Department of Mathematics

Sub Title: PROBABILITY AND QUEUING THEORY

Sub Code: 18MAB 204T

UNIT -IV - PRINCIPLES OF QUEUEING THEORY

Model I (M/M/1) : (∞ / FIFO) (Single server , infinite capacity queue)

This is a simple queue with poisson arrival, exponential service time , single server , infinite capacity and First in First out queue discipline.

The arrival rate and service rate are constant (λ *and* μ res.) To find the steady state probabilities, put $\lambda_n = \lambda$, $\mu_n = \mu$ for all n in $P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_{n-1}\mu_n}}$ and $P_n = \frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_{n-1}\mu_n} P_0$

=> $P_0 = 1 - \frac{\lambda}{\mu}$ *and* $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$ => $P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

Let , the average arrival rate of customer = λ and the average service rate of the server = μ

and $\lambda < \mu$.

(a) P(server is busy) = Traffic intensity = $\rho = \frac{\lambda}{\mu}$ = P(system is busy)

(b) P (server is idle) = P(No customer in the system) = P(system is empty) = $P_0 = 1 - \rho$ => $P_0 = 1 - \frac{\lambda}{\mu}$

(c) Steady state Probabilities :
$P_0 = 1 - \frac{\lambda}{\mu}$ => $P_0 = 1 - \rho$ and $P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$ => $P_n = \rho^n (1 - \rho)$

Characteristics of the queue :

1. Average number of customer in the system = E(n) = $L_s = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$ and $L_s = \lambda W_s = L_q + \frac{\lambda}{\mu}$
   (where n is the number of customer in the system which includes the one being served)

2. Average number of customers in the queue = E(n - 1) = $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

(where n is the number of customer in the system which doesn't includes the one being served)

3. Average number of customers in non empty queue or (Average length of the queue formed from time to time)

If n is the number of customers in the system , then (n-1) is the number of customers in the queue and for a non-empty queue we must have (n-1) > 0

Mean number of customers in a nonempty queue = $L_w = E\{(n-1)/(n-1)>0\} = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\rho}$

4. P(Number of customers in the system exceeds k) : P(N>k) = $\left(\frac{\lambda}{\mu}\right)^{k+1} = (\rho)^{k+1}$
5. Probability density function of the waiting time in the system (W_s) is f(w) = $(\mu-\lambda)e^{-(\mu-\lambda)w}$ which is the p.d.f of an exponential distribution with parameter $(\mu-\lambda)$
6. Probability density function of the waiting time in the queue (W_q) is $g(w) = \begin{cases} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)w}, & w>0 \\ 1-\frac{\lambda}{\mu}, & w=0 \end{cases}$
7. Average waiting time in the queue $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$
8. Average waiting time of a customer in the system $W_s = \frac{1}{(\mu-\lambda)}$ => $W_s = \frac{L_s}{\lambda} = W_q + \frac{1}{\mu}$
9. P(waiting time of a customer exceeds t) :

   In the system : $P(W_s > t) = e^{-(\mu-\lambda)t}$

   In the queue $P(W_q > t) = \frac{\lambda}{\mu}\, e^{-(\mu-\lambda)t}$

10. Total Cost per unit period of time

    = Server facility cost per unit period + Customer idle time cost per unit period

    = Server facility cost per unit period + (L_s * Idle time cost per customer per unit period)

11. Average waiting time of a customer in the queue, if he has to wait :

    $E(W_q/W_q>0) = \frac{E(Wq)}{P(Wq>0)} = \frac{1}{(\mu-\lambda)}$

Model II (M/M/1) ; (K / FIFO) (Single server , finite capacity queue)

In general steady state $P_n = \frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_{n-1}\mu_n} P_0$ and $P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_{n-1}\mu_n}}$

put $\mu_n = \mu$, , n = 1,2,3,4,… $\lambda_n = \begin{cases} \lambda, & if\ n = 0,1,2,3, \dots k-1 \\ 0 & n = k \end{cases}$

=> $P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_{n-1}\mu_n}}$

=> $P_0 = \frac{1}{1+\sum_{n=1}^{k}\left(\frac{\lambda}{\mu}\right)^n} = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$ if $\frac{\lambda}{\mu} \neq 1$ and If $\frac{\lambda}{\mu} = 1$, $P_0 = \frac{1}{k+1}$

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n P_0, & \lambda \neq \mu \\ P_0, & \lambda = \mu \end{cases} \quad \text{and} \quad P_0 = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \lambda \neq \mu \\ \frac{1}{k+1}, & \lambda = \mu \end{cases}$$

Characteristics of the queue :

1. Average number of customer in the system

$L_s = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$ if $\lambda \neq \mu$ and $L_s = \frac{k}{2}$ if $\lambda = \mu$

2. Effective arrival rate is $\lambda' = \lambda\,(1 - P_k)$ This is also written as $\lambda' = \mu\ (1 - P_0)$
3. Average waiting time in the queue $W_q = \frac{1}{\lambda'}\ L_q$
4. Average waiting time of a customer in the system $W_s = \frac{1}{\lambda'}\ L_s$ and $L_s = L_q + \frac{\lambda'}{\mu}$

PART-A

1. The use of F-distribution is to test the

(a) Mean of two small samples (b) Variance of two small samples

(c) Mean of two large samples (d) Variance of two large samples **Ans: (b)**

2. The value of test statistic F is

(a) $F > 1$ (b) $F < 1$ (c) $F = 1$ (d) $F = 0$ **Ans: (a)**

3. Chi square distribution is used to

(a) To test the mean of two small samples (b) To test the mean of two large samples

(c) To test the goodness of fit (d) To test the variance of two populations

Ans: (c)

4. In Chi square test , the number of observations in the sample is

(a) ≥ 50 (b) ≤ 50 (c) 10 (d) 100 **Ans: (a)**

5. In Chi square test , the condition to choose small n is

(a) $4 \leq$ n (b) $4 \leq n \leq 16$ (c) $n \geq 16$ (d) $n \leq 4$ **Ans: (b)**

6. The statistic of chi square test is

(a) $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$ (b) $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i^2}$

(c) $\chi^2 = \sum (O_i - E_i)$ (d) $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$ **Ans: (a)**

7. The number of degrees of freedom of Chi square test is

(a) n-2 (b) n-3 (c)n-4 (d) n-1 **Ans: (d)**

8. The value of χ^2 for 2 x 2 contingency table is

(a) $\chi^2 = \frac{N(ad - bc)}{(a+b)(c+d)(a+c)(b+d)}$ (b) $\chi^2 = \frac{N(ad + bc)^2}{(a+b)(c+d)(a+c)(b+d)}$

(c) $\chi^2 = \frac{N(ad - bc)^2}{(a+b)(c+d)(a+c)(b+d)}$ (d) $\chi^2 = \frac{N(ad - bc)^2}{(a-b)(c+d)(a+c)(b+d)}$ **Ans: (c)**

9. In Queueing system,the Number of arrivals per unit time always follows _____ distribution.

(a) poisson (b) exponential (c) Binomial d)Normal **Ans: (a)**

10. In the model M/M/1 , the first M represents ___________

a) server b) arrival c) no. of servers d)departure **Ans: (b)**

11. In the model M/M/1 , then 1 represents ___________

a)single server b) multiple server c) single arrival d) multiple arrival **Ans: (a)**

12. The average waiting time of a customer in the (M/M/1):$(\infty/FIFO)$ system is

a) $\frac{1}{\mu - \lambda}$ b) $\frac{\lambda}{\mu - \lambda}$ c) $\frac{\mu}{\mu - \lambda}$ d) $\frac{\mu}{\mu + \lambda}$ **Ans: (a)**

13 . Mean arrival rate is denoted by

a) $\frac{1}{\lambda}$ b) λ c) μ d) $\frac{1}{\mu}$ **Ans: (a)**

14. The number of arrivals per unit time has a poisson distribution with mean _____

a) $\frac{1}{\lambda}$ b)λ c)μ d) $\frac{1}{\mu}$ **Ans: (b)**

15. The number of customers in the system in M/M/1 model is

a) $\frac{1}{\mu-\lambda}$ b) $\frac{\lambda}{\mu-\lambda}$ c) $\frac{\mu}{\mu-\lambda}$ d) $\frac{\mu}{\mu+\lambda}$ **Ans: (b)**

16. The probability that the arrival enter the service without wait is

a)1 + P(arrival has to wait) b)P(arrival has to wait) – 1

c)1 - P(arrival has to wait) d)zero **Ans: (c)**

17. Average number of customer in the system when ρ=1 in (M/M/1) :(K/FIFO) is ________

a) K/2 b)2K c) K d)0 **Ans: (a)**

18. The number of customer in the system are always ________

a) mutually exclusive b) mutually exhaustive

c) mutually exclusive and exhaustive d) unique **Ans: (c)**

19. The relation between $E(N_s)$ and $E(N_q)$ is

a) $E(N_s) = E(N_q) + \frac{\lambda}{\mu}$ b) $E(N_s) = E(N_q) - \frac{\lambda}{\mu}$

c) $E(N_s) = E(N_q) + \frac{1}{\mu}$ d) $E(N_s) = E(N_q) + \lambda\mu$ **Ans: (a)**

F – Test

$F = \frac{S_X^2}{S_Y^2}$, if $S_X^2 > S_Y^2$ (or) $F = \frac{S_Y^2}{S_X^2}$, if $S_X^2 < S_Y^2$,

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1}$, $S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1}$, $S_1^2 = \frac{n_1 s_1^2}{n_1-1}$, $S_2^2 = \frac{n_2 s_2^2}{n_2-1}$,

degree of freedom : $(\nu_1 = n_1 - 1,\ \nu_2 = n_2 - 1)$.

Applications: (i) To test whether two independent samples have been drawn from the normal populations with the same variance σ^2 (ii) To test whether two independent estimates of the population variance are homogeneous or not.

PROBLEMS

12. ***If one sample of 8 observations the sum of the squares of deviations of the sample values from the sample mean was 84.4 and in the other sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level.***

Solution: $n_1 = 8,\ n_2 = 10,\ \sum(x-\bar{x})^2 = 84.4,\ \sum(y-\bar{y})^2 = 102.6$

Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1} = \frac{84.4}{8-1} = 12.057,\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1} = \frac{102.6}{10-1} = 11.4,\quad F = \frac{S_X^2}{S_Y^2} = \frac{12.057}{11.4} = 1.057$,

d.f. = $(n_1 - 1,\ n_2 - 1) = (7, 9)$ at 5% LOS = 3.29. Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**.

13. ***Two random samples of 11 and 9 items show the sample standard deviations of their weights as 0.8 and 0.5 respectively. Assuming that the weight distributions are normal, test the hypothesis that the true variances are equal, against the alternative hypothesis that they are not.***

Solution: $n_1 = 11,\ n_2 = 9,\ s_1^2 = 0.8,\ s_2^2 = 0.5$, Null Hypothesis: $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_1^2 = \frac{n_1 s_1^2}{n_1-1} = \frac{11(0.8)^2}{11-1} = 0.704,\ S_2^2 = \frac{n_2 s_2^2}{n_2-1} = \frac{9(0.5)^2}{9-1} = 0.28125,\ F = \frac{S_1^2}{S_2^2} = \frac{0.704}{0.28125} = 2.503$,

d.f. = $(n_1 - 1,\ n_2 - 1) = (10,\ 8)$ at 5% LOS = 3.34. Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**.

14. ***The time taken by workers in performing a job by Method I and Method II is given below***

Method I	***20***	***16***	***26***	***27***	***23***	***22***	***-***
Method II	***27***	***33***	***42***	***35***	***32***	***34***	***38***

Do the data show that the variances of time distribution form population from which these samples are drawn do not differ significantly?

Solution: $n_1 = 6,\ n_2 = 7,\ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n_1} = \frac{134}{6} = 22.3,\ \bar{y} = \frac{\sum_{j=1}^{n} y_j}{n_2} = \frac{241}{7} = 34.4$

x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
20	-2.3	5.29	27	-7.4	54.76
16	-6.3	39.69	33	-1.4	1.96
26	3.7	13.69	42	7.6	57.76
27	4.7	22.09	35	0.6	0.36
23	0.7	0.49	32	-2.4	5.76
22	-0.3	0.09	34	-0.4	0.16
$\sum x = 134$		$\sum(x-\bar{x})^2 = 81.34$	38	3.6	12.96
			$\sum y = 241$		$\sum(y-\bar{y})^2 = 132.72$

Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1} = \frac{81.34}{6-1} = 16.268,\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1} = \frac{133.72}{7-1} = 22.29,\ F = \frac{S_Y^2}{S_X^2} = \frac{22.29}{16.268} = 1.37$,

d.f. = $(n_2 - 1,\ n_1 - 1) = (6,\ 5)$ at 5% LOS = 4.95. Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**.

15. Two horses A and B were tested according to the time (in seconds) to run a particular track with the following

Horse A	***28***	***30***	***32***	***33***	***33***	***29***	***34***
Horse B	***29***	***30***	***30***	***24***	***27***	***29***	***-***

Test whether the two horses have the same running capacity.

Solution: $n_1 = 7,\ n_2 = 6,\ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n_1} = \frac{219}{7} = 31.28,\ \bar{y} = \frac{\sum_{j=1}^{n} y_j}{n_2} = \frac{169}{6} = 28.2$

x	$x - \bar{x}$	$(x - \bar{x})^2$	y	$y - \bar{y}$	$(y - \bar{y})^2$
28	-3.28	10.75	29	0.8	0.64
30	-1.28	1.64	30	1.8	3.24
32	0.72	0.52	30	1.8	3.24
33	1.72	2.96	24	-4.2	17.64
33	1.72	2.96	27	-1.2	1.44
29	-2.28	5.20	29	0.8	0.64
34	2.72	7.40	-	-	-
$\sum x = 219$		$\sum(x - \bar{x})^2 = 21.2$	$\sum y = 169$		$\sum(y - \bar{y})^2 = 108$

F - Test

Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1} = \frac{31.43}{7-1} = 5.238,\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1} = \frac{26.84}{6-1} = 5.368\ ,\ F = \frac{S_Y^2}{S_X^2} = \frac{5.368}{5.238} = 1.02$,

degree of freedom $= (n_2 - 1,\ n_1 - 1) = (5,\ 6)$ at 5% LOS = 4.39.

Calculate value F < Tabulated F. $\mathbf{H_0}$ **is accepted**. ***Conclusion:*** The two horses have the same running capacity.

t – test and F – Test

16. The nicotine contents in milligrams in two samples of tobacco were found to be as follows

Sample A	***24***	***27***	***26***	***21***	***25***	***-***
Sample B	***27***	***30***	***28***	***31***	***22***	***36***

Can it be said that two samples come from normal populations.

Solution: $n_1 = 5,\ n_2 = 6,\ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n_1} = \frac{123}{5} = 24.6,\ \bar{y} = \frac{\sum_{j=1}^{n} y_j}{n_2} = \frac{174}{6} = 29$

x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
24	-0.6	0.36	27	-2	4
27	2.4	5.76	30	1	1
26	1.4	1.96	28	-1	1
21	-3.6	12.96	31	2	4
25	0.4	0.16	22	-7	49
$\sum x = 123$		$\sum(x-\bar{x})^2 = 21.2$	36	7	49
			$\sum y = 174$		$\sum(y-\bar{y})^2 = 108$

Student's t test

Null Hypothesis : $H_0: \bar{x_1} = \bar{x_2}$, ***Alternate Hypothesis :*** $H_1: \bar{x_1} \neq \bar{x_2}$ (Two tailed)

$s^2 = \frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2} = \frac{21.2+108}{5+6-2} = 14.35,\ t = \frac{\bar{x}-\bar{y}}{\sqrt{s^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{24.6-29}{\sqrt{14.35\left(\frac{1}{5}+\frac{1}{6}\right)}} = -1.92 \Rightarrow |t| = 1.92,$

degree of freedom = $n_1 + n_2 - 2 = 9$ at 5% LOS for two tailed test = 2.26.

Calculate value t < Tabulated t. **H_0 is accepted.**

F - Test

Null Hypothesis : $H_0: \sigma_X^2 = \sigma_Y^2$, Alternate Hypothesis : $H_1: \sigma_X^2 \neq \sigma_Y^2$

$S_X^2 = \frac{\sum(x-\bar{x})^2}{n_1-1} = \frac{21.2}{5-1} = 5.3,\ S_Y^2 = \frac{\sum(y-\bar{y})^2}{n_2-1} = \frac{108}{6-1} = 21.6\ ,\ F = \frac{S_Y^2}{S_X^2} = \frac{21.6}{5.3} = 4.07\ ,$

degree of freedom $= (n_2 - 1,\ n_1 - 1) = (5,\ 4)$ at 5% LOS = 6.26.

Calculate value F < Tabulated F. **H_0 is accepted**.

Conclusion: We conclude that the two samples come from same normal population.

χ^2 – test

The χ^2distribution function is one of the most extensively used distribution functions in statistics.

Application (or uses) of χ^2distribution

1. To test the goodness of fit. 2. To test the independence of attributes.
3. To test if the hypothetical value of the population variance is σ^2.
4. To test the homogeneity of independent estimates of the population variance.
5. To test the homogeneity of independent estimates of the population correlation coefficient.

Condition for validity of χ^2 – test

1. The sample observations should be independent.
2. Constraints on the cell frequencies, if any, should be linear, e.g., $\sum O_i = \sum E_i$
3. N, the total frequency should be reasonably large, say, greater than 50.
4. No theoretical cell frequency should be less than 5. (The chi square distribution is frequency is less than t) . If any theoretical cell frequency is less than 5, then for the application of χ^2 test, it is pooled with the preceding or succeeding frequency so that the pooled frequency is more than 5 and finally adjust for the degree of freedom lost in pooling.

χ^2 – test of Goodness of fit

This is a powerful test for testing the significance of the discrepancy between theory and experiment – discovered by Karl Pearson in 1900. It helps us to find if the deviation of the experiment from theory is just by chance or it is due to the in adequacy of the theory to fit the observed data.

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$, where O_i – set of observed frequencies, E_i- Set of expected frequencies. d.f. $= n - 1$

1. ***The following table gives the number of aircraft accidents that occurred during the various days of the week. Test whether the accidents are uniformly distributed over the week.***

Days	*Mon*	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
No. of Accidents	*14*	*18*	*12*	*11*	*15*	*14*

Solution: H_0 : Accidents occur uniformly over the week. The expected no. of accidents on any day $= \frac{83}{6} = 14$.

O	E	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
14	14	0	0	0
18	14	4	16	1.143
12	14	-2	4	0.286
11	14	-3	9	0.643
15	14	1	1	0.071
14	14	0	0	0
				2.143

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{2.143}$, d.f. $= n - 1 = 5$ at 5% LOS = 11.07, Calculated $\chi^2 <$ tabulated χ^2,

$\boldsymbol{H_0}$ **is accepted.** The accidents are uniformly distributed over the week.

2. ***The theory predicts the proportion of beans, in the four groups A, B, C, and D should be 9 : 3 : 3 : 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287, and 118. Does the experimental result support the theory.***

Solution: H_0 : The experimental result support the theory.

O	E	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
882	$\frac{9}{16} \times 1600 = 900$	-18	324	0.36
313	$\frac{3}{16} \times 1600 = 300$	13	169	0.563
287	$\frac{3}{16} \times 1600 = 300$	-13	169	0.563
118	$\frac{1}{16} \times 1600 = 100$	18	324	3.24
				4.726

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{4.726}$, d.f. $= n - 1 = 4 - 1 = 3$ at 5% LOS = 7.81

Calculated $\chi^2 <$ tabulated χ^2, $\boldsymbol{H_0}$ **is accepted.** The experimental results support the theory.

3. *Fit a binomial distribution for the following data and also test the goodness of fit. Find the parameters of the distribution.*

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80

Solution: To find the binomial frequency distribution $N(q+p)^n$, which fits the given data, we require N, n and p. We assume N = total frequency = 80 and n = no. of trials = 6 from the given data.

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80
fx	0	18	56	36	28	30	24	192

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{192}{80} = 2.4$, $np = 2.4 \Rightarrow 6p = 2.4$, $p = 0.4$, $q = 0.6$, N $P(X = x) = N\, nC_x p^x q^{n-x}$, $x = 0,1, \dots n$

$80\,P(X=0) = 80 \times 6C_0\,(0.4)^0(0.6)^{6-0} = 3.73$, $80\,P(X=1) = 80 \times 6C_1\,(0.4)^1(0.6)^{6-1} = 14.93$

$80\,P(X=2) = 80 \times 6C_2\,(0.4)^2(0.6)^{6-2} = 24.88$, $80\,P(X=3) = 80 \times 6C_3\,(0.4)^3(0.6)^{6-3} = 22.12$

$80\,P(X=4) = 80 \times 6C_4\,(0.4)^4(0.6)^{6-4} = 11.06$, $80\,P(X=5) = 80 \times 6C_5\,(0.4)^5(0.6)^{6-5} = 2.95$

$80\,P(X=6) = 80 \times 6C_6\,(0.4)^6(0.6)^{6-6} = 0.33$

X	0	1	2	3	4	5	6	***Total***
E_i	4	15	25	22	11	3	0	***80***

The 1st class is combined with the second and the last 2 classes are combined with the last but 2nd class in order to make the expected frequency in each class ≥ 5. Thus, after regrouping, we have

O_i	*23*	*28*	*12*	*17*
E_i	19	25	22	14

H_0 : The given distribution is approximately binomial distribution.

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
23	19	4	16	0.8421
28	25	3	9	0.36
12	22	-10	100	4.5455
17	14	3	9	0.6429
				6.39

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i-E_i)^2}{E_i} = \mathbf{6.39}$, d.f. $= n - k = 4 - 2 = 2$ at 5% LOS = 5.99,

Calculated $\chi^2 >$ tabulated χ^2, $\boldsymbol{H_0}$ **is rejected.** The binomial fit for the given distribution is not satisfactory.

4. ***Fit a Poisson distribution for the following distribution and also test the goodness of fit.***

x	***0***	***1***	***2***	***3***	***4***	***5***	***Total***
f	***142***	***156***	***69***	***27***	***5***	***1***	***400***

Solution : $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1, \dots \infty$; $\lambda = \bar{x} = \frac{\sum fx}{\sum f} = \frac{400}{400} = 1$, $N = 400$

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400
fx	0	156	138	81	20	5	400

Theoretical frequencies are given by $N\,P(X = x) = \frac{N\,e^{-\lambda}\lambda^x}{x!} = \frac{400\,e^{-1}1^x}{x!}, x = 0,1, \dots \infty$

$400\,P(X = 0) = \frac{400\,e^{-1}1^0}{0!} = 147.15$, $400\,P(X = 1) = \frac{400\,e^{-1}1^1}{1!} = 147.15$

$400\,P(X = 2) = \frac{400\,e^{-1}1^2}{2!} = 73.58$, $400\,P(X = 3) = \frac{400\,e^{-1}1^3}{3!} = 24.53$

$400\,P(X = 4) = \frac{400\,e^{-1}1^4}{4!} = 6.13$, $400\,P(X = 5) = \frac{400\,e^{-1}1^5}{5!} = 1.23$

O_i	142	156	69	27	5	1
E_i	147	147	74	25	6	1

The last 3 classes are combined into one, so that the expected frequency in that class may be ≥ 10.

O_i	142	156	69	33
E_i	147	147	74	32

$\boldsymbol{O}$	$\boldsymbol{E}$	$\boldsymbol{O - E}$	$(\boldsymbol{O - E})^2$	$\frac{(O-E)^2}{E}$
142	147	-5	25	0.17
156	147	9	81	0.55
69	74	-5	25	0.34
33	32	1	1	0.03
				1.09

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{1.09}$, d.f. $= n - k = 4 - 2 = 2$ at 5% LOS = 5.99

Calculated $\chi^2 >$ tabulated χ^2, $\boldsymbol{H_0}$ **is accepted.** The Poisson fit for the given distribution is satisfactory.

χ^2 – test of Independence of Attributes

Literally, an attribute means a quality or characteristic. E.g.: drinking, smoking, blindness, honesty, etc.

a	b	$(a + b)$
c	d	$(c + d)$
$(a + c)$	$(b + d)$	N

$E(a) = \frac{(a + c)(a + b)}{N}$	$E(b) = \frac{(b + d)(a + b)}{N}$	$(a + b)$
$E(c) = \frac{(a + c)(c + d)}{N}$	$E(d) = \frac{(b + d)(c + d)}{N}$	$(c + d)$
$(a + c)$	$(b + d)$	N

Degree of freedom = $(r - 1)(c - 1)$, where r – number of rows, c – number of colums.

5. ***On the basis of information given below about the treatment of 200 patients suffering from a disease, state whether the new treatment is comparatively superior to the conventional treatment.***

	Favourable	*Not Favourable*	*Total*
New	***60***	***30***	***90***
Conventional	***40***	***70***	***110***
	100	***100***	***200***

Solution: H_0 : New and conventional treatment are independent.

$E(60) = \frac{90 \times 100}{200} = 45$	$E(30) = \frac{90 \times 100}{200} = 45$	***90***
$E(40) = \frac{110 \times 100}{200} = 55$	$E(70) = \frac{110 \times 100}{200} = 55$	***110***
100	***100***	***200***

O	E	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
60	45	15	225	5
30	45	-15	225	5
40	55	-15	225	4.09
70	55	15	225	4.09
				18.18

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \mathbf{18.18}$, d.f. $= (r-1)(c-1) = (2-1)(2-1) = 1$ ***at*** 5% LOS = 3.841.

Calculated χ^2 > tabulated χ^2. ***H_0* is rejected.** New and conventional treatment are not independent.

6. ***Given the following contingency table for hair colour and eye colour. Find the value of chi square. Is there good association between the two.***

		Hair Colour			***Total***
		Fair	***Brown***	***Black***	
Eye Colour	***Blue***	***15***	***5***	***20***	***40***
	Grey	***20***	***10***	***20***	***50***
	Brown	***25***	***15***	***20***	***60***
	Total	***60***	***30***	***60***	***150***

Solution: H_0 : The two attributes Hair colour and eye colour are independent.

O	E	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
15	16	-1	1	0.0625
5	8	-3	9	1.125
20	16	4	16	1
20	20	0	0	0
10	10	0	0	0
20	20	0	0	0
25	24	1	1	0.042
15	12	3	9	0.75
20	24	-4	16	0.666
				3.6458

$E(15)=\frac{60\times 40}{150}=16$	$E(5)=\frac{30\times 40}{150}=8$	$E(20)=\frac{60\times 40}{150}=16$	***40***
$E(20)=\frac{60\times 50}{150}=20$	$E(10)=\frac{30\times 50}{150}=10$	$E(20)=\frac{60\times 50}{150}=20$	***50***
$E(25)=\frac{60\times 60}{150}=24$	$E(15)=\frac{30\times 60}{150}=12$	$E(20)=\frac{60\times 60}{150}=24$	***60***
60	***30***	***60***	***150***

$\chi^2=\sum_{i=1}^{n}\frac{(O_i-E_i)^2}{E_i}=\mathbf{3.6458}$, d.f. $=(r-1)(c-1)=(3-1)(3-1)=4$ *at* 5% LOS = 9.488.

Calculated χ^2 < tabulated χ^2, $\boldsymbol{H_0}$ **is accepted.** The hair colour and eye colour are independent.

7. A group of 10 rats fed on diet A and another group of 8 rats fed on diet B recorded the following increase in weight

Diet A	5	6	8	1	12	4	3	9	6	10
Diet B	2	3	6	8	10	1	2	8	-	-

Show that the estimates of the population variance from the samples are not significantly different

Solution:

Null hypothesis H_0 : There is no significant difference between the variance

increase in weight due to diets A & B (i.e) $S_1^2 = S_2^2$

Alternative hypothesis H_1 : $S_1^2 \neq S_2^2$

Test Statistic :

$$F=\frac{S_1^2}{S_2^2} \text{ or } F=\frac{S_2^2}{S_1^2}$$

To calculate sample means and variance:

x	$x-\bar{x}$ (x -6.4)	$(x-\bar{x})^2$	y	y- $\bar{y}$ (y-5)	(y- $\bar{y}$ $)^2$
5	-1.4	1.96	2	-3	9
6	0.4	0.16	3	-2	4
8	1.6	2.56	6	1	1

1	-5.4	29.16	8	3	9
12	5.6	31.36	10	5	25
4	-2.4	5.76	1	-4	16
3	-3.4	11.56	2	-3	9
9	2.6	6.76	8	3	9
6	-0.4	0.16	-	-	-
10	3.6	12.96	-	-	-
Σ= 64		102.40	40		82

Mean of diet A = $\bar{x} = \frac{\sum x}{n_1}$

Here $n_1 = 10$, $\bar{x} = \frac{64}{10}$

$\bar{x} = 6.4$

Mean of diet B $\bar{y} = \frac{\sum y}{n_2}$

Here $n_2 = 8$, $\bar{y} = \frac{40}{8}$

$\bar{y} = 5$

$$S_2^2 = \frac{\sum(y-\bar{y})^2}{n_2 - 1}$$

$$S_2^2 = \frac{82}{8-1}$$

$S_2^2 = 11.7143$

$$S_1^2 = \frac{\sum(x-\bar{x})^2}{n_1 - 1}$$

$$\frac{102.40}{10-1}$$

$S^2_1 = 11.3778$

$$\therefore F = \frac{S_2^2}{S_1^2} \quad (\because S_2^2 > S_2^2)$$

$$= \frac{11.7143}{11.3778}$$

F_{cal} = 1.0296 with degrees of freedom $\upsilon = (n_2-1, n_1-1)$

$$\upsilon = (7,9)$$

Tabulated value of $F_{(7,9)}=3.12$

Since $F_{cal} < F_{tab}$, H_0 is a accepted.

(ie) there is no significant difference in population variance from the samples.

8. Test if the variances are significantly different for

X_1	24	27	26	21	25	
X_2	27	30	32	36	28	23

Solution:

To test the variance are significantly different , we use F - test

Given $n_1 = 5, n_2=6$

Calculation for means and S.D of the samples

x	$x - \bar{x}$ (x -24.6)	$(x - \bar{x})^2$	y	y- $\bar{y}$ (y-29.33)	(y- $\bar{y}$)^2
24	-0.6	0.36	27	-2.33	5.4289
27	2.4	5.76	30	0.67	0.4489
26	1.4	1.96	32	2.67	7.1289
21	-3.6	12.96	36	6.67	44.4889
25	0.4	0.16	28	-1.33	1.7689
-	-	-	23	-6.33	40.0689
Σ=123		21.20	176		99.3334

$\bar{x} = \frac{\sum x}{n_1} = \frac{123}{5} = 24.6$ $\qquad$ $\bar{y} = \frac{\sum y}{n_2} = \frac{176}{6} = 29.33$

$\sum(x-\bar{x})^2 = 21.2$ $\qquad$ $\sum(y-\bar{y})^2 = 99.3334$

$S_1^2 = \frac{\sum(x-\bar{x})^2}{n_1-1}$ $\qquad$ $S_2^2 = \frac{\sum(y-\bar{y})^2}{n_2-1}$

$= \frac{21.20}{5-1}$

$S_1^2 = 5.3$

$= \frac{99.3334}{6-1}$

$S_2^2 = 19.8667$

Null hypothesis H_0: $\sigma_1^2 = \sigma_2^2$

Test statistic $F = \frac{S_2^2}{S_1^2} = \frac{19.8667}{5.3}$

$F_{cal} =$ 3.7484 with degrees of freedom $\upsilon = (n_2-1, n_1-1)$

$\upsilon = (5,4)$

Tabulated value of F for (5,4) d.f at 5% level of significance is 6.26,

Since $F_{cal} < F_{tab}$,we accept H_o (ie) the variances are equal.

9. The number of accident in a certain locality was 12,8,20,2,4,10,15,6,9,4,. Are these frequencies in agreement with the belief that accident conditions were the same during this 10 week period.

Solution:

Expected frequency of a accidents each week = $\frac{100}{10} = 10$

Null hypothesis H_o : The accident conditions were the same during the 10 week period

Observed frequency (0)	Expected frequency (E)	(0-E)	$\frac{(0-E)^2}{E}$
12	10	2	0.4
8	10	-2	0.4
20	10	10	10.0
2	10	-8	6.4
14	10	4	1.6
10	10	0	0
15	10	5	2.5
6	10	-4	1.6

9	10	-1	0.1
4	10	-6	3.6
Σ=100	100		26.6

Now $\Psi^2 = \sum\left[\frac{(0-E)^2}{E}\right]$

$\Psi^2 = 26.6$

(ie) calculated value $\Psi^2 = 26.6$

Degrees of freedom d.f= υ = n-1 = 10-1

$\upsilon = 9$

Since $\Psi^2{}_{cal} > \Psi^2{}_{tab,}$ we reject the null hypothesis (ie) The accident conditions were not the same during the 10 week period.

UNIT – IV PRINCIPLES OF QUEUING THEORY (1)

PART – B

(1) For $(M/M/1):(\infty/FIFO)$ model, write down the Little's Formula.

(i) $L_s = \lambda W_s$

(ii) $L_q = \lambda W_q$

(iii) $W_s = W_q + \frac{1}{\mu}$

iv) $L_s = L_q + \frac{\lambda}{\mu}$.

(2) what is the probability that a customer has to wait more than 15 minutes to get his service completed in a $(M/M/1)$ queuing system, if $\lambda = 6$ per hour, $\mu = 10$ per hour?

Probability that the waiting time in the system exceeds t is

$$\int_t^{\infty} (\mu-\lambda)\, e^{-(\mu-\lambda)w}\, dw = e^{-(\mu-\lambda)t}$$

$$P\left(W_s > \frac{15}{60}\right) = e^{-(10-6)\frac{15}{60}}$$

$$= e^{-1} = 0.3679.$$

(3) In a given $(M/M/1):(\infty/FIFO)$ queue, $\rho = 0.6$, what is the probability that the queue contain 5 or more students?

(2)

$$P(x \geq 5) = \rho^5 = (0.6)^5 = 0.0467.$$

(4) What is the effective arrival rate for $(M/M/1):(4/FIFO)$ queueing model when $\lambda = 2$ & $\mu = 5$

$$\lambda' = \mu(1 - P_0), \quad P_0 = \frac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}$$

here $K = 4$, $\lambda = 2$ & $\mu = 5$

$$P_0 = \frac{1 - \frac{2}{5}}{1 - \left(\frac{2}{5}\right)^5} = 0.607$$

$$\lambda' = \mu(1 - P_0) = 5(1 - 0.607) = 5 \times 0.393 = 1.965$$

$$\boxed{\lambda' \simeq 2}$$

(5) a) In $(M|M|1):(K|FIFO)$ write down the expression for P_0.

b) In $(M|M|1):(K|FIFO)$, $\lambda = 3/hr$, $\mu = 4/hr$, $K = 7$, Calculate P_0.

Soln:

a)
$$P_0 = \begin{cases} \dfrac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} & \text{if } \lambda \neq \mu \\[2ex] \dfrac{1}{K+1} & \text{if } \lambda = \mu. \end{cases}$$

(5) The tpm of a Markov chain $(X_n : n = 1, 2, 3 \dots)$ with three states 0, 1 & 2 is (3)

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$ with initial distribution

$P(0) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Find $P(X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 2)$.

Given
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix} \end{array} = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{pmatrix}.$$

and $P(0) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$P(X_0 = 0) = P(X_0 = 1) = P(X_0 = 2) = \frac{1}{3}$$

$$P(X_3 = 1, X_2 = 1, X_1 = 1, X_0 = 2)$$
$$= P(X_3 = 1 / X_2 = 1)\, P(X_2 = 1 / X_1 = 1)\, P(X_1 = 1 / X_0 = 2)\, P(X_0 = 2)$$
$$= P_{11}^{(1)} P_{11}^{(1)} P_{21}^{(1)} P(X_0 = 2)$$
$$= \frac{1}{2} \frac{1}{2} \frac{\cancel{3}}{4} \frac{1}{\cancel{3}} = \frac{1}{16}.$$

(6) If the initial state distribution of a Markov chain is $p^{(0)} = \left(\frac{5}{6}\ \frac{1}{6}\right)$ and the tpm of the chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, find the probability distribution of the chain after 2 steps.

Given, $p^{(0)} = \left(\frac{5}{6}\ \frac{1}{6}\right)$

and $P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(9) In a railway marshalling yard goods train arrive at rate of 30 trains per day. Assuming that the inter arrival time follows an exponential distribution and the service time is also exponential with an average of 36 minutes. Calculate expected queue size (line length). (4)

Given $\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$ trains/minute.

$\mu = \frac{1}{36}$ trains/minute

Expected queue length $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$

$$L_q = \frac{\left(\frac{1}{48}\right)^2}{\frac{1}{36}\left(\frac{1}{36} - \frac{1}{48}\right)}.$$

$$= \frac{0.000434}{0.0001929}$$

$$= 2.24$$

$$\boxed{L_q \simeq 2}$$

(10) Consider an M/M/1 queuing system, if $\lambda = 6$ & $\mu = 8$ find the probability ~~that~~ of atleast 10 customers in the system.

Probability that atleast n customers in the System $= \left(\frac{\lambda}{\mu}\right)^n = \left(\frac{6}{8}\right)^{10}$

$$= \left(\frac{3}{4}\right)^{10}.$$

(11). In an $(M|M|1):(K|FIFO)$ queue with $K=5$, $\lambda = 5/hr$, Service time $= 10$ min/person find the percentage of time the system is idle. (5)

Probability that the system is idle

$$= P_0$$

$$= \frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}, \quad \lambda \neq \mu$$

~~= 1 − (5~~

Given, $\lambda = 5/hr$, $\mu = \frac{1}{10}/min = 6/hr.$

$$P_0 = \frac{1-\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)^6} = \frac{\frac{1}{6}}{1-0.335}$$

$$= 0.2556$$

$$= 25.56\%$$

(12) People arrive to purchase cinema tickets at the average rate of 6 per minute, and it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and it takes exactly 1.5 min to reach the correct seat after purchasing a ticket. Can he expect to be seated for the start of the picture?

$\lambda = 6/min$ $\quad \mu = \frac{60}{7.5} = 8/min$

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{8-6} = 0.5 min$$

$\therefore$ Total time required to purchase the ticket and to reach the seat $= 0.5+1.5 = 2$ min.

PART - C

(6)

(1) Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 minutes.

(i) Find the average number of persons waiting in the system.

(ii) What is the probability that a person arriving at the booth will have to wait in the queue?

(iii) What is the probability that it will take him more than 10 minutes altogether to wait for the phone and completes his call?

iv) Estimate the fraction of the day when the phone will be in use.

Given $\lambda = \frac{1}{12}$ /min $\mu = \frac{1}{4}$ /min.

(i) $$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} = \frac{\frac{1}{12}}{\frac{8}{48}} = \frac{1}{12} \times \frac{48}{8}$$

$$= \frac{1}{2} = 0.5$$

(ii) Probability that a person arriving at the booth will have to wait in the queue

$$= P(n \geq 1) = 1 - \left(\frac{\lambda}{\mu}\right)^1$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

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(iii) what is the probability that it will take ~~him~~ him more than 10 min altogether ~~to~~ to wait for the phone and complete his call?

$$P(W_q > 10) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$$

$$= \frac{\frac{1}{12}}{\frac{1}{4}} e^{-\left(\frac{1}{4}-\frac{1}{12}\right)10}$$

$$= \frac{1}{3} e^{-\frac{1}{6}\times 10}$$

$$= \frac{1}{3} e^{-\frac{10}{6}} = 0.063.$$

iv) Fraction of a day that the phone will be in use $= \frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3} = 0.33$

② A departmental store has a single cashier. During the rush hours, customer arrive at the rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour.

(i) what is the probability that the cashier is idle?

(ii) what is the average number of customer in the system?

(iii) what is the average time a customer spends in the system?

iv) what is the average number of customers in the queue?

v) what is the average time a customer spends in the queue; waiting for service?

The model is $(M|M|1):(\infty|FIFO)$. (8)

Given, $\lambda = 20/\text{hour}$ & $\mu = 24/\text{hour}$.

(i) Probability that the cashier is idle $= P_0$

$$= 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{20}{24} = 0.1674$$

(ii) Average number of customers in the queuing system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{20}{24-20}$$

$$\boxed{L_s = 5}$$

(iii) The average time a customer spends in the system

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{4} \text{ hour}$$

$$W_s = 15 \text{ min}$$

(iv) Average number of customers in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{20^2}{24(24-20)}$$

$$= 4.167$$

$$\simeq 4 \text{ customers.}$$

v) The average time a customer spends in the queue waiting for service

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{24(24-20)}$$

$$= \frac{20}{96} \text{ hr} = 12.5 \text{ min}.$$

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(3) Customers arrive at a watch repair shop according to a Poisson Process at a rate of every 10 minutes and the Service time is an exponential random variable with mean 8 minutes. (i) Find the average number of customers L_s in the shop.

(ii) Find the average time a customer spends in the shop.

(iii) Find the average number of customers in the queue L_q.

iv) What is the probability that the server is idle?

Soln:

Given $\lambda = 6/\text{hour}$, $\mu = \frac{60}{8}/\text{hour}$
$= \frac{15}{2}/\text{hour}$.

(i) $L_s = \dfrac{\lambda}{\mu-\lambda} = \dfrac{6}{\frac{15}{2}-6} = 6 \times \frac{2}{3} = 4$

(ii) $W_s = \dfrac{1}{\mu-\lambda} = \dfrac{1}{\frac{15}{2}-6} = \frac{2}{3}$ hour
$= 40$ minutes.

(iii) $L_q = \dfrac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{\lambda}{\mu} \cdot L_s$
$= \dfrac{6}{15/2} \times 4$
$= \dfrac{\cancel{12}^{4}}{\cancel{15}_{5}} \times 4 = \dfrac{16}{5}$
$= 3.2 \simeq 3$.

iv) P(system is empty) $= P_0 = 1 - \dfrac{\lambda}{\mu}$
$= 1 - \dfrac{6}{15/2} = \dfrac{1}{5}$.

(4) A self-Service store one cashier at its counter. (10)
8 customers arrive on an average every 5 minutes while the cashier can serve 10 customers in the same time. Assuming Poisson distribution for arrival and exponential distribution for service rate. Determine

(i) Average number of customers in the system.
(ii) Average number of customers in queue or average queue length.
(iii) Average time a customer spends in the system.
iv) Average time a customer waits before being served.

Soln:

Given arrival rate $\lambda = \frac{8}{5} = 1.6$ customer/min

Service rate $\mu = \frac{10}{5} = 2$ customers /min

(i) Average number of customers in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1.6}{2 - 1.6} = 4$$

(ii) Average number of customers in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(1.6)^2}{2(2-1.6)} = 3.2 \simeq 3$$

(iii) Average time a customer spends in the system

$$W_s = \frac{L_s}{\lambda} = \frac{4}{1.6} = 2.5 \text{ minutes.}$$

iv) Average time a customer spends in the queue

$$W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right) = 0.8\left(\frac{1}{2-1.6}\right) = 2 \text{ minutes.}$$

(5) On an average 96 patients per 24 hour day require the service of an emergency clinic. Also an average a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs 100/- per patient treated to obtain an average servicing time of 10 minutes and that each minute of decrease in this average time would cost Rs 10/- per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patient to $\frac{1}{2}$ patient. (11)

Soln: Given $\lambda = \frac{96}{24} = 4$ / hour.

$\mu = 6$ / hour

Average number of patients in the queue

$$= L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{16}{6(6-4)} = \frac{4}{3} = 1\frac{1}{3}$$

The new length L_q is $\frac{1}{2}$.

Let μ' be the new service rate

$$L_q' = \frac{\lambda^2}{\mu'(\mu'-\lambda)}$$

$$\frac{1}{2} = \frac{16}{\mu'(\mu'-4)} \qquad \therefore \mu'(\mu'-4) = 32$$

(12)

$$\mu'^2 - 4\mu' - 32 = 0$$

$$(\mu' - 8)(\mu' + 4) = 0$$

$$\mu' = 8 \quad \text{or} \quad \mu' = -4$$

Since μ' cannot be -4, we have $\mu' = 8$/hour.

$\therefore$ To decrease the size of the queue to $\frac{1}{2}$ patient, the new service rate should be $\mu' = 8$/hour.

$\therefore$ Average time required to attend a patient $= \frac{1}{8}$ hr

$= \frac{15}{2}$ minutes.

$\therefore$ Decrease in time required to attend a patient

$= 10 - \frac{15}{2} = \frac{5}{2}$ minutes.

$\therefore$ Increase in treatment cost of a patient

$= \frac{5}{2} \times 10 = $ Rs 25.

$\therefore$ Amount to be budgeted per patient to decrease the size of the queue $=$ Rs $(100 + 25)$

$=$ Rs 125/-.

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MODEL II : (M/M/1) : (K/FIFO) (13)

(6) The local one-person barber shop can accommodate a maximum of 5 people at a time (4 waiting and 1 getting hair cut). customers arrive according to a Poisson distribution with mean 5 per hour. The barber cuts hair at an average rate of 4 per hour (Exponential Service time)

a) what percentage of time is the barber idle?

b) what fraction of the potential customers are turned away?

c) what is the expected number of customers waiting for a hair-cut?

d) How much time can a customer expect to spend in the barber shop?

$\lambda = 5, \quad \mu = 4 \quad K =$ capacity of the System $= 5.$

a) P(the barber is idle) $= P(N=0)$

$$= P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}$$

$$= \frac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^{6}}$$

$$= 0.0888.$$

$\therefore$ Percentage of time when the barber is idle $= 9\,\%$

b) P (a customer is turned away) = $P(N>5)$ (74)

$$= \left(\frac{\lambda}{\mu}\right)^K \left[\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right]$$

$$= \left(\frac{5}{4}\right)^5 \left[\frac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^6}\right]$$

$$= \frac{3125}{11529} = 0.2711$$

$\therefore$ 0.2711 × Potential customers are turned away.

c) $E(L_q) = L_s - (1-P_0)$

$$= \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} - (1-P_0)$$

$$= -5 - \frac{6\times\left(\frac{5}{4}\right)^6}{1-\left(\frac{5}{4}\right)^6} - (1-0.0888)$$

$$= \frac{6\times\frac{15625}{4096}}{\frac{11529}{4096}} - 5.9112$$

$= 2.2$ customers.

d) $W_s = \frac{L_s}{\lambda'} = \frac{L_s}{\mu(1-P_0)}$

$$= \frac{3.1317}{3.6448} = 0.8592\ h$$

$= 51.5$ min. //

7) Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour. (15)

a) Find the effective arrival rate at the clinic.

b) what is the probability that an arriving patient will not wait?

c) what is the expected waiting time until a patient is discharged from the clinic?

Given, $\lambda = 30/\text{hour}$, $\mu = 20/\text{hour}$

$K = 1+14 = 15$

a) Since $\lambda \neq \mu$, $P_0 = \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}$

$$= \frac{1-\frac{3}{2}}{1-\left(\frac{3}{2}\right)^{16}} = 0.00076$$

$\therefore$ Effective arrival rate $= \mu(1-P_0)$

$$\lambda' = 20(1-0.00076)$$

$$= 19.98/\text{hour}.$$

b) $P(\text{a patient will not wait})$

$$= P_0 = 0.00076$$

(16)

c) $$L_s = \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}$$

$$= -3 - \frac{16 \times \left(\frac{3}{2}\right)^{16}}{1-\left(\frac{3}{2}\right)^{16}}$$

$= 13$ patients nearly.

$$\therefore W_s = \frac{L_s}{\lambda'} = \frac{13}{19.98}$$

$= 0.65\, h$ or 39 min.

(8) At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Train arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 6 per hour. Assuming Poisson arrivals and exponential service distribution, find the probabilities for the number of trains in the system. Also find the average waiting time of a new train coming in to the yard. If the handling rate is doubled, how will the above results get modified?

Given, $\lambda = 6/\text{hour}$, $\mu = 6/\text{hr}$

$K = 1+2 = 3$

Since $\lambda = \mu$, $P_0 = \frac{1}{K+1}$

(17)

$$\boxed{P_0 = \frac{1}{4}}$$

$$L_s = \frac{K}{2}$$
$$= 1.5 \text{ trains}$$

$$W_s = \frac{L_s}{\lambda'} = \frac{1.5}{\mu(1-P_0)} = \frac{1.5}{6(1-\frac{1}{4})}$$
$$= \frac{1}{3} h = 20 \text{ min.}$$

(ii) $\lambda = 6,\ \mu = 12,\ K = 3$

Since $\lambda \neq \mu$, $P_0 = \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}$

$$= \frac{1-\frac{1}{2}}{1-\left(\frac{1}{2}\right)^4} = \frac{8}{15}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right)$$
$$= \left(\frac{8}{15}\right)\left(\frac{1}{2}\right)^n, \text{ for } n = 1, 2, 3 \ldots$$

$$L_s = \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}$$
$$= 1 - \frac{4}{15} = \frac{11}{15}.$$
$$= 0.73 \text{ train.}$$

(18)

$$W_s = \frac{L_s}{\lambda'}$$

$$= \frac{L_s}{\mu(1-P_0)}$$

$$= \frac{\frac{11}{15}}{12\left(1-\frac{8}{15}\right)}$$

$$= \frac{11}{84}\ h \quad \text{or } 7.9 \text{ min.}$$

—— x ——.