Module – 3 Laplace Transforms

Laplace Transforms of standard functions – Transforms properties – Transforms of Derivatives and Integrals – Initial value theorems (without proof) and verification for some problems – Final value theorems (without proof) and verification for some problems – Inverse Laplace transforms using partial fractions – Inverse Laplace transforms using second shifting theorem – LT using Convolution theorem – problems only – ILT using Convolution theorem – problems only – LT of periodic functions – problems only – Solve linear second order ordinary differential equations with constant coefficients only – Solution of Integral equation and integral equation involving convolution type – Application of Laplace Transform in Engineering.

Periodic function:

A function $f(t)$ is said to be periodic function if $f(t + p) = f(t)$ for all t. The least value of $p > 0$ is called the period of $f(t)$. For example, $\sin t$ and $\cos t$ are periodic functions with period 2π .

Laplace Transform:

Let f(t) be a given function which is defined for all positive values of t, if

$$L\{f(t)\}= F(s) = \int_0^\infty e^{-st} f(t)\, dt$$ exists, then F(s) is called *Laplace transform* of f(t).

Sufficient condition for the existence of Laplace transform:

The Laplace transform of $f(t)$ exists if

- **i.** $f(t)$ is piecewise continuous in $[a, b]$ where $a > 0$.
- **ii.** $f(t)$ is of exponential order.

Laplace transform for some basic functions

S.No	**f(t)**	**L{f(t)}**
1	e^{at}	$\frac{1}{s-a},\ s-a>0$
2	e^{-at}	$\frac{1}{s+a},\ s+a>0$
3	$\sin at$	$\frac{a}{s^2+a^2},\ s>0$
4	$\cos at$	$\frac{s}{s^2+a^2},\ s>0$

5	$\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
6	$\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$
7	1	$\frac{1}{s}$
8	t	$\frac{1}{s^2}$
9	t^n	$\frac{n!}{s^{n+1}}$
10	Periodic function with period ‘p’	$\frac{1}{1-e^{-ps}}\int_0^p e^{-at} f(t)\,dt$

Properties of Laplace transform:

Sl. No.	Property	Laplace Transform
1	Linear Property	$L\big(af(t)\pm bg(t)\big)=aL\big(f(t)\big)\pm bL\big(g(t)\big)$
2	First shifting theorem	$L\big(e^{-at}f(t)\big)=F(s+a)$ $L\big(e^{at}f(t)\big)=F(s-a)$
3	Change of scale property	$L\big(f(at)\big)=\frac{1}{a}F\left(\frac{s}{a}\right),\ a>0$
4	Multiplication by t	$L\big(t^n f(t)\big)=(-1)^n\frac{d^n}{ds^n}F(s)$
5	Division by t	$L\left(\frac{f(t)}{t}\right)=\int_s^\infty F(s)\,ds$, provided $\lim_{t\to 0}\frac{f(t)}{t}$ exists
6	Transforms of integrals	$L\left(\int_0^t f(t)dt\right)=\frac{L[f(t)]}{s}$

Inverse Laplace transform for some basic functions:

S.No	F(s)	$f(t)=L^{-1}(F(s))$
1	$\frac{1}{s-a},\ s-a>0$	e^{at}
2	$\frac{1}{s+a},\ s+a>0$	e^{-at}
3	$\frac{a}{s^2+a^2},\ s>0$	$\sin at$
4	$\frac{s}{s^2+a^2},\ s>0$	$\cos at$
5	$\frac{a}{s^2-a^2},\ s>\lvert a\rvert$	$\sinh at$
6	$\frac{s}{s^2-a^2},\ s>\lvert a\rvert$	$\cosh at$
7	$\frac{1}{s}$	1
8	$\frac{1}{s^2}$	t
9	$\frac{n!}{s^{n+1}}$	t^n

Initial Value theorem:

If $L(f(t))=F(s)$ then $\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$

Final value theorem:

If $L(f(t))=F(s)$ then $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$

Convolution:

The convolution of two functions f(t) and g(t) is defined as $\int_0^t f(u)g(t-u)\,du = f(t)*g(t)$

Convolution theorem:

The Laplace transform of convolution of two functions is equal to the product of their Laplace transforms.

(i.e) $L\left[f(t)*g(t)\right]=L\left[f(t)\right]L\left[g(t)\right].$

1. **Obtain the Laplace transform of $\boldsymbol{sin\,2t-2t\,cos\,2t}$.**

Solution: $L[\sin 2t-2t\cos 2t] = L[\sin 2t]-2L[t\cos 2t]=L[\sin 2t]-2\left(-\frac{d}{ds}L[\cos 2t]\right)$

$$=\frac{2}{s^2+4}+2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4}+2\left(\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right)$$

$$=\frac{2(s^2+4)+2(4-s^2)}{(s^2+4)^2}$$

$\therefore L[\sin 2t-2t\cos 2t] = \dfrac{16}{(s^2+4)^2}$

2. **Find the Laplace transform $\boldsymbol{sin^3(2t)}$**

Solution: $L\left[sin^3(2t)\right]=\frac{1}{4}\text{L[3sin2t – sin6t]} = \frac{3}{4}\text{L[sin2t]} – \frac{1}{4}\text{L[sin6t]}$

$\left(\because \sin^3 t=\frac{1}{4}[3\sin t-\sin 3t]\right)$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right)-\frac{1}{4}\left(\frac{6}{s^2+36}\right) = \frac{6}{4}\left(\frac{1}{s^2+4}-\frac{1}{s^2+36}\right).$$

Find the Laplace transform of $\boldsymbol{f(t)=cos^2(3t)}$.

3.

Solution: $\text{L}[\cos^2 3\text{t}] = L\left[\frac{1+\cos 6t}{2}\right]=\frac{L(1)+L(\cos 6t)}{2}\ \because \cos^2 t=\frac{1+\cos 2t}{2}$

$$= \frac{1}{2s}+\frac{s}{2(s^2+36)}\ \because L(1)=\frac{1}{s}, L(\cos at)=\frac{s}{s^2+a^2}$$

$\therefore$ $\text{L}[\cos^2 3\text{t}] = \dfrac{s^2+18}{s(s^2+36)}$

4. **Find the Laplace transform of unit step function**

Solution: The Unit step function is $u_a(t)=\begin{cases}0, & t<a\\ 1, & t>a,\ a\geq 0\end{cases}$

The Laplace transform $L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt=\int_a^\infty e^{-st}(1)\,dt=\left[\frac{e^{-st}}{-s}\right]_a^\infty=-\frac{1}{s}\left[e^{-\infty}-e^{-as}\right]=\frac{e^{-as}}{s}.$

5. **Find the Laplace transform of the following functions (i) $\frac{e^{-t}\sin t}{t}$ (ii) $t^2\cos t$**

Solution:

(i) To find $\frac{e^{-t}\sin t}{t}$

$$L\left(\frac{e^{-t}\sin t}{t}\right)=\int_s^\infty L\left(e^{-t}\sin t\right)ds$$

$$=\int_s^\infty L(\sin t)_{s+1}\,ds=\int_s^\infty\left(\frac{1}{s^2+1}\right)_{s+1}ds=\int_s^\infty\frac{1}{(s+1)^2+1}ds$$

$$=\left[\tan^{-1}(s+1)\right]_s^\infty=\frac{\pi}{2}-\tan^{-1}(s+1)=\cot^{-1}(s+1)$$

(ii) $t^2\cos t$

$$L\left[t^2\cos t\right]=\left[\frac{d^2}{ds^2}L[\cos t]\right]=\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$=\frac{d}{ds}\left(\frac{(s^2+1).1-1.2s.s}{(s^2+1)^2}\right)=\frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$$

$$=\frac{(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)2s}{(s^2+1)^3}=\frac{-2s(3-s^2)}{(s^2+1)^3}$$

6. **Find the Laplace transform of $e^{-2t}t^{1/2}$.**

Solution: $L\left[e^{-2t}t^{1/2}\right]=L\left[t^{1/2}\right]_{s\to s+2}$

$\because$ *If* $L[f(t)]=F(s)$, then $L\left[e^{-at}f(t)\right]=F(s)\big|_{s\to s+a}$

$$= \left[\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}\right]_{s\to s+2} = \left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}\right]_{s\to s+2}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{(s+2)^{3/2}} \left(\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \Gamma n+1 = n\Gamma n\right)$$

7. **Find $L\left[t^2 e^{-t}\cos t\right]$**

Solution:

$$L\left[t^2e^{-t}\cos t\right] = L\left[t^2\cos t\right]_{s\to s+1}$$

$$= \left[(-1)^2\frac{d^2}{ds^2}L\left[\cos t\right]\right]_{s\to s+1} = \left[\frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{\left(s^2+1\right)1 - s.2s}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{1-s^2}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$= \left[\frac{2s^3-6s}{\left(s^2+1\right)^3}\right]_{s\to s+1}$$

$$= \frac{2(s+1)^3 - 6(s+1)}{\left((s+1)^2+1\right)^3}$$

8. **Find $L\left[t^2 e^{t}\sin t\right]$**

Solution:

$$L\left[t^2e^{t}\sin t\right] = (-1)^2\frac{d^2}{ds^2}L\left[e^t\sin t\right] \quad \ldots (1)$$

Now $L\left[e^{t}\sin t\right]=\left[L[\sin t]\right]_{s\to(s-1)}=\dfrac{1}{(s-1)^{2}+1}$ ... (2)

Substituting (2) in (1) we get

$$L\left[t^{2}e^{t}\sin t\right]=\frac{d}{ds}\left[\frac{0-2(s-1)}{\left((s-1)^{2}+1\right)^{2}}\right]=\frac{d}{ds}\left[\frac{-2(s-1)}{\left(s^{2}-2s+2\right)^{2}}\right]$$

$$=\frac{\left(s^{2}-2s+2\right)^{2}(-2)+2(s-1)2\left(s^{2}-2s+2\right)(2s-2)}{\left(s^{2}-2s+2\right)^{4}}$$

$$=\frac{2\left(s^{2}-2s+2\right)\left[-\left(s^{2}-2s+2\right)+4(s-1)^{2}\right]}{\left(s^{2}-2s+2\right)^{4}}$$

$$=\frac{2\left(s^{2}-2s+2\right)\left[-s^{2}+2s-2+4s^{2}+4-8s\right]}{\left(s^{2}-2s+2\right)^{4}}$$

$$\therefore F(s)=\frac{2\left(s^{2}-2s+2\right)\left[3s^{2}-6s+2\right]}{\left(s^{2}-2s+2\right)^{4}}=\frac{2\left(3s^{2}-6s+2\right)}{\left(s^{2}-2s+2\right)^{3}}$$

9. **Find** $\boldsymbol{L\left[\dfrac{\sin^{2}t}{t}\right]}$

Solution:

$$L\left[\frac{\sin^{2}t}{t}\right]=L\left[\frac{1-\cos 2t}{2t}\right]=\frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right]=\frac{1}{2}\int_{s}^{\infty}L[1-\cos 2t]\;ds$$

$$=\frac{1}{2}\int_{s}^{\infty}\left\{L[1]-L[\cos 2t]\right\}ds=\frac{1}{2}\int_{s}^{\infty}\left[\frac{1}{s}-\frac{s}{s^{2}+4}\right]ds$$

$$=\frac{1}{2}\left[\log s-\frac{1}{2}\log\left(s^{2}+4\right)\right]_{s}^{\infty}=\frac{1}{2}\left[\log\frac{s}{\sqrt{s^{2}+4}}\right]_{s}^{\infty}$$

$$= \frac{1}{2}\left[\log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right] = \frac{1}{2}\left[0 - \log\frac{s}{\sqrt{s^2+4}}\right]$$

$$F(s) = \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1} = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

10. **Using Laplace transform, Evaluate** $\int_0^\infty t\,e^{-2t}\sin t\,dt$

Solution: $\int_0^\infty e^{-2t} f(t)dt = \left[\int_0^\infty e^{-st} f(t)dt\right]_{s=2} = [L[t\sin t]]_{s=2} = \left[-\frac{d}{ds}L[\sin t]\right]_{s=2}$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{-2s}{(s^2+1)^2}\right) = \frac{4}{25}$$

11. **Evaluate** $\int_0^t \sin u\cos(t-u)\,du$ **using Laplace Transform.**

Solution: Let $L\left[\int_0^t \sin u\cos(t-u)du\right] = L[\sin t * \cos t]$

$= \mathrm{L}[\sin t]\quad \mathrm{L}[\cos t]$ (by Convolution theorem)

$$= \frac{1}{(s^2+1)}\frac{s}{(s^2+1)} = \frac{s}{(s^2+1)^2}$$

$$\int_0^t \sin u\cos(t-u)du = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = \frac{t}{2}\sin t \quad \left(\because L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right] = t\sin at\right)$$

12. **Find the Laplace transform of** $\int_0^t t\,e^{-t}\sin t\,dt$

Solution:

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L\left[t\sin t\right] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{\left(s^2+1\right)0-2s}{\left(s^2+1\right)^2}\right) = \frac{2s}{\left(s^2+1\right)^2}$$

$$\therefore\ L\left[te^{-t}\ \sin t\right] = \left.\frac{2s}{\left(s^2+1\right)^2}\right|_{s\to s+1} = \frac{2(s+1)}{\left((s+1)^2+1\right)^2} = \frac{2(s+1)}{\left(s^2+2s+2\right)^2}$$

$$L\left[\int_0^t t\,e^{-t}\sin t\,dt\right] = \frac{1}{s}L\left[te^{-t}\ \sin t\right]$$

$$\therefore\quad = \frac{1}{s}\frac{2(s+1)}{\left(s^2+2s+2\right)^2}$$

13. **Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$**

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1} = \left[\frac{1}{s}L\left(t\cos t\right)\right]_{s\to(s+1)}$$

$$= \left[\frac{1}{s}\left(-\frac{d}{ds}L\left(\cos t\right)\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to(s+1)}$$

$$= \left[-\frac{1}{s}\left(\frac{s^2+1-2s^2}{\left(s^2+1\right)^2}\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)\right]_{s\to(s+1)}$$

$$\therefore F(s) = \left[\frac{s^2-1}{s\left(s^2+1\right)^2}\right]_{s\to(s+1)} = \left[\frac{(s+1)^2-1}{(s+1)\left[(s+1)^2+1\right]^2}\right] = \frac{s^2+2s}{(s+1)\left(s^2+2s+2\right)^2}$$

14. **Find the Laplace transform of** $e^{-4t}\int_0^t t\ \sin 3t dt$

Solution:

$$L\left[\sin 3t\right] = \frac{3}{s^2+9}$$

$$L\left[t\sin 3t\right]=-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)=-\left(\frac{\left(s^2+9\right)0-3(2s)}{\left(s^2+9\right)^2}\right)=\frac{6s}{\left(s^2+9\right)^2}$$

$$L\left(\int_0^t t\sin 3t\,dt\right)=\frac{L(t\sin 3t)}{s}=\frac{6}{(s^2+9)^2}$$

$$L\left(e^{-4t}\int_0^t t\sin 3t\,dt\right)=L\left(\int_0^t t\sin 3t\,dt\right)\Bigg|_{s\to s+4}=\frac{6}{\left(\left(s+4\right)^2+9\right)^2}=\frac{6}{\left(s^2+8s+16+9\right)^2}$$

$$\therefore\ L\left(e^{-4t}\int_0^t t\sin 3t\,dt\right)=\frac{6}{\left(s^2+8s+25\right)^2}$$

15. **Verify initial and final value theorems for the function f(t) = 1 + e^{-t} (sin t + cos t)**

Solution:

Initial value theorem states that $\lim\limits_{t\to 0} f\left(t\right)=\lim\limits_{s\to\infty} sF\left(s\right)$

$$L\left[f\left(t\right)\right]=F\left(s\right)$$

$$=\frac{1}{s}+L\left[\sin t+\cos t\right]_{s\to s+1}$$

$$=\frac{1}{s}+\frac{1}{\left(s+1\right)^2+1}+\frac{s+1}{\left(s+1\right)^2+1}=\frac{1}{s}+\frac{s+2}{\left(s+1\right)^2+1}$$

L.H.S. $=\lim\limits_{t\to 0} f\left(t\right)=1+1=2$

R.H.S $=\lim\limits_{s\to\infty} s\left[\frac{1}{s}+\frac{s+2}{\left(s+1\right)^2+1}\right]=\lim\limits_{s\to\infty}\left[1+\frac{s\left(s+2\right)}{\left(s+1\right)^2+1}\right]$

$$=\lim_{s\to\infty}\left[1+\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right]=\lim_{s\to\infty}\left[1+\frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right]=1+1=2$$

L.H.S=R.H.S

Initial value theorem verified.

Final value theorem states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\text{L.H.S.} = \lim_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right] = 1+0 = 1$$

$$\text{R.H.S} = \lim_{s\to 0}\left[1+\frac{s(s+2)}{(s+1)^2+1}\right] = 1+0 = 1$$

L.H.S.=R.H.S

Hence final value theorem verified

16. **Find the Laplace transform of the square wave function defined by** $\boldsymbol{f(t)=\begin{cases} E, & 0<t<\frac{a}{2} \\ -E, & \frac{a}{2}<t<a \end{cases} \ \& \ f(t+a)=f(t)}$

Solution:

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} Ee^{-st}\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right] = \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-\left(e^{-\frac{as}{2}}-1\right)+\left(e^{-as}-e^{-\frac{as}{2}}\right)\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-e^{-\frac{as}{2}}+1+e^{-as}-e^{-\frac{as}{2}}\right]$$

$$= \frac{E}{s\left(1-e^{-\frac{as}{2}}\right)\left(1+e^{-\frac{as}{2}}\right)}\left(1-e^{-\frac{as}{2}}\right)^2 = \frac{E}{s}\left(\frac{1-e^{-\frac{as}{2}}}{1+e^{-\frac{as}{2}}}\right)$$

$$\therefore F(s) = \frac{E}{s}\left[\frac{e^{sa/4} - e^{-sa/4}}{e^{sa/4} + e^{-sa/4}}\right] = \frac{E}{s}\tanh\left(\frac{sa}{4}\right)$$

17. **Find the Laplace transform of the rectangular wave given by** $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic in the interval $(0, 2b)$ with period $2b$.

$$L[f(t)] = \frac{1}{1 - e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1 - e^{-2bs}}\left[\int_0^{b} e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1 - e^{-2bs}}\left[\int_0^{b} e^{-st}\,dt + \int_b^{2b} e^{-st}(-1)\,dt\right] = \frac{1}{1 - e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{b} - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s\left(1 - e^{-2bs}\right)}\left[-\left(e^{-bs} - 1\right) + \left(e^{-2bs} - e^{-bs}\right)\right]$$

$$= \frac{1}{s\left(1 - e^{-2bs}\right)}\left[-e^{-bs} + 1 + \left(e^{-bs}\right)^2 - e^{-bs}\right]$$

$$= \frac{1}{s\left(1 - e^{-bs}\right)\left(1 + e^{-bs}\right)}\left(1 - e^{-bs}\right)^2 = \frac{1}{s}\left(\frac{1 - e^{-bs}}{1 + e^{-bs}}\right)$$

$$\therefore F(s) = \frac{1}{s}\left[\frac{e^{sb/2} - e^{-sb/2}}{e^{sb/2} + e^{-sb/2}}\right] = \frac{1}{s}\tanh\left(\frac{sb}{2}\right)$$

18. **Find the Laplace transform of f(t) =** $\begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a \end{cases}$ **and** $f(t+2a) = f(t)$ **for all** t

Solution:

$$L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\, t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[-t\left(\frac{e^{-st}}{s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[-(2a-t)\left(\frac{e^{-st}}{s}\right)+\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[\left(-a\,\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\frac{e^{-2as}}{s^2}-\left(-\frac{ae^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right] = \frac{\left(1-e^{-sa}\right)^2}{s^2\left(1-e^{-as}\right)\left(1+e^{-as}\right)}$$

$$\therefore\ F(s) = \frac{1-e^{-sa}}{s^2\left(1+e^{-as}\right)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

19. **Find the Laplace transform of the rectangular wave given by** $f(t)=\begin{cases}\sin\omega t, & 0<t<\dfrac{\pi}{\omega}\\[2mm] 0, & \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega}\end{cases}$

Solution:

This function is periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0,\frac{2\pi}{\omega}\right)$

$$L\left[f(t)\right] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\, dt + 0\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left[-s\sin\omega t - \omega\cos\omega t\right]\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)} = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

20. **Find $L^{-1}\left(\frac{s}{s^2+4s+5}\right)$**

Solution:

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right) = L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right) = e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$= e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= e^{-2t}\left[\cos t - 2\sin t\right]$$

21. **Find $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]$**

Solution: $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right] \because L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$\therefore L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = e^{-t}(\cos t + \sin t)$$

22. **Find $L^{-1}\left(\frac{s}{(s+2)^3}\right)$**

Solution: $L^{-1}\left(\frac{s}{(s+2)^3}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^3}\right)$

$$= L^{-1}\left(\frac{1}{(s+2)^2}\right) - 2\ L^{-1}\left(\frac{1}{(s+2)^3}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) - e^{-2t}L^{-1}\left(\frac{2}{s^3}\right)$$

$$= e^{-2t}\left(t - t^2\right).$$

23. **Find $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$**

Solution: Let F(s) = $\tan^{-1}\left(\frac{1}{s}\right)$

$$F'(s) = \frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right) = \frac{-1}{s^2+1}$$

By property $L^{-1}\left[F'(s)\right] = -L^{-1}\left[\frac{1}{s^2+1}\right] = -\sin t$

$$\therefore L^{-1}\left(F'(s)\right) = -\sin t\,;\ L^{-1}\left(F(s)\right) = \frac{-1}{t}L^{-1}\left[F'(s)\right]$$

$$\therefore L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = \frac{\sin t}{t}$$

24. **Find the inverse Laplace transform of $\frac{s}{(s+2)^2}$**

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right) = \frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= \frac{d}{dt}\left(e^{-2t}\,t\right) = e^{-2t} + t\left(-2e^{-2t}\right) = e^{-2t}\left(1-2t\right)$$

25. **Find $L^{-1}\left[\cot^{-1}(s+1)\right]$**

Let $L^{-1}\left[\cot^{-1}(s+1)\right] = f(t)$

$\therefore\ L\left[f(t)\right] = \cot^{-1}(s+1)$

$$L\left[tf(t)\right] = -\frac{d}{ds}\left[\cot^{-1}(s+1)\right] = \frac{1}{(s+1)^2+1}$$

$$tf(t) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{-t}\sin t$$

$$\therefore\ f(t) = \frac{e^{-t}\sin t}{t}$$

26. **Find the inverse Laplace transform of $\log\left(\frac{1+s}{s^2}\right)$**

Solution:

Let $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = f(t)$

$$\therefore\ L\left[f(t)\right] = \log\left(\frac{1+s}{s^2}\right)$$

$$L\left[t\,f(t)\right] = \frac{-d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right] = \frac{-d}{ds}\left[\log(1+s) - \log\left(s^2\right)\right] = -\frac{1}{1+s} + \frac{1}{s^2}2s$$

$$L\left[t\,f(t)\right] = \frac{2}{s} - \frac{1}{s+1}$$

$$t\,f(t) = L^{-1}\left[\frac{2}{s} - \frac{1}{s+1}\right] = 2L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] = 2(1) - e^{-t}$$

$$\therefore\ f(t) = \frac{2-e^{-t}}{t}$$

$$\therefore\ L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = \frac{2-e^{-t}}{t}$$

27. **Find** $\mathbf{L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

Put $s=-1 \Rightarrow \boxed{A=-\frac{1}{3}}$

Equating the coefficients of $s^3 \Rightarrow \boxed{B=\frac{1}{3}}$

Put $s=2 \Rightarrow \boxed{D=-7}$

Put $s=0 \Rightarrow \boxed{C=4}$

$$\therefore\ \frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{-1/3}{s+1} + \frac{1/3}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = -\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right] - 7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}\,t - \frac{7}{2}e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

$$\therefore\ f(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}\,t - \frac{7}{2}e^{2t}t^2$$

28. **Using Convolution theorem find** $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos at*\frac{1}{a}\sin at=\frac{1}{a}\left[\cos at*\sin at\right]$$

$$=\frac{1}{a}\int_0^t \cos au\ \sin a\left(t-u\right)\,du=\frac{1}{a}\int_0^t \sin\left(at-au\right)\cos au\ \ du$$

$$=\frac{1}{a}\int_0^t \frac{\sin\left(at-au+au\right)+\sin\left(at-au-au\right)}{2}\ du$$

$$=\frac{1}{2a}\int_0^t \left[\sin at+\sin a\left(t-2u\right)\right]\ du$$

$$=\frac{1}{2a}\left[\sin at\ u+\left(\frac{-\cos a\left(t-2u\right)}{-2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[u\ \sin at+\left(\frac{\cos a\left(t-2u\right)}{2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[t\ \sin at+\left(\frac{\cos at}{2a}\right)-\left(0+\frac{\cos at}{2a}\right)\right]$$

$$f\left(t\right)=\frac{1}{2a}\left[t\sin at+\frac{\cos at}{2a}-\frac{\cos at}{2a}\right]=\frac{1}{2a}t\sin at$$

29. **Find the inverse Laplace transform of** $\frac{s}{(s^2+a^2)(s^2+b^2)}$ **using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$\therefore\ L^{-1}\left[\frac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$=\frac{1}{b}\cos at*\sin bt$$

$$=\frac{1}{b}\int_0^t \cos au\ \ \sin b(t-u)\ du$$

$$=\frac{1}{2b}\int_0^t \left[\sin(au+bt-bu)-\sin(au-bt+bu)\right]\ du$$

$$=\frac{1}{2b}\int_0^t \left[\sin((a-b)u+bt)-\sin((a+b)u-bt)\right]\ du$$

$$=\frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{a-b}+\frac{\cos((a+b)u-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2b}\left[\left(\frac{-\cos(bt+at-bt)}{a-b}+\frac{\cos(at+bt-bt)}{a+b}\right)-\left(\frac{-\cos bt}{a-b}+\frac{\cos bt}{a+b}\right)\right]$$

$$=\frac{1}{2b}\left[\left(\frac{-\cos(at)}{a-b}+\frac{\cos(at)}{a+b}\right)-\left(\frac{-\cos bt}{a-b}+\frac{\cos bt}{a+b}\right)\right]$$

$$=\frac{1}{2b}\left(\frac{-2b\cos at}{a^2-b^2}+\frac{2b\cos bt}{a^2-\mathrm{b}^2}\right)$$

$$f(t)=\frac{\cos bt-\cos at}{a^2-b^2}$$

30. **Find the inverse Laplace transform of** $\boldsymbol{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}}$ **using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{s}{s^2+b^2}\right]=\cos at*\cos bt$$

$$=\int_0^t \cos au\ \cos b\left(t-u\right)\,du$$

$$=\frac{1}{2}\int_0^t\left[\cos\left(au+bt-bu\right)+\cos\left(au-bt+bu\right)\right]\,du$$

$$=\frac{1}{2}\int_0^t\left[\cos\left(\left(a-b\right)u+bt\right)+\cos\left(\left(a+b\right)u-bt\right)\right]\,du$$

$$=\frac{1}{2}\left[\frac{\sin\left(bt+\left(a-b\right)u\right)}{a-b}+\frac{\sin\left(\left(a+b\right)u-bt\right)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\left(\frac{\sin\left(bt+at-bt\right)}{a-b}+\frac{\sin\left(at+bt-bt\right)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\sin\left(at\right)}{a-b}+\frac{\sin\left(at\right)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left(\frac{2a\sin\left(at\right)}{a^2-b^2}-\frac{2b\sin\left(bt\right)}{a^2-\mathrm{b}^2}\right)$$

$$f\left(t\right)=\frac{a\sin\left(at\right)-b\sin\left(bt\right)}{a^2-b^2}$$

31. **Find the inverse Laplace transform of** $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)}$

Solution:

$$L^{-1}\left[\frac{s}{\left(s^2+1\right)\left(s^2+4\right)}\right]=L^{-1}\left[\frac{s}{s^2+1}\ \frac{1}{s^2+4}\right]=L^{-1}\left[\frac{s}{s^2+1}\right]*\frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=\frac{1}{2}\cos t*\sin 2t$$

$$= \frac{1}{2}\int_0^t \cos u \, \sin 2(t-u)\,du$$

$$= \frac{1}{4}\int_0^t \left[\sin(u+2t-2u) - \sin(u-2t+2u)\right]du \qquad \left(\because 2\cos A \sin B = \sin(A+B) - \sin(A-B)\right)$$

$$= \frac{1}{4}\int_0^t \left[\sin(2t-u) - \sin(u-2t)\right]du$$

$$= \frac{1}{4}\left[\frac{-\cos(2t-u)}{-1} + \frac{\cos(u-2t)}{1}\right]_0^t$$

$$= \frac{1}{4}\left[\cos t - \cos 2t + \cos t - \cos 2t\right]$$

$$= \frac{1}{4}\left[2\cos t - 2\cos 2t\right]$$

$$\therefore\ f(t) = \frac{1}{2}\left[\cos t - \cos 2t\right]$$

32. **Using Convolution theorem find the inverse Laplace transform of** $\boldsymbol{\dfrac{2}{(s+1)(s^2+4)}}$

Solution:

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{1}{s+1}\ \frac{2}{s^2+4}\right] = L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= e^{-t} * \sin 2t$$

$$= \int_0^t e^{-u}\ \sin 2(t-u)\,du$$

$$= \int_0^t e^{-u}\ \sin(2t-2u)\,du$$

$$= \int_0^t e^{-u}\ \left[\sin 2t\ \cos 2u - \cos 2t\ \sin 2u\right]du$$

$$= \int_0^t e^{-u}\ \sin 2t\ \cos 2u\ du - \int_0^t e^{-u}\ \cos 2t\ \sin 2u\ du$$

$$= \sin 2t \int_0^t e^{-u} \cos 2u \; du - \cos 2t \int_0^t e^{-u} \sin 2u \; du$$

$$= \sin 2t \left[\frac{e^{-u}}{1+4} \left(-\cos 2u + 2\sin 2u \right) \right]_0^t - \cos 2t \left[\frac{e^{-u}}{1+4} \left(-\sin 2u - 2\cos 2u \right) \right]_0^t$$

$$= \sin 2t \left[\left(\frac{e^{-t}}{5} \left(-\cos 2t + 2\sin 2t \right) \right) - \left(\frac{1}{5}(-1) \right) \right] - \cos 2t \left[\frac{e^{-t}}{5} \left(-\sin 2t - 2\cos 2t \right) - \left(\frac{1}{5}(-2) \right) \right]$$

$$= \sin 2t \left[\frac{e^{-t}}{5} \left(-\cos 2t + 2\sin 2t \right) + \frac{1}{5} \right] - \cos 2t \left[\frac{e^{-t}}{5} \left(-\sin 2t - 2\cos 2t + \frac{2}{5} \right) \right]$$

$$= \frac{e^{-t}}{5} \left[-\sin 2t \cos 2t + 2\sin^2 2t + \sin 2t \; \cos 2t + 2\cos^2 2t \right] + \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t$$

$$= \frac{e^{-t}}{5} \left[2(1) \right] + \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t$$

$$f(t) = \frac{1}{5} \left[2e^{-t} + \sin 2t - 2\cos 2t \right]$$

33. **Find the inverse Laplace transform of** $\dfrac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}$

Solution:

$$L^{-1}\left[F(s) G(s) \right] = L^{-1}\left[F(s) \right] * L^{-1}\left[G(s) \right]$$

$$\therefore \; L^{-1}\left[\frac{s^2}{\left(s^2+1^2\right)\left(s^2+2^2\right)} \right] = L^{-1}\left[\frac{s}{s^2+1^2} \right] * L^{-1}\left[\frac{s}{s^2+2^2} \right]$$

$$= \cos t * \cos 2t$$

$$= \int_0^t \cos u \; \cos 2(t-u) \; du$$

$$= \frac{1}{2} \int_0^t \left[\cos(u + 2t - 2u) + \cos(u - 2t + 2u) \right] \; du$$

$$= \frac{1}{2} \int_0^t \left[\cos(-u + 2t) + \cos(3u - 2t) \right] \; du$$

$$= \frac{1}{2}\left[\frac{\sin(2t-u)}{-1} + \frac{\sin(3u-2t)}{3}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin t}{-1} + \frac{\sin t}{3}\right) - \left(\frac{\sin 2t}{-1} - \frac{\sin 2t}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\sin t}{-3} - \frac{4\sin 2t}{-3}\right)$$

$$f(t) = \frac{\sin t - 2\sin 2t}{-3}$$

34. Find $L^{-1}\left(\frac{e^{-2s}}{\left(s^2+s+1\right)^2}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{\left(s^2+s+1\right)^2}\right) = L^{-1}\left(\frac{e^{-s}}{s^2+s+1}\,\frac{e^{-s}}{s^2+s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1}$$

$$= L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1} * L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1}$$

$$= e^{-t/2}L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1} * e^{-t/2}L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1}$$

$$= \left[e^{-t/2}\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} * e^{-t/2}\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}\right]_{t\to t-1}$$

$$= \frac{2}{\sqrt{3}} e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * \frac{2}{\sqrt{3}} e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right)$$

$$= \frac{4}{3}\left[e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right) * e^{-(t-1)/2} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right]$$

$$= \frac{4}{3}\int_0^t e^{-\frac{u-1}{2}} e^{-\frac{t-u-1}{2}} \sin\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}\right) \ \sin\left(\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}\right) du$$

$$= \frac{4}{3}\int_0^t e^{-\left(\frac{t}{2}-1\right)} \frac{1}{2} \ \cos\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}t\right) - \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right) du$$

$$= \frac{2}{3} e^{-\left(\frac{t-2}{2}\right)} \left[\frac{\sin\left(\frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}} - \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right) u \right]_0^t$$

$$= e^{-\left(\frac{t-2}{2}\right)} \left[\frac{4}{3\sqrt{3}} \sin\frac{\sqrt{3}}{2}t - \frac{2}{3} t \cos\left(\frac{\sqrt{3}}{2}t - \sqrt{3}\right)\right]$$

35. **Solve using Laplace transform $\boldsymbol{\frac{dy}{dt} + y = e^{-t}}$ given that y(0) = 0.**

Solution: Taking L.T. on both sides, we get $L\left[y'(t)\right] + L\left[y(t)\right] = L\left[e^{-t}\right]$

$$sL\left[y(t)\right] - y(0) + L\left[y(t)\right] = L\left[e^{-t}\right]$$

$$sL\left[y(t)\right] - 0 + L\left[y(t)\right] = \frac{1}{s+1}$$

$$(s+1)\ L[y(t)] = \frac{1}{s+1}$$

$$L[y(t)] = \frac{1}{(s+1)^2}$$

$$\therefore y(t) = L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t} L\left(\frac{1}{s^2}\right) = e^{-t}\, t \qquad \left(\because\ L\left[e^{-at} f(t)\right] = F(s+a)\right)$$

36. **Using Laplace transform to solve the differential equation $y'' + y' = t^2 + 2t$, given $y = 4,\ y' = -2$ when $t = 0$**

Solution:

Given $y'' + y' = t^2 + 2t$

$$L\left[y'' + y'\right] = L\left[t^2 + 2t\right]$$

$$\left[s^2 L\left[y(t)\right] - sy(0) - y'(0)\right] + \left[sL\left[y(t)\right] - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L\left[y(t)\right]\left(s^2 + s\right) = \frac{2}{s^3} + \frac{2}{s^2} + 4s - 2 + 4$$

$$L\left[y(t)\right]s(s+1) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2$$

$$L\left[y(t)\right] = \frac{2 + 2s + 4s^4 + 2s^3}{s^4(s+1)}$$

$$L\left[y(t)\right] = \frac{2}{s} + \frac{2}{s^4} + \frac{2}{s+1}$$

$$y(t) = L^{-1}\left[\frac{2}{s} + \frac{2}{s^4} + \frac{2}{s+1}\right]$$

$$= 2 + 2\frac{t^3}{6} + 2e^{-t}$$

$$y(t) = 2 + \frac{t^3}{3} + 2e^{-t}$$

37. **Solve $\left(D^2 + 3D + 2\right)y = e^{-3t}$, given $y(0) = 1$, and $y'(0) = -1$ using Laplace Transforms**

Solution:

Given $y'' + 3y' + 2y = e^{-3t}$

Taking Laplace transforms on both side

$$L\left(y'' + 3y' + 2y\right) = L\left(e^{-3t}\right)$$

$$L\left[y''(t)\right] + 3L\left[y'(t)\right] + 2L\left[y(t)\right] = \frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y^{'}(0)\right]+3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-s(1)-(-1)\right]+3\left[sL\left[y(t)\right]-1\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$L\left[y(t)\right]\left[s^2+3s+2\right]=\frac{1}{s+3}+s+2$$

$$L\left[y(t)\right]=\frac{s^2+5s+7}{(s+3)\left(s^2+3s+2\right)},\ y(t)=L^{-1}\left[\frac{s^2+5s+7}{(s+1)(s+2)(s+3)}\right]$$

$$y(t)=L^{-1}\left[\frac{3/2}{s+1}-\frac{1}{s+2}+\frac{1/2}{s+3}\right]$$

$$y(t)=\frac{3}{2}L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s+2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t)=\frac{3}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t}$$

38. **Solve $y^{''}+2y^{'}-3y=\sin t$, given $y(0)=0,\ y^{'}(0)=0$**

Solution:

Given $y^{"}+2y^{'}-3y=\sin t$

$$L\left[y^{"}(t)+2y^{'}(t)-3y(t)\right]=L[\sin t]$$

$$L\left[y^{"}(t)\right]+2L\left[y^{'}(t)\right]-3L\left[y(t)\right]=L[\sin t]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y^{'}(0)\right]+2\left[sL\left[y(t)\right]-y(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$\left[s^2L\left[y(t)\right]-s(0)-0\right]+2\left[sL\left[y(t)\right]-(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$s^2L\left[y(t)\right]+2sL\left[y(t)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]\left(s^2+2s-3\right)=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]=\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}$$

$$y(t)=L^{-1}\left[\frac{1}{(s^2+1)(s^2+2s-3)}\right]==L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$$

Now

$$\frac{1}{(s-1)(s+3)(s^2+1)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{(s^2+1)}$$

$$1=A(s+3)(s^2+1)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)$$

Put $s=1\Rightarrow \boxed{A=\frac{1}{8}}$

Put $s=-3\Rightarrow \boxed{B=\frac{-1}{40}}$

Equating coeff. of $s^3\Rightarrow \boxed{C=\frac{-1}{10}}$

Equating the constant terms $\Rightarrow \boxed{D=\frac{-1}{5}}$

$$\therefore\ \frac{1}{(s-1)(s+3)(s^2+1)}=\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{(s^2+1)}$$

$$L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]=L^{-1}\left[\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{(s^2+1)}\right]$$

$$=\frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right]-\frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right]-\frac{1}{10}L^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\left[L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{2}{s^2+1}\right]\right]$$

$$=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}[\cos t+2\sin t]$$

39. **Solve the equation y″+ 9y = cos2t with y(0) =1, y $\left(\frac{\pi}{2}\right)$ = –1**

Solution:

Given $\left(D^2+9\right)y=\cos 2t$

Taking Laplace transforms on both sides

$$L\left[y^{''}(t)\right]+9L\left[y(t)\right]=L\left[\cos 2t\right]$$

$$s^2L\left[y(t)\right]-sy(0)-y^{'}(0)+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

Using the initial conditions

$y(0)=1,$ and taking $y^{'}(0)=k$

We have

$$s^2L\left[y(t)\right]-(s)(1)-k+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

$$\Rightarrow L\left[y(t)\right]=\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s+k}{s^2+9}$$

$$=\frac{s}{5\left(s^2+4\right)}-\frac{s}{5\left(s^2+9\right)}+\frac{s}{s^2+9}+\frac{k}{s^2+9}$$

$$\therefore\ y(t)=\frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{1}{5}L^{-1}\left[\frac{s}{s^2+9}\right]+L^{-1}\left[\frac{s}{s^2+9}\right]+k\,L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{k}{3}\sin 3t$$

Put $t=\frac{\pi}{2}$ we get $y\left(\frac{\pi}{2}\right)=\frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{k}{3}(-1)=-\frac{1}{5}-\frac{k}{3}$

But given $y\left(\frac{\pi}{2}\right)=-1$

$$\therefore\ -1=-\frac{1}{5}-\frac{k}{3}$$

$$\Rightarrow k=\frac{12}{5}$$

$$\therefore\ y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{4}{5}\sin 3t$$

$$y(t)=\frac{4}{5}\left[\cos 3t+\sin 3t\right]+\frac{1}{5}\cos 2t$$

40. **Solve $x'' + 2x' + 5x = e^{-t}\sin t$, where $x(0) = 0,\ x'(0) = 1$ using Laplace Transforms**

Solution:

Given $x'' + 2x' + 5x = e^{-t}\sin t$

Taking Laplace transforms on both side

$$L\left[x'' + 2x' + 5x\right] = L\left[e^{-t}\sin t\right]$$

$$L\left[x''(t)\right] + 2L\left[x'(t)\right] + 5L\left[x(t)\right] = \frac{1}{s^2 + 2s + 2}$$

$$\left[s^2 L\left[x(t)\right] - sx(0) - x'(0)\right] + 2\left[sL\left[x(t)\right] - x(0)\right] + 5L\left[x(t)\right] = \frac{1}{s^2 + 2s + 2}$$

$$\left[s^2 L\left[x(t)\right] - s(0) - 1\right] + 2\left[sL\left[x(t)\right] - (0)\right] + 5L\left[x(t)\right] = \frac{1}{s^2 + 2s + 2}$$

$$L\left[x(t)\right]\left[s^2 + 2s + 5\right] = \frac{1}{s^2 + 2s + 2} + 1$$

$$L\left[x(t)\right]\left[s^2 + 2s + 5\right] = \frac{s^2 + 2s + 3}{s^2 + 2s + 2}$$

$$L\left[x(t)\right] = \frac{s^2 + 2s + 3}{\left(s^2 + 2s + 2\right)\left(s^2 + 2s + 5\right)} = \frac{(s+1)^2 + 2}{\left((s+1)^2 + 1\right)\left((s+1)^2 + 4\right)}$$

$$x(t) = L^{-1}\left[\frac{(s+1)^2 + 2}{\left((s+1)^2 + 1\right)\left((s+1)^2 + 4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{s^2 + 2}{\left(s^2 + 1\right)\left(s^2 + 4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{1/3}{s^2 + 1} + \frac{2/3}{s^2 + 4}\right]$$

$$= e^{-t}\left[\frac{1}{3}\sin t + \frac{1}{3}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}\left[\sin t + \sin 2t\right]$$

41. **Using Laplace transform to solve the differential equation**

$y'' - 3y' + 2y = 4t + e^{3t}$, where $y(0) = 1,\ y'(0) = -1$

Solution:

Given $y'' - 3y' + 2y = 4t + 3e^{t}$

$$L\left[y'' - 3y' + 2y\right] = L\left[4t + 3e^{t}\right]$$

$$L\left[y''(t)\right] - 3L\left[y'(t)\right] + 2L\left[y(t)\right] = 4L[t] + 3L\left[e^{3t}\right]$$

$$\left[s^2 L\left[y(t)\right] - sy(0) - y'(0)\right] - 3\left[sL\left[y(t)\right] - y(0)\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2 L\left[y(t)\right] - s(1) - (-1)\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2 L\left[y(t)\right] - s + 1\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2 - 3s + 2\right) = s - 4 + \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2 - 3s + 2\right) = \frac{(s-4)s^2(s-3) + 4(s-4) + 3s^2}{s^2(s-3)}$$

$$L\left[y(t)\right] = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{\left(s^2 - 3s + 2\right)s^2(s-3)}$$

$$L\left[y(t)\right] = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-2)(s-1)s^2(s-3)}$$

$$y(t) = L^{-1}\left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-2)(s-1)s^2(s-3)}\right]$$

$$= L^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{s-2}+\frac{E}{s-3}\right]$$

$$= L^{-1}\left[\frac{3}{s}+\frac{2}{s^2}+\frac{-1/2}{s-1}+\frac{-2}{s-2}+\frac{1/2}{s-3}\right]$$

$$y(t) = 3+2t-\frac{1}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}$$

42. **Solve $y''-3y'+2y=e^{2t}$, $y(0)=-3,\ y'(0)=5$**

Solution:

Given $y''-3y'+2y=e^{2t}$

$$L\left[y''-3y'+2y\right]=L\left[e^{2t}\right]$$

$$L\left[y''\right]-3L\left[y'\right]+2L[y]=L\left[e^{2t}\right]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]-3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$\left[s^2L\left[y(t)\right]-s(-3)-5\right]-3\left[sL\left[y(t)\right]-(-3)\right]+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$s^2L\left[y(t)\right]+3s-5-3sL\left[y(t)\right]-9+2L\left[y(t)\right]=\frac{1}{s-2}$$

$$L\left[y(t)\right]\left[s^2-3s+2\right]+3s-14=\frac{1}{s-2}$$

$$\therefore\quad L\left[y(t)\right]\left[s^2-3s+2\right]=\frac{1}{s-2}-3s+14$$

$$\therefore\quad L\left[y(t)\right]=\frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}$$

$$y(t)=L^{-1}\left[\frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}\right]$$

$$y(t)=L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]$$

$$\frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$-3s^2+20s-27=A(s-2)^2+B(s-1)(s-2)+C(s-1)$$

Put $s=1 \Rightarrow \boxed{A=-10}$

Put $s=2 \Rightarrow \boxed{C=1}$

Equating the coeff. of $s^2 \Rightarrow \boxed{B=7}$

$$\therefore \frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{-10}{s-1}+\frac{7}{s-2}+\frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]=L^{-1}\left[\frac{-10}{s-1}\right]+L^{-1}\left[\frac{7}{s-2}\right]+L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$=-10e^t+7e^{2t}+e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=-10e^t+7e^{2t}+te^{2t}$$

43. **Solve $\frac{dx}{dt}-2x+3y=0;\ \frac{dy}{dt}-y+2x=0$ with $x(0)=8,\ y(0)=3$**

The given differential equation canbe written as

$$x'(t)-2x+3y=0 \quad y'(t)-y+2x=0$$

Taking Laplace transforms weget,

$$L\left[x'(t)-2x+3y\right]=L[0]$$

$$sL\left[x(t)\right]-x(0)-2L\left[x(t)\right]+3L\left[y(t)\right]=0$$

$$sL\left[x(t)\right]-8-2L\left[x(t)\right]+3L\left[y(t)\right]=0$$

$$L\left[x(t)\right](s-2)+3L\left[y(t)\right]=8 \qquad (1)$$

And $L\left[y'(t)-y+2x\right]=L[0]$

$$sL\left[y(t)\right]-y(0)-L\left[y(t)\right]+2L\left[x(t)\right]=0$$

$$sL\left[y(t)\right]-3-L\left[y(t)\right]+2L\left[x(t)\right]=0$$

$$2L\left[x(t)\right]+(s-1)L\left[y(t)\right]=3 \qquad (2)$$

Solving (1) and (2) we get,

$$L\left[x(t)\right]=\frac{8s-17}{(s+1)(s-4)}=\frac{5}{s+1}+\frac{3}{s-4},$$

$$\therefore\ x(t)=L^{-1}\left[\frac{5}{s+1}+\frac{3}{s-4}\right],$$

$$\boxed{x(t)=5e^{-t}+3e^{4t}}$$

and $$L\left[y(t)\right]=\frac{3s-22}{(s+1)(s-4)}=\frac{5}{s+1}-\frac{2}{s-4}$$

$$y(t)=L^{-1}\left[\frac{5}{s+1}-\frac{2}{s-4}\right]=5e^{-t}-2e^{4t}$$

44. **Determine y which satisfies the equation** $\boldsymbol{\frac{dy}{dt}+2y+\int_0^t y\,dt=2\cos t,\ \ y(0)=1}$

Solution:

Given $y'(t)+2y(t)+\int_0^t y(t)\,dt=2\cos t,\ \ y(0)=1$

$$L\left[y'(t)\right]+2L\left[y(t)\right]+L\left[\int_0^t y(t)\,dt\right]=L\left[2\cos t\right]$$

$$sL\left[y(t)\right]-y(0)+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$sL\left[y(t)\right]-1+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$L\left[y(t)\right]=\frac{s}{s^2+1}$$

$$y(t)=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$