3/4/17 Unit - V MARKOV CHAINS

Definition:

A random process in which the future value depends only on the present value and not on the past value is called MARKOV PROCESS.

Markov chain:

if $\forall n$,

$$P[x_n = a_n / x_{n-1} = a_{n-1}, x_{n-2} = a_{n-2}, \dots x_0 = a_0]$$
$$= P[x_n = a_n / x_{n-1} = a_{n-1}]$$

Then the random variables $\{x_n\}$ is called Markov chain, $n = 0, 1, 2 \dots$ etc and $a_1, a_2 \dots a_n$ are called states of Markov chain.

Transition probability:

$P[X_m = a_i] = P_i(M)$ represent the probability that at time $t = t_m$, the system occupies the state a_i (ie)

$P[x_n = a_j / x_m = a_i]$, $P_{ij}(m, n)$ represent the prob that the system goes into state a_j at $t = t_n$ given that it was in state a_i at $t = t_m$. The no's $P_{ij}(m, n)$ represent the transition property of the Markov chain from state a_i to a_j.

One step transition probability

The conditional probability

$P[x_{n+1} = a_j / x_n = a_i]$ is called one step transition property.

from a_i at time t_n to state a_j at time t_{n+1} in one step.

gets denoted by $P_{ij}(n, n+1)$.

A markov chain is set to be homogenous in time if one step transition prob is independent of the step.

n step-transition probability.

The conditional prob $P[X_n = a_j / X_0 = a_i]$ is called n step transition prob. and it is denoted by $P_{ij}(n)$ (or) $P_{ij}^{(n)}$.

Probability vector and stochastic Matrix

A vector (row Matrix) $[P_1, P_2 \ldots P_n]$ is called prob. vector

if (i) $P_i \geq 0, \forall i$
ii) $\sum_{i=1}^{n} P_i = 1$.

A square matrix in $P = [P_{ij}]$ is called a stochastic matrix if each row of P is a prob. vector.

Chapman-kolmogorov Equation:

The n^{th} step transition prob. can be computed using CK equation which can be stated as $P_{ij}(M+n) = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(M)}$,

$\forall\ n, m, i, j \geq 0$

Note: If P is a transition prob matrix of a regular Markov chain then $\pi P = \pi$ where $\pi = (\pi_1, \pi_2)$ and $\pi_1 + \pi_2 = 1$ given. The invariant prob or the stationary probability.

Remark:
When Markov chain shows steady state or stationary behaviour then system in the long run has an invariant prob.

Classification of Markov chain:

i) Irreducible Markov chain: A markov chain is said to irreducible if every state is reachable from every other state
ie $P_{ij}^{(n)} > 0$, for some n and $\forall\, i, j$.

ii) Non-null persistant: A finite and irreducible markov chain is non-null persistant markov chain.

iii) A-periodic (or) periodic:
The period d_i of a return state i is the GCD of all integer m such that $P_{ii}^{(m)} > 0$, $d_i = GCD \{ m \mid P_{ii}^{(m)} > 0 \}$.
The state i is periodic with period d_i state i is A-periodic if $d_i = 1$.

iv) Ergodic:-
A non-null persistance, A-periodic markov chain is called Ergodic Markov chain.

5|2|7

Problems:

1) Find the invariant probability for Markov chain $\{X_n\}$, $n \geq 1$ with the state space $\{0,1\}$ and one step TPM is given by $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Solution:

The invariant prob is $\pi P = \pi$ where $\pi = (\pi_1, \pi_2)$ — (1)

and $\pi_1 + \pi_2 = 1$

$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2) \qquad \pi P = \pi$$

$$(\pi_1, \pi_2) P = (\pi_1, \pi_2)$$

$$\left(\frac{\pi_2}{2}, \pi_1 + \pi_2/2\right) = (\pi_1, \pi_2)$$

$$\pi_1 = \frac{\pi_2}{2} \qquad \pi_2 = \pi_1 + \frac{\pi_2}{2}$$

$$\pi_2 = 2\pi_1 \qquad \Rightarrow \pi_1 = \frac{\pi_2}{2} \Rightarrow 2\pi_1 = \pi_2$$

$$\pi_1 + 2\pi_1 = 1$$

$$3\pi_1 = 1$$

$$\pi_1 = 1/3.$$

$$\pi_1 + \pi_2 = 1$$

$$1/3 + \pi_2 = 1$$

$$\pi_2 = 2/3.$$

2) Let $P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$ be the tpm of 2 state Markov chain. Find the stationary prob. of the chain.

Solution

The stationary prob $\pi P = \pi$ where $\pi = (\pi_1, \pi_2)$ and $\pi_1 + \pi_2 = 1$.

$$(\pi_1 \ \ \pi_2) \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2)$$

$$\left(\frac{3\pi_1}{4} + \frac{\pi_2}{2} \quad \frac{\pi_1}{4} + \frac{\pi_2}{2} \right) = (\pi_1, \pi_2)$$

$$\pi_1 = \frac{3\pi_1}{4} + \frac{\pi_2}{2} \qquad \pi_2 = \frac{\pi_1}{4} + \frac{\pi_2}{2}$$

$$\pi_2 = \frac{\pi_1}{4} + \frac{\pi_2}{2}$$

$$(\pi_1 - \pi_2) = \frac{3\pi_1}{4} - \frac{\pi_1}{4} = \frac{2\pi_1}{4} = \frac{\pi_1}{2}$$

$$\pi_1 - \pi_2 = \frac{\pi_1}{2}$$

$$-\pi_2 = \frac{\pi_1}{2} - \pi_1$$

$$\pi_2 = \frac{\pi_1}{2} \Rightarrow \pi_1 = 2\pi_2$$

$$\pi_1 + \pi_2 = 1 \qquad \boxed{\pi_1 = 2/3}$$

$$2\pi_2 + \pi_2 = 1$$

$$\boxed{\pi_2 = 1/3}$$

$$\pi = (\pi_1, \pi_2)$$

$$= (2/3 \quad 1/3)$$

3) A college student x has the foll study habit. If he studies one night he is 70% sure not no next night. If he doesnt study one night he is only 60% sure not to study next night also.

i) find tpm.

ii) How often he studies in a long run.

Solution:

Since the study pattern depends on the present not on the past it is a markov chain. Where the states are studying (s) and not studying (n).

i) ~~First~~ tpm:

$$P = \begin{array}{c c} & \begin{array}{cc} s & n \end{array} \\ \begin{array}{c} s \\ n \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.4 \end{bmatrix} \end{array}$$

i) tpm:

$$\begin{array}{c|cc} & S & N \\ \hline S & 0.3 & 0.7 \\ N & 0.4 & 0.6 \end{array}$$

Let $\pi = (\pi_1, \pi_2)$ be the long run then $\pi p = \pi$.

where $\pi_1 + \pi_2 = 1$.

for long run we have to find stationary.

$$(\pi_1, \pi_2)\begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix} = (\pi_1, \pi_2)$$

$$0.3\pi_1 + 0.4\pi_2 \quad 0.7\pi_1 + 0.6\pi_2 = (\pi_1, \pi_2)$$

$$\pi_1 = 0.3\pi_1 + 0.4\pi_2$$

$$\pi_1 - 0.3\pi_1 = 0.4\pi_2$$

$$0.7\pi_1 = 0.4\pi_2$$

$$\frac{0.7\pi_1}{0.4} = \pi_2 \implies \pi_1 = \frac{0.4}{0.7}\pi_2$$

$$0.7\pi_1 + 0.6\pi_2 = \pi_2$$

$$0.7\pi_1 = 0.4\pi_2$$

$$\pi_1 = \frac{0.4}{0.7}\pi_2$$

$$\frac{0.4\pi_2}{0.7} + \pi_2 = 1$$

$$0.571\pi_2 + \pi_2 = 1$$

$$1.571 = 1 \implies \pi_2 = 0.6364$$

$$\pi_1 = 0.571\pi_2$$

$$= 0.571 \times 0.636$$

$$= 0.363$$

$$\pi = (\pi_1, \pi_2) = (0.363, 0.636)$$

4) A man either drives a car or catches a train to go to office every day. He never goes in a row by train, but if he ~~fix~~ drive one day, then the next day is just as slightly to drive again as he is to travel by train. Suppose that on the first day of the week the man tossed a fair die, and ~~try~~ drive to work if 6 appeared.

find i) the prob that it takes a train on the third day. Prob that he drives to work in the long run.

Solution

The travel pattern of tomorrow depends on today. ∴ It is a Markov chain.

TPM
$$P = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

Initial state probability distribution is obtained by throwing a die.

$$P^{(1)} = \begin{array}{cc} T & C \\ (5/6 & 1/6) \end{array}$$

ii) P(takes a train on the 3rd day)

$$P^{(2)} = P^{(1)} \cdot P.$$

$$= \left(\frac{5}{6} \quad 1/6\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}.$$

$$= \left(1/12 \quad 5/6 + 1/12\right) = \left(\frac{1}{12} \quad \frac{11}{12}\right)$$

$P^{(3)} = P^{(2)} \cdot P$

$= \left(\frac{1}{12} \quad \frac{11}{12}\right) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$

$= \left(\frac{11}{24} \quad \frac{1}{12} + \frac{11}{24}\right).$

$= \left(\frac{11}{24} \quad \frac{13}{24}\right)$

P (train in the 3rd day) $= \frac{11}{24}$.

Let $\pi = (\pi_1, \pi_2)$ be the Prob. for long run.

$\therefore \pi P = \pi$ where $\pi_1 + \pi_2 = 1$. ——— (1)

$(\pi_1 \quad \pi_2) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2)$

$\left(\frac{\pi_2}{2} \quad \pi_1 + \frac{\pi_2}{2}\right) = (\pi_1 \quad \pi_2)$

$\frac{\pi_2}{2} = \pi_1$ and $\frac{\pi_2}{2} + \pi_1 = \pi_2$

$\pi_2 = 2\pi_1$ $\qquad \pi_2 = 2\pi_1$

from (1). $\pi_1 + \pi_2 = 1$.

$\pi_1 + 2\pi_1 = 1$

$\pi_1 = 1/3$. $\quad \therefore \pi_2 = 2/3$.

$\therefore \pi = \left(\frac{1}{3} \quad \frac{2}{3}\right) =$ P(drives) $= 2/3$ //

5) Suppose that the probability of a dry day following rainy day is 1/3 and the prob of rainy day following a dry day is 1/2. If May 1st is a dry day, find the prob. that

i) May 3rd is a dry day again.

ii) May 5th is a dry day.

$$\text{TPM } P = \begin{array}{c|cc} & D & R \\ \hline D & \left[\begin{matrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{matrix}\right] \\ R & \end{array}$$

Initial state probability distribution is obtained by May 1st is a dry day.

$$p^{(1)} = \begin{array}{cc} D & R \\ (1 & 0) \end{array}$$

$p^2 = p^0 \cdot p$

$p^3 = p^2 p$

$p^4 = p^3 p$

(i) P (May 3rd is a dry day)

$$p^{(2)} = p^{(1)} \cdot p$$

$$= (1 \quad 0) \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= (1/2 \quad 1/2).$$

$$p^{(3)} = p^{(2)} \cdot p.$$

$$= \left(\frac{1}{2} \quad \frac{1}{2}\right) \begin{pmatrix} \frac{1}{2} & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= \begin{array}{cc} D & R \\ \left(\frac{5}{12}\right. & \left.7/12\right) \end{array}$$

P (May 3rd is dry day) = 5/12.

ii) P(May 5^{th} is a dry day)

$$p^{(4)} = p^{(3)} \cdot P$$

$$= \begin{pmatrix} 5/12 & 7/12 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 29/72 & 43/72 \end{pmatrix}$$

$$p^{(5)} = p^{(4)} \cdot P$$

$$= \begin{pmatrix} \frac{29}{72} & \frac{43}{72} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} \overset{D}{\frac{173}{432}} & \overset{R}{\frac{259}{432}} \end{pmatrix}$$

P(May 5^{th} is dry).

$$= \frac{173}{432} //$$

6) The TPM of a Markov process of $\{x_n\}, n = 1, 2, 3 \ldots$ having 3 states is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \text{ and}$$

initial distribution $p^{(0)} = (\overset{1}{0.7} \quad \overset{2}{0.2} \quad \overset{3}{0.1})$

find.

i) $P[x_2 = 3, x_1 = 3, x_0 = 2]$

ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

iii) $P[X_2 = 3]$.

Solution

i) $P[X_2 = 3, X_1 = 3, X_0 = 2]$

$= P[X_2 = 3 / X_1 = 3] \cdot P[X_1 = 3 / X_0 = 2] \cdot P[X_0 = 2]$

$= P_{33}^{(1)} \cdot P_{23}^{(1)} \cdot P[X_0 = 2]$

$= 0.3 \times 0.2 \times 0.2$

$= 0.012$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

$= P[X_3 = 2 / X_2 = 3] \cdot P[X_2 = 3 / X_1 = 3] \cdot P[X_1 = 3 / X_0 = 2] \cdot P[X_0 = 2]$

$= P_{32}^{(1)} \cdot P_{33}^{(1)} \cdot P_{23}^{(1)} \cdot P[X_0 = 2]$

$= 0.4 \cdot 0.3 \cdot 0.2 \times 0.2 = 0.0048$

(iii) $P[X_2 = 3]$

$= \sum_{i=1}^{3} P(X_2 = 3 / X_0 = i] \cdot P[X_0 = i]$

$= P[X_2 = 3 / X_0 = 1] \cdot P[X_0 = 1] + P[X_2 = 3 / X_0 = 2] \cdot P[X_0 = 2] + P[X_2 = 3 / X_0 = 3] \cdot P[X_0 = 3]$.

$= P_{13}^{(2)} \cdot P[X_0 = 1] + P_{23}^{(2)} \cdot P[X_0 = 2] + P_{33}^{(2)} \cdot P[X_0 = 3]$

now

$P^{(2)} = P \cdot P$

$$= \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$P^{(2)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} \end{array}$$

$(1) \Rightarrow \; = 0.26(0.7) + (0.34)(0.2) + 0.29(0.1)$

$= 0.279.$ //

7) The tpm of a Markov process of $\{X_n\}$, $x = 1, 2, 3 \ldots$ having 3 states $i = 0, 1, 2$ is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$

and the initial distribution

$$P^{0} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ (1/3 \;\; 1/3 \;\; 1/3) \end{array}$$

find:

i) $P[X_3 = 2 / X_2 = 1]$

ii) $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$

iii) $P[X_2 = 2, X_1 = 1, X_0 = 2]$

iv) $P[X_2 = 2]$

<u>Solution</u>

i) $P[X_3 = 2 / X_2 = 1]$

$P^{(1)}_{12} = 1/4.$

ii) $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$

$= P[X_3 = 1 / X_2 = 2] \cdot P[X_2 = 2 / X_1 = 1] \cdot P[X_1 = 1 / X_0 = 2] \cdot P[X_0 = 2]$

$= P^{1}_{21} \cdot \cancel{Pf}\, P^{1}_{12} \cdot P^{1}_{21} \cdot P[X_0 = 2]$

$= 3/4 \cdot 1/4 \cdot 3/4 \cdot 1/3$

$= \dfrac{9}{64 \times 3} = \dfrac{3}{64}$ //

iii) $P[X_2 = 2, X_1 = 1, X_0 = 2]$

$= P[X_2 = 2 / X_1 = 1] \cdot P[X_1 = 1 / X_0 = 2] \cdot P[X_0 = 2]$

$= P_{12}^{(1)} \cdot P_{21}^{(1)} \cdot P[X_0 = 2]$

$= 1/4 \cdot 3/4 \cdot 1/3$

$= \frac{3}{16 \times 3} = \frac{1}{16}$

iv) $P[X_2 = 2]$

$= \sum_{i=0}^{2} P[X_2 = 2 / X_0 = i] \; P[X_0 = i]$

$= P[X_2 = 2 / X_0 = 0] \cdot P[X_0 = 0] +$

$P[X_2 = 2 / X_0 = 1] \cdot P[X_0 = 1] +$

$P[X_2 = 2 / X_0 = 2] \cdot P[X_0 = 2]$

$= P_{02}^{2} \cdot P[X_0 = 0] + P_{12}^{2} \cdot P[X_0 = 1] + P_{22}^{2} \cdot P[X_0 = 2]$

$\Rightarrow P^2 = P \times P$

$$= \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.625 & 0.312 & 0.0625 \\ 0.3125 & 0.5 & 0.187 \\ 0.187 & 0.562 & 0.25 \end{bmatrix} \end{array}$$

(1) $= 0.0625 \times 1/3 + 0.187 \times 1/3 + 0.25 \times 1/3$

$= 0.166$ //

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8) A gambler has Rs 2. He bets Rs 1 at a time and wins Rs 1 with prob 1/2. He stops playing if he if he loses Rs 2 or wins Rs 4.

i) what is the tpm related to Markov chain

ii) what is the prob that he has lost his money at the end of 5th play.

iii) what is the prob that the game last more than 7 plays.

Sol states are $\{\overset{\text{losses}}{0} \; 1 \; \overset{2}{2} \; 3 \; 4 \; \overset{\text{win}}{5}\}$

Initial prob dist is $p(0) = \{0 \; 0 \; 1 \; 0 \; 0 \; 0 \; 0\}$

TPM

$$
\begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
2 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\
3 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\
4 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\
5 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

ii) P[that he lost money at the end of 5th play]

$P^{(1)} = P^{(0)} \cdot P$

$= [0 \; 1/2 \; 0 \; 1/2 \; 0 \; 0 \; 0]$

$P^{(2)} = P^{(1)} \cdot P$

$= [1/4 \; 0 \; 1/2 \; 0 \; 1/4 \; 0 \; 0]$

$p^{(3)} = p^{(2)} P$

$= [1/4 \quad 1/4 \quad 0 \quad 3/8 \quad 0 \quad 1/8 \quad 0]$

$p^{(4)} = p^{(3)} P$

$= [3/8 \quad 0 \quad 5/16 \quad 0 \quad 1/4 \quad 0 \quad 1/16]$

$p^{(5)} = p^{(4)} \cdot P$

$= [3/8 \quad 5/32 \quad 0 \quad 9/32 \quad 0 \quad 1/8 \quad 1/16]$

P[He lost his money at the end of 5^{th} play]

$= 3/8$

iii) P[game last more than 7 plays]

$p^{(6)} = p^{(5)} P$

$= (29/64 \quad 0 \quad 14/64 \quad 0 \quad 13/64 \quad 0 \quad 1/8]$

$= [29/64 \quad 0 \quad 7/32 \quad 0 \quad 13/64 \quad 0 \quad 1/8]$

$p^{(7)} = p^{(6)} P$

$= (29/64 \quad 7/64 \quad 0 \quad 27/128 \quad 0 \quad 13/128 \quad 1/8)$

P[game last more than the 7 play] ie > 7 $= 7/64 + 0 + \frac{27}{128} + 0 + \frac{13}{128}$

$= \frac{54}{128} \Rightarrow \frac{27}{64}$

Q) A die is tossed repeatedly. If $\{X_n\}$ denotes the maximum of the numbers occuring in the tosses, i) find also P^{2} such that $P[X_2 = 6]$.

Sol States are = $\{1, 2, 3, 4, 5, 6\}$.

Individual prob p^{0} = $[1/6, 1/6, 1/6, 1/6, 1/6, 1/6]$

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\Rightarrow P(X_2 = 6)$.

$= \sum_{i=1}^{6} (X_2 = 6 / X_0 = i)\, P(X_0 = i)$.

$\Rightarrow P(X_2 = 6 / X_0 = 1) \cdot P(X_0 = 1) +$
$P(X_2 = 6 / X_0 = 2) \cdot P(X_0 = 2) \cdots +$
$P(X_2 = 6 / X_0 = 6) \cdot P(X_0 = 6)$.

$\Rightarrow P_{16}^{2} \cdot P(X_0 = 1) + P_{26}^{2}\, P(X_0 = 2) \cdots$
$P_{66}^{2}\, P(X_0 = 6)$

$P^{2} = P \cdot P$

$$= \frac{1}{36} \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

$$+ \frac{11}{36}\cdot\frac{1}{6} + \frac{11}{36}\cdot\frac{1}{6}$$

$$= \frac{11}{36}\cdot\frac{1}{6} + \frac{11}{36}\cdot\frac{1}{6}$$

$$+ \frac{11}{36}\cdot\frac{1}{6} + \frac{11}{36}\cdot\frac{1}{6}$$

$$P[X_2 = 6] = \frac{91}{216}$$ //

Classification:

1) Irreducible:

$P_{ij}^{(m)} > 0, \forall i,j$ and for some m.

ii) Non-null persistent should be of irreducible and finite.

iii) Aperiodic (or) periodic $P_{ij}^{(m)} > 0$.

$$d_i = GCD\{m / P_{ij}^{(m)} > 0\}$$

$$= GCD\{2, 4\}.$$

$d_i > 1 \rightarrow$ periodic

$d_i = 1 \Rightarrow$ Aperiodic

iv) Ergodic:

2^{nd} type and Aperiodic

10) find the nature of state of markov chain with TPM given

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 1 & 0 \end{bmatrix}$$ and the

states are $\{0,1,2\}$

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \quad P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \quad P^5 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

and so on.

$P^2 = P^{2n}$, $n = 1, 2, 3$.

$P = P^{2n+1}$

i)

$P_{00}^{2} > 0$, $P_{01}^{1} > 0$, $P_{11}^{2} > 0$,

$P_{10}^{1} > 0$, $P_{11}^{2} > 0$, $P_{12}^{1} > 0$,

$P_{20}^{2} > 0$, $P_{21}^{1} > 0$, $P_{22}^{2} > 0$,

$\therefore$ It is irreducible. since states are $\{0,1,2\}$

ii) $P_{00}^{2} > 0 =$, $P_{00}^{4} > 0$, $= P_{00}^{6} > 0$.

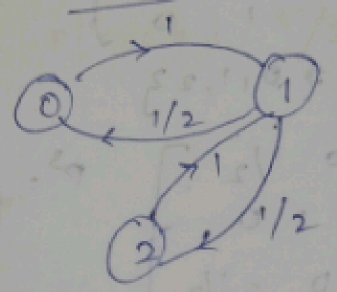
$\therefore$ for all i, states of chain are periodic with 2. $\therefore d_i > 1$.

State : 1 $\Rightarrow$ $P_{11}^{(2)} > 0 = P_{11}^{(4)} > 0 = P_{11}^{(6)} > 0$. $d_i = 2$

State 2 : $P_{22}^{2n} > 0$ $n = 1, 2$ $\therefore$ periodic

iv) Non null persistent but its not Aperiodic as it not ergodic.

Transition diagram



11) Three boys A, B, C are throwing a ball to eachother. A always throws to B, B always throws to C. But C is just as likely to throw ball to A or B. Show that process is not ergodic. find TPM and classify state.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \quad P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \quad P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

80 . m

$P_{AA}^{5} > 0 \quad P_{AB}^{5} > 0 \quad P_{AC}^{5} > 0$

$P_{BA} > 0 \quad P_{BB} > 0 \quad P_{BC} > 0$

$P_{CA} > 0 \quad P_{CB} > 0 \quad P_{CC} > 0.$

Irreducible.

ii) Irreducible, states $\{A, B, C\}$

$\therefore$ non-null persistent.

$1/4^5$

iii) State A.

$P_{AA}^{3} > 0, \; P_{AA}^{5} > 0, \; P_{AA}^{6} > 0, \; P_{AA}^{7} > 0.$

$d_i = GCD \{ M \mid P_{ij}^{(m)} > 0)$

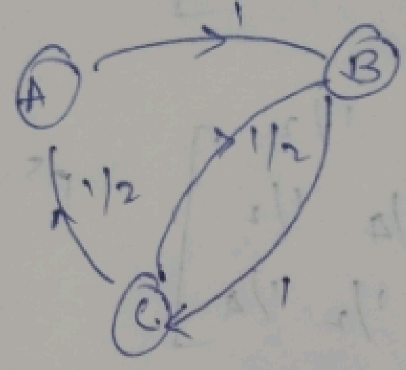
$= 1 \Rightarrow$ A periodic.

state B : $d_i = 1$ A-periodic

state C : $d_i = 1$ A-periodi

iv) Non-null A-periodic.

It is ergodic



x ———————— x