SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 1 – Multiple Integrals

Part – B (Each question carries 3 Marks)

1. Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy$.

Solution

$$\int_2^3 \int_1^2 \frac{1}{xy}\, dx\, dy = \left[\int_2^3 \frac{1}{y} dy\right]\left[\int_1^2 \frac{1}{x} dx\right] = [\log y]_2^3 [\log x]_1^2$$

$$= (\log 3 - \log 2)(\log 2 - \log 1) = \left(\log \frac{3}{2}\right)(\log 2)$$

2. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\, dr\, d\theta$.

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{(\sin\theta)^2}{2}\right] d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta \quad = \frac{1}{2} * \frac{1}{2} * \frac{\pi}{2} = \frac{\pi}{8}$$

3. Evaluate $\int_0^2 \int_0^2 dx\, dy$.

Solution

$$\int_0^2 \int_0^2 dx\, dy = \int_0^2 [x]_0^2 dy = \int_0^2 [2-0] dy = [2\,y]_0^2 = (2)(2) - 0 = 4$$

4. Evaluate $\int_0^3 \int_0^2 (x^2 + y^2)dx\, dy$.

Solution

$$I = \int_0^3 \int_0^2 (x^2+y^2)dxdy = \int_0^3 \left[\left(\frac{x^3}{3}\right) + xy^2\right]_0^2 dy = \int_0^2 \left[\frac{8}{3} + 2y^2\right] dy$$

$$= \left[\frac{8\mathrm{y}}{3} + \frac{2\mathrm{y}^3}{3}\right]_0^3 = \frac{8*3}{3} + \frac{2*3^3}{3} = 8 + 18 = 26$$

5. Evaluate $\int_0^a \int_0^b \int_0^c dx\, dy\, dz$.

Solution

$$\int_0^a \int_0^b \int_0^c dx\, dy\, dz = \int_0^a \int_0^b (x)_0^c\, dy\, dz = \int_0^a \int_0^b (c-0)dy\, dz$$

$$= c\int_0^a (y)_0^b\, \mathrm{dz} = c\int_0^a (b-0)dz = \ b\ c\int_0^a dz = bc\ (z)_0^a$$

$$= \mathrm{bc(a-0) = abc}$$

6. Evaluate $\int_{\mathbf{0}}^{\boldsymbol{\pi}} \int_0^a r\, dr\, d\theta$.

Solution

$$\int_0^\pi \int_0^a r\, dr\, d\theta = \int_0^\pi \left(\frac{r^2}{2}\right)_0^a d\theta = \int_0^\pi \left[\frac{(a)^2}{2} - 0\right] d\theta = \frac{a^2}{2}\int_0^\pi d\theta = \frac{a^2}{2}(\theta)_0^\pi$$

$$= \frac{a^2}{2}(\pi - 0) = \frac{\pi a^2}{2}$$

7. Evaluate $\int_0^2 \int_0^2 e^{x+y}\, dx\, dy$.

Solution

$$\int_0^2 \int_0^2 e^{x+y} dxdy \ = \int_0^2 e^x dx \int_0^2 e^y dy = [e^x]_0^2 [e^y]_0^2$$

$$= (e^2 - e^0)(e^2 - e^0) = (e^2 - 1)^2$$

8. Evaluate $\int_1^2 \int_0^{2-y} xy\, dx\, dy$.

Solution

$$\int_1^2 \int_0^{2-y} xy\, dx\, dy = \int_1^2 \left(y\left(\frac{x^2}{2}\right)\right)_0^{2-y} dy = \left(\frac{1}{2}\right) \int_1^2 (y(2-y)^2 dy$$

$$= \frac{1}{2}\int_1^2 y(4 + y^2 - 4y)dy = \frac{1}{2}\int_1^2 (4y + y^3 - 4y^2)dy$$

$$= \frac{1}{2}\left[4\left(\frac{y^2}{2}\right) + \left(\frac{y^4}{4}\right) - 4\left(\frac{y^3}{3}\right)\right]_1^2$$

$$= \frac{1}{2}\left\{\left[4\left(\frac{4}{2}\right) + \left(\frac{16}{4}\right) - 4\left(\frac{8}{3}\right)\right] - \left[4\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) - 4\left(\frac{1}{3}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{5}{12}\right\} = \frac{5}{24}$$

9. Evaluate $\int_0^1 \int_y^1 \frac{x}{x^2+y^2} dx\, dy$.

Solution

$$\int_0^1 \int_y^1 \frac{x}{x^2 + y^2} dxdy = \int_0^1 \int_0^x \frac{x}{x^2 + y^2} dydx = \int_0^1 \left(tan^{-1}\left(\frac{y}{x}\right)_{y=0}^{y=x}\right) dx$$

$$= \int_0^1 (tan^{-1}(1) - tan^{-1}(0))\ dx$$

$$= \int_0^1 \left(\frac{\pi}{4} - 0\right) dx = \frac{\pi}{4}\int_0^1 dx = \frac{\pi}{4}(x)_0^1 = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$$

10. Evaluate $\int_0^3 \int_0^2 x\,y(x+y)\,dy\,dx$.

Solution

$$\int_0^3 \int_0^2 x\,y(x+y)\,dy\,dx = \int_0^3 \int_0^2 \left(x^2\,y + x\,y^2\right) dy\,dx$$

$$= \int_0^3 \left(\frac{x^2\,y^2}{2} + x\frac{y^3}{3}\right)_0^2 dx$$

$$= \int_0^3 \left(2\,x^2 + \frac{8}{3}x\right) dx$$

$$= \left(2\frac{x^3}{3} + \frac{8}{3}\frac{x^2}{2}\right)_0^3 = 30$$

11. Evaluate $\int_0^1 \int_0^1 (x+y)\,dx\,dy$.

Solution

$$\int_0^1 \int_0^1 (x+y)\,dx\,dy = \int_0^1 \left[\left(\frac{x^2}{2} + xy\right)\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} + y\right) dy$$

$$= \left(\frac{y}{2} + \frac{y^2}{2}\right)_0^1$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) - (0+0)$$

$$= 1$$

12. Find the value of $\int_0^{\pi} \int_0^1 (x^2 \sin y)\,dx\,dy$.

Solution

$$\int_0^{\pi} \int_0^1 (x^2 \sin y)\,dx\,dy = \int_0^1 x^2\,dx \int_0^{\pi} \sin y\,dy$$

$$= \left(\frac{x^3}{3}\right)_0^1 (-\cos y)_0^{\pi}$$

$$= \left(\frac{1}{3} - 0\right)(-\cos\pi + \cos 0)$$

$$= \left(\frac{1}{3} - 0\right)(1 + 1)$$

$$= \frac{2}{3}$$

13. Evaluate $\int_0^c \int_0^b \int_0^a (x+y+z)\,dx\,dy\,dz$.

Solution

$$\int_0^c \int_0^b \int_0^a (x+y+z)\,dx\,dy\,dz = \int_0^c \int_0^b \left(\frac{x^2}{2} + xy + xz\right)_0^a dy\,dz$$

$$= \int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az\right) dy\,dz$$

$$= \int_0^c \left(\frac{a^2}{2} b + a\frac{b^2}{2} + azb\right) dz$$

$$= \int_0^c \left(\frac{a^2}{2} y + a\frac{y^2}{2} + azy\right)_0^b dz$$

$$= \left(\frac{a^2}{2} bz + a\frac{b^2}{2} z + ab\frac{z^2}{2}\right)_0^c$$

$$= \frac{abc(a+b+c)}{2}$$

14. Evaluate $\int_0^4 \int_0^x \int_0^{\sqrt{x+y}} z\;dx\,dy\,dz$.

Solution

$$I = \int_{x=0}^{4} \int_{y=0}^{x} \int_{z=0}^{\sqrt{x+y}} z\;dz\,dy\,dx$$

$$= \int_0^4 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dydx$$

$$= \frac{1}{2}\int_0^4\int_0^x (x+y)\,dy\,dx$$

$$= \frac{1}{2}\int_0^4\left(xy+\frac{y^2}{2}\right)_0^x dx \quad = \frac{1}{2}\int_0^4\left(x^2+\frac{x^2}{2}\right)dx \quad = \frac{3}{4}\int_0^4 x^2\,dx = \frac{3}{4}\left(\frac{x^3}{3}\right)_0^4 = 16$$

15. Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$.

Solution

$$I = \int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+y^2}}\tan^{-1}\left(\frac{x}{\sqrt{1+y^2}}\right)\right)_0^{\sqrt{1+y^2}} dy$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+y^2}}\left(\tan^{-1}(1)-\tan^{-1}(0)\right)\right)dy$$

$$= \int_0^1 \frac{\pi}{4}\frac{dy}{\sqrt{1+y^2}} = \frac{\pi}{4}\log(1+\sqrt{2})$$

16. Evaluate $\int_0^a \int_0^{\sqrt{ay}} x\,y\;dx\,dy$.

Solution

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\;dy\,dx = \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^a y\,a\,y\;dy = \frac{a^4}{6}$$

17. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$.

Solution

Let $\displaystyle I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}} \right) \right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

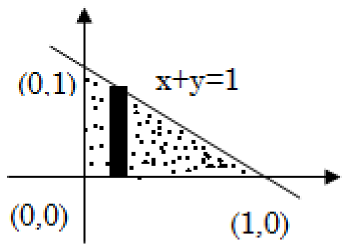
$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{\pi}{2} - 0\right] dy\,dx = \frac{\pi}{2}\int_0^a [y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a = \frac{\pi}{2}\left[\left(0+\frac{a^2}{2}\frac{\pi}{2}\right) - (0+0)\right] = \frac{\pi^2 a^2}{8}$$

18. Evaluate $\displaystyle\iint_R (\mathbf{x}^2 + \mathbf{y}^2)\,\mathbf{dy\,dx}$ over the region R for which $x, y \geq 0,\ x + y \leq 1$.

Solution

The region of integration is the triangle bounded by the lines $x = 0,\ y = 0,\ x + y = 1$



Limits of y : 0 to $1 - x$; Limits of x : 0 to 1

$$\iint_R (x^2+y^2)\,dy\,dx = \int_0^1 \int_0^{1-x} \left(x^2+y^2\right) dy\,dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1$$

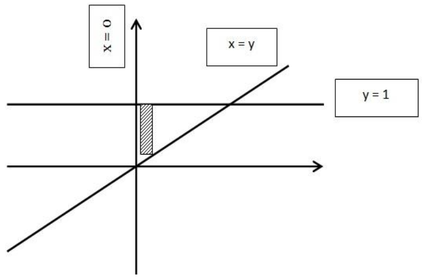
$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1$$

$$=\frac{1}{3}-\frac{1}{4}+\frac{1}{12}$$
$$=\frac{1}{6}$$

19. Find the area bounded by the lines $x=0,\ y=1$ and $y=x$ using double integration.

Solution



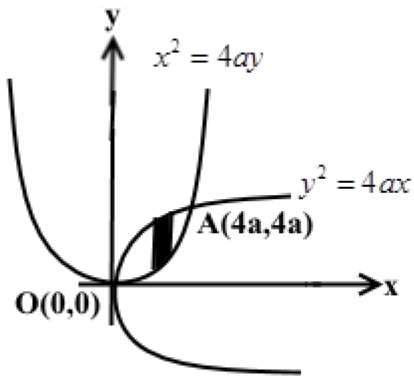
Given $x=0,\ y=1$ *and* $y=x$.

Hence x *varies from* 0 *to* 1 *and* y *varies from* x *to* 1.

$$I=\int_0^1\int_x^1 dydx=\int_0^1 [y]_x^1 dx=\int_0^1 (1-x)dx=\left[x-\frac{x^2}{2}\right]_0^1=1-\frac{1}{2}=\frac{1}{2}$$

20. **Find by double integration the area between the parabolas $y^2=4ax$ and $x^2=4ay$.**

Solution



$$\therefore \textbf{Area}=\int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dydx=\int_0^{4a}[y]_{\frac{x^2}{4a}}^{\sqrt{4ax}}dx=\int_0^{4a}\left[\sqrt{4ax}-\frac{x^2}{4a}\right]dx$$

$$=\int_0^{4a}\left[2\sqrt{a}\,x^{1/2}-\frac{1}{4a}x^2\right]dx=\left[2\sqrt{a}\,\frac{x^{3/2}}{3/2}-\frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$=\frac{4\sqrt{a}}{3}(4a)^{3/2}-\frac{1}{12a}(4a)^3$$

$$=\frac{4\sqrt{a}}{3}(4)^{3/2}(a)^{3/2}-\frac{1}{12a}64a^3=\frac{4^{5/2}}{3}a^{4/2}-\frac{1}{12a}64a^3$$

$$=\frac{(2^2)^{5/2}}{3}a^2-\frac{16}{3}a^2=\frac{32}{3}a^2-\frac{16}{3}a^2$$

$$=\frac{16}{3}a^2$$

21. **Find the area of the circle $x^2 + y^2 = a^2$ using double integration.**

Solution

Area of circle = 4 × Area in first quadrant

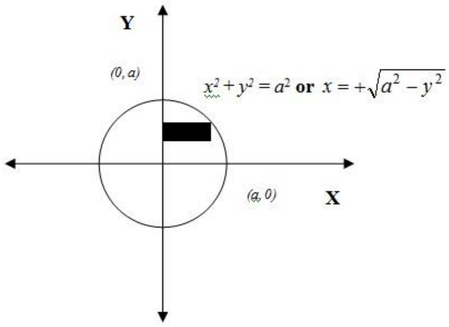
$$= 4\int_0^a \int_0^{\sqrt{a^2-y^2}} dx\,dy$$

$$= 4\int_0^a (x)_0^{\sqrt{a^2-y^2}}\, dy$$

$$= 4\int_0^a \sqrt{a^2-y^2}\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a^2}{2}\frac{\pi}{2}\right] = \pi a^2$$



22. Find the area of the circle $x^2 + y^2 = a^2$ using polar coordinates.

Solution

$x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$

$r^2 = a^2$

$$\text{Area} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{a^2}{2}\,d\theta$$

$$= \pi a^2$$

23. Find the area of the cardioid $\mathbf{r = a(1+\cos\theta)}$ by using double integration.

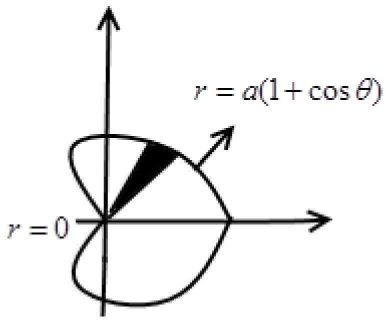
Solution

Given the curve in polar co ordinates $\mathbf{r = a(1+\cos\theta)}$

$\therefore$ **Area of the cardioid = 2(Area above the initial line)**

θ varies from 0 to π

r varies from 0 to $\mathbf{r = a(1+\cos\theta)}$



$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\left(\frac{1+\cos\theta}{2}\right)\right]d\theta \qquad = a^2\int_0^{\pi}\left[\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right]d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\pi} \qquad \because \sin n\pi = 0,\ \forall n$$

$$= a^2\left[\frac{3}{2}\pi\right] \qquad = \frac{3\pi a^2}{2}$$

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