0^{bk}dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(k^2\frac{b^2k^2}{2}+\frac{k^4b^4}{4b^2}-\frac{2k^4b^3}{3b}\right)_0^{bk}dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{b^2k^4}{2}+\frac{k^4b^2}{4}-\frac{2k^4b^2}{3}\right)dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{6b^2k^4 + 3b^2k^4 - 8b^2k^4}{12}\right)dx$$

$$= \frac{c^2}{24}\int_0^a x^2 b^2 k^4 dx$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1 - \frac{x}{a}\right)^4 dx \qquad \because k = 1 - \frac{x}{a}$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left[1 - \frac{4x}{a} + \frac{4(4-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{4(4-1)(4-2)}{3!}\left(\frac{x}{a}\right)^3 + \frac{4(4-1)(4-2)(4-3)}{4!}\left(\frac{x}{a}\right)^4\right]dx$$

$$\left[\because (1+x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\right]$$

$$= \frac{b^2c^2}{24}\int_0^a\left[x^2 - \frac{4x^3}{a} + \frac{6x^4}{a^2} - \frac{4x^5}{a^3} + \frac{x^6}{a^4}\right]dx$$

$$= \frac{b^2c^2}{24}\left[\frac{x^3}{3} - \frac{4x^4}{4a} + \frac{6x^5}{5a^2} - \frac{4x^6}{6a^3} + \frac{x^7}{7a^4}\right]_0^a$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} - \frac{a^4}{a} + \frac{6a^5}{5a^2} - \frac{4a^6}{6a^3} + \frac{a^7}{7a^4}\right]$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} - a^3 + \frac{6a^3}{5} - \frac{2a^3}{3} + \frac{a^3}{7}\right]$$

$$= \frac{b^2c^2a^3}{24}\left[\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right]$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} - a^3 + \frac{6a^3}{5} - \frac{2a^3}{3} + \frac{a^3}{7}\right]$$

$$= \frac{b^2c^2a^3}{24}\left[\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left[-\frac{4}{3} + \frac{1}{7} + \frac{6}{5}\right]$$

$$= \frac{a^3b^2c^2}{24}\left[\frac{-140+15+126}{105}\right] = \frac{a^3b^2c^2}{24}\left[\frac{1}{105}\right]$$

$$= \frac{a^3b^2c^2}{2520}$$

PROBLEM : 6

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz\, dy\, dx$

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz dy dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz dy dx}{\sqrt{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1} \frac{z}{\sqrt{a^2-x^2-y^2}} \right)_0^{\sqrt{a^2-x^2-y^2}} dydx \qquad \because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}} \right) - \sin^{-1}(0) \right] dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - 0 \right] dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\frac{\pi}{2} \right) dydx \qquad \left[\because \sin^{-1}(1) = \frac{\pi}{2} \right]$$

$$= \int_0^a \frac{\pi}{2} (y)_0^{\sqrt{a^2-x^2}} \, dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2 - x^2} \, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a}{2} \sin^{-1}\left(\frac{x}{a} \right) \right]_0^a$$

$$= \frac{\pi}{2} \left[0 + \frac{a}{2} \sin^{-1}\left(\frac{a}{a} \right) - 0 - \frac{a}{2} \sin^{-1}(0) \right]$$

$$= \frac{\pi}{2}\left[\frac{a}{2}\sin^{-1}(1)\right] = \frac{\pi a}{4}.\frac{\pi}{2} \qquad \because \sin^{-1}(1) = \frac{\pi}{2}$$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}} = \frac{a\pi^2}{8}$$