① INITIAL VELOCITY → ZERO

A tightly stretched string with fixed end point $x=0$ & $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l-x)$. Find the displacement.

Sol:

The wave equation is $\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2}$

The solution is $y = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$ → ①

We have the Boundary conditions (BC)

(1) $y=0$ when $x=0$ ie) $y(0,t)=0$

(2) $y=0$ when $x=l$ ie) $y(l,t)=0$

(3) $\frac{\partial y}{\partial t}=0$ when $t=0$ (Initial Velocity is zero) ie) $\frac{\partial y}{\partial t}(x,0)=0$

(4) $y=3x(l-x)$ when $t=0$ ie) $y(x,0)=3x(l-x)$ when $t=0$.

Applying BC (1) in equation ① ie) Put $y=0$ when $x=0$ in ①

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$\Rightarrow 0 = [A(1) + B(0)][C\cos pat + D\sin pat]$$

$$\Rightarrow A(C\cos pat + D\sin pat) = 0$$

$$\Rightarrow C\cos pat + D\sin pat \neq 0, \quad \boxed{A=0}$$

∴ Sol ① becomes [Put $A=0$ in ①]

$$y = B\sin px\,(C\cos pat + D\sin pat) \rightarrow ②$$

Applying BC (2) in equation ② ie) (Put $y=0$ when $x=l$) in ②

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

$$\Rightarrow C\cos pat + D\sin pat \neq 0 \quad \& \quad B\sin pl = 0$$

$$\Rightarrow B \neq 0, \quad \sin pl = 0$$

∴ $\sin pl = 0 = \sin n\pi$

$$\Rightarrow pl = n\pi \quad \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub $P = \frac{n\pi}{L}$ in eq ②

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{L}\right) + D\sin\left(\frac{n\pi at}{L}\right)\right] \rightarrow ③$$

Diff eq ③ w.r.to t (partially)

$$\frac{\partial y}{\partial t} = B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi at}{L}\right)\times\left(\frac{n\pi a}{l}\right) + D\cos\left(\frac{n\pi at}{L}\right)\times\left(\frac{n\pi a}{L}\right)\right]$$

Now apply BC ③ in the above equation ies $\frac{\partial y}{\partial t} = 0$ at $t = 0$

$$0 = B\sin\left(\frac{n\pi x}{L}\right)\left[-C\sin(0)\times\left(\frac{n\pi a}{L}\right) + D\cos 0\times\left(\frac{n\pi a}{L}\right)\right]$$

$$\Rightarrow \quad 0 = B\sin\left(\frac{n\pi x}{L}\right)\left[0 + D\left(\frac{n\pi a}{L}\right)\right]$$

$$\Rightarrow \quad \underset{\substack{\downarrow \\ \text{already said}}}{B \neq 0} \quad \sin\left(\frac{n\pi x}{l}\right) \neq 0, \quad \frac{n\pi a}{l} \neq 0 \quad \therefore \boxed{D = 0}$$

$\therefore$ Sub $D=0$ in eq ③

$$y = B\sin\left(\frac{n\pi x}{l}\right)\times C\cos\left(\frac{n\pi at}{L}\right)$$

$$y = BC\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

$$y = b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right) \quad (BC = b_n)$$

$\therefore$ The General Solution is

$$y = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \longrightarrow ④$$

Now apply BC ④ in equation ④ ies $y = 3x(l-x)$ when $t=0$

$$3x(l-x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\cos 0. \quad \Rightarrow f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

RHS is Half range sine series in $(0,l)$

$$\therefore \quad b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l 3x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{6}{l}\int_0^l (lx-x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$u = lx - x^2$ $\qquad$ $v = \sin\frac{n\pi x}{l}$

$u' = l - 2x$ $\qquad$ $v_1 = \dfrac{-\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}$

$u'' = -2$

$v_2 = \dfrac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3$

$v_3 = \dfrac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$.

(i)

$$b_n = \frac{6}{l}\left[\frac{-(lx-x^2)\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)} + (l-2x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2} - 2\frac{\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{6}{l}\left[\left\{0+0-\frac{2\cos\left(\frac{n\pi \cancel{l}}{\cancel{l}}\right)}{\left(\frac{n\pi}{l}\right)^3}\right\} - \left\{0+0-\frac{2\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right\}\right]$$

$$= \frac{6}{l}\left[\frac{-2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} + \frac{2\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{-12}{\cancel{l}}\left[\{\cos n\pi - \cos 0\}\times\frac{l^{\cancel{3}}}{n^3\pi^3}\right] = -12\left[\{(-1)^n - 1\}\frac{l^2}{n^3\pi^3}\right] = \frac{-12l^2}{n^3\pi^3}\left[(-1)^n-1\right]$$

$\therefore\ b_n = \dfrac{-12l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$.

When n is even, $b_n = \left[(-1)^2 - 1\right]\left(\dfrac{-12l^2}{n^3\pi^3}\right) = 0$. $(n=2)$

When n is odd $\quad b_n = -\dfrac{12l^2}{n^3\pi^3}\left[-1-1\right] = \dfrac{-12l^2}{n^3\pi^3}(-2) = \dfrac{24l^2}{n^3\pi^3}$.

$\therefore$ The final solution is (ie) The displacement is given by)

$$y = \sum_{n=1,3,5\ldots}^{\infty} \frac{24l^2}{n^3\pi^3}\sin\left(\frac{n\pi x}{l}\right).\cos\left(\frac{n\pi a t}{l}\right) \quad //$$

[Another way of Question]

A string of length L is fixed at both ends $x=0$ and $x=l$, It starts vibrating from the position $y=\lambda x(l-x)$ at $t=0$ while the initial velocity is zero. Find the displacement function at any time t and at any distance x.

Here $\boxed{\lambda = 3}$

(2) A tightly stretched string of length l has its end fastened at $x=0$, $x=l$. At $t=0$, the string is in the form $f(x) = Kx(l-x)$ and then released. Find the displacement at any point on the string at a distance x from one end and at any time $t>0$.

[another way of the same question.]

A tightly stretched string with fixed end points $x=0$ and $x=L$ is initially in a position given by $y(x,0) = K(lx - x^2)$. If it is released from rest from this position, find the displacement y at any time and at any distance from the end $x=0$

$\left[\begin{array}{l} f(x) = Kx(l-x) \\ y(x,0) = K(lx-x^2) \\ 3x(l-x) \end{array}\right]$

Type:2 Trignometric Function (initial velocity $=0$)

(3) A uniform string is stretched and fastend to two points l apart. Motion is started by displacing the string into the form of the curve $y = K\sin^3\left(\frac{\pi x}{l}\right)$ and then releasing it from this position at time $t=0$. Find the displacement of the point of the string at a distance from one end at time t.

Sol: The wave equation is $\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2}$

The solution is $y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow (1)$

Boundary conditions are

(1) $y=0$ when $x=0$ i.e. $y(0,t)=0$

(2) $y=0$ when $x=l$ i.e. $y(l,t)=0$

(3) $\frac{\partial y}{\partial t}=0$ when $t=0$ i.e. $\frac{\partial y}{\partial t}(x,0)=0$ (IV=0)

(4) $y = K\sin^3\left(\frac{\pi x}{l}\right)$ when $t=0$ i.e. $y(x,0) = K\sin^3\left(\frac{\pi x}{l}\right)$ when $t=0$, $0 \le x \le l$

Applying Boundary condition (1) in eq (1) i.e. put $y=0$ when $x=0$ in (1)

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$\Rightarrow 0 = (A(1) + 0)(C\cos pat + D\sin pat)$$

$$\Rightarrow A(C\cos pat + D\sin pat) = 0$$

$$\Rightarrow (C\cos pat + D\sin pat) \neq 0, \quad \boxed{A=0}$$

$\therefore$ Sol. (1) becomes [Put $A=0$ in (1)]

$$y = B\sin px\,(C\cos pat + D\sin pat) \longrightarrow (2)$$

Applying Boundary condition (2) in eq (2) i.e. (Put $y=0$ when $x=l$) eq (2)

$$\Rightarrow 0 = B\sin pl\,(C\cos pat + D\sin pat)$$

$$\Rightarrow C\cos pat + D\sin pat \neq 0 \;\&\; B\sin pl = 0$$

$$\Rightarrow B \neq 0 \quad \sin pl = 0$$

$\sin pl = 0 = \sin n\pi$

$\Rightarrow pl = n\pi$

$\Rightarrow \boxed{p = \frac{n\pi}{l}}$

Put $p = \frac{n\pi}{l}$ in eq ②

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \rightarrow ③$$

Diff. Partially eq ③ w.r.to t

$$\frac{\partial y}{\partial t} = B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi at}{l}\right)\times\left(\frac{n\pi a}{l}\right) + D\cos\left(\frac{n\pi at}{l}\right)\times\left(\frac{n\pi a}{l}\right)\right]$$

Now applying Boundary condition ③ in the above equation

i.e. $\frac{\partial y}{\partial t} = 0$ at $t = 0$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin(0)\times\left(\frac{n\pi a}{l}\right) + D\cos(0)\times\left(\frac{n\pi a}{l}\right)\right]$$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\left[0 + D\left(\frac{n\pi a}{l}\right)\right]$$

$$0 = BD\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$\Rightarrow B \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$ $\quad \frac{n\pi a}{l} \neq 0$ $\quad \therefore \boxed{D = 0}$

$\therefore$ Sub $D = 0$ in eq ③

$$\therefore y = B\sin\left(\frac{n\pi x}{l}\right) \times C\cos\left(\frac{n\pi at}{l}\right)$$

$$y = BC\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

$$y = b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \qquad BC = b_n$$

$\therefore$ The general solution is

$$y = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \longrightarrow ④$$

Now we apply boundary condition (A) in eq (H)

i.e. $y = K \sin^3\left(\frac{\pi x}{l}\right)$ when $t=0$

$$\therefore \quad K \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos 0 .$$

$$\Rightarrow K \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$\Rightarrow \sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$

$\Rightarrow \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$\therefore \sin^3\left(\frac{n\pi x}{l}\right) = \frac{3}{4}\sin\left(\frac{n\pi x}{l}\right) - \frac{1}{4}\sin\left(\frac{3n\pi x}{l}\right)$

$$\Rightarrow K\left[\frac{3}{4}\sin\left(\frac{n\pi x}{l}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{l}\right)\right] = b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + b_3 \sin\left(\frac{3\pi x}{l}\right)$$

Equating the like co-efficients

$$b_1 = \frac{3K}{4}, \quad b_2 = 0, \quad b_3 = -\frac{K}{4}, \quad b_4 = 0 \quad \dots$$

$\therefore$ Sol. is

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) \qquad \text{Eq (4)}$$

$$= b_1 \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi at}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)\cos\left(\frac{2\pi at}{l}\right) + b_3 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi at}{l}\right) + \dots$$

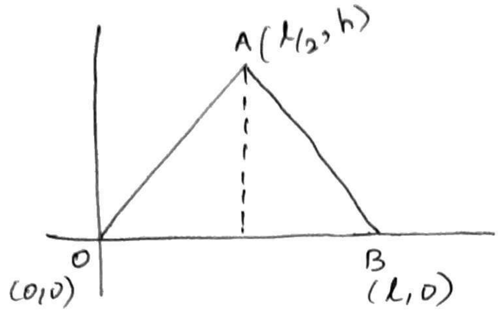
$$= \frac{3K}{4}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi at}{l}\right) + 0 + \left(-\frac{K}{4}\right)\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi at}{l}\right)$$

$$= \frac{3K}{4}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi at}{l}\right) - \frac{K}{4}\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi at}{l}\right) \quad //$$

(4) (mid point)

A string of length l is fixed at the ends $x=0$ and $x=l$. If the midpoint of it is displaced to a small height h and in this position the string starts vibrating, find the displacement function at any time t and at any distance x from one end.

Sol:



The initial position of the string is OAB whose equation can be found by the equation of OA and AB seperately.

Equation of a line joining pts (x_1,y_1) & (x_2,y_2) is $\dfrac{y-y_1}{y_1-y_2} = \dfrac{x-x_1}{x_1-x_2}$

Equation of OA $(0,0)$ $(l/2,h)$ $\dfrac{y-0}{0-h} = \dfrac{x-0}{0-l/2} \Rightarrow \dfrac{y}{-h} = \dfrac{x}{-l/2}$

$\Rightarrow \dfrac{y}{h} = \dfrac{2x}{l} \Rightarrow y = \dfrac{2hx}{l}$

ie) $y = \dfrac{2hx}{l}$ in $(0, l/2)$

Equation of BA $(l,0)$ $(l/2,h)$ $\dfrac{y-0}{0-h} = \dfrac{x-l}{l-l/2} \Rightarrow \dfrac{y}{-h} = \dfrac{x-l}{\frac{2l-l}{2}} = \dfrac{2(x-l)}{l}$

$\Rightarrow \dfrac{y}{-h} = -\dfrac{2(l-x)}{l}$

$\Rightarrow y = \dfrac{2h(l-x)}{l}$ $(l/2, l)$

$$\therefore \quad y = f(x) = \begin{cases} \dfrac{2hx}{l} & 0 \le x \le l/2 \\ \dfrac{2h(l-x)}{l} & l/2 \le x \le l. \end{cases}$$

The wave equation is $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{a^2}\dfrac{\partial^2 y}{\partial t^2}$

The solution is $y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow (1)$

Boundary Conditions are

(1) $y=0$ when $x=0$

(2) $y=0$ when $x=l$

(3) $\frac{\partial y}{\partial t}=0$, when $t=0$

(4) $y=f(x)$ when $t=0$

Applying Boundary condition ① in eq ① [Put $y=0$ when $x=0$] in eq ①

$$0=(A\cos 0+B\sin 0)(C\cos pat+D\sin pat)$$

$$\Rightarrow 0=A(C\cos pat+D\sin pat)$$

$$\Rightarrow C\cos pat+D\sin pat\neq 0 \quad \therefore \boxed{A=0}$$

Sub $A=0$ in ①

$$\therefore y=B\sin px\,(C\cos pat+D\sin pat) \rightarrow ②$$

Applying Boundary condition (2) in eq ② [Put $y=0$ and $x=l$ in eq ②]

$$0=B\sin pl\,(C\cos pat+D\sin pat)$$

$$\Rightarrow C\cos pat+D\sin pat\neq 0 \qquad B\sin pl=0$$

$$\Rightarrow B\neq 0,\ \sin pl=0.$$

$$\sin pl=0=\sin n\pi$$

$$\Rightarrow pl=n\pi$$

$$\Rightarrow \boxed{p=\left(\frac{n\pi}{l}\right)}$$

Put $p=n\pi/l$ in eq ②

$$\therefore y=B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right)+D\sin\left(\frac{n\pi at}{l}\right)\right] \rightarrow ③$$

Diff partially eq ③ w.r.to t

$$\frac{\partial y}{\partial t}=B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi at}{l}\right)\times\left(\frac{n\pi a}{l}\right)+D\cos\left(\frac{n\pi at}{l}\right)\times\left(\frac{n\pi a}{l}\right)\right]$$

Now applying Boundary condition (3) in eq ③ $\left(\frac{\partial y}{\partial t}=0 \text{ at } t=0\right)$

$$0=B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin(0)\times\left(\frac{n\pi a}{l}\right)+D\cos(0)\times\left(\frac{n\pi a}{l}\right)\right]$$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\left[0 + D\left(\frac{n\pi a}{l}\right)\right]$$

$$\cancel{0 = BD\sin} \qquad 0 = BD\sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right)$$

$$\Rightarrow B \neq 0 \quad \sin\left(\frac{n\pi x}{l}\right) \neq 0 \quad \frac{n\pi a}{l} \neq 0 \qquad \therefore \boxed{D = 0}.$$

Sub $D=0$ in eq (3)

$$\therefore\ y = B\sin\left(\frac{n\pi x}{l}\right) \times C\cos\left(\frac{n\pi at}{l}\right)$$

$$y = BC\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

$$y = b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \qquad (BC = b_n)$$

$\therefore$ The general sol. is

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \rightarrow (4)$$

Now apply Boundary conditions (4) in eq (4) when $y = f(x),\ t=0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos(0)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad \text{RHS (Half Range Sine Series)}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2}\left(\frac{2hx}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}\frac{2h(l-x)}{l}\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l} \times \frac{2h}{l}\left[\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{4h}{l^2}\left[I_1 + I_2\right] \rightarrow (6)$$

$$I_1 = \int_0^{l/2} x \sin\left(\frac{n\pi x}{l}\right) dx$$

$u = x \quad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = 1 \quad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$u'' = 0 \quad v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$uv_1 - u'v_2$

$$= \left[\frac{-x\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} + (1)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_0^{l/2}$$

$$= \left[\left\{ \frac{-l/2 \cos\left(\frac{n\pi l/2}{l}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi l/2}{l}\right)}{(n\pi/l)^2} \right\} - \{0+0\} \right]$$

$$= -\frac{l}{2} \times \frac{l}{n\pi} \cos(n\pi/2) + \frac{l^2}{n^2\pi^2} \sin(n\pi/2)$$

$$= \frac{-l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$uv_1 - u'v_2$

$$I_2 = \int_{l/2}^{l} (l-x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$u = (l-x) \quad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = -1 \quad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$u'' = 0 \quad v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$$= \left[\frac{-(l-x)\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right) \right]_{l/2}^{l}$$

$$= \left[-(l-x)\left(\frac{l}{n\pi}\right) \cos\left(\frac{n\pi x}{l}\right) - \left(\frac{l^2}{n^2\pi^2}\right) \sin\left(\frac{n\pi x}{l}\right) \right]_{l/2}^{l}$$

$$= \left[\{0+0\} - \left\{ -(l - l/2)(l/n\pi) \cos\left(\frac{n\pi l/2}{l}\right) - \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi l/2}{l}\right) \right\} \right]$$

$$= -\left\{ -\left(\frac{l}{2}\right)\left(\frac{l}{n\pi}\right) \cos\left(\frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right\}$$

$$= \frac{l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$\therefore$ (5) $\Rightarrow$

$$b_n = \frac{4h}{l^2} \left[\cancel{\frac{-l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right)} + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right)} + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{4h}{\cancel{l^2}} \left[\frac{2\cancel{l^2}}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right] = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$\therefore$ Sol. is

$$y = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right)$$

$$\Rightarrow \quad y = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \quad //$$