SRM Institute of Science and Technology
Ramapuram Campus
Department of Mathematics
18MAB101T - Calculus And Linear Algebra

Year/Sem: I/I

Branch: Common to ALL B.Tech. except B.Tech. (Business Systems)

Unit – V

SEQUENCE AND SERIES

Part – C

Question 1

Show that the series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots$ **is convergent.**

Solution

$u_n=\frac{1}{n(n+1)(n+2)}=\frac{1}{n\,n\left(1+\frac{1}{n}\right)n\left(1+\frac{2}{n}\right)}$, and let $v_n=\frac{1}{n^3}$.

Now $\frac{u_n}{v_n}=\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$\lim\limits_{n\to\infty}\frac{u_n}{v_n}=1$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

But $\sum v_n=\sum\frac{1}{n^3}$ is convergent by Harmonic series test or p - series test.

Hence by comparison test, $\Sigma\, u_n$ is convergent.

Question 2:

Show that the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^3}+\cdots$ **is divergent.**

Solution

Neglect the first term. Then the series is $\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^3}+\cdots$

$u_n=\frac{n^n}{(n+1)^{n+1}}=\frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^1}$ and let $v_n=\frac{1}{n}$.

Now $\frac{u_n}{v_n}=\frac{1}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$

$\lim_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{e}$, which is finite and non-zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

But $\sum v_n=\sum\frac{1}{n}$ is divergent by Harmonic series test or p-series test.

Hence by comparison test, $\Sigma\, u_n$ is divergent.

Question 3:

Show that the series $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ is divergent.

Solution

$u_n=\sin\left(\frac{1}{n}\right)$

Let $v_n=\frac{1}{n}$

Now $\frac{u_n}{v_n}=\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{\frac{1}{n}\to 0}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=1$, which is finite and non-zero. **Formula :** $\lim_{x\to 0}\frac{\sin x}{x}=1$

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \frac{1}{n}$ is divergent by Harmonic series test or p-series test.

Hence by comparison test, $\Sigma\, u_n$ is divergent.

Question 4:

Show that the series $1+\frac{2^p}{2\,!}+\frac{3^p}{3\,!}+\frac{4^p}{4\,!}+\cdots$ **is convergent for all values of *p*.**

Solution

$$u_n = \frac{n^p}{n\,!}, \quad u_{n+1} = \frac{(n+1)^p}{(n+1)\,!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \times \frac{(n+1)!}{(n+1)^p} = \frac{n+1}{\left(1+\frac{1}{n}\right)^p} = \frac{n\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^p}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty > 1$$

Hence by Ratio test, $\Sigma\, u_n$ is convergent.

Question 5:

Establish the convergence of the series $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots$

Solution

$$u_n = \left(\frac{n}{2n+1}\right)^n$$

$$\lim_{n\to\infty} \left(u_n\right)^{1/n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$$

Hence by Cauchy's root test, $\Sigma\, u_n$ is convergent.

Question 6:

Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

Solution

$f(n) = \frac{1}{n \log n}$ Clearly $f(n)$ is a monotonic decreasing sequence.

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x \log x} dx = \int_2^{\infty} \frac{1/x}{\log x} dx = \left(\log(\log(x)\right)_2^{\infty} = \infty$$

By Cauchy's integral test, the given series is divergent.

Question 7:

Test the convergence of the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$.

Solution

$u_n = \frac{1}{\sqrt{n}}$ $\qquad$ $u_{n+1} = \frac{1}{\sqrt{n+1}}$

(i) Clearly $u_n > u_{n+1}$.

(ii) $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

$\therefore$ By Leibnitz's test, the given series is convergent.

Question 8:

Test the convergence of the series $\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \cdots$.

Solution

$u_n = \frac{1}{(2n-1)(2n)}$ $\qquad$ $u_{n+1} = \frac{1}{(2n+1)(2n+2)}$

(i) Since $\frac{1}{(2n-1)(2n)} > \frac{1}{(2n+1)(2n+2)}$ always, clearly $u_n > u_{n+1}$.

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\left(2-\frac{1}{n}\right)2n} = 0$

$\therefore$ By Leibnitz's test, the given series is convergent.

Question 9:

Prove that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ **is conditionally convergent.**

Solution

$u_n = \frac{1}{n}$ $\qquad u_{n+1} = \frac{1}{n+1}$

(i) Clearly $u_n > u_{n+1}$.

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By Leibnitz's test, the given series is *convergent*.

Also $\Sigma\,|\,u_n\,| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum\frac{1}{n}$ is *divergent* by Harmonic series test (or) p-series test.

Hence the given series is conditionally convergent.

Question 10:

Test the convergence of the series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

Solution

The series of absolute terms $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \sum\frac{1}{n^2}$ is clearly *convergent* by Harmonic series test (or) p-series test.

$\therefore$ The series is absolutely convergent.

Since every absolutely convergent series is convergent, the given series is convergent.

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