Solution of one-dimensional wave equation

The one dimensional wave equation is $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \rightarrow (*)$

This can be solved by variable seperable method.

Assume the solution to be $y = X(x)\,T(t)$

$y = XT \rightarrow (1)$

X - function of x alone
T - function of t alone

Diff. (1) Par. w.r.to x, $\frac{\partial y}{\partial x} = X'T$

Diff again P. w.r.to x, $\frac{\partial^2 u}{\partial x^2} = X''T \rightarrow (2)$

Diff (1) par. w.r.to t, twice $\frac{\partial^2 y}{\partial t^2} = XT'' \rightarrow (3)$

Sub. (2) & (3) in $(*)$ $X''T = \frac{1}{c^2} XT''$

Dividing by XT on both sides of the above eqn

$$\frac{X''T}{XT} = \frac{1}{c^2}\frac{XT''}{XT}$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = K \text{ (Constant)}$$

The constant K can be positive, Negative or zero (ie $K>0$, $K<0$, $K=0$)

<u>case (i)</u> <u>Let $K>0$</u> (be positive)

$K = p^2$ (when $p>0$ or $p<0$, K is +ve)

$$\therefore \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = p^2 \Rightarrow \frac{X''}{X} = p^2 \quad \& \quad \frac{1}{c^2}\frac{T''}{T} = p^2$$

when $\frac{X''}{X} = p^2 \Rightarrow X'' = p^2 X$

$\Rightarrow D^2X = p^2X$

$\Rightarrow (D^2 - p^2)X = 0$

$\therefore$ AE is $m^2 - p^2 = 0$

$\Rightarrow m^2 = p^2$

$\Rightarrow m = \pm p$. $\therefore$ Sol is $Ae^{px} + Be^{-px}$ (or) $X = Ae^{px} + Be^{-px}$

When $\frac{1}{c^2}\frac{T''}{T} = p^2$

$\Rightarrow T'' = p^2c^2T$

$\Rightarrow D^2T = p^2c^2T$

$\Rightarrow (D^2 - p^2c^2)T$

AE is $m^2 - p^2c^2 = 0$

$\Rightarrow m^2 = p^2c^2$

$\Rightarrow m = \pm pc$

$\therefore$ Sol. is $T = Ce^{pct} + De^{-pct}$

$\therefore$ The solution is

$y = XT$

$= (Ae^{px} + Be^{-px})(Ce^{pct} + De^{-pct})$

Case (ii) $K < 0$ (be negative)

$K = -p^2$ (p = +ve or -ve, K is negative)

$\therefore \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -p^2 \Rightarrow \frac{X''}{X} = -p^2, \quad \frac{1}{c^2}\frac{T''}{T} = -p^2$

$\frac{X''}{X} = -p^2$

$X'' = -Xp^2$

$\Rightarrow D^2X = -Xp^2$

$\Rightarrow D^2X + p^2X = 0$

$\Rightarrow (D^2+p^2)X = 0$

$\therefore$ AE is $m^2+p^2 = 0$

$\Rightarrow m^2 = -p^2$

$\Rightarrow m = \pm pi$

$\therefore$ Sol: is

$X = A\cos px + B\sin px$

$\frac{1}{c^2}\frac{T''}{T} = -p^2$

$T'' = -p^2c^2T$

$D^2T = -p^2c^2T$

$D^2T + p^2c^2T = 0$

$\Rightarrow (D^2+p^2c^2)T = 0$

AE $m^2+p^2c^2 = 0$

$m^2 = -p^2c^2$

$m = \pm pci$

$m = \pm ipc$

$\therefore$ Sol. is

$T = C\cos pct + D\sin pct$.

$\therefore$ The sol. is $y = XT$

$= (A\cos px + B\sin px)(C\cos pct + D\sin pct)$

Case (iii) $K=0$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = 0$$

$\Rightarrow \frac{X''}{X} = 0$ $\qquad \frac{1}{c^2}\frac{T''}{T} = 0$

$\Rightarrow X'' = 0$ $\qquad \Rightarrow T'' = 0$

$\Rightarrow D^2 X = 0$ $\qquad \Rightarrow D^2 T = 0$

AE $m^2 = 0$ $\qquad$ AE $\Rightarrow m^2 = 0.$

$m = 0, 0$ $\qquad \Rightarrow m = 0, 0$

Sol:

$X = Ax + B.$ $\qquad T = ct + D.$

$\therefore$ The sol. is

$y = XT$

$= (Ax+B)(ct+D)$ //

Hence we have 3 solutions for one dimensional wave equation

(1) $y = (Ae^{px} + Be^{-px})(Ce^{pct} + De^{-pct})$

(2) $y = (A\cos px + B\sin px)(C\cos pct + D\sin pct)$ $\rightarrow$ we will be using this as it is suitable for our problems.

(3) $y = (Ax+B)(ct+D)$