Unit IV

Queueing Theory

Queueing Theory Objective:

The objective is to have good economic balance b/w waiting time cost and service cost

It is used to find optimum Solution for minimising the waiting time & Service Cost.

General Queuing Model:

Arrival → Queue → Service → Departure

Queue descipline:

* FIFO (FCFS) (First in first out)
* LIFO (LCFS)
* Service in Random Order
* priority

Customer Behaviour:

Balking:- Customers has no intention for join Queue

Raneging:- Customers may leave the Queue due to impatiance

Jockeying:- Customers jump from one queue to other queue.

Arriving / Arrival rate of Customers:

For poisson distribution, arrival rate = λ (no of customers)

For exponential distribution, arrival rate = $\frac{1}{\lambda}$ (one customer during an interval)

Service Cost:

For poisson, Service rate = μ (no of customers service)

For exponential, service rate = $\frac{1}{\mu}$ (No of customers Served in an interval) instant

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For poisson distibution, arrival rate $= \lambda$ (no of customers)

For exponential distribution, arrival rate $= \frac{1}{\lambda}$ (one customer during an interval)

Service Cost:

For poisson, service rate $= \mu$ (no of customers service)

For exponential, service rate $= \frac{1}{\mu}$ (No of customers served in an interval instant)

System Capacity: Maximum no of customers accomodated for service.

Kendal's Notation for a queuing System:

Notation: $(a/b/c):(d/e)$

a – no. of arrivals per unit time

b – no of services per unit time

c – No of ~~services~~ server in the system

d – capacity of the system

e – queueing discipline

Type I: $(M/M/1):(\infty/FIFO)$

$P_0 \rightarrow$ the system is Idle

$P_1 \rightarrow$ the system is busy

$n \rightarrow$ no. of customers

$N \rightarrow$ Queuing capacity (avg no of customers).

$L_s \rightarrow$ Average Queue length in System

$L_q \rightarrow$ Average Queue length

$W_s \rightarrow$ waiting time in system

$W_q \rightarrow$ waiting time in queue

System $\rightarrow$ queue + service

(13)

Little's Formulae:

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$W_s = \frac{L_s}{\lambda}$$

$$W_q = W_s - \frac{1}{\mu}$$

Type I Formulae

1) $P = \frac{\lambda}{\mu}$ (Traffic Intensity)
(row)

2) $L_s = \frac{\lambda}{\mu - \lambda}$

3) $L_q = \lambda^2 / \mu(\mu - \lambda)$

4) $W_s = \frac{L_s}{\lambda} = \frac{1}{\mu - \lambda}$

5) $W_q = W_s - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)}$

6) P (the system is busy)

$= P_1 = \frac{\lambda}{\mu}$

7) P (the system is Idle) $= P_0 = 1 - P_1$

$P_0 = 1 - \frac{\lambda}{\mu}$

8) $P[W_s > t] = e^{-(\mu - \lambda)t}$

9) $P[W_q > t] = \frac{\lambda}{\mu} e^{-(\mu - \lambda)t}$

10) $P[N > k] = \left(\frac{\lambda}{\mu}\right)^{k+1}$

11) $P[N \geq k] = \left(\frac{\lambda}{\mu}\right)^{K}$

Problems On Type I $(M/M/1) : (\infty/FIFO)$ → (Single Server, infinite capacity)

Q) A departmental store has a single cashier during the rush hrs. The avg no of Customers customers arrive at the rate of 20 per hr. The avg no of Customers than can be process by the Cashier is 24 per hr.

(i) What does the prob that cashier is idle

(ii) What is the avg no of customer in the queuing System

(iii) What does the avg time the customer spend in the System

(iv) What is the avg. no of customers in a queue

(v) Avg time spends by customer in queue ~~(exp)~~ waiting for Server

(vi) What is the probability that waiting time in system exceeds 30hrs.

Sol:

$\lambda = 20/hr. \rightarrow$ Arrival rate

$\mu = 24/hr \rightarrow$ Server rate.

Traffic Intensity $\rho = \frac{20}{24} = 0.83$

(i) $P_0 = 1 - P_1$

$= 1 - \frac{\lambda}{\mu}$

$= 1 - 0.83$

$= 0.17$

(ii) System = Queue + Service

Avg customer $L_s = \frac{\lambda}{\mu - \lambda}$

$= \frac{20}{4}$

$L_s = 5$

Avg queue length $L_q = L_s - \frac{\lambda}{\mu}$

$= 5 - 0.83$

$= 4.17 \simeq 4$

(iii) $W_s = \frac{L_s}{\lambda} = \frac{5}{20} = \frac{1}{4} = 0.25$

(iv) $L_s = \frac{\lambda}{\mu-\lambda}$

$= 5$

$L_q = 4$

(v) $W_q = W_s - \frac{1}{\mu}$

$= 0.25 - \frac{1}{24}$

$= 0.25 - 0.04$

$= 0.208$ (or) $5/24$ hrs

(vi) $P(W_s > 30)$

$P(W_s > t) = e^{-(\mu-\lambda)t}$

$= e^{-(24-20)t}$

$= e^{-4t}$

$= e^{-4(30)}$

$= e^{-120}$

Q) Arrivals at telephone booth are considered to be poisson with the avg time 12 min b/w the arrivals and length of the phone call is consumed to be distributed exponentially with a mean 4min.

(i) Find Avg no of persons waiting in a system

(ii) What is the prob that a person arriving at a booth will have to wait in the queue.

(iii) What is the prob that if he takes more than 12min all together wait for the phone and complete this call

(iv) Estimate the fraction of day with the phone is been use

(v) The telephone department will instal a second booth when convince that arrival has to wait on the avg atleast 3min per phone

By how much flow of arrival increase in order to justify a 2nd booth

Sol: $\lambda = \frac{1}{12}$, $\mu = \frac{1}{4}$

Mean of Exponential $= \frac{1}{\mu}$

Traffic intensity, $\rho = \frac{\lambda}{\mu}$

$= \frac{1}{12} \times 4$

$= \frac{1}{3}$

$= 0.33$

(i) L_s (Avg no of customers)

$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/12}{0.25 - 0.08}$

$=$

(ii) P(system is busy)

$P_1 = \rho = 0.33$ (row)

(iii) $P(W_s > 10) = e^{-(\mu-\lambda)t}$

$= e^{-(1/4 - 1/12)10}$

$= e^{-(2/12)10}$

$= e^{-5/3}$

$= e^{-1.67}$

(iv) phone is use $\rightarrow$ system is busy

$P_1 = \rho = \frac{\lambda}{\mu} = 0.33$

(v) $W_q > 3$

$$\frac{\lambda}{\mu(\mu-\lambda)} > 3$$

$$\frac{\lambda}{\frac{1}{4}(\frac{1}{4}-\lambda)} > 3$$

$$4\lambda > 3(\tfrac{1}{4}-\lambda)$$

$$4\lambda > \tfrac{3}{4} - 3\lambda$$

$$7\lambda > \tfrac{3}{4}$$

$$\lambda > \tfrac{3}{28}$$

Diff b/w $\frac{3}{28} - \frac{1}{12}$

$$= \frac{3-2}{42}$$

$$= \frac{1}{42}$$

The flow of arrival when the customers atleast 3 min $= \frac{1}{42}$ //

(Q) Customers arrive at single terminal computer centre to check their mail according to a poisson process the mean interval arrival time of 20 min. The customers spend avg of 15 min in the computer. If an hr is used as unit of time

(i) What is the probability that the customer need not to wait to check the mail

(ii) what is the expected number of customers and in queue

(iii) How much time can a customer expect to spend in ~~queue~~ centre

(iv) find the avg time the customer spend in queue

(v) find the prob that centre is idle

sol: $\lambda = 1/20$, $\mu = 1/15$

(i) $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{1}{20} \times 15 = 1 - \frac{3}{4} = 0.\cancel{7}25$

(ii) $L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/20}{1/15 - 1/20} = \frac{1/20}{\frac{\cancel{60}\,4-3}{60}}$

$= 1/20 \times 60$

$= 3$

$\boxed{L_s = 3}$

$L_q = L_s - \frac{\lambda}{\mu} = 3 - 0.75$

$= 2.25 \simeq 2$ $\boxed{L_q = 2}$

(iii) $W_s = \frac{L_s}{\lambda} = \frac{3}{1/20} = 60\,\text{min} = 1\,\text{hr}$ $\boxed{W_s = 1\,\text{hr}}$

(iv) $W_q = W_s - \frac{1}{\mu}$

$= 60 - \frac{1}{1/15}$

$= \cancel{60} - \cancel{0.064}\ 60 - 15$

$= \cancel{59.93}\ 45 = \frac{45}{60} = \frac{3}{4}\,\text{hrs}$ $\boxed{W_q = 3/4\,\text{hrs}}$

(v) $P_0 = 1 - \frac{\lambda}{\mu}$

$= 0.25$

$\therefore P_0 = 0.25$

$L_s = 3$

$L_q = 2$

$W_s = 1\,hr$

$W_q = 3/4\,hrs$

(Q) Customers arrive at 1 man barber shop according to poisson's process with the mean

customers spend an avg of 10 ms in the barber chair.

(i) What is the ~~accpected~~ expected number of customers in barber shop and in queue. ?

(ii) Calculate the % of time an arrival can walk straightly to barber chair ~~in the~~ without having to wait

(iii) How much time can a customer ~~accept~~ expected to spend in barber shop. ?

(iv) Managment will provide other chair and ~~heir~~ hire other barber when the customers waiting time exceeds 1.25hr. How much

(v) what is the avg time customers spend in queue ?

(vi) what is the probability that waiting time exceeds t.

(vii) calculate the % of customers

(viii) What is the probability that more than 3 customers in system.

Sol $\lambda = \frac{1}{12}$, $\mu = \frac{1}{10}$

Traffic intensity $\rho = \frac{\lambda}{\mu} = 0.83$

(i) $L_s = \frac{\lambda}{\mu - \lambda} = 5$

$L_q = L_s - \frac{\lambda}{\mu} = 5 - 0.83$

$= 4.17 \simeq 4$

(ii) P(customer need not wait) = P(system is idle)

$\Rightarrow P_0 = 1 - P_1$

$= 1 - 0.83$

$= 0.17$

percentage $= 0.17 \times 100$

$= 17\%$

(iii) $W_s = \frac{L_s}{\lambda} = \frac{5}{1/12} = 60 \text{ min} = 1 \text{hr}$

(iv) $W_s > 1.25 \text{ hr}$

$W_s > (1.25)(60)$

$W_s > 75 \text{ min}$

$\frac{1}{\mu - \lambda_s} > 75$

$\frac{1}{\frac{1}{10} - \lambda_r} > 75$

$\frac{10}{1 - 10\lambda_r} > 75 \Rightarrow$

$-750\lambda_r < 10 - 75$

$-750\lambda_r < -65$

$\lambda_r > \frac{65}{750}$

$10 > 75 - 750\lambda_r$

$\frac{65}{750} > \lambda_r \Rightarrow \lambda_r < \frac{-65}{750} = \frac{-13}{150}$

(v) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = W_s - \frac{1}{\mu} = 60 - \frac{1}{1/10} = 50$ min

(vi) $P(W_s > t) = e^{-(\mu-\lambda)t}$

$= e^{-(1/10 - 1/12)t}$

$P(W_s > 60) = e^{-(0.3)}$

$= 0.6065$

(vii) P(customer to wait) = P(system is busy)

$P_1 = 1 - P_0$

$= 1 - 0.17$

$= 0.83$

percentange = 83%

(viii) $P(N > k) = (\lambda/\mu)^{k+1}$

$P(N > 3) = (0.83)^{3+1}$

$= (0.83)^4$

$= 0.4823$ //

(Q) In a city airport flights arrive at the rate of 24 per day. It is known that inter arrival time follows an Exponential ~~time~~ distribution and service type also follow ED. with an avg of 30 mins.

(i) what is the prob. that airport is idle

(ii) find the avg no of flights in queue

(iii) find prob that queue size exceeds 7

(iv) If the input of flight increases to an avg 30 time per day.

What will be the changes of question 1, 2, 3.

Sol $\lambda = \frac{24}{24} = 1$ per hr $\mu = 1/30$ per min $= 1/30 \times 60 = 2$ per hr

$\rho = \frac{1}{1/30 \; 2} = \frac{1}{2} = 0.5$ (Traffic Intensity)

(I) P(system is Idle) = P_0

$P_0 = 1 - P_1$

$= 1 - 0.5$

$= 0.5$

(II) $L_q = \frac{\lambda}{(\lambda - \mu)} = \frac{1}{2-1} = 1$

$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = L_s - \frac{\lambda}{\mu}$

$= 1 - 0.5$

$= 0.5$

(III) $P(N \geq 7) = \left(\frac{\lambda}{\mu}\right)^{k+1} = (0.5)^8 = 3.90 \times 10^{-3}$

(IV) $\lambda = \frac{30}{24} = 1.25$ per hr $\mu = \frac{1}{30}$ per min

$\lambda = 1.25$ $\mu = \frac{1}{30} \times 60 = 2$ per hr

$\rho = \frac{\lambda}{\mu} = 0.675$ hr

(i) $P_0 = 1 - 0.675$

$= 0.325$

(II) $L_q = 1.041 \simeq 1$

(III) $P(N \geq 7) = (0.675)^8$

$= 0.0232$ //

Type II : (M/M/1) : (k/FIFO) (single server, finite capacity)

1. $P_1 = \frac{\lambda}{\mu} P_0$ (system busy)

2. Traffic Intensity $\rho = \frac{\lambda}{\mu}$

$$P_1 = \rho P_0$$

3. $$P_0 = \begin{cases} \frac{1-\rho}{1-\rho^{k+1}} & , \lambda \neq \mu \\ \frac{1}{k+1} & , \lambda = \mu \end{cases}$$

4. P_0 = the customer need not wait

$$P_n = \begin{cases} \rho^n P_0 & , \lambda \neq \mu \\ \frac{1}{k+1} & , \lambda = \mu \end{cases}$$

5. $$L_s = \begin{cases} \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} & , \lambda \neq \mu \\ k/2 & , \lambda = \mu \end{cases}$$

6. P(customer turned away) $= P_k = \rho^k P_0$

Little's formulae:

1. Effective arrival state $\lambda' = \mu(1-P_0)$

2. $L_q = L_s - \frac{\lambda'}{\mu}$

3. $W_s = \frac{L_s}{\lambda'}$

4. $W_q = W_s - \frac{1}{\mu} = \frac{L_q}{\lambda'}$

(Q) The local one person barber shop can accomodate a maximum of 5 people at a time. Customer arrives according to poisson distribution with mean 5p/hr. The barber cuts hair at average rate of 4 per hr.

(1) What is the % of time is barber is idle.

(2) what fraction of day the potential customers turned away.

(3) What is the expected no. of customers waiting for haircut

(4) How much time can a customer expected to spend in barber shop.

Sol: The given prblm is type II

$(M|M|1):(k|FIFO)$

$k=5$, $\lambda \neq \mu$

$\lambda = 5$ $\quad \mu = 4$

$$\rho = \frac{\lambda}{\mu} = \frac{5}{4} = 1.25$$

(1) $$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{1-1.25}{1-(1.25)^6} = \frac{-0.25}{1-(1.25)^6} = \frac{-0.25}{-2.81}$$

$$= 0.088$$

percentage $= \cancel{88}\ \% = 8.8\%$

(II) P(customers turned away)

$$P_5 = \rho^5 P_0$$

$$= (1.25)^5 (0.088)$$

$$= (3.05)(0.088)$$

$$= 0.2685$$

percentage $=$ 27 %.

(iii) $Lq = Ls - \frac{\lambda'}{\mu}$

$$Ls = \frac{\rho}{1-\rho} - \frac{(k+1)\,\rho^{k+1}}{1-\rho^{k+1}}$$

$$= \frac{1.25}{1-1.25} - \frac{6(1.25)^6}{1-(1.25)^6}$$

$$= 3.13$$

$$\lambda' = \mu(1-P_0)$$

$$= 4(1-1.25)$$

$$= 4(1-0.088)$$

$$= 4(0.912)$$

$$= 3.65$$

$$Lq = 3.13 - \frac{3.65}{4}$$

$$= 3.13 - 0.912$$

$$= 2.218 \simeq 2 \text{ per hr}$$

(iv) $W_s = \frac{Ls}{\lambda'}$

$$= \frac{3.13}{3.65}$$

$$= 0.857 \text{ hr}$$

$$= 51.5 \text{ mints.}$$

(Q) The patients arrived at clinic with 1 consultant according to poisson distribution 30 per hr. The waiting queue does not accomodate more than 14 patients. Examination time for paitent is exponential with mean rate 20 per her

(1) find Effective Arrival rate

(2) what is the probability that at arriving patients will not wait

(3) How much time can a patient expect to spend in centre

(4) Find the average time the patient spent in waiting room

(5) What is the expected waiting time until the patient is

discharge from the clinic

(6) what is the probability that patient has to wait

sol: The given problem is of (M|M|1):(k|FIFO)

$\lambda = 30 \quad \mu = 20 \quad k = 14+1 = 15 \quad , \lambda \neq \mu$

(i) $\rho = \frac{\lambda}{\mu} = \frac{30}{20} = 1.5$

Effective arrival rate $\lambda' = \mu(1 - P_0)$

$P_0 = \frac{1-\rho}{1-\rho^{k+1}}$ $\quad$ $\cancel{P_0 + \lambda + P_k}$

$\Rightarrow \frac{1-1.5}{1-(1.5)^{16}} = \frac{-0.5}{1-(1.5)^{16}} = \frac{-0.5}{-655.84} = 0.00078$

$\lambda' = 20(1 - 0.00078) = 19.9844 \simeq 20$ per hr

(ii) P(customer need not wait)

$P_0 = 0.00078$

(iii) $W_s = \frac{L_s}{\lambda'}$

$L_s = \frac{\rho}{1-\rho} \cdot \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$

$= \frac{1.5}{1-1.5} \cdot \frac{(16)(1.5)^{16}}{1-(1.5)^{16}}$

$\Rightarrow \frac{1.5}{-0.5} \cdot \frac{(16)(656.84)}{1-656.84} = \frac{1.5}{-0.5} \cdot \frac{10509.45}{1-656.84}$

$= 13$

$$W_s = \frac{13}{20} = 0.65 \text{ hrs}$$

(IV) $W_q = W_s - \frac{1}{\mu} = 0.65 - 0.05$

$= 0.6$

(V) $W_s = 0.65 hr$

(VI) P (patient has to wait) $= 1 - P_0$

$= 1 - 0.00078$

$= 0.99922$ //

(Q) At railway station only 1 train is ~~ad~~ handled at a time the railway yard is sufficient only for 2 trains to wait while other is given signal to leave the station. Trains arrives at a station at a avg rate of 6 per hr and the railway station can handle them on the avg of 6 per hr. assume poisson arrivals and exponential distribution Find
(1) Find the probability that no of trains in the system
(2) Find avg waiting time of new train coming into the yard
3. If the handling wait is double how will the above results got modified.

$\lambda = 6$ per hr $\quad \mu = 6/hr \quad k = 2+1 = 3$

$\boxed{\lambda = \mu}$

$$\rho = \frac{\lambda}{\mu} = 1$$

(i) $P_n = \frac{1}{K+1} = \frac{1}{4} = 0.25$

(II) $W_s = \frac{L_s}{\lambda'}$ $\qquad \lambda' = \mu(1 - P_0)$ $\qquad P_0 = \frac{1}{K+1}$

$= \frac{1.5}{4.5} = 1/3 hr$ $\qquad = 6 \times \frac{3}{4}$ $\qquad = \frac{1}{4}$

$= 20 mints$ $\qquad = 4.5/hr.$

(iii) $\mu = 12/hr$, $\lambda = 6$ per hr $k = 3$

$\boxed{\lambda \neq \mu}$

(i) $P_n = \rho^n P_0$ $\rho = \frac{6}{12} = \frac{1}{2} = 0.5/hr$

$= (0.5)^n (P_0)$ $\lambda' = \mu(1-P_0)$

$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = 8/15$ $= 12(1-8/15)$

$= 12(7/15)$

$= 84/15$

P (no of trains) $= P_n$ $= 5.6/hr$

$P_n = \rho^n P_0$

$= (0.5)^n \frac{8}{15}$

(ii) $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} = \frac{11}{15} = 0.73$

$W_s = \frac{L_s}{\lambda'}$

$\Rightarrow \frac{0.73}{5.6} = \frac{11}{84}$ hrs $= 7.9$ mins //

Arithematic Progression:

$a, a+d, a+2d, a+3d$ are said to be in A.P

last term $= t_n =$ n^{th} term

→ Sum of n terms $= \frac{n}{2}[2a+(n-1)d]$

→ sum of n terms $= \frac{n}{2}[a+l]$ where l is last term.

Geometric progression:

a, ar^2, ar^3. are said to be in G.P in which

first term $= a$

common ratio $= r$

→ n^{th} term $= a.r^{n-1}$

→ sum of n terms $= \frac{a(1-r^n)}{1-r}$, where $r<1$

$\frac{a(r^n-1)}{r-1}$, $r>1$

* How many natural numbers b/w 78 and 80 are divisible by 6.

$a+(n-1)d = 78.$

$\cancel{a=7}, \cancel{l=80}$

$12+(11)6 = 78$

$= 12+66$

$= 78$

Arithematic Progression:

$a, a+d, a+2d, a+3d$ are said to be in A.P

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→ Sum of n terms $= \frac{n}{2}[a+l]$ where l is last term.

Geometric progression:

a, ar^2, ar^3. are said to be in G.P in which

first term $= a$

common ratio $= r$

→ n^{th} term $= a \cdot r^{n-1}$

→ sum of n terms $= \frac{a(1-r^n)}{1-r}$, where $r<1$

$\frac{a(r^n-1)}{}$, $r>1$

* How many natural numbers b/w 7 and 80 are divisible by 6.

$a+(n-1)d = 78$.

$\cancel{a=7}$, $\cancel{l=80}$

$12+(11)6 = 78$

$= 12+66$

$= 78$