18MAB204T-U1-Lecture Notes

18MAB204T-Probability and Queuing Theory

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Contents

DEAR ALL, HERE IT HAS BEEN SOLVED FEW PROBLEMS ONLY AND SOME TOPICS MAY NOT BE COVERED. SO, YOU CAN FOLLOW THE REGULAR CLASSWORK NOTES TO HAVE ALL THE TOPICS FOR YOUR PREPARATION. TAKE EXERCISE PROBLEMS GIVEN AT THE END FOR YOUR PRACTICE. APART FROM EXERCISE, YOU CAN FOLLOW ANY REFERENCE BOOK IN RELATED TOPICS FOR YOUR PRACTICE.

SOME OF THE SECTIONS/TOPICS IN THIS NOTES ARE PRELIMINARIES WHICH ARE THE BASIC IDEAS NEEDED TO DO OUR REGULAR COURSE EXAMPLES AND EXERCISES.

TOPICS:

- ⋆ Probability Concepts and Problems
- ⋆ Random Variables (Discrete and Continuous)
- ⋆ General Probability Distributions and Problems
- ⋆ Moments and Moment Generating Functions

1 Probability

1.1 Defintion of Probability and Basic Theorems

1.1.1 Random Experiment

An Experiment whose outcome or result can be predicted with certainty is called a deterministic experiment.

An Experiment whose all possible outcomes may be known in advance, the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown cases is called a random experiment.
Definition 1.1.1 (Sample Space). *Set of all possible outcomes which are assumed to be equally likely is called the sample space and is usually denoted by S. Any subset A of S containing favorable outcomes is called an event.*

1.1.2 The Approaches to Define the Probability

Definition 1.1.2 (Mathematical or Apriori Definition of Probability). *Let S be a sample space and A be an event associated with a random experiment. Let n(S) and n(A) be the number of elements of S and A. Then the probability of event A occurring, denoted by P(A) and defined as*

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favorable to A}}{\text{Exhaustive number of cases in S(Set of all possible cases)}}$$

Definition 1.1.3 (Statistical or Aposteriori Definition of Probability). *Let a random experiment be repeated n times and let an event A occur n_1 times out of n trials. The ratio $\frac{n_1}{n}$ is called the relative frequency of the event A. As n increases, $\frac{n_1}{n}$ shows a tendency to stabilize and to approach a constant value and is denoted by P(A) is called the probability of the event A.* i.e.,

$$P(A) = \lim_{n\to\infty} \frac{n_1}{n}$$

Definition 1.1.4 (Axiomatic Approach/Definition of Probability). *Let S be a sample space and A be an event associated with a random experiment. Then the probability of the event A is denoted by P(A) is defined as a real number satisfying the following conditions/axioms:*

(i) $0 \le P(A) \le 1$

(ii) P(S)=1

(iii) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Axiom (iii) can be extended to arbitrary number of events. i.e., *If $A_1, A_2, \dots, A_n, \dots$ be mutually exclusive, then $P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n \dots) = P(A_1) + P(A_2) + P(A_2) + P(A_3) + \cdots + P(A_n) + \dots$*

Theorem 1. *The probability of the impossible event is zero.* i.e., *if ϕ is a subset of S containing no event, then $P(\phi) = 0$.*

Theorem 2. *If $\overline{A}$ be the complimentary event of A, then $P(\overline{A}) = 1 - P(A) \leq 1$.*

Theorem 3 (Addition Theorem of Probability)**.** *If A and B be any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$.*

The above Theorem can be extended for 3 events which is given by $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Theorem 4. *If $B \subset A$, then $P(B) \leq P(A)$.*

1.2 Conditional Probability and Independents Events

The conditional probability of an event B, assuming that the event A has happened, is denoted by $P(B/A)$ and is defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

Theorem 5 (Product Theorem of Probability)**.** $P(A \cap B) = P(A) \cdot P(B/A)$

The product theorem can be extended to three events such as

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \text{ and } B)$$

There are few properties for the conditional distribution. Please go through it in the textbook.

When two events A and B are independent, we have $P(B/A) = P(B)$ and the Product Theorem takes the form $P(A \cap B) = P(A) \cdot P(B)$. The converse of this statement is also. *i.e.,* If $P(A \cap B) = P(A) \cdot P(B)$, then the events A and B are said to be independent.

This result can be extended to any number of events. *i.e.,* If $A_1, A_2, \ldots, A_n$ be n no. of events, then

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdots P(A_n)$$

Theorem 6. *If the events A and B are independent, then the events $\overline{A}$ and $\overline{B}$ (and similarly A and $\overline{B}$, $\overline{A}$ and B) are also independent.*

1.3 Total Probability and Bayes' Theorem

Theorem 7 (Theorem of Total Probability). *If $B_1, B_2, \ldots, B_n$ be the set of exhaustive and mutually exclusive events and A is another event associated with(or caused by)B_i, then*

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

Theorem 8 (Bayes' Theorem of Probability of Causes). *If $B_1, B_2, \ldots, B_n$ be the set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with(or caused by)B_i, then*

$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum\limits_{i=1}^{n} P(B_i)P(A/B_i)},\ i = 1, 2, \ldots, n$$

2 Discrete and Continuous Distributions

For the problems on Discrete and Continuous Random Variables(RV's) we have to note down the following points from the table for understanding.

	Discrete ♯RV (Probability Distribution Function)	Continuous ♯RV (Probability Density Function)
Operator	Σ	$\int$
Takes Values of the form (Eg.)	$1, 2, 3, \ldots$	$0 < x < \infty$
Standard Notation	$P(X = x)$	$f(x)$
CDF*	$F(X = x) = P(X \leq x)$	$F(X = x) = P(X \leq x)$
Total Probability	$\sum_{-\infty}^{\infty} P(X = x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$

♯RV-Random Variable, *CDF-Cumulative Distribution Function or Distribution Function

3 Mathematical Expectation

Definition 3.0.1 (Expectation). *An average of a probability distribution of a random variable X is called the expectation or the expected value or mathematical expectation of X and is denoted by $E(X)$.*

Definition 3.0.2 (Expectation for Discrete Random Variable). *Let X be a discrete random variable taking values $x_1, x_2, x_3, \ldots$ with probabilities $P(X = x_1) = p(x_1), P(X = x_2) = p(x_2), P(X = x_3) = p(x_3), \ldots$, the expected value of X is defined as $E(X) = \sum_{i=-\infty}^{\infty} x_i p(x_i)$, if the right hand side sum exists.*

Definition 3.0.3 (Expectation for Continuous Random Variable). *Let X be a continuous random variable with pdf $f(x)$ defined in $(-\infty, \infty)$, then the expected value of X is defined as*

$$E(X) = \int_{-\infty}^{\infty} x f(x)dx.$$

Note: $E(X)$ is called the mean of the distribution or mean of X and is denoted by $\overline{X}$ or μ.

3.1 Properties of Expectation

1. If c is any constant, then $E(c) = c$.

2. If a, b are constants, then $E(aX + b) = aE(X) + b$.
3. If (X, Y) is two dimensional random variable, then $E(X + Y) = E(X) + E(Y)$.
4. If (X, Y) is two dimensional **independent** random variable, then $E(XY) = E(X)E(Y)$.

3.2 Variance and its properties

Definition 3.2.1. *Let X be a random variable with mean $E(X)$, then the variance of X is defined as $E(X^2) - [E(X)]^2$ and id denoted by $Var(X)$ or σ_X^2.*

Properties of Variance

1. $Var(aX) = a^2 Var(X)$
2. $Var(aX \pm b) = a^2 Var(X)$.

4 Moments and Moment Generating Function

4.1 Moments

Definition 4.1.1 (Moments about origin)**.** *The r^{th} moment of a random variable X about the origin is defined as $E(X^r)$ and is denoted by μ'_r. Moments about origin are known as raw moments.*

Note: By moments we mean the moments about origin or raw moments.

The first four moments about the origin are given by

1. $\mu'_1 = E(X)$=Mean
2. $\mu'_2 = E(X^2)$
3. $\mu'_3 = E(X^3)$
4. $\mu'_4 = E(X^4)$

Note: $Var(X) = E(X^2) - [E(X)]^2 = \mu'_2 - {\mu'_1}^2$=Second moment - square of the first moment.

Definition 4.1.2 (Moments about mean or Central moments)**.** *The r^{th} moment of a random variable X about the mean μ is defined as $E[(X - \mu)^r]$ and is denoted by μ_r.*

The first four moments about the mean are given by

1. $\mu_1 = E(X - \mu) = E(X) - E(\mu) = \mu - \mu = 0$
2. $\mu_2 = E[(X - \mu)^2] = Var(X)$
3. $\mu_3 = E[(X - \mu)^3]$

4. $\mu_4 = E[(X-\mu)^4]$

Definition 4.1.3 (Moments about any point a)**.** *The r^{th} moment of a random variable X about any point a is defined as $E[(X-a)^r]$ and we denote it by m'_r.*

The first four moments about a point 'a' are given by

1. $m'_1 = E(X-a) = E(X) - a = \mu - a$
2. $m'_2 = E[(X-a)^2]$
3. $m'_3 = E[(X-a)^3]$
4. $m'_4 = E[(X-a)^4]$

Relation between moments about the mean and moments about any arbitrary point a

Let μ_r be the r^{th} moment about mean and m'_r be the r^{th} moment about any point a. Let μ be the mean of X.

$$
\begin{aligned}
\therefore\ \mu_r &= E[(X-\mu)^r] \\
&= E[(X-a)-(\mu-a)]^r \\
&= E[(X-a)-m'_1]^r \\
&= E\left[(X-a)^r - {}^rC_1(X-a)^{r-1}m'_1 + {}^rC_2(X-a)^{r-2}(m'_1)^2 - \cdots + (-1)^r(m'_1)^r\right] \\
&= E(X-a)^r - {}^rC_1E(X-a)^{r-1}m'_1 + {}^rC_2E(X-a)^{r-2}(m'_1)^2 \\
&\quad - {}^rC_3E(X-a)^{r-3}(m'_1)^3 + {}^rC_4E(X-a)^{r-4}(m'_1)^4 - \cdots + (-1)^r(m'_1)^r \\
&= m'_r - {}^rC_1m'_{r-1}m'_1 + {}^rC_2m'_{r-2}(m'_1)^2 - {}^rC_3m'_{r-3}(m'_1)^3 + {}^rC_4m'_{r-4}(m'_1)^4 \\
&\quad - \cdots + (-1)^r(m'_1)^r
\end{aligned}
$$

We define(fix) $m'_0 = 1$, then we have

$$
\begin{aligned}
\mu_1 &= m'_1 - m'_0m'_1 = 0 \\
\mu_2 &= m'_2 - {}^2C_1m'_1\cdot m'_1 + (m'_1)^2 \\
&= m'_2 - (m'_1)^2 \\
\mu_3 &= m'_3 - {}^3C_1m'_2\cdot m'_1 + {}^3C_2m'_1\cdot(m'_1)^2 - (m'_1)^3\cdot \\
&= m'_3 - 3m'_2\cdot m'_1 + 2(m'_1)^3 \\
\mu_4 &= m'_4 - {}^4C_1m'_3\cdot m'_1 + {}^4C_2m'_2\cdot(m'_1)^2 - {}^4C_3m'_1\cdot(m'_1)^3 + (m'_1)^4\cdot \\
&= m'_4 - 4m'_3\cdot m'_1 + 6m'_2\cdot(m'_1)^2 - 3(m'_1)^4
\end{aligned}
$$

4.2 Moment Generating Function (MGF)

Definition 4.2.1 (Moment Generating Function (MGF))**.** *The moment generating function of a random variable X is defined as $E(e^{tX})$ for all$t \in (-\infty,\infty)$. It is denoted by $M(t)$ or $M_X(t)$.*

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum\limits_{x=0}^{n} e^{tx} P(X = x) \text{ if } X \text{ is discrete} \\ \\ \int\limits_{x=0}^{\infty} e^{tx} f(x) dx \text{ if } X \text{ is continuous} \end{cases}$$

If X is a random variable. Its MGF $M_X(t)$ is given by

$$\begin{aligned} M_X(t) = E(e^{tX}) &= E\left(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots + \frac{(tX)^r}{r!} + \ldots\right) \\ &= 1 + \frac{t}{1!}E(X) + \frac{(t)^2}{2!}E(X^2) + \frac{(t)^3}{3!}E(X^3) + \cdots + \frac{(t)^r}{r!}E(X^r) + \ldots \\ &= 1 + \frac{t}{1!}\mu_1' + \frac{(t)^2}{2!}\mu_2' + \frac{(t)^3}{3!}\mu_3' + \cdots + \frac{(t)^r}{r!}\mu_r' + \ldots \end{aligned}$$

i.e.

$$\begin{aligned} \mu_1' &= \text{Coefft. of } t \text{ in the expansion of } M_X(t) \\ \mu_2' &= \text{Coefft. of } \frac{t^2}{2!} \text{ in the expansion of } M_X(t) \\ &\vdots \\ \mu_r' &= \text{Coefft. of } \frac{t^r}{r!} \text{ in the expansion of } M_X(t) \\ &\vdots \end{aligned}$$

$\therefore$ $M_X(t)$ generates all the moments about the origin. (That is why we call it as Moment Generating Function (MGF)).

Another phenomenon which we often use to find the moments is given below:

$$\mu_r' = M_X^{(r)}(0)$$

We have

$$M_X(t) = E(e^{tX}) = 1 + \frac{t}{1!}\mu_1' + \frac{(t)^2}{2!}\mu_2' + \frac{(t)^3}{3!}\mu_3' + \cdots + \frac{(t)^r}{r!}\mu_r' + \ldots$$

Differentiate with respect to t, we get

$$\begin{aligned}
M'_X(t) &= \mu'_1 + t\mu'_2 + \frac{(t)^2}{2!}\mu'_3 + \cdots + \frac{(t)^{r-1}}{(r-1)!}\mu'_r + \ldots \\
M''_X(t) &= \mu'_2 + t\mu'_3 + \frac{(t)^2}{2!}\mu'_4 + \cdots + \frac{(t)^{r-2}}{(r-2)!}\mu'_r + \ldots \\
&\vdots \\
M^{(r)}_X(t) &= \mu'_r + \text{terms of higher powers of } t \\
&\vdots
\end{aligned}$$

Putting $t = 0$, we get

$$\begin{aligned}
M'_X(0) &= \mu'_1 \\
M''_X(0) &= \mu'_2 \\
&\vdots \\
M^{(r)}_X(0) &= \mu'_r \\
&\vdots
\end{aligned}$$

The Maclaurin's series expansion given below will give all the moments.

$$M_X(t) = M_X(0) + \frac{t}{1!}M'_X(0) + \frac{t^2}{2!}M''_X(0) + \ldots$$

The MGF of X about its mean μ is $M_{X-\mu}(t) = E\left[e^{t(X-\mu)}\right]$.

Similarly, the MGF of X about any point a is $M_{X-a}(t) = E\left[e^{t(X-a)}\right]$.

Properties of MGF

1. $M_{cX}(t) = M_X(ct)$
2. $M_{X+c}(t) = e^{ct}M_X(t)$
3. $M_{aX+b}(t) = e^{bt}M_X(at)$
4. If X and Y are independent RV's then, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

5 Example Worked out Problems

Example: 1. *From 6 positive and 8 negative numbers, 4 are chosen at random(without replacement) and multiplied. What is the probability that the product is negative?*

Hints/Solution: If the product is negative(-ve), any one number is -ve (or) any 3 numbers are -ve.

No. of ways choosing 1 negative no. = ${}^6C_3 \cdot {}^8 C_1 = 20 \cdot 8 = 160$.
No. of ways choosing 3 negative nos. = ${}^6C_1 \cdot {}^8 C_3 = 6 \cdot 56 = 336$.

Total no. of ways choosing 4 nos. out of 14 nos. $= {}^{14}C_4 = 1001$.

$$\therefore \quad P(\text{the product is -ve}) = \frac{160+336}{1001} = \frac{496}{1001}.$$

Example: 2. *A box contains 6 bad and 4 good apples. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good.*

Hints/Solution: Let A be an event of first drawn apple is good and let B be an event that the other one is also good.

$\therefore$ The required probability is $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{{}^4C_2/{}^{10}C_2}{\frac{4}{10}} = \dfrac{6/45}{4/10} = \dfrac{1}{3}$

Aliter: Probability of the first drawn apple is good$=\dfrac{4}{10}$.
Once the first apple is drawn, the box contains only 9 apples in total and 3 good apples.

$\therefore$ Probability of the second drawn apple is also good$=\dfrac{3}{9} = \dfrac{1}{3}$.

Example: 3. *A random variable X has the following distribution*

X	0	1	2	3	4	5	6	7
P[X=x]	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) the value of k,(ii) the Cumulative Distribution Function (CDF) (iii) $P(1.5 < X < 4.5/X > 2)$ and (iv) the smallest value of α for which $P(X \leq \alpha) > \dfrac{1}{2}$.

Hints/Solution:

(i) We know that $\displaystyle\sum_{-\infty}^{\infty} P[X=x] = 1$.

Here,

$$\begin{aligned}
P[X=0]+P[X=1]+P[X=2]+P[X=3]+ & \\
P[X=4]+P[X=5]+P[X=6]+P[X=7] &= 1\\
\text{i.e. } 0+k+2k+2k+3k+k^2+2k^2+7k^2+k &= 1\\
\text{i.e. } 10k^2+9k-1 &= 0\\
\text{i.e. } (10k-1)(k+1) &= 0\\
\implies \quad k=\frac{1}{10} \text{ or } k=-1
\end{aligned}$$

Since the probability is not negative, we take the value $k=\frac{1}{10}$.

$\therefore$ The probability distribution is given by

X	0	1	2	3	4	5	6	7
$P[X=x]$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$
$P[X=x]$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(ii) The Cumulative Distribution Function (CDF)

$$\begin{aligned}
F(x) &= P(X\le x)\\
F(0) &= P(X\le 0)=P(X=0)=0\\
F(1) &= P(X\le 1)=P(X=0)+P(X=1)=0+\frac{1}{10}=\frac{1}{10}\\
F(2) &= P(X\le 1)=P(X=0)+P(X=1)+P(X=2)=0+\frac{1}{10}+\frac{2}{10}=\frac{3}{10}\\
&\vdots\\
F(7) &= P(X\le 7)=P(X=0)+P(X=1)+P(X=2)+P(X=2)\\
&\quad +P(X=4)+P(X=5)+P(X=6)+P(X=7)\\
&= \frac{100}{100}=1
\end{aligned}$$

The detailed CDF $F(X=x)$ is given in the table.

X	0	1	2	3	4	5	6	7
$P[X=x]$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$
$P[X=x]$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$
$F[X=x]$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	$\frac{100}{100}=1$

(iii)

$$
\begin{aligned}
P(1.5 < X < 4.5/X > 2) &= \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P(X > 2)} \\
P(1.5 < X < 4.5) &= P(X = 3) + P(X = 4) = \frac{5}{10} \\
P(X > 2) &= 1 - P(X \leq 2) \\
&= 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\} \\
&= 1 - \frac{3}{10} = \frac{7}{10} \\
P(1.5 < X < 4.5/X > 2) &= \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}
\end{aligned}
$$

(iv) The smallest value of α for which $P(X \leq \alpha) > \frac{1}{2} = 0.5$.

From the CDF table, we found that $P(X \leq 3) = \frac{5}{10} = 0.5$. But we need the probability which is more than 0.5, for this we have $P(X \leq 4) = \frac{8}{10} = 0.8 > 0.5$. $\therefore \; \alpha = 4$ satisfies the given condition.

Example: 4. *A random variable X has the following distribution*

X	0	1	2	3	4	5	6	7	8
P[X=x]	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$

Find (i) the value of k*,(ii) the Distribution Function (CDF) (iii)* $P(0 < X < 3/X > 2)$ *and (iv) the smallest value of* α *for which* $P(X \leq \alpha) > \frac{1}{2}$.

Hints/Solution:

X	0	1	2	3	4	5	6	7	8
$P[X = x]$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$
$P[X = x]$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

(i) The value of $k = \frac{1}{81}$

(ii) the Distribution Function (CDF)

X	0	1	2	3	4	5	6	7	8
$P[X = x]$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$
$P[X = x]$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$
$F[X = x]$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	$\frac{81}{81} = 1$

(iii) $P(0 < X < 3/X > 2)$

$$P(0 < X < 3/X > 2) = \frac{P[(0 < X < 3) \cap (X > 2)]}{P(X > 2)} = 0(\because \textit{ there is no common elemen}$$

(iv) The smallest value of α for which $P(X \leq \alpha) > \frac{1}{2} = 0.5$.

From the CDF table, we found that $P(X \leq 5) = \frac{36}{81} < 0.5$. But we need the probability which is more than 0.5, for this we have $P(X \leq 6) = \frac{49}{81} > 0.5$. $\therefore \quad \alpha = 6$ satisfies the given condition.

Example: 5. *Find the value of k for the pdf* $f(x) = \begin{cases} kx, & \textit{when } 0 \leq x \leq 1 \\ k, & \textit{when } 1 \leq x \leq 2 \\ 3k - kx, & \textit{when } 2 \leq x \leq 3 \\ 0, & \textit{otherwise} \end{cases}$. *Also find (a) the CDF of X (b) $P(1.5 < X < 3.2/0.5 < X < 1.8)$*

Hints/Solution:

$$\text{We know that } \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\text{i.e. } \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\text{i.e. } \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1} kx \cdot dx + \int_{1}^{2} k \cdot dx + \int_{2}^{3} (3k - kx)dx + \int_{3}^{\infty} 0 \cdot dx = 1$$

$$\text{i.e. } 0 + k\left[\frac{x^2}{2}\right]_0^1 + k\,[x]_1^2 + \left[3k \cdot x - k \cdot \frac{x^2}{2}\right]_2^3 + 0 = 1$$

$$\text{i.e. } 2k = 1$$

$$\implies \boxed{k = \frac{1}{2}}$$

(a) Since $k = \frac{1}{2}$, $\quad f(x) = \begin{cases} \frac{x}{2}, & \text{when } 0 \le x \le 1 \\ \frac{1}{2}, & \text{when } 1 \le x \le 2 \\ \frac{1}{2}(3-x), & \text{when } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$

The CDF of X is $F(x) = P(X \le x) = \int\limits_{-\infty}^{x} f(x)dx$

$$\text{When } \ x < 0, \ F(x) = 0, \ \text{since } \ f(x) = 0 \ \text{ for } \ x < 0$$

$$\begin{aligned} \text{When } \ 0 \le x < 1, \ F(x) &= \int\limits_{-\infty}^{x} f(x)dx \\ &= \int\limits_{0}^{x} f(x)dx = \int\limits_{0}^{x} \frac{x}{2}\,dx = \boxed{\frac{x^2}{4}} \end{aligned}$$

$$\begin{aligned} \text{When } \ 1 \le x < 2, \ F(x) &= \int\limits_{0}^{x} f(x)dx \\ &= \int\limits_{0}^{1} \frac{x}{2}\,dx + \int\limits_{0}^{x} \frac{1}{2}\,dx = \boxed{\frac{1}{4}(2x-1)} \end{aligned}$$

$$\begin{aligned} \text{When } \ 2 \le x < 3, \ F(x) &= \int\limits_{0}^{x} f(x)dx \\ &= \int\limits_{0}^{1} \frac{x}{2}\,dx + \int\limits_{0}^{x} \frac{1}{2}\,dx + \int\limits_{0}^{x} \frac{1}{2}(3-x)\,dx = \boxed{\frac{1}{4}(-x^2+6x-5)} \end{aligned}$$

$$\text{When } \ x \ge 3, \ F(x) = 1$$

$$\therefore \quad F(x) = \begin{cases} 0 & \text{when} x < 0 \\ \frac{x^2}{2}, & \text{when } 0 \le x \le 1 \\ \frac{1}{4}(2x-1), & \text{when } 1 \le x \le 2 \\ \frac{1}{4}(-x^2+6x-5), & \text{when } 2 \le x \le 3 \\ 1, & \text{when } x \ge 3 \end{cases}$$

(b) $P(1.5 < X < 3.2/0.5 < X < 1.8) = \dfrac{P[(1.5 < X < 3.2) \cap (0.5 < X < 1.8)]}{P(0.5 < X < 1.8)}$

$$\begin{aligned}
P[(1.5 < X < 3.2) \cap (0.5 < X < 1.8)] &= P(1.5 < X < 1.8) \\
&= \int_{1.5}^{1.8} \frac{1}{2}\, dx = \boxed{\frac{3}{20}} \\
P(0.5 < X < 1.8) &= \int_{0.5}^{1} \frac{x}{2}\, dx + \int_{1}^{1.8} \frac{1}{2}\, dx = \boxed{\frac{3}{16}} \\
P(1.5 < X < 3.2/0.5 < X < 1.8) &= \frac{P[(1.5 < X < 3.2) \cap (0.5 < X < 1.8)]}{P(0.5 < X < 1.8)} \\
&= \boxed{\frac{16}{20}}
\end{aligned}$$

Example: 6. *Experience has shown that while walking in a certain park, the time X (in minutes) duration between seeing two people smoking has a density function*

$$f(x) = \begin{cases} kxe^{-x}, & \textit{when } x > 0 \\ 0, & \textit{otherwise} \end{cases}$$

Find the (a) value of k (b) distribution function of X (c) What is the probability that a person, who has just seen a person smoking will see another person smoking in 2 to 5 minutes, in at least 7 minutes?

Hints/Solution: Given the pdf of X is

$$f(x) = \begin{cases} kxe^{-x}, & \text{when } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$\therefore \quad f(x) \geq 0 \ \forall \ x \quad \Longrightarrow \quad k > 0.$

(a) We know that $\int\limits_{-\infty}^{\infty} f(x)\,dx = 1$

$$\begin{aligned} i.e. \int\limits_{0}^{\infty} kxe^{-x}\,dx &= 1 \\ i.e.\ k\left[x\frac{e^{-x}}{-1} - \frac{e^{-x}}{(-1)^2}\right]_0^{\infty} &= 1 \\ \implies \boxed{k=1} \end{aligned}$$

$$\therefore\ f(x) = \begin{cases} xe^{-x}, & \text{when } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) The distribution function (CDF) is given by

$$F(x) = P(X \le x) = \int\limits_{-\infty}^{x} f(x)\,dx$$

Here we have $x > 0$, $F(x) = \int\limits_{0}^{x} xe^{-x}\,dx = 1 - (1+x)e^{-x},\ \ x > 0$

(c)

$$\begin{aligned} P(2 < X < 5) &= \int\limits_{2}^{5} f(x)\,dx \\ &= \int\limits_{2}^{5} xe^{-x}\,dx \\ &= -6e^{-5} + 3e^{-2} \\ &= -0.04 + 0.406 = 0.366 \end{aligned}$$

and

$$\begin{aligned} P(X > 7) &= \int\limits_{7}^{\infty} f(x)\,dx \\ &= \int\limits_{7}^{\infty} xe^{-x}\,dx \\ &= 8e^{-7} \\ &= 0.007 \end{aligned}$$

Example: 7. *The sales of convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a distribution function*

$$F(x) = \begin{cases} 0, & \text{when } x < 0 \\ \dfrac{x^2}{2}, & \text{when } 0 \le x < 1 \\ k(4x - x^2) - 1, & \text{when } 1 \le x < 2 \\ 1, & \text{when } x \ge 2 \end{cases}$$

Suppose that this convenience store's total sales on any given day is less than 2000 units(Dollars or Pounds or Rupees). Find the (a) value of k (b) Let A and B be the events that tomorrow the store's total sales between 500 and 1500 units respectively. Find $P(A)$ and $P(B)$. (c) Are A and B are independent events?

Hints/Solution: The pdf of X is given by

$$f(x) = F'(x) = \frac{d}{dx}F(x) = \begin{cases} 0, & \text{when } x < 0 \\ x, & \text{when } 0 \le x < 1 \\ k(4 - 2x), & \text{when } 1 \le x < 2 \\ 0, & \text{when } x \ge 2 \end{cases}$$

Since $f(x)$ is a pdf,

$$\text{we have } \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\text{i.e. } \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$\text{i.e. } \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1} x\,dx + \int_{1}^{2} k(4 - 2x)\,dx + \int_{2}^{\infty} 0 \cdot dx = 1$$

$$\text{i.e. } 0 + \left[\frac{x^2}{2}\right]_0^1 + k\left[4x - x^2\right]_1^2 + 0 = 1$$

$$\implies \boxed{k = \frac{1}{2}}$$

$$f(x) = \begin{cases} 0, & \text{when } x < 0 \\ x, & \text{when } 0 \le x < 1 \\ (2-x), & \text{when } 1 \le x < 2 \\ 0, & \text{when } x \ge 2 \end{cases}$$

Since the total sales X is in thousands of units, the sales between 500 and 1500 units is the event A which stands for $\frac{1}{2} = 0.5 < X < \frac{3}{2} = 1.5$ and the sales over 1000 units is the event B which stands for $X > 1$. $\implies \quad A \cap B = 1 < X < 1.5$

Now

$$\begin{aligned} P(A) &= P(0.5 < X < 1.5) = \int_{0.5}^{1.5} f(x)dx \\ &= \int_{0.5}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx = \boxed{\frac{3}{4}} \end{aligned}$$

$$\begin{aligned} P(B) &= P(X > 1) = \int_{1}^{2} f(x)dx \\ &= \int_{1}^{2} (2-x)\,dx = \boxed{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} P(A \cap B) &= P(1 < X < 1.5) = \int_{1}^{1.5} f(x)dx \\ &= \int_{1}^{1.5} (2-x)\,dx = \boxed{\frac{3}{8}} \end{aligned}$$

The condition for independent events: $P(A) \cdot P(B) = P(A \cap B)$
Here, $P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$
$\therefore \quad A$ and B are independent events.

Example: **8.** *If a random variable* X *has the following probability distribution, find* $E(X)$, $E(X^2)$, $Var(X)$, $E(2X+1)$, $Var(2X+1)$.

x	-1	0	1	2
$p(x)$	0.3	0.1	0.4	0.2

Hints/Solution: Here X is a discrete RV. $\therefore$

$$\begin{aligned} E(X) &= \sum_{i=-\infty}^{\infty} x_i p(x_i) \\ &= (-1)\times 0.3 + 0\times 0.1 + 1\times 0.4 + 2\times 0.2 \\ &= -0.3 + 0 + 0.4 + 0.4 = \boxed{0.5} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum_{i=-\infty}^{\infty} x_i^2 p(x_i) \\ &= (-1)^2\times 0.3 + 0^2\times 0.1 + 1^2\times 0.4 + 2^2\times 0.2 \\ &= 0.3 + 0 + 0.4 + 0.8 = \boxed{1.5} \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= (1.5) - (0.5)^2 \\ &= 1.5 - 0.25 = \boxed{1.25} \end{aligned}$$

$$\begin{aligned} E(2X+1) &= 2E(X) + 1 \\ &= 2\times(0.5) + 1 \\ &= 1 + 1 = \boxed{2} \end{aligned}$$

$$\begin{aligned} Var(2X+1) &= 2^2 Var(X) \\ &= 4\times(1.25) = \boxed{5} \end{aligned}$$

Example: **9.** *If a random variable* X *has the following probability distribution, find* $E(X)$, $E(X^2)$, $Var(X)$, $E(3X-4)$, $Var(3X-4)$.

x	0	1	2	3	4
$p(x)$	$\frac{1}{25}$	$\frac{3}{25}$	$\frac{5}{25}$	$\frac{7}{25}$	$\frac{9}{25}$

Hints/Solution: Here X is a discrete RV. $\therefore$

$$\begin{aligned} E(X) &= \sum_{i=-\infty}^{\infty} x_i p(x_i) \\ &= \boxed{\frac{14}{5}} \end{aligned}$$

$$
\begin{aligned}
E(X^2) &= \sum_{i=-\infty}^{\infty} x_i^2 p(x_i) \\
&= \boxed{\frac{46}{5}}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{46}{5} - \left(\frac{14}{5}\right)^2 \\
&= \boxed{\frac{34}{25}}
\end{aligned}
$$

$$
\begin{aligned}
E(3X-4) &= 3E(X) - 4 \\
&= = \boxed{\frac{22}{5}}
\end{aligned}
$$

$$
\begin{aligned}
Var(3X-4) &= 3^2 Var(X) \\
&= \boxed{\frac{306}{25}}
\end{aligned}
$$

Example: 10. *A test engineer found that the distribution function of the lifetime X (in years) of an equipment follows is given by*

$$
F(x) = \begin{cases} 0, & \text{when } x < 0 \\ 1 - e^{-\frac{x}{5}}, & \text{when } x \geq 0 \end{cases}
$$

Find the pdf, mean and variance of X.

Hints/Solution: The pdf of X is given by

$$
f(x) = F'(x) = \frac{d}{dx}F = \begin{cases} 0, & \text{when } x < 0 \\ \frac{1}{5}e^{-\frac{x}{5}}, & \text{when } x \geq 0 \end{cases}
$$

$$\text{The Mean } \ E(X) = \int_{-\infty}^{\infty} xf(x)dx$$
$$= \int_{-\infty}^{0} x\frac{1}{5}e^{-\frac{x}{5}}dx$$
$$= -\frac{1}{5}\left[(5x+25)e^{-\frac{x}{5}}\right]_0^{\infty} = \boxed{5}$$

$$Var(X) = E(X^2) - [(X)]^2$$

$$\text{Now } \ E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$
$$= \int_{-\infty}^{0} x^2\frac{1}{5}e^{-\frac{x}{5}}dx$$
$$= -\frac{1}{5}\left[(5x^2+50x+250)e^{-\frac{x}{5}}\right]_0^{\infty} = \boxed{50}$$

$$\therefore \quad Var(X) = 50 - [5]^2 = \boxed{25}$$

Example: 11. *Find the mean and standard deviation of the distribution*

$$f(x) = \begin{cases} kx(2-x), & \textit{when } 0 <\leq x \leq 2 \\ 0, & \textit{otherwise} \end{cases}$$

Hints/Solution: Given that the continuous RV X whose pdf is given by

$$f(x) = \begin{cases} kx(2-x), & \text{when } 0 <\leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Since $f(x)$ is a pdf,

$$\begin{aligned}
\text{we have } \int_{-\infty}^{\infty} f(x)dx &= 1 \\
i.e. \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx &= 1 \\
i.e. \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{2} kx(2-x)\,dx + \int_{2}^{\infty} 0 \cdot dx &= 1 \\
i.e.\ 0 + k\left[x^2 - \frac{x^3}{3}\right]_0^2 + 0 &= 1 \\
\implies \boxed{k = \frac{3}{4}}
\end{aligned}$$

$$\begin{aligned}
\text{The Mean } E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-\infty}^{0} x\frac{3}{4}x(2-x)dx \\
&= -\frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \boxed{1}
\end{aligned}$$

$$Var(X) = E(X^2) - [(X)]^2$$

$$\begin{aligned}
\text{Now } E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_{-\infty}^{0} x^2\frac{3}{4}x(2-x)dx \\
&= -\frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \boxed{\frac{6}{5}}
\end{aligned}$$

$$\therefore \quad Var(X) = \frac{6}{5} - [1]^2 = \boxed{\frac{1}{5}} \quad \implies \quad S.D. = \sqrt{\frac{1}{5}}$$

Example: 12. *If X has the distribution function*

$$F(x) = \begin{cases} 0, & \textit{when } x < 1 \\ \frac{1}{3}, & \textit{when } 1 \le x < 4 \\ \frac{1}{2}, & \textit{when } 4 \le x < 6 \\ \frac{5}{6}, & \textit{when } 6 \le x < 10 \\ 1, & \textit{when } x \ge 10 \end{cases}$$

Find (a) the probability distribution of X (b) Find the mean and variance of X.

Hints/Solution: Given that the CDF of X is

$$F(x) = \begin{cases} 0, & \text{when } x < 1 \\ \frac{1}{3}, & \text{when } 1 \le x < 4 \\ \frac{1}{2}, & \text{when } 4 \le x < 6 \\ \frac{5}{6}, & \text{when } 6 \le x < 10 \\ 1, & \text{when } x \ge 10 \end{cases}$$

We know that $P(X = x_i) = F(x_i) - F(x_{i-1}), \quad i = 1, 2, 3, \ldots,$ where F is constant in $x_{i-1} \le x \le x_i$.

The CDF changes its values at $x = 1, 4, 6, 10$. $\therefore$ The probability distribution takes its values as follows:

$$\begin{aligned} P(X = 1) &= F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3} \\ P(X = 4) &= F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\ P(X = 6) &= F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3} \\ P(X = 10) &= F(10) - F(6) = 1 - \frac{5}{6} = \frac{1}{6} \end{aligned}$$

$\therefore$ The probability distribution is given by

x	1	4	6	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

Mean=$E(X) = \Sigma x p(x) = \dfrac{14}{3}$.

$E(X^2) = \Sigma x^2 p(x) = \dfrac{95}{3}$.

$Var(X) = E(X^2) - [E(X)]^2 = \dfrac{285}{9} - \dfrac{196}{9} = \dfrac{89}{9}$.

Example: 13. *If X has the distribution function*

$$F(x) = \begin{cases} 0, & \text{when } x < 0 \\ \frac{1}{6}, & \text{when } 0 \leq x < 2 \\ \frac{1}{2}, & \text{when } 2 \leq x < 4 \\ \frac{5}{8}, & \text{when } 4 \leq x < 6 \\ 1, & \text{when } x \geq 6 \end{cases}$$

Find (a) the probability distribution of X (b) Find the mean and variance of X.

Hints/Solution: Given that the CDF of X is

$$F(x) = \begin{cases} 0, & \text{when } x < 0 \\ \frac{1}{6}, & \text{when } 0 \leq x < 2 \\ \frac{1}{2}, & \text{when } 2 \leq x < 4 \\ \frac{5}{8}, & \text{when } 4 \leq x < 6 \\ 1, & \text{when } x \geq 6 \end{cases}$$

We know that $P(X = x_i) = F(x_i) - F(x_{i-1}), \quad i = 1, 2, 3, \ldots,$ where F is constant in $x_{i-1} \leq x \leq x_i$.

The CDF changes its values at $x = 1, 4, 6, 10$. $\therefore$ The probability distribution takes its values as follows:

$$\begin{aligned}
P(X=0) &= \frac{1}{6} \\
P(X=2) &= F(2)-F(0)=\frac{1}{2}-\frac{1}{6}=\frac{1}{3} \\
P(X=4) &= F(4)-F(2)=\frac{5}{8}-\frac{1}{2}=\frac{1}{8} \\
P(X=6) &= F(6)-F(4)=1-\frac{5}{8}=\frac{3}{8}
\end{aligned}$$

$\therefore$ The probability distribution is given by

x	0	2	4	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{3}{8}$

Mean=$E(X) = \Sigma xp(x) = \frac{37}{12}$.
$E(X^2) = \Sigma x^2 p(x) = \frac{101}{6}$.
$Var(X) = E(X^2) - [E(X)]^2 = \frac{101}{6} - \left[\frac{37}{12}\right]^2 = 16.83 - 9.51 = 7.32.$

Example: 14. *The first four moments about $X = 5$ are 2,5,12 and 48. Find the first four central moment.*

Hints/Solution: Let m'_1, m'_2, m'_3, m'_4 be the first four moments about $X = 5$. Given $m'_1 = 1, m'_2 = 5, m'_3 = 12, m'_4 = 48$ be the first four moments about $X = 5$ $m'_1 = E(X-5) = 1 \implies E(X) - 5 = 1 \implies E(X) = \mu_1 = 6$

$$\begin{aligned}
\mu_2 &= m'_2 - (m'_1)^2 = 5 - 1 = 4 \\
\mu_3 &= m'_3 - 3m'_2 \cdot m'_1 + 2(m'_1)^3 \\
&= 12 - 3 \times 5 \times 1 + 2(1) = -1 \\
\mu_4 &= m'_4 - 4m'_3 \cdot m'_1 + 6m'_2 \cdot (m'_1)^2 - 3(m'_1)^4 \\
&= 48 - 4(12)(1) + 6(5)(1) - 3(1) = 27
\end{aligned}$$

Example: 15. *The first four moments about $x = 4$ are 1,4,10 and 45. Find the first four moments about the mean.*

Hints/Solution:

Let m_1', m_2', m_3', m_4' be the first four moments about $X = 4$. Given $m_1' = 1, m_2' = 4, m_3' = 10, m_4' = 45$ be the first four moments about $X = 4$ $m_1' = E(X-4) = 1 \implies E(X) - 4 = 1 \implies E(X) = \mu_1 = 5$

$$\begin{aligned}
\mu_2 &= m_2' - (m_1')^2 = 4 - 1 = 3 \\
\mu_3 &= m_3' - 3m_2' \cdot m_1' + 2(m_1')^3 \\
&= 10 - 3 \times 4 \times 1 + 2(1) = 0 \\
\mu_4 &= m_4' - 4m_3' \cdot m_1' + 6m_2' \cdot (m_1')^2 - 3(m_1')^4 \\
&= 45 - 4(10)(1) + 6(4)(1) - 3(1) = 26
\end{aligned}$$

Example: 16. *A random variable X has the pdf $f(x) = kx^2e^{-x}$, $x \geq 0$. Find the r^{th} moment and hence find the first four moments.*

Hints/Solution: We know that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1 \\
\text{i.e., } \int_0^{\infty} kx^2e^{-x}\,dx &= 1 \\
\text{i.e., } k\int_0^{\infty} e^{-x}x^{3-1}\,dx &= 1 \\
\text{i.e., } k\Gamma(3) &= 1 \qquad \left[\because \Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}\,dx\right] \\
\text{i.e., } k \cdot 2! &= 1 \implies k = \frac{1}{2}. \qquad [\because\ \Gamma(n) = (n-1)!]
\end{aligned}$$

Now, The r^{th} moment is given by

$$\begin{aligned}
\mu'_r &= E(X^r) \\
&= \int_{-\infty}^{\infty} x^r f(x)\,dx \\
&= \int_{0}^{\infty} x^r k x^2 e^{-x}\,dx \\
&= k\int_{0}^{\infty} e^{-x} x^{r+3-1}\,dx \\
&= \frac{1}{2}\Gamma(r+3) = \frac{1}{2}(r+2)!
\end{aligned}$$

$$\begin{aligned}
\text{Now,}\quad \text{First Moment}\quad \mu'_1 &= \frac{1}{2}(3)! = 3 \\
\text{Second Moment}\quad \mu'_2 &= \frac{1}{2}(4)! = 12 \\
\text{Third Moment}\quad \mu'_3 &= \frac{1}{2}(5)! = 60 \\
\text{Fourth Moment}\quad \mu'_4 &= \frac{1}{2}(6)! = 360
\end{aligned}$$

Example: 17. *A random variable X has the pdf $f(x) = \frac{1}{2}e^{-\frac{x}{2}}, \quad x \geq 0$. Find the MGF(Moment Generating Function) and hence find its mean and variance.*

Hints/Solution: The MGF of X is given by

$$
\begin{aligned}
M_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\
&= \frac{1}{2}\int_{0}^{\infty} e^{tx} e^{-\frac{x}{2}}\, dx \\
&= \frac{1}{2}\int_{0}^{\infty} e^{-\frac{1}{2}(1-2t)x}\, dx \\
&= \frac{1}{2}\left[\frac{e^{-\frac{1}{2}(1-2t)x}}{-\frac{1}{2}(1-2t)}\right]_0^{\infty} \\
\therefore\ M_X(t) &= \frac{1}{1-2t} \text{ if } t < \frac{1}{2}. \\
&= (1-2t)^{-1} = 1 + 2t + 4t^2 + 8t^3 + \ldots
\end{aligned}
$$

Now, Differentiating w.r.to t, we get

$$
\begin{aligned}
M'_X(t) &= 2 + 8t + 24t^2 + \ldots \\
M''_X(t) &= 8 + 48t + \ldots \\
\text{Now, First Moment=Mean= } E(X) = \mu'_1 &= M'_X(0) = 2 \\
\text{Second Moment= } E(X^2) = \mu'_2 &= M''_X(0) = 8 \\
\therefore\ Var(X) &= E(X^2) - [E(X)]^2 = 8 - 4 = 4.
\end{aligned}
$$

Example: 18. *A random variable* X *has the pdf* $p(x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$. *Find the MGF(Moment Generating Function) and hence find its mean and variance.*

Hints/Solution: The MGF of X is given by

$$\begin{aligned}
M_X(t) = E(e^{tX}) &= \sum_{x=-\infty}^{\infty} e^{tx} p(x) \\
&= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x \\
&= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \dots\right] \\
&= \frac{e^t}{2}\left[1 - \left(\frac{e^t}{2}\right)\right]^{-1} \\
\therefore\ M_X(t) &= \frac{e^t}{2-e^t} \text{ if } e^t \neq 2.
\end{aligned}$$

Now, Differentiating w.r.to t, we get

$$\begin{aligned}
M'_X(t) &= \frac{2e^t}{(2-e^t)^2} \\
M''_X(t) &= \frac{2[(2-e^t)e^t + 2e^{2t}]}{(2-e^t)^3}
\end{aligned}$$

$$\begin{aligned}
\text{Now, First Moment=Mean= } E(X) = \mu'_1 &= M'_X(0) = 2 \\
\text{Second Moment= } E(X^2) = \mu'_2 &= M''_X(0) = 6 \\
\therefore\ Var(X) &= E(X^2) - [E(X)]^2 = 6 - 4 = 2.
\end{aligned}$$

Example: 19. *A random variable X has the r^{th} moment of the form $\mu'_r = (r+1)!2^r$. Find the MGF(Moment Generating Function) and hence find its mean and variance.*

Hints/Solution: The MGF of X is given by

$$\begin{aligned}
\text{Given } \mu_r' &= (r+1)!2^r \\
\therefore \mu_1' &= 2!2 \\
\mu_2' &= 3!2^2 \\
\mu_3' &= 4!2^3 \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\therefore M_X(t) = E(e^{tX}) &= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \ldots \\
&= 1 + \frac{t}{1!}2 + \frac{t^2}{2!}3!2^2 + \frac{t^3}{3!}4!2^3 + \ldots \\
&= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \ldots \\
\therefore M_X(t) &= (1-2t)^{-2}.
\end{aligned}$$

Now, Differentiating w.r.to t, we get

$$\begin{aligned}
M_X'(t) &= -2(1-2t)^{-3}(-2) \\
M_X''(t) &= 6(1-2t)^{-4}(-2)^2 \\
\text{Now, First Moment=Mean= } E(X) = \mu_1' &= M_X'(0) = 4 \\
\text{Second Moment= } E(X^2) = \mu_2' &= M_X''(0) = 24 \\
\therefore Var(X) &= E(X^2) - [E(X)]^2 = 24 - 16 = 8.
\end{aligned}$$

Example: 20. *A random variable X takes the values -1,0,1 with probabilities $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluate $P\{|X-\mu| \geq 2\sigma\}$ and compare it with the upper bound given by Tchebycheff's(Chebychev's) inequality.*

Hints/Solution:

$$\begin{aligned} E(X) &= 0 \\ E(X^2) &= \frac{1}{4} \\ Var(X) &= \frac{1}{4} \\ \therefore\ P\{|X-\mu| \geq 2\sigma\} &= P\{|X| \geq 1\} \\ &= P\{X = -1 \text{ or } X = 1\} = \frac{1}{4} \\ \text{By Chebychev's inequality} & \\ P\{|X-\mu| \geq c\} &\leq \frac{\sigma^2}{c^2} \\ \text{Choosing } c = 2\sigma & \\ P\{|X-\mu| \geq 2\sigma\} &\leq \frac{1}{4} \end{aligned}$$

In this problem both the values coincide.

Note: Alternative form of Chebychev's inequality: If we put $c = k\sigma$, where $k > 0$, then it takes the form

$$P\left\{\frac{|X-\mu|}{k} \geq \sigma\right\} \leq \frac{1}{k^2}$$

$$P\left\{\frac{|X-\mu|}{k} \leq \sigma\right\} \geq 1 - \frac{1}{k^2}$$

Example: 21. *A fair dice is tossed 720 times. Use Tchebycheff's(Chebychev's) inequality to find a lower bound for the probability of getting 100 to 140 sixes.*

Hints/Solution:

$$\begin{aligned} p &= P\{\text{getting '6' in a single toss}\} = \frac{1}{6} \\ q &= 1 - \frac{1}{6} = \frac{5}{6} \text{ and} \\ n &= 720 \end{aligned}$$

X follows binomial distribution with mean $np = 120$ and variance $npq = 100$. *i.e.* $\mu = 120$ and $\sigma = 100$.

By Alternate form of Chebychev's inequality

$$
\begin{aligned}
P\{|X-\mu| \leq k\sigma\} &\geq 1-\frac{1}{k^2} \\
P\{|X-120| \leq 10k\} &\geq 1-\frac{1}{k^2} \\
P\{120-10k \leq X \leq 120+10k\} &\geq 1-\frac{1}{k^2} \\
\text{Choosing } k=2, &\quad \text{we get,} \\
P\{100 \leq X \leq 140\} &\geq \frac{3}{4}
\end{aligned}
$$

$\therefore$ Required lower bound is $\frac{3}{4}$.

5.1 Problems based on one dimensional Functions of Random Variables

Example: 22. *Let X be a random variable with pdf $f(x) = \frac{2}{9}(x+1),\ -1 < x < 2$. Find the pdf of the random variable $Y = X^2$.*

Hints/Solution:
Divide the interval into two parts (-1,1) and (1,2).

$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x)$

When $-1 < x < 1,\ \ f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1+\sqrt{y}) + \frac{2}{9}(1-\sqrt{y})\right] = \frac{2}{9\sqrt{y}},\ \ 0 < y < 1.$

When $1 < x < 2,\ \ f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1+\sqrt{y})\right],\ \ 1 < y < 4.$

Example: 23. *Let X be a random variable with pdf $f(x) = \frac{x}{12},\ 1 < x < 5$. Find the pdf of the random variable $Y = 2X - 3$.*

Hints/Solution:

$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x) = \frac{1}{2}\cdot\frac{x}{12}$

When $1 < x < 5,\ \ f_Y(y) = \frac{y+3}{48},\ \ -1 < y < 7.$

6 Exercise/Practice/Assignment Problems

1. Find the probability of drawing two red balls in succession from a bag containing 3 red and 6 black balls when (i) the ball that is drawn first is replaced, (ii) it is not replaced.

2. A bag contains 3 red and 4 white balls. Two draws are made without replacement; what is the probability that both the balls are red.

3. A random variable X has the following distribution

X	-2	-1	0	1	2	3
P[X=x]	0.1	k	0.2	$2k$	0.3	$3k$

Find (i) the value of k,(ii) the Distribution Function (CDF) (iii) $P(0 < X < 3/X < 2)$ and (iv) the smallest value of α for which $P(X \le \alpha) > \frac{1}{2}$.

Ans: $k = \frac{1}{15}$

4. A random variable X has the following distribution

X	0	1	2	3	4
P[X=x]	k	$2k$	$5k$	$7k$	$9k$

Find (i) the value of k,(ii) the Distribution Function (CDF) (iii) $P(0 < X < 3/X < 2)$ and (iv) the smallest value of α for which $P(X \le \alpha) > \frac{1}{3}$.

Ans: $k = \frac{1}{24}$

5. A random variable X has a pdf

$$f(x) = \begin{cases} 3x^2, & \text{when } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) Find a, b such that $P(X \le a) = P(X > a)$ and $P(X > b) = 0.05$,(ii) the Distribution Function (CDF) (iii) $P(0 < X < 0.5/X < 0.8)$. Ans: $a = \sqrt[3]{\frac{1}{2}}$, $b = \sqrt[3]{\frac{19}{20}}$

6. The amount of bread (in hundred kgs) that a certain bakery is to sell in a day is a random variable X with a pdf

$$f(x) = \begin{cases} Ax, & \text{when } 0 \le x < 5 \\ A(10 - x), & \text{when } 5 \le x < 10 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the value of A,(ii) the Distribution Function (CDF) (iii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$. (iv) the probability that in a day the sales is (a) more than 500 kgs (b) less than 500 kgs (c) between 250 and 750 kgs. Ans: $A = \frac{1}{25}$

7. The cumulative distribution function (CDF) of a random variable X is given by

$$F(x) = \begin{cases} 1 - \dfrac{4}{x^2}, & \text{when } x > 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the pdf of X,(ii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$ (iii) $P(X < 3)$, $P(3 < X < 5)$.

8. A coin is tossed until a head appears. What is the expected value of the number of tosses?. Also find its variance.

9. The pdf of a random variable X is given by

$$f(x) = \begin{cases} a + bx, & \text{when } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the value of a, b if the mean is 1/2,(ii) the variance of X (iii) $P(X > 0.5/X < 0.5)$

10. The first three moments about the origin are 5,26,78. Find the first three moments about the value x=3. Ans: 2,5,-48

11. The first two moments about x=3 are 1 and 8. Find the mean and variance. Ans: 4,7

12. The pdf of a random variable X is given by

$$f(x) = \begin{cases} k(1 - x), & \text{when } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the the value of k,(ii) the r^{th} moment about origin (iii) mean and variance. Ans: $k = 2$

13. An unbiased coin is tossed three times. If X denotes the number of heads appear, find the MGF of X and hence find the mean and variance.

14. Find the MGF of the distribution whose pdf is $f(x) = ke^{-x}$, $x > 0$ and hence find its mean and variance.

15. The pdf of a random variable X is given by

$$f(x) = \begin{cases} x, & \text{when } 0 \leq x \leq 1 \\ 2 - x, & \text{when } 1 < x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

For this find the MGF and prove that mean and variance cannot be find using this MGF and then find its mean and variance using expectation.

16. The pdf of a random variable X is given by $f(x) = ke^{-|x|}$, $-\infty < x < \infty$ Find the (i) the the value of k,(ii) the r^{th} moment about origin (iii) the MGF and hence mean and variance. Ans: $k = \dfrac{1}{2}$

17. Find the MGF of the RV whose moments are given by $\mu'_r = (2r)!$. Find also its mean and variance.

18. Use Tchebycheff's(Chebychev's) inequality to find how many times a fair coin must be tossed in order to the probability that the ratio of the number of heads to the number of tosses will lie between 0.45 and 0.55.

19. If X denotes the sum of the numbers obtained when 2 dice are thrown, obtain an upper bound for $P\{|X - 7| \geq 4\}$ (Use Chebychev's inequality). Compare with the exact probability.

SOLVE MORE PROBLEMS FROM THE TEXTBOOK AND REFERENCE BOOKS

VALUES OF $e^{-\lambda}$										
λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$e^{-\lambda}$	1.000	.905	.819	.741	.670	.607	.549	.497	.449	.407
λ	1	2	3	4	5	6	7	8	9	10
$e^{-\lambda}$	.368	.135	.0498	.0183	.00674	.00248	.000912	.000335	.000123	.000045

Figure 6.1: Values of $e^{-\lambda}$.

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