Unit - II

PROBABILITY DISTRIBUTIONS

→ probability distributions

Discrete distribution → Binomial, poisson, geometric } part c

Continous Distribution → Exponential, Uniform } Part c (or) GM, Normal

* poisson called limiting case of Binomial

Binomial Distribution:

The Pmf is $\boxed{P(X=x) = nC_x p^x q^{n-x}, \quad x=0,1,2,\dots n}$

$n \to$ no. of trials

parameters: n & p

p = probability of success

q = probability of failure

$\boxed{p+q=1}$, $q = 1-p$, $0 \le p \le 1$, $0 \le q \le 1$

(1) Find MGF, Mean & Variance of B.D

The pmf of BD is

$$P(X=x) = nC_x p^x q^{n-x}, \quad x=0,1,2,\dots n$$

MGF:

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} [nC_x p^x q^{n-x}]$$

$$= q^n \left[\sum_{x=0}^{n} nC_x \left(\frac{pe^t}{q}\right)^x \right]$$

$$= q^n \left[nC_0 \left(\frac{pe^t}{q}\right)^0 + nC_1 \left(\frac{pe^t}{q}\right) + \dots nC_n \left(\frac{pe^t}{q}\right)^n \right]$$

$$= q^n \left[1 + nC_1 (pe^t/q) + \dots nC_n \left(\frac{pe^t}{q}\right)^n \right]$$

$$= q^n \left[1 + \frac{pe^t}{q}\right]^n$$

$$= q^n \left[\frac{q + pe^t}{q}\right]^n$$

$$= \cancel{q^n} \frac{(q + pe^t)^n}{\cancel{q^n}}$$

$$= (q + pe^t)^n$$

$$\boxed{M_x(t) = (q + pe^t)^n}$$

<u>Mean</u>: $E(x) = \mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$

$$= \left[\frac{d}{dt} [q + pe^t]^n\right]_{t=0}$$

$$= \left[n(q + pe^t)^{n-1} pe^t\right]_{t=0}$$

$$= n[q + p]^{n-1} p$$

$$= np$$

$$\boxed{\text{Mean} = np}$$

<u>Variance</u>: $V(x) = \mu_2' - \mu_1'^2$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(n(pe^t + q)^{n-1} pe^t\right)\right]_{t=0}$$

$$= np \left[\frac{d}{dt} e^t (q + pe^t)^{n-1}\right]_{t=0}$$

$$= np\left[e^t (q + pe^t)^{n-1} + e^t (n-1)(q + pe^t)^{n-2} pe^t\right]_{t=0}$$

Scanned by CamScanner

$$= np[(q+p)+(n-1)(q+p)p]$$

$$= np[1+(n-1)p]$$

$$= np[1+np-p]$$

$$= np+n^2p^2-np^2$$

$$= np[1-p]+n^2p^2$$

$$\mu_2' = npq+n^2p^2$$

$$V(X) = n^2p^2+npq-(np)^2$$

$$= npq \text{ //}$$

$$\boxed{V(X) = npq}$$

Problems On BD :

(1) Out of 800 families with 4 childrean each, How many families would be expected to have 2boys, 2 girls atleast 1 boy, atmost 2 girls.

(i) 2 Boys, 2 girls

ii) atleast 1 boy

(iii) atmost 2 girls

(iv) children of both gender

Assume equal prob. of girls & boys.

Sol: Let X be a RV denoting the family having a boy

$$p = 1/2,\ q = 1/2,\ n = 4$$

The pmf,

$$P(X=x) = nC_x p^x q^{n-x}, \quad x = 0,1,2 \ldots n$$

$$P(X=x) = 4C_x(1/2)^x(1/2)^{4-x}, \quad x = 0,1,2,3,4$$

$$\boxed{P(X=x) = 4C_x(1/16)} \quad x = 0,1,2,3,4$$

Scanned by CamScanner

(i) $P(X=2) = \frac{1}{16}\, 4C_2$

$= \frac{1}{16} \cdot \frac{4!}{2(2)!}$

$= \frac{1}{16} \cdot \frac{\cancel{4}^{2} \times 3 \times \cancel{2}}{\cancel{2} \times \cancel{2}}$

$\Rightarrow \frac{6}{16} = \frac{3}{8}$

$\left.\begin{array}{l}\text{No of families with} \\ \text{2 boys \& 2girls}\end{array}\right\} = 800 \times \frac{3}{8}$

$= 300$

(ii) $P(X = \text{atleast 1 boy}) = P(X \geq 1)$

$= 1 - P(X < 1)$

$= 1 - P(X=0)$

$= 1 - \frac{1}{16}\, 4C_0$

$= 1 - \frac{1}{16}$

$= \frac{15}{16}$

$\left.\begin{array}{l}\text{No of families with} \\ \text{atleast 1 boy}\end{array}\right\} = \frac{15}{\cancel{16}} \times \cancel{800}^{50}$

$= 750$

(iii) atmost 2 girls $\rightarrow P(X=4)$

$= \frac{1}{16}\, 4C_4 + \frac{1}{16}\, 4C_3 + \frac{1}{16}\, 4C_2$

$= \frac{1}{16} \cdot \left(\frac{4!}{4(0)!} \right.$

$= \frac{1}{16} \left[4C_4 + 4C_3 + 4C_2 \right]$

$= \frac{1}{16}[1+4+6]$

$= \frac{1}{16}(11)$ $\Rightarrow$ $800 \times \frac{11}{16} = 550$ (with 800 and 16 cancelled to 50)

$= \frac{11}{16}$.

No of families with atmost 2 girls $\}$ = 550

(iv) P(both gender)

$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$ (with $+P(X=4)$ struck through)

$= \frac{1}{16}[4C_1 + 4C_2 + 4C_3]$

$= \frac{1}{16}[4+6+4]$

$\Rightarrow \frac{14}{16} = \frac{7}{8}$

No of families with both gender $\}$ $= 800 \times \frac{7}{8}$

$= 700$

→ Six dice are thrown 729 times. How many times can you except atleast three dice to show 5 or 6?

The Pmf is

$P(X=x) = nCx\, p^x q^{n-x}, \quad x=0,1,2 \dots n$

$p = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$ (for getting 5 or 6)

$q = 1-p = 1-1/3 = 2/3$

$n=6$

$P(X=x) = 6Cx \left(\frac{1}{3}\right)^x (2/3)^{6-x} \quad x=0,1,2,\dots 6$

$P(x \geq 3) = 1 - P(X<3)$

$= 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - [6C_0 (1/3)^0 (2/3)^6 + 6C_1 (1/3)(2/3)^5 + 6C_2 (1/3)^2 (2/3)^4]$

$= 1 - [1 \times \frac{64}{729} + 6(1/3)\left(\frac{32}{243}\right) + \frac{240}{729}]$

$= 1 - \frac{496}{729} = \frac{233}{729}$

No. of times getting 5 or 6 in 729 trials $\Big\} = 729 \times \frac{233}{729}$

$= 233$ times

→ In a BD, mean = 6 and SD = 2. find the first two terms of BD

Mean = $np = 6$

Variance = $npq = 2^2 = 4$

$SD = \sqrt{Var}$

$Var = (SD)^2$

$npq = 4$

$np = 6$

$\frac{npq}{np} = \frac{4}{6} = \frac{2}{3}$

$\boxed{q = 2/3 , \ p = 1/3}$

$np = 6$

$n(1/3) = 6$

$\boxed{n = 18}$

The pmf is

$P(X=x) = 18C_x (1/3)^x (2/3)^{18-x}, \ x = 0, 1, 2, \dots 18$

$P(X=0) = 18C_0 (1/3)^0 (2/3)^{18} = (2/3)^{18}$

$P(X=1) = 18C_1 \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{17} = 6 \, (2/3)^{17}$

→ Fit a BD for the following data

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80
f(x)	0	18	56	36	28	30	24	$\Sigma f(x) = 192$

Mean $= \frac{\Sigma f(x)}{\Sigma f} = \frac{192}{80} = 2.4$

$np = 2.4$

$n = 6$

$np = 2.4$

$n = 0.4$

$p = 0.4$, $q = 0.6$, $n = 6$

The pmf is $P(X=x) = nC_x p^x q^{n-x}$, $x = 0, 1, 2 \ldots n$

$P(X=x) = 6C_x (0.4)^x (0.6)^{6-x}$, $x = 0, 1, 2 \ldots 6$

Expected frequencies

$NP(X=x) = 80P(X=x)$

X	$P(X=x) = 6C_x(0.4)^x(0.6)^{6-x}$	Expected frequencies $80P(X=x)$
0	$6C_0(0.6)^6 = 0.0456$	$80 \times 0.46 = 3.68$
1	$6C_1(0.4)^1(0.6)^5 = 0.18$	14.4
2	$6C_2(0.4)^2(0.6)^4 = 0.31$	24.88
3	$6C_3(0.4)^3(0.6)^3 = 0.276$	22.08
4	$6C_4(0.4)^4(0.6)^2 = 0.138$	11.04
5	$6C_5(0.4)^5(0.6) = 0.37$	2.96
6	$6C_6(0.4)^6 = 0.004$	0.32
	Total	$79.2 \simeq 80$

Hence B.D is the best fit.

→ <u>Poisson Distribution</u> (Limitting case of BD)

The Pmf is

$$\boxed{P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, 3 \ldots}$$

parameter → λ

$\boxed{\lambda = np}$ where n is large
p is small

Mean = Variance = λ

NOTE :

Additive property : $X_1 + X_2$ follows P.D with $\lambda_1 + \lambda_2$

But $X_1 - X_2$ do not follow PD with $\lambda_1 - \lambda_2$

(1) Find MGF, mean & Variance of PD

Sol : The Pmf is

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,2 \ldots$$

MGF : $M_x(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$

$$= \sum_{x=0}^{\infty} e^{tx}\left(\frac{e^{-\lambda}\lambda^x}{x!}\right)$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right]$$

$$= e^{-\lambda}\left[e^{\lambda e^t}\right] \qquad \left[\because 1 + \frac{x}{1!} + \frac{x^2}{2!} = e^x\right]$$

$$\boxed{M_x(t) = e^{\lambda[e^t - 1]}}$$

Mean : $\mu_1' = \frac{d}{dt}\left[M_x(t)\right]_{t=0}$

$$= \left(\frac{d}{dt}\left[e^{\lambda(e^t - 1)}\right]\right)_{t=0}$$

$$= \left[e^{\lambda(e^t-1)} \cdot \lambda e^t\right]_{t=0}$$

$$= \left[e^{\lambda(1-1)} \cdot \lambda e^0\right]$$

$$= \lambda$$

$$\boxed{\text{Mean} = \lambda}$$

Variance :

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$\mu_2' = \left[\frac{d^2}{dt^2}\left[e^{\lambda(e^t-1)}\right]\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[e^{\lambda(e^t-1)} \lambda e^t\right]\right\}_{t=0}$$

$$= \lambda\left[e^{\lambda(e^t-1)} \cdot \lambda e^t e^t + e^{\lambda(e^t-1)} \cdot e^t\right]_{t=0}$$

$$= \lambda[\lambda+1]$$

$$= \lambda^2+\lambda$$

$$\text{Variance} = \mu_2' - \mu_1'^2$$

$$= \lambda^2+\lambda-(\lambda)^2$$

$$= \lambda$$

$$\boxed{\text{Variance } (V(x)) = \lambda}$$

→ The no of monthly ~~ta~~break down of a computer follows. PD with mean 1.8. Find the probability that the computer will function for a month

(i) without breaking

(ii) with only one break down

(iii) with atleast one

Sol : Let x be the no. of monthly breakdowns of a computer

Given,

$$\lambda = \text{mean} = 1.8$$

Pmf is $$P(X=x) = \frac{e^{-1.8} \cdot (1.8)^x}{x!}$$

(i) $$P(X=0) = \frac{e^{-1.8} \cdot (1.8)^0}{0!} = e^{-1.8} = 0.1653$$

(ii) $P(x=1) = \dfrac{e^{1.8}.(1.8)^1}{1!} = e^{1.8}(1.8) = 0.2975$

(iii) $P(X \geq 1) = 1 - P(X<1)$

$= 1 - 0.1653$

$= 0.8347$

→ ~~An~~ travel companies has 2 cars for hiring. The demand for a car on each day is distributed as poisson variable with param 1.5. Calculate the proportion of the day on which

(i) neither car used

(ii) Some demand is refused

Sol: let X be the no. of cars for demand

$\lambda = 1.5$

$$P(X=x) = \frac{e^{-1.5}.(1.5)^x}{x!}$$

(i) $P(X=0) = e^{-1.5} = 0.2$

(ii) $P(X \geq 2) = 1 - P(X \leq 2)$

$= 1 - P(X=0) + P(X=1) + P(X=2)$

$= 1 - [0.2 + \dfrac{e^{-1.5}(1.5)}{1!} + \dfrac{(0.2)(1.5)^2}{2}]$

$= 0.8 + (1.5)e^{-1.5} + \dfrac{(0.2)(2.25)}{2}$

$= 0.8 + (1.5)(0.2) + \dfrac{1.0125}{2}$

$= 0.8 + 0.3 + 0.50$

$= 1.1 + 0.5$

$= 1.6$

→ A manufacture produces IC chips out of which 1% are of defective. Find the probability that ~~good~~ defective we ~~count~~ no box contains 100 chips

$p = 1\% = 0.01, \ n = 100$

$\lambda = np$

$\lambda = 100 \times (0.01)$

$= 1$

$$\text{pmf} = \frac{e^{-1}(1)^x}{x!} = \frac{e^{-1}}{x!}, \ x = 0, 1, 2 \cdots$$

$$P(X=0) = \frac{e^{-1}}{1} = e^{-1}$$

$= 0.3679$

→ X follows PD with $P(X=2) = 9P(X=4) + 90P(X=6)$. find mean & Variance.

Sol: $P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}, \ x = 0, 1, 2 \cdots$

given, $P(X=2) = 9P(X=4) + 90P(X=6)$

$$\frac{e^{-2}(2)^x}{x!} = 9\left(\frac{e^{-4}\cdot(4)^x}{x!}\right) + 90\frac{e^{-6}\cdot(6)^x}{x!}$$

$$\Rightarrow \frac{e^{-\lambda}\cdot(\lambda)^2}{2!} = 9\left(\frac{e^{-\lambda}\cdot(\lambda)^4}{4!}\right) + 90\cdot\frac{e^{-\lambda}(\lambda)^6}{6!}$$

$\div \lambda^2$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$3\lambda^2 + \lambda^4 - 4 = 0$

$\cancel{3\lambda^2} - \cancel{3\lambda} + \cancel{4\lambda} \cancel{4}$

$\lambda^4 + 3\lambda^2 - 4 = 0$

$(\lambda^2 - 1)(\lambda^2 + 4) = 0$

$\lambda^2 - 1 = 0 \quad , \quad \lambda^2 + 4 = 0$

$\lambda = \pm 1 \quad , \quad \lambda = \pm 2i$

$\lambda \rightarrow$ is always +ve & real $\Rightarrow \boxed{\lambda = 1}$

Mean = Variance = 1

→ Fit a PD to the following Data

x	0	1	2	3	4	Total
f	43	38	22	9	1	113

sol: Mean $= \lambda = \dfrac{\Sigma f x}{\Sigma f}$

$\Sigma f(x) = 0 + 38 + 44 + 27 + 4 = 113$

$\lambda = \dfrac{113}{113}$

$\boxed{\lambda = 1}$

Pmf is $P(X = x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0, 1, 2 \ldots$

$= \dfrac{e^{-1}}{x!}, \quad x = 0, 1, 2, 4$

Fitting PD

X	$P(X = x) = \dfrac{e^{-1}}{x!}$	Expected frequency $NP(x) = 113 P(x)$
0	$\dfrac{e^{-1}}{0!} = e^{-1} = 0.3678$	41.5614
1	$e^{-1}/1! = e^{-1} = 0.3678$	41.5614
2	$e^{-2}/2! = 0.1839$	20.7807
3	$e^{-3}/3! = 0.0613$	6.9269
4	$e^{-4}/4! = 0.0152$	1.7176

Total = 112.548 $\simeq$ 113

Geometric Distribution: (first success)

The Pmf is

$$P(X=x) = q^{x-1}p, \quad x=1,2,3\ldots$$

(or)

$$P(X=x) = q^{x}p, \quad x=0,1,2\ldots$$

Parameter is p

→ find MGF, Mean and Variance of GD

Sol: Pmf is $P(X=x)=q^{x-1}p, \; x=1,2,3\ldots$

$$M_x(t) = E(e^{tx})$$

$$= \sum e^{tx}\, p(x)$$

$$= \sum_{1}^{\infty} e^{tx}(q^{x-1}p)$$

$$= \frac{p}{q}\left[\sum_{1}^{\infty}(qe^t)^x\right]$$

$$= \frac{p}{q}\left[qe^t + (qe^t)^2 + \ldots\right]$$

$$= \frac{p}{q}[qe^t]\left[1+qe^t+(qe^t)^2+\ldots\right]$$

$$= pe^t\left[1-qe^t\right]^{-1}$$

$$\boxed{M_x(t) = \frac{pe^t}{1-qe^t}}$$

Mean:

$$\mu_1' = \left[\frac{d}{dt}M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\frac{pe^t}{1-qe^t}\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}\right]_{t=0}$$

$$= \frac{p}{(1-q)^2}$$

$$= p/p^2$$

$$= 1/p$$

$$\boxed{\text{Mean} = \mu_1' = 1/p}$$

<u>Variance</u> : $\mu_2' = \left(\frac{d^2}{dt^2} M_x(t)\right)$

$$= \left(\frac{d}{dt}\left(\frac{pe^t}{(1-qe^t)^2}\right)\right)_{t=0}$$

$$= \left(\frac{(1-qe^t)^2 pe^t - pe^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right)_{t=0}$$

$$= \frac{(1-q)^2 p + 2p(1-q)(-q)}{(1-q)^4}$$

$$= \frac{(p^2)p + 2p(p)(+q)}{(1-q)^4}$$

$$= \frac{p^3 + 2p^2 q}{(p)^4}$$

$$= \frac{p^2[p+2q]}{p^4}$$

$$= \frac{p+2q}{p^2}$$

Now, $\mu_2 = \left(\frac{p+2q}{p^2}\right) - \frac{1}{p^2}$

$$\Rightarrow p+2q-1/p^2 = \frac{p+q+q-1}{p^2} = q/p^2 \quad //$$

→ Suppose that a training soilder shoots a target in a independent fashion. The probability that the target is short or anyone that is 0.8.

(i) target would be hit on 6^{th} attempt

(ii) taken in less than 5 shots

(iii) taken in even no of shots

$$p = 0.8$$

The pmf is

$$P(X=x) = q^{x-1} p, \quad x = 1, 2, 3 \ldots$$

(i) $P(X=6) = q^5 p$

$= (0.2)^5 \times 0.8$

$= 2.56 \times 10^{-4}$

(ii) $P(X<5) = P(X=1) + P(X=2) + P(X=4) + P(X=3)$

$= p + qp + q^2 p + q^3 p$

$= 0.8 + (0.2)(0.8) + (0.2)^2(0.8) + (0.2)^3(0.8)$

$= 0.8 + 0.16 + 0.032 + 0.0064$

$= 0.9984$

(iii) $P(X = \text{even})$

$= P(X=2) + P(X=4) + P(X=6) + \ldots$

$= qp + q^3 p + q^5 p + \ldots$

$= qp[1 + q^2 + q^4 + q^6 + \ldots]$

$= qp[1-q^2]^{-1}$

$= \dfrac{qp}{1-q^2} = \dfrac{(0.2)(0.8)}{1-(0.2)^2}$

$= 0.1667$

→ The probability of a student passing a Subject is 0.8

(i) 3rd attempt ?

(ii) before 3rd attempt ?

$p = 0.8$, $q = 0.2$

(i) $P(X=3) = q^2 p = (0.2)^2 (0.8) = 0.032$

(ii) $P(X<3) = p(X=2) + P(X=1)$

$= qp + p$

$= (0.8)(0.2) + (0.8)$

$= 0.96$ //

→ If one copy of magazine out of 10 copies has the special prize following GD. Determine mean & Variance

for GD;

$\text{Mean} = E(X) = \frac{1}{p} = 10$

$\text{Variance} = V(X) = \frac{q}{p^2} = \frac{9}{10} \times 100$

$= 90$ //

→ A dice is tossed until the occurrence of 6 what is the probability that it must be tossed more than 5 times

$p = (\text{probability getting } 6) = \frac{1}{6}$; $q = \frac{5}{6}$

$P(X>5) = 1 - P(X \leq 5)$

$= 1 - [P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)]$

$= 1 - \left[\frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) \right]$

$= 1 - 0.598$

$= 0.401$

→ State & prove Memoryless property of GD. If X is a RV following a geometric distribution and $s,t>0$ are integers then

$$\boxed{P[X>s+t/X>s] = P[X>t]}$$

Proof: The PMF of GD is

$P(X=x) = q^{x-1}p, \quad x=1,2,3 \dots$

consider,

$P[X>k] = P[X=k+1] + P(X=k+2] + \dots$

$= q^k p + q^{k+1} p + \dots$

$= q^k p [1+q+q^2+ \dots]$

$= q^k p [1-q]^{-1}$

$= \dfrac{q^k p}{1-q} \qquad [1-q=p]$

$= \dfrac{q^k p}{p}$

$\boxed{P[X>k] = q^k}$ —(1)

$P[A/B] = \dfrac{P(A\cap B)}{P(B)}$

L.H.S: $P[X>s+t/X>s]$

$= \dfrac{P[X>s+t \cap X>s]}{P[X>s]}$

$= \dfrac{P[X>s+t]}{P[X>s]}$

$= \dfrac{q^{s+t}}{q^s} = q^t$

$= P[X>t] = RHS$

Hence proved //

Continuous Distributions:

1. Exponential distribution:

The Pdf of ED is.

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \quad \lambda > 0$$

$\lambda \rightarrow$ parameter

① Find MGF & Variance of ED

sol: The Pdf is

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

MGF: $M_X(t) = E(e^{tx})$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_{0}^{\infty} e^{tx} e^{-\lambda x} dx$$

$$= \lambda \int_{0}^{\infty} e^{x(t-\lambda)} dx$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$M_X(t) = \lambda \left[0 - \frac{1}{-(\lambda - t)} \right]$$

$$\boxed{M_X(t) = \frac{\lambda}{\lambda - t}}$$

Scanned by CamScanner

Mean : $\mu_1' = E(X)$

$$= \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[\frac{\lambda}{\lambda-t}\right]\right]_{t=0}$$

$$= \left[\frac{0-\lambda(-1)}{(\lambda-t)^2}\right]_{t=0}$$

$$= \frac{\lambda}{\lambda^2}$$

$$\boxed{E(X) = 1/\lambda}$$

Variance : $V(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \mu_2' = \left(\frac{d^2}{dt^2} M_X(t)\right)_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{\lambda}{(\lambda-t)^2}\right)\right]_{t=0}$$

$$= \left[\frac{0+2\lambda(\lambda-t)}{(\lambda-t)^4}\right]_{t=0}$$

$$= \frac{2\lambda^2}{\lambda^4} = \frac{2}{\lambda^2}$$

$$V(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = 1/\lambda^2$$

$$M_X(t) = \frac{\lambda}{\lambda-t}, \quad E(X) = \frac{1}{\lambda}, \quad V(X) = \frac{1}{\lambda^2}$$

In ED, $V(X) = \text{Mean}^2$

Memoryless Properties of ED:

If X is a CRV following E.D and s, t are real numbers then

$$\boxed{P[X>s+t / X>s] = P[X>t]}$$

Proof: The pdf of ED is

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

consider,

$$P[X>k] = \int_k^{\infty} \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^{\infty}$$

$$= e^{-\infty} + e^{-\lambda k}$$

$$P[X>k] = e^{-\lambda k} \quad \text{--- (1)}$$

Now consider,

$$P[X>s+t / X>s] = \frac{P[X>s+t \cap X>s]}{P[X>s]}$$

$$= \frac{P[X>s+t]}{P[X>s]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = RHS$$

$$\therefore P[X>t] = P[X>s+t / X>s]$$

Hence proved //

Problems:

1. The time required to repair a machine is ED with $\lambda = 1/2$. What is probability that (i) the repair time exceeds 9hrs (ii) conditional probability that it takes atleast 10hrs given that it already exceeds 9hrs.

Sol: The pdf is

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$= \frac{1}{2} e^{-x/2} , x \geq 0$

x - Time required to repair machine

(i) $P(X>9) = \int_{9}^{\infty} \frac{1}{2} e^{-x/2} dx$

$= \frac{1}{2} \int_{9}^{\infty} e^{-x/2} dx$

$= \frac{1}{2} \left[\frac{e^{-x/2}}{-1/2} \right]_{9}^{\infty}$

$= e^{-9/2}$

$\Rightarrow e^{-4.5}$

(2) $P(X \geq 10 / X>9) = ?$

$= \frac{P(X \geq 10)}{P(X>9)}$ → by memoryless property

$= P[X>1]$

$= \frac{1}{2} \int_{1}^{\infty} e^{-x/2} dx$

$= e^{-1/2}$

//

2. The milage which car owners get with the certain kind of radial tire. It's a random variable following a ED with mean 40000km. Find the probobility that one of the tyre be last atmost 30000 km.

(ii) Atleast 20,000 km

Sol: $\lambda = \frac{1}{40,000} \Rightarrow \frac{1}{\lambda} = 4000$

The pdf is

$f(x) = \lambda e^{-\lambda x} , x \geq 0$

$= \frac{1}{40000} e^{-x/40000} , x \geq 0$

$X \to$ milage given by car

(i) P (atmost 30,000)

$= P(X \leq 30{,}000)$

$$= \int_0^{30000} f(x)dx = \frac{1}{40000}\int_0^{30{,}000} e^{-x/40000}\,dx$$

$$= \frac{1}{40{,}000}\left[\frac{e^{-x/40000}}{-(1/40000)}\right]_0^{30000}$$

$$= -e^{-3/4}+1$$

$$= 1-e^{-3/4}$$

$$= 0.527$$

(ii) P(atleast 20,000) $= P(X \geq 20000)$

$$\int_{20000}^{\infty} f(x)dx = \frac{1}{40{,}000}\int_{20000}^{\infty} e^{-x/40000}\,dx$$

$$= \frac{1}{40000}\left[\frac{e^{-x/40000}}{-1/40000}\right]_{20000}^{\infty}$$

$$= 0+e^{-1/2}$$

$$= 0.6065$$

$\rightarrow$ The daily consumption of milk 20,000 galloons is appx distributed exponentially with parameter $\frac{1}{3000}$, the city has a daily stock of 35,000 galloons. what is the probability that if 2 days selected at random, the stock is insufficient for both the days

sol: parameter $\lambda = \frac{1}{3000}$

Pdf $f(x) = \lambda e^{-\lambda x}, \; x \geq 0$

$$= \frac{1}{3000} e^{-x/3000}, \; x \geq 0$$

$X \to$ be the daily consumption of milk

P(stock is insufficient for one day)

$P(X + 20{,}000 > 35{,}000)$

$= P(X > 35{,}000 - 20000) = P(X > 15000)$

$$= \int_{15000}^{\infty} f(x)\,dx$$

$$= \frac{1}{3000} \int_{15000}^{\infty} e^{-x/3000}\,dx$$

$$= 1/3000 \left[\frac{e^{-x/3000}}{-1/3000} \right]_{15000}^{\infty}$$

$$= -e^{-\infty} + e^{-5}$$

P(stock is insufficient for a day) $= e^{-5}$

For two days,

$P(X > 15000) = e^{-5} \times e^{-5}$

$= e^{-10}$

$=$

$\checkmark$

* Uniform Distribution:

The pdf is

$$f(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0 & , \text{otherwise} \end{cases}$$

$\to$ Find MGF, Mean & Variance of UD. The pdf is $f(x) = \frac{1}{b-a}$

$, a < x < b$

sol: MGF: $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$$= \int_a^b \frac{e^{tx}}{b-a}\,dx = \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left(\frac{e^{tx}}{t}\right)_a^b$$

$$= \frac{e^{bt}-e^{at}}{(b-a)(t)}$$

$$\boxed{MGF = \frac{e^{bt}-e^{at}}{(b-a)t}}$$ //

Mean:

$$E(x) = \int_a^b x f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \frac{b+a}{2}$$

$$\boxed{\therefore E(x) = \frac{a+b}{2}}$$

$$E(x^2) = \int_a^b x^2 f(x)\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{1}{3(b-a)}(b-a)(a^2+ab+b^2)$$

$$E(x^2) = \frac{a^2+ab+b^2}{3}$$

Scanned by CamScanner

Variance : $V(x) = E(x^2) - [E(x)]^2$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2+3b^2+6ab}{12}$$

$$\boxed{V(x) = \frac{(a-b)^2}{12}}$$

→ Buses arrives at a specified stop at 15mins in 1hr starting at 7am arrives at 7:00, 7:15 . . . If a passenger arrives at a random time. which is Uniformly distributed that the passenger waits for (i) Less than 5min for a bus
(ii) More than 10mins for a bus

Sol: pdf is $f(x) = \frac{1}{b-a}, \; a<x<b$

$$\begin{cases} \frac{1}{30}, & 0<x<30 \\ 0, & \text{otherwise} \end{cases}$$

$x \rightarrow$ be arrival time of passenger during $1/2$ an hour

(i) P(passenger wait for < 5mins)

$= P(10<x<15) + P(25<x<30)$

$$\int_{10}^{15}\frac{1}{30}dx + \int_{25}^{30}\frac{1}{30}dx = \frac{1}{30}[15-10] + \frac{1}{30}[30-25]$$

$$= \frac{5}{30}+\frac{5}{30} = \frac{10}{30} = 1/3$$

(ii) P(passenger wait for > 10min)

$= P(0<x<5) + P(15<x<20)$

$$= \frac{1}{30}\int_0^5 dx + \frac{1}{30}\int_{15}^{20} dx$$

$$= \frac{1}{30}(5-0) + \frac{1}{30}(20-15)$$

$$\Rightarrow \frac{5}{30}+\frac{5}{30} = \frac{10}{30} = 1/3 \;//$$

Scanned by CamScanner

→ If X is uniformly distributed with mean 1 and variance $4/3$ find $P(X<0)$.

Sol: For Uniformly distribution,

$$\text{Mean} = E(X) \Rightarrow \frac{b+a}{2} = 1$$

$$b+a=2 \quad \text{--- (1)}$$

$$\text{Variance } V(X) = \frac{(b-a)^2}{12} = 4/3$$

$$(b-a)^2 = 16$$

$$b-a = \pm 4 \quad \text{--- (2)}$$

solve (1) & (2)

$$\begin{array}{ll} b+a=2 & b+a=2 \\ b-a=4 & b-a=-4 \\ \hline 2b=6 & 2b=-2 \\ \boxed{b=3} & \boxed{b=-1} \end{array}$$

$$a+b=2 \qquad\qquad a+b=2$$

$$a=2-3 \Rightarrow \boxed{a=-1} \qquad \boxed{a=3}$$

For UD, $a<x<b$

$$\boxed{a=-1,\ b=3}$$

The pdf is $f(x) = \frac{1}{b-a}$, $a<x<b$

$$= \frac{1}{4}, \quad -1<x<3$$

$$P(X<0) = \int f(x)\,dx$$

$$= \int_{-1}^{0} \frac{1}{4}\,dx$$

$$= \frac{1}{4}(0+1) = 1/4$$

-1 0 3
f(x)

→ A RV X has UD over $(-\alpha, \alpha)$, $\alpha > 0$ find α such that

(i) $P(X>1) = 1/3$

(ii) $P(|X|<1) = P(|X|>1)$

The pdf of x is

$$f(x) = \frac{1}{b-a}, \quad a<x<b$$

$$= \frac{1}{2\alpha}, \quad -\alpha < x < \alpha$$

(i) $P(X>1) = 1/3$

$$\int f(x)\,dx = 1/3$$

$$\frac{1}{2\alpha}\int_1^{\infty} dx = 1/3$$

$$\frac{1}{2\alpha}[x]_1^{\alpha} = 1/3$$

$$\frac{1}{2\alpha}[\alpha-1] = 1/3$$

$$3[\alpha-1] = 2\alpha$$

$$-3+3\alpha-2\alpha = 0$$

$$-3+\alpha = 0$$

$$\boxed{\alpha = 3}$$

(ii) $P(|X|<1) = P(|X|>1)$

$$= 1-P(|X|\le 1)$$

$$2P(|X|\le 1) = 1$$

$$P(|X|\le 1) = 1/2$$

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{2}$$

$$\frac{1}{2\alpha}\int_{-1}^{1} dx = 1/2$$

$$\Rightarrow \frac{1}{2\alpha}(x)_{-1}^{1} = \boxed{\alpha = 2}$$

Normal Distribution (or) Gaussian Distribution (part-c)

→ It is a limiting case of B.D under CRV

* The pdf of N.D is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

→ parameter are μ & σ

$E(x) = \mu$, standard deviation $\sigma = \sqrt{V(x)}$

Symbolically,

$$X \sim N(\mu, \sigma)$$

↓
(follows)

eg: $X \sim N(3, 6)$

Here, $\mu = 3$, $\sigma = 6$

→ Find MGF, Mean, Variance of ND [HW]

Standard Normal Distribution:

The transformation is

$$\boxed{Z = \frac{X-\mu}{\sigma}}$$

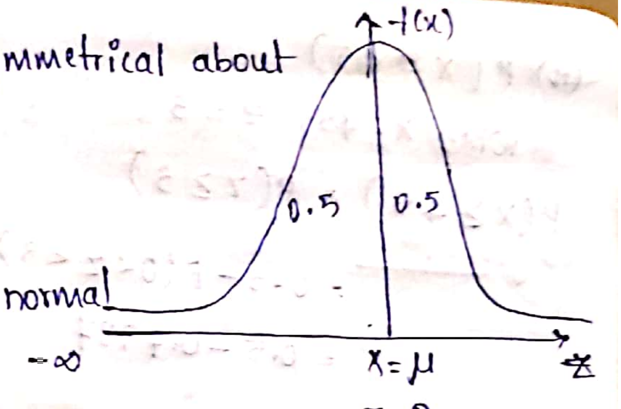
The pdf of z is

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}}, \quad -\infty < z < \infty$$

$E(Z) = 0$, $V(Z) = 1$

$\boxed{Z \sim N(0,1)}$ → MCQ

properties of Normal curve :



1) The curve is bell shaped and Symmetrical about the line $x=\mu$ i.e $z=0$

2) $P(0<z<z_1)=\int_0^{z_1}\phi(z)dz$

3) The total area curved by the normal curve is 1

$P(-\infty<Z<\infty)=1$

(4) Since it is symmetric about z-axis

$P(-\infty<z<0)=P(0<z<\infty)=0.5$

(5) $P(0<z<\infty)=P(-\infty<z<0)$
$=0.5$

(6) $P(0<z<z_1)=P(-z<z<0)$ (7) $P(-a<z<a)=2P(0<z<a)$

→ If x is a NRV with mean 30 and SD 5 find $P(26<X\leq 40)$

(ii) $P(x\geq 45)$

(iii) $P(|x-30|>5)$

$\mu=30$, $\sigma=5$

$X\sim N(30,5)$

$$Z=\frac{x-\mu}{\sigma}$$

$$Z=\frac{x-30}{5}$$

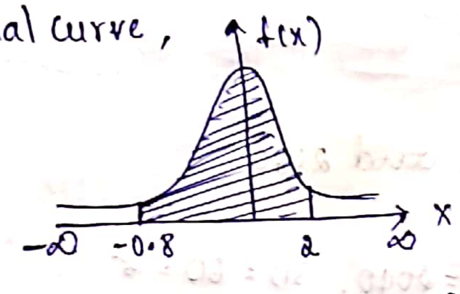
(i) $P(26\leq x\leq 40)$

when $x=26$, then $z=\frac{-4}{5}=-0.8$

$x=40$ then $z=\frac{10}{5}=2$

$\Rightarrow P(-0.8\leq Z\leq 2)$

from Normal curve,



$=P(-0.8<z<0)+P(0<z<2)$

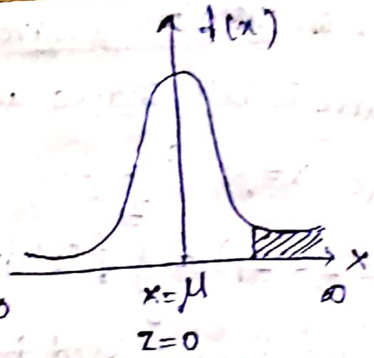
$=P(0<z<0.8)+P(0<z<2)$

$\Rightarrow 0.2881+0.4772=0.7653$ //

Scanned by CamScanner

(ii) $P(X \geq 45)$

when $X = 45$, $Z = 3$



$P(X \geq 45) = P(Z \geq 3)$

$= 0.5 - P(0 < Z < 3)$

$= 0.5 - 0.4987$

$= 0.0013$

iii) $P(|X-30| > 5) = 1 - P[|X-30| \leq 5)$

$= 1 - P(-5 \leq X-30 \leq 5)$

$= 1 - P(25 \leq X \leq 35)$

when

$X = 25$, $Z = -1$

$X = 35$, $Z = 1$

$P(|X-30| > 5) = 1 - P(-1 \leq Z \leq 1)$

$= 1 - 2P[0 \leq Z \leq 1)$

$= 1 - 2(0.3413)$

$= 0.3174$ //

→ In a test on 2000 electric bulbs, the average life time of particular bulb is 2040 hrs and SD is 60 hrs which is normally distributed. Find the probability that

(i) The life time of bulb exceeds 2150

(ii) atmost 1950

(iii) between 1920 and 2160

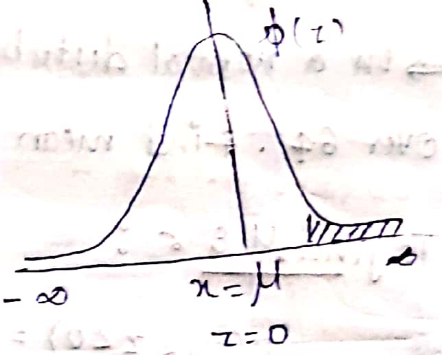
Given, $\mu = 2040$, $SD = 60 = \sigma$

$X \rightarrow$ be the particular bulb life time

Let, $Z = \frac{x-\mu}{\sigma}$

Scanned by CamScanner

$$z = \frac{x - 2040}{60}$$



(i) $P(x > 2150)$

when $x = 2150$,

$$z = \frac{2150 - 2040}{60} = \frac{110}{6} = 1.833 - \infty$$

$$P(x > 2150) = P(z > 1.833)$$
$$= 0.5 - P(0 < z < 1.833)$$
$$= 0.5 - 0.4664$$
$$= 0.0336$$

No of bulbs with life time $> 2150 = 2000 \times 0.0336$
$= 67.2$
$\approx$ ~~67 bulbs~~

(ii) $P(x < 1950)$

when, $x = 1950$, $z = \frac{1950 - 2040}{60} = -1.5$

$$P(x < 1950) = P(z < -1.5)$$
$$= 0.5 - P(-1.5 < z < 0)$$
$$= 0.5 - 0.4332$$
$$= 0.0668$$

No of bulbs life time $> 1950 = 2000 \times 0.068$
$= 133.6 \simeq 134$ bulbs

(iii) $P(1920 < x < 2160)$

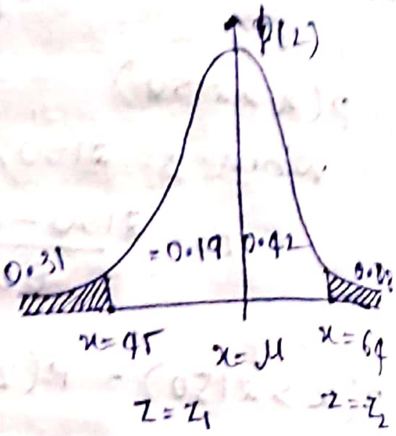
when $x = 1920$, $z = \frac{1920 - 2040}{60} = -2$

when $x = 2160$, $z = \frac{2160 - 2000}{60} = 2$

$$P(1920 < x < 2160) = P(-2 < z < 0)$$
$$= 2P(0 < z < 2)$$
$$= 2(0.4773)$$
$$= 0.9546$$

No of bulbs with life time $(1920 - 2160) = 2000 \times 0.9546 = 1908.8$ bulbs

→ In a normal distribution, 31% items are under 45 and 8% are over 64. Find mean and SD



To find μ & σ :

$$P(z_1 < z < 0) = 0.19$$

$$P(0 < z < z_1) = 0.19$$

-According to table

$$z_1 = 0.5 \text{ (since it is } < 0)$$

$$z_1 = -0.5$$

when $x = 45$

$$z_1 = -0.5 \Rightarrow \frac{x-\mu}{\sigma} = z_1$$

$$\frac{45-\mu}{\sigma} = -0.5$$

$$45 - \mu = -0.5\sigma$$

$$\boxed{\mu - 0.5\sigma = 45} \quad — (1)$$

$$P(0 < z < z_2) = 0.42$$

$$z_2 = 1.41$$

when $x = 64$

$$= \frac{64-\mu}{\sigma} = 1.41$$

$$\Rightarrow \boxed{\mu + 1.41\sigma = 64} \quad — (2)$$

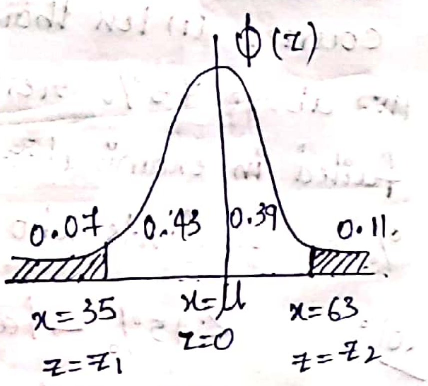
Solving 1 & 2

$$\mu - 0.5\sigma = 45$$
$$\mu + 1.41\sigma = 64$$

$$\boxed{\mu \simeq 50, \sigma \simeq 10}$$

In a normal distribution, 7% of the items are under 35 and 89% are under 63. Find Mean & SD.

To find mean & SD



$P(z_1 < z < 0) = 0.43$

$P(0 < z < z_1) = 0.43$

$z_1 = -1.48$

when $x = 35$

$$\frac{35-\mu}{\sigma} = -1.48$$

$$\boxed{\mu - 1.48\sigma = 35} \quad \text{— (1)}$$

$P(0 < z < z_2) = 0.39$

$z_2 = 1.23$

when $x = 63$

$$\frac{63-\mu}{\sigma} = 1.23$$

$$\boxed{\mu + 1.23\sigma = 63} \quad \text{— (2)}$$

Solving (1) & (2)

$$\mu + 1.23\sigma = 63$$
$$\underset{(+)}{-\mu} - 1.48\sigma = \underset{(-)}{35}$$

$$2.71\sigma = 28$$

$$\sigma = \frac{28}{2.71}$$

$$\boxed{\sigma = 10.3}$$

$\mu = 63 - 1.23\sigma = 50.7$ (or) $\mu = 35 + 1.48\sigma$
$= 50.24$ //

Scanned by CamScanner

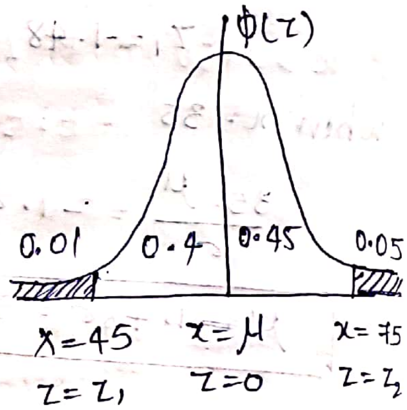
→ In an engineering examination, a student is considered to have failed in, second class, first class, destination according as his courses (i) less than 45% (ii) b/w 45-60% (iii) b/w 60-75% (iv) above 75% respectively. In a particular year 10% of students failed in examination & 5% of students got destinction. Find % of students who have got first class & second class

sol: $< 45\% \rightarrow$ failed

$45\% - 60\% \rightarrow 2^{nd}$ class

$60\% - 75\% \rightarrow 1^{st}$ class

$> 75\% \rightarrow$ distinction



$P(z_1 < z < 0) = 0.4$

$P(0 < z < z_1) = 0.4$

$z_1 = -1.3$

when $x = 45$, $z = z_1$

$\frac{45-\mu}{\sigma} = -1.3$

$\boxed{\mu - 1.3\sigma = 45}$ — ①

$P(0 < z < z_2) = 0.45$

$z_2 = 1.7$

when $x = 75$

$\boxed{\mu + 1.75 = 75}$ — ②

Solving ① & ②

$\mu = 58.18, \quad \sigma = 10.28$

$z = \frac{x - 58.15}{10.28}$

$P(45 < x < 60) = ?$

when $x = 45$, $z = \frac{45 - 58.15}{10.28} = -0.2$

when $x = 60$, $z = \dfrac{60 - 58.15}{10.28} = 0.17$

$$P(45 < x < 60) = P(-1.27 < z < 0.17)$$

$$= P(-1.27 < z < 0) + P(0 < z < 0.17)$$

$$= P(0 < z < 1.27) + P(0 < z < 0.17)$$

$$= 0.0169 + 0.4554$$

$$= 0.4723$$

The % of students second class b/w 45% - 60% $= 100 \times 0.4723 \simeq 47\%$

$\left.\begin{array}{l}\text{percentage of} \\ \text{students got } 60+75\%\end{array}\right\} = 100 - 10 - 5 - 47$

$\simeq 38\%$