## One-dimensional Heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Where $\alpha^2 = \frac{K}{\rho C}$

$K$ - Thermal conductivity
$\rho$ - density
$c$ - specific heat

## Solutions of one-dimensional heat flow equation

$$u = (Ae^{px} + Be^{-px})(C e^{\alpha^2 p^2 t})$$

$$u = (A\cos px + B\sin px)(C e^{-\alpha^2 p^2 t}) \rightarrow$$ we use this solution as it is suitable for the problems.

$$u = (Ax + B)c$$

**Def:** **Steady state condition:**

The condition at which the temparature remains constant when time increases is called steady state condition.

(ie) $u(x,t) \rightarrow$ is a function of $x$ alone

$\therefore \frac{\partial u}{\partial t} = 0 \rightarrow$ under steady state condition.

### Condition for steady state

$$\frac{\partial u(x,t)}{\partial t} = 0 \quad \text{(ie)} \quad \frac{\partial u}{\partial t} = 0$$

1-D Heat Equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$

In steady state $\frac{\partial u}{\partial t} = 0 \quad \therefore (1) \Rightarrow \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \; ; \alpha^2 \neq 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{d^2 u}{dx^2} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{du}{dx}\right) = 0$$

$$\frac{d^2u}{dx^2} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{du}{dx}\right) = 0$$

Integrate w.r.to $x$

$$\frac{du}{dx} = a \text{ (constant)}$$

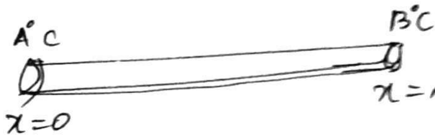
Integrate w.r.to $x$

$$\boxed{u = ax + b}$$

In steady state
$u(x,t) \to u(x)$

① one end of the rod of length $l$ is kept at $A°c$ and the other end is $B°c$ until steady state condition prevails. Find the steady state temperature on the rod.

Sol.:




The steady state temperature is

$$u = ax + b \longrightarrow ①$$

when $x = 0$, $u = A$ $\therefore$ ① $\Rightarrow A = 0 + b$

$$\Rightarrow \boxed{A = b}$$

$$\therefore u = ax + A \longrightarrow ②$$

when $x = l$, $u = B$ $\therefore$ ② $\Rightarrow B = al + A \Rightarrow B - A = al$

$$\Rightarrow a = \frac{B-A}{l}$$

$$\therefore ② \Rightarrow u = \frac{(B-A)}{l}x + A$$

② one end of the rod of length 10cm is kept at 30°c and other end of the rod is kept at 50°c until steady state condition prevails. Find the steady state temperature

Sol:


Steady state temperature is

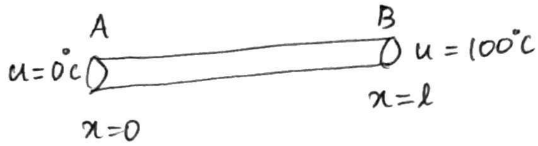
$$u = \left(\frac{B-A}{l}\right)x + A$$

$$= \left(\frac{50° - 30°}{10}\right)x + A = \frac{20}{10}x + A \Rightarrow \boxed{u = 2x + 30}$$

(3) A rod of length $l$ has its ends A and B, kept at $0°c$ and $100°c$ respectively until steady state condition prevails. If the temperature at B is reduced suddenly to $0°c$ and kept so, while that of A is maintained. Find the temperature $u(x,t)$

Sol:

The steady state temperature




$$u=\left(\frac{B-A}{l}\right)x+A$$

$$\Rightarrow \quad u=\left(\frac{100-0}{l}\right)x+0$$

$$\Rightarrow \quad u=\frac{100}{l}x \quad ; \quad 0<x<l$$

$u(x,t) \rightarrow$ temperature distribution of the rod of length $l$ from the origin.

one-dim. heat equation is $\dfrac{\partial u}{\partial t}=\alpha^2\dfrac{\partial^2 u}{\partial x^2}$

Boundary conditions are

(1) $u=0$ when $x=0$

(2) $u=0$ when $x=l$

(3) $u=\dfrac{100x}{l}$ when $t=0$

$0<x<l$.

Boundary conditions can also be written as

(1) $u(0,t)=0 \quad \forall t>0$

(2) $u(l,t)=0 \quad \forall t>0$

(3) $u(x,0)=100x \quad ; \quad 0<x<l$

Solution is $u=(A\cos px+B\sin px)\,e^{-\alpha^2p^2t} \longrightarrow (1)$

Apply BC (1) to eq (1) $(u=0,\ x=0)$

$$0=(A\cos 0+B\underbrace{\sin 0}_{0})\,e^{-\alpha^2p^2t}$$

$$\Rightarrow \quad 0=A\,e^{-\alpha^2p^2t}$$

$$\Rightarrow \quad e^{-\alpha^2p^2t}\neq 0 \quad \therefore \boxed{A=0}$$

Sub $A=0$ in eq (1)

$$\therefore \quad u=(0\cdot\cos px+B\sin px)\,e^{-\alpha^2p^2t}$$

$$\Rightarrow \quad u=(B\sin px)\,e^{-\alpha^2p^2t} \longrightarrow (2)$$

Apply BC (2) to eqn (2) $(u=0$ when $x=l)$

$$0 = B\sin pl \, e^{-\alpha^2 p^2 t}$$

$$\Rightarrow e^{-\alpha^2p^2t} \neq 0 \qquad B \neq 0 \qquad \therefore \sin pl = 0$$

(already said)

$$\Rightarrow \sin pl = \sin n\pi$$

$$\Rightarrow pl = n\pi$$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub $p = \frac{n\pi}{l}$ in eqn (2)

$$u = B\sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}$$

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t} \qquad (B = b_n)$$

$\to$ (3)

Apply BC (3) to eqn (3) $u = \frac{100x}{l}$, $t=0$.

$$\therefore \frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^0$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \to \text{Half Range Sine Series in } (0,l)$$

$$\therefore b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l \frac{100x}{l}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{200}{l^2}\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx$$

$uv_1 - u'v_2$

$u = x \qquad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = 1 \qquad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$$= \frac{200}{l^2}\left[\frac{-x\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} + (1)\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[\left\{-l\cos\left(\frac{n\pi l}{l}\right)\times\frac{l}{n\pi} + 0\right\} - \{0+0\}\right]$$

4

$$= -\frac{200}{l^2}\left\{\frac{l^2}{n\pi}\cos n\pi\right\}$$

$$b_n = -\frac{200}{n\pi}(-1)^n$$

Sub. $b_n$ in eq ③

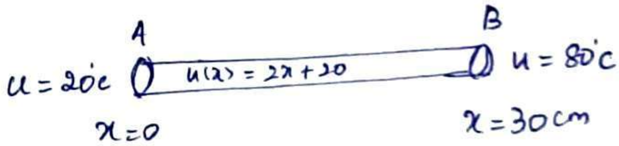
$\therefore$ eq ③ becomes

$$u = \sum_{n=1}^{\infty}\left(-\frac{200(-1)^n}{n\pi}\right)\sin\left(\frac{n\pi x}{l}\right)e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}$$

$$\text{i.e } u(x,t) = \sum_{n=1}^{\infty}\left(-\frac{200(-1)^n}{n\pi}\right)\sin\left(\frac{n\pi x}{l}\right)e^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}$$

② A rod 30cm long, has its ends A and B kept at 20°c and 80°c respectively until steady state conditions prevails. The temperature at each end is suddenly reduced to 0°c and kept so. Find the resulting temperature function $u(x,t)$, taking $x=0$ at A.

Sol:




The steady state temperature

$$u = \left(\frac{B-A}{l}\right)x + A$$

$$\Rightarrow u = \left(\frac{80-20}{30}\right)x + 20$$

$$\Rightarrow u = \left(\frac{60}{30}\right)x + 20$$

$$u = 2x + 20 \quad ; \quad 0 < x < 30$$

one-dim heat equation is $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$

Boundary conditions are

(1) $u=0$ when $x=0$

(2) $u=0$ when $x=30$

(3) $u=2x+20$ ; $0<x<30$ , $t=0$

Solution is $u=(A\cos px+B\sin px)\,e^{-\alpha^2p^2t} \rightarrow ①$

Apply BC (1) to eq ①

$$0=(A\cos 0+B\sin 0)\,e^{-\alpha^2p^2t}$$

$$\Rightarrow \quad 0 = A e^{-\alpha^2p^2t}$$

$$\Rightarrow \quad e^{-\alpha^2p^2t} \neq 0 \quad \therefore \boxed{A=0}$$

Sub. $A=0$ in eq ①

$\therefore$ ① becomes $u=(0+B\sin px)\,e^{-\alpha^2p^2t}$

$$\Rightarrow \quad u=B\sin px\; e^{-\alpha^2p^2t} \rightarrow ②$$

Apply BC (2) to eq ②

$$0=B\sin 30p \cdot e^{-\alpha^2p^2t}$$

$$\Rightarrow e^{-\alpha^2p^2t}\neq 0 \;,\; B\neq 0 \quad \therefore \sin 30p=0$$

$$\Rightarrow \sin 30p=\sin n\pi$$

$$\Rightarrow 30p=n\pi$$

$$\Rightarrow \boxed{p=\frac{n\pi}{30}}$$

Sub $p=\frac{n\pi}{30}$ in eq ②

$$\therefore u=B\sin\left(\frac{n\pi x}{30}\right)e^{-\alpha^2\left(\frac{n\pi}{30}\right)^2t} \rightarrow ③$$

Put BC (3) to eq ③

$$2x+20 = B\sin\left(\frac{n\pi x}{30}\right)e^{0}$$

ie $$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right) \quad \text{where } B=b_n$$

↳ Half Range sine series in (0,30)

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Here $l = 30$

$$\therefore \quad b_n = \frac{2}{30}\int_0^{30} (2x+20)\sin\left(\frac{n\pi x}{30}\right)dx$$

$uv_1 - u'v_2$

$u = 2x+20 \qquad v = \sin\left(\frac{n\pi x}{30}\right)$

$u' = 2 \qquad v_1 = \dfrac{-\cos\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)}$

$v_2 = \dfrac{-\sin\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)^2}$

$$= \frac{2}{30}\left[\frac{-(2x+20)\cos\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)} + \frac{(2)\sin\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)^2}\right]_0^{30}$$

$$= \frac{2}{30}\left[\left\{\frac{-(2(30)+20)\cos\left(\frac{n\pi(30)}{30}\right)}{\left(\frac{n\pi}{30}\right)} + \frac{2\sin\left(\frac{n\pi(30)}{30}\right)}{\left(\frac{n\pi}{30}\right)^2}\right\}\right.$$

$$\left. - \left\{\frac{-20\cos 0}{\left(\frac{n\pi}{30}\right)} + \frac{2\sin 0}{\left(\frac{n\pi}{30}\right)^2}\right\}\right]$$

$$= \frac{2}{\cancel{30}}\left[-80\cos n\pi \times\left(\frac{\cancel{30}}{n\pi}\right) + 20\times\left(\frac{30}{n\pi}\right)\right]$$

$$= -2\left[\frac{80(-1)^n - 20}{n\pi}\right]$$

$$= -\frac{40}{n\pi}\left[4(-1)^n - 1\right]$$

$$b_n = \frac{40}{n\pi}\left[1 - 4(-1)^n\right]$$

Sub. $b_n$ in eq ③

$$u = \sum_{n=1}^{\infty}\frac{40}{n\pi}\left[1-4(-1)^n\right]\sin\left(\frac{n\pi x}{30}\right)e^{-\alpha^2\left(\frac{n\pi}{30}\right)^2 t}$$

$u(x,0) =$