UNIT - 3

16/11/2021.

Application of PDE

Types - { Parabolic, Hyperbolic, Elliptic } (2nd Order).

Consider the PDE $\frac{A\partial^2 f}{\partial x^2} + \frac{B\partial^2 f}{\partial x \partial y} + \frac{C\partial^2 f}{\partial y^2} + \cdots = 0 \longrightarrow (1)$

PDE of (1) is said to be

- Elliptic $(B^2 - 4AC < 0)$
- Parabolic $(B^2 - 4AC = 0)$
- Hyperbolic $(B^2 - 4AC > 0)$

Classify the Pole:

(i) $u_{xx} + u_{yy} = 0$

$\Rightarrow A = 1,\ B = 0,\ C = 1$

$\Rightarrow B^2 - 4AC$

$\Rightarrow 0^2 - 4(1)(1)$

$\Rightarrow 0 - 4$

$\Rightarrow -4\ (<0)$

$\therefore$ It is elliptic

(ii) $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

$a^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0$

$\Rightarrow A = a^2,\ B = 0,\ C = -1$

$\Rightarrow B^2 - 4AC$

$\Rightarrow 0^2 - 4(a^2)(-1)$

$\Rightarrow 4a^2 \ (>0)$

$\therefore$ It is hyperbolic

(iii) $f_{xx} + 2f_{xy} + f_{yy} = 0$

$\Rightarrow A = 1,\ B = 2,\ C = 1$

$\Rightarrow B^2 - 4AC$

$\Rightarrow (2)^2 - 4(1)(1)$

$\Rightarrow 4 - 4$

$\Rightarrow 0 \ (=0)$

$\therefore$ It is parabolic.

One Dimensional Wave Equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Zero Initial Velocity	Non-Zero Velocity
Boundary Condn.	Boundary Condn:
(i) $y(0,t) = 0,\ \forall t$	(i) $y(0,t) = 0,\ \forall t$
(ii) $y(l,t) = 0,\ \forall t$	(ii) $y(l,t) = 0,\ \forall t$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \; 0<x<l$ | (iii) $y(x,0) = 0, \; 0<x<l$

(iv) $y(x,0) = f(x), \; 0<x<l$ | (iv) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = f(x), \; 0<x<l$

The various possible solutions are,

(i) $y(x,t) = \left(Ae^{px} + Be^{-px}\right)\left(Ce^{pat} + De^{-pat}\right)$

(ii) $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$ → ①

Suitable Solution

(iii) $y(x,t) = (Ax+B)(Ct+D)$.

The hints we get after substituting the boundary conditions are,

*. Apply (i), we get, $A=0$

*. Apply (ii), we get, $p = \frac{n\pi}{l}$

*. Apply (iii), we get, $D=0$

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \times \cos\frac{n\pi at}{l} \rightarrow ②$$

*. Apply (iv), we get, $y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = f(x)$ → ③

where, $C_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Sub C_n in ②, we get the general solution:

TYPE I - Zero Velocity

~~Sub c_n in ⓐ~~, we get the ~~general solution~~

Sums: Type I - Algebraic:

1. A string is stretched and fastened to two points $x=0$ and $x=l$ apart. Motion is started by displacing the string to the form $y = k(lx - x^2)$ from which it is released at time $t=0$. Find the displacement at any point on the string at a distance of x from one end at time t.

Soln: The displacement of a string at a time t and at a distance $x=l$ from the ~~equation~~ origin is given by,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Boundary Conditions are,

(i) $y(0,t) = 0, \ \forall t$

(ii) $y(l,t) = 0, \ \forall t$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \quad 0 < x < l$

(iv) $y(x,0) = k(lx - x^2), \quad 0 < x < l$

The suitable solution which satisfies our boundary conditions is given by,

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$$

Apply condn. (i) in (1), $y(0,t) = 0$

$y(0,t) = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat) = 0$

$= A(C\cos pat + D\sin pat) = 0$

$\therefore C\cos pat + D\sin pat \neq 0$

$\Rightarrow \boxed{A = 0}$ $[\because t > 0]$

Sub in ①

$$y(x,t) = B \sin px \,(C \cos pat + D \sin pat) \longrightarrow ②$$

Apply condn. (ii) in ②, $y(l,t) = 0, \forall t > 0$

$$y(l,t) = B \sin pl \,(C \cos pat + D \sin pat) = 0$$

here, $C \cos pat + D \sin pat \neq 0, \because t > 0$

$$\therefore B \sin pl = 0$$

$$B \neq 0, \ \sin pl = 0 \quad [\text{Already, } A = 0]$$

$$pl = \sin^{-1}(0)$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Put $p = \frac{n\pi}{l}$ in ②

$$y(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right) \longrightarrow ③$$

Apply ~~condn (iii) in ②~~

Partially Differentiate ③ w.r.t. 't'

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = B \sin \frac{n\pi x}{l} \left(-C \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi at}{l} + D \left(\frac{n\pi a}{l}\right) \cos \frac{n\pi at}{l} \right)$$

Apply condn. (iii) in , $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \ 0 < x < l$

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = B \sin \frac{n\pi x}{l} \left(-C \left(\frac{n\pi a}{l}\right) \sin 0 + D \left(\frac{n\pi a}{l}\right) \cos 0 \right) = 0$$

$= 0$

$$= B \sin \frac{n\pi x}{l} \left(D \cdot \frac{n\pi a}{l} \right) = 0$$

$B \neq 0$, $\cancel{\frac{n\pi x}{l}}$, $D = 0$ $\left[\because \sin \frac{n\pi x}{l} \neq 0, \; \frac{n\pi a}{l} \neq 0 \right]$

Already $A = 0$

Put $D = 0$ in (3)

$$y(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} \right)$$

$$= BC \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi at}{l}$$

$$y(x,t) = C_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi at}{l} \longrightarrow (4)$$

where, $BC = C_n$

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \longrightarrow (5)$$

Apply condn. (iv) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = k(lx - x^2) \longrightarrow (6)$$

To find C_n expand $k(lx-x^2)$ in a half range fourier sine series $(0, l)$

$$K(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \longrightarrow (7)$$

From (6) & (7), we get,

$$b_n = C_n$$

$$\therefore C_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$u = (lx-x^2)$

$u' = l-2x$

$u'' = -2$

$v = \sin\frac{n\pi x}{l}$

$v_1 = -\cos\frac{n\pi x}{l} \Big/ \frac{n\pi}{l}$

$v_2 = -\sin\frac{n\pi x}{l} \Big/ \frac{n^2\pi^2}{l^2}$

$v_3 = \cos\frac{n\pi x}{l} \Big/ \frac{n^3\pi^3}{l^3}$

$$= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) + (l-2x)\left(\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) - 2\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\left((l^2-l^2)0 - 0 - 2\frac{\cos n\pi}{(n\pi/l)^3}\right) - \left(0-0-2\frac{\cos 0}{(n\pi/l)^3}\right)\right]$$

$$= \frac{2k}{l}\left[-2\left(\frac{l}{nx}\right)^3(-1)^n + 2\left(\frac{l}{n\pi}\right)^3(1)\right]$$

$$= \frac{2k}{l} \times \frac{2l^{3}}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\,[\,]$$

$$C_n = \begin{cases} \dfrac{8kl^2}{n^3\pi^3} & \text{, if } n \text{ is odd} \\ 0 & \text{, if } n \text{ is even.} \end{cases}$$

Sub C_n in (5)

$$y(x,t) = \sum_{n=1,3,5} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$\boxed{y(x,t) = \frac{8kl^2}{\pi^3} \sum_{n=1,3,5} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l}}$$

Type II - Trigonometric:

2. A tightly stretched string of length l with fixed end points is in a position by $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest in that position, find the displacement.

Soln: Repeat the steps from the previous problem by applying your boundary conditions (i), (ii) and (iii)

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \longrightarrow (5)$$

Apply condn. (iv) in (5)

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = y_0 \sin^3\left(\frac{\pi x}{l}\right) \longrightarrow (6)$$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = y_0 \sin^3\left(\frac{\pi x}{l}\right) \rightarrow (7)$$

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = b_1 \sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l}$$

Equating the coefficients

$$b_1 = \frac{3y_0}{4}, \quad b_2 = 0, \quad b_3 = \frac{y_0}{4}$$

$$\boxed{y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}}$$

Non-zero velocity

3. A tightly stretched string with fixed point $x=0$ & $x=l$ is initially at rest at its equilibirium position. each point a velocity ~~λx to~~ $\lambda x(l-x)$. Find the displacement.

Soln: Wave equation is,

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are,

(i) $y(0,t) = 0, \ \forall t$

(ii) $y(l,t) = 0, \ \forall t$

(iii) $y(x,0) = 0$

(iv) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = f(x) = \lambda x(l-x)$

The suitable solution is,

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$

Now apply condn(i) in (1)

$y(0,t) = (A+0)(C\cos pat + D\sin pat) = 0$

$= A(C\cos pat + D\sin pat) = 0$

$\therefore \boxed{A = 0}$ $(\because C\cos pat + D\sin pat > 0)$

Sub in (1)

$y(x,t) = B\sin px (C\cos pat + D\sin pat) \longrightarrow (2)$

Apply condn.(ii) in (2)

$y(l,t) = B\sin pl (C\cos pat + D\sin pat) = 0$

$C\cos pat + D\sin pat > 0$

$B \neq 0$, $A = 0$

$\therefore \sin pl = 0$

$pl = \sin^{-1} 0$

$pl = n\pi$

$\boxed{P = \frac{n\pi}{l}} \rightarrow$ Sub in (2)

From eqn (2) $\rightarrow$

$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow (3)$

Apply condn (iii) in (3)

$y(x,t) = B\sin\frac{n\pi x}{l}(C) = 0$

$y(x,t) = BC\sin\frac{n\pi x}{l} = 0$

$\Rightarrow B \neq 0, \frac{n\pi x}{l} \neq 0,$

$\therefore \boxed{C = 0} \Rightarrow$ Sub in ③

$$y(x,t) = B \sin \frac{n\pi x}{l} \left(D \sin \frac{n\pi at}{l} \right) \longrightarrow ④$$

~~Partially Differentiate ④ w.r.t. 't'~~

$$y(x,t) = BD \sin \frac{n\pi x}{l} \left(\sin \frac{n\pi at}{l} \right) \cancel{\longrightarrow ⑤}$$

~~Partially Differentiate ④ w.r.t. 't'~~

$\cancel{\frac{\partial y}{\partial t} = }$

$$y(x,t) = B_n \sin \frac{n\pi x}{l} \left(\sin \frac{n\pi at}{l} \right) \longrightarrow ⑤$$

P.D. ⑤ w.r.t. 't'

$$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = B_n \sin \frac{n\pi x}{l} \left(\frac{n\pi a}{l}\right)\left(\cos \frac{n\pi at}{l}\right) \longrightarrow ⑥$$

Apply condn. (iv) in ⑥.

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = B_n \sin \frac{n\pi x}{l} \left(\frac{n\pi a}{l}\right) = \lambda x(l-x)$$

where, $b_n = B_n \left(\frac{n\pi a}{l}\right) \longrightarrow ⑦$

$$b_n = \frac{2}{\pi} \int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n \frac{n\pi a}{l} = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{4\lambda l^2}{n^3\pi^3}\left(\frac{l}{n\pi a}\right)\left[1-(-1)^n\right]$$

$$B_n = \frac{4\lambda l^3}{n^4\pi^4 a}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \dfrac{8\lambda l^3}{n^4\pi^4 a} & , \text{ if } n \text{ is odd} \end{cases}$$

$$\Rightarrow f(x) = 3x(l-x) \Rightarrow B_n = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \dfrac{24 l^3}{n^4\pi^4 a} & , \text{ if } n \text{ is odd.} \end{cases}$$

- One dimensional Heat Equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Steady State Temperature (SST):

when a rod is heated, after a certain stage, the temperature remains constant. That temperature is called steady state temperature.

Condition for steady state temparature:-

$$u = ax + b$$

$$u = \left(\frac{b-a}{l}\right)x + a$$

1. An insulated rod of length 60 cm has its end at A and B maintained at $20^\circ c$ & $80^\circ c$ respectively. Find the steady state solution of the rod.

Soln: Condn for steady state:

$$a = 20, \quad b = 80, \quad l = 60$$

$$u = \left(\frac{b-a}{l}\right)x + 20$$

$$u = \left(\frac{80-20}{60}\right)x + 20$$

$$u = \left(\frac{60}{60}\right)x + 20$$

$$\boxed{u = x + 20}$$

2. A rod of length 30 cm has its end A and B kept at 20°C and 80°C respectively until steady state condition prevailed. The temperature at each end is then suddenly reduced to 0° and kept so. Find the resulting temperature function $u(x,t)$ taking $x = 0$ at A.

Soln: One dimensional heat eqn. is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Boundary condns of the rod:

(i) $u(0,t) = 0, \forall t$

(ii) $u(l,t) = 0, \forall t$

(iii) $u(x,0) = f(x)$

In steady state condn. eqn. (1) will be $\frac{\partial^2 u}{\partial x^2} = 0$ and $u(x) = ax + b \longrightarrow (2)$

The boundary conditions for the steady state are,

(I) $u(0) = 20$

(II) $u(l) = 80$

Apply condn. (I) in (2)

$$u(0) = 0 + b = 20$$

$$\boxed{b = 20}$$

Apply condn. (II) in (2)

$$u(l) = al + b = 80 \longrightarrow (3)$$

Sub $b = 20$ in (3)

$$u(l) = al + 20 = 80$$

$$al = 80 - 20$$

$$al = 60$$

$$\boxed{a = \frac{60}{l}}$$

Sub 'a' and 'b' in (2)

$$u(x) = \left(\frac{60}{l}\right) x + 20$$

Thus, the temperature function in steady state is,

$$\boxed{u(x) = \frac{60}{l} x + 20}$$

After the steady state condn. is over we have the boundary conditions.

(i) $u(0,t) = 0, \forall t$
(ii) $u(l,t) = 0, \forall t$
(iii) $u(x,0) = f(x)$

The most suitable solution of (1) is

~~$u(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$~~

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \longrightarrow (4)$$

Apply (i) in (4)

$$u(0,t) = (A(1) + B(0))\, e^{-\alpha^2 p^2 t} = 0$$

$$u(0,t) = A e^{-\alpha^2 p^2 t} = 0$$

$$\therefore \boxed{A = 0}, \quad e^{-\alpha^2 p^2 t} \neq 0, \forall t$$

Sub in (4)

$$u(x,t) = (B\sin px)\, e^{-\alpha^2 p^2 t} \longrightarrow (5)$$

Apply (ii) in (5)

$$u(l,t) = (B\sin pl)\, e^{-\alpha^2 p^2 t} = 0$$

$$\therefore B \neq 0, \quad \sin pl = 0, \quad e^{-\alpha^2 p^2 t} \neq 0, \forall t$$

$$pl = \sin^{-1}(0)$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}} \Rightarrow \text{Sub in (5)}$$

Sub $\boxed{p = n\pi/l}$ in (5)

$$u(x,t) = B\sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 (n\pi/l)^2 t} \longrightarrow (6)$$

~~Apply (iii)~~

General Solution is given by,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t}} \longrightarrow (7)$$

Apply condn. (iii) in (7)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{60}{l}x + 20$$

$$\text{where, } B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\int_0^l \left(\frac{60}{l} + 20\right)\sin\frac{n\pi x}{l}\,dx$$

Sub $l = 30$

$$= \frac{\cancel{2}}{\cancel{30}_{15}}\int_0^{30}\left(\frac{\cancel{60}^{2}}{\cancel{30}}x + 20\right)\sin\frac{n\pi x}{30}\,dx$$

$$= \frac{1}{15}\int_0^{30}(2x+20)\sin\frac{n\pi x}{30}\,dx$$

$u = 2x + 20$ $\qquad v = \sin\frac{n\pi x}{30}$

$u' = 2$

$$v_1 = \frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}$$

$$v_2 = \frac{-\sin\frac{n\pi x}{30}}{\left(\frac{n\pi}{30}\right)^2}$$

$$B_n = \frac{1}{15}\left[(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{n\pi x/30}\right) + 2\left(\frac{\sin\frac{n\pi x}{30}}{(n\pi x/30)^2}\right)\right]$$

$$= \frac{1}{15}\left[-(2x+20)\left(\frac{30}{n\pi}\right)\left(\cos\frac{n\pi x}{30}\right) + 2\left(\frac{30}{n\pi}\right)^2\left(\sin\frac{n\pi x}{30}\right)\right]_0^{30}$$

$$\cancel{\frac{1}{15}\left[-(2(30)+20)\left(\frac{30}{30 n\pi}\right)\right.}$$

$$= \frac{1}{15}\cancel{\left[-(2(30)+20)\left(\frac{30}{n\pi\times 30}\right)\left(\cos\frac{n\pi\times 30}{30}\right) + 20\left(\frac{30}{}\right)\right.}$$

$$= \frac{1}{15}\left[-(2(30)+20)\left(\frac{30}{n\pi}\right)\left(\cos\frac{n\pi\times\cancel{30}}{\cancel{30}}\right) + 20\left(\frac{30}{n\pi}\right)(\cos 0)\right]$$

$$= \frac{1}{15}\left[-\left(\frac{80\times 30}{n\pi}\right)(-1)^n + \frac{600}{n\pi}\,\cancel{(-1)^{n+1}}\right]$$

$$= \frac{1}{15}\left[\frac{2400}{n\pi}(-1)^{n+1} + \frac{600}{n\pi}\right]$$

$$= \frac{1}{\cancel{15}} \times \overset{40}{\cancel{600}}\left[\frac{4}{n\pi}(-1)^{n+1} + \frac{1}{n\pi}\right]$$

$$B_n = \frac{40}{n\pi}\left[1 + 4(-1)^{n+1}\right]$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left[1 + 4(-1)^{n+1}\right]\sin\frac{n\pi x}{l} \times e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$$

$$= \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + 4(-1)^{n+1}\right]\sin\frac{n\pi x}{l}\; e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$$

3. <u>Type 2</u>: Steady State with non-zero boundary conditions:

1. The ends A and B of a bar l cm long are kept at $0°$ & $100°$ respectively until steady state condn. prevailed. The temp. at A is raised to $50°$ & temp. at B is raised to $150°C$. Find the temp $u(x,t)$.

<u>Soln</u>:
$$u(x) = \left(\frac{b-a}{l}\right)x + a$$
$$= \left(\frac{100-0}{l}\right)x + 0$$
$$\boxed{u(x) = \frac{100}{l}x}$$

Heat equation is,
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions are,

(i) $u(0,t) = 50, \forall t$

(ii) $u(l,t) = 150$

(iii) $u(x,0) = \frac{100}{l}(x) \Rightarrow f(x)$

The steady state temperature using the boundary conditions,
$$u = \left(\frac{b-a}{l}\right)x + a$$
$$= \left(\frac{150-50}{l}\right)x + 50$$
$$\boxed{u(x) = \frac{100}{l}x + 50}$$

The suitable solution is,

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t^2}$$

$$u(x,t) = \frac{100x}{l} + 50 + (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t^2} \longrightarrow (1)$$

Apply condn. (i) in (1)

$$u(0,t) = \cancel{50} + A\, e^{-\alpha^2 p^2 t} = \cancel{50}$$

$$= A\, e^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow \boxed{A = 0} \Rightarrow \text{Sub in } (1)$$

$$u(x,t) = \frac{100x}{l} + 50 + (B\sin px)\, e^{-\alpha^2 p^2 t} \longrightarrow (2)$$

Apply condn. (ii) in (2)

$$u(l,t) = \frac{100\cancel{l}}{\cancel{l}} + 50 + (B\sin pl)\, e^{-\alpha^2 p^2 t} = 150$$

$$= 100 + 50 + (B\sin pl)\, e^{-\alpha^2 p^2 t} = 150$$

$$= \cancel{150} + B\sin pl\; e^{-\alpha^2 p^2 t} = \cancel{150}$$

$$= B\sin pl\; e^{-\alpha^2 p^2 t} = 0$$

$$B \neq 0,\quad e^{-\alpha^2 p^2 t} \neq 0 \quad [\because t > 0]$$

$$\therefore \sin pl = 0$$

$$pl = \sin^{-1}(0)$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}} \Rightarrow \text{Sub in } (2)$$

$$u(x,t) = \frac{100x}{l} + 50 + \left(B\sin\frac{n\pi x}{l}\right) e^{-\alpha^2 \left(\frac{n\pi}{l}\right)^2 t} \longrightarrow (3)$$

The most general solution is,

$$u(x,t) = \frac{100x}{l} + 50 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 p^2 t} \rightarrow (4)$$

Apply condn. (iii) in (4)

$$u(x,0) = \cancel{\frac{100x}{l}} + 50 + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = \cancel{\frac{100x}{l}}$$

$$50 + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = 0$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = -50$$

$$B_n = \frac{100}{n\pi}\left((-1)^n - 1\right)$$

$$B_n = \begin{cases} \frac{200}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$