UNIFORM DISTRIBUTION

Definition:- A continuous random variable x is said to follow uniform distribution, if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{where } a<x<b \\ 0, \text{otherwise} \end{cases}$$

Here a,b where $(b>a)$ are the parameters of the distribution.

Moment generating function, Mean, variance.

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot F(x)dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left(\frac{e^{tx}}{t}\right)_a^b$$

$$= \frac{1}{t(b-a)} \left(e^{tb} - e^{ta}\right)$$

$$M_x(t) = \frac{e^{t(b-a)}}{t(b-a)}$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x\, F(x)dx$$

$$= \int_a^b x \cdot \frac{1}{b-a} dx.$$

$$= \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b$$

$$= \frac{1}{2(b-a)}(b^2-a^2)$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 . F(x)\,dx$$

$$= \int_a^b x^2 . \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{1}{3(b-a)}(b^3-a^3)$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\therefore var(x) = E(x^2) - (E(x))^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

↳ Mean

$$var(x) = \frac{b^2+ab+a^2}{3} - \left(\frac{a+2ab+b^2}{4}\right)$$

$$= \frac{4(b^2+ab+a^2) - 3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

↳ variance.

33.) If x is uniformly distributed with mean 1 and variance $4/3$. find $P(x < 0)$.

Soln→33.) Mean $= E(x) = \frac{a+b}{2} = 1$

$\Rightarrow a+b = 2 \rightarrow (1)$

variance $= \frac{(a-b)^2}{12} = \frac{4}{3}$

$\Rightarrow (a-b)^2 = \frac{48}{3}$

$\Rightarrow (a-b)^2 = 16$

$\Rightarrow a-b = 4 \rightarrow (2)$

or

$a-b = -4 \rightarrow (3)$

Solve (1) & (2)

$a+\cancel{b} = 2$

$a-\cancel{b} = 4$

$2a = 6$

$a = 3$

Put a in (1)

$\boxed{b = -1}$

Solve (1) & (3)

$a+\cancel{b} = 2$

$a-\cancel{b} = -4$

$2a = -2$

$a = -1$

Put $a = -1$ in (1)

$-1+b = 2$

$\boxed{b = 3}$

$$f(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{3-(-1)} & , -1 < x < 3 \\ 0 & , \text{otherwise.} \end{cases}$$

$$f(x) = \begin{cases} 1/4 & , -1 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$$

$$P(x<0) = \int_{-1}^{0} F(x)\,dx$$

$$= \int_{-1}^{0} \frac{1}{4}\,dx$$

$$= \frac{1}{4}(x)_{-1}^{0}$$

$$= \frac{1}{4}(0-(-1))$$

$$= \frac{1}{4}$$

34.) A random variable x has an uniform distribution over the interval $(-3, 3)$. find.

(i.) $P(x<2)$

(ii.) $P(|x|<2)$

(iii.) $P(|x-2|<2)$

(iv.) find 'k' such that

$$P(x>k) = 1/3$$

Soln→34→

$$F(x) = \begin{cases} \frac{1}{b-a} & , a<x<b \\ 0 & , \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 1/6 & , -3<x<3 \\ 0 & , \text{otherwise} \end{cases}$$

i.) $$P(x<2) = \int_{-3}^{2} F(x)\,dx$$

$$= \int_{-3}^{2} 1/6\,dx = \frac{1}{6}(x)_{-3}^{2} = \frac{5}{6}.$$

ii) $P(|x|<2) = P(-2<x<2)$

$$= \int_{-2}^{2} \frac{1}{6} dx = \frac{1}{6}(x)_{-2}^{2} = 4/6.$$

iii) $P(|x-2|<2)$

$= P(-2<x-2<2)$

$= P(0<x<4)$

$= \int_0^4 F(x)' dx$

$= \int_0^3 1/6\, x + \int_3^4 0\, dx$

$= 3/6 = \frac{1}{2}.$

iv) $P(x>k) = 1/3$

$\int_k^3 F(x) dx = 1/3$

$= \int_k^3 \frac{1}{6} dx = 1/3$

$= \frac{1}{6}(x)_k^3 = \frac{1}{3}$

$= \frac{1}{6}(3-k) = 1/3$

$= 3-k = 2$

$= \boxed{k = 1}$

35) A bus arrives every 20 minutes at a specified stop begining at 6:40 am and continuing until 8:40 am. A passenger does not know the schedule, but arrives randomly (uniformly distributed) between 7 am and 7:30 am every morning. What is the probability that the passenger waits more than 5 mins for a bus?

Soln→35→ x is uniformly distributed in the interval $(0,30)$.

The passenger has to wait more than 5 minutes only if his arrival is between 7 and 7:15 am (or) between 7:20 and 7:30 am.

7 to 7:30

6:40, 7, 7:20, 7:40, 8

$$\therefore P(x \geq 5) = \int_5^{30} f(x)\,dx$$

$$= \int_5^{30} \frac{1}{30}\,dx = \frac{1}{30}(x)_5^{30}$$

$$= \frac{25}{30} = \frac{5}{6}.$$

36.) Buses arrive at a specified stop at 15 min interval starting at ~~a~~ 7am. If a passenger arrives at the stop at a random time that is uniformly distributed between 7 and 7:30 am, Find the probability that he is waiting for

i) less than 5 mins for a bus.

ii) at least 12 mins for a bus.

Soln→36→ x is uniformly distributed in $(0,30)$ (a,b)

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{7, 7:15, 7:30, 7:45}$

$$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$$

i.) The passenger has to wait less than 5 mins only if he comes 7:10 to 7:15 (or) 7:25 to 7:30

∴ Required Probability

$$= \int_{10}^{15} f(x)\,dx + \int_{25}^{30} f(x)\,dx$$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30}(x)_{10}^{15} + \frac{1}{30}(x)_{25}^{30}$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{1\cancel{0}}{3\cancel{0}} = \frac{1}{3}$$

ii.) The passenger has to wait ≥12 atleast 12 mins only if he comes 7 to 7:03 or 7:18 7:15 - 7:18

$\boxed{7, 7:15, 7:30,}$ 7:45

∴ Required probability $= \int_{0}^{3} f(x)\,dx + \int_{15}^{18} f(x)\,dx$

$$= \int_{0}^{3} \frac{1}{30}\,dx + \int_{15}^{18} \frac{1}{30}\,dx$$

$$= \frac{1}{30}(x)_{0}^{3} + \frac{1}{30}(x)_{15}^{18}$$

$$= \frac{1}{30}(3-0) + \frac{1}{30}(18-15)$$

$$= \frac{3}{30} + \frac{3}{30} = \frac{\cancel{6}}{3\cancel{6}} = \frac{1}{5}.$$