Poisson distribution:-

Definition: A discrete random variable x is said to follow poisson distribution, if its probability mass distribution function is given by

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

where

$x = 0, 1, 2, 3, \ldots$

$\lambda > 0.$

Moment generating function, Mean, variance of Probability Distribution:-

$$M_x(t) = E\left(e^{tx}\right)$$

$$= \sum e^{tx} \cdot p(x)$$

$$= \sum_{x=0}^{\infty} \frac{e^{tx} \cdot e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot (\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

[Taking λ out because it is independent of x]

$$= e^{-\lambda}\left[\frac{1}{0!} + \frac{(\lambda e^{t})^{1}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \frac{(\lambda e^{t})^{3}}{3!} + \cdots\right]$$

$$\boxed{M_x(t) = e^{-\lambda} \cdot e^{\lambda e^{t}}}$$

$$\left[e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots\right]$$

Mean

$$\mu_1' = E(x) = \left[\frac{d}{dt} M_x(t)\right]_{at\ t=0}$$

[Differentiate with respect to t]

$$= \left(\frac{d}{dt} e^{-\lambda} \cdot e^{\lambda e^{t}}\right)_{at\ t=0}$$

$$= \left[e^{-\lambda} \cdot e^{\lambda e^{t}} (\lambda e^{t})\right]_{at\ t=0}$$

$$= \left[\lambda e^{-\lambda} \cdot e^{t} \cdot e^{\lambda e^{t}}\right]_{at\ t=0}$$

$$= \lambda e^{-\lambda} \cdot e^{0} \cdot e^{\lambda e^{0}}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda}$$

$$\boxed{\text{Mean} = \lambda}$$

Variance

$$\mu_2' = E(x^2) = \left[\frac{d^2}{dt} M_x(t)\right]_{\text{at } t=0}$$

$$= \left(\frac{d^2}{dt^2} e^{-\lambda} \cdot e^{\lambda e^t}\right)_{\text{at } t=0}$$

$$= \left[\frac{d}{dt} \lambda e^{-\lambda} \cdot e^t \cdot e^{\lambda e^t}\right]_{\text{at } t=0.}$$

$$= \lambda e^{-\lambda}\left[e^t \left(e^{\lambda e^t} \cdot \lambda e^t\right) + e^{\lambda e^t} \cdot e^t\right]_{\text{at } t=0}$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} \cdot \lambda + e^{\lambda}\right]$$

$$E(x^2) = \lambda^2 + \lambda$$

$$\text{variance} = E(x^2) - (E(x))^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{variance} = \lambda}$$

Problems Under Poisson distribution:-

11.) If x is a poisson variate such that $P(x=0) = 0.5$. Find $var(x)$.

Soln→11→ For a poisson's distribution

$$P(x=x) = \frac{e^{-\lambda}.\lambda^{x}}{x!},$$

$x = 0, 1, 2, \ldots$

$$P(x=0) = \frac{e^{-\lambda}.\lambda^{0}}{0!} = 0.5$$

$\Rightarrow e^{-\lambda} = 0.5$

$\Rightarrow \ln(e^{-\lambda}) = \ln 0.5$ [taking ln on b.s]

$\Rightarrow -\lambda \ln e = \ln 0.5$

$\Rightarrow -\lambda = -0.6931$

$\Rightarrow \lambda = 0.6931.$

12.) If x is a poisson variate with $\lambda = 1.5$. Find $P(x=3)$.

Soln→12→ For a P. D

$$P(x=x) = \frac{e^{-\lambda}.\lambda^{x}}{x!}, x = 0, 1, 2, 3, \ldots$$

$$P(x=3) = \frac{e^{-\lambda}.\lambda^{3}}{3!}$$

$$= \frac{e^{-1.5}(1.5)^3}{3\times2\times1} = \frac{0.2231\times3.3750}{6} = 0.1255.$$

13.) If x is a poison variate and if $(P(x=1))_2$ $P(x=2)$. find λ.

Soln→13→ For a P.D

$$P(X=x) = \frac{e^{-\lambda}.\lambda^{x}}{x!}$$

$x = 0, 1, 2, 3, \ldots$

Given:-

$P(x=1) = P(x=2)$

$$\Rightarrow \frac{e^{-\lambda}.\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

$$\Rightarrow 1 = \frac{\lambda}{2}$$

$$\Rightarrow \lambda = 2.$$

14.) If x is a poison variate such that $P(x=1)=0.3$ and $P(x=2)=0.2$. find $P(x=0)$

Soln→14→ For a P.D

$P(x=1) = 0.3$

$$\frac{e^{-\lambda}.\lambda^1}{1!} = 0.3$$

$$e^{-\lambda} = \frac{3}{10\lambda}$$

$P(x=2) = 0.2$

$$\frac{e^{-\lambda}.\lambda^2}{2!} = 0.2$$

$$e^{-\lambda} = \frac{4}{10\lambda^2}$$

$$\frac{3}{\cancel{10\lambda}} = \frac{4}{\cancel{10\lambda}^2}$$

$$3\lambda = 4$$

$$\boxed{\lambda = \frac{4}{3}}$$

Now

$$P(x=0) = \frac{e^{-4/3}.(-4/3)^0}{0!} = 0.2636$$

$P(x=0) = 0.2636.$

15) The monthly breakdown of a computer is a random variable having Poison's distribution with mean 1.5. Find the probability that the computer will function for a month.

i) without breakdown.

ii) with only one breakdown

iii) with atleast one breakdown.

Soln→15→ For a P.D

$$P(X=x) = \frac{e^{-\lambda}.\lambda^{x}}{x!}, x = 0, 1, 2, 3, \ldots$$

i) $P(x=0) = \frac{e^{-1.5}(1.5)^{0}}{0!} = 0.2231$

ii) $P(x=1) = \frac{e^{-1.5}(1.5)^{1}}{1!} = 0.3346$

iii) $P(x \geq 1) = P(x=1) + P(x=2) + P(x=3) + \ldots$

$= 1 - (P(x<1))$

$= 1 - P(x=0)$

$= 1 - 0.2231 = 0.7769.$

16) The number of accidents in a year to taxi-driver in a city follows p.D with mean 3. out of 100 taxi drivers, find appr. the number of drivers with

a) No accidents in a year.

b) more than 3 accident in year.

Soln→16→ For a P.D,

$$P(x=0) = \frac{e^{(-3)}.(3)^{0}}{0!} = 0.04978.$$

$\therefore$ out of 100 taxi drivers, no. of drivers having

No. accidents $= 100 \times 0.04378$

$= 4.97$

$\simeq 5$

b) $P(x>3) = P(x=4) + P(x=5) + \cdots + \infty$

$= 1 - P(x \leq 3)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3)]$

$= 1 - \left[\frac{e^{-3}(3)^0}{0!} + \frac{e^{-3}(3)^1}{1!} + \frac{e^{-3}(3)}{2!} + \frac{e^{-3}(3)^3}{3!}\right]$

$= 1 - [0.0497 + 0.1494 + 0.2240 + 0.2240]$

$= 1 - 0.6472$

$= 0.3528$

$\therefore$ out of 100 taxi drivers, no. of drivers making into more than 3 accidents

$= 100 \times 0.3528$

$= 35.2 \simeq 35$ drivers.

17.) R.V.Q Find the P.D for a following data.

x	0	1	2	3	4
f	43	38	22	9	1

x	f	fx	Theoretical frequency or Expected frequency $= N \times \frac{e^{-\lambda} \cdot \lambda^x}{x!}$
0	43	0	41.5704 $\triangleq$ 41
1	38	38	41.5704 $\triangleq$ 42
2	22	44	20.7852 $\triangleq$ 21
3	9	27	6.9284 $\triangleq$ 7
4	1	4	1.7321 $\triangleq$ 2

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{113}{113} = 1$$

$$\lambda = 1.$$

x	0	1	2	4
observed frequency	43	38	22	1
Expected frequency	41.	42	21	2.

18.) State and prove additive property of Poisson distribution.

Statement:-

If x_1 and x_2 are independent poissons variates with parameters λ_1 and λ_2 then x_1+x_2 is also a poisson variate with parameter $\lambda_1+\lambda_2$.

Proof:-

$$P(x_1+x_2=n) = P(x_1=r) \cdot P(x_2=n-r)$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1}(\lambda_1)^r}{r!} \cdot \frac{e^{-\lambda_2}(\lambda_2)^{n-r}}{(n-r)!}$$

$$= \frac{e^{(-\lambda_1+\lambda_2)}}{n!} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} (\lambda_1)^r (\lambda_2)^{n-r}$$

$$= \frac{e^{(-\lambda_1+\lambda_2)}}{n!} \sum_{r=0}^{n} nc_r (\lambda_1)^r (\lambda_2)^{n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \left[nc_0 (\lambda_1)(\lambda_2)^n + nc_1 (\lambda_1)(\lambda_2)^{n-1} + \dots \dots + nc_n (\lambda_1)^n (\lambda_2)^0 \right]$$

$$\Rightarrow P(x_1+x_2=n) \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n$$

$\therefore x_1+x_2$ is a poison variate with parameter $\lambda_1+\lambda_2$.

19.) A car hire firm has two cars, which it hires out day by day. The numbers of demands for a car on each day is distributed as Poisson variate with mean 1.5. calculate the probability on which

i) neither car is used.

ii) some demand is refused.

Soln → 19 → $P(x=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}, \quad x=0,1,2,\ldots$

i) $P(x=0) = \frac{e^{-1.5}(1.5)^0}{0!}$

$= 0.2231$

ii) $P(x>2) = P(x=3)+P(x=4)+\ldots$

$= 1-P(x\le 2)$

$= 1-[P[(x=0)]+P(x=1)+P(x=2)]$

$= 1-\left[\frac{e^{-1.5}(1.5)^0}{0!}+\frac{e^{-1.5}(1.5)^1}{1!}+\frac{e^{-1.5}(1.5)^2}{2!}\right]$

$= 1-(0.2231+0.3347+0.2510)$

$= 0.1914.$