UNIT 3 LAPLACE TRANSFORMS

4.1 INTRODUCTION

Laplace transform is an integral transform named after its inventor Pierre-Simon Laplace. It transforms a function of a real variable t (often time) to a function of a complex variable s (complex frequency). The Laplace transform is an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

4.1.1 DEFINITION

Let a function f(t) be continuous and defined for positive values of 't'. The Laplace Transformation of f(t) is the function F(s), which is a unilateral transform defined by equation (1) where 's' is a complex number frequency parameter

$$L\left[f(t)\right] = F(s) = \int_0^\infty e^{-st} f(t)\,dt, t > 0 \qquad \dots (1)$$

Here F(s) ia said to be the Laplace Transform of f(t) and it is written as $L\left[f(t)\right]$ *or* $L[f]$. Thus $F(s) = L\left[f(t)\right]$

4.1.2 SUFFICIENT CONDITION FOR EXISTENCE OF LAPLACE TRANSFORM

If f(t) is piecewise continuous function in $[0,\infty)$ and is of exponential order then the Laplace transform F(s) of f(t) exists for all $s > a$, where 'a' is a real number that depends on f(t).

(OR)

F(t) should be continuous or piecewise continuous in the given closed interval [a, b] where a > 0 and f(t) should be of exponential order.

4.1.3 EXPONENTIAL ORDER

A function f(t) is said to be of exponential order 'a' as $t \to \infty$ if $\lim_{t\to\infty} e^{-st} f(t) = \text{a finite quantity.}$

Note: Functions for which Laplace transform does not exist

(i) $L[e^{t^3}]$ does not exist since e^{t^3} is not of any exponential order.

(ii) $L[\cot t]$ does not exist since $\cot t$ is not piecewise continuous.

(iii) $L[\tan t]$ does not exist since $\tan t$ is not piecewise continuous.

(iv) example of a function which has a Laplace Transform but it is not continuous, since it is discontinuous at π.

$$f(t) = \begin{cases} \cos t \ , & 0 < t < \pi \\ \sin t & , t > \pi \end{cases}$$

4.1.4 APPLICATIONS OF LAPLACE IN REAL LIFE

A) LAPLACE TECHNIQUE IN CIRCUITS

Laplace transform methods can be employed to study circuits in the *s*-domain. Laplace techniques convert circuits with voltage and current signals that change with time to the *s* domain, so you can analyse the circuit's action using only algebraic techniques.

Laplace transformations of circuit elements are similar to phasor representations, but they are not the same. Laplace transformations are more general than phasors, and can be easier to use in some instances.

B) LAPLACE TRANSFORM IS USED IN CONTROL THEORY

We are using Laplace transform in control theory to transform convolution of two functions into product of two functions.

$$L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

And also The Final Value Theorem in Laplace transform allows us to determine the value of the time domain equation, as the time approaches infinity, from the S domain equation. In Control Theory, The Final Value Theorem is used most frequently to determine the steady-state value of a system. The real part of the poles of the function must be <0

C) LAPLACE TRANSFORM IS USED IN ANALYSIS OF NETWORK

An electrical network is an Linear time invariant system if all the elements are linear time in variant. A network consists of elements like resistors, capacitors and inductors. A network can be modeled using fundamental laws. The model of the network can be obtained by writing integral differential equations. Then the network can be analyzed by solving these equations using Laplace transform.

4.2 TRANSFORMS OF ELEMENTARY FUNCTIONS

Basic Results in Laplace Transforms:

1. $L[1] = \frac{1}{s}$
2. $L[t^n] = \frac{n!}{s^{n+1}}$
3. $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$, n is not a integer
4. $L[e^{at}] = \frac{1}{s-a}$
5. $L[e^{-at}] = \frac{1}{s+a}$
6. $L[\sin at] = \frac{a}{s^2+a^2}$
7. $L[\cos at] = \frac{s}{s^2+a^2}$
8. $L[\sinh at] = \frac{a}{s^2-a^2}$
9. $L[\cosh at] = \frac{s}{s^2-a^2}$

WORKED EXAMPLE – 4.2 (A)

Example 1

Evaluate $L\left[e^{-at}\right]$

Solution:

$$L\left[f(t)\right]=F(s)=\int_0^\infty e^{-st}f(t)dt, t>0$$

$$L\left[e^{-at}\right]=\int_0^\infty e^{-st}e^{-at}dt$$

$$=\int_0^\infty e^{-(s+a)t}dt$$

$$=\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$=-\frac{1}{(s+a)}\left[e^{-\infty}-e^{0}\right]$$

$$=-\frac{1}{(s+a)}[0-1]$$

$$=\frac{1}{(s+a)}, where(s+a)>0$$

Example 2

Evaluate $L[\sin at]$

Solution:

$$L\left[f(t)\right]=F(s)=\int_0^\infty e^{-st}f(t)dt, t>0$$

$$L[\sin at]=\int_0^\infty e^{-st}\sin at dt \qquad \because \int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$$

$$=\left[\frac{e^{-st}}{s^2+a^2}\left(-s\sin at-a\cos at\right)\right]_0^\infty$$

$$=0-\left[\frac{1}{s^2+a^2}\left(-s\sin(0)-a\cos(0)\right)\right]$$

$$=\frac{a}{s^2+a^2}$$

Example 3

Evaluate $L\left[t^n\right]$

Solution: $L\left[f(t)\right]=F(s)=\int_0^\infty e^{-st}f(t)dt, t>0$

$$L\left[t^n\right]=\int_0^\infty e^{-st}t^n dt$$

$$=\int_0^\infty t^n d\left[\frac{e^{-st}}{-s}\right]$$

$$=t^n\left[\frac{e^{-st}}{-s}\right]_0^\infty-\int_0^\infty\left[\frac{e^{-st}}{-s}\right]nt^{n-1}dt$$

$$=0-\left[-\frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt\right]$$

$$=\frac{n}{s}L\left[t^{n-1}\right]$$

Similarly,

$$L\left[t^{n-1}\right]=\frac{n-1}{s}L\left[t^{n-2}\right]$$

$$L\left[t^{n-2}\right]=\frac{n-2}{s}L\left[t^{n-3}\right]$$

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$$L\left[t^{n-(n-1)}\right]=\frac{n-(n-1)}{s}L\left[t^{n-(n-1)-1}\right]$$

$$=\frac{1}{s}L\left[t^0\right]$$

$$=\frac{1}{s}L[1]$$

$$=\frac{1}{s^2}$$

$$\therefore L\left[t^n\right]=\frac{n}{s}\frac{n-1}{s}\cdots\frac{2}{s}\frac{1}{s}L[1]=\frac{n!}{s^n}\frac{1}{s}$$

$$L\left[t^n\right]=\frac{n!}{s^{n+1}}$$

5Example 4

Evaluate $L[\cosh at]$

Solution: we know that $\cosh \mathrm{at}=\dfrac{\mathrm{e}^{\mathrm{at}}+\mathrm{e}^{-\mathrm{at}}}{2}$

$$\mathrm{L}[\cosh \mathrm{at}]=\mathrm{L}\left[\frac{\mathrm{e}^{\mathrm{at}}+\mathrm{e}^{-\mathrm{at}}}{2}\right]$$

$$=\frac{1}{2}\mathrm{L}\left[\mathrm{e}^{\mathrm{at}}+\mathrm{e}^{-\mathrm{at}}\right]$$

$$=\frac{1}{2}\left(\mathrm{L}\left[\mathrm{e}^{\mathrm{at}}\right]+\mathrm{L}\left[\mathrm{e}^{-\mathrm{at}}\right]\right)$$

$$=\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) \quad \because L[e^{at}]=\frac{1}{s-a},\; L[e^{-at}]=\frac{1}{s+a}$$

$$=\frac{1}{2}\left(\frac{s+a+s-a}{s^2-a^2}\right)$$

$$=\frac{1}{2}\left(\frac{2s}{s^2-a^2}\right)$$

$$L[\cosh at]=\frac{s}{s^2-a^2},\, s>|a|$$

Example 5

Find the unilateral Laplace transform of the signal $x(t)=\sin\Omega_0 t$

Solution: W.K.T $L[f(t)]=\int_0^\infty e^{-st}f(t)dt$

Here $f(t)=x(t)=\sin\Omega_0 t$

W.K.T $L[\sin at]=\frac{a}{s^2+a^2}$

here $a=\Omega_0$

$$\boxed{L[\sin\Omega_0 t]=\frac{\Omega_0}{s^2+\Omega_0{}^2}}$$

Example 6

Find the Laplace transform of $\mathbf{L}\left[\frac{\mathbf{1}}{\sqrt{\mathbf{t}}}\right]$

Solution: $L\left[\frac{1}{\sqrt{t}}\right]=L\left[t^{-\frac{1}{2}}\right]$ $\qquad w.k.t\; L\left[t^n\right]=\frac{\Gamma n+1}{s^{n+1}}(if\; n>-1)$

$$L\left[t^{-\frac{1}{2}}\right]=\frac{\Gamma-\frac{1}{2}+1}{s^{-\frac{1}{2}+1}}\left(here\, n=-\frac{1}{2}\right)$$

$$=\frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}}=\frac{\sqrt{\pi}}{\sqrt{s}} \quad \left(\because \Gamma\frac{1}{2}=\sqrt{\pi}\right)$$

$$L\left[\frac{1}{\sqrt{t}}\right]=\frac{\sqrt{\pi}}{\sqrt{s}}$$

EXERCISE: 4.2

1. **Find** $\mathbf{L}\left[\mathbf{e}^{\mathbf{3t+5}}\right]$ **Ans:** $\frac{e^5}{s-3}$
2. **Find** $\mathbf{L}\left[\mathbf{cos}\left(\mathbf{at+b}\right)\right]$ **Ans:** $\frac{s\cos b-a\sin b}{s^2+a^2}$

3. **Find** $L\left[2e^{-3t}\right]$ **Ans:** $\frac{2}{s+3}$

4. **Find** $L\left[\sin 5t\cos 2t\right]$ **Ans:** $\frac{1}{2}\left[\frac{7}{s^2+7^2}+\frac{3}{s^2+3^2}\right]$

5. **Find** $L\left[\cosh 2t-\cos 2t\right]$ **Ans:** $\left[\frac{s}{s^2-2^2}-\frac{s}{s^2+2^2}\right]$

4.3 PROPERTIES

I) LINEAR PROPERTY:

If c_1 **and** c_2 **are constant and f and g are functions of t then** $L\left\{c_1 f(t)+c_2 g(t)\right\}=c_1 L\left\{f(t)\right\}+c_2 L\left\{g(t)\right\}$**.**

Proof: We know that, $L[f(t)]=\int_0^\infty e^{-st}f(t)dt$

$$L[c_1 f(t)\pm c_2 g(t)]=\int_0^\infty e^{-st}\left[c_1 f(t)\pm c_2 g(t)\right]dt$$

$$=\int_0^\infty e^{-st}c_1 f(t)dt\pm\int_0^\infty e^{-st}c_2 g(t)dt$$

$$=c_1\int_0^\infty e^{-st}f(t)dt\pm c_2\int_0^\infty e^{-st}g(t)dt$$

$$\boxed{L[c_1 f(t)\pm c_2 g(t)]=c_1 L\left[f(t)\right]\pm c_2 L\left[g(t)\right]}$$

(II) SHIFTING PROPERTY

First Shifting Theorem:

(i) $L[f(t)]=F(s), \text{then } L[e^{at}f(t)]=F(s-a)$

(ii) $L[f(t)]=F(s), \text{then } L[e^{-at}f(t)]=F(s+a)$

Proof:

We know that, $L[f(t)]=F(s)=\int_0^\infty e^{-st}f(t)dt$

$$L\left[e^{at}f(t)\right]=\int_0^\infty e^{-st}e^{at}f(t)dt$$

$$=\int_0^\infty e^{-(s-a)t}f(t)dt \qquad \because \int_0^\infty e^{-st}f(t)dt=F(s)$$

$$L\left[e^{at}f(t)\right]=F(s-a)$$

$$L\left[e^{-at}f(t)\right]=\int_0^\infty e^{-st}e^{-at}f(t)dt$$

$$=\int_0^\infty e^{-(s+a)t}f(t)dt \qquad \because \int_0^\infty e^{-st}f(t)dt=F(s)$$

$$\boxed{L\left[e^{-at}f(t)\right]=F(s+a)}$$

Second Shifting Theorem:

If $L[f(t)] = F(s)$ and $G(t) = \begin{cases} f(t-a), t > a \\ 0, \quad\quad t < a \end{cases}$ then $L[G(t)] = e^{-as}F(s)$

Proof:

$L[G(t)]=\int_0^\infty e^{-st}G(t)dt$

$=\int_0^a e^{-st}0dt + \int_a^\infty e^{-st}f(t-a)dt \;\; =\int_0^\infty e^{-st}f(t-a)dt$

Put $t-a=u$

$dt = du$

$t \to a \Rightarrow u \to 0$

$t \to \infty \Rightarrow u \to \infty$

$=\int_0^\infty e^{-s(u+a)}f(u)du$

$= e^{-sa}\int_0^\infty e^{-su}f(u)du \qquad \because$ If $L[f(t)] = F(s)$ then $L[e^{at}f(t)] = F(s-a)$

$= e^{-sa}\int_0^\infty e^{-st}f(t)dt \qquad \because$ Replace u by t since u is dummy variable

$= e^{-sa}L[f(t)] \qquad \because L[f(t)]=\int_0^\infty e^{-st}f(t)dt$

$\boxed{L[G(t)] = e^{-as}F(s)}$

(III) CHANGE OF SCALE PROPERTY

If L[f(t)]=F(s) then prove that $\mathbf{L}\left[\mathbf{f}\left(a\mathbf{t}\right)\right] = \frac{1}{a}\mathbf{F}\left(\frac{\mathbf{s}}{a}\right)$

Proof: By definition we have

$L\left\{f\left(at\right)\right\} = \int_0^\infty e^{-st}f(at)dt$

Put $at = x \Rightarrow t = \frac{x}{a}$ so that $dt = \frac{dx}{a}$

$=\int_0^\infty e^{s(x/a)}f(x)\frac{dx}{a}$

$=\frac{1}{a}\int_0^\infty e^{(s/a)x}f(x)dx$

$=\frac{1}{a}\int_0^\infty e^{(s/a)t}f(t)dt \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad \because F(s) = \int_0^\infty e^{-st}f(t)dt$

(IV) DERIVATIVE PROPERTY

If $L\left[f(t)\right] = F(s)$ then $L[tf(t)] = -\frac{d}{ds}F(s) = -F'(s)$

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Differentiating (1) w.r.t. 's', we get

$$\frac{d}{ds}\left[F(s)\right] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)dt$$

$$= \int_0^\infty (-te^{-st}) f(t)dt$$

$$= \int_0^\infty e^{-st}\left[-t.f(t)\right]dt$$

$$= L\left[-t\, f(t)\right]$$

$$\boxed{\therefore L\left[t\, f(t)\right] = -\frac{d}{ds}\left[F(s)\right]}$$

Calculating succesive derivativies we derive that $\mathbf{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)}$

UNIT STEP FUNCTION

The Unit step function is defined as $U(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t > a \end{cases}$.

The Laplace transform of Unit Step function is $\frac{e^{-as}}{s}$

$$L\left[u_a(t)\right] = \int_0^\infty e^{-st} u_a(t)dt$$

$$= \int_0^a e^{-st} u_a(t)dt + \int_a^\infty e^{-st} u_a(t)dt$$

$$= \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}(1)dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{-1}{s}\left[e^{-\infty} - e^{-as}\right] \quad \left(\because e^{-\infty} = 0\right)$$

$$= \frac{-1}{s}\left[0 - e^{-as}\right] = \frac{e^{-as}}{s}$$

$$\boxed{L\left[u_a(t)\right] = \frac{e^{-as}}{s}}$$

UNIT IMPULSE FUNCTION OR DIRAC DELTA FUNCTION

Unit Impulse Function exists only at t = 0 at which it is infinitely great.

$\lim\limits_{h\to 0}\{f(t)\}$, where $f(t)$ is defined by

$$f(t)=\begin{cases}\dfrac{1}{h}, \text{ when } a-\dfrac{h}{2}\le t\le \dfrac{h}{2}\\ 0, \qquad \text{otherwise}\end{cases}$$

is called Unit Impulse Function or Dirac Delta Function and is denoted by $\delta_a(t)$ or $\delta(t-a)$.

LAPLACE TRANSFORM OF UNIT IMPULSE FUNCTION

$$L\{\delta_a(t)\}=L\left[\lim_{h\to 0}\{f(t)\}\right], \text{ where f(t) is taken as given in the definition}$$

$$=\lim_{h\to 0}L\{f(t)\}$$

$$=\lim_{h\to 0}\int_0^{\infty}e^{-st}f(t)dt$$

$$=\lim_{h\to 0}\int_{a-\frac{h}{2}}^{a+\frac{h}{2}}e^{-st}\frac{1}{h}dt$$

$$=\lim_{h\to 0}\left[\frac{1}{h}\left(\frac{e^{-st}}{-s}\right)_{a-\frac{h}{2}}^{a+\frac{h}{2}}\right]$$

$$=\lim_{h\to 0}\left[\frac{e^{-s\left(a-\frac{h}{2}\right)}-e^{-s\left(a+\frac{h}{2}\right)}}{sh}\right]$$

$$=e^{-as}\lim_{h\to 0}\left[\frac{e^{\frac{sh}{2}}-e^{-\frac{sh}{2}}}{sh}\right]$$

$$=e^{-as}\lim_{h\to 0}\left[\frac{2\sinh\left(\frac{sh}{2}\right)}{sh}\right]$$

$$=e^{-as}\lim_{h\to 0}\left[\frac{s\cosh\left(\frac{sh}{2}\right)}{s}\right], \text{by L' Hospital's rule}$$

$$=e^{-as}\lim_{h\to 0}\left[\cosh\left(\frac{sh}{2}\right)\right]$$

$$=e^{-as}$$

WORKED EXAMPLE 4.3 (A)

Example 1

Find L[f(t)] if $f(t)=\begin{cases} e^{-t}, 0<t<4 \\ 0, t>4 \end{cases}$

Solution:

WKT $L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt$

$$= \int_0^4 e^{-st} e^{-t} dt + \int_4^{\infty} e^{-st} 0dt = \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^4$$

$$= \frac{-1}{(s+1)}\left[e^{-(s+1)t}\right]_0^4 = \frac{-1}{(s+1)}[e^{-4(s+1)} - 1] = \frac{1}{s+1}[1-e^{-4(s+1)}]$$

Example 2

Find the Laplace transform of f(t) if $f(t)=\begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

Solution: we know that $L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt = \int_0^{\pi} e^{-st} \sin t dt$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^{\pi}$$

$$= \left[\frac{e^{-s\pi}}{s^2+1}(-s\sin\pi - \cos\pi)\right] + \frac{1}{s^2+1}$$

$$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} \qquad [\because \sin\pi = 0; \cos\pi = -1]$$

$$= \frac{1+e^{-s\pi}}{s^2+1}$$

Example 3

If $f(t)=\begin{cases} 3, 0<t<2 \\ -1, 2<t<4 \\ 0, t\geq 4 \end{cases}$ **, find L[f(t)].**

Solution: we know that $L[F(t)] = \int_0^{\infty} e^{-st} f(t)dt$

$$= \int_0^2 e^{-st}\ 3\ dt + \int_2^4 e^{-st}\ (-1)\ dt + \int_4^{\infty} e^{-st}\ 0\ dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_0^2 - \left[\frac{e^{-st}}{-s}\right]_2^4$$

$$= \frac{1}{-s}\left[3(e^{-2s} - e^{0}) - (e^{-4s} - e^{-2s})\right]$$

$$= \frac{1}{-s}\left[3(e^{-2s} - e^{0}) - (e^{-4s} - e^{-2s})\right]$$

Example 4

Find the Laplace transform of $\mathbf{f(t) = \begin{cases} 0, & t < 2\pi/3 \\ \cos\left(t - \frac{2\pi}{3}\right), & t > 2\pi/3 \end{cases}}$

Solution:

By second shifting theorem.

If L[f(t)]=F(s) and $G(t) = \begin{cases} f(t-a), t > a \\ 0, t < a \end{cases}$

Then $L[G(t)] = e^{-as} L[f(t)]$

Take $G(t) = \begin{cases} 0, & t < 2\pi/3 \\ \cos\left(t - \frac{2\pi}{3}\right), & t > 2\pi/3 \end{cases}$

$$here \quad a = \frac{2\pi}{3} \quad and \quad f(t) = \cos t$$

$$hence \ L[G(t)] = e^{\frac{-2\pi}{3}s} L[\cos t]$$

$$= e^{\frac{-2\pi s}{3}} \frac{s}{s^2+1} \qquad \because L[\cos at] = \frac{s}{s^2+a^2}$$

$$\boxed{L[G(t)] = \frac{se^{\frac{-2\pi s}{3}}}{s^2+1}}$$

Example 5

Is the linearity property applicable to $\mathbf{L\left[\frac{1-\cos t}{t}\right]}$**?**

Solution: Given : $L\left[\frac{1-\cos t}{t}\right]$

$= L\left[\frac{1}{t}\right] - L\left[\frac{\cos t}{t}\right]$ by linearity property.

$L\left[\frac{1}{t}\right]$ does not exist. Since $\lim_{t\to\infty} \frac{1}{t} = \frac{1}{0} = \infty$

$\therefore$ Linearity property is not applicable to $L\left[\frac{1-\cos t}{t}\right]$

Example 6

Find $\mathbf{L\left[a+bt+\frac{c}{\sqrt{t}}\right]}$**.**

Solution:

$$L\left(a+bt+\frac{c}{\sqrt{t}}\right)=L(a)+L(bt)+L\left(\frac{c}{\sqrt{t}}\right) \qquad \because \text{Linearity property}$$

$$=aL(1)+bL(t)+cL\left(t^{-1/2}\right) \qquad \because L[1]=\frac{1}{s},\ L\left[t^{n}\right]=\frac{n!}{s^{n+1}}$$

$$=a\left(\frac{1}{s}\right)+b\left(\frac{1}{s^{2}}\right)+c\left(\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\right) \ \because L(t^{n})=\frac{\Gamma(n+1)}{s^{n+1}},\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\boxed{L\left(a+bt+\frac{c}{\sqrt{t}}\right)=\frac{a}{s}+\frac{b}{s^{2}}+c\sqrt{\frac{\pi}{s}}}$$

Example 7

Find $\mathbf{L(\cos 4t \sin 2t)}$

Solution:

$$\text{we know that } \cos A\sin B=\frac{1}{2}[\sin(A+B)-\sin(A-B)]$$

$$\cos 4t\sin 2t=\frac{1}{2}[\sin(4t+2t)-\sin(4t-2t)]$$

$$=\frac{1}{2}[\sin(6t)-\sin(2t)]$$

$$=\frac{1}{2}[\sin(6t)-\sin(2t)]$$

$$L\left(\cos 4t\sin 2t\right)=L\left[\frac{1}{2}[\sin(6t)-\sin(2t)]\right]$$

Example 8

Find the Laplace transform of the signal $x(t)=e^{-3t}u(t)+e^{-2t}u(t)$

Solution: We know that

$$u(t)=\begin{cases}0, & t<0\\ 1, t>a, a\geq 0\end{cases}$$

$$L[x(t)]=\int_{t=0}^{\infty} x(t)e^{-st}dt$$

$$=\int_{t=0}^{\infty} e^{-3t}u(t)e^{-st}dt+\int_{t=0}^{\infty} e^{-2t}u(t)e^{-st}dt$$

$$=\int_{t=0}^{\infty} e^{-(s+3)t}dt+\int_{t=0}^{\infty} e^{-(s+2)t}dt$$

$$=\left[\frac{-1}{s+3}e^{-(s+3)t}\right]_{0}^{\infty}+\left[\frac{-1}{s+2}e^{-(s+2)t}\right]_{0}^{\infty}=\frac{1}{s+3}+\frac{1}{s+2}=\frac{(s+2)+(s+3)}{(s+3)(s+3)}$$

$$\boxed{L[x(t)] = \frac{2s+5}{s^2+5s+6}}$$

Example 9

Find $L[e^{-3t}\sin t \cos t]$.

Solution:

$$L[e^{-3t}\sin t\cos t] = \frac{1}{2}L[e^{-3t}(2)\sin t\cos t]$$

$$= \frac{1}{2}L\left[e^{-3t}\sin 2t\right] \qquad \because \sin 2t = 2\sin t\cos t$$

$$= \frac{1}{2}L\left[\sin 2t\right]_{s\to s+3} \qquad \because L[e^{-at}f(t)] = F(s+a)$$

$$= \frac{1}{2}\left[\frac{2}{s^2+2^2}\right]_{s\to s+3} \qquad \because L[\sin at] = \frac{a}{s^2+a^2}$$

$$= \frac{1}{2}\left[\frac{2}{(s+3)^2+2^2}\right]$$

$$\boxed{L[e^{-3t}\sin t\cos t] = \left[\frac{1}{(s+3)^2+2^2}\right]}$$

Example 10

If $L\left[f(t)\right] = \frac{s}{s^2-4}$, find L[f(2t)].

Solution: If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right), a>0$

Therefore $L[f(2t)] = \frac{1}{2}\frac{\frac{s}{2}}{\left(\left(\frac{s}{2}\right)^2-4\right)}$

And hence $L[f(2t)] = \frac{s}{\left(s^2-16\right)}$

Example 11

Find the Laplace transform of $\left[\frac{t}{e^t}\right]$

Solution:

$$L\left[\frac{t}{e^t}\right] = L\left[te^{-t}\right] = -\frac{d}{ds}\left[L\left(e^{-t}\right)\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s+1}\right] \qquad \because L[t^n f(t)] = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$= \frac{1}{(s+1)^2}$$

$$\boxed{L\left[\frac{t}{e^t}\right]=\frac{\mathbf{1}}{(s+\mathbf{1})^{\mathbf{2}}}}$$

Example 12

Find $L\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)$

Solution:

$$L\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)=L[\sqrt{t}]-L\left[\frac{1}{\sqrt{t}}\right]$$

$$=L\left[t^{1/2}\right]-L\left[t^{-1/2}\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}-\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}\qquad\left[\because L[t^n]=\frac{n!}{s^{n+1}}\right]$$

$$=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}-\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\qquad\left[\because \Gamma(n+1)=n\Gamma(n);\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

$$=\frac{\frac{\sqrt{\pi}}{2}}{s^{3/2}}-\frac{\sqrt{\pi}}{s^{1/2}}$$

$$=\sqrt{\pi}\left[\frac{1}{2s^{3/2}}-\frac{1}{s^{1/2}}\right]$$

Example 13

Find the Laplace transform of f(t)= tcosht.

Solution: $L[f(t)]=L[t\cosh t]$.

$$=-\frac{d}{ds}L[\cosh t]\qquad \because L[t^n f(t)]=(-1)^n\frac{d^n}{ds^n}F(s)$$

$$=-\frac{d}{ds}\left[\frac{s}{s^2-1}\right]\qquad \because L[\cosh t]=\frac{s}{s^2-1}$$

$$=-\left[\frac{(s^2-1)-s(2s)}{(s^2-1)^2}\right]$$

$$=-\left[\frac{s^2-1-2s^2)}{(s^2-1)^2}\right]$$

$$=-\left[\frac{-1-s^2}{(s^2-1)^2}\right]$$

$$\boxed{L[f(t)]=\left[\frac{1+s^2}{\left(s^2-1\right)^2}\right]}$$

Example 14

Find the Laplace transform of $[t\cos t\sinh 2t]$**.**

Solution: Given $[t\cos t\sinh 2t] \;=L\left[t\cos t\left(\frac{e^{2t}-e^{-2t}}{2}\right)\right]$

$$=\frac{1}{2}L[te^{2t}\cos t]-\frac{1}{2}L[te^{-2t}\cos t] \qquad ....(1)$$

$$L[te^{2t}\cos t]=L[t\cos t]_{s\to(s-2)} \qquad \because L\left[e^{at}f(t)\right]=L\left[f(t)\right]_{s\to(s-a)}$$

$$=\left[-\frac{d}{ds}L(\cos t)\right]_{s\to(s-2)} \qquad \left[\because L[tf(t)]=-\frac{d}{ds}L[f(t)]\right.$$

$$=\left[-\frac{d}{ds}\left[\frac{s}{s^2+1}\right]\right]_{s\to(s-2)}$$

$$=\left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to(s-2)}$$

$$=\frac{(s-2)^2-1}{[(s-2)^2+1]^2}=\frac{s^2-4s+4-1}{[s^2-4s+4+1]^2}$$

$$=\frac{s^2-4s+3}{(s^2-4s+3)^2}$$

$$Similarly, L[te^{-2t}\cos t]=L[t\cos t]_{s\to(s+2)} \qquad \because L\left[e^{-at}f(t)\right]=L\left[f(t)\right]_{s\to(s+a)}$$

$$=\left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to(s+2)}$$

$$=\frac{(s+2)^2-1}{[(s+2)^2+1]^2}=\frac{s^2+4s+4-1}{[s^2+4s+4+1]^2}$$

$$=\frac{s^2+4s+3}{(s^2+4s+3)^2}$$

$$(1)\Rightarrow \quad [t\cos t\sinh 2t]=\frac{1}{2}\left[\frac{s^2-4s+3}{(s^2-4s+3)^2}\right]-\frac{1}{2}\left[\frac{s^2+4s+3}{(s^2+4s+3)^2}\right]$$

Example 15

Find the Laplace transform of $\mathbf{t^2e^{-3t}\sin 2t}$.

Solution:

$$L[t^n\ f(t)]=(-1)^n\frac{d^n}{ds^n}F(s)=(-1)^n F^{(n)}(s)$$

$$L[t^2e^{-3t}\sin 2t] = (-1)^2\frac{d^2}{ds^2}L[e^{-3t}\sin 2t]$$

$$= \frac{d^2}{ds^2}L[e^{-3t}\sin 2t]$$

$$= \frac{d^2}{ds^2}L[\sin 2t]_{s\to(s+3)}$$

$$= \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]_{s\to(s+3)} \qquad \because L[\sin at] = \frac{a}{s^2+a^2}$$

$$= \frac{d^2}{ds^2}\left[\frac{2}{(s+3)^2+4}\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{2}{s^2+6s+9+4}\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{2}{s^2+6s+13}\right]$$

$$= \frac{d}{ds}\left[-\frac{2(2s+6)}{(s^2+6s+13)^2}\right]$$

$$= -4\frac{d}{ds}\left[\frac{s+3}{(s^2+6s+13)^2}\right]$$

$$= 4\left[\frac{(s+3)[2(s^2+6s+13)(2s+6)]-(s^2+6s+13)^2}{(s^2+6s+13)^4}\right]$$

$$= 4\left[\frac{(s+3)[2(2s+6)]-(s^2+6s+13)}{(s^2+6s+13)^3}\right]$$

$$= 4\left[\frac{4(s+3)^2-(s^2+6s+13)}{(s^2+6s+13)^3}\right]$$

$$= 4\left[\frac{4[s^2+6s+9]-(s^2+6s+13)}{(s^2+6s+13)^3}\right]$$

$$= 4\left[\frac{4s^2+24s+36-s^2-6s-13}{(s^2+6s+13)^3}\right]$$

$$= 4\left[\frac{3s^2+18s+23}{(s^2+6s+13)^3}\right]$$

$$\boxed{L[t^2e^{-3t}\sin 2t] = \frac{12s^2+72s+92}{(s^2+6s+13)^3}}$$

Example 16

Evaluate $\int_0^\infty te^{-2t}\sin 3t\,dt$ using Laplace transform.

Solution: $\int_0^\infty e^{-2t}t\sin 3t\,dt = L\left[t\sin 3t\right]_{s=2} \to (1)$ $\qquad \left[\because \int_0^\infty e^{-st}f(t)dt = L[f(t)]\right]$

$$\therefore L\left[t\sin 3t\right]=(-1)\frac{d}{ds}\left[\frac{3}{s^2+9}\right] \qquad \left[\because L\left[tf(t)\right]=(-1)\frac{d}{ds}\left[L[f(t)]\right]\right]$$

$$=\frac{6s}{(s^2+9)^2}\rightarrow(2)$$

Using (1) in (2), we have

$$\int_0^\infty e^{-2t}t\sin 3t\,dt=\left[\frac{6s}{(s^2+9)^2}\right]_{s=2}=\frac{12}{169}$$

Example 17

Evaluate $L(t^2e^{-t}\cos t)$

Solution:

$$\text{(i)}L(t^2e^{-t}\cos t)=(-1)^2\frac{d^2}{ds^2}L\left[e^{-t}\cos t\right]$$

$$=\frac{d^2}{ds^2}\left[L\left[\cos t\right]_{s\to(s+1)}\right]$$

$$=\frac{d^2}{ds^2}\left[\left(\frac{s}{s^2+1}\right)_{s\to(s+1)}\right]$$

$$=\frac{d^2}{ds^2}\left[\frac{s+1}{(s+1)^2+1}\right]=\frac{d^2}{ds^2}\left[\frac{s+1}{s^2+2s+2}\right]$$

$$=\frac{d}{ds}\left[\frac{\left(s^2+2s+2\right)(1)-\left(s+1\right)(2s+2)}{\left(s^2+2s+2\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{s^2+2s+2-\left(2s^2+2s+2s+2\right)}{\left(s^2+2s+2\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{s^2+2s+2-2s^2-4s-2}{\left(s^2+2s+2\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{-s^2-2s}{\left(s^2+2s+2\right)^2}\right]$$

$$=\left[\frac{\left(s^2+2s+2\right)^2(-2s-2)-2(-s^2-2s)(s^2+2s+2)(2s+2)}{\left(s^2+2s+2\right)^4}\right]$$

$$=\frac{\left(s^2+2s+2\right)(-2s-2)-2(-s^2-2s)(2s+2)}{\left(s^2+2s+2\right)^3}$$

$$=\frac{(2s+2)\left[-s^2-2s-2+2s^2+4s\right]}{\left(s^2+2s+2\right)^3}$$

$$= \frac{(2s+2)\left[s^2+2s-2\right]}{\left(s^2+2s+2\right)^3}$$

Example 18

Find the Laplace transform of $te^{-2t}\cos 3t$.

Solution:

$L[te^{-2t}\cos 3t\ ]$

$$= -\frac{d}{ds}\left[L(e^{-2t}\cos 3t)\right]$$

$$= -\frac{d}{ds}\left\{\left[L(\cos 3t)\right]_{s\to(s+2)}\right\}$$

$$= -\frac{d}{ds}\left\{\left[\frac{s}{s^2+3^2}\right]_{s\to(s+2)}\right\}$$

$$= -\frac{d}{ds}\left[\frac{s+2}{(s+2)^2+3^2}\right]$$

$$= -\left[\frac{\left[(s+2)^2+3^2\right]-(s+2)(2)(s+2)}{\left[(s+2)^2+3^2\right]^2}\right]$$

$$= -\left[\frac{s^2+2s+4+9-2[(s+2)^2]}{\left[(s+2)^2+3^2\right]^2}\right]$$

$$= -\left[\frac{s^2+2s+4+9-2s^2-8-8s}{\left[(s+2)^2+3^2\right]^2}\right].$$

$$= -\left[\frac{-s^2-6s+5}{\left[(s+2)^2+3^2\right]^2}\right]$$

$$\boxed{L[te^{-2t}\cos 3t\ ] = \frac{s^2+6s-5}{\left[(s+2)^2+3^2\right]^2}}$$

EXERCISE: 4.3

1. **Find Laplace transform of $F(t)=(1+te^{-2t})^3$.** **Ans:** $\frac{1}{2}+\frac{12}{15}e^{3t}-\frac{3}{10}e^{-2t}$
2. **Find the Laplace transform of $t\sin 3t\cos 2t$** **Ans:** $\frac{5s}{\left(s^2+25\right)^2}+\frac{s}{\left(s^2+1\right)^2}$
3. **Find $L\left[te^{-t}\sin t\right]$** **Ans:** $\frac{2(s+1)}{\left(s^2+2s+2\right)^2}$

4. Find $L\left[e^{at}\sinh\ bt\right]$ **Ans:** $\frac{b}{(s-a)^2-b^2}$

4.4 TRANSFORM OF INTEGRALS

If $L\left[f(t)\right]=F(s)$ and $\frac{1}{t}f(t)$ has a limit as $t \to 0$ then $L\left[\frac{1}{t}f(t)\right]=\int_s^\infty F(s)\,ds$

WORKED EXAMPLE 4.4 (A)

Example 1

Evaluate $\int_0^\infty te^{-2t}\cos t\,dt$ using Laplace transform.

Solution:

$$\text{Consider}\int_0^\infty t\,e^{-2t}\cos t\,dt=\left[L[t\cos t]\right]_{s=2}$$

$$=\left[-\frac{d}{ds}L[\cos t]\right]_{s=2}$$

$$\textit{Now consider } L[\cos t]=\int_0^\infty e^{-st}\cos t\,dt$$

$$=\left[\frac{e^{-st}}{s^2+1}(-s\cos t+\sin t)\right]_0^\infty$$

$$=\left[0-\frac{1}{s^2+1}(-s+0)\right]_0^\infty$$

$$=\frac{s}{s^2+1}$$

$$\therefore\ L[\cos t]=\frac{s}{s^2+1}$$

$$\text{Therefore}\quad \int_0^\infty te^{-2t}\cos t\,dt=\left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s=2}$$

$$=\left[-\left[\frac{(s^2+1)-s(2s)}{(s^2+1)^2}\right]\right]_{s=2}$$

$$=\left[-\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]\right]_{s=2}$$

$$=\left[-\left[\frac{1-s^2}{(s^2+1)^2}\right]\right]_{s=2}$$

$$=-\left(\frac{1-4}{(4+1)^2}\right)=-\left(\frac{-3}{25}\right)$$

$$=\frac{3}{25}$$

Example 2

Evaluate $\int_0^\infty e^{-t}\left(\frac{\cos 2t - \cos 3t}{t}\right)dt$

Solution:

$$\int_0^\infty e^{-t}\left(\frac{\cos 2t-\cos 3t}{t}\right)dt = L\left[\frac{\cos 2t-\cos 3t}{t}\right]_{s=1}$$

$$=\left[\int_0^\infty L[\cos 2t-\cos 3t]ds\right]_{s=1} = \left[\frac{1}{2}\int_0^\infty\left[\frac{2s}{s^2+4}ds-\frac{1}{2}\frac{2s}{s^2+9}\right]ds\right]_{s=1}$$

$$=\left[\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)_s^\infty\right]_{s=1} = \left[\frac{1}{2}\log 1-\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)\right]_{s=1}$$

$$=\left[\frac{-1}{2}\log\frac{s^2+4}{s^2+9}\right]_{s=1} = \frac{-1}{2}\log\left(\frac{5}{10}\right)=\frac{1}{2}\log_e 2$$

Example 3

Find the Laplace transform of $\mathbf{e^{-t}\int_0^t \frac{\sin t}{t}dt}$.

Solution:

$$L\left(\int_0^t \frac{\sin t}{t}dt\right)=\frac{1}{s}L\left(\frac{\sin t}{t}\right)$$

$$\text{But } L\left(\frac{\sin t}{t}\right)=\int_s^\infty L(\sin t)ds$$

$$=\int_s^\infty \frac{1}{s^2+1}ds$$

$$=\left[\tan^{-1}s\right]_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}s \qquad \because \frac{\pi}{2}-\tan^{-1}x=\cot^{-1}x$$

$$=\cot^{-1}s$$

$$L\left(\int_0^t \frac{\sin t}{t}dt\right)=\frac{1}{s}\cot^{-1}(s)$$

$$\text{Hence } L\left(e^{-t}\int_0^t \frac{\sin t}{t}dt\right)=\left(\frac{1}{s}\cot^{-1}s\right)_{s\to s+1}$$

$$=\frac{\cot^{-1}(s+1)}{s+1}$$

Example 4

Evaluate $\int_0^\infty \mathbf{e^{-t}\cos t\,dt}$ **using Laplace transform.**

Solution: We know that, $\int_0^\infty e^{-st} f(t)dt = L[f(t)]$

$$\int_0^\infty e^{-t}\cos t dt = L[\cos t]_{s=1}$$

$$= \frac{s}{s^2+1}$$

$$= \frac{1}{1+1}$$

$$\boxed{\int_0^\infty e^{-t}\cos t dt = \frac{1}{2}}$$

Example 5

Find $L\left[\int_0^t te^{-t}dt\right]$

Solution:

$$L\left[\int_0^t te^{-t}dt\right] = \frac{1}{s}L\left[te^{-t}\right] \qquad \because L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}L\left(e^{-t}\right)\right] \qquad \because L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{1}{s+1}\right)\right]$$

$$= \frac{1}{s}\left[-\left(\frac{-1}{[s+1]^2}\right)\right]$$

$$\boxed{L\left[\int_0^t te^{-t}dt\right] = \frac{1}{s(s+1)^2}}$$

Example 6

Find the Laplace Transform of $\int_0^t (u^2 - u + e^{-u})du.$

Solution:

$$L\left[\int_0^t (u^2 - u + e^{-u})du\right] = \frac{L\left[t^2 - t + e^{-t}\right]}{s} \qquad \because L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}L[f(t)]$$

$$= \frac{L\left[t^2\right] + L[t] + L\left[e^{-t}\right]}{s}$$

$$= \frac{\frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s+1}}{s}$$

$$= \frac{s^3 + s^2 + 3s + 2}{s^4(s+1)}$$

Example 7

Find $\mathbf{L\left[\dfrac{\cos at-\cos bt}{t}\right]}$

Solution: We know that $L\left[\dfrac{f(t)}{t}\right]=\int_s^\infty L\left[f(t)\right]ds$

$$f(t)=\cos at-\cos bt$$

$$\therefore L\left[\frac{\cos at-\cos bt}{t}\right]=\int_s^\infty L\left[\cos at-\cos bt\right]ds$$

$$=\int_s^\infty L\left[\cos at\right]ds-\int_s^\infty L\left[\cos bt\right]ds \qquad \because L[\cos at]=\frac{s}{s^2+a^2}$$

$$=\left[\int_s^\infty \frac{s}{s^2+a^2}ds-\int_s^\infty \frac{s}{s^2+b^2}ds\right]$$

$$=\frac{1}{2}\left[\int_s^\infty \frac{2s}{s^2+a^2}ds-\int_s^\infty \frac{2s}{s^2+b^2}ds\right]$$

$$=\frac{1}{2}\left[\log(s^2+a^2)-\log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log 1-\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]$$

$$=\frac{1}{2}\left[-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \qquad \because \log 1=0$$

$$\boxed{=\frac{1}{2}\left[\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]}$$

Example 8

Find the Laplace transform of $\mathbf{\dfrac{e^{at}-e^{-bt}}{t}}$**.**

Solution: $\mathrm{L}\left[\dfrac{e^{at}-e^{-bt}}{t}\right]=\int_s^\infty\left[\mathrm{L}[e^{at}]-\mathrm{L}[e^{-bt}]\right]ds$

$$= \int_s^\infty \left[\frac{1}{s-a} - \frac{1}{s+b} \right] ds \qquad \because \int \frac{dx}{x} = \log x$$

$$= \left[\log(s-a) - \log(s+b) \right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{s+b} \right) \right]_s^\infty$$

$$= 0 - \log\left(\frac{s-a}{s+b} \right)$$

$$\boxed{L\left[\frac{e^{at} - e^{-bt}}{t} \right] = \log\left(\frac{s+b}{s-a} \right)}$$

Example 9

Find Laplace transform of $F(t) = \dfrac{1-\cos t}{t^2}$.

Solution:

$$L\left[\frac{1-\cos t}{t^2} \right] = \int_s^\infty \int_s^\infty L[1-\cos t] \, ds ds$$

$$= \int_s^\infty \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1} \right] ds ds$$

$$= \int_s^\infty \left[\log s - \frac{1}{2} \log(s^2+1) \right]_s^\infty ds$$

$$= \int_s^\infty \left[\log \frac{s}{\sqrt{(s^2+1)}} \right]_s^\infty ds$$

$$= \int_s^\infty \left[0 - \log \frac{s}{\sqrt{(s^2+1)}} \right] ds$$

$$= \int_s^\infty \left[\log \frac{\sqrt{(s^2+1)}}{s} \right] ds$$

$$= \frac{1}{2} \int_s^\infty \log\left(1 + \frac{1}{s^2} \right) ds$$

$$= \frac{1}{2} \left[s \log\left(1 + \frac{1}{s^2} \right) \right]_s^\infty - \frac{1}{2} \int_s^\infty s \frac{1}{1 + \frac{1}{s^2}} \left(\frac{-2}{s^3} \right) ds$$

$$= 0 - \frac{1}{2} s \log\left(\frac{s^2+1}{s^2} \right) + \int_s^\infty \frac{ds}{s^2+1}$$

$$= -\frac{1}{2} s \log\left(\frac{s^2+1}{s^2} \right) + \left[\tan^{-1}(s) \right]_s^\infty$$

$$= s\log\left(\frac{s}{\sqrt{s^2+1}}\right)+\tan^{-1}(\infty)-\tan^{-1}(s)$$

$$= s\log\left(\frac{s}{\sqrt{s^2+1}}\right)+\frac{\pi}{2}-\tan^{-1}(s)$$

$$= s\log\left(\frac{s}{s^2+1}\right)+\cot^{-1}(s)$$

EXERCISE: 4.4

1. **Find the Laplace transform of $\frac{e^{at}-e^{-bt}}{t}$.** **Ans:** $\log\left(\frac{s+b}{s-a}\right)$
2. **Find $L\left[\frac{\sin at}{t}\right]$** **Ans:** $\cot^{-1}\left(\frac{s}{a}\right)$
3. **Find $L\left[\frac{\sin^2 t}{t}\right]$** **Ans:** $\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s^2}\right)$
4. **Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$** **Ans:** $\frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$
5. **Using Laplace Transform prove that $\int_0^\infty \frac{1-\cos 2t}{t^2}dt=\pi$**

4.5 INITIAL AND FINAL VALUE THEOREMS

Initial value theorem of Laplace transforms.

The initial value theorem of Laplace transforms states that:

If L[f(t)]=F[s] then $\lim_{t\to 0} f(t)=\lim_{s\to\infty}\left[sF(s)\right]$

Final value theorem of Laplace transforms.

The final value theorem of Laplace transforms states that:

If L[f(t)]=F[s] then $\lim_{t\to\infty} f(t)=\lim_{s\to 0}\left[sF(s)\right]$

Example 1

Verify the initial value theorem for $f(t)=ae^{-bt}$

Solution: Initial value theorem is

$$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s) \qquad \dots (1)$$

Given $f(t)=ae^{-bt}$

$$F(s)=L\left[f(t)\right]=L\left[ae^{-bt}\right]=aL[e^{-bt}]=a\left(\frac{1}{s+b}\right) \qquad \because L[e^{-at}]=\frac{1}{s+a}$$

$$F(s)=\frac{a}{s+b}$$

$$\text{LHS of } (1)=\lim_{t\to 0} f(t)$$

$$=\lim_{t\to 0} ae^{-bt}=ae^0=a(1)$$

$$\text{LHS of } (1)=a$$

$$\text{RHS of } (\mathbf{1}) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty}\left(s\frac{a}{s+b}\right)$$

$$= \lim_{s\to\infty}\left(s\frac{a}{s(\mathbf{1}+b/s)}\right)$$

$$= \lim_{s\to\infty}\left(\frac{a}{(1+b/s}\right) = \frac{a}{1+0} = a$$

RHS of equation $(\mathbf{1}) = a$

LHS = RHS

Therefore initial value theorem is verified

Example 2

If $\mathbf{L[f(t)] = \frac{1}{s(s^2+a^2)}}$, find $\mathbf{\lim_{t\to 0} f(t)}$ and $\mathbf{\lim_{t\to\infty} f(t)}$

Solution:

Given $L[f(t)] = F(s) = \dfrac{1}{s(s^2+a^2)}$

By Initial value theorem,

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}\left[sF(s)\right] = \lim_{s\to\infty} s\frac{1}{s(s^2+a^2)} = \lim_{s\to\infty}\frac{1}{(s^2+a^2)}$$

$$= 0$$

By Final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[sF(s)\right] = \lim_{s\to 0} s\frac{1}{s(s^2+a^2)} = \lim_{s\to 0}\frac{1}{(s^2+a^2)}$$

$$= \frac{1}{a^2}$$

Example 3

If $\mathbf{L\left[f(t)\right] = \frac{1}{s(s+1)}}$, find $\mathbf{\lim_{t\to 0} f(t)}$ and $\mathbf{\lim_{t\to\infty} f(t)}$ using initial and final value theorems.

Solution:

Initial value theorem,

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}\left[sF(s)\right] = \lim_{s\to\infty}\left[s\frac{1}{s(s+1)}\right] = \lim_{s\to\infty}\frac{1}{s+1}$$

$$= 0$$

Final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[sF(s)\right] = \lim_{s\to 0}\left[s\frac{1}{s(s+1)}\right] = \lim_{s\to 0}\frac{1}{s+1}$$

$$= 1$$

Example 4

Verify initial value theorem for $\mathbf{f(t) = 1+e^{-t}(\sin t+\cos t)}$

Solution: Given that $f(t) = 1+e^{-t}\left(\sin t+\cos t\right)$

$$L\left[f(t)\right] = L\left[1 + e^{-t}\sin t + e^{-t}\cos t\right]$$

$$= L[1] + L\left[e^{-t}\sin t\right] + L\left[e^{-t}\cos t\right] \qquad \because L[1] = \frac{1}{s}, L[\sin at] = \frac{a}{s^2 + a^2}$$

$$= \frac{1}{s} + \left[\frac{1}{s^2+1}\right]_{s \to s+1} + \left[\frac{s}{s^2+1}\right]_{s \to s+1} \qquad \because L[\cos at] = \frac{s}{s^2 + a^2}$$

$$\because L^{-1}[e^{-at} f(t)] = F(s+a)$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2 + 1} + \frac{(s+1)}{(s+1)^2 + 1}$$

$$= \frac{1}{s} + \frac{1}{s^2 + 2s + 2} + \frac{(s+1)}{s^2 + 2s + 2}$$

$$F(s) = \frac{1}{s} + \frac{s+2}{s^2 + 2s + 2}$$

Initial value theorem,

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

$$LHS = \lim_{t \to 0} f(t) = \lim_{t \to 0}\left[1 + e^{-t}(\sin t + \cos t)\right]$$

$$= 1 + e^{0}[\sin 0 + \cos 0] \qquad \because \sin 0 = 0, \cos 0 = 1$$

$$= 1 + 1(1)$$

$$= 2$$

$$RHS = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} sL\left[f(t)\right]$$

$$= \lim_{s \to \infty} s\left[\frac{1}{s} + \frac{s+2}{s^2 + 2s + 2}\right]$$

$$= \lim_{s \to \infty}\left[1 + \frac{s^2 + 2s}{s^2 + 2s + 2}\right] = \lim_{s \to \infty}\left[1 + \frac{1 + \frac{2}{s}}{1 + \frac{2}{s} + \frac{2}{s^2}}\right] \quad \left[\because \frac{1}{\infty} = 0\right]$$

$$= 1 + 1$$

$$= 2$$

LHS=RHS

Hence initial value theorem is verified.

EXERCISE: 4.5

Verify initial and final value theorem for the following functions

1. $f(t) = 3e^{-2t}$
2. $f(t) = 1 - e^{-at}$

4.6 TRANSFORMS OF PERIODIC FUNCTIONS

A function f(x) is said to be periodic iff f(x + p) = f(x) is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of a function.

The formula for Laplace transform of a periodic function $L[f(t)] = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t)\,dt$

WORKED EXAMPLE 4.6 (A)

Example 1

Find the Laplace transform of $|\sin t|$.

Solution: $|\sin t|$ **is a periodic function with period** π

By definition, $L[f(t)]=\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt,$ where $p=\pi$ is period

$$L\left[|\sin t|\right]=\frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st}\sin t dt$$

$$=\frac{1}{1-e^{-\pi s}}\left[\frac{e^{-st}}{s^2+1}\left(-s\sin t-\cos t\right)\right]_0^{\pi}$$

$$=\frac{1}{1-e^{-\pi s}}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] \qquad \tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$=\frac{1+e^{-\pi s}}{1-e^{-\pi s}}\left[\frac{1}{s^2+1}\right] \qquad \coth\left(\frac{x}{2}\right)=\frac{1+e^{-x}}{1-e^{-x}}$$

$$=\coth\left[\frac{\pi s}{2}\right]\left[\frac{1}{s^2+1}\right]$$

$$\boxed{=\left[\frac{1}{s^2+1}\right]\coth\left[\frac{\pi s}{2}\right]}$$

Example 2

Obtain the Laplace transform of the periodic saw-tooth wave function given by
$f(t)=\frac{kt}{\omega},\quad 0<t<\omega$ **and** $f(t+\omega)=f(t).$

Solution: Since f(t) is a periodic function with period T we have $L[f(t)]=\frac{1}{1-e^{-s\omega}}\int_0^{\omega} e^{-st}f(t)dt$

$$=\frac{1}{1-e^{-s\omega}}\int_0^{T}\frac{kt}{\omega}e^{-st}dt$$

$$=\frac{1}{\omega\left(1-e^{-s\omega}\right)}\int_0^{\omega} te^{-st}dt$$

$$=\frac{1}{\omega\left(1-e^{-sT}\right)}\left\{t\left(\frac{e^{-st}}{-s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right\}_0^{\omega}$$

$$=\frac{1}{\omega\left(1-e^{-s\omega}\right)}\left[\left(\frac{-\omega e^{-s\omega}}{s}-\frac{e^{-s\omega}}{s^2}\right)-\left(0-\frac{1}{s^2}\right)\right]$$

$$=\frac{1}{\omega\left(1-e^{-s\omega}\right)}\left[\frac{-\omega e^{-s\omega}}{s}+\frac{1}{s^2}\left(1-e^{-s\omega}\right)\right]$$

$$=\frac{k}{\omega s^2}-\frac{ke^{-s\omega}}{s(1-e^{-s\omega})}$$

Example 3

Find the Laplace transform of the of $f(t)=\begin{cases} a\sin\omega t & ,0\le t\le \pi/\omega \\ 0 & ,\pi/\omega \le t \le 2\pi/\omega \end{cases}$ **and** $f(t+2\pi/\omega)=f(t)$ **for all t.**

Solution:

Given $f(t)=\begin{cases} a\sin\omega t & ,0\le t\le \pi/\omega \\ 0 & ,\pi/\omega \le t \le 2\pi/\omega \end{cases}$

This function is a periodic function with period $2\pi/\omega$ in the interval (0, $2\pi/\omega$).

$$L[f(t)]=\frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}a\sin\omega t dt+0\right]$$

$$=\frac{a}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t dt\right]$$

$$=\frac{a}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}\left[-s\sin\omega t-\omega\cos\omega t\right]\right]_0^{\omega}$$

$$=\frac{a}{1-\left(e^{-\pi s/\omega}\right)^2}\left[\frac{e^{-s\pi/\omega}\omega+\omega}{s^2+\omega^2}\right]$$

$$=\frac{a\omega\left[1+e^{-s\pi/\omega}\right]}{\left(1-e^{-s\pi/\omega}\right)\left(1+e^{-s\pi/\omega}\right)\left(s^2+\omega^2\right)}$$

$$=\frac{a\omega}{\left(1-e^{-s\pi/\omega}\right)\left(s^2+\omega^2\right)}$$

Example 4

Find the Laplace transform of the rectangular wave given by $f(t)=\begin{cases} 1, & 0\le t\le b \\ -1, & b\le t\le 2b \end{cases}$

$f(t+2b)=f(t)$.

Solution:

$$L[f(t)]=\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt \qquad \text{where p =2b}$$

$$=\frac{1}{1-e^{-s(2b)}}\left[\int_0^{2b} e^{-st}f(t)dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}dt+\int_b^{2b} e^{-st}\left(-1\right)dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}dt-\int_b^{2b} e^{-st}dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\left[\frac{e^{-st}}{-s}\right]_0^b+\left[\frac{e^{-st}}{s}\right]_b^{2b}\right]$$

$$=\frac{1}{s(1-e^{-2bs})}\left[-\left[e^{-bs}-1\right]+\left[e^{-2bs}-e^{-sb}\right]\right]$$

$$=\frac{1}{s(1-e^{-2bs})}\left[-e^{-bs}+1+(e^{-bs})^2-e^{-bs}\right]$$

$$\because \tanh\theta=\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$$

$$=\frac{1}{s(1-e^{-bs})(1+e^{-bs})}\left[1-e^{-bs}\right]^2$$

$$=\frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$=\frac{1\left[e^{\frac{bs}{2}}-e^{\frac{-bs}{2}}\right]}{s\left[e^{\frac{bs}{2}}+e^{\frac{-bs}{2}}\right]}$$

$$L\left[f\left(t\right)\right]=\frac{1}{s}\tanh\left(\frac{bs}{2}\right)$$

Example 5

Find the Laplace transform of a square wave function given by $f\left(t\right)=\begin{cases} E\,,\ 0\le t\le\dfrac{a}{2} \\ -E\,,\ \dfrac{a}{2}\le t\le a\end{cases}$

$f\left(t+a\right)=f\left(t\right)$.

Solution:

Given f(t) is a periodic function with period 'a'

$$\mathrm{L}\left[\mathrm{f}(\mathrm{t})\right]=\frac{1}{1-\mathrm{e}^{-\mathrm{ps}}}\int_0^{\mathrm{p}}\mathrm{e}^{-\mathrm{st}}\mathrm{f}(\mathrm{t})\mathrm{dt}\quad \text{where p=a}$$

$$=\frac{1}{1-\mathrm{e}^{-\mathrm{as}}}\int_0^{\mathrm{a}}\mathrm{e}^{-\mathrm{st}}\mathrm{f}(\mathrm{t})\mathrm{dt}$$

$$=\frac{1}{1-\mathrm{e}^{-\mathrm{as}}}\left[\int_0^{\mathrm{a}/2}\mathrm{e}^{-\mathrm{st}}(\mathrm{E})\mathrm{dt}+\int_{\mathrm{a}/2}^{\mathrm{a}}\mathrm{e}^{-\mathrm{st}}(-\mathrm{E})\mathrm{dt}\right]$$

$$= \frac{E}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right]$$

$$= \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-as/2}}{-s}+\frac{1}{s}\right)+\left(\frac{e^{-as}}{s}-\frac{e^{-as/2}}{s}\right)\right]$$

$$= \frac{E}{1-e^{-as}}\left[-\frac{e^{-as/2}}{s}+\frac{1}{s}+\frac{e^{-as}}{s}-\frac{e^{-as/2}}{s}\right]$$

$$= \frac{E}{1-e^{-as}}\left[\frac{1-2e^{-as/2}+e^{-as}}{s}\right]$$

$$= \frac{E}{1-e^{-as}}\left[\frac{1-2e^{-as/2}+e^{-as/2}e^{-as/2}}{s}\right]$$

$$= \frac{E}{1-e^{-as}}\left[\frac{1-2e^{-as/2}+\left(e^{-as/2}\right)^2}{s}\right]$$

$$= \frac{E}{1-\left(e^{-as/2}\right)^2}\frac{\left[1-e^{-as/2}\right]^2}{s}$$

$$= \frac{E\left[1-e^{-as/2}\right]^2}{\left(1+e^{-as/2}\right)\left(1-e^{-as/2}\right)s}$$

$$= \frac{E\left[1-e^{-as/2}\right]}{\left(1+e^{-as/2}\right)s}$$

$$= \frac{E}{s}\frac{\left[e^{as/4}-e^{-as/4}\right]}{\left(e^{as/4}+e^{-as/4}\right)} \qquad \because \tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$$

$$= \frac{E}{s}\tanh\left[\frac{as}{4}\right]$$

$$\boxed{L\left[f(t)\right] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)}$$

Example 6

Find the Laplace transform of the of the Half wave rectifier $f(t)=\begin{cases}\sin\omega t & ,0<t<\pi/\omega \\ 0 & ,\pi/\omega<t<2\pi/\omega\end{cases}$

and $f(t+2\pi/\omega)=f(t)$ **for all t.**

Solution:

Given $f(t)=\begin{cases}\sin\omega t & 0<t<\pi/\omega \\ 0 & \pi/\omega<t<2\pi/\omega\end{cases}$

The Laplace transformation of a periodic function f(t) with period p is given by

$L[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$ this function is a periodic function with the period $p=\frac{2\pi}{\omega}$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\int_0^{\frac{2\pi}{\omega}} e^{-st}f(t)\,dt$$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\ dt+0;\ \because \int e^{at}\sin bt\ dt=\frac{e^{at}}{a^2+b^2}\left[a\sin bt-b\cos bt\right]$$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t-\omega\cos\omega t\right)\right]_0^{\frac{\pi}{\omega}}$$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2+\omega^2}\left(0-\omega(-1)\right)-\left\{\frac{1}{s^2+\omega^2}\left(0-\omega\right)\right\}\right];\ \text{w.k.t}\sin\pi=0,\cos\pi=-1$$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2+\omega^2}\left(0-\omega(-1)\right)-\left\{\frac{1}{s^2+\omega^2}\left(0-\omega\right)\right\}\right]$$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2+\omega^2}\left(\omega\right)-\left\{\frac{1}{s^2+\omega^2}\left(-\omega\right)\right\}\right]$$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-\frac{s\pi}{\omega}}}{s^2+\omega^2}\left(\omega\right)+\left\{\frac{\omega}{s^2+\omega^2}\right\}\right]$$

$$=\frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-\frac{s\pi}{\omega}}\omega+\omega}{s^2+\omega^2}\right]$$

$$=\frac{\omega}{1-e^{-\left(\frac{\pi}{\omega}s\right)^2}}\left[\frac{e^{-\frac{s\pi}{\omega}}+1}{s^2+\omega^2}\right]$$

$$=\frac{\omega}{1^2-e^{-\left(\frac{\pi}{\omega}s\right)^2}}\left[\frac{e^{-\frac{s\pi}{\omega}}+1}{s^2+\omega^2}\right]$$

$$=\frac{\omega}{\left(1+e^{-\left(\frac{\pi}{\omega}s\right)}\right)\left(1-e^{-\left(\frac{\pi}{\omega}s\right)}\right)}\left[\frac{e^{-\frac{s\pi}{\omega}}+1}{s^2+\omega^2}\right]$$

$$\boxed{L[f(t)]=\frac{\omega}{\left(1-e^{-\left(\frac{\pi}{\omega}s\right)}\right)}\left[\frac{1}{s^2+\omega^2}\right]}$$

Example 7

Find the Laplace transform of a Triangular wave function $f(t)=\begin{cases} t, & 0\le t\le a \\ 2a-t, & a\le t\le 2a \end{cases}$

$f(t+2a)=f(t)$

Solution:

Given $f(t)=\begin{cases} t, & 0\le t\le a \\ 2a-t, & a\le t\le 2a \end{cases}$, $f(t+2a)=f(t)$

This function is periodic with period $2a$

$$L[f(t)]=\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt \qquad \text{where p is period.}$$

$$=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,t\,dt+\int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$\because \int udv = uv-u'v_1+u''v_2-......$

$$=\frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a+\left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$=\frac{1}{1-e^{-2as}}\left[\left[-t\frac{e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_0^a+\left[-(2a-t)\frac{e^{-st}}{s}+\frac{e^{-st}}{s^2}\right]_a^{2a}\right]$$

$$=\frac{1}{1-e^{-2as}}\left[\left[\left(-a\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\left(\frac{e^{-2as}}{s^2}\right)-\left(-a\frac{e^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right]$$

$$=\frac{1}{1-e^{-2as}}\left[-\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$=\frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right]$$

$$=\frac{[1-e^{-as}]^2}{s^2(1-e^{-as})(1+e^{-as})} \qquad \because (a-b)^2=a^2-2ab-b^2$$

$$\&(a^2-b^2)=(a+b)(a-b)$$

$$=\frac{1-e^{-as}}{s^2(1+e^{-as})}=\frac{1}{s^2}\tanh\left(\frac{as}{2}\right) \qquad \because \tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$\boxed{\therefore L(f(t))=\frac{1}{s^2}\tanh\left(\frac{as}{2}\right)}$$

EXERCISE: 4.6

1. **Find the Laplace transform of f(t) = t,** $0\le t\le 4$**, if f(t + 4) = f(t) for all** $t\ge 0$**.**

Ans: $\frac{1}{s\left(1-e^{-4s}\right)}\left[\frac{1}{s}-\frac{e^{-4s}}{s}-4e^{-4s}\right]$

2. **Find L[f(t)] if** $f(t)=\begin{cases}1, 0<t<1\\ 0, 1<t<2\end{cases}$ **given f(t+2) = f(t).** **Ans:** $\frac{1}{s\left(1+e^{-s}\right)}$

4.7 INVERSE LAPLACE TRANSFORMS

Inverse Operator

Given F(s) we obtain the function f(t). This we call it as Inverse Laplace transform. In other words, we say that the Inverse Laplace Transform of F(s) is f(t).

If L[f(t)] = F(s), then $L^{-1}\left[F(s)\right]=f(t)$ where L^{-1} is called the Inverse Laplace Transform operator.

Important Formulas

1. $L^{-1}[1]=\delta(t)$
2. $L^{-1}\left[F(s+a)\right]=e^{-at}L^{-1}\left[F(s)\right]$
3. $L^{-1}\left[F(s-a)\right]=e^{at}L^{-1}\left[F(s)\right]$
4. $L^{-1}\left[\frac{1}{s}\right]=1$
5. $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$
6. $L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$
7. $L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{1}{a}\sinh at$
8. $L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$
9. $L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$
10. $L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$
11. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right]=\frac{1}{b}e^{at}\sin bt$
12. $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right]=\frac{1}{b}e^{at}\sinh bt$
13. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos bt$
14. $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right]=e^{at}\cosh bt$

WORKED EXAMPLE 4.7 (A)

Example 1

Find $L^{-1}[(1/2s-3)^4]$**.**

Solution: $L^{-1}\left[\frac{1}{(2s-3)^4}\right]=\frac{1}{2^4}L^{-1}\left[\frac{1}{\left(s-\frac{3}{2}\right)^4}\right]$

$$=\frac{1}{2^4}e^{\frac{3}{2}t}L^{-1}\left(\frac{1}{s^4}\right)$$

$$=\frac{1}{2^4}e^{\frac{3}{2}t}\frac{1}{3!}L^{-1}\left(\frac{3!}{s^4}\right)$$

$$= \frac{1}{2^4} e^{\frac{3}{2}t} \frac{1}{3!} t^3 \qquad \left[\because L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n \right.$$

$$= \frac{1}{96} t^3 e^{\frac{3}{2}t}$$

Example 2

Evaluate $\mathbf{L^{-1}\left[\frac{1}{s^2+6s+13}\right]}$.

Solution:

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right]$$

$$= L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$$

$$= e^{-3t} L^{-1}\left[\frac{1}{s^2+2^2}\right] \qquad \because e^{-at} f(t) = L^{-1}[F(s+a)]$$

$$= \frac{e^{-3t}}{2} L^{-1}\left[\frac{2}{s^2+2^2}\right] \qquad \because \sin at = L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$\boxed{L^{-1}\left[\frac{1}{s^2+6s+13}\right] = \frac{e^{-3t}}{2} \sin 2t}$$

Example 3

Find $\mathbf{L^{-1}\left(\frac{s+2}{s^2+4s+8}\right)}$.

Solution:

$$L^{-1}\left(\frac{s+2}{s^2+4s+8}\right) \qquad \because L^{-1}[F(s+a)] = e^{-at} f(t),\ L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$$

$$= L^{-1}\left(\frac{s+2}{[s+2]^2+4}\right)$$

$$= e^{-2t} L^{-1}\left(\frac{s}{s^2+2^2}\right)$$

$$\boxed{= e^{-2t} \cos 2t}$$

Example 4

Find the inverse Laplace transform of $\mathbf{\frac{e^{-\pi s}}{(s-1)^2}}$. (M/J 14) (K)

Solution:
$$L^{-1}\left[\frac{1}{(s-1)^2}\right] = e^t L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t\, t$$

$$\because L^{-1}\left[F(s+a)\right]=e^{-at}f(t),\ L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$$

$$\boxed{L^{-1}\left[\frac{e^{-\pi s}}{(s-1)^2}\right]=e^{(t-\pi)}(t-\pi)U(t-\pi)}$$

$$\because L^{-1}\left[e^{-as}F(s)\right]=f(t-a)U(t-a)$$

Example

Find $L^{-1}\left[\cot^{-1}s\right]$.

Solution: $L^{-1}[\cot^{-1}s]=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}[\cot^{-1}s]\right\}$ $\quad\because L^{-1}[F(s)]=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}[F(s)]\right\}$

$$=\frac{-1}{t}L^{-1}\left[\frac{-1}{s^2+1}\right] \qquad \because \frac{d}{dx}\left[\cot^{-1}x\right]=\frac{-1}{1+x^2}$$

$$=\frac{1}{t}L^{-1}\left[\frac{1}{s^2+1^2}\right] \qquad \because \sin at=L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$\boxed{L^{-1}\left[\cot^{-1}(s)\right]=\frac{1}{t}(\sin t)}$$

4.7.1 INVERSE LAPLACE TRANSFORMS USING PARTIAL FRACTIONS

WORKED EXAMPLE 4.7 (B)

Example 1

Evaluate $L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)}\right]$.

Solution:

Let $\frac{3s^2+16s+26}{s(s^2+4s+13)}=\frac{A}{s}+\frac{Bs+C}{s^2+4s+13}$

$$3s^2+16s+26=A\left(s^2+4s+13\right)+\left(Bs+C\right)s$$

Put s = 0,	13A = 26	**A = 2**
Coef of s =16,	16 = 4A + C	**C = 8**
Coef of s^2 = 3,	A + B = 3	**B = 1**

$$\frac{3s^2+16s+26}{s(s^2+4s+13)}=\frac{2}{s}+\frac{s+8}{s^2+4s+13}$$

$$L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)}\right]=L^{-1}\left[\frac{2}{s}\right]+L^{-1}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$=2L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{(s+2)+6}{(s+2)^2+9}\right]$$

$$=2L^{-1}\left[\frac{1}{s}\right]+e^{-2t}L^{-1}\left[\frac{s+6}{s^2+9}\right]$$

$$=2L^{-1}\left[\frac{1}{s}\right]+e^{-2t}\left\{L^{-1}\left[\frac{s}{s^2+9}\right]+\frac{6}{3}L^{-1}\left[\frac{3}{s^2+9}\right]\right\}$$

$$=2(1)+e^{-2t}\left[\cos 3t+2\sin 3t\right]$$

$$\boxed{L^{-1}\left[\frac{3s^2+16s+26}{s(s^2+4s+13)}\right]=2+e^{-2t}\left[\cos 3t+2\sin 3t\right]}$$

Example 2

Find $\mathbf{L^{-1}\left[\dfrac{s^2-s+2}{s^2(s+2)(s-3)}\right]}$.

Solution:

To find $L^{-1}\left[\dfrac{s^2-s+2}{s^2(s+2)(s-3)}\right]$

$$\frac{s^2-s+2}{s^2(s+2)(s-3)}=\frac{A}{s^2}+\frac{B}{s+2}+\frac{C}{s-3}+\frac{D}{s}$$

$$s^2-s+2=A(s+2)(s-3)+B(s^2)(s-3)+C(s^2)(s+2)+Ds(s+2)(s-3)$$

Put $s=-2$

$$4+2+2=B(4)(-5)$$

$$8=-20B$$

$$B=-\frac{8}{20}$$

$$B=-\frac{2}{5}$$

Put $s=3$

$$9-3+2=C(9)(5)$$

$$45C=8$$

$$C=\frac{8}{45}$$

Put $s=0$

$$2=-6A$$

$$A=-\frac{2}{6}$$

$$A=-\frac{1}{3}$$

Put $s=1$

$$1-1+2=A(3)(-2)+B(-2)+C(3)+D(3)(-2)$$

$$2=-6A-2B+3C-6D$$

$$2=-6(-1/3)-2(-2/5)+3(8/45)-6D$$

$$2=2+\frac{4}{5}+\frac{8}{15}-6D$$

$$6D=\frac{4}{5}+\frac{8}{15}=\frac{12+8}{15}=\frac{20}{15}=\frac{4}{3}$$

$$D=\frac{4}{3}\text{x}\frac{1}{6}=\frac{2}{9}$$

$$\frac{s^2-s+2}{s^2(s+2)(s-3)}=\frac{-\frac{1}{3}}{s^2}+\frac{-\frac{2}{5}}{s+2}+\frac{\frac{8}{45}}{s-3}+\frac{\frac{2}{9}}{s}$$

$$L^{-1}\left[\frac{s^2-s+2}{s^2(s+2)(s-3)}\right]=-\frac{1}{3}L^{-1}\left[\frac{1}{s^2}\right]-\frac{2}{5}L^{-1}\left[\frac{1}{s+2}\right]+\frac{8}{45}L^{-1}\left[\frac{1}{s-3}\right]+\frac{2}{9}L^{-1}\left[\frac{1}{s}\right] \qquad \because L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$$

$$=-\frac{1}{3}t-\frac{2}{5}e^{-2t}(1)+\frac{8}{45}e^{3t}(1)+\frac{2}{9}(1)$$

$$\because e^{at}f(t)=L^{-1}\left[F(s-a)\right], \because e^{-at}f(t)=L^{-1}\left[F(s+a)\right]$$

$$\boxed{L^{-1}\left[\frac{s^2-s+2}{s^2(s+2)(s-3)}\right]=-\frac{1}{3}t-\frac{2}{5}e^{-2t}+\frac{8}{45}e^{3t}+\frac{2}{9}}$$

Example 3

Find $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$ using partial fraction method.

Solution:

By Partial fraction method

$$\frac{1-s}{(s+1)(s^2+4s+13)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+4s+13)} \rightarrow (1)$$

$$1-s=A(s^2+4s+13)+(Bs+C)(s+1)$$

$$Put\ s=-1$$

$$1-(-1)=A[(-1)^2+4(-1)+13]+[B(-1)+C](-1+1)$$

$$\Rightarrow 2=A[1-4+13]$$

$$\Rightarrow 2=10A$$

$$\Rightarrow A=\frac{1}{5}$$

Equate the coefficient of S,

$$-1=4A+B+C$$

$$-1=\frac{4}{5}+B+C$$

$$-1-\frac{4}{5}=B+C$$

$$B+C=-\frac{9}{5} \qquad .....(2)$$

Equate the coefficient of Constant

$$1=13A+C$$

$$\Rightarrow C=1-13A$$

$$\Rightarrow C=1-13A$$

$$\Rightarrow C=1-13\left(\frac{1}{5}\right)$$

$$\Rightarrow C=\frac{5-13}{5}$$

$$\Rightarrow C = \frac{-8}{5}$$

$$(2) \Rightarrow \text{B} - \frac{8}{5} = -\frac{9}{5}$$

$$\text{B} = -\frac{9}{5} + \frac{8}{5}$$

$$= \frac{-1}{5}$$

$sub\, A, B, C in\, (1)$

$$\therefore \frac{1-s}{(s+1)(s^2+4s+13)} = \frac{\frac{1}{5}}{(s+1)} + \frac{-\frac{1}{5}s - \frac{8}{5}}{(s^2+4s+13)}$$

Taking Inverse Laplace transform on both sides

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right] = \frac{1}{5}L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{1}{5}L^{-1}\left[\frac{s+8}{(s^2+4s+13)}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2+6}{(s+2)^2+9}\right] \{\because s^2+4s+13 = s^2+4s+4+9\}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - \frac{1}{5}L^{-1}\left[\frac{6}{(s+2)^2+9}\right]$$

$$\because L^{-1}\left[F(s+a)\right] = e^{-at}L^{-1}[F(S)]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - \frac{2}{5}e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$\because \cos at = L^{-1}\left[\frac{s}{s^2+a^2}\right] \quad \because \sin at = L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right] = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t$$

Example 4

Find the inverse Laplace transform of the following function $\mathbf{\frac{1}{(s^2+a^2)(s^2+b^2)}}$

SOLUTION:

By Partial fraction method

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{A}{(s^2+a^2)} + \frac{B}{(s^2+a^2)} \rightarrow (1)$$

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{A(s^2+b^2)+B(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}$$

$$1 = A(s^2+b^2) + B(s^2+a^2)$$

$$Put\, s^2 = -a^2$$

$$1 = A(-a^2 + b^2)$$

$$A = \frac{-1}{a^2 - b^2}$$

$$Put\, s^2 = -b^2$$

$$1 = B(-b^2 + a^2)$$

$$B = \frac{1}{a^2 - b^2}$$

$$Sub\, A, B\, in\, (1)$$

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{\frac{-1}{a^2-b^2}}{(s^2+a^2)} + \frac{\frac{1}{a^2-b^2}}{(s^2+b^2)}$$

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{-1}{(s^2+a^2)} + \frac{1}{(s^2+b^2)}\right]$$

$$Take\, L^{-1}\, on\, both\, sides$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2} L^{-1}\left[\frac{-1}{(s^2+a^2)} + \frac{1}{(s^2+b^2)}\right]$$

$$= \frac{1}{a^2-b^2}\left\{L^{-1}\left[\frac{-1}{(s^2+a^2)}\right] + L^{-1}\left[\frac{1}{(s^2+b^2)}\right]\right\}$$

$$= \frac{1}{a^2-b^2}\left\{-\frac{1}{a} L^{-1}\left[\frac{a}{(s^2+a^2)}\right] + \frac{1}{b} L^{-1}\left[\frac{b}{(s^2+b^2)}\right]\right\}$$

$$= \frac{1}{a^2-b^2}\left[\frac{-1}{a}\sin at + \frac{1}{b}\sin bt\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{1}{b}\sin bt - \frac{1}{a}\sin at\right]$$

$$\boxed{\therefore L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left[\frac{1}{b}\sin bt \frac{-1}{a}\sin at\right]}$$

EXERCISE: 4.7.1

1. **Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$ using partial fractions.** **Ans:** $\frac{1}{2}\left[1 + e^{-2t} - 2e^{-t}\right]$

2. **Find $L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right]$ using partial fractions.**

   Ans: $-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4te^{2t} - \frac{7}{2}e^{2t}t^2$

3. **Find $L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right]$** **Ans:** $\frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} + 3te^{t}$

4. **Find** $L^{-1}\left[\dfrac{3\left(s^2+2s+3\right)}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}\right]$ **Ans:** $e^{-t}\sin t+e^{-t}\sin 2t$

5. **Find** $L^{-1}\left[\dfrac{\left(s^2+2\right)}{\left(s^2+10\right)\left(s^2+20\right)}\right]$ **Ans:** $\dfrac{-4}{5\sqrt{10}}\sin\sqrt{10}t+\dfrac{9}{10\sqrt{5}}\sin 2\sqrt{5}t$

4.7.2 INVERSE LAPLACE TRANSFORMS USING PROPERTIES

1. **Omitting s**

If $L^{-1}\left[F(s)\right]=f(t)$ then $L^{-1}\left[sF(s)\right]=f'(t)$, provided $f(0)=0$

$$=\frac{d}{dt}f(t)$$

$$=\frac{d}{dt}\left[L^{-1}F(s)\right]$$

2. **Omitting** $\frac{1}{s}$

If $L\left[f(t)\right]=F(s)$ then $L\left[\int_0^t f(t)dt\right]=\frac{1}{s}F(s)$

i.e., $L^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t f(t)dt=\int_0^t L^{-1}\left[F(s)\right]dt$

3. **Inverse of Derivative**

If $L^{-1}\left[F(s)\right]=f(t)$ then $L^{-1}\left[F'(s)\right]=-tf(t)=-t\,L^{-1}\left[F(s)\right]$

4. **Inverse of Integrals**

$L^{-1}\left[\int_s^\infty F(s)ds\right]=\frac{1}{t}f(t)=\frac{1}{t}L^{-1}\left[F(s)\right]$ **(or)**

$tL^{-1}\left[\int_s^\infty F(s)ds\right]=L^{-1}\left[F(s)\right]$

5. **Change of Scale Property**

If If $L^{-1}\left[F(s)\right]=f(t)$ then $L^{-1}\left[F(cs)\right]=\frac{1}{c}f\left(\frac{t}{c}\right)$

WORKED EXAMPLE 4.7 (C)

Example 1

Find the inverse Laplace transform of $\log\left(\dfrac{s+1}{s-1}\right)$.

Solution:

we know that, $L^{-1}[F(s)]=-\frac{1}{t}L^{-1}\left[F'(s)\right]$

$$L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{t}\left[e^{-t} - e^{t}\right]$$

$$= \frac{1}{t}\left[e^{t} - e^{-t}\right]$$

$$= \frac{2}{t}\left[\frac{e^{t} - e^{-t}}{2}\right] \quad \because \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\boxed{L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{2}{t}\sinh t}$$

Example 2

Find $\mathbf{L^{-1}\left[\frac{2s-9}{s^2+6s+34}\right]}$.

Solution: $L^{-1}\left[\frac{2s-9}{s^2+6s+34}\right] = L^{-1}\left[\frac{2s-9}{s^2+6s+9+25}\right]$

$$= 2L^{-1}\left[\frac{s - \frac{9}{2}}{s^2+6s+9+25}\right]$$

$$= 2L^{-1}\left[\frac{s+3-3-\frac{9}{2}}{(s+3)^2+25}\right]$$

$$= 2e^{-3t}L^{-1}\left[\frac{s-\frac{15}{2}}{s^2+25}\right]$$

$$= 2e^{-3t}L^{-1}\left[\frac{s}{s^2+25} + \frac{\left(-\frac{15}{2}\right)}{s^2+25}\right]$$

$$= 2e^{-3t}\left\{L^{-1}\left[\frac{s}{s^2+25}\right] - \frac{15}{2}L^{-1}\left[\frac{1}{s^2+25}\right]\right\}$$

$$= 2e^{-3t}\left\{L^{-1}\left[\frac{s}{s^2+25}\right] - \frac{3}{2}L^{-1}\left[\frac{5}{s^2+25}\right]\right\}$$

$$= 2e^{-3t}\left\{\cos 5t - \frac{3}{2}\sin 5t\right\}$$

$$\boxed{L^{-1}\left[\frac{2s-9}{s^2+6s+34}\right] = e^{-3t}\left[2\cos 5t - 3\sin 5t\right]}$$

Example 3

Find $L^{-1}\left[\log\left(\frac{s^2+4}{(s-2)^2}\right)\right]$

Solution:

WKT $L^{-1}[F'(s)] = -tL^{-1}[F(s)]$

$(or)\, L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F'(s)]$

$$L^{-1}\left[\log\left(\frac{s^2+4}{(s-2)^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{ds}\left[\log(s^2+4) - s\log(s-2)\right]\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+4} - \frac{2}{s-2}\right]$$

$$= -\frac{2}{t}\cos 2t + \frac{2}{t}e^{2t}$$

Example 4

Find $\mathbf{L^{-1}\left(\log\frac{s^2+1}{s(s+1)}\right)}$**.**

Solution:

$$\text{Let } L^{-1}\left[\log\frac{s^2+1}{s(s+1)}\right] = f(t)$$

$$L[f(t)] = \log\left[\frac{s^2+1}{s(s+1)}\right]$$

$$= \log(s^2+1) - \log s(s+1)$$

$$= \log(s^2+1) - \log s - \log(s+1)$$

$$\therefore L[tf(t)] = -\frac{d}{ds}\left[\log(s^2+1) - \log s - \log(s+1)\right]$$

$$= -\left[\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right]$$

$$= \frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+1}$$

$$\therefore tf(t) = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= 1 + e^{-t} - 2\cos t$$

$$\therefore f(t) = \frac{1+e^{-t}-2\cos t}{t}$$

$$\boxed{\therefore L^{-1}\left[\log\frac{s^2+1}{s(s+1)}\right]=\frac{1+e^{-t}-2\cos t}{t}}$$

Example 5

Find $\mathbf{L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right]}$.

Solution:

$$L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right]=L^{-1}\left[\tan^{-1}\left(\frac{s+1}{2}\right)\right]$$

$$=\frac{1}{t}L^{-1}\left[(-1)\frac{d}{ds}\tan^{-1}\left(\frac{s+1}{2}\right)\right] \qquad L\left[tf(t)\right]=(-1)\frac{d}{ds}F(s)$$

$$=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{s+1}{2}\right)\right] \qquad tf(t)=L^{-1}\left[(-1)\frac{d}{ds}F(s)\right]$$

$$=-\frac{1}{t}L^{-1}\left(\frac{1}{2}\left[\frac{1}{1+\left(\frac{s+1}{2}\right)^2}\right]\right)$$

$$f(t)=\frac{1}{t}L^{-1}\left[(-1)\frac{d}{ds}F(s)\right]$$

$$=-\frac{1}{t}L^{-1}\left(\frac{1}{2}\left[\frac{4}{4+(s+1)^2}\right]\right)$$

$$=-\frac{1}{t}L^{-1}\left(\left[\frac{2}{4+(s+1)^2}\right]\right)$$

$$=-\frac{1}{t}e^{-t}L^{-1}\left(\left[\frac{2}{4+s^2}\right]\right)$$

$$f(t)=-\frac{1}{t}e^{-t}\sin 2t$$

Example 6

Find the inverse Laplace transform of $\frac{14s+10}{49s^2+28s+13}$

Solution:

$$L^{-1}\left(\frac{14s+10}{49s^2+28s+13}\right)=L^{-1}\left(\frac{14s+10}{(49s^2+28s+4)+9}\right)$$

$$=L^{-1}\left(\frac{14s+10}{(49s^2+28s+4)+9}\right)$$

$$=L^{-1}\left(\frac{2(7s+2)+6}{(7s+2)^2+9}\right)$$

$$= L^{-1}\left(\frac{14(s+\frac{2}{7})+6}{49(s+\frac{2}{7})^2+9}\right)$$

$$= e^{-\frac{2}{7}t} L^{-1}\left(\frac{14s+6}{49s^2+9}\right)$$

$$= \frac{e^{-\frac{2}{7}t}}{49} L^{-1}\left(\frac{14s+6}{s^2+\frac{9}{49}}\right)$$

$$= \frac{e^{-\frac{2}{7}t}}{49}\left[14L^{-1}\left(\frac{s}{s^2+\frac{9}{49}}\right)+6L^{-1}\left(\frac{1}{s^2+\frac{9}{49}}\right)\right]$$

$$= \frac{e^{-\frac{2}{7}t}}{49}\left[14L^{-1}\left(\frac{s}{s^2+\frac{9}{49}}\right)+6.\frac{7}{3}L^{-1}\left(\frac{\frac{3}{7}}{s^2+\frac{9}{49}}\right)\right]$$

$$= \frac{e^{-\frac{2}{7}t}}{49}\left[14L^{-1}\left(\frac{s}{s^2+\frac{9}{49}}\right)+14L^{-1}\left(\frac{\frac{3}{7}}{s^2+\frac{9}{49}}\right)\right]$$

$$= \frac{e^{-\frac{2}{7}t}}{49}\left[14\sin\frac{3}{7}t+14\cos\frac{3}{7}t\right]$$

$$= \frac{14}{49}e^{-\frac{2}{7}t}\left[\sin\frac{3}{7}t+\cos\frac{3}{7}t\right]$$

$$= \frac{2}{7}e^{-\frac{2}{7}t}\left[\sin\frac{3}{7}t+\cos\frac{3}{7}t\right]$$

EXERCISE: 4.7.2

1. **Find** $L^{-1}\left[\frac{s}{s^2a^2+b^2}\right]$ **Ans:** $\frac{1}{a^2}\cos\frac{b}{a}t$
2. **Find** $L^{-1}\left[\log\frac{s(s+1)}{s^2+1}\right]$ **Ans:** $\frac{1}{t}\left[2\cos t-e^{-t}-1\right]$
3. **Find** $L^{-1}\left[\frac{s}{\left(s^2-a^2\right)^2}\right]$ **Ans:** $\frac{t}{2a}\sinh at$
4. **Find** $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$ **Ans:** $\frac{1}{t}\sin t$

5. **Find** $L^{-1}\left[\dfrac{s}{(s+2)^2+4}\right]$ **Ans:** $e^{-2t}\left[\cos 2t-\sin 2t\right]$

6. **Find** $L^{-1}\left[\dfrac{s}{4s^2-25}\right]$ **Ans:** $\dfrac{1}{4}\cosh\dfrac{5}{2}t$

4.7.3 INVERSE LAPLACE TRANSFORMS USING CONVOLUTION THEOREM

Convolution:

If f(t) and g(t) are two function of t, then the convolution of f(t) and g(t) denoted by f * g, is defined by the relation

$$f*g=\int_0^t f(x)g(t-x)\,dx \qquad \& \qquad g*f=\int_0^t g(x)f(t-x)\,dx$$

$$\therefore f*g=g*f$$

Convolution Theorem on Laplace transforms.

If f(t) and g(t) are functions defined for $t\geq 0$, then $L\left[f(t)*g(t)\right]=L\left[f(t)\right].L[g(t)]$

and $L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$ where $F(s)=L[f(t)], G(s)=L[g(t)]$

Example 1

Using convolution theorem, find $L^{-1}\left[\dfrac{1}{s^3(s+1)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{S^3(S+1)}\right]=L^{-1}\left[\frac{1}{S^3}\right]*L^{-1}\left[\frac{1}{S+1}\right]$$

$$=\frac{t^2}{2}*e^{-t}$$

$$=\frac{1}{2}\int_0^t u^2e^{-(t-u)}du$$

$$=\frac{e^{-t}}{2}\int_0^t u^2e^{u}du$$

$$=\frac{e^{-t}}{2}\left[u^2e^u-2ue^u+2e^u\right]$$

$$=\frac{e^{-t}}{2}\left[\left(t^2e^t-2te^t+2e^t\right)-2\right]$$

$$L^{-1}\left[\frac{1}{s^3(s+1)}\right]=\frac{t^2}{2}-t+1-e^{-t}$$

Example 2

Using convolution theorem, find $L^{-1}\left[\dfrac{1}{s^2(s+1)^2}\right]$

Solution: $L^{-1}\left[\dfrac{1}{s^2(s+1)^2}\right]=L^{-1}\left[\dfrac{1}{s^2}\right]*L^{-1}\left[\dfrac{1}{(s+1)^2}\right]$

$=t*te^{-t}$

$$= \int_0^t u.(t-u)\, e^{-(t-u)}\, du$$

$$= \int_0^t ut du - \int_0^t u^2 e^{-(t-u)} du$$

$$= t\left[\frac{u^2}{2}\right]_0^t - e^{-t}\int_0^t u^2 e^u du$$

$$= t.\frac{t^2}{2} - e^{-t}\int_0^t u^2 de^u$$

$$= \frac{t^3}{2} - e^{-t}\left[u^2.e^u - 2ue^u + 2e^u\right]_0^t$$

$$= \frac{t^3}{2} - e^{-t}\left[e^u\left(u^2 - 2u + 2\right)\right]_0^t$$

$$= \frac{t^3}{2} - e^{-t}\left[e^t\left(t^2 - 2t + 2\right) - 2\right]$$

$$= \frac{t^3}{2} - \left(t^2 - 2t + 2\right) + 2e^{-t}$$

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = \frac{t^3}{2} - \left(t^2 - 2t + 2\right) + 2e^{-t}$$

Example 3

Using convolution theorem, find $\mathbf{L^{-1}\left[\frac{s^2}{\left(s^2+4\right)^2}\right]}$

Solution:

$$L^{-1}\left[\frac{s^2}{\left(s^2+4\right)^2}\right] = L^{-1}\left[\frac{s}{s^2+2^2}.\frac{s}{s^2+2^2}\right]$$

$$\because L^{-1}\left[F(s)\,G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$= L^{-1}\left[\frac{s}{s^2+2^2}\right] * L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= \cos 2t * \cos 2t \qquad \because \cos at = L^{-1}\left[\frac{s}{s^2+a^2}\right]$$

$$= \int_0^t \cos 2u\, \cos 2(t-u)\, du \qquad \because f(t)*g(t) = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t \cos 2u\, \cos\,(2t-2u)\, du \quad \because \cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$=\frac{1}{2}\int_0^t \left[\cos(2u+2t-2u)+\cos(2u-2t+2u)\right]du$$

$$=\frac{1}{2}\int_0^t \left[\cos(2t)+\cos(4u-2t)\right]du$$

$$=\frac{1}{2}\left[(\cos 2t)u+\frac{\sin(4u-2t)}{4}\right]_0^t$$

$$=\frac{1}{2}\left[\left(t\cos 2t+\frac{\sin 2t}{4}\right)-\left(0-\frac{\sin 2t}{4}\right)\right]$$

$$=\frac{1}{2}\left[t\cos 2t+\frac{\sin 2t}{4}+\frac{\sin 2t}{4}\right]$$

$$=\frac{1}{2}\left[t\cos 2t+\frac{\sin 2t}{2}\right]$$

$$=\frac{1}{4}\left[\sin 2t+2t\cos 2t\right]$$

$$\boxed{L^{-1}\left[\frac{s^2}{\left(s^2+4\right)^2}\right]=\frac{1}{4}\left[\sin 2t+2t\cos 2t\right]}$$

Example 4

Using Convolution theorem find the inverse Laplace transform of $\mathbf{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}}$

Solution: We know that $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]=L^{-1}\left[\dfrac{s}{s^2+a^2}\right]*L^{-1}\left[\dfrac{s}{s^2+b^2}\right]$

$$=\cos at*\cos bt \because L^{-1}\left[F(s)*G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$=\int_0^t \cos au\ \cos b(t-u)\,du \quad \because CosACosB=\frac{1}{2}\left\{\cos(A+B)+\cos(A-B)\right\}$$

$$=\int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2}du$$

$$=\frac{1}{2}\int_0^t \cos\left\{(a-b)u+bt)\right\}+\cos\left\{(a+b)u-bt)\right\}du$$

$$=\frac{1}{2}\left[\frac{\sin\left\{bt+(a-b)u\right\}}{a-b}+\frac{\sin\left\{(a+b)u-bt\right\}}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\frac{\sin(bt+at-bt)}{a-b}+\frac{\sin(at+bt-bt)}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin bt}{a+b}\right]$$

$$=\frac{1}{2}\left[\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2 - b^2}\right]$$

$$\boxed{L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}}$$

Example 5

Using Convolution theorem find the inverse Laplace transform of the function $\frac{s}{(s^2+1^2)^2}$.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+1^2)^2}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+1^2)}\frac{1}{(s^2+1^2)}\right] \qquad \because L^{-1}\left[F(s)*G(s)\right] = L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$= L^{-1}\left[\frac{s}{s^2+1^2}\right]*L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= \cos t * \sin t$$

$$= \int_0^t \cos u \sin(t-u)du$$

$$= \int_0^t \sin(t-u)\cos u\, du$$

$$= \frac{1}{2}\int_0^t \left[\sin t + \sin\left(t-2u\right)\right]du$$

$$= \frac{1}{2}\left[\left(\sin t\right)\left(u\right) - \frac{\cos\left(t-2u\right)}{-2}\right]_0^t$$

$$= \frac{1}{2}\left[\left(t\sin t + \frac{\cos(-t)}{2}\right) - \left(0 + \frac{\cos t}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(t\sin t + \frac{\cos(t)}{2} - \frac{\cos(t)}{2}\right)\right]$$

$$\boxed{L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2}\sin t}$$

Example 6

Using convolution theorem find inverse Laplace transform of $\frac{s^2+s}{\left(s^2+4\right)\left(s^2+2s+10\right)}$.

Solution: $L^{-1}\left[\frac{s^2+s}{\left(s^2+4\right)\left(s^2+2s+10\right)}\right]$

$$= L^{-1}\left[\frac{s+1}{s^2+2s+10}\cdot\frac{s}{s^2+2^2}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2+3^2}\cdot\frac{s}{s^2+2^2}\right]$$

$$= L^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} * L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= e^{-t}L^{-1}\left[\frac{s}{s^2+3^2}\right] * L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= (e^{-t}\cos 3t) * (\cos 2t)$$

$$= \int_0^t e^{-u}\cos 3u \cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}\left[\cos(u+2t)+\cos(5u-2t)\right]du$$

$$= \frac{1}{2}\left[\int_0^t e^{-u}\cos(u+2t)\,du + \int_0^t e^{-u}\cos(5u-2t)\,du\right]$$

$$= \frac{1}{2}\left\{\frac{e^{-u}}{1+1}\left[-\cos(u+2t)+\sin(u+2t)\right]\right\}_0^t$$

$$+\frac{1}{2}\left\{\frac{e^{-u}}{1+25}\left[-\cos(5u-2t)+5\sin(5u-2t)\right]\right\}_0^t$$

$$= \frac{1}{2}\left\{\frac{e^{-t}}{2}\left[-\cos(u+2t)+\sin(u+2t)\right]-\frac{1}{2}\left[-\cos 2t+\sin 2t\right]\right\}$$

$$+\frac{1}{2}\left\{\frac{e^{-t}}{26}\left[-\cos 3t+5\sin 3t\right]-\frac{1}{26}\left[-\cos 2t-5\sin 2t\right]\right\}$$

$$= \frac{1}{2}\left\{\begin{array}{l} -e^{-t}\left[\dfrac{-\cos 3t}{2}+\dfrac{\sin 3t}{2}-\dfrac{\cos 3t}{26}+\dfrac{5}{26}\sin 3t\right] \\ \qquad\qquad +\left[\dfrac{\cos 2t}{2}-\dfrac{\sin 2t}{2}+\dfrac{\cos 2t}{26}+\dfrac{5}{26}\sin 2t\right] \end{array}\right\}$$

$$= \frac{1}{2}\left\{\begin{array}{l} \dfrac{-e^{-t}}{26}\left[-13\cos 3t+13\sin 3t-\cos 3t+5\sin 3t\right] \\ \qquad\qquad +\dfrac{1}{26}\left[13\cos 2t-13\sin 2t+\cos 2t+5\sin 2t\right] \end{array}\right\}$$

$$= \frac{1}{52}\left\{\begin{array}{l} -e^{-t}\left[-13\cos 3t+13\sin 3t-\cos 3t+5\sin 3t\right] \\ \qquad\qquad +\left[13\cos 2t-13\sin 2t+\cos 2t+5\sin 2t\right] \end{array}\right\}$$

$$= \frac{1}{52}\left(1+e^{-t}\right)\left(13\cos 2t-13\sin 2t+\cos 2t+5\sin 3t\right)$$

Example 7

Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Solution:

We know that $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\frac{1}{(s^2+a^2)}\right] \qquad \because L^{-1}[F(s)*G(s)] = L^{-1}[F(s)]*L^{-1}[G(s)]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= \cos at * \frac{1}{a}\sin at \qquad \left[\because \cos at = L^{-1}\left[\frac{s}{s^2+a^2}\right],\ \sin at = L^{-1}\left[\frac{a}{s^2+a^2}\right]\right.$$

$$= \int_0^t \cos au \frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{a}\int_0^t \sin(at-au)\cos au\, du$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin\left(at-2au\right)\right]du$$

$$\left[\because \sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]\right.$$

$$= \frac{1}{2a}\left[\left(\sin at\right)\left(u\right) - \frac{\cos\left(at-2au\right)}{-2a}\right]_0^t$$

$$= \frac{1}{2a}\left[\left(t\sin at + \frac{\cos(-at)}{2a}\right) - \left(0+\frac{\cos at}{2a}\right)\right]$$

$$= \frac{1}{2a}\left[\left(t\sin at + \frac{\cos(at)}{2a} - \frac{\cos(at)}{2a}\right)\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

Example 8

Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\frac{1}{(s^2+a^2)}\right]$$

$$\left[\because L^{-1}[F(s)*G(s)] = L^{-1}[F(s)]*L^{-1}[G(s)]\right]$$

$$= L^{-1}\left[\frac{1}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{a}\sin at * \frac{1}{a}\sin at$$

$$= \frac{1}{a^2}\int_0^t \sin au \sin(at - au)du$$

$$\left[\because \sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]\right.$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(au - at + au) - \cos(au + at - au)\right]du$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2au - at) - \cos\left(at\right)\right]du$$

$$= \frac{1}{2a^2}\left[\frac{\sin\left(2au - at\right)}{2a} - (\cos at)u\right]_{u=0}^{u=t}$$

$$= \frac{1}{2a^2}\left[\left(\frac{\sin at}{2a} - t\cos at\right) - \left(-\frac{\sin at}{2a} - 0\right)\right]$$

$$= \frac{1}{2a^2}\left[\left(\frac{\sin at}{a} - t\cos at\right)\right]$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}\left[\sin at - at\cos at\right] \ldots (1)$$

EXERCISE: 4.7.3

1. **Using convolution theorem, find** $L^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]$ **Ans: 1 – cos t**
2. **Using convolution theorem, find** $L^{-1}\left[\frac{1}{s^2\left(s+5\right)}\right]$ **Ans:** $\frac{1}{25}\left[e^{-5t} + 5t - 1\right]$
3. **Using convolution theorem, find** $L^{-1}\left[\frac{1}{\left(s+a\right)\left(s+b\right)}\right]$ **Ans:** $\frac{1}{a-b}\left[e^{-bt} - e^{-at}\right]$

4.8 SOLVING LINEAR ODE OF SECOND ORDER USING LAPLACE TRANSFORM

$$L\left[y'(t)\right] = sL\left[y(t)\right] - y(0)$$

$$L\left[y''(t)\right] = s^2L\left[y(t)\right] - sy(0) - y'(0)$$

$$L\left[y'''(t)\right] = s^3L\left[y(t)\right] - s^2y(0) - sy'(0) - y''(0)$$

Example 1

Solve $(D^2 - D - 2)y = 20\sin 2t$ **given that y = -1 , Dy = 2 when t= 0 by using Laplace transform methods.**

Solution: The given differential equation can be written as

$y'' - y' - 2y = 20\sin 2t$

Taking Laplace Transform on both sides

$$L\left[y''(t)\right]-L\left[y'(t)\right]-2L\left[y(t)\right]=20L\left[\sin 2t\right]$$

$$[s^2L(y)-sy(0)-y'(0)]-[sL(y)-y(0)]-2\,L(y)=20\frac{2}{s^2+4}$$

Given condition $y(0)=-1, y'(0)=2$

$$[s^2L(y)-s(-1)-2]-[sL(y)+1]-2\,L(y)=20\frac{2}{s^2+4}$$

$$(s^2-s-2)\,L(y)+s-2-1=\frac{40}{s^2+4}$$

$$(s^2-s-2)\,L(y)=\frac{40}{s^2+4}-s+3$$

$$=\frac{40-s^3-4s+3s^2+12}{s^2+4}$$

$$L(y)=\frac{-s^3+3s^2-4s+52}{(s^2-s-2)(s^2+4)}$$

$$\frac{-s^3+3s^2-4s+52}{(s^2-s-2)(s^2+4)}=\frac{As+B}{(s^2-s-2)}+\frac{Cs+D}{(s^2+4)}$$

$$-s^3+3s^2-4s+52=As+B(s^2+4)+Cs+D(s^2-s-2)$$

Equating the coefficient of s^3 on both sides, we get

$-1=A+C$ ...(1)

Equating the coefficient of s^2 on both sides, we get

$3=B+D-C$ …..(2)

Equating the coefficient of s on both sides, we get

$-4=4A-D-2C$ …..(3)

Equating the constants on both sides, we get

$52 = 4B- 2D$ ....(4)

From (1), A = -C - 1

$(3)\Rightarrow 4(-C-1)-D-2C=-4$

$\Rightarrow -6C-D=0$

$\Rightarrow D=-6C$

$(2)\Rightarrow B-6C-C=3$

$\Rightarrow B-7C=3$ ...(5)

$(4)\Rightarrow 2B+6C=26$

$\Rightarrow B+3C=13$ ....(6)

On solving (5) and (6) we get, c =1

$(1)\Rightarrow A=-2$

$(3)\Rightarrow 4(-2)-D-2(1)=-4$

$\Rightarrow -8-D-2=-4$

$\Rightarrow D=-6$

$(4)\Rightarrow 2B+6=26$

$\Rightarrow 2B=20$

$\Rightarrow B=10$

$$L[y(t)]=\frac{-2s+10}{s^2-s-2}+\frac{s-6}{s^2+4}$$

$$=\frac{-2s+10}{\left(s-\frac{1}{2}\right)^2-\frac{9}{4}}+\frac{s-6}{s^2+4}$$

$$=\frac{-2\left(s-\frac{1}{2}\right)+9}{\left(s-\frac{1}{2}\right)^2-\frac{9}{4}}+\frac{s-6}{s^2+4}-\frac{6}{s^2+4}$$

$$y(t)=L^{-1}\left(\frac{-2\left(s-\frac{1}{2}\right)+9}{\left(s-\frac{1}{2}\right)^2-\frac{9}{4}}\right)+L^{-1}\left(\frac{s}{s^2+4}\right)-L^{-1}\left(\frac{6}{s^2+4}\right)$$

$$=e^{\frac{1}{2}t}L^{-1}\left[\frac{-2s+9}{s^2-\frac{9}{4}}\right]+\cos 2t-3\sin 2t$$

$$=e^{\frac{1}{2}t}\left[L^{-1}\left[\frac{-2s}{s^2-\frac{9}{4}}\right]+L^{-1}\left[\frac{9}{s^2-\frac{9}{4}}\right]\right]+\cos 2t-3\sin 2t$$

$$=-2e^{\frac{1}{2}t}L^{-1}\left[\frac{s}{s^2-\frac{9}{4}}\right]+9.\frac{2}{3}e^{\frac{1}{2}t}L^{-1}\left[\frac{\frac{3}{2}}{s^2-\frac{9}{4}}\right]+\cos 2t-3\sin 2t$$

$$y(t)=e^{\frac{1}{2}t}\left[-2\cosh\frac{3}{2}t+6\sinh\frac{3}{2}t\right]-\cos 2t-3\sin 2t$$

Example 2

Solve $\mathbf{y''-6y'+9y=t^2e^{3t}, y(0)=2, y'(0)=6}$ by Laplace transform methods

Solution:

$y''-6y'+9y=t^2e^{3t}$, given $y(0)=2, y'(0)=6$

Taking Laplace transform on both sides

$$L[y''(t)]-6L[y'(t)]+9L[y(t)]=L\left[t^2e^{3t}\right]$$

$$\left\{s^2L[y(t)]-sy(0)-y'(0)\right\}-6\left\{sL[y(t)]-y(0)\right\}+9L[y(t)]=L\left[t^2\right]_{s\to s-3}$$

$$\because L\left[t^n\right]=\frac{n!}{s^{n+1}}$$

$$\left\{s^2L[y(t)]-2s-6\right\}-6\left\{sL[y(t)]-2\right\}+9L[y(t)]=\left[\frac{2}{s^3}\right]_{s\to s-3}$$

$$s^2L[y(t)]-2s-6-6sL[y(t)]+12+9L[y(t)]=\frac{2}{(s-3)^3}$$

$$L[y(t)][s^2-6s+9]=\frac{2}{(s-3)^3}+2s+6-12$$

$$L[y(t)](s-3)^2=\frac{2}{(s-3)^3}+2s-6$$

$$=\frac{2}{(s-3)^5}+\frac{2s-6}{(s-3)^2}$$

$$=\frac{2}{(s-3)^5}+\frac{2(s-3)}{(s-3)^2}$$

$$=\frac{2}{(s-3)^5}+\frac{2}{(s-3)}$$

$$y(t)=2L^{-1}\left[\frac{1}{(s-3)^5}\right]+2L^{-1}\left[\frac{1}{(s-3)}\right]\quad \because e^{at}f(t)=L^{-1}\left[F(s-a)\right]$$

$$y(t)=2e^{3t}L^{-1}\left[\frac{1}{s^5}\right]+2e^{3t}L^{-1}\left[\frac{1}{s}\right]\qquad \because L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$$

$$=2e^{3t}\frac{t^4}{4!}+2e^{3t}(1)$$

$$\boxed{y(t)=e^{3t}\frac{t^4}{12}+2e^{3t}}$$

Example 3

Using Laplace transform solve $y''+2y'-3y=3, y(0)=4, y'(0)=-7$.

Solution:

Given: $y''(t)+2y'(t)-3y(t)=3$

$$L[y''(t)]+2L[y'(t)]-3L[y(t)]=L[3]$$

$$\left[s^2L[y(t)]-s\,y(0)-y'(0)\right]+2\left[s\,L[y(t)]-y(0)\right]-3L[y(t)]=3L[1]\because L[1]=\frac{1}{s}$$

$$\left[s^2L[y(t)]-s(4)+7\right]+2\left[s\,L[y(t)]-4\right]-3L[y(t)]=3\frac{1}{s}$$

$$(s^2+2s-3)\,L[y(t)]-4s-1=\frac{3}{s}$$

$$(s^2+2s-3)\,L[y(t)]=\frac{3}{s}+4s+1$$

$$(s^2+2s-3)\;L[y(t)]=\frac{3+4s^2+s}{s}$$

$$L[y(t)] = \frac{3+4s^2+s}{s(s^2+2s-3)}$$

$$= \frac{4s^2+s+3}{s(s-1)(s+3)} \quad ...(1)$$

$$\frac{4s^2+s+3}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s+3)}$$

$$4s^2+s+3 = A(s-1)(s+3) + Bs\ (s+3) + Cs(s-1)$$

Put s = 0 we get

3 = A (-1) (3)

A = -1

Put s = 1 we get

4+1+3 = B (1) (1+3)

8 = B (1) (4)

B = 2

Put s = -3 we get

4 (9) – 3+3 = C(-3) (-3-1)

36 = C (12)

C = 3

$$\therefore (1) \Rightarrow L[y(t)] = \frac{-1}{s} + \frac{2}{s-1} + \frac{3}{s+3} \quad y(t) = L^{-1}\left[\frac{-1}{s}\right] + L^{-1}\left[\frac{2}{s-1}\right] + L^{-1}\left[\frac{3}{s+3}\right] \qquad \because L^{-1}\left[\frac{1}{s}\right] = 1$$

$$= -L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{s+3}\right]$$

$$\left[\because L^{-1}\left[\frac{1}{s-a}\right] = e^{at}, L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}\right. \qquad = -1 + 2e^{t} + 3e^{-3t}$$

$$\boxed{\mathbf{y(t) = -1 + 2e^{t} + 3e^{-3t}}}$$

Example 4

Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \sin t, \text{if } \frac{dy}{dt} = 0$ and y = 2 when t = 0 using Laplace Transforms.

Solution:

Given: $y'' + 4y' + 4y = \sin t$

Taking Laplace transforms on both sides

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[\sin t]$$

$$s^2 L[y(t)] - s\,y(0) - y'(0) + 4[s\,L[y(t)] - y(0)]] + 4L[y(t)] = L[\sin t]$$

$$s^2 L[y(t)] - s(2) - 0 + 4[s\,L[y(t)] - 2]] + 4L[y(t)] = L[\sin t]$$

$$(s^2+4s+4)\,L[y(t)] - 2s - 8 = \frac{1}{s^2+1} \qquad \because L[\sin at] = \frac{a}{s^2+a^2}$$

$$(s^2+4s+4)\,L[y(t)] = \frac{1}{s^2+1} + 2s + 8$$

$$(s^2+4s+4)\,L[y(t)] = \frac{1+2s(s^2+1)+8(s^2+1)}{s^2+1}$$

$$(s^2+4s+4)\,L[y(t)] = \frac{1+2s^3+2s+8s^2+8}{s^2+1}$$

$$L[y(t)] = \frac{2s^3+8s^2+2s+9}{(s^2+1)(s^2+4s+4)}$$

$$= \frac{2s^3+8s^2+2s+9}{(s+2)^2(s^2+1)}$$

$$\frac{2s^3+8s^2+2s+9}{(s+2)^2(s^2+1)} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{Cs+D}{(s^2+1)} \qquad ...(I)$$

$$2s^3+8s^2+2s+9 = A(s+2)(s^2+1) + B(s^2+1) + (Cs+D)(s+2)^2$$

Put s = -2 we get

2 (-8) + 8 (4) + 2 (-2) + 9 = B (4+1)

-16+32-4+9 = 5B

21 = 5B $\qquad B = 21/5$

Put s = 0 we get

9 = A (2) (1) + B (1) + D (4)

9 = 2 A + B + 4 D

9 = 2 A + 21/5 + 4 D

9 -21/5 = 2 A + 4 D

2 A + 4 D = 24/5

A + 2 D = 12/5 …… (1)

Equating coefficient of s^3 on both sides we get

A + C =2 … (2)

Equating coefficient of s^2 on both sides we get

2A + B+ 4C +D = 8

2A + 21/5+ 4 (2-A)+D = 8

2A + 21/5+ 8-4A+D = 8

-2A + 21/5+D = 8-8

-2A +D = - 21/5 … (3)

(1) $\Rightarrow$ A + 2 D = 12/5 … (4)

(3) × 2 $\Rightarrow$ -4A +2D = - 42/5 …(5)

(4) – (5) $\rightarrow$ 5A = 54/5 $\qquad A = 54/25$

Substitute A = 54/25 in (1) we get

(54/25) +2 D = 12/5

$$2D = \frac{12}{5} - \frac{54}{25} = \frac{60-54}{25} = \frac{6}{25} \qquad D = \frac{3}{25}$$

Substitute A = 54/25 in (2) we get

54/25 + C = 2

C = 2 - 54/25 = -4/25 $\qquad C = -\frac{4}{25}$

$$(1) \Rightarrow L[y(t)] = \frac{54}{25(s+2)} + \frac{21}{5(s+2)^2} + \frac{-4s}{25(s^2+1)} + \frac{3}{25(s^2+1)}$$

$$y(t)=L^{-1}\left[\frac{54}{25(s+2)}\right]+L^{-1}\left[\frac{21}{5(s+2)^2}\right]$$

$$+L^{-1}\left[\frac{-4s}{25(s^2+1)}\right]+L^{-1}\left[\frac{3}{25(s^2+1)}\right]$$

$$=\frac{54}{25}e^{-2t}+\frac{21}{5}t\,e^{-2t}-\frac{4}{25}\cos t+\frac{3}{25}\sin t$$

$$\left[\begin{array}{l}\because e^{-at}=L^{-1}\left[\frac{1}{s+a}\right],\cos at=L^{-1}\left[\frac{s}{s^2+a^2}\right],\\ \sin at=L^{-1}\left[\frac{a}{s^2+a^2}\right],L^{-1}\left[F(s+a)\right]=e^{-at}L^{-1}[F(s)]\end{array}\right.$$

$$\boxed{\mathbf{y(t)=\frac{1}{25}\left[54e^{-2t}+105\,t\,e^{-2t}-4\cos t+3\sin t\right]}}$$

Example 5

Using Laplace Transform solve the differential equation $\mathbf{\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{-t}}$ **with** $y(0)=1$ **and** $y'(0)=0$.

Solution

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{-t}\ with\ y(0)=1 and\ y'(0)=0$$

$$ie\ y''-3y'+2y=e^{-t}$$

Take laplace transform on b.s

$$L[y''(t)]-3L[y'(t)]+2L[y(t)]=L[e^{-t}]\qquad \because L[e^{-at}]=\frac{1}{s+a}$$

$$\left[s^2L[y(t)]-s\,y(0)-y'(0)\right]-3\left[sL[y(t)]-y(0)\right]+2L[y(t)]=\frac{1}{s+1}$$

$$s^2L[y(t)]-s(1)-0-3sL[y(t)]+3(1)+2L[y(t)]=\frac{1}{s+1}$$

$$L[y(t)]\left[s^2-3s+2\right]-s+3=\frac{1}{s+1}$$

$$L[y(t)]\left[(s-1)(s-2)\right]=\frac{1}{s+1}+s-3$$

$$L[y(t)]=\frac{1}{(s+1)(s-1)(s-2)}+\frac{s-3}{(s-1)(s-2)}$$

$$=\frac{1+(s+1)(s-3)}{(s+1)(s-1)(s-2)}$$

$$=\frac{1+s^2-3s+s-3}{(s+1)(s-1)(s-2)}$$

$$=\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}$$

$$y(t)=L^{-1}\left[\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}\right]$$

By Partial fraction method

$$\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}=\frac{A}{(s+1)}+\frac{B}{(s-1)}+\frac{C}{(s-2)}\to(1)$$

$$\Rightarrow s^2-2s-2=A(s-1)(s-2)+B(s+1)(s-2)+C(s+1)(s-1)$$

$Put\ s=1, we\ get$

$$1-2-2=A(0)+B(2)(-1)+C(0)$$

$$-3=-2B$$

$$\therefore B=\frac{3}{2}$$

$Put\ s=2, we\ get$

$$4-4-2=C(3)(1)$$

$$-2=3C$$

$$\therefore C=\frac{-2}{3}$$

$Put\ s=-1, we\ get$

$$1+2-2=A(-2)(-3)$$

$$1=6A$$

$$\therefore A=\frac{1}{6}$$

$Sub\ the\ values\ of\ A,B,C\ in\ (1)$

$$\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}=\frac{\frac{1}{6}}{(s+1)}+\frac{\frac{3}{2}}{(s-1)}+\frac{\frac{-2}{3}}{(s-2)}$$

$$L^{-1}\left[\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}\right]=L^{-1}\left[\frac{\frac{1}{6}}{(s+1)}+\frac{\frac{3}{2}}{(s-1)}+\frac{\frac{-2}{3}}{(s-2)}\right]$$

$$=\frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right]+\frac{3}{2}L^{-1}\left[\frac{1}{(s-1)}\right]-\frac{2}{3}L^{-1}\left[\frac{1}{(s-2)}\right]\qquad \because e^{-at}=L^{-1}\left[\frac{1}{s+a}\right]$$

$$\boxed{\mathrm{y}(\mathrm{t})=\frac{1}{6}e^{-t}+\frac{3}{2}e^{t}-\frac{2}{3}e^{2t}}$$

Example 6

Solve $y''+4y'+3y=e^{-t}$, $y(0)=1$, $y'(0)=1$.

Solution:

$$y''+4y'+3y=e^{-t}$$

$$L[y''(t)]+4L[y'(t)]+3L[y(t)]=L[e^{-t}]$$

$$[s^2L[y(t)]-\mathrm{sy}(0)-\mathrm{y}'(0)]+4[\mathrm{sL}(\mathrm{y})-\mathrm{y}(0)]+3\,\mathrm{L}[\mathrm{y}(\mathrm{t})]=\frac{1}{s+1}$$

$$(s^2+4s+3)L[y(t)]=\frac{1}{s+1}+s+5$$

$$(s^2+4s+3)L[y(t)]=\frac{s^2+6s+6}{s+1}$$

$$L(y(t))=\frac{s^2+6s+6}{(s+1)(s^2+4s+3)}$$

$$L[y(t)]=\frac{s^2+6s+6}{(s+1)^2(s+3)}$$

$$\frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s+3}$$

$$s^2+6s+6=A(s+3)(s+1)+B(s+3)+C(s+1)^2$$

Putting s=-1, we get $B=\frac{1}{2}$

Putting s=-3, $C=-\frac{4}{3}$

Putting s = 0, $\frac{5}{6}=3A\Rightarrow A=\frac{5}{18}$

$$[y(t)]=\frac{5}{18}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right]-\frac{4}{3}L^{-1}\left[\frac{1}{s+3}\right]$$

$$[y(t)]=\frac{5}{18}e^{-t}+\frac{1}{2}te^{-t}-\frac{4}{3}e^{-3t}$$

Example 7

Solve $y''-3y'+2y=1$**, given that** $y(0)=0$**,** $y'(0)=1$ **by using Laplace Transform method**

Solution:

Given $y''-3y'+2y=1, y(0)=0, y'(0)=1$

$$L[y''(t)]-3L[y'(t)]+2Ly(t)=L(1)$$

$$[s^2L[y(t)]-sy(0)-y'(0)]-3[sL(y(t)]-y(0)]+2L[y(t)]=\frac{1}{s}$$

$$[s^2L[y(t)]-1]-3s[L(y(t)]+2L[y(t)]=\frac{1}{s}$$

$$[3s^2-3s+2]L[y(t)]-1=\frac{1}{s}$$

$$[3s^2-3s+2]L[y(t)]=1+\frac{1}{s}=\frac{s+1}{s}$$

$$L[y(t)]=\frac{s+1}{s(s^2-3s+2)}=\frac{s+1}{s(s-1)(s-2)}$$

$$=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2} \qquad \text{....... (1)}$$

$$s+1=A(s-1)(s-2)+Bs(s-2)+Cs(s-1)$$

Put $s=0$, we get $1=2A$ $\Rightarrow$ A=$\frac{1}{2}$

Put $s=1$, we get $2=-B$ $\Rightarrow$ B=-2

Put $s=2$, we get $3=2C$ $\Rightarrow$ A=$\frac{3}{2}$

$$(1) \Rightarrow L[y(t)] = \frac{1/2}{s} + \frac{(-2)}{s-1} + \frac{3/2}{s-2}$$

$$\Rightarrow \quad y(t)=\frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s-1}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{2}(1) - 2e^{t} + \frac{3}{2}e^{2t}$$

$$y(t) = \frac{1}{2}[1-4e^{t}+3e^{2t}]$$

Example 8

Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$ given $x=0$, $\frac{dx}{dt}=5$ for $t=0$ using Laplace transform method.

Solution:

Given $x''(t) - 3x'(t) + 2x(t) = 2,\ x(0)=0 \text{ and } x'(0)=5$

Take L.T on both sides

$$L\left[x''(t)\right] - 3L\left[x'(t)\right] + 2L\left[x(t)\right] = L(2)$$

$$s^2L\left[x(t)\right] - sx(0) - x'(0) - 3sL\left[x(t)\right] + 3x(0) + 2L\left[x(t)\right] = \frac{2}{s}$$

$$L\left[x(t)\right](s^2-3s+2) - 5 = \frac{2}{s}$$

$$L\left[x(t)\right] = \frac{2+5s}{s(s^2-3s+2)}$$

$$\left[x(t)\right] = L^{-1}\left[\frac{5s+2}{s(s-1)(s-2)}\right]\text{.................}(1)$$

$$\frac{5s+2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}\text{.............}(2)$$

$$5s+2 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

$Put\ s=1,\ B=-7$

$Put\ s=0,\ A=1$

$Put\ s=2,\ c=6$

$$(2) \Rightarrow \frac{5s+2}{s(s-1)(s-2)} = \frac{1}{s} + \frac{-7}{s-1} + \frac{6}{s-2}$$

$$(1) \Rightarrow \left[x(t)\right] = L^{-1}\left[\frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}\right]$$

$$\boxed{x(t) = 1 - 7e^{t} + 6e^{2t}.}$$

Example 9

Solve by Laplace transform $\mathbf{\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=6t^2e^{-3t}, y(0)=0, y^{'}(0)=0}$ **.**

Solution:

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=6t^2e^{-3t} \quad y(0)=0, y^{'}(0)=0$$

$$ie\ y^{"}(t)+6y^{'}(t)+9y(t)=6t^2e^{-3t}$$

Taking Laplace transform on both sides

$$L[y^{"}(t)]+L[6y^{'}(t)]+9L[y(t)]=6L[t^2e^{-3t}]$$

$$\left[s^2L[y(t)]-s\,y(0)-y^{'}(0)\right]+6\left[sL[y(t)]-y(0)\right]+9L[y(t)]=6\left(\frac{2}{s^3}\right)_{s\to s+3}$$

$$\left[\because L\left[t^n\right]=\frac{n!}{s^{n+1}}\right.$$

$$s^2L[y(t)]+6sL[y(t)]+9L[y(t)]=\left(\frac{12}{s^3}\right)_{s\to s+3}$$

$$L[y(t)]\left[s^2+6s+9\right]=\frac{12}{(s+3)^3}$$

$$L[y(t)]=\frac{12}{\left[s^2+6s+9\right](s+3)^3}$$

$$=\frac{12}{(s+3)^3(s+3)^2}$$

$$L[y(t)]=\frac{12}{(s+3)^5}$$

$$y(t)=L^{-1}\frac{12}{(s+3)^5}$$

$$=12L^{-1}\frac{1}{(s+3)^5} \qquad \because L^{-1}\left[F(s+a)\right]=e^{-at}L^{-1}[F(S)]$$

$$=12e^{-3t}L^{-1}\left[\frac{1}{(s)^5}\right] \qquad \because L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$$

$$=12e^{-3t}\left(\frac{t^4}{4!}\right)=12e^{-3t}\left(\frac{t^4}{24}\right)$$

$$=e^{-3t}\left(\frac{t^4}{2}\right)$$

$$\boxed{\therefore y(t)=\frac{1}{2}e^{-3t}t^4}$$

Example 10

Using Laplace transform solve $y''+y'=t^2+2t$**,** $y(0)=4$ **and** $y'(0)=-2$**.**

Solution:

$$y'' + y' = t^2 + 2t,\ y(0) = 4 \text{ and } y'(0) = -2$$

$$L[y''(t)] + L[y'(t)] = L[t^2] + 2L[t]$$

$$\left\{s^2 L[y(t) - sy(0) - y'(0)\right\} + sL[y(t)] - y(0) = \frac{2!}{s^3} + 2\left(\frac{1}{s^2}\right)$$

$$s^2 L[y(t)] - 4s + 2 + sL[y(t)] - 4 = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L[y(t)][s^2 + s] = \frac{2}{s^3} + \frac{2}{s^2} + 4s - 2 + 4$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2$$

$$= \frac{2 + 2s + 4s^4 + 2s^3}{s^3}$$

$$\Rightarrow L[y(t)] = \frac{2 + 2s + 2s^3 + 4s^4}{s^3(s^2 + s)}$$

$$= \frac{2 + 2s + 2s^3 + 4s^4}{s^4(s+1)}$$

$$= \frac{2 + 2s}{s^4(s+1)} + \frac{2s^3}{s^4(s+1)} + \frac{4s^4}{s^4(s+1)}$$

$$= \frac{2(s+1)}{s^4(s+1)} + \frac{2}{s(s+1)} + \frac{4}{(s+1)}$$

$$= \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s(s+1)}$$

$$= 2L^{-1}\left[\frac{1}{s^4}\right] + 4L^{-1}\left[\frac{1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s(s+1)}\right]$$

$$= 2\left(\frac{t^3}{3!}\right) + 4e^{-t}(1) + 2\int_0^t L^{-1}\left[\frac{1}{s+1}\right]dt$$

$$= 2\left(\frac{t^3}{3!}\right) + 4e^{-t} + 2\int_0^t e^{-t}dt$$

$$= \frac{t^3}{3} + 4e^{-t} + 2\left[\frac{e^{-t}}{-1}\right]_0^t$$

$$= \frac{t^3}{3} + 4e^{-t} - 2\left[e^{-t} - 1\right]$$

$$= \frac{t^3}{3} + 4e^{-t} - 2e^{-t} + 2$$

$$\boxed{L[y(t)] = \frac{t^3}{3} + 2e^{-t} + 2}$$

EXERCISE 4.8

1. **Solve the differential equation $y'' - 3y' + 2y = 4t + e^{3t}$, where y(0) = 1 and $y'(0) = -1$ using Laplace transforms**
2. **Using Laplace Transform solve the differential equation $\frac{d^2x}{dt^2} + 9x = t + \frac{1}{2}$, given $x(0) = \frac{1}{18}$ and $x'(0) = \frac{1}{9}$.**
3. **Using Laplace Transform solve the differential equation $(D^2 + 3D + 2)y = e^{-3t}$ with $y(0) = 1$ and $y'(0) = -1$.**
4. **Solve $y'' + 9y = \cos 2t, y(0) = 1 \,\&\, y(\pi/2) = -1$ by using Laplace Transform**
5. **Solve by Laplace transform $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}, y(0) = 0, y'(0) = 0$**