PROBABILITY
&
QUEUEING THEORY

(As per SRM INSTITUTE OF SCIENCE AND TECHNOLOGY Syllabus)

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PROBABILITY AND QUEUEING THEORY

UNIT – I : RANDOM VARIABLES

Syllabus

- Review of Probability Concepts - Types of Events, Axioms
- Conditional Probability, Multiplication Theorem, Applications
- Discrete Random Variable
- Continuous Random Variable
- Expectation and Variance
- Moment Generating Function
- Function of Random Variable (One Dimensional Only)
- Chebychev's Inequality

PROBABILITY

Probability (or) Chance: Probably, Chances, Likely, Possible - The terms convey the same meaning.

Example:

1. ***Probably*** your method is correct
2. The ***chances*** of getting ranks Ram and Gothai are equal.
3. It is ***likely*** that Ram may not come for taking his classes today.
4. It is ***possible*** to reach the college by 8.30am.

Ordinary Language: The word probability means uncertainty about happening.

Mathematics or Statistics : A numerical measure of uncertainty is practiced by the important branch of statistics is called the **Theory of Probability.**

Day to Day Life:

- ***Certainty*** - Every day the sun rises in the east
- ***Impossibility*** - It is possible to live without water
- ***Uncertainty*** - Probably Raman gets that job.

In the theory of probability, we represent certainty by 1, impossibility by 0 and uncertainty by a positive fraction which lies between 0 and 1.

Applications : There is no area in **social, physical (or) natural sciences** where the probability theory is not used.

- It is the base of the fundamental laws of statistics.
- It gives solutions to betting of games.
- It is extensively used in business situations characterized by uncertainty.
- It is essential tool in statistical inference and forms the basis of the Decision Theory.

Random Experiment (or) Trial and Event (or) Cases:

An experiment in which the outcome cannot be predicted with certainty is called a random experiment, even though all possible outcomes are known in advance. Tossing a coin is a **random experiment** and getting a head or tail is an **event.**

Favourable Events :

The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event.

Example: In tossing 2 coins the cases favourable to the event of getting a head are HT, TH, and HH.

Exhaustive Events : The total number of possible outcomes in any **trial** is known as exhaustive events.

Example: In tossing a coin the possible outcomes are getting a head or tail. Hence we have 2 exhaustive events in throwing a coin.

Mutually Exclusive Event:

Two events are said to be mutually exclusive when the occurrence of one affects the occurrence of the other. In other words, if A & B are mutually exclusive events and if A happens then B will not happen and vice versa.

Example: In tossing a coin the events head or tail are mutually exclusive, since both tail & head cannot appear in the same time.

Equally Likely Events: Two events are said to be equally likely if one of them cannot be expected in preference to the other. ***Example:*** In tossing a coin, head or tail are equally likely events.

Independent Event : Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other. ***Example :*** In tossing a coin, the event of getting a head in the 1^{st} toss is independent of getting a head in the 2^{nd} toss, 3^{rd} toss, etc.

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Mathematical Definition of Probability:

If P is the notation for probability of happening of the event, then $P(A) = \frac{Number\ of\ Favourable\ Cases}{Total\ Number\ of\ Exhaustive\ Cases} = \frac{m}{n}$

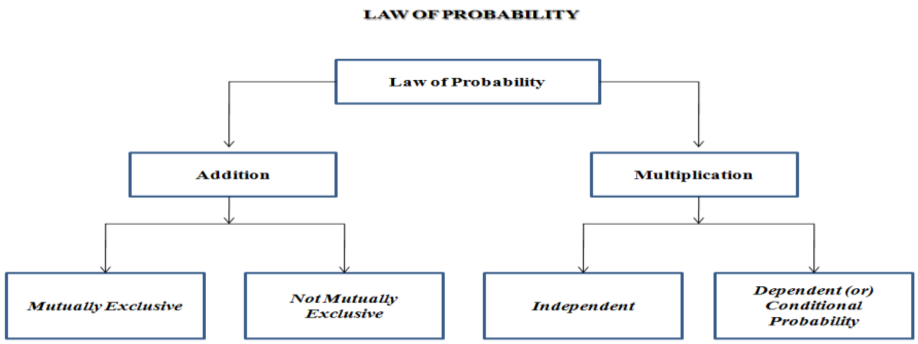
Statistical Definition of Probability:

If in n trials, an event E happens m times, then $P(E) = \lim_{n\to\infty} \frac{m}{n}$

Axiomatic Definition of Probability:

1. For any event A, $P(A) \geq 0$.
2. $P(S) = 1$
3. If $A_1, A_2, A_3, \dots, A_n$ are finite number of disjoint events of S, then
   $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \cdots = \sum P(A_i)$

LAW OF PROBABILITY



ADDITION LAW OF PROBABILITY

Case (i): When events are mutually exclusive

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Case (ii): When events are not mutually exclusive

If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

MULTIPLICATION LAW OF PROBABILITY

Case (i): When events are independent : The probability that both independent events, A and B will occur is equal to product of the probabilities of each event, then $P(A \cap B) = P(A)\,P(B)$.

Case (ii): When events are dependent (or) conditional probability: If the occurrence of an event A is affected by the occurrence of the another event B, then the events A and B are dependent. $P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)$

RANDOM VARIABLE

The outcomes of many random experiments may be non-numerical. It is inconvenient to deal with these descriptive outcomes mathematically.

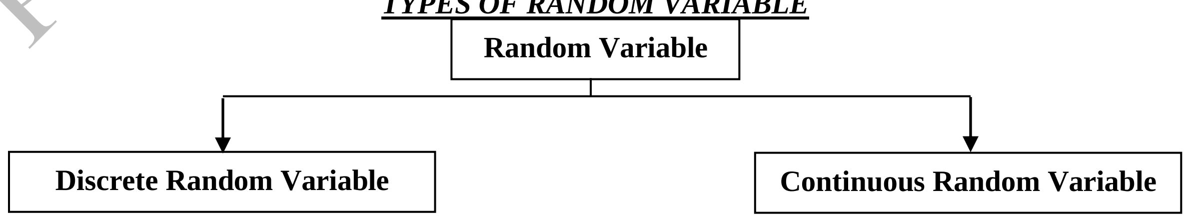
Example: When toss a coin we get two outcomes, namely head or tail. We can assign numerical values; say 1 to head and 0 to tail. This interpretation is easy and attractive from mathematical point of view and also practically meaningful.

Example: Three students sat for an examination & X denotes the number of students who passed. Describe the RV X.

Sample Space S	None	S_1	S_2	S_3	S_1S_2	S_2S_3	S_3S_1	$S_1S_2S_3$
No. of Students who passed X	0	1	1	1	2	2	2	3

$n(S) = 8, \quad P(X=0) = \frac{1}{8}, \quad P(X=1) = \frac{3}{8}, \quad P(X=2) = \frac{3}{8}, \quad P(X=3) = \frac{1}{8}$

TYPES OF RANDOM VARIABLE



DISCRETE RANDOM VARIABLE

A random variable X is discrete, if it assumes only finite number or countably infinite number of values.

Example : (i) *The mark obtained by a student in an examination. It's possible values are 0, 85 or 100.*

(ii) The number of students who are absent for a particular period.

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1. Probability Mass Function (p.m.f.) $\sum_{i=1}^{\infty} P(x_i) = 1$ 2. Mean $E(X) = \sum_{i=1}^{\infty} x_i P(x_i), E(X^2) = \sum_{i=1}^{\infty} {x_i}^2 P(x_i)$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \sum_{i=1}^{x} P(x_i)$

CONTINUOUS RANDOM VARIABLE

A RV X is continuous, if it takes all possible values between certain limits or in an interval which may be finite or infinite. ***E.g:***(i)The density of milk taken for testing at a farm.(ii)The operating time between two failures of a computer.

1. Probability Density Function (p.d.f.) $\int_{-\infty}^{\infty} f(x)dx = 1$ 2. Mean E$(X) = \int_{-\infty}^{\infty} x\, f(x)dx,\ E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$

PROPERTIES OF EXPECTATION

If X and Y are random variables and a, b are constants, then

1. $E(a) = a$ 2. $E(aX) = aE(X)$ 3. $E(aX + b) = aE(X) + b$ 4. $E(X - \bar{X}) = 0$ 5. $|E(X)| \le E(|X|)$
6. $E(X) \ge 0, if\ X \ge 0$ 7. $E(X + Y) = E(X) + E(Y)$ ***(Additive Theorem)***
8. $E(XY) = E(X)E(Y)$ ***($\because$ A and B are independent)***
9. $E\big(a\, g(X)\big) = aE\big(g(X)\big)$ 10. $E(g(X) + a) = E\big(g(X)\big) + a$
11. $(E[g(X)\,]) = g[E(X)]$ $[\boldsymbol{g(X)\ is\ linear\ in\ X}]$
12. $P(X \ge a) \le \frac{E(X)}{a}, a > 0$ ***(Markov Inequality)***
13. $P\{|X - E(X)| \ge k\} \ge \frac{\sigma_X^2}{\epsilon^2}$ ***(Chebyshev's Inequality)***

PROPERTIES OF VARIANCE

1. $Var(X) \ge 0$ 2. $E(X^2) \ge [E(X)]^2$ 3. $Var(b) = 0, b\ constant$
4. If X is a random variables, a is constants then $Var(aX) = a^2Var(X)$
5. If a and b are constants, $Var(aX \pm b) = a^2Var(X)$
6. If X and Y are two independent RV, a and b are constants then $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$

PROPERTIES OF CUMULATIVE DISTRIBUTION FUNCTION

1. If F is the distribution function of the RV X and if $a < b$, then
   $P(a < X \le b) = P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = F(b) - F(a)$
2. If F is the distribution function of one dimensional RV X, then (i) $0 \le F(X) \le 1$ (ii) $F(X) \le F(Y)$, if $x < y$
   In other words, all distribution functions are monotonically non-decreasing and lie between 0 and 1.
3. If F is the distribution function of one dimensional random variable X, then
   $F(-\infty) = \lim_{x\to-\infty} F(X) = 0$ and $F(\infty) = \lim_{x\to\infty} F(X) = 1$ 4. $f(x) = \frac{d}{dx}\big(F(x)\big)$

MOMENT GENERATING FUNCTION

Definition : Moment generating function of a random variable about the origin is defined as
Discrete : $M_X(t) = E(e^{tX}) = \sum_x e^{tx}\, p(x)$, ***Continuous :*** $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$
Where the integration or summation is taken over the entire range of X, t being a real parameter, assuming that integration or summation is absolutely convergent.

$$M_X(t) = 1 + t\,\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' ,\quad \text{Where } \mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

Note: 1. $\mu_r' = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$ 2. $M_{CX}(t) = M_X(Ct),\ C$ being a constant. 3. $M_{X=a}(t) = e^{-at} M_X(t)$
1. If $X_1, X_2, \ldots X_n$ are n independent RVs, then $M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t).M_{X_2}(t) \ldots M_{X_n}(t)$

PROBLEMS IN DISCRETE RANDOM VARIABLE

1. ***A discrete RV X has the following probability distribution***

$\boldsymbol{x}$	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***
$\boldsymbol{p(x)}$	$\boldsymbol{a}$	$\boldsymbol{3a}$	$\boldsymbol{5a}$	$\boldsymbol{7a}$	$\boldsymbol{9a}$	$\boldsymbol{11a}$	$\boldsymbol{13a}$	$\boldsymbol{15a}$	$\boldsymbol{17a}$

(i) Find the value of a (ii) $\boldsymbol{P(X < 3)}$ ***(iii)*** $\boldsymbol{P(X \ge 3)}$ ***(iv)*** $\boldsymbol{P(0 < X < 3)}$ ***(v) Find the distribution function of X.***
Solution

(i) $\sum_{x=0}^{8} P(x) = 1 \Rightarrow P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) = 1$
$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1 \Rightarrow 81a = 1 \Rightarrow \boldsymbol{a = \frac{1}{81}}$
(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = a + 3a + 5a = 9a = 9 \times \frac{1}{81} = \frac{1}{9}$

(iii) $P(X \geq 3) = 1 - P(X < 3) = 1 - \frac{1}{9} = \frac{8}{9}$

(iv) $P(0 < X < 3) = P(X = 1) + P(X = 2) = 3a + 5a = 8a = \frac{8}{81}$

(v)

x	0	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

2. ***A discrete random variable X has the probability function given below:***

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
$p(x)$	*0*	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

Find (i) The value of K (ii) $P(1.5 < X < 4.5/X > 2)$ ***(iii) The smallest value of*** λ ***for which*** $P(X \leq \lambda) > 1/2$.

Solution:

(i) $\sum_{x=0}^{7} P(x) = 1 \Rightarrow P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$

$$0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1 \Rightarrow 10K^2 + 9K = 1$$

$$(10K - 1)(K + 1) = 0 \Rightarrow K = \frac{1}{10}, -1 \Rightarrow \boldsymbol{K = \frac{1}{10}}$$ ***($\because$ K = −1, which is meaningless)***

(ii) $P(1.5 < X < 4.5/X > 2) = \frac{P[(1.5<X<4.5)\cap(X>2)]}{P(X>2)}$ $\quad \because \boldsymbol{P(A/B) = \frac{P(A\cap B)}{P(B)}}$

$$P(1.5 < X < 4.5/X > 2) = \frac{P(3)+P(4)}{P(3)+P(4)+P(5)+P(6)+P(7)} = \frac{\left(\frac{5}{10}\right)}{\left(\frac{7}{10}\right)} = \boldsymbol{\frac{5}{7}}$$

(iii) $P(X \leq \lambda) > \frac{1}{2}$, $\lambda = 0$, $P(X \leq 0) = 0 \ngtr \frac{1}{2}$; $\lambda = 1$, $P(X \leq 1) = \frac{1}{10} \ngtr \frac{1}{2}$;

$\lambda = 2$, $P(X \leq 2) = \frac{3}{10} \ngtr \frac{1}{2}$; $\lambda = 3$, $P(X \leq 3) = \frac{5}{10} \ngtr \frac{1}{2}$; $\lambda = 4$, $P(X \leq 4) = \frac{8}{10} > \frac{1}{2}$

The smallest value of λ for which $P(X \leq \lambda) > 1/2$ is 4.

3. ***If the RV X takes the values 1, 2, 3 & 4 such that*** $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$, ***find the probability distribution and cumulative distribution function of X.***

Solution: Let $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = 30K$

x	1	2	3	4
$p(x)$	$15K$	$10K$	$30K$	$6K$

$\sum_{x=1}^{4} P(x) = 1 \Rightarrow P(1) + P(2) + P(3) + P(4) = 1 \Rightarrow 15K + 10K + 30K + 6K = 1 \Rightarrow 61K = 1 \Rightarrow \boldsymbol{K = \frac{1}{61}}$

Cumulative distribution function of X

x	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$F(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

4. ***A discrete RV X has the following probability distribution***

x	*-2*	*-1*	*0*	*1*	*2*	*3*
$p(x)$	*0.1*	K	*0.2*	$2K$	*0.3*	$3K$

Find (i) K ***(ii)*** $P(X < 2)$ ***(iii)*** $P(-2 < X < 2)$ ***(iv) the cdf of X (v) the mean of X.***

Solution

(i) $\sum_{x=-2}^{3} P(x) = 1 \Rightarrow P(-2) + P(-1) + P(0) + P(1) + P(2) + P(3) = 1 \Rightarrow 6K + 0.6 = 1 \Rightarrow \boldsymbol{K = \frac{1}{15}}$

	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$

(ii) $P(X < 2) = P(-2) + P(-1) + P(0) + P(1) = \frac{1}{10} + \frac{1}{15} + \frac{2}{10} + \frac{2}{15} = \frac{1}{2}$

(iii) $P(-2 < X < 2) = P(-1) + P(0) + P(1) = \frac{1}{15} + \frac{2}{10} + \frac{2}{15} = \frac{2}{5}$

(iv)

x	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$
$F(X)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{11}{30}$	$\frac{1}{2}$	$\frac{4}{5}$	1

(v) *Mean of X*

$$E(X) = \sum_{x=-2}^{3} x\,P(x) = (-2)P(-2) + (-1)P(-1) + 0\,P(0) + 1\,P(1) + 2\,P(2) + 3\,P(3)$$
$$= \left(-2 \times \frac{1}{10}\right) + \left(-1 \times \frac{1}{15}\right) + \left(0 \times \frac{2}{10}\right) + \left(1 \times \frac{2}{15}\right) + \left(2 \times \frac{3}{10}\right) + \left(3 \times \frac{3}{15}\right) = \frac{16}{15}$$

5. *If X is RV having the density function* $(x) = \begin{cases} \frac{x}{6} & for\ x = 1,2,3 \\ 0, & otherwise \end{cases}$. *Find* $E(X^3 + 2X + 7)$ *and* $Var\,(4X + 5)$.

Solution

x	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$$E(X) = \sum_{x=1}^{3} x\,P(x) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{2}{6}\right) + \left(3 \times \frac{3}{6}\right) = \frac{7}{3}$$
$$E(X^2) = \sum_{x=1}^{3} x^2\,P(x) = \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{2}{6}\right) + \left(9 \times \frac{3}{6}\right) = 6$$
$$E(X^3) = \sum_{x=1}^{3} x^3\,P(x) = \left(1 \times \frac{1}{6}\right) + \left(8 \times \frac{2}{6}\right) + \left(27 \times \frac{3}{6}\right) = \frac{49}{3}$$
$$E(X^3 + 2X + 7) = E(X^3) + 2E(X) + 7 = \frac{84}{3}$$
$$Var(X) = E(X^2) - [E(X)]^2 = \frac{5}{9}, \qquad Var\,(4X + 5) = 4^2 Var(X) = 16 \times \frac{5}{9} = \frac{80}{9}$$

6. *If X has the distribution function* $F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 4 \\ \frac{1}{2}, & 4 \le x < 6 \\ \frac{5}{6}, & 6 \le x < 10 \\ 1, & x \ge 10 \end{cases}$

Find (i) The probability distribution of X (ii)* $P(2 < X < 6)$ *(iii) Mean of X (iv) Variance of X.

Solution

(i) For the given c.d.f., the probability distribution of X is

$$P(X = 1) = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}, \qquad P(X = 4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$
$$P(X = 6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}, \qquad P(X = 10) = F(10) - F(6) = 1 - \frac{5}{6} = \frac{1}{6}$$

x	1	4	6	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(ii) $P(2 < X < 6) = P(X = 4) = \frac{1}{6}$

(iii) $E(X) = \sum_i x_i\,P(x_i) = \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{2}{6}\right) + \left(10 \times \frac{1}{6}\right) = \frac{14}{3}$

$$E(X^2) = \sum_i x_i^2\,P(x_i) = \left(1 \times \frac{1}{3}\right) + \left(16 \times \frac{1}{6}\right) + \left(36 \times \frac{2}{6}\right) + \left(100 \times \frac{1}{6}\right) = \frac{95}{3}$$
$$Var(X) = E(X^2) - [E(X)]^2 = \frac{89}{3}$$

7. *When a die is thrown, X denotes the number that turns up. Find* $E(X), E(X^2), Var(X)$ *and standard deviation.*

Solution: $p = \frac{1}{6}$, X = 1, 2, 3, 4, 5, 6 Here X is a discrete RV

$$E(X) = \sum_i x_i\,P(x_i) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = 3.5$$
$$E(X^2) = \sum_i x_i^2\,P(x_i) = \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(9 \times \frac{1}{6}\right) + \left(16 \times \frac{1}{6}\right) + \left(25 \times \frac{1}{6}\right) + \left(36 \times \frac{1}{6}\right) = \frac{91}{6} = 15.167$$
$$Var(X) = E(X^2) - [E(X)]^2 = 2.9166, \qquad S.D. = \sigma_X = \sqrt{Var(X)} = 1.7078$$

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8. ***A coin is tossed until a head appears. What is the expectation of the number of tosses required?***
Solution: Let X – No. of tosses required to get the 1st head. The 1st head may appear in the 1st or 2nd … and so on.
The events are H, TH, TTH, TTTH, ... $p = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

x	1	2	3	4	5	…
$p(x)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	…

$$E(X) = \sum_i x_i P(x_i) = \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} = 2 \qquad [\because (\mathbf{1} - \boldsymbol{x})^{-2} = \mathbf{1} + \mathbf{2}\boldsymbol{x} + \mathbf{3}\boldsymbol{x}^{\mathbf{2}} + \cdots]$$

9. ***By throwing a fair dice, a player gains Rs. 20 if 2 turns up, gains Rs. 40 if 4 turns up and loses Rs. 30 if 6 turns up. He never loses or gains if any other number turns up. Find the expected value of money he gains.***
Solution: Let X – money won on an trial. $x_i =$ Amount of money won, if the faces show $i = 1, 2,3,4,5,6$.

	1	2	3	4	5	6
x	0	20	0	40	0	-30
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \sum_i x_i P(x_i) = \left(0 \times \frac{1}{6}\right) + \left(20 \times \frac{1}{6}\right) + \left(0 \times \frac{1}{6}\right) + \left(40 \times \frac{1}{6}\right) + \left(0 \times \frac{1}{6}\right) + \left(-30 \times \frac{1}{6}\right) = 5$$

10. ***A RV X has the probability function $f(x) = \frac{1}{2^x}, x = 1, 2, 3\ldots$ Find the (i) moment generating function (ii) Mean***
Solution :

(i) $M_X(t) = \sum_x e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \cdots\right] = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2 - e^t}$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{e^t}{2 - e^t}\right)\right]_{t=0} = \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0} = \frac{(2-e^0)e^0 - e^0(-e^0)}{(2-e^0)^2} = 2$

11. ***If a RV X has moment generating function $M_X(t) = \frac{3}{3-t}$, obtain the standard deviation of X.***

Solution : $M_X(t) = \frac{3}{3-t} = \frac{3}{3\left(1 - \frac{t}{3}\right)} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \left(\frac{t}{3}\right) + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{3}\right) + \frac{t^2}{2!}\left(\frac{2}{9}\right) + \frac{t^3}{3!}\left(\frac{6}{27}\right) + \cdots$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}$, $\mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{3}$, $\mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{2}{9}$

Variance $= \mu_2' - (\mu_1')^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$, Standard deviation $= \sqrt{\text{Variance}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

PROBLEMS IN CONTINUOUS RANDOM VARIABLE

1. ***If $p(x) = \begin{cases} x\, e^{-\frac{x^2}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (i) Show that $p(x)$ is a p.d.f. (ii) Find its distribution function $P(x)$.***

Solution

(i) $\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{0} p(x)dx + \int_0^{\infty} p(x)dx = \int_{-\infty}^{0} 0\, dx + \int_0^{\infty} x\, e^{-\frac{x^2}{2}} dx = \int_0^{\infty} x\, e^{-\frac{x^2}{2}} dx$

Put $x^2 = t$, $2x\, dx = dt \Rightarrow x\, dx = \frac{dt}{2}$, $x = 0,\ t = 0$ and $x = \infty,\ t = \infty$

$\int_{-\infty}^{\infty} p(x)dx = \int_0^{\infty} e^{-\frac{t}{2}} \frac{dt}{2} = \frac{1}{2}\int_0^{\infty} e^{-\frac{t}{2}} dt = \frac{1}{2}\left[\frac{e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{\infty} = -e^{-\infty} + e^0 = 1 \qquad (\because \boldsymbol{e^{-\infty} = 0,\ e^0 = 1})$

$\therefore p(x)$ is a p.d.f. of a RV X.

(ii) $F(X) = P(X \leq x) = \int_0^x p(x)dx = \int_0^x x\, e^{-\frac{x^2}{2}} dx = 1 - e^{-\frac{x^2}{2}},\ x \geq 0$

2. ***If the density function of a continuous RV X is given by $(x) = \begin{cases} ax, & 0 \leq x \leq 1 \\ a, & 1 \leq x \leq 2 \\ 3a - ax, & 2 \leq x \leq 3 \\ 0, & otherwise \end{cases}$. Find (i) a (ii) c.d.f.***

Solution: (i) $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^1 ax\, dx + \int_1^2 a\, dx + \int_2^3 (3a - ax)dx = 1$

$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1 \Rightarrow \frac{a}{2} + a(2-1) + \left(9a - \frac{9a}{2}\right) - \left(6a - \frac{4a}{2}\right) = 1 \Rightarrow \boldsymbol{a} = \frac{\mathbf{1}}{\mathbf{2}}$

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$$f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{3-x}{2}, & 2 \le x \le 3 \\ 0, & otherwise \end{cases}$$

(ii) ***c.d.f. of X:*** $F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$

If $x < 0$, *then* $F(X) = 0$, *since* $f(x) = 0$ *for* $x < 0$

If $0 \le x \le 1$, *then* $F(X) = \int_0^x \frac{x}{2}\, dx = \left[\frac{x^2}{4}\right]_0^x = \frac{x^2}{4}$

If $1 \le x \le 2$, *then* $F(X) = \int_0^x f(x)\, dx = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^x \left(\frac{1}{2}\right) dx = \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^x = \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{1}{4}(2x - 1)$

If $2 \le x \le 3$, *then* $F(X) = \int_0^x f(x)\, dx = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^2 \left(\frac{1}{2}\right) dx + \int_2^x (3a - ax)\, dx$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \left[\frac{3x}{2} - \frac{x^2}{4}\right]_2^x = \frac{1}{4} + \frac{2}{2} - \frac{1}{2} + \left(\frac{3x}{2} - \frac{x^2}{4}\right) - \left(\frac{6}{2} - \frac{4}{4}\right) = \frac{1}{4}(6x - x^2 - 5)$$

If $x \ge 3$, *then* $F(X) = 1$

$$F(x) = \begin{cases} \frac{x^2}{4}, & 0 \le x \le 1 \\ \frac{1}{4}(2x - 1), & 1 \le x \le 2 \\ \frac{1}{4}(6x - x^2 - 5), & 2 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

3. ***A continuous RV X has a pdf*** $\boldsymbol{f(x) = 3x^2, 0 \le x \le 1}$***. Find a and b such that***

(i) $\boldsymbol{P(X \le a) = P(X > a)}$ ***(ii)*** $\boldsymbol{P(X > b) = 0.05}$

Solution:

(i) $P(X \le a) = P(X > a) \Rightarrow \int_{-\infty}^{a} f(x)dx = \int_a^{\infty} f(x)dx \Rightarrow \int_0^a 3\,x^2 dx = \int_a^1 f(x)dx \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = 3\left[\frac{x^3}{3}\right]_a^1$

$a^3 = 1 - a^3 \Rightarrow 2a^3 = 1 \Rightarrow a^3 = \frac{1}{2} \Rightarrow \boldsymbol{a} = \left(\frac{1}{2}\right)^{\frac{1}{3}} = \mathbf{0.7937}$

(ii) $P(X > b) = 0.05 \Rightarrow \int_b^1 3\,x^2 dx = 0.05 \Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05 \Rightarrow \boldsymbol{b} = (\mathbf{0.95})^{\frac{1}{3}} = \mathbf{0.9830}$

4. ***A Continuous RV X that can assume any value between*** $\boldsymbol{x = 2}$ ***and*** $\boldsymbol{x = 5}$ ***has a density function given by*** $\boldsymbol{f(x) = k(1 + x)}$***. Find*** $\boldsymbol{P(X < 4)}$***.***

Solution : $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_2^5 k(1 + x)dx = 1 \Rightarrow k\left[x + \frac{x^2}{2}\right]_2^5 = 1 \Rightarrow k\left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1 \Rightarrow \boldsymbol{k} = \frac{\mathbf{2}}{\mathbf{27}}$

$$P(X < 4) = \frac{2}{27}\int_2^4 (1 + x)dx = \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4 = \frac{2}{27}\left[\left(4 + \frac{16}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = \frac{16}{27}$$

5. ***A RV X has a pdf*** $\boldsymbol{f(x) = kx^2e^{-x}, x \ge 0}$***. Find k, mean, variance and*** $\boldsymbol{E(3X^2 - 2X)}$***.***

Solution: $\int_{-\infty}^{\infty} f(x)dx = 1, \quad \int_0^{\infty} kx^2e^{-x}dx = 1$

$\boldsymbol{Differentiation}: u = x^2, \quad u' = 2x, \quad u'' = 2, \quad u''' = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, \; v_3 = \frac{e^{-x}}{(-1)^3}$ $\quad (\because \int \boldsymbol{uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots})$

$k\left[x^2\frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty} = 1 \Rightarrow k[(0 - 0 + 0) - (0 - 0 + 2)] = 1 \Rightarrow \boldsymbol{k} = \frac{\mathbf{1}}{\mathbf{2}}$ $\quad (\because \boldsymbol{e^{-\infty} = 0, \; e^0 = 1})$

Mean of $\boldsymbol{X}$ $\; E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_0^{\infty} x\left(\frac{1}{2}x^2e^{-x}\right)dx = \frac{1}{2}\int_0^{\infty} x^3e^{-x}dx$

$\boldsymbol{Differentiation}: u = x^3, \quad u' = 3x^2, \quad u'' = 6x, \quad u''' = 6, \quad u'^{v} = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, \; v_3 = \frac{e^{-x}}{(-1)^3}, v_4 = \frac{e^{-x}}{(-1)^4}$ $(\because \int \boldsymbol{uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots})$

$E(X) = \frac{1}{2}\left[x^3\frac{e^{-x}}{(-1)} - 3x^2\frac{e^{-x}}{(-1)^2} + 6x\frac{e^{-x}}{(-1)^3} - 6\frac{e^{-x}}{(-1)^4}\right]_0^{\infty} = 3$ $\quad (\because \boldsymbol{e^{-\infty} = 0, \; e^0 = 1})$

$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_0^{\infty} x^2\left(\frac{1}{2}x^2e^{-x}\right)dx = \frac{1}{2}\int_0^{\infty} x^4e^{-x}dx$

$\boldsymbol{Differentiation}: u = x^4, \quad u' = 4x^3, \quad u'' = 12x^2, \quad u''' = 24x, \quad u'^{v} = 24, \quad u^{v} = 0$

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Integration : $\quad v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, \quad v_2 = \frac{e^{-x}}{(-1)^2}, \quad v_3 = \frac{e^{-x}}{(-1)^3}, \; v_4 = \frac{e^{-x}}{(-1)^4}, v_5 = \frac{e^{-x}}{(-1)^5}, \; v_6 = \frac{e^{-x}}{(-1)^6}$

$$E(X^2) = \frac{1}{2}\left[x^4 \frac{e^{-x}}{(-1)} - 4x^3 \frac{e^{-x}}{(-1)^2} + 12x^2 \frac{e^{-x}}{(-1)^3} - 24x \frac{e^{-x}}{(-1)^4} + 24x \frac{e^{-x}}{(-1)^5}\right]_0^\infty = 12$$

$$V(X) = E(X^2) - [E(X)]^2 = 12 - 9 = 3, \quad E(3X^2 - 2X) = 3E(X^2) - 2E(X) = 3(12) - 2(3) = 30$$

6. ***The prob. distribution function of a RV X is*** $(\boldsymbol{x}) = \begin{cases} \boldsymbol{x}, & \mathbf{0} < \boldsymbol{x} < 1 \\ \mathbf{2} - \boldsymbol{x}, & \mathbf{1} < \boldsymbol{x} < 2 \\ \mathbf{0}, & \boldsymbol{x} > 2 \end{cases}$. ***Find the mean and variance.***

Solution

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_{-\infty}^{0} x\, f(x)dx + \int_0^1 x\, f(x)dx + \int_1^2 x\, f(x)dx + \int_2^\infty x\, f(x)dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_0^1 x\,(x)dx + \int_1^2 x\,(2-x)dx + \int_2^\infty 0\, dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx$$

$$E(X) = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_{-\infty}^{0} x^2\, f(x)dx + \int_0^1 x^2\, f(x)dx + \int_1^2 x^2\, f(x)dx + \int_2^\infty x^2\, f(x)dx$$

$$= \int_0^1 x^3 x + \int_1^2 (2x^2 - x^3)dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = \frac{1}{4} + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{7}{6}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{6}$$

7. ***The distribution function of a RV X is given by*** $\boldsymbol{F(x) = 1 - (1 + x)e^{-x}}$, $\boldsymbol{x \geq 0}$. ***Find the density function, mean and variance of X.***

Solution

$$f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}[1 - (1 + x)e^{-x}] = [0 - (1 + x)(-e^{-x}) - e^{-x}] = e^{-x} + xe^{-x} - e^{-x} = xe^{-x}, x \geq 0$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_0^\infty x^2 e^{-x} dx = \left[x^2 \frac{e^{-x}}{(-1)} - 2x \frac{e^{-x}}{(-1)^2} + 2 \frac{e^{-x}}{(-1)^3}\right]_0^\infty = 2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_0^\infty x^3 e^{-x} dx = \left[x^3 \frac{e^{-x}}{(-1)} - 3x^2 \frac{e^{-x}}{(-1)^2} + 6x^2 \frac{e^{-x}}{(-1)^3} - 6 \frac{e^{-x}}{(-1)^4}\right]_0^\infty = 6$$

$$V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$$

8. ***The cdf of a continuous RV X is given by*** $\boldsymbol{F(x)} = \begin{cases} \mathbf{0}, & \boldsymbol{x} < 0 \\ \boldsymbol{x^2}, & \mathbf{0} \leq \boldsymbol{x} < \frac{1}{2} \\ \mathbf{1} - \frac{3}{25}(\mathbf{3} - \boldsymbol{x})^2, & \frac{1}{2} \leq \boldsymbol{x} < 3 \\ \mathbf{1}, & \boldsymbol{x} \geq \mathbf{3} \end{cases}$

Find the p.d.f. of X and evaluate $\boldsymbol{P(|X| \leq 1)}$ ***and*** $\boldsymbol{P}\left(\frac{1}{3} \leq \boldsymbol{X} < 4\right)$ ***using both the pdf and cdf.***

Solution: $\quad f(x) = \frac{d}{dx}[F(x)]$

$$f(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 \leq x < \frac{1}{2} \\ \frac{6}{25}(3 - x), & \frac{1}{2} \leq x < 3 \\ 0, & x \geq 3 \end{cases}$$

pdf : $P(|X| \leq 1) = P(-1 \leq X \leq 1) = \int_{-1}^{0} 0\, dx + \int_0^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3 - x)dx = 2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1} = \frac{13}{25}$

cdf: $P(|X| \leq 1) = P(-1 \leq X \leq 1) = F(1) - F(-1) = \frac{13}{25}$

pdf : $P\left(\frac{1}{3} \leq X < 4\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{3} \frac{6}{25}(3 - x)dx + \int_3^4 0\, dx = 2\left[\frac{x^2}{2}\right]_{\frac{1}{3}}^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{3} = \frac{8}{9}$

cdf: $P\left(\frac{1}{3} \leq X < 4\right) = F(4) - F\left(\frac{1}{3}\right) = 1 - \frac{1}{9} = \frac{8}{9}$

9. ***A RV X has density function given by*** $\boldsymbol{f(x)} = \begin{cases} \mathbf{2}\, \boldsymbol{e^{-2x}}, & \boldsymbol{x} \geq \mathbf{0} \\ \mathbf{0}, & \boldsymbol{x} < 0 \end{cases}$. ***Obtain the (i) moment generating function (ii) Four moments about the origin (iii) Mean (iv) Variance.***

Solution: $\quad M_X(t) = \int_{x=-\infty}^{\infty} e^{tx} f(x)dx = \int_{x=0}^{\infty} e^{tx}\, 2\, e^{-2x}\, dx = \int_{x=0}^{\infty} 2\, e^{-(2-t)x}\, dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^\infty = \frac{2}{2-t}$

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$$M_X(t) = \frac{2}{2-t} = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{2}\right) + \frac{t^2}{2!}\left(\frac{1}{2}\right) + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\left(\frac{3}{2}\right) + \cdots$$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}$, $\quad r = 1,\ \mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{2}$

$r = 2,\ \mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{1}{2}$, $\quad r = 3,\ \mu_3' = coefficient\ of\ \frac{t^3}{3!} = \frac{3}{4}$

$r = 4,\ \mu_4' = coefficient\ of\ \frac{t^4}{4!} = \frac{3}{2}$, Mean $= \mu_1' = \frac{1}{2}$, Variance $= \mu_2' - (\mu_1')^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

10. ***Find the moment generating function of the RV whose moments are given by $\mu_r' = (r+1)!\,2^r$. Find also mean and variance.***

Solution: $\mu_1' = 2!\,2^1$, $\mu_2' = 3!\,2^2$, $\mu_3' = 4!\,2^3$, $M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\mu_3' + \cdots$

$M_X(t) = 1 + \frac{t}{1!}(2!\,2^1) + \frac{t^2}{2!}(3!\,2^2) + \frac{t^3}{3!}(4!\,2^3) + \cdots = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots = (1-2t)^{-2}$

Mean $= \mu_1' = 4$, $\mu_2' = 24$, Variance $= \mu_2' - (\mu_1')^2 = 24 - 16 = 8$

FUNCTION OF ONE DIMENSITIONAL RANDOM VARIABLE

One to One Transformation of Random Variables:

Consider that a random variable X is linearly transformed into an another random variable Y. Let Y be $T(x)$.

A monotonically increasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 < x_2$. For example, $y = ax, a > 0$

A monotonically decreasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 > x_2$. For example, $y = ax, a < 0$

If the transformation is monotonically increasing $f_Y(y) = f_X(x)\ \frac{dx}{dy}$

If the transformation is monotonically decreasing $f_Y(y) = f_X(x)\ \left(-\frac{dx}{dy}\right)$

In general, for a linear transformation $\boldsymbol{f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|, where\ x = g^{-1}(y)}$

Non - One to One Transformation of Random Variables:

For a transformation which is non - one to one, the transformation will be broken up into transformations each of which one to one. $\quad \boldsymbol{f_Y(y) = f_X(x_1)\ \left|\frac{dx_1}{dy}\right| + f_X(x_2)\ \left|\frac{dx_2}{dy}\right| + \cdots + f_X(x_n)\ \left|\frac{dx_n}{dy}\right|}$

PROBLEMS IN FUNCTION OF RANDOM VARIABLE

1. ***Consider a RV X with p.d.f. $f(x) = e^{-x}, x \geq 0$ with transformation $y = e^{-x}$. Find the transformed density function.***

Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right| = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{e^{-x}}{|-e^{-x}|} = \frac{y}{y} = 1, 0 < y \leq 1$

2. ***Let X be a RV with p.d.f. $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$. Find the p.d.f. of $Y = 8X^3$.***

Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = 8x^3 \Rightarrow x^3 = \frac{y}{8} \Rightarrow x = \left(\frac{y}{8}\right)^{\frac{1}{3}}$, $\quad \frac{dx}{dy} = \frac{1}{3}\left(\frac{y}{8}\right)^{\frac{1}{3}-1}\frac{1}{8} = \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}}$

$f_Y(y) = 2x\ \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = 2\left(\frac{y}{8}\right)^{\frac{1}{3}}\frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = \frac{1}{12}\left(\frac{y}{8}\right)^{-\frac{1}{3}}$, ***Range:*** $0 < x < 1 \Rightarrow 0 < \left(\frac{y}{8}\right)^{\frac{1}{3}} < 1 \Rightarrow 0 < y < 8$

3. ***If X is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ find the pdf of $Y = \tan X$.***

Solution: Given $f_X(x) = \frac{1}{b-a} = \frac{1}{\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} = \pi$, $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = \tan x \Rightarrow x = \tan^{-1} y$, $\frac{dx}{dy} = \frac{1}{1+y^2}$

$f_Y(y) = \pi\frac{1}{1+y^2}$ $\quad$ ***Range:*** $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow -\infty < y < \infty$

4. ***If X has an exponential distribution with parameter 1, find the pdf of $Y = \sqrt{X}$.***

Solution: *Given* $= 1$, $f_X(x) = \lambda\, e^{-\lambda x}, x > 0 = e^{-x}, x > 0$, $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = \sqrt{x} \Rightarrow x = y^2$, $\frac{dx}{dy} = 2y$, $\quad f_Y(y) = 2y\ e^{-y^2}$ $\quad$ ***Range:*** $x > 0 \Rightarrow y > 0$

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TCHEBYCHEFF INEQULAITY

Statement :

If X is a RV with $E(X)=\mu$ and $V(X)=\sigma^2$, then $P\{|X-\mu|\geq c\}\leq \frac{\sigma^2}{c^2}$ or $P\{|X-\mu|<c\}\geq 1-\frac{\sigma^2}{c^2}$, $c>0$.

Alternative Form : **If we put $c=k\sigma$, where $k>0$ then Tchebycheff inequality takes the form**

$$P\left\{\left|\frac{X-\mu}{k}\right|\geq \sigma\right\}\leq \frac{1}{k^2} \text{ or } P\left\{\left|\frac{X-\mu}{k}\right|\leq \sigma\right\}\geq 1-\frac{1}{k^2}$$

PROBLEMS IN TCHEBYCHEFF INEQULAITY

1. ***A RV X has mean $\mu=12$ and variance $\sigma^2=9$ and an unknown probability distribution. Find $P(6<X<18)$.***

Solution: Since the probability distribution of X is not known, we can not find the value of the required probability. We can find only a lower bound for the probability using Tchebycheff inequality.

$$P\{|X-\mu|\geq c\}\leq \frac{\sigma^2}{c^2}, c>0 \quad \text{(or)} \quad P\{|X-\mu|<c\}\geq 1-\frac{\sigma^2}{c^2},\ c>0$$

$$P\{-c<(X-\mu)<c\}\geq 1-\frac{\sigma^2}{c^2} \Rightarrow P\{\mu-c<X<\mu+c\}\geq 1-\frac{\sigma^2}{c^2}$$

$$\mu=12,\ \sigma^2=9,\ P\{12-c<X<12+c\}\geq 1-\frac{9}{c^2}$$

$$Put\ c=6,\ P\{12-6<X<12+6\}\geq 1-\frac{9}{6^2}$$

$$\boldsymbol{P\{6<X<18\}\geq \frac{3}{4}}$$

2. ***A fair die is tossed 720 times. Use Tchebycheff inequality to find a lower bound for the probability of getting 100 to 140 sixes.***

Solution: Let X – no. of sixes obtained when a fair die is tossed 720 times. $p=\frac{1}{6},\ q=\frac{5}{6},\ n=720$

X follows a binomial distribution with mean $np=120$ and variance $npq=100$, that is $\mu=120, \sigma=10$

By Tchebycheff inequality $P\{|X-\mu|\leq k\sigma\}\geq 1-\frac{1}{k^2}$

$$P\{|X-120|\leq 10k\}\geq 1-\frac{1}{k^2}$$

$$P\{-10k<(X-120)<10k\}\geq 1-\frac{1}{k^2}$$

$$P\{120-10k<X<120+10k\}\geq 1-\frac{1}{k^2}$$

$$Put\ k=2,\ P\{100<X<140\}\geq 1-\frac{1}{4}$$

$$\boldsymbol{P\{100<X<140\}\geq \frac{3}{4}}$$

3. ***A discrete RV X takes the values $-1,0,1$ with probabilities $\frac{1}{8},\frac{3}{4},\frac{1}{8}$ respectively. Evaluate $P\{|X-\mu|\geq 2\sigma\}$ and compare it with the upper bound given by Tchebycheff inequality.***

Solution:

$$E(X)=\sum_{x=-1}^{1} x\,P(x)=\left(-1\times\frac{1}{8}\right)+\left(0\times\frac{3}{4}\right)+\left(1\times\frac{1}{8}\right)=0$$

$$E(X^2)=\sum_{x=-1}^{1} x^2\,P(x)=\left(1\times\frac{1}{8}\right)+\left(0\times\frac{3}{4}\right)+\left(1\times\frac{1}{8}\right)=\frac{1}{4}$$

$$V(X)=E(X^2)-[E(X)]^2=\frac{1}{4}-0=\frac{1}{4}$$

$$P\{|X-\mu|\geq 2\sigma\}=P\{X\geq 1\}=P(X=-1\ or\ X=1)=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

By Tchebycheff inequality, $P\{|X-\mu|\geq k\sigma\}\leq \frac{1}{k^2}$

$$\boldsymbol{P\{|X-\mu|\geq 2\sigma\}\leq \frac{1}{4}}$$

All the Best

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