Unit : 2 Theoritical Distribution

Discrete distributions :

- Binomial distribution
- Poison distribution
- Geometric distribution
- continous distribution :
- 1. exponential distribution
  2. Normal distribution

Binomial distribution :

A random variable X is said to follow binomial distribution if its Probability mass function is given by.

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

Note : $p+q=1$

Moment generating function (MGF) mean variance of binomial distribution.

$$M_X(t) = E\left[e^{tx}\right] = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \left({}^nC_x\, p^x q^{n-x}\right)$$

$$= \sum_{x=0}^{\infty} {}^nC_x \left(pe^t\right)^x \cdot q^{n-x}$$

Substituting $x=0$ ${}^nC_0 (pe^t)^0 q^{n-0} + {}^nC_1 (pe^t)^1 q^{n-1} + {}^nC_2 (pe^t)^2 q^{n-2} + \dots$

$$= q^n + {}^nC_1 (pe^t) q^{n-1} + \dots$$

$$= (q+pe^t)^n$$

Mean = $E(X) = \frac{d}{dt} M_X(t) \Big]_{t=0}$

$= \frac{d}{dt}\left[q+pe^t\right]^n \Big\}_{t=0}$

differentiating

$= n\left[q+pe^t\right]^{n-1} \cdot pe^t \Big]_{t=0}$

$= n\left[q+pe^0\right]^{n-1} \cdot pe^0$

$= n\left[q+p\right]^{n-1} \cdot p$

$= n[1] \cdot p$

$E(X) = np$

differentiating this

$E[X^2] = \frac{d^2}{dt^2} M_X(t) \Big\}_{t=0}$

$= \frac{d}{dt}\left[npe^t(q+pe^t)^{n-1}\right]_{t=0}$

$= npe^t\left((n-1)\cdot(q+pe^t)^{n-2} \cdot pe^t\right)$
$\quad + (q+pe^t)^{n-1} \cdot npe^t \Big\}_{t=0}$

$d(uv) = uv' + vu'$

$= npe^0\left[(n-1)(q+p)^{n-2} \cdot pe^0\right]$
$\quad + (q+pe^0)^{n-1} \cdot npe^0$

$= np(n-1)p + np$

$= np((n-1)p+1)$

Varience.

$E(X^2) - [E(X)]^2$

$= np[1+(n-1)p] - (np)^2$

$= np + np^2(n-1) - n^2p^2$

$= np + n^2p^2 - np^2 - n^2p^2$

$= np(1-p) = npq$

Poisson Distribution :

Random variable .

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}, \quad x = 0, 1, \ldots n, \lambda = 0$$

MGF, mean, varience of poisson's distribution

sudden occurence .

<u>MGF</u>:

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \cdot \frac{(e^{-\lambda} \cdot \lambda^{x})}{x!}$$

$$= \sum_{x=0}^{n} \frac{e^{-\lambda} \cdot (\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda} \left[\sum_{x=0}^{n} \frac{(\lambda e^{t})^{x}}{x!} \right]$$

$$= e^{-\lambda} \left[\frac{(\lambda e^{t})^{0}}{0!} + \frac{(\lambda e^{t})^{1}}{1!} + \cdots \right]$$

$$= e^{-\lambda} \left[1 + \frac{(\lambda e^{t})}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \cdots \right]$$

$$= e^{-\lambda} \cdot e^{\lambda e^{t}}$$

Note : $e^{\theta} = \left[1 + \frac{\theta}{1!} + \frac{\theta^{2}}{2!} + \cdots \right]$

<u>Mean</u>:

$$\text{Mean} = E(x) = \frac{d}{dt}\Big[M_x(t)\Big]_{t=0}$$

$$= \frac{d}{dt}\Big[e^{\lambda[e^t-1]}\Big]$$

$$= e^{\lambda[e^t-1]} \times \lambda e^t\Big]_{t=0} \rightarrow ①$$

$$= e^{\lambda[1-1]} \times \lambda(1)$$

$$= \lambda$$

diff ① wrt t

$$\frac{d}{dt}\, e^{\lambda[e^t-1]} \cdot \lambda e^t$$

$$= e^{\lambda[e^t-1]} \cdot (\lambda e^t)^2 + e^{\lambda[e^t-1]} \cdot \lambda e^t\Big]_{t=0}$$

$$= 1\cdot(\lambda)^2 + \lambda$$

$$= \lambda(\lambda+1)$$

$$\text{variance} = \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda$$

$$\cancel{E(x^2) = \frac{d^2}{dt^2}\Big[M_x(t)\Big]_{t=0}}$$

$$\cancel{= \frac{d}{dt}\Big[\lambda e^{-\lambda} \cdot e^{\lambda e^t} e^t\Big]}$$

Problems under binomial distribution.

Q:1 For a binomial distribution mean is 6 standard deviation $\sqrt{2}$. Find first 2 terms of the deviation, and the PMF

Sol:

$np = 6 \qquad p = \frac{6}{n}$

$\sqrt{npq} = \sqrt{2}$

$npq = 2$

$n\left(\frac{6}{n}\right)q = 2$

$6q = 2$

$\boxed{q = \frac{1}{3}} \quad \boxed{q = \frac{1}{3}} \quad \boxed{q = 1/3}$

$n\frac{1}{3} = 6 \qquad p + \frac{1}{3} = 1 \qquad np = 6 \quad \boxed{np = 6}$

$\frac{n}{3} = 6 \qquad \boxed{p = \frac{2}{3}} \qquad n\left(\frac{2}{3}\right) = 6$

$n =$

$n = \frac{6 \times 3}{2}$

$\boxed{n = 9}$

$Pmf = P(X=x) = \sum_{x=0}^{\infty} n C_x p^x q^{n-x}$

Put $x=0$

$9C_0 \left(\frac{2}{3}\right)^0 \cdot \left(\frac{1}{3}\right)^9$

$= 9C_0 (1) \left(\frac{1}{3}\right)^9$

Put $x=1$

$9C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^8 = \frac{18}{3^9}$

If X is a binomial with n = 6 random variable and P(x)

$$9P(x=4) = P(x=2). \text{ find "p"}$$

we know that

B.D

Pmf $nC_x p^x q^{n-x}$

$$9\left[6C_4 (P)^4 q^{6-4}\right] = 6C_2 p^2 q^{6-2}$$

$$9\left[\frac{6\times5\times4\times3}{1\times2\times3\times4} \cdot p^{4-2} q^2\right] = \frac{6\times5}{1\times2} p^2 \cdot q^{4-2}$$

$$= 9p^2 = q^2$$

$$\boxed{3p = q}$$

$$p+q = 1$$

$$p + 3p = 1$$

$$4p = 1$$

$$\boxed{p = 1/4}$$

If x is binomial distributed random variable with $E[x] = 2$ & variance of $x = \frac{4}{3}$. Find $P(x) = 5$.

sol:

$E[x] = 2$, Varience $x = \frac{4}{3}$

$$npq = \frac{4}{3} \qquad 2q = \frac{4}{3} \qquad q = \frac{2}{3}$$

$p+q=1$

$\cancel{q}=p=\frac{1}{3}$

$np=2$

$\boxed{n=6}$

$P(x)=5=6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^{6.5}$

$=6\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)$

$=\frac{4}{3^5}$ //

If 4 coins are tossed simultaneously what is the probability of 2 getting (i) 2 heads (ii) atleast 2 heads.

Let x be head or tail

i) $P(x)=2$

$p=\frac{1}{2}\quad q=\frac{1}{2}$

$P(x=2)$

$4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$

$=6\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)=\frac{3}{8}$

(ii) $p(x\geq 2)$

$P(x<2)=P(0)+P(1)$

$P(x=0)=4C_0p^0q^4+4C_1p^1q^3$

$=(1)(1)q^4+4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3$

$=\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^4_4=\frac{1}{16}+\frac{1}{4}=\frac{5}{16}\Rightarrow 1-\frac{5}{16}=\frac{11}{16}$ //

X: Out of 800 families with 4 children Each, how many families would be expected to have

(i) 2 boys and 2 girls

(ii) Atleast one boy

(iii) Atmost 2 girls

(iv) Children of both gender.

Assume each probabilities for. boys and girls

Total $N = 800$

$n = 4$

In this sum we are taking X is no of boys for solving.

$p = \frac{1}{2}$ $q = \frac{1}{2}$

$P(X=x) = nc_x p^x q^{n-x}$

(i) 2 boys and 2 girls.

So in this case $X = 2$ (Total 4 so 2b 2G)

$P(X=2) = 4c_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$

$= \frac{\cancel{4}\times 3}{1\times 2} \times \frac{1}{\cancel{4}} \times \frac{1}{4}$

$= \frac{3}{8}$

out of 800 2b 2g so multiply with 800. $\frac{3}{\cancel{8}} \times \cancel{800}^{100} = 300$

800 families having 2 boys and 2 girls

(ii) Atleast one boy.

$P(x \geq 1) \Rightarrow$ i.e; $\underline{1 - P(x<1) = 0}$

$x = 1, 2, 3 \cdot\cdot 4 \ldots\ldots$ so Totalprob $- P(x) = 0$

Stop territory

$$P(x \geq 1) = 1 - P(x<1)$$

$$= 1 - P(x=0)$$

$$= 1 - \underset{1 \times 1}{\underline{4C_0 (1/2)^0}} (1/2)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

Out of 800

$$800 \times \frac{15}{16} = 750.$$

(iii) Atmost 2 girls (max 2 girls)

$= P(0 \text{ Girl}) + P(1 \text{ girl}) + P(2 \text{ girl})$.

$= P(x=4) + P(x=3) + P(x=2)$

$= 4C_4 (1/2)^4 (1/2)^0 + 4C_3 (1/2)^3 (1/2)$

$+ 4C_2 (1/2)^2 (1/2)^2$

$$= \frac{11}{16}$$

Out of 800

$$800 \times \frac{11}{16} = 550.$$

iv) children of both gender.

$X=1,\ X=2,\ X=3$

$P(X=1)+P(X=2)+P(X=3)$

$= \frac{7}{8}$

Out of 800 $= \frac{7}{\cancel{8}} \times \cancel{800}^{100} = 700$

6 dice are thrown 729 times. How many times do we expect atleast 3 dice to show 5 or 6.

6dice are thrown, so, $n=6$

$N = 729$

$P(X \geq 3) \rightarrow$ atleast 3dice.

$P(X=3)+P(X=4)+P(X=5)+P(X=6)$

$p = \frac{2}{6} = \frac{1}{3}$ → getting 5 or 6

$q = \frac{2}{3}$ $\quad 1-p = 1-\frac{1}{3} = \frac{2}{3}$

$\frac{1}{6}+\frac{1}{6}$

getting 5 or getting 6

$P(X \geq 3) = 1 - P(X<3)$

$= 1 - [P(X=0)+P(X=1)+P(X=2)]$

$= 1 - [P[x=0) + P(x=1) + P(x=2)]$

$= nC_x p^x q^{n-x}.$

$P(x=0) = 6C_0 \left(\frac{1}{3}\right)^0 \cdot \left(\frac{2}{3}\right)^6.$

$= 1 \cdot 1 \cdot \frac{64}{729} = \frac{64}{729}$

$6C_1$

$nC_1 = n.$

$P(x=1) = 6C_1 \left(\frac{1}{3}\right)^1 \cdot \left(\frac{2}{3}\right)^5$

$nC_0 = 1$

$= 6 \cdot \frac{1}{3} \cdot \frac{32}{\cancel{1093}\ 729}$

$P(x=2) = 6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$

$= 15 \left(\frac{1}{9}\right)\left(\frac{16}{34}\right)]$

$= 1 - \left[\frac{64 + 192 + 240}{729}\right] = \frac{729 - 496}{729}$

$= \frac{233}{729}$

In 729 times

$\frac{\cancel{729} \times 233}{\cancel{729}} = 233$

P:1 If x is a poisson variable with $\lambda = 1.5$

Find $P(X) = 3$.

Sol:

Poisson distribution:

$$\boxed{Pmf = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}}$$

$\lambda = 1.5$; find $P(X=3)$

$$= \frac{e^{-1.5} \cdot (1.5)^{3}}{3!} \qquad e^{-1.5} = 0.2231$$

$$= \underline{0.1255}$$

P:2 If x is a poisson ~~variable~~ & if $P(x=1) = P(x=2)$

Find the value of λ

$$Pmf = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$\frac{\cancel{e^{-\lambda}}\, \lambda^{1}}{1!} = \frac{\cancel{e^{-\lambda}}\, \lambda^{2}}{2!}$$

$$\boxed{\lambda = 2}$$

P:3 The monthly breakdown of a computer is a random variable having poisson distribution with mean $1.5 (\lambda)$. Find the probab. that the computer will fn for a month

i) without a breakdown $x=0$

(ii) with only one breakdown $x=1$

(iii) atleast one breakdown

Sol:

(i) $Pmf = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$x=0 \quad \lambda=1.5$

$\frac{e^{-1.5} \cdot (1.5)^{0}}{0!}$

$= 0.223$

(ii) $pmf = \frac{e^{-1.5} \cdot (1.5)^{1}}{1!}$

$= 0.3347$

(iii) $P(x \geq 1) = 1 - P(x<1)$

$= 1 - P(x=0)$

$= 1 - \left[\frac{e^{-1.5} \cdot (1.5)^{0}}{0!}\right]$

$= 1 - 0.223$

$= 0.777$

$$\begin{array}{r} \cancel{1.000} \\ 0.223 \\ \hline 0.777 \end{array}$$

Ex: P: 4 The no of accidents in a year to taxi drivers in city follows poisson distribution. with mean = 3. out of 100 taxi drivers

Find the approximate

i) no accidents $X = 0$

ii) more than 3 accidents $X > 3$

(i) Pmf $= \dfrac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$= \dfrac{e^{-3} \cdot 3^{0}}{0!}$

$= 0.4\cdot96 \times 100 \rightarrow$ out of 100 $\Rightarrow 49.6$ drivers.

(ii) $P(X>3)$

$= 1 - P(X \leq 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - [0.049 + 0.1494 + 0.2240 + 0.2240]$

$= 1 - 0.6473$

$= 0.3528$

$\sim = 35$

Fit a poisson distribution for the following set of obs. calculate the theoritical frequencies.

x	0	1	2	3	4
f	43	38	22	9	1

$\lambda = 1$

Sol:

x	f	$xf(x)$	$N \times e^{-\lambda} \cdot \frac{\lambda^x}{x!}$
0	43	0	$\frac{113 \times e^{-1}(1)^0}{0!} = 42$
1	38	38	$\frac{113 \times e^{-1}(1)}{1!} = 42$
2	22	44	$\frac{113 \times e^{-1}(2)}{2!} = 21$
3	9	27	$\frac{113 \times e^{-1}(3)}{3!-1} = 7$
4	1	4	$\frac{113 \times e(4)}{4!} = 2$
	$N = 113$	113	

$$\lambda = \frac{\sum xf(x)}{f(x)} = \frac{113}{113} = 1$$

x	0	1	2	3	4
Observed frequency	43	38	44	27	4
Theoritical frequency	42	41	21	7	2

Geometric Distribution

A random variable X is said to follow geometric distribution if probability mass function is given by.

$$P(X=x) = \sum_{x=1}^{n} q^{x-1} p$$

MGF, & Mean, variance of geometric distribution.:

MGF:

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot \frac{q^x}{q} \cdot p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

Substitute $x=1$,

$$\frac{p}{q}\left[qe^t + (qe^t)^2 + (qe^t)^3 + \ldots\right]$$

$$= \frac{p}{q}(qe^t)\left[1 + (qe^t) + (qe^t)^2 + \ldots\right]$$

$$\text{MGF} = \frac{p \cdot qe^t}{q}\left[1 - qe^t\right]^{-1} \qquad \because \left[(1-x)^{-1} = 1 + x + x^2 + \ldots\right]$$

Mean :- $E(x)$

$$E(x) = \sum_{x=1}^{\infty} x\,p(x)$$

$$= \sum_{x=1}^{\infty} x\left[q^{x-1} \cdot p\right]$$

$$= \sum_{x=1}^{\infty} x\left[\frac{q^{x}}{q} \cdot p\right]$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} x q^{x}$$

$$= \frac{p}{q}\left[q + 2q^2 + 3q^3 + \cdots\right]$$

$$= \frac{p}{q} \cdot q\left[1 + 2q + 3q^2 + \cdots\right]$$

$$= p\left[1-q\right]^{-2} \qquad \left[(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots\right]$$

$$= p p^{-2} = \frac{1}{p}$$

Variance :

$$\text{variance} = E(x^2) - \left[E(x)\right]^2$$

$$E(x^2) = \sum_{x=1}^{\infty} x^2 p(x)$$

$$= \sum_{x=1}^{\infty} \left[x(x+1) - x\right] p(x)$$

$$= \sum_{x=1}^{\infty} \left[x(x+1)\,p(x)\right] - \sum_{x=1}^{\infty} x\,p(x)$$

$$= \sum_{x=1}^{\infty} \left(x(x+1)\right) q^{x-1} p - \sum_{x=1}^{\infty} x\, q^{x-1} p$$

$$= \frac{p}{\cancel{q}}\left[\sum_{x=1}^{\infty} x(x+1).q^{x-1}\right] - \frac{1}{p}$$

$$= p\left[1.2.q^0 + 2.3.q^1 + 3.4.q^2 + \dots\right] - \frac{1}{p}$$

$$= 2p\left[q^0 + 3q^1 + 6q^2 + \dots\right] - \frac{1}{p}$$

$$= 2p\left[1-q\right]^{-3} - \frac{1}{p}$$

$$(1-x)^{-8} = 1 + 3x + 6x^2 + \dots$$

$$= 2pp^{-3} - \frac{1}{p}$$

$$= \frac{2p}{p^3} - \frac{1}{p}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

$$\text{Variance} = E(x^2) - \left[E(x)\right]^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$$

$$= \frac{q}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

Ex: If the probability of the target to be destroyed on any one shot is 0.5 what is the probability that it could be destroyed on 6th attempt

~~$\not{p}$~~ $x = 6 \quad P = 0.5 \quad q = 0.5$

$$P(x=6) = q^{6-1} \cdot p$$
$$= (0.5)^5 \cdot (0.5)$$
$$= 0.0156.$$

~~The app~~ If the prob. that the applicant for drivers liscence to pass road test on any given trail is 0.8. what is the probability that he will finally pass the test.

i) on the 4th trail $(x=4)$

(ii) less than 4 trails $(x = 1, 2, 3)$

(i) $p = 0.8. \quad q = 0.2$

$$P(x=4) = (0.2)^{4-1} \cdot 0.8$$
$$= (0.2)^3 \cdot 0.8.$$
$$=$$

(ii) $P(x<4) = q^0 \cdot p + q^1 \cdot p + q^2 \cdot p$

$$= (0.2)^0 \cdot (0.8) + (0.2)^1 (0.8) + (0.2)^2 (0.8)$$

$$= 0.9920$$

X. State and prove memory less ~~problem~~ property of geometric distribution.

If X follows geometric distribution, for any 2 +ve int 'm' & 'n'

we are going to prove

$$P\left(X>m+n / X>m\right) = P(X>n)$$

Proof:

we know that

$$P(X=x) = \sum_{x=1}^{\infty} q^{x-1} \cdot p$$

$$P(X>n) = \sum_{x=n+1}^{\infty} q^{x-1} \cdot p$$

$$= \frac{p}{q} \sum_{x=n+1}^{\infty} q^{x}$$

$$= \sum_{x=n+1}^{\infty} \frac{q^{x}}{q} \cdot p$$

$$= \frac{p}{q} \sum_{x=n+1}^{\infty} q^{x}$$

$$= \frac{p}{q} \left[q^{n+1} + q^{n+2} + \cdots \right]$$

$$= \frac{p}{q} \cdot q^{n+1} \left[1 + q + q^{2} + \cdots \right]$$

$$= \frac{p}{\cancel{q}} \cdot q^{\cancel{n+1}} \left[1 - q \right]^{-1} = p \cdot q^{n} [p]^{-1}$$

$$= q^{n}$$

$P(X>n) = q^n$ — (1)

$$P[X>m+n / X>m] = \frac{P[X>m+n \cap X>m]}{P(X>m)}$$

$$= \frac{P[X>m+n]}{P(X>m)} = \frac{q^{m+n}}{q^m}$$

$$= \frac{q^m \cdot q^n}{q^m} = q^n = P(X>n)$$

= RHS.

= RHS.

continuous distribution

A random variable X is said to follow exponential distribution if its prob. density fn is given by

$$\lambda e^{-\lambda x}, \quad \lambda > 0, \; 0 < x < \infty$$

Moment Generating Function;

Mean, variance of exponential function:

MGF:

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda - t)} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{-\lambda}{(\lambda - t)} \left[-1 \right]$$

$$= \frac{\lambda}{\lambda - t}$$

$$\text{Mean} = E(x) = \frac{d}{dt} M_x(t) \Big]_{t=0}$$

$$= \frac{d}{dt} \left[\lambda (\lambda - t)^{-1} \right]_{t=0}$$

$$= (-\lambda)(\lambda - t)^{-2} \cdot (-1) \Big]_{t=0}$$

$$= \lambda (\lambda - t)^{-2} \Big]_{t=0}$$

$$= \lambda (\lambda)^{-2}$$

$$= \frac{1}{\lambda}$$

$$E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big]_{t=0}$$

$$= \frac{d}{dt} \left[\lambda (\lambda - t)^{-2} \right]_{t=0}$$

$$= -2\lambda(\lambda-t)^{-3}(-1)\Big]_{t=0}$$

$$= 2\lambda(\lambda-t)^{-3}\Big]_{t=0}$$

$$= \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

P:1 Suppose the duration x in mins of long distance calls from your home follows exponential distribution with PDF

$$f(x) = \begin{cases} \frac{1}{5}\cdot e^{-(x/5)}, & x>0 \\ 0 & \text{o.w} \end{cases}$$

Find i) $P(x>5)$ iii) Mean of X

ii) $P(3 \le x \le 6)$ iv) variance of X.

S: i) $P(x>5) = \int_5^\infty \lambda e^{-\lambda x}\,dx$

$$= \int_5^\infty \frac{1}{5}\cdot e^{-\frac{x}{5}}\,dx$$

$$= \frac{1}{\cancel{5}}\left[\frac{e^{-x/5}}{-1/\cancel{5}}\right]_5^\infty = -\left[e^{-\infty} - e^{-1}\right] \qquad e^{-1} =$$

$$= -[-0.3679]$$

$$= 0.3679 //$$

ii) $P(3 \le x \le 6)$

$$= \int_3^6 \lambda e^{-\lambda x} dx$$

$$= \frac{1}{5} \int_3^6 e^{-x/5} dx = \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_3^6$$

$$= -\left[e^{-6/5} - e^{-3/5} \right] = 0.2476.$$

iii) Mean of x $E(x) = \frac{1}{\lambda} = \frac{1}{1/5} = 5$

$$= \frac{1}{1/5} = 5$$

iv) variance of x

$$var(x) = \frac{1}{\lambda^2} = \frac{1}{(\frac{1}{5})^2} = 25$$

※ The time required to repair a machine is exponentially distributed with $\lambda = 1/2$ what is the probability that the required time exceeds 2hrs.

Sol: $\lambda = 1/2$

$$\cancel{\int_0^2 x e^{-\lambda x}} \quad \int_0^2$$

$$P(x \geq 2) = \int_2^{\infty} \lambda e^{-\lambda t} dt$$

$$= \frac{1}{2} \int_2^{\infty} e^{-x/2} dx$$

$$= \frac{1}{2} \left[\frac{e^{-x/4}}{1/2} \right]_2^{\infty}$$

$$= \left[e^{-\infty} - e^{-1} \right]$$

$$= 0.6321$$

The mileage which car owners get with a certain radial tyres exponentially mean 40,000 kms. To find the probability that one of the tyres will lost.

(i) atleast 20,000 km

(ii) atmost 30,000 km

$$f(x) = \lambda \cdot e^{-\lambda x}$$

$$\lambda = \frac{1}{40,000}$$

$$\text{mean} = \frac{1}{\lambda} = 40,000$$

a) $P(X > 20,000) = \int_{20,000}^{\infty} \frac{1}{40,000} e^{-\frac{x}{40,000}} dx$

$$= \frac{1}{40,000} \left[\frac{e^{-x/40,000}}{-1/40,000} \right]_{20,000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{\frac{-20,000}{40,000}} \right] = 0.6065$$

(ii) $P(0 < x < 30,000)$

$$= \int_{0}^{30,000} \frac{1}{40,000} e^{-x/40,000} dx = 0.5276$$

∴ The daily consumption of milk in excess of 20,000 gallons is approximately exponentially distributed with $\lambda = \frac{1}{3000}$. The city has daily stock of 35,000 gallons. What is the probability that two days selected at random when stock is insufficient for both the days.

X- daily stock   Y → daily consumption

$$X = Y + 20000$$

$$Y = X - 20,000$$

Y follows exponential distribution with $\lambda = \frac{1}{3000}$

$f(y) = \lambda e^{-\lambda y}$.

Probability of the stock is insufficient for ~~the~~ a day = $P(X > 35,000)$

$P(Y + 20,000 > 35,000)$

$\Rightarrow P(Y > 15,000)$

$$\Rightarrow \int_{15,000}^{\infty} \frac{1}{3000} e^{-x/3000} dx$$

$$= \frac{1}{3000} \left[\frac{e^{-x/3000}}{-1/3000} \right]_{15,000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-5} \right] = e^{-5}$$

~~= 0.0067~~

Then for two days $= e^{-5} \times e^{-5} = e^{-10}$ //

2. State and Prove memory less property of exponential distribution.

If X distributed exponentially with parameter λ, for any 2 +ve integers m and n

$$P(X > m+n / X > m) = P(X > n)$$

Proof:

$P(X > m+n)$

we know that $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$, $0 < x < \infty$

Now,

$$P(x>n) = \int_{n}^{\infty} f(x)\,dx$$

$$= \int_{n}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_{n}^{\infty} e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_{n}^{\infty}$$

$$= -\left[-e^{-\infty} - e^{-n\lambda}\right]$$

$$= -\left[0 - e^{-n\lambda}\right]$$

$$P(x>n) = e^{-n\lambda} \quad \text{--- (1)}$$

$$P(x>m+n / x>m) = \frac{P(x>m+n \cap x>m)}{P(x>m)}$$

$$= \frac{P(x>m+n)}{P(x>m)} = \frac{e^{-\lambda(m+n)}}{e^{-\lambda m}}$$

$$= \frac{e^{-\lambda m} \cdot e^{-\lambda n}}{e^{-\lambda m}} = e^{-\lambda n}$$

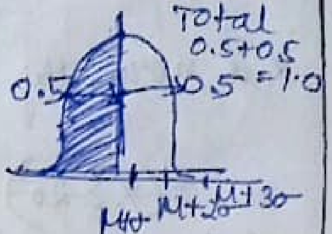
$$= P(x>n)$$

An random variable X is said to follow normal distribution if its prob. density fn is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{e^{-(x-\mu)^2}}{2\sigma^2}, -\infty < x < \infty$$

Standarised ~~mean~~

Symmetric about mean.



$$\frac{e^{-\lambda m} \cdot e^{-\lambda n}}{e^{-\lambda m}} = e^{-\lambda n}$$

$= P(x > n)$

$$\boxed{Z = \frac{x-\mu}{\sigma}}$$

Formula:

$$M_x(t) = e^{\mu + \frac{\sigma^2 t^2}{2}}$$

Mean $= E(x) = \mu$

$E(x^2) = \sigma^2 + \mu^2$

Variance $= \sigma^2$

Properties of normal distribution:

- The curve is bell shaped.
- $F(x)$ approaches 0 as x tends to $+\infty / -\infty$
- The curve is symmetric about $x = \mu$
- Mean = median = mode

- Skewness = 0
- Total area under the curve is 1

X is normally distributed with mean 12

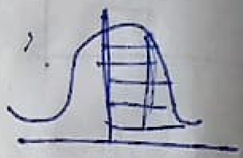
~~$P(X \geq 20)$~~, and standard is 4

find $P(X \geq 20)$

Sol:

$P(X \geq 20)$ when $x = 20$, $\Rightarrow Z = \frac{20-12}{4} = \frac{8}{4} = 2$

$P(X \geq 20) = P(Z \geq 2)$



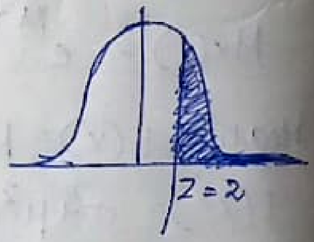
$P(X \geq 20) = P(Z \geq 2)$

$P(Z > 2)$

$= 0.5 - P(0 < Z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$ //



X is normally distributed and mean of X is 12 (μ). Standard devi is 4 (σ). Find

i) $P(X \geq 20)$

(ii) $P(X \leq 20)$

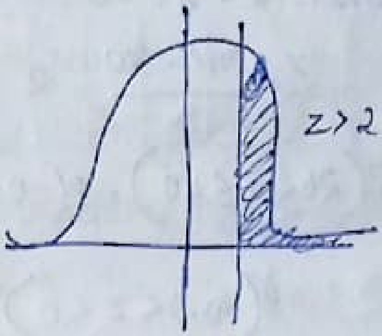
(iii) $P(6 \leq X \leq 12)$

(i) $P(x \geq 20)$

when $x=20$, $z = \frac{x-\mu}{\sigma}$

$= \frac{20-12}{4}$

$z = 2$



$P(x \geq 20) = P(z \geq 2)$

$P(x \leq 20) = P(z \leq 2)$

$P(z>2) = 0.5 - P(0<z<2) = 0.5 - 0.4772 = 0.0228$

$P(z<2) = 0.5 + P(0<z<2)$

$= 0.5 + 0.4772$

$= 0.9772$ //

$P(0 \leq x \leq 12)$

when $x=0 \Rightarrow z = \frac{0-12}{4} = -3$

when $x=12 \Rightarrow z=0$

$P(0 \leq x \leq 12) = P(-3<z<0)$

$= P(0<z<3)$

$= 0.4987$ //

If x is normally distributed with mean 30 and variance 25, compute

i) $P(26 \leq x \leq 40)$

(ii) $P(|x-30|>6)$

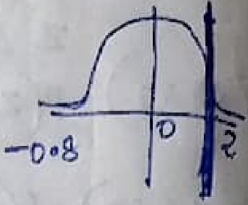
i) when $x = 26$

$z = \frac{26-30}{5 \rightarrow \sqrt{25}} = -0.800$

when $x = 40$

$$z = \frac{40-30}{5} = 2$$

$$P(26 \le x \le 40) = P(-0.8 < z < 2)$$



$$= P(-0.8 < z < 0) + P(0 < z < 2)$$

$$= 0.2881 + 0.4772 = 0.7653$$

$$P(|X-30| > 6) = 1 - P(|X-30| \le 6)$$

$$= 1 - P(-6 < X-30 < 6)$$

$$= 1 - P(30-6 < X < 30+6)$$

$$1 - P(24 < x < 36)$$

<u>when $x = 24$</u>

$$z = \frac{24-30}{5} = \frac{-6}{5} = -1.2$$

<u>when $x = 36$</u>

$$z = \frac{36-30}{5} = \frac{6}{5} = 1.2$$

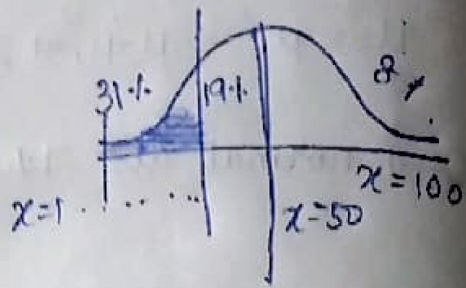
$$= P(-1.2 < z < 1.2)$$

$$= 1 - 2(0 < z < 1.2)$$

$$= 1 - 2(0.3849)$$

$$= 1 - 2(0.3849) = 0.2302$$

In normal distribution 21% of products are under 45 8% are over 64. Find mean and variance of distribution.



$$Z = \frac{45-\mu}{\sigma}$$

Given:

$P(x < 45) = 31\%$

when $x = 45$

$$Z \Rightarrow \frac{45-\mu}{\sigma}$$

$P\left(Z < \frac{45-\mu}{\sigma}\right) = 31\%$

$P(x > 64) = 8\%$

when $x = 64$ = $Z = \frac{64-\mu}{\sigma} = 8\%$

$P\left(\frac{45-\mu}{\sigma} < Z < 0\right) = 19\%$

$P\left(0 < Z < \frac{\mu-45}{\sigma}\right) = 19\%$

$= 0.19$

Search 0.19 inside the table.

$$\frac{\mu - 45}{\sigma} = 0.5$$

$\Rightarrow \mu - 45 = 0.5\sigma$

$\Rightarrow \mu - 0.5\sigma = 45$ — (1)

$$P\left(0 < z < \frac{64-\mu}{\sigma}\right) = 0.42$$

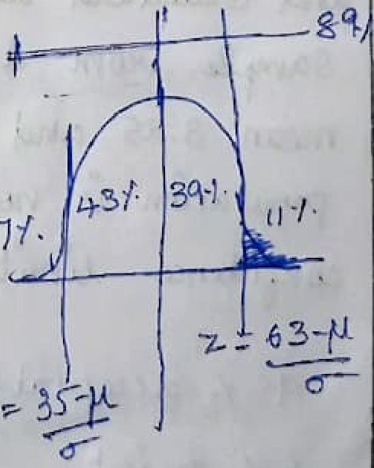
$$\Rightarrow \frac{64 \cdot \mu}{\sigma} = 1.41$$

$\Rightarrow 64 = 1.41\sigma + \mu$ — (2)

Solving (1) and (2)

$\sigma \cong 10$, $\mu = 50$

In normal distribution exactly 7% ~~of~~ & 89% under 63.
Find mean and standard deviation of the distribution



$$P\left(0 < z < \frac{\mu - 35}{\sigma}\right) = 0.43$$

$$\frac{\mu - 35}{\sigma} = 1.48 \text{ — (1)}$$

Unit-II (Uniform distribution)

A continuous random Variable X is said to follow a uniform distribution over an interval a,b if its PDF is given by $f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 & o.w \end{cases}$

$$\text{mean} = \frac{a+b}{2}, \quad \text{Variance} = \frac{(b-a)^2}{12}$$

An electric train at certain lin run every half an hr between midnight & 6 in mrng. what is the prob. that a man entering the station at a random time during this period will have to wait atleast 20 mins

Sol:

$X \rightarrow$ waiting time in minutes

$(a,b) = (0, 30) \rightarrow$ every half an hr

$$f(x) = \frac{1}{b-a} = \frac{1}{30}$$

$$P(X \geq 20) = \int_{20}^{30} f(x)\,dx$$

$$= \int_{20}^{30} \frac{1}{30}\, dx$$

$$= \frac{1}{30} \left[x \right]_{20}^{30}$$

$$= \frac{30-20}{30} = \frac{10}{30} = \frac{1}{3}$$

Buses arrive at a specific bustop at 15 min interval at 7am if a passenger arrives at a stop at a random time that is uniformly distributed btn 7 to 7.30 am! Find the probability that he waits..

i) less than 5 mins for a bus

ii) atleast 12 mins for a bus

Sol:

$X \rightarrow$ time.

$(a, b) = (0, 30)$

$$f(x) = \frac{1}{b-a} = \frac{1}{30}$$

(i) $P(X < 5 \text{ minutes})$

= 7.10 to 7.15 + 7.25 to 7.30

$$= \int_{10}^{15} f(x)\, dx + \int_{25}^{30} f(x)\, dx$$

$$= \frac{1}{30}\left[(15-10) + (30-25)\right]$$

$$= \frac{1}{30}\left[5+5\right] = \frac{10}{30} = \frac{1}{3}$$

ii) $p(x \geq 12)$ = between he arrives 7 to 7.03

+ between 7.15 to 7.18

$$= \int_0^3 f(x)dx + \int_{15}^{18} f(x)dx$$

$$= \frac{1}{5}.$$

If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$. Find $P(x<0)$

$$\frac{a+b}{2} = 1$$

$\Rightarrow a+b = 2$ —— (1)

$$\frac{(b-a)^2}{12} = \frac{4}{3}$$

$$(b-a)^2 = \frac{4 \times \cancel{12}^{4}}{\cancel{3}} = 16$$

$b-a = 4$ —— (2)

From (1) and (2)

$a = -1, b = 3$

Now $P(x<0) = \int_{-1}^{0} f(x)dx = \int_{-1}^{0} \frac{1}{3+1} dx$

$$= \frac{1}{4}[x]_{-1}^{0}$$

$$= \frac{1}{4} \times 1 = \frac{1}{4}$$