Unit - V

Markov process :-

A random process in which future value Depends only on the present value not on past is called Markov process.

if $\forall n$

state a_{n-1} to a_n

$$P\left[x_n = a_n / x_{n-1} = a_{n-1}, x_{n-2} = a_{n-2} = \cdots x_0 = a_0\right]$$

x_{n-1} to x_n 1 step

$$= P\left[x_n = a_n / x_{n-1} = a_{n-1}\right]$$

The random variable $\{x_n\}$ is called Markov Chain, $n = 0, 1, 2, \ldots$ and $a_1, a_2, \ldots$ are called states of Markov chains

Transition probability :

$$P[X_m = a_i] = P_i(M)$$

The Probability that on time $t = t_m$ the system occupies the state a_i

$$P\left[x_n = a_j / x_m = a_i\right] = P_{ij}(m,n)$$

represent the probability that the system goes to state a_j from $t = t_n$ a_i at $t = t_m$

The numbers $P_{ij}(m,n)$ represent that the transition property of the markov from state a_i to a_j

One step Transition probability

The conditional probability

$$P[x_{n+1} = a_j \mid x_n = a_i]$$

one step (n to n+1)

is called one step Transition property from a_i and time t_n to state a_j at time t_{n+1} is one step. It is denoted by $P_{ij}(n, n+1)$.

A markov chain is said to be homogeneous in time if one step transition probability is independent of the step.

N step Transition probability:

$P[x_n = a_j \mid x_0 = a_i]$ and denoted by

n - 0 = n so (n step)

$P_{ij}(n)$ or $P_{ij}^{(n)}$

Probability vector and stochastic matrix

A vector $[P_1, P_2 \ldots P_n]$ is probability vector if &

(i) $P_i \geq 0 \;\forall i$

(ii) $\sum_{i=1}^{n} P_i = 1$

A square matrix $P = [P_{ij}]$ is called Stochastic nature, if each row of ~~matrix~~ P is a probability vector

Chapman Kolmogorov

The nth step transition probability can be computed using

$$P_{ij}(m+n) = \sum_{k=0}^{\infty} P_{ik}(n)\, P_{kj}(n) + n, m, i, j \geq 0$$

Note: If P is a transition prob matrix of regular markov chain then

$$\boxed{\pi P = \pi}$$

where $\pi = (\pi_1, \pi_2 \ldots)$

and $\boxed{\pi_1 + \pi_2 = 1}$ given stationery / invariable probability.

Remark :-

when markov chain shows steady state or stationery behaviour system in the long run has an invariant probability

classification of markov chain

→ Irreducible markov chain :-

A markov chain is said to be irreducible we can reach any state from any other state.

$$P_{ij}(n) > 0 \quad \text{for some } n, \forall i, j$$

→ Non-null persistance

A finite and irreducible markov chain is non-null persistant.

→ periodic and aperiodic

The period d_i of return stage it is greatest common divisor of integer m such that

$$P_{ij}(m) > 0, \quad d_i = GCD$$

$$\{m, P_{ij}(m) > 0\}$$

The state i is periodic with period d_i state i is aperiodic if $d_i = 1$.

Ergodic process :-

A non null persistent aperiodic markov chain is called ergodic markov chain

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P:1. Find the invariant probability for markov chain $\{X_n\}$ $n \geq 1$ with the state space $\{0, 1\}$ and one step TPM (Transition probability matrix) is given by $P = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$\pi P = \pi$$

$$\pi = (\pi_1, \pi_2)$$

$$\Rightarrow (\pi_1, \pi_2) \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} = [\pi_1, \pi_2]$$

$$\left[\pi_1 \times 0 + \pi_2 \times \frac{1}{2} \quad \pi_1 \times 1 + \pi_2 \times 1/2\right]$$

$$= [\pi_1, \pi_2]$$

$$\left[\frac{\pi_2}{2} \quad \pi_1 + \frac{\pi_2}{2}\right] = [\pi_1, \pi_2]$$

$$\begin{bmatrix} \frac{\pi_2}{2} & \pi_1 + \frac{\pi_2}{2} \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

$$\underbrace{\frac{\pi_2}{2} = \pi_1}_{1}, \quad \underbrace{\pi_1 + \frac{\pi_2}{2} = \pi_2}_{2}$$

$$\boxed{\pi_1 + \pi_2 = 1} \quad \boxed{\pi_1 = 1/3, \ \pi_2 = 2/3}$$

A college student X has the following study habit. If he studies one night 70% sure not to next night. If he doesn't study 1 night, he is 60% sure not to study next night also. Find i) TPM

ii) How often he studies in long run

Soln:

Since the study pattern depends on present not on past, it is markov chain. So the 2 states are study, not study.

$$\text{state space} = \{S \quad NS\}$$

(i)
$$\begin{array}{c|c} & \begin{array}{cc} S & NS \end{array} \\ \begin{array}{c} S \\ NS \end{array} & \begin{bmatrix} 30\% & 70\% \\ & 60\% \end{bmatrix} \end{array} = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \rightarrow \text{TPM}$$

colum wise today

row wise yest

ii) $\pi = (\pi_1, \pi_2)$

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

also $\boxed{\pi_1 + \pi_2 = 1}$

$\pi P = \pi$

$$[\pi_1, \pi_2] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [\pi_1, \pi_2]$$

$$0.3\pi_1 + 0.4\pi_2 = \pi_1 \quad \text{—} (1)$$

$$0.7\pi_1 + 0.6\pi_2 = \pi_2 \quad \text{—} (2)$$

$\pi_1 = 0.363$

$\pi_2 = 0.636$

$$\pi = \begin{array}{cc} S & NS \\ (0.363, & 0.636) \end{array}$$

A man either drives a car catches a train to office He never goes in a row by train. But if he drives one day then the next day he just as likely to drive again by train suppose that, the first day of the week, a man tossed a fair die, and drive to work if 6 appears. Find

(i) probability that he takes a train on third day.

(ii) Find the probability that he drives to work in long run.

Sol:

State space = $\{T, C\}$

$$TPM = \begin{array}{c|cc} & T & C \\ \hline T & 0 & 1 \\ C & 1/2 & 1/2 \end{array}$$

Initial state probability

$$P = \begin{array}{cc} T & C \\ \{5/6, & 1/6\} \end{array}$$

(i) P[takes train on 3rd day]

$$= P^{(2)} = P^{(1)} \cdot P$$

$$= \left(\frac{5}{6}, \frac{1}{6}\right) \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$p^{(3)} = p^{(2)} . p$

$$= \left(\frac{1}{12}, \frac{11}{12}\right) \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{array}{cc} T & C \\ \left(\frac{11}{24}\right. & \left.\frac{13}{24}\right) \end{array}$$

Train car

$P[\text{train on } 3^{rd} \text{ day}] = \frac{11}{24}$

by train

(i) ~~If~~ find $P(3)$

$$\text{statespace} \quad \begin{array}{cc} T & C \\ \multicolumn{2}{c}{\begin{bmatrix} \quad & \quad \end{bmatrix}} \end{array}$$

$P[\text{train on } 3^{rd} \text{ day}] = \frac{11}{24}$

(ii) $\pi p = \pi$ $\qquad \pi_1 + \pi_2 = 1$ — (1)

Let $\pi = (\pi_1, \pi_2)$

$$\pi p = \pi$$

$$(\pi_1, \pi_2) \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = (\pi_1, \pi_2)$$

$\pi_1 + \frac{\pi_2}{2} = \pi_1$ — (2)

$\pi_1 + \frac{\pi_2}{2} = \pi_2$ — (3)

solve (1), (2) & (3) $\Rightarrow \pi_1 = \frac{1}{3}, \pi_2 = \frac{2}{3}$

$$\pi_1 \ (\pi_2) : \begin{array}{cc} T & C \\ \left(\frac{1}{3} \right. & \left. \frac{2}{3}\right) \end{array}$$

Probability that he drives for long run is $\frac{2}{3}$ //.

Suppose that the prob of a dry day following rainy day is $1/3$. And prob of rainy day following dry day is $1/2$.

If May 1st is a dry day. Find the prob that

i) May 3rd is dry day again

ii) May 5th is dry day.

State Space = {Dry day, Rainy day}

$$\begin{array}{c|cc} & D & R \\ \hline D & 1/2 & 1/2 \\ R & 1/3 & 2/3 \end{array}$$

Initial probability is obtained

by may is a dry day $p^{(1)} = (1, 0)$

(i) P[may 3rd be a dry day]

$$p^{(3)} = p^{(2)} \cdot p^{(1)}$$

$$p^{(2)} = p^{(1)} \cdot P$$

$$= (1, 0)\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1/2 & 2/3 \end{bmatrix}$$

$$= \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$p^{(3)} = p^{(2)} = \left(\frac{1}{2}, \frac{1}{2}\right)\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} =$$

$$\left[\frac{5}{12}, \frac{7}{12}\right]$$

(ii)