Unit - 5

Z-Transforms

Definition -

Let $\{x(n)\}$ be a sequence defined on $n = 0, 1, 2 \ldots$ and $x(n) = 0$ for $n < 0$ then its z-transform is defined as

$$z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

where z is an arbitrary complex number.

1) Find $z\{1\}$.

WKT $z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$ $\qquad (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$$z\{1\} = \sum_{n=0}^{\infty} 1 . z^{-n}$$

$$z\{1\} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$z\{1\} = \frac{z}{z-1}$$

2) Find $z\{a^n\}$.

$$z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{z^n}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$= \frac{z}{z-a}$$

3) Find $z\left\{\frac{1}{n!}\right\}$

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots=e^x$$

$$z\left\{\frac{1}{n!}\right\}=\sum_{n=0}^{\infty}\frac{1}{n!}z^{-n}$$

$$=1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}+\dots$$

$$=1+\frac{\left(\frac{1}{z}\right)}{1!}+\frac{\left(\frac{1}{z}\right)^2}{2!}+\frac{\left(\frac{1}{z}\right)^3}{3!}+\dots$$

$$=e^{\frac{1}{z}}$$

4) Find $z\left\{\frac{1}{n}\right\}$.

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots=-\log(1-x)$$

$$z\left\{\frac{1}{n}\right\}=\sum_{n=0}^{\infty}\frac{1}{n}z^{-n}$$

$$=\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z^2}\right)+\frac{1}{3}\left(\frac{1}{z^3}\right)+\dots$$

$$=\frac{1}{z}+\frac{\left(\frac{1}{z}\right)^2}{2}+\frac{\left(\frac{1}{z}\right)^3}{3}+\dots$$

$$=-\log\left(1-\frac{1}{z}\right)$$

$$=-\log\left(\frac{z-1}{z}\right)$$

$$=\log\left(\frac{z}{z-1}\right)$$

5) Find $z\left\{\frac{1}{n-1}\right\}$.

$$z\left\{\frac{1}{n-1}\right\}=\sum_{n=0}^{\infty}\frac{1}{(n-1)}z^{-n}$$

$$=-1+\frac{1}{z^2}+\frac{1}{2}\left(\frac{1}{z^3}\right)+\frac{1}{3}\left(\frac{1}{z^4}\right)\dots$$

Linear Property

i) $z\{ax(n)+by(n)\} = az\{x(n)\} + bz\{y(n)\}$

ii) $z\{af(t)+bg(t)\} = az\{f(t)\} + bz\{g(t)\}$

Find $z\left\{\frac{2}{n}+3\left(\frac{1}{2}\right)^n\right\} = 2z\left\{\frac{1}{n}\right\} + 3z\left\{\left(\frac{1}{2}\right)^n\right\}$

$= 2\log\left(\frac{z}{z-1}\right) + 3\left(\frac{z}{z-\frac{1}{2}}\right)$

$z\left\{\left(\frac{1}{2}\right)^n\right\} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}$

$= 1+$

Find $z\left\{\frac{1}{3^n}+e^{-3n}\right\} = z\left\{\left(\frac{1}{3}\right)^n+(e^{-3})^n\right\}$

$= z\left\{\left(\frac{1}{3}\right)^n\right\} + z\left\{(e^{-3})^n\right\}$

$= \frac{z}{z-\left(\frac{1}{3}\right)} + \frac{z}{z-(e^{-3})}$

$= \frac{3z}{3z-1} + \frac{z}{z-e^{-3}}$

Find $z\{\cos n\theta\}$ and $z\{\sin n\theta\}$.

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{in\theta} = \cos n\theta + i\sin n\theta$

$(e^{i\theta})^n = e^{in\theta}$

$z\{e^{in\theta}\} = z\{(e^{i\theta})^n\} = \frac{z}{z-e^{i\theta}}$

But, $z\{\cos n\theta + i\sin n\theta\} = \frac{z}{z-(\cos\theta+i\sin\theta)}$

$z\{\cos n\theta\} + iz\{\sin n\theta\} = \frac{z}{(z-\cos\theta)-i\sin\theta}$

$= \frac{z((z-\cos\theta)+i\sin\theta)}{((z-\cos\theta)-i\sin\theta)((z-\cos\theta)+i\sin\theta)}$

$= \frac{z(z-\cos\theta)+iz\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$z\{\cos n\theta\} + iz\{\sin n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

comparing real and imaginary sides on both sides,

$$z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

z- Transform basic -
Formulae -

1) $z(1) = \frac{z}{z-1}$

2) $z(a^n) = \frac{z}{z-a}$

3) $z(n) = \frac{z}{(z-1)^2}$

4) $z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$

5) $z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)$

6) $z\left(\frac{1}{n-1}\right) = \frac{1}{z}\log\left(\frac{z}{z-1}\right)$

7) $z\left(\frac{1}{n!}\right) = e^{\frac{1}{z}}$

8) $z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

9) $z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$

Find $z(r^n \cos n\theta)$ and $z(r^n \sin n\theta)$.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

$$r^n e^{in\theta} = r^n \cos n\theta + i r^n \sin n\theta$$

$$z\{(re^{i\theta})^n\} = \frac{z}{z - re^{i\theta}}$$

$$z(r^n e^{in\theta}\} = \frac{z}{z-(r\cos\theta + ir\sin\theta)}$$

$$= \frac{z \,(z+(r\cos\theta + ir\sin\theta))}{z-(r\cos\theta + ir\sin\theta)(z+(r\cos\theta + ir\sin\theta))}$$

$$= \frac{z^2 + zr\cos\theta + zir\sin\theta}{z^2 - (r\cos\theta + ir\sin\theta)^2}$$

$$z(r^n\cos n\theta + ir^n \sin n\theta) = \frac{z}{z-r\theta} \frac{((z-r\cos\theta)+ir\sin\theta)}{((z-r\cos\theta)-ir\sin\theta)((z-r\cos\theta)+ir\sin\theta)}$$

$$= \frac{z(z-r\cos\theta) + izr\sin\theta}{z^2 + r^2\cos^2\theta - 2zr\cos\theta + r^2\sin^2\theta}$$

$$= \frac{z(z-r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$z(r^n\cos\theta) + iz(r^n\sin\theta) = \frac{z(z-r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} + i\frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

comparing real and imaginary parts on both sides, we get

$$z(r^n\cos\theta) = \frac{z(z-r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

$$z(r^n\sin\theta) = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Find z transform of $z\left\{\cos \frac{n\pi}{2}\right\}$, $z\{a^{n-1}\}$, $z\left\{\frac{1}{n(n-1)}\right\}$.

i) WKT $z\{\cancel{z}\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

$$z\left\{\cos \frac{n\pi}{2}\right\} = \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z(z-0)}{z^2+1} = \frac{z^2}{z^2+1}$$

ii) $z\{a^{n-1}\} =$

WKT $z\{a^n\} = \frac{z}{z-a}$

$$z\{a^{n-1}\} = z\{a^n . a^{-1}\}$$

$$= \frac{1}{a} z\{a^n\}$$

$$= \frac{1}{a} \times \frac{z}{z-a}$$

iii) $z\left\{\frac{1}{n(n-1)}\right\} = z\left\{\frac{1}{n} \times \frac{1}{(n-1)}\right\}$

$$\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$$

$$1 = A(n-1) + Bn$$

$$1 = An - A + Bn$$

$$1 = (A+B)n - A$$

comparing coeff on both sides,

$$-A = 1$$

$$A + B = 0$$

$$-1 + B = 0$$

$$B = 1$$

$$\frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$$

$$z\left\{\frac{1}{n(n-1)}\right\} = z\left\{\frac{-1}{n} + \frac{1}{n-1}\right\}$$

First Shifting Theorem -

If $z\{f(n)\} = F(z)$, then $z\{a^n f(n)\} = F(\frac{z}{a})$

Find z-transform of $z\{a^n \cos n\theta\}$.

$z\{a^n \cos n\theta\} = z\{\cos n\theta\}$, after finding z-transform of this; replace z by $(\frac{z}{a})$

$$= \left[\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right]_{z \to \frac{z}{a}}$$

$$= \frac{\frac{z}{a}\left(\frac{z}{a} - \cos\theta\right)}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

Find $z\{\frac{2^{n+1}}{n!}\}$.

$$= z\left\{\frac{2^n \cdot 2}{n!}\right\} = 2z\left\{\frac{2^n}{n!}\right\} = 2z\left\{\frac{1}{n!}\right\}_{z \to \frac{z}{2}}$$

$$= 2e^{\frac{1}{z}}$$

$$= 2e^{\frac{2}{z}}$$

Find $z\{a^n n\}$.

$$z\{n\} = \left[\frac{z}{(z-1)^2}\right]_{z \to \frac{z}{a}}$$

$$= \frac{\left(\frac{z}{a}\right)}{\left(\frac{z}{a} - 1\right)^2}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{z}{\cancel{a}} \times \frac{a^{\cancel{2}}}{(z-a)^2} = \frac{za}{(z-a)^2}$$

Find $z\{\frac{a^n}{n}\}$.

$$z\{\frac{1}{n}\} = \left[\log\left(\frac{z}{z-1}\right)\right]_{z \to \frac{z}{a}}$$

$$= \left(\log \frac{\frac{z}{a}}{(\frac{z}{a}-1)}\right)$$

$$= \log\left(\frac{\frac{z}{a}}{\frac{z-a}{a}}\right)$$

$$= \log\left(\frac{z}{\cancel{a}} \times \frac{\cancel{a}}{z-a}\right)$$

$$= \log\left(\frac{z}{z-a}\right)$$

Find $z\{\frac{a^n}{n!}\}$.

$(a^2+b^2) = (a+b)^2 - 2ab$

$$z\{\frac{1}{n!}\}_{z \to \frac{z}{a}} = [e^{\frac{1}{z}}]_{z \to \frac{z}{a}}$$

$$= e^{\frac{a}{z}}$$

$-z((z+2)^2 - 2$

Find z (or) Differentiation Derivative Property -

$$z\{nf(n)\} = -z\frac{d}{dz}F(z)$$

Find $z\{n^2\}$ and $z\{n^3\}$.

$$z\{n^2\} = z\{n \times \underbrace{n}_{f(n)}\}$$

$$= -z\frac{d}{dz}z\{n\}$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left[\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z^2 - \cancel{2z} + 1 + \cancel{2z} + 1}{(z-1)^4}\right]$$

$$= \frac{-z(z^2+2)}{(z-1)^4} \qquad = \frac{z(z+1)}{(z-1)^3}$$

$$z(n^3) = z(n.n^2)$$

$$= -z\frac{d}{dz}\, z(n^2)$$

$$= -z\frac{d}{dz}\left(\frac{z^2+zu}{(z-1)^3}\right)$$

$$= -z\left[\frac{(z-1)^3(2z+1)-(z^2+z)\,3(z-1)^2}{(z-1)^6}\right]$$

$$= -z\left[\frac{\cancel{(z-1)^2}\left[(2z+1)(z-1)-(z^2+z)3\right]}{(z-1)^{\cancel{6}\,4}}\right]$$

$$= -z\left[\frac{2z^2-2z+z-1-3z^2-3z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right] \qquad = +z\left[\frac{z^2+4z+1}{(z-1)^4}\right]$$

Initial Value Theorem -

If $Z\{f(t)\} = F(z)$

then $f(0) = \lim_{z \to \infty} F(z)$

Final Value Theorem -

If $Z\{f(t)\} = F(z)$

then $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1)F(z)$

If $F(z) = \dfrac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$, find $f(0)$ and $\lim_{t \to \infty} f(t)$.

By initial value theorem,

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{z(z-\cos at)}{z^2 - 2z\cos at + 1}$$

L' hospital rule, (differentiate num and den separately)

$$= \lim_{z \to \infty} \frac{2z - \cos at}{2z - 2\cos at}$$

$$= \lim_{z \to \infty} \frac{2}{2} = 1$$

By final value theorem,

$$\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1)F(z)$$

$$= \lim_{z \to 1} \frac{(z-1)z(z-\cos at)}{z^2 - 2z\cos at + 1}$$

$$= 0$$

Inverse Z-Transform-

The inverse z-transform of $z\{x(n)\} = X(z)$ is defined as $z^{-1}\{x(z)\} = x(n)$.

$$z(a^n) = \frac{z}{z-a}$$

$$z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$$

Basic Formulae-

1) $z^{-1}\left\{\frac{z}{z-1}\right\} = 1$, $z^{-1}\left\{\frac{z}{z+1}\right\} = (-1)^n$

2) $z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$, $z^{-1}\left\{\frac{z}{z+a}\right\} = (-a)^n$, $z^{-1}\left\{\frac{1}{z+a}\right\} = a^{n-1}$

3) i) $z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = (n+1)a^n$

ii) $z^{-1}\left\{\frac{z}{(z-a)^2}\right\} = na^{n-1}$

iii) $z^{-1}\left\{\frac{1}{(z-a)^2}\right\} = (n-1)a^{n-2}$

iv) $z^{-1}\left\{\frac{z^2}{(z-1)^2}\right\} = n+1$

v) $z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n$

vi) $z^{-1}\left\{\frac{1}{(z-1)^2}\right\} = n-1$

4) $z^{-1}\left\{\frac{z^2}{z^2+a^2}\right\} = a^n \cos\frac{n\pi}{2}$

5) $z^{-1}\left\{\frac{z}{z^2+a^2}\right\} = a^n \sin\frac{n\pi}{2}$

Partial Fraction Method-

1) Find $z^{-1}\left\{\frac{10z}{(z-1)(z-2)}\right\}$.

Here, $X(z) = \frac{10z}{(z-1)(z-2)}$

$$\frac{X(z)}{z} = \frac{10}{(z-1)(z-2)}$$

Partial Fractions to be done for $\frac{X(z)}{z}$ and not $X(z)$.

$$\frac{10}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2}$$

$$10 = a(z-2) + b(z-1)$$

For $z=1$,

$$10 = a(1-2)$$

$$10 = a(-1)$$

$$a = -10$$

For $z=2$,

$$10 = b(2-1)$$

$$\frac{10}{1} = b$$

$$b = 10$$

$$\frac{X(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2}$$

$$x(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$z^{-1}\{x(z)\} = z^{-1}\left\{\frac{-10z}{z-1}\right\} + z^{-1}\left\{\frac{10z}{z-2}\right\}$$

$$= -10\, z^{-1}\left\{\frac{z}{z-1}\right\} + 10\, z^{-1}\left\{\frac{z}{z-2}\right\}$$

$$= -10(1) + 10(2^n)$$

$$= -10 + 10 . 2^n$$

$$= 10(2^n - 1)$$

Find $z^{-1}\left\{\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right\}$

$\uparrow$
$x(z)$

$$\frac{x(z)}{z} = \frac{z^2-z+2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{\cancel{Cz+D}\, C}{(z-1)^2}$$

$$z^2-z+2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

For $z=1$,

$$2 = C(2)$$

$$C = 1$$

For $z=-1$,

$$4 = A(4) + 1(0)$$

$$A = 1$$

For $z = 0$,

$$2 = 1(1) + B(1)(-1) + 1(1)$$

$$2 = 2 - B$$

$$B = 0$$

$$\frac{x(z)}{z} = \frac{1}{z+1} + 0 + \frac{1}{(z-1)^2}$$

$$x(z) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$$z^{-1}\{x(z)\} = z^{-1}\left\{\frac{z}{z+1}\right\} + z^{-1}\left\{\frac{z}{(z-1)^2}\right\}$$

$$= (-1)^n + n$$

Long Division Method -

1) Find $z^{-1}\left\{\dfrac{z^2+2z}{z^2+2z+4}\right\}$ by long division method.

$$1-4z^{-2}+8z^{-3}-32z^{-5}$$

$$z^2+2z+4 \,\big)\, \cancel{z^2}+\cancel{2z}$$

$$\underset{(-)}{\cancel{z^2}} \underset{(-)}{+} \cancel{2z} \underset{(-)}{+} 4$$

$$-\cancel{4}$$

$$\underset{(-)}{-\cancel{4}} \underset{(-)}{-} 8z^{-1} \underset{(-)}{-} 16z^{-2}$$

$$8\cancel{z^{-1}}+16\cancel{z^{-2}}$$

$$\underset{(-)}{8\cancel{z^{-1}}} \underset{(-)}{+} 16\cancel{z^{-2}} \underset{(-)}{+} 32z^{-3}$$

$$-32\cancel{z^{-3}}$$

$$\underset{(-)}{-32\cancel{z^{-3}}} \underset{(-)}{-} 64z^{-4} \underset{(-)}{-} 128z^{-5}$$

$$64z^{-4}+128z^{-5}$$

Here, $\dfrac{z^2+2z}{z^2+2z+4} = 1-4z^{-2}+8z^{-3}-32z^{-5}+\dots$

$\{x(n)\} = \{x(0), x(1), x(2), x(3), x(4), \dots\}$

$x(0)=1,\ x(1)=0,\ x(2)=-4,\ x(3)=8,\ x(4)=0,\ x(5)=-32 \dots$

The sequence of the inverse z-transform is,

$\{1, 0, -4, 8, 0, -32, \dots\}$

2) Find $z^{-1}\left\{\dfrac{1}{1+4z^{-2}}\right\}$.

$$1-4z^{-2}+16z^{-4}-64z^{-6}$$

$$1+4z^{-2} \,\big)\, \cancel{1}$$

$$\underset{(-)}{\cancel{1}} \underset{(-)}{+} 4z^{-2}$$

$$-4\cancel{z^{-2}}$$

$$\underset{(-)}{-4\cancel{z^{-2}}} \underset{(-)}{-} 16z^{-4}$$

$$16\cancel{z^{-4}}$$

$$\underset{(-)}{16\cancel{z^{-4}}} \underset{(-)}{+} 64z^{-6}$$

$$-64\cancel{z^{-6}}$$

$$\underset{(-)}{-64\cancel{z^{-6}}} \underset{(-)}{-} 256z^{-8}$$

$$256z^{8}$$

$$\frac{1}{1+4z^{-2}} = 1-4z^{-2}+16z^{-4}-64z^{-6}+\dots$$

(with $2^0, 2^2, 2^4, 2^6$ written above the terms)

$x(0)=1,\ x(1)=0,\ x(2)=-4,\ x(3)=0,$

$x(4)=16,\ x(5)=0,\ x(6)=-64$

sequence of inverse z-transform is,

$\{1, 0, -4, 0, 16, 0, -64, \dots\}$

$$x(n) = \sum_{n=0}^{\infty} 2^n \cos\frac{n\pi}{2} z^{-n} \quad \text{(Sequence)}$$

$$x(n) = 2^n \cos\frac{n\pi}{2}$$

Finding z-inverse using Residue Method-

1) Find $f(z)$

2) construct $f(z)$ into $f(z).z^{n-1}$

3) Equate denominator of $f(z).z^{n-1}$ to zero to find poles

4) Write poles and its order

5) Find the residues at each poles

6) $x(n)$ = Sum of residues

Find $z^{-1}\left\{\dfrac{10z}{(z-1)(z-2)}\right\}$.

$$f(z) = \frac{10z}{(z-1)(z-2)}$$

Now, $f(z).z^{n-1} = \dfrac{10z.z^{n-1}}{(z-1)(z-2)}$

$$= \frac{10.z^n}{(z-1)(z-2)}$$

consider, $(z-1)(z-2)=0$

$z=1,2$ are poles

Poles are 1, 2 of order 1.

Residue of $f(z).z^{n-1}]$ at $z=1$ $= \lim\limits_{z\to 1} \cancel{(z-1)} \dfrac{10z^n}{\cancel{(z-1)}(z-2)}$

$$= \cancel{10z} \;\frac{10}{-1} = -10$$

Residue of $f(z).z^{n-1}]$ at $z=2$ $= \lim\limits_{z\to 2} \cancel{(z-2)} \dfrac{10z^n}{\cancel{(z-1)}\cancel{(z-2)}}$ $(z-1)$

$$= \frac{10.2^n}{1} = 10.2^n$$

$z^{-1}\left\{\dfrac{10z}{(z-1)(z-2)}\right\}$ = sum of residues

$= 10.2^n - 10$

$= 10(2^n - 1)$

Find $z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$.

$$f(z) = \frac{z^2}{(z-a)(z-b)}$$

$$f(z).\, z^{n-1} = \frac{z^2}{(z-a)(z-b)} . z^{n-1}$$

$$= \frac{z^{n+1}}{(z-a)(z-b)}$$

Den = 0,

$(z-a)(z-b) = 0$

$z = a, b$ of order 1

$$\text{Res } f(z).\, z^{n-1}\Big]_{\text{at } z=a} \Rightarrow \lim_{z\to a} \cancel{(z-a)} \frac{z^{n+1}}{\cancel{(z-a)}(z-b)}$$

$$= \frac{a^{n+1}}{a-b}$$

$$\text{Res } f(z).\, z^{n-1}\Big] \text{ at } z=b \Rightarrow \lim_{z\to b} \cancel{(z-b)} \frac{z^{n+1}}{(z-a)\cancel{(z-b)}}$$

$$= \frac{b^{n+1}}{b-a}$$

$$z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}$$

Find $z^{-1}\left\{\frac{2z}{(z-2)(z^2+1)}\right\}$.

$$f(z) = \frac{2z}{(z-2)(z^2+1)}$$

$$f(z).\, z^{n-1} = \frac{2.z^n}{(z-2)(z^2+1)} = \frac{2.z^n}{(z-2)(z-i)(z+i)}$$

$(z-2)(z^2+1) = 0$

$z = 2, \pm i$ are poles of order 1

$$\text{Res at } z=2 \Rightarrow \lim_{z\to 2} \cancel{(z-2)} \frac{2.z^n}{\cancel{(z-2)}(z-i)(z+i)}$$

$$= \frac{2^{n+1}}{(2-i)(2+i)} = \frac{2^{n+1}}{2^2+1} = \frac{2^{n+1}}{5}$$

Res at $z=i \Rightarrow \lim_{z\to i} \cancel{(z-i)} \dfrac{2 \cdot z^n}{(z-2)(\cancel{z-i})(z+i)}$

$$= \frac{\cancel{2} \cdot i^n}{(i-2)(\cancel{2i})} = \frac{i^{n-1}}{(i-2)}$$

Res at $z=-i \Rightarrow \lim_{z\to -i} \cancel{(z+i)} \dfrac{2 \cdot z^n}{(z-2)(z-i)(\cancel{z+i})}$

$$= \frac{2 \cdot (-i)^n}{(-i-2)(-2i)}$$

$$= \frac{\cancel{2} \cdot (-i)^n}{(2+i)(\cancel{2}i)} =$$

$$z^{-1}\left\{\frac{2z}{(z-2)(z^2+1)}\right\} = \frac{2^{n+1}}{(2-i)(2+i)} + \frac{2 \cdot i^n}{(i-2)(2i)} + \frac{2 \cdot (-i)^n}{(2+i)(2i)}$$

Find $z^{-1}\left\{\dfrac{z(z+1)}{(z-1)^3}\right\}$ by residue method.

Pole of order $n \to$

$$\text{Res } f(z)]_{z=a} \Rightarrow \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

simple pole $\to$

$$\text{Res } f(z)]_{z=a} = \lim_{z\to a} (z-a) f(z)$$

$$f(z) = \frac{z(z+1)}{(z-1)^3}$$

$$f(z) \cdot z^{n-1} = \frac{z^n(z+1)}{(z-1)^3}$$

$z=1$ is a pole of order 3

$$\text{Res } f(z) \cdot z^{n-1}]_{\text{at } z=1} \Rightarrow \frac{1}{(3-1)!} \lim_{z\to 1} \frac{d^{3-1}}{dz^{3-1}} \cancel{(z-1)^3} \frac{z^n(z+1)}{\cancel{(z-1)^3}}$$

$$\Rightarrow \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2} (z^{n+1} + z^n)$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz} [\underset{u}{(n+1)} \underset{v}{z^n} + \underset{u}{n} \underset{v}{z^{n-1}}]$$

$$= \frac{1}{2} \lim_{z\to 1} \left[(n+1) n z^{n-1} + n(n-1) z^{n-2}\right]$$

$$= \frac{1}{2}[n(n+1)(1)^{n-1} + n(n-1)(1)^{n-2}]$$

$$= \frac{1}{2}[n(n+1) + n(n-1)]$$

$$= \frac{n}{2}[n+\cancel{1} + n-\cancel{1}] = \frac{\cancel{2}n^2}{\cancel{2}} = n^2$$

By residue method, $x(n) = n^2$

convolution of two series -

$$f(n) * g(n) = \sum_{K=0}^{n} f(K)\, g(n-K)$$

convolution Theorem -

$$z\{f(n) * g(n)\} = z\{f(n)\} * z\{g(n)\}$$

$$= F(z).G(z)$$

$$z^{-1}\{F(z).G(z)\} = f(n) * g(n)$$

$$\boxed{1 + a + a^2 + a^3 + \dots + a^n = \frac{a^{n+1} - 1}{a - 1}}$$

Find $z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\}$ by convolution theorem.

Given $z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = z^{-1}\left\{\left(\frac{z}{z-a}\right)\cdot\left(\frac{z}{z-a}\right)\right\}$

$$= z^{-1}\left\{\frac{z}{z-a}\right\} * z^{-1}\left\{\frac{z}{z-a}\right\}$$

$$= \underset{f(n)}{a^n} * \underset{g(n)}{a^n}$$

$$= \sum_{K=0}^{n} f(K)\, g(n-K)$$

$$= \sum_{K=0}^{n} a^k . a^{n-k}$$

$$= \sum_{K=0}^{n} a^n$$

$$= a^n \sum_{K=0}^{n} 1$$

$$= a^n (n+1)$$

Find $z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\}$ using convolution theorem.

$$z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = z^{-1}\left\{\frac{\cancel{8}z^2}{\cancel{8}(z-\frac{1}{2})(z-\frac{1}{4})}\right\}$$

$$= z^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\} * z^{-1}\left\{\frac{z}{z-\frac{1}{4}}\right\}$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$
$$\uparrow \qquad \uparrow$$
$$f(n) \qquad g(n)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{4}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{2(n-k)}$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{2n-2k}$$

$$= \sum_{k=0}^{n} \cancel{\left(\frac{1}{2}\right)^{3n-k}} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{2n-2k}$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{2n-k}$$

$$= \left(\frac{1}{2}\right)^{2n} \sum_{k=0}^{\infty} 2^k$$

$$= \frac{1}{2^{2n}} \left(1 + 2 + 2^2 + \dots + 2^n\right)$$

$$= \frac{1}{2^{2n}} \quad \frac{2^{n+1} - 1}{2-1}$$

$$= \frac{2^{n+1} - 1}{2^{2n}}$$

Find $z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\}$ using convolution theorem.

$$z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\} = z^{-1}\left\{\frac{z}{z-1}\right\} * z^{-1}\left\{\frac{z}{z-3}\right\}$$

$$= 1 * (3)^n$$
$$\uparrow \qquad \uparrow$$
$$f(n) \quad g(n)$$

$$= \sum_{k=0}^{n} (1)^k (3)^{n-k}$$

$$= \sum_{k=0}^{n} 3^{n-k}$$

$$= 3^n \sum_{k=0}^{n} 3^{-k}$$

$$= 3^n \sum_{k=0}^{n} \frac{1}{3^k}$$

$$= 3^n\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots+\frac{1}{3^n}\right)$$

$$= 3^n \frac{\left(\frac{1}{3}\right)^{n+1}-1}{\left(\frac{1}{3}-1\right)}$$

$$= 3^n\left(\left(\frac{1}{3}\right)^{n+1}-1\right) \times -\frac{3}{2}$$

$$= -\frac{3^{n+1}}{2}\left(\left(\frac{1}{3}\right)^{n+1}-1\right)$$

$$= \frac{-3^{n+1}}{2}\left(\frac{1}{3^{n+1}}-\frac{3^{n+1}}{3^{n+1}}\right)$$

$$= -\frac{\cancel{3^{n+1}}}{2}\left(\frac{1-3^{n+1}}{\cancel{3^{n+1}}}\right)$$

$$= \frac{3^{n+1}-1}{2}$$

Find $z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$ using convalution theorem -

$$z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = z^{-1}\left\{\frac{z}{z-a}\right\} * z^{-1}\left\{\frac{z}{z-b}\right\}$$

$$= \underset{f(n)}{a^n} * \underset{g(n)}{b^n}$$

$$= \sum_{k=0}^{n} a^k b^{n-k}$$

$$= b^n \sum_{k=0}^{n} a^k b^{-k}$$

$$= b^n \sum_{k=0}^{n} \frac{a^k}{b^k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n\left[1+\frac{a}{b}+\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^3+\dots+\left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1}-1}{\left(\frac{a}{b}\right)-1}\right] = \frac{\cancel{b^n \times b}}{a-b}\left[\frac{a^{n+1}-b^{n+1}}{\cancel{b^{n+1}}}\right]$$

$$= \frac{a^{n+1}-b^{n+1}}{a-b}$$

Applications of z-transforms to difference equations -

1) $z[y_n] = z[y(n)] = y(z)$

~~2) $z[y_{n+1}]$ =~~

2) $z[y_{n+1}] = z[y(n+1)] = zy(z) - zy(0)$

3) $z[y_{n+2}] = z[y(n+2)] = z^2 y(z) - z^2 y(0) - zy(1)$

4) $z[y_{n+3}] = z[y(n+3)] = z^3 y(z) - z^3 y(0) - z^2 y(1) - zy(2)$

Solve using z-transform, the difference equation

$$y_{n+2} + 4y_{n+1} + 3y_n = 3^n \text{ with } y_0 = 0, y_1 = 1.$$

Taking z-transform on both sides,

$$z\{y_{n+2}\} + 4z\{y_{n+1}\} + 3z\{y_n\} = z\{3^n\}$$

$$z^2 y(z) - z^2 y(0) - zy(1) + 4(zy(z) - zy(0)) + 3y(z) = \frac{z}{z-3}$$

$y(0) = y_0$; $y(1) = y_1$

$$z^2 y(z) - z^2(0) - z(1) + 4(zy(z) - z(0)) + 3y(z) = \frac{z}{z-3}$$

$$z^2 y(z) - z + 4zy(z) + 3y(z) = \frac{z}{z-3}$$

$$y(z)(z^2 + 4z + 3) = \frac{z}{z-3} + z$$

$$y(z)(z^2 + 3z + z + 3) = \frac{z + z(z-3)}{z-3}$$

$$y(z)((z+3)(z+1)) = \frac{z + z^2 - 3z}{z-3}$$

$$y(z) = \frac{z^2 - 2z}{(z-3)(z+3)(z+1)}$$

We have to find the sequence, so we apply inverse z-transform

$$z^{-1}\{y(z)\} = z^{-1}\left\{\frac{z^2 - 2z}{(z-3)(z+3)(z+1)}\right\}$$

$$z^{-1}\left\{\frac{z(z-2)}{(z-3)(z+3)(z+1)}\right\}$$

$\uparrow$

$x(z)$

$$\frac{x(z)}{z} = \frac{z-2}{(z-3)(z+3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+3} + \frac{C}{z+1}$$

$$z-2 = A(z+3)(z+1) + B(z-3)(z+1) + C(z-3)(z+3)$$

using $z=3$,

$$1 = A(6)(4)$$

$$\frac{1}{24} = A$$

using $z=-3$,

$$-5 = B(-6)(-2)$$

$$\frac{-5}{12} = B$$

using $z=-1$,

$$-3 = C(-4)(2)$$

$$\frac{+3}{+8} = C$$

$$\frac{z-2}{(z-3)(z+3)(z+1)} = \frac{1}{24(z-3)} - \frac{5}{12(z+3)} + \frac{3}{8(z+1)}$$

$$x(z) = \frac{z}{24(z-3)} - \frac{5z}{12(z+3)} + \frac{3z}{8(z+1)}$$

$$z^{-1}\{x(z)\} = z^{-1}\left\{\frac{z}{24(z-3)} - \frac{5z}{12(z+3)} + \frac{3z}{8(z+1)}\right\}$$

$$= \frac{1}{24}(3)^n - \frac{5}{12}(-3)^n + \frac{3}{8}(-1)^n$$

$$z^{-1}\{y(z)\} = z^{-1}\left\{\frac{x(z)}{z}\right\}$$
$$= z^{-1}\{x(z)\}$$

Solve $y_{n+2} - 3y_{n+1} - 10y_n = 0$ given $y_0 = 1, y_1 = 0$ using z-transforms.

$$z\{y_{n+2}\} - 3z\{y_{n+1}\} - 10z\{y_n\} = 0$$

$$z^2 y(z) - z^2 y(0) - z y(1) - 3(z y(z) - z y(0)) - 10 y(z) = 0$$

$$z^2 y(z) - z^2(1) - z(0) - 3(z y(z) - z(1)) - 10 y(z) = 0$$

$$z^2 y(z) - z^2 - 3z y(z) + 3z - 10 y(z) = 0$$

$$y(z)(z^2 - 3z - 10) = z^2 - 3z$$

$$y(z) = \frac{z^2 - 3z}{(z-5)(z+2)}$$

$$y(z) = \frac{z(z-3)}{(z-5)(z+2)}$$

$$z^{-1}\{y(z)\} = z^{-1}\left\{\frac{z(z-3)}{(z-5)(z+2)}\right\}$$

$$z^{-1}\left\{\frac{y(z)}{z}\right\} = z^{-1}\left\{\frac{(z-3)}{(z-5)(z+2)}\right\}$$

$$z-3 = A(z+2) + B(z-5)$$

For $z = -2$,

$$-5 = B(-7)$$

$$B = \frac{5}{7}$$

For $z = 5$,

$$2 = A(7)$$

$$\frac{2}{7} = A$$

$$y(z) = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

$$z^{-1}\{y(z)\} = z^{-1}\left\{\frac{2z}{7(z-5)} + \frac{5z}{7(z+2)}\right\}$$

$$= \frac{2}{7}(5)^n + \frac{5}{7}(-2)^n$$

Solve $y_{n+2} - 3y_{n+1} + 2y_n = 2^n$ with $y_0 = y_1 = 0$.

$$z\{y_{n+2}\} - 3z\{y_{n+1}\} + 2z\{y_n\} = z\{2^n\}$$

$$z^2y(z) - z^2y(0) - zy(1) - 3(zy(z) - zy(0)) + 2y(z) = \frac{z}{z-2}$$

$$z^2y(z) - 3zy(z) + 2y(z) = \frac{z}{z-2}$$

$$y(z)(z^2 - 3z + 2) = \frac{z}{z-2}$$

$$y(z)((z-1)(z-2)) = \frac{z}{z-2}$$

$$y(z) = \frac{z}{(z-1)(z-2)^2}$$

$$z^{-1}\{y(z)\} = z^{-1}\left\{\frac{z}{(z-1)(z-2)^2}\right\}$$

$$z^{-1}\left\{\frac{y(z)}{z}\right\} = z^{-1}\left\{\frac{1}{(z-1)(z-2)^2}\right\}$$

$$\frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

For $z = 2$,

$1 = C(1)$

$C = 1$

For $z = 1$,

$1 = A(1-2)^2$

$A = 1$

For $z = 0$,

$1 = 1(-2)^2 + B(0-1)(0-2) + 1(0-1)$

$1 = 4 + 2B - 1$

$1 = 3 + 2B$

$-2 = 2B$

$B = -1$

$$\frac{1}{(z-1)(z-2)^2} = \frac{1}{z-1} + \frac{-1}{z-2} + \frac{1}{(z-2)^2}$$