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UNIT - 2

FOURIER SERIES

PERIODIC FUNCTION:

A function $f(x)$ is said to be periodic if and only if $f(x+p) = f(x)$ is true for some values of t and every value of x. The smallest value of P is called periodic of that function.

Eg 1) $f(x) = \sin x$

$f(x+2\pi) = \sin(2\pi + x)$

$= \sin x = f(x)$

2) $f(x) = \cos x$

$f(x+2\pi) = \cos(2\pi + x) = \cos x = f(x)$

3) $f(x) = \tan x$

$f(x+\pi) = \tan(\pi + x) = \tan x = f(x)$

CONTINUOUS FUNCTION:

A function $f(x)$ is said to be continuous at $x = a$, if ~~limit~~ $\lim_{x \to a} f(x) = f(a)$

NOTE:

$f(x)$ is continuous in an interval (a, b), if it is continuous at every point of that interval.

DISCONTINUOUS FUNCTION:

A function $f(x)$ is said to be discontinuous function if it is not continuous.

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right)=\frac{3\pi^2}{12}$$

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right)=\frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8} \quad - (4)$$

Parseval's Identity

Let $f(x)$ be a periodic function in the interval (a, b) then the parseval's identity is

$$\frac{1}{b-a}\int_a^b [f(x)]^2\,dx = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

In interval $(-\pi, \pi)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Root Mean Square value (RMS)

Let $f(x)$ be a function defined in an interval (a, b) then

$\sqrt{\left(\frac{\int_a^b [f(x)]^2\,dx}{b-a}\right)}$ is called RMS & its denoted by $\bar{y}$

$$\bar{y}=\sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

$$\bar{y}^2=\frac{1}{b-a}\int_a^b [f(x)]^2\,dx$$

$$\bar{y}^2=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Half range sine series

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Half range cosine series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

Q) Find the RMS value of $f(x) = l-x$ in $(0, l)$

$$\bar{y}=\sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

$$\bar{y}^2=\frac{1}{l}\int_0^l (l-x)^2\,dx$$

$$=\frac{1}{l}\int_0^l (l^2+x^2-2lx)\,dx$$

$$=\frac{1}{l}\left[l^2x+\frac{x^3}{3}-\frac{2lx^2}{2}\right]_0^l$$

$$=\frac{1}{l}\left[\cancel{l^3}+\frac{l^3}{3}-\cancel{l^3}\right]$$

$$=\frac{l^3}{3l}=\frac{l^2}{3}$$

$$\bar{y}^2=l^2/3$$

$$\bar{y}=l/\sqrt{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi^2 - 2\pi x + x^2) \cos nx \, dx$$

Let $u = \pi^2 - 2\pi x + x^2$ $\quad$ $v = \cos nx$

$u' = -2\pi + 2x$ $\quad$ $v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad$ $v_2 = -\frac{\cos nx}{n^2}$

$v_3 = -\frac{\sin nx}{n^3}$

$uv_1 - u'v_2 + u''v_3$

$$a_n = \frac{1}{\pi}\left[(\pi^2 - 2\pi x + x^2)\frac{\sin nx}{n} - (-2\pi + 2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right.$$

$$a_n = \frac{1}{\pi}\left[(\pi^2 - 2\pi(2\pi) + (2\pi)^2)\frac{\sin 2n\pi}{n} + (-2\pi + 2(2\pi))\frac{\cos 2n\pi}{n^2} - \frac{2\sin 2n\pi}{n^3}\right.$$

$$\left. - \left[\pi^2 \frac{\sin 0}{n} + (-2\pi)\left(\frac{\cos 0}{n^2}\right) - \frac{2\sin 0}{n^3}\right]\right]$$

$$= \frac{1}{\pi}\left[2\pi\frac{1}{n^2} + 2\pi\frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi^2 - 2\pi x + x^2) \sin nx \, dx$$

Let $u = \pi^2 - 2\pi x + x^2$ $\quad$ $v = \sin nx$

$u' = -2\pi + 2x$ $\quad$ $v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad$ $v_2 = -\frac{\sin nx}{n^2}$

$v_3 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi}\left[(\pi^2 - 2\pi x + x^2)\left(-\frac{\cos nx}{n}\right) - (-2\pi + 2x)\left(-\frac{\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2 - 4\pi^2 + 4\pi^2)\left(-\frac{1}{n}\right) + \frac{2}{n^3}\right] - \left[\pi^2\left(-\frac{1}{n}\right) + \frac{2}{n^3}\right]\Big\}$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$$\boxed{b_n = 0}$$

Sub a_0, a_n & b_n in ①

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$f(x) = \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

$x = 0$ in above

$$\frac{f(0) + f(2\pi)}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\frac{(\pi - 0)^2 + (\pi - 2\pi)^2}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\cancel{2}\pi^2}{3} \times \frac{1}{\cancel{4}_2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum \frac{1}{n^2}$$

NOTE:

If $f(x)$ is even ~~sol~~ if Fourier expression or series contains only cosine ($\cos$) terms if $f(x)$ is odd, it contains only sin terms.

CONVERGENCE OF FOURIER SERIES

★ If $f(x)$ is continuous at x_0, then the sum of fourier series at $x = x_0$ is equal to $f(x_0)$

$$f(x_0) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

★ If $f(x)$ is discontinuous at $x = x_0$ where x_0 is a middle point of the given interval, then the sum of F.S is equal to the average of L.H.L & R.H.L

★ If $f(x)$ is discontinuous at $x = x_0$, then x_0 is at an extreme position, then the sum of F.S is the average of $f(x)$ at extreme point.

BASIC FORMULAE

1) Bernoulli's formula

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 \cdots$$

2) $$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

3) $$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

4) $(-1)^{n+1} = (-1)^{n-1}$

5) $\cos n\pi = (-1)^n$ & $\cos 2n\pi = (-1)^{2n} = 1$

6) $\sin n\pi = \sin 2n\pi = 0$

8) Find fourier series for function $(\pi - x)^2$ in $0 < x < 2\pi$ and also find value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$

Sol $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad -①$$

where,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi^2 - 2x\pi + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\pi^2 x - \cancel{2}\pi \frac{x^2}{\cancel{2}} + \frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\pi^2(2\pi) - \pi(2\pi)^2 + \frac{(2\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{6\pi^3 - 12\pi^3 + 8\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = \frac{2\pi^2}{3} \Rightarrow a_0 = \frac{2\pi^2}{3}$$

4m

Dirichlet condition

Suppose that,

i) $f(x)$ is periodic, single valued and finite function.

ii) $f(x)$ has a finite number of discontinuous in any one period & no infinite discontinuity

iii) $f(x)$ has at the most finite numbers of maximum or minimum

SINGLE VALUED FUNCTION:

If $f(x)$ has only one value for a given x then $f(x)$ is said to be single valued function, otherwise $f(x)$ is called multi-valued function.

Eg

$f(x) = x^2 + 2$

$f(2) = 4 + 2 = 6$

$f(x) = \sqrt{x}$

$f(4) = \sqrt{4} = \pm 2$

NOTE :

Fourier Series can be used to represent a continuous function, discontinuous function and periodic function, whereas Taylor's series, Maclurian's series expansion is valid only for function which are continuous and differentiable

FOURIERS SERIES :

If $f(x)$ is periodic function and satisfies Dirichlet condition, then it can be expressed as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where a_0, a_n, b_n, are called fourier coefficients.

In the interval, $(c, c+2\pi)$

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{c+2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\,dx$$

① In $(0, 2\pi)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx$$

② In $(-\pi, \pi)$ (neither even nor odd)

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$f(x)$ is even

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin x\,dx$$

$$b_n = 0$$

$f(x)$ is odd

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx$$

x	$\theta = \frac{2\pi x}{6}$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	doubt $y\sin\theta$	$\cos 2\theta$
0	0 0°	9	1	9	0	0	1
1	$2\pi/6$ 60°	18	0.5	9	0.8660	15.588	-0.5
2	$2\pi/3$ 120°	24	-0.5	-12	0.8660	20.784	-0.5
3	π 180°	28	-1	-28	0	0	1
4	$4\pi/3$ 240°	26	-0.5	-13	-0.8660	-22.516	-0.5
5	$5\pi/3$ 300°	20	0.5	10	-0.8660	-17.32	-0.5
		115		-30		-12.124	

$y = \sin 2\theta$

doubt $y\cos 2\theta$	$\sin 2\theta$
9	0
-9	0.8660
-12	-0.8660
28	0
-13	0.8660
-10	-0.8660
-2	-8.66

$a_0 = \frac{2\Sigma y}{n} = 38.3333$ $\quad a_2 = \frac{2\Sigma y\cos 2\theta}{n} = -0.6607$

$a_1 = \frac{2\Sigma y\cos\theta}{n} = -10$ $\quad b_1 = \frac{2\Sigma y\sin\theta}{n} = -4.0413$

$b_2 = -2.8867$

29.836

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Classificatio

General form of PDE A

Form of std PDE A

1) Find th

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2) Classif

Sol A

Wave

$uv_1 - u'v_2 + u''v_3$

$= \frac{2}{\pi}\left[(\pi^2+x^2-2\pi x)\left(\frac{\sin nx}{n}\right) - (2x-2\pi)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$

$= \frac{2}{\pi}\left[-(2x-2\pi)\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$

$= \frac{2}{\pi}\left[-(2\pi-2\pi)\left(-\frac{\cos n\pi}{n^2}\right) + (-2\pi)\left(-\frac{\cos n0}{n^2}\right)\right]$

$= \frac{2}{\pi}\left[\frac{2\pi \cos n0}{n^2}\right]$

$\boxed{a_n = \frac{4}{n^2}}$

$\therefore f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$

P.I

$\frac{1}{b-a}\int_a^b [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$

$\cancel{\frac{1}{\pi}\int_0^\pi (\pi^2+x^2-2\pi x)}$

$\frac{1}{\pi}\int_0^\pi (\pi-x)^4 dx = \frac{4\pi^4}{9 \times 4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{n^4}$

$= \frac{1}{\pi}\left[\frac{(\pi-x)^5}{5\times(-1)}\right]_0^\pi = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$

$= \left(\frac{1}{\pi}\left[\frac{\pi^5}{5}\right] - \frac{\pi^4}{9}\right)\frac{1}{8} = \sum_{n=1}^{\infty}\frac{1}{n^4}$

$\pi^4\left(\frac{1}{5}-\frac{1}{9}\right)\times\frac{1}{8} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \infty$

$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \infty = \frac{\pi^4}{90}$

9) H.R.C.S $f(x) = x$ in $(0, l)$ S.T $\sum_{n=odd} \frac{1}{n^4} = \frac{\pi^2}{96}$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx$

$= \frac{2}{l}\int_0^l f(x)dx$

$= \frac{2}{l}\int_0^l x\,dx$

$= \frac{2}{l}\left(\frac{x^2}{2}\right)_0^l$

$= \frac{2}{l}\frac{l^2}{2} = l$

$a_0 = l$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$u = x$ $\quad v = \cos\left(\frac{n\pi x}{l}\right)$

$u' = 1$ $\quad v_1 = \sin\left(\frac{n\pi x/l}{n\pi/l}\right)$

$v_2 = -\frac{\cos(n\pi x/l)}{(n\pi/l)^2}$

$uv_1 - u'v_2$

$a_n = \cancel{\left(x\cos\left(\frac{n\pi x}{l}\right)\right)}$ $\frac{2}{l}\left[\left(x \cancel{\sin}\left(\frac{n\pi x/l}{n\pi/l}\right)^{0} + \left(\frac{\cos(n\pi x/l)}{(n\pi/l)^2}\right)\right)\right]_0^l$

$a_n = \frac{2}{l}\frac{\cos(n\pi)-1}{(n\pi/l)^2} = \frac{2}{l}\left[\frac{((-1)^n-1)\,l^2}{n^2\pi^2}\right] = \frac{2l[(-1)^n-1]}{n^2\pi^2}$

$f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

$\therefore f(x)$ is even, then $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left(\frac{\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$u = x^2$ $\quad v = \cos nx$

$u' = 2x$ $\quad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad v_2 = \frac{-\cos nx}{n^2}$

$v_3 = \frac{-\sin nx}{n^3}$

$u v_1 - u' v_2 + u'' v_3$

$$a_n = \frac{2}{\pi}\left[\cancel{x^2\frac{\sin nx}{n}}^{\,0} + 2x\frac{\cos nx}{n^2} - \cancel{\frac{2\sin nx}{n^3}}^{\,0}\right]_0^{\pi}$$

$$a_n = \frac{2}{\cancel{\pi}}\left[2\cancel{\pi}\frac{\cos n\pi}{n^2} - 0\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx \quad — (1)$$

Put $x = \pi$ in (1)

$$\frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}(-1)^n$$

$$\frac{\pi^2+\pi^2}{2} - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\left(\pi^2 - \frac{\pi^2}{3}\right)\frac{1}{4} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty$$

$$\frac{2\pi^2}{12} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty \quad — (2)$$

Put $x = 0$ in (1)

$$\therefore f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}$$

$$0 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\infty$$

$$-\left(\frac{\pi^2}{12}\right) = -\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\infty\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\infty \quad - (3)$$

(2) + (3)

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty\right) = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$\begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4l}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{l}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4l}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$$

By P.I

$$\frac{1}{b-a}\int_a^b [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

$$\frac{1}{l}\int_0^l x^2 dx = \frac{l^2}{4} + \frac{1}{2}\sum_{n=odd} \frac{16l^2}{n^4\pi^4}$$

$$\frac{l^2}{3} - \frac{l^2}{4} = \frac{8l^2}{\pi^4}\sum_{n=odd}\frac{1}{n^4}$$

$$\frac{l^2}{12} \times \frac{\pi^4}{8l^2} = \sum_{n=1,3,5}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \infty$$

9) Find fourier series for $f(x) = x^2, -l < x < l$

S.T $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \infty = \frac{\pi^4}{90}$

$f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

$\therefore f(x)$ is even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x^2 dx$$

$$= \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l$$

$$= \frac{2}{l}\left(\frac{l^3}{3}\right)$$

$$= \frac{2l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$v = \cos\left(\frac{n\pi x}{l}\right)$

$u = x^2$ $\quad v_1 = \sin\left(\frac{n\pi x/l}{n\pi/l}\right)$

$u' = 2x$

$u'' = 2$ $\quad v_2 = -\cos\left(\frac{n\pi x/l}{(n\pi/l)^2}\right)$

$v_3 = -\sin\left(\frac{n\pi x/l}{(n\pi/l)^3}\right)$

$uv_1 - u'v_2 + u''v_3$

$$= \frac{2}{l}\left[\cancelto{0}{x^2 \sin\left(\frac{n\pi x/l}{n\pi/l}\right)} + 2x\cos\frac{(n\pi x/l)}{(n\pi/l)^2} - \cancelto{0}{2\sin\left(\frac{n\pi x/l}{(n\pi/l)^3}\right)}\right]_0^l$$

$$= \frac{2}{l}\left[2\cancel{l}\frac{\cos(n\pi)}{(n\pi/l)^2}\right]_0^l$$

$$= 4\frac{\cos(n\pi)}{(n\pi/l)^2} = \frac{4(-1)^n l^2}{n^2\pi^2}$$

$$\therefore f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

Q) $f(x) = 2x - x^2$ in $(0, 3)$

$$\bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2 dx}$$

$$\bar{y}^2 = \frac{1}{3}\int_0^3 (2x - x^2)^2 dx$$

$$\bar{y}^2 = \frac{1}{3}\int_0^3 \left[4x^2 + x^4 - 4x^3\right]_0^3 \cancel{dx}$$

$$\bar{y}^2 = \frac{1}{3}\left[\frac{4x^3}{3} + \frac{x^5}{5} - \frac{4x^4}{4}\right]_0^3$$

$$\bar{y}^2 = \frac{1}{3}\left[4 \times 9 + \frac{243}{5} - \frac{4(81)}{4}\right]$$

$$\bar{y}^2 = \frac{1}{3}\left[36 + \frac{243}{5} - 81\right]$$

$$= \frac{1}{3}\left[\frac{180 + 243 - 405}{5}\right]$$

$$\bar{y}^2 = \frac{18}{15} = \frac{6}{5}$$

Q) Find H.R.SS for $f(x) = x$ in $(0, \pi)$

4m Sol

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx \, dx$$

$u = x$ $\qquad v = \sin nx$

$u' = 1$ $\qquad v' = -\frac{\cos nx}{n}$

$\qquad v'' = -\frac{\sin nx}{n^2}$

$uv_1 - u'v_2$

$$b_n = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}\right]_0^\pi$$

$$b_n = \frac{2}{\cancel{\pi}}\left[-\frac{\cancel{\pi}\cos n\pi}{n}\right]$$

$$b_n = \frac{-2(-1)^n}{n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx \ //$$

$\begin{array}{r} 27 \\ \times 3 \\ \hline 81 \end{array}$

Q) H.R.C.S for $f(x) = (\pi - x)^2$ in $(0, \pi)$ find $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \infty$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x) dx$$

$$= \frac{2}{\pi}\int_0^\pi (\pi - x)^2 dx$$

$$= \frac{2}{\pi}\int_0^\pi (\pi^2 + x^2 - 2\pi x) dx$$

$$= \frac{2}{\pi}\left[\pi^2 x + \frac{x^3}{3} - \frac{2\pi x^2}{2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\cancel{\pi^3} + \frac{\pi^3}{3} - \cancel{\pi^3}\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx \, dx$$

$$\frac{2}{\pi}\int_0^\pi (\pi - x)^2 \cos nx \, dx$$

$u = \pi^2 + x^2 - 2\pi x$ $\qquad v = \cos nx$

$u' = 2x - 2\pi$ $\qquad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\qquad v_2 = -\frac{\cos nx}{n^2}$

$\qquad v_3 = -\frac{\sin nx}{n^3}$

Sin x	y Sin x	Sin 2x	y Sin 2x	Sin 3x	y Sin 3x
0	0	0	0	0	0
0.2660	1.2124	0.8660	1.2124	0	0
0.8660	1.6454	0.8660	-1.6454	0	0
0	0	0	0	0	0
-0.8660	-1.217	0.8660	1.299	0	0
-0.8660	-1.0372	-0.8660	-1.0392	0	0
	0.5196		0.1732		0

$a_0 = \frac{2\Sigma y}{n} = \frac{2(8.7)}{6} = 2.9$ $\quad a_1 = \frac{2\Sigma y \cos x}{a} = -0.337$

$a_2 = -0.1 \quad a_3 = 0.03$

$b_1 = 0.1732 \quad b_2 = -0.06 \quad b_3 = 0$

$f(x) = \frac{2.9}{2} + (-0.337 \cos x + 0.1732 \sin x) + (-0.1 \cos x - 0.06 \sin 2x)$

$+ 0.03 \cos 3x$ //

Q) The values of x and corresponding values of f(x) over a period T are given below. Show that

$f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$ where $\theta = 2\pi x / T$

x	0	T/6	T/3	T/2	2T/3	5T/6	T
f(x) y	1.48	1.30	1.05	1.30	-0.88	-0.25	1.98

Sol when $x : 0$ to T

$\theta = 0$ to 2π

x	$\theta = \frac{2\pi x}{T}$	$y = f(x)$	Cos θ	y cos θ	Sin θ	y Sin θ
0	0	1.98	1	1.98	0	0
T/6	$\frac{2\pi}{6} = \pi/3$	1.30	0.5	0.65	0.8660	1.1258
T/3	$\frac{2\pi}{3}$	1.05	-0.5	-0.525	0.8660	0.9893
T/2	π	1.30	-1	-1.30	0	0
2T/3	$\frac{4\pi}{3}$	-0.88	-0.5	0.44	-0.8660	0.7621
5T/6	$\frac{5\pi}{3}$	-0.25	0.5	-0.425	-0.8660	0.2165
		4.5		1.12		3.013

$a_0 = \frac{2\Sigma y}{n} = 1.5 \quad a_1 = \frac{2\Sigma y \cos\theta}{n} = 0.3733$

$b_1 = \frac{2\Sigma y \sin\theta}{n} = 1.0046$

$f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$

$= 0.75 + 0.3733 \cos\theta + 1.0046 \sin\theta$

3) Find 1st two harmonic for $f(x) = y$ in $(0, 6)$

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

Sol By taking $\theta = 2\pi x / l$, here $l = 6$

$f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta$

where $\theta = 2\pi x / l$

By P.I

$$\frac{1}{b-a}\int_a^b [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

$$\frac{1}{2l}\int_{-l}^{l} x^4 dx = \frac{\cancel{x}l^4}{9\times\cancel{x}} + \frac{1}{\cancel{2}}\sum_{n=1}^{\infty}\frac{16l^4}{n^4\pi^4}$$

$$= \frac{2}{2l}\int_0^l x^4 dx = \frac{l^4}{9} + \frac{8l^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\left(\frac{l^4}{5} - \frac{l^4}{9}\right)\frac{\pi^4}{8l^4} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\left(\frac{9l^4 - 5l^4}{45}\right)\frac{\pi^4}{8l^4} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{l^4}{90}\times\frac{\pi^4}{l^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \infty$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \infty = \frac{\pi^4}{90}$$

HARMONIC ANALYSIS :

The process of finding F.S for a function given by numerical values is called Harmonic Analysis.

The fourier constants are given by,

$a_0 = \frac{2\Sigma y}{n}$ $a_n = \frac{2\Sigma y \cos nx}{n}$ $b_n = \frac{2\Sigma y \sin nx}{n}$, where $n \to$ no. of intervals.

$\therefore$ The Fourier series is, $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos^2 x + \dots$

Note : i) The term $a_1 \cos x + b_1 \sin x$ is called the 1st harmonic ~~as~~ or ~~find~~ fundamental harmonic.

2) The term $a_2 \cos 2x + b_2 \sin 2x$ is the 2nd Harmonic and so on.

Types :

i) Given datas are in π form $\to$ radius

ii) Given datas are in degree form $\to$ degree

iii) Given datas are in f form $\to$ radius

iv) Given datas are in l form $\to$ radius

HARMONIC ANALYSIS

1) Find the F.S upto 3rd harmonic for $y = f(x)$ in $(0, 2\pi)$. Define by the value given below :

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y :	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Sol

The F.S upto 3rd harmonic is

$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x)$

1st harmonic (or) fundamental

$b_1 = \frac{2\Sigma y \sin x}{n}$, $b_2 = \frac{2\Sigma y \sin 2x}{n}$, $b_3 = \frac{2\Sigma y \sin 3x}{n}$

x	y	$\cos x$	$y \cos x$	$\cos 2x$	$y \cos 2x$	$\cos 3x$	$y \cos 3x$
0	1	1	1	1	1	1	1
60° $\pi/3$	1.4	0.5	0.7	-0.5	-0.7	-1	-1.4
120° $2\pi/3$	1.9	-0.5	-0.95	-0.5	-0.95	1	1.9
180° π	1.7	-1	-1.7	1	1.7	-1	-1.7
240 $4\pi/3$	1.5	-0.5	-0.75	-0.5	-0.75	1	+1.5
300 $5\pi/3$	1.2	0.5	0.6	-0.5	-0.6	-1	-1.2
	8.4		-0.1		-0.3		0.1

$$\frac{\pi^2}{12} = 1\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right)$$

Put $x = \pi$

$$\frac{f(-\pi) + f(\pi)}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos n\pi$$

$$\frac{-\cancel{\pi} + \pi^2 + \cancel{\pi} + \pi^2}{2} - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$$\left(\pi^2 - \frac{\pi^2}{3}\right) \times \frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3} \times \frac{1}{4_2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$$

Q) $f(x) = l - x^2$ in $(-l, l)$

$$f(-x) = l - (-x)^2$$
$$= l - x^2$$
$$= f(x)$$

$\therefore f(x)$ is even.

$\Rightarrow b_n = 0$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos(n\pi x/l)\, dx$$

$$a_0 = \frac{2}{l} \int_0^l (l - x^2)\, dx$$

$$= \frac{2}{l}\left[lx - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[l^2 - \frac{l^3}{3}\right] = \frac{2l^{\cancel{2}}}{\cancel{l}}\left[1 - \frac{l}{3}\right]$$

$$a_0 = 2l\left[1 - l/3\right]$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos(n\pi x/l)\, dx$$

~~u = f(x)~~

$$a_n = \frac{2}{l} \int_0^l (l - x^2) \cos(n\pi x/l)\, dx$$

$u = l - x^2$ $\quad v = \cos(n\pi x/l)$

$u' = -2x$ $\quad v_1 = \sin\left(\frac{n\pi x/l}{n\pi/l}\right)$

$u'' = -2$ $\quad v_2 = -\frac{\cos(n\pi x/l)}{(n\pi/l)^2}$

$v_3 = -\frac{\sin(n\pi x/l)}{(n\pi/l)^3}$

$uv_1 - u'v_2 + u''v_3$

$$a_n = \frac{2}{l}\left[(l - x^2)\left(\frac{\cancel{\sin(n\pi x/l)}}{(n\pi/l)}\right) - (+2x)\left(+\frac{\cos(n\pi x/l)}{(n\pi/l)^2}\right) + 2\frac{\cancel{\sin(n\pi x/l)}}{(n\pi/l)^3}\right]$$

$$a_n = \frac{2}{\cancel{l}}\left[-\frac{2\cancel{l}\cos n\pi}{(n\pi/l)^2}\right]$$

$$a_n = -\frac{4\cos n\pi}{(n\pi/l)^2}$$

$$\boxed{a_n = -\frac{4(-1)^n l^2}{n^2\pi^2}}$$

$$f(x) = l\left[1 - \frac{l}{3}\right] + \sum_{n=1}^{\infty} \frac{-4(-1)^n l^2}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$$

Q) $f(x) = x^2$ in $(-\pi, \pi)$ & deduce that

i) $1/1^2 + 1/2^2 + 1/3^2 + \cdots \infty = \pi^2/6$

ii) $1/1^2 - 1/2^2 + 1/3^2 - \cdots \infty = \pi^2/12$

iii) $1/1^2 + 1/3^2 + 1/5^2 + \cdots \infty = \pi^2/8$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3} \right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3} \right) \right]$$

$$= \frac{1}{\pi} \left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} \right]$$

$$= \frac{1}{\pi} \left[\frac{2\pi^3}{3} \right]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx\,dx$$

$u = x + x^2$ $\quad v = \cos nx$

$u' = 1 + 2x$ $\quad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\cos nx}{n^2}$

$v_3 = -\frac{\sin nx}{n^3}$

$$uv_1 - u'v_2 + u''v_3$$

$$a_n = \frac{1}{\pi} \left[(x + x^2) \cancel{\frac{\sin nx}{n}}^{0} + (1 + 2x) \frac{\cos nx}{n^2} - \cancel{\frac{2 \sin nx}{n^3}}^{0} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[(1 + 2\pi) \frac{\cos n\pi}{n^2} - (1 - 2\pi) \frac{\cos n\pi}{n^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi(-1)^n}{n^2} \right] \Rightarrow \boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx\,dx$$

$v = \sin nx$

$u = x + x^2$ $\quad v_1 = -\frac{\cos nx}{n}$

$u' = 1 + 2x$ $\quad v_2 = -\frac{\sin nx}{n^2}$

$u'' = 2$ $\quad v_3 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi} \left\{ -(x + x^2) \frac{\cos nx}{n} + (1 + 2x) \cancel{\frac{\sin nx}{n^2}}^{0} + \frac{2 \cos nx}{n^3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left(-(\pi + \pi^2) \frac{\cos n\pi}{n} + \frac{2 \cos n\pi}{n^3} \right) - \left(-(-\pi + \pi^2) \frac{\cos n\pi}{n} + \frac{2 \cos n\pi}{n^3} \right) \right]$$

$$= \frac{1}{\pi} \left[(-\pi - \pi^2) \frac{\cos n\pi}{n} + \cancel{\frac{2 \cos n\pi}{n^3}} + (-\pi + \pi^2) \frac{\cos n\pi}{n} - \cancel{\frac{2 \cos n\pi}{n^3}} \right]$$

$$= \frac{1}{\cancel{\pi}} \left[-2\cancel{\pi} \frac{\cos n\pi}{n} \right]$$

$$b_n = \frac{-2(-1)^n}{n}$$

$$f(x) = \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2} \cos nx - \frac{2(-1)^n}{n} \sin nx \right]$$

Put $x = 0$

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} \times \frac{1}{4} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{1^3} + \cdots$$

Q) Find the fourier series for $f(x) = \begin{cases} x & 0 < x < \pi \\ 2\pi - x & \pi < x < 2\pi \end{cases}$ &

deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$

Sol

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx$$

$$a_0 = \frac{1}{\pi} \left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right) \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 2\pi^2 - \frac{3\pi^2}{2} \right] = \frac{1}{\pi} \left[\frac{\pi^2 + 4\pi^2 - 3\pi^2}{2} \right]$$

$$= \frac{1}{\pi} (\pi^2)$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx\,dx + \int_0^{2\pi} (2\pi - x) \cos nx\,dx \right]$$

$u = 2\pi - x$ $\quad V = \cos nx$ $\qquad u = x$ $\quad V = \cos nx$

$u' = -1$ $\quad V_1 = \frac{\sin nx}{n}$ $\qquad u' = 1$ $\quad V_1 = \frac{\sin nx}{n}$

$V_2 = -\frac{\cos nx}{n^2}$ $\qquad V_2 = -\frac{\cos nx}{n^2}$

$$a_n = \frac{1}{\pi} \left\{ \left[x \cancel{\frac{\sin nx}{n}}^{0} + \frac{\cos nx}{n^2} \right]_0^{\pi} + \left((2\pi - x) \cancel{\frac{\sin nx}{n}}^{0} - \frac{\cos nx}{n^2} \right)_{\pi}^{2\pi} \right\}$$

$$a_n = \frac{1}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} - \frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n - 1 - 1 + (-1)^n}{n^2} \right] = \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right]$$

$$b_n = \frac{1}{\pi} \left\{ \int_0^{\pi} x \sin nx\,dx + \int_0^{2\pi} (2\pi - x) \sin nx\,dx \right\}$$

$u = x$ $\quad V = \sin nx$ $\qquad u = 2\pi - x$ $\quad V = \sin nx$

$u' = 1$ $\quad V_1 = -\frac{\cos nx}{n}$ $\qquad u' = -1$ $\quad V_1 = -\frac{\cos nx}{n}$

$V_2 = -\frac{\sin nx}{n^2}$ $\qquad V_2 = -\frac{\sin nx}{n^2}$

$$b_n = \frac{1}{\pi} \left\{ \left(-\frac{x \cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}^{0} \right]_0^{\pi} + \left[-(2\pi - x) \frac{\cos nx}{n} - \cancel{\frac{\sin nx}{n^2}}^{0} \right]_0^{2\pi} \right\}$$

$$= \frac{1}{\pi} \left[-\frac{\pi \cos n\pi}{n} + \frac{\pi \cos n\pi}{n} \right] = \boxed{b_n = 0}$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{\pi n^2} \cos nx$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4}{\pi n^2} \cos nx$$

Sub $x = 0$ in above eq,

$$\frac{f(0) + f(2\pi)}{2} = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} -\frac{4}{\pi n^2}$$

$$0 \quad -\frac{\pi}{2} = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1,3,5}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8} \quad //$$

Q) $f(x) = x(2l-x)$ in $(0, 2l)$ and deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} (2lx - x^2)\,dx$$

$$= \frac{1}{l}\left[\left(lx^2 - \frac{x^3}{3}\right)_0^{2l}\right]$$

$$= \frac{1}{l}\left[4l^3 - \frac{8l^3}{3}\right]$$

$$= \frac{4l^3}{3l}$$

$$\boxed{a_0 = \frac{4l^2}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\int_0^{2l} (2lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$u = 2lx - x^2$ $\quad v = \cos\left(\frac{n\pi x}{l}\right)$

$u' = 2l - 2x$ $\quad v_1 = \frac{\sin\left(\frac{\cancel{\pi} n x}{l}\right)}{n\pi/l}$

$u'' = -2$

$$v_2 = -\frac{\cos(n\pi x/l)}{(n\pi/l)^2}$$

$$v_3 = \frac{-\sin(n\pi x/l)}{(n\pi/l)^3}$$

$$a_n = \frac{1}{l}\left[(2lx - x^2)\overset{0}{\cancel{\sin}}\frac{(n x\pi/l)}{n\pi/l} + (2l - 2x)\frac{\cos(n\pi x/l)}{(n\pi/l)^2} + \frac{2\sin^{0}(n\pi x/l)}{(n\pi/l)^2}\right]_0^{2\pi}$$

$$a_n = \frac{1}{l}\left[-2l\,\frac{\cos 2n\pi}{(n\pi/l)^2} - \frac{2l\cos 0}{(n\pi/l)^2}\right]$$

$$a_n = \frac{1}{l}\left[\frac{-4l}{(n\pi/l)^2}\right]$$

$$\boxed{a_n = \frac{-4l^2}{n^2\pi^2}}$$

$$\boxed{b_n = 0}$$

$$\therefore f(x) = \frac{2l^2}{3} + \sum_{n=1}^{\infty} \frac{-4l^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

Sub $x = l$ in above equation

$$f(l) = \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi$$

$$l(2l - l) - \frac{2l^2}{3} = -\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\left(l^2 - \frac{2l^2}{3}\right)\times\frac{\pi^2}{-4l^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{l^2}{3}\times\frac{\pi^2}{-4l^2} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \infty$$

$$-\frac{\pi^2}{12} = -\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \infty\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \infty$$

Q) $f(x) = x + x^2$ $(-\pi, \pi)$ and deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$$

Sol

$$f(x) = x + x^2$$

$$f(-x) = -x + (-x)^2$$

$$= -x + x^2$$

neither even nor odd.