21/8/18

UNIT-3

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATION

Classification of 2nd order PDE

General form of PDE $\quad A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) = 0$

Form of std PDE $\quad A U_{xx} + B U_{xy} + C U_{yy} + f\left(x, y, u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) = 0$

$B^2 - 4AC > 0$ hyperbolic

$B^2 - 4AC = 0$ parabolic

$B^2 - 4AC < 0$ elliptic

1) Find the nature of the PDE $4U_{xx} + 4U_{xy} + U_{yy} + 2U_x - U_y = 0$

Solu

Here, $A = 4$

$B = 4$

$C = 1$

$B^2 - 4AC = 16 - 16 = 0$

2) Classify the equation $4U_{xx} + 4U_{xy} + U_x - U_y = 0$

Sol $\quad A = 4, \; B = 4, \; C = 0$

$B^2 - 4AC = 16 - 0 = 16 > 0$

Wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \Rightarrow c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0$$

$A = c^2, \quad B = 0, \quad C = -1$

$B^2 - 4AC = 4c^2 > 0$

$\therefore$ W.E is H.B

21/8/18

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APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATION

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Form of std PDE $\quad A U_{xx} + B U_{xy} + C U_{yy} + f\left(x, y, u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) = 0$

$B^2 - 4AC > 0$ hyperbolic

$B^2 - 4AC = 0$ parabolic

$B^2 - 4AC < 0$ elliptic

$B^2 - 4AC$

1) Find the nature of the PDE $4U_{xx} + 4U_{xy} + U_{yy} + 2U_x - U_y = 0$

Solu

Here, $A = 4$

$B = 4$

$C = 1$

$B^2 - 4AC = 16 - 16 = 0$

2) Classify the equation $4U_{xx} + 4U_{xy} + U_x - U_y = 0$

Sol. $A = 4,\ B = 4,\ C = 0$

$B^2 - 4AC = 16 - 0 = 16 > 0$

Wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (=) \quad c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0$$

$A = c^2,\ B = 0,\ C = -1$

$B^2 - 4AC = 4c^2 > 0$

$\therefore$ W.E is H.B

$0 = A(C\cos cpt + D\sin cpt)$

From this, $A = 0$ since $C\cos cpt + D\sin cpt \neq 0$

$\therefore$ (1) $\Rightarrow y(x,t) = B\sin px\,(C\cos cpt + D\sin cpt)$ — (2)

From (ii) $x = l$ and sub (2)

$y(l,t) = B\sin pl\,(C\cos cpt + D\sin cpt)$

$0 = B\sin pl\,(C\cos cpt + D\sin cpt)$

$\sin pl = 0$ since $B \neq 0$

$\sin pl = \sin n\pi$

$pl = n\pi$

$$\boxed{p = \frac{n\pi}{l}}$$

~~From 2~~

(2) $\Rightarrow y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\frac{cn\pi t}{l} + D\sin\frac{cn\pi t}{l}\right]$ — (3)

Partially differentiate (3) wrt 't'.

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right) + D\cos\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right)\right]$$

$$y_t(x,t) = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right) + D\cos\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right)\right]$$

(iii) $\Rightarrow t = 0$ sub above equation

$$y_t(x,0) = B\sin\frac{n\pi x}{l}\left[0 + D(1)\frac{cn\pi}{l}\right]$$

$$0 = B\sin\frac{n\pi x}{l}\left[D\frac{cn\pi}{l}\right]$$

$D = 0$

(3) $\Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{cn\pi t}{l}\right)$

$y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l}$ — (4)

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l} \quad \text{— (5)}$$

(iv) $\Rightarrow t = 0$ sub (5)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \quad \text{— (6)}$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$u = lx$ $\quad v = x^2$

$u' = l$ $\quad v' = 2x$

$u'' = 0$ $\quad v'' = 2$

$$B_n = \frac{2k}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$u = lx - x^2$ $\quad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = l - 2x$ $\quad v_1 = -\frac{\cos(n\pi x/l)}{n\pi/l}$

$u'' = -2$ $\quad v_2 = -\frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$

$v_3 = \frac{\cos(n\pi x/l)}{(n\pi/l)^3}$

$uv_1 - u'v_2 + u''v_3$

$$B_n = \frac{2k}{l}\left[-(lx - x^2)\left(\frac{\cos n\pi x/l}{n\pi/l}\right) + (l - 2x)\cancel{\sin\left(\frac{n\pi x}{l}\right)}^{0} - 2\frac{\cos(n\pi x/l)}{(n\pi/l)^3}\right]_0^l$$

1D Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

$A = c^2 \quad B = C = 0$

$B^2 - 4AC = 0.$

$\therefore$ 1D Heat equation is parabolic

2D Heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1, \; B = 0, \; C = 1$

$B^2 - 4AC = -4 < 0$

$\therefore$ 2D Heat equation is elliptic.

Solution of wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Ans

i) $y(x,t) = (Ae^{px} + Be^{-px})(Ce^{cpt} + De^{-cpt})$

ii) $y(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$

iii) $y(x,t) = (Ax + B)(Ct + D)$

Type - I

i) Vibrating string with zero initial velocity

The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Boundary condition

i) $y(0,t) = 0$ iii) $\frac{\partial y}{\partial t}(x,0) = 0$

ii) $y(l,t) = 0$ iv) $y(x,0) = f(x)$

Suitable solution $y(x,t) = (\cancel{A\cos px + B\sin px})(\cancel{C\cos cpt + D\sin cpt})$

$\downarrow$

$(A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$

Type - II

i) Vibrating string with non-zero initial velocity

The wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

B.C

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = g(x)$

iv) $y(x,0) = 0$

Suitable soln :

$y(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$

Q) A string is stretched and fastened to two pts $x = 0$, $x = l$ apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of 'x' from one end at time 't'.

Solution :

The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Suitable solution $y(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$ — (1)

B.C

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y_t(x,0) = \frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = k(lx - x^2)$

From (i) $x = 0$ sub. (1)

$y(0,t) = [A(1) + B(0)][C\cos cpt + D\sin cpt]$

$0 = A\,(C\cos cpt + D\sin cpt)$

From this, $A = 0$ since $C\cos cpt + D\sin cpt \neq 0$

$\therefore (1) \Rightarrow y(x,t) = B\sin px\,(C\cos cpt + D\sin cpt)$ — (2)

From (ii) $x = l$ and sub (2)

$y(l,t) = B\sin pl\,(C\cos cpt + D\sin cpt)$

$0 = B\sin pl\,(C\cos cpt + D\sin cpt)$

$\sin pl = 0$ since $B \neq 0$

$\sin pl = \sin n\pi$

$pl = n\pi$

$$\boxed{p = \frac{n\pi}{l}}$$

From 2

(2) $\Rightarrow y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\frac{cn\pi t}{l} + D\sin\frac{cn\pi t}{l}\right]$ — (3)

Partially differentiate (3) w.r.t. 't'.

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right) + D\cos\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right)\right]$$

$$y_t(x,t) = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right) + D\cos\frac{cn\pi t}{l}\left(\frac{cn\pi}{l}\right)\right]$$

(iii) $\Rightarrow t = 0$ sub above equation

$$y_t(x,0) = B\sin\frac{n\pi x}{l}\left[0 + D(1)\frac{cn\pi}{l}\right]$$

$$0 = B\sin\frac{n\pi x}{l}\left[D\frac{cn\pi}{l}\right]$$

$D = 0$

(3) $\Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left\{C\cos\frac{cn\pi t}{l}\right\}$

$$y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l} \quad \text{— (4)}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l} \quad \text{— (5)}$$

(iv) $\Rightarrow t = 0$ sub (5)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \quad \text{— (6)}$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l k\,(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$u = lx$ $\quad v = x^2$

$u' = l$ $\quad v' = 2x$

$u'' = 0$ $\quad v'' = 2$

$$B_n = \frac{2k}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$u = lx - x^2$ $\quad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = l - 2x$ $\quad v_1 = -\frac{\cos(n\pi l/l)}{n\pi/l}$

$u'' = -2$

$v_2 = -\frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$

$v_3 = \frac{\cos(n\pi x/l)}{(n\pi/l)^3}$

$uv_1 - u'v_2 + u''v_3$

$$B_n = \frac{2k}{l}\left[-(lx - x^2)\left(\frac{\cos n\pi l/l}{n\pi/l}\right) + (l - 2x)\cancel{\sin\left(\frac{n\pi x}{l}\right)} - 2\,\frac{\cos(n\pi x/l)}{(n\pi/l)^3}\right]_0^l$$

The most general solution:

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{an\pi t}{l} \quad \text{--- (4)}$$

P.D (4) w.r.t 't'.

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{an\pi t}{l} \left(\frac{an\pi}{l}\right)$$

(iv) ⇒ $t = 0$ sub above equation.

$$y_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\left(\frac{an\pi}{l}\right)$$

$$y_t(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where } b_n = B_n\frac{an\pi}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2\lambda}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$u = lx - x^2$ $\qquad v = \sin\frac{n\pi x}{l}$

$u' = l - 2x$ $\qquad v_1 = -\frac{\cos(n\pi x/l)}{n\pi/l}$

$u'' = -2$ $\qquad v_2 = -\frac{\sin(n\pi x/l)}{(n\pi/l)^2}$

$\qquad v_3 = \frac{\cos(n\pi x/l)}{(n\pi/l)^3}$

$$b_n = \frac{2\lambda}{l}\left\{-(lx-x^2)\frac{\cos(n\pi x/l)}{n\pi/l} + (l-2x)\frac{\sin(n\pi x/l)}{(n\pi/l)^2} - 2\frac{\cos(n\pi x/l)}{(n\pi/l)^3}\right\}_0^l$$

$$b_n = \frac{2\lambda}{l}\left[-\frac{2\cos n\pi}{(n\pi/l)^3} + \frac{2}{(n\pi/l)^3}\right]$$

$$= \frac{4\lambda l^3}{l n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8\lambda l^2}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases}$$

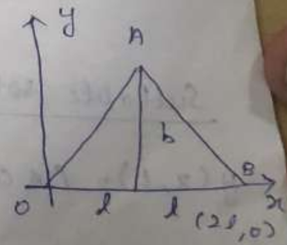
w.k.t, $b_n = B_n \frac{an\pi}{l} \Rightarrow B_n = \frac{b_n l}{an\pi}$

$$B_n = \frac{8\lambda l^2}{n^3\pi^3} \times \frac{l}{an\pi} \quad \text{if } n \text{ is odd.}$$

$$= \frac{8\lambda l^3}{a n^4 \pi^4} \quad \text{if } n \text{ is odd}$$

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8\lambda l^3}{a\pi^4 n^4} \sin\frac{n\pi x}{l} \sin\frac{an\pi t}{l} \;//$$

9) Give the boundary condition to find the displacement of the string of length '2l' which is fastend of at both ends and the mid point of the string is taken to a height 'b' and then released from the rest in that position. Find the displacement.



Solution

Boundary condition

i) $y(0,t) = 0$

ii) $y(2l,t) = 0$

iii) $y_t(x,0) = 0$

iv) $y(x,0) = \begin{cases} OA, & 0 < x < l \\ AB, & l < x < 2l \end{cases}$

$= \frac{2k}{\ell}\left[\left(0 - \frac{2\cos n\pi}{(n\pi/\ell)^3}\right) - \left(0 - \frac{2\cos 0}{(n\pi/\ell)^3}\right)\right]$

$= \frac{2k}{\ell}\left[-\frac{2(-1)^n}{(n\pi/\ell)^3} + \frac{2}{(n\pi/\ell)^3}\right]$

$= \frac{4k}{\ell} \times \frac{\ell^3}{n^3\pi^3}\left[1-(-1)^n\right]$

$B_n = \frac{4k\ell^2}{n^3\pi^3}\left[1-(-1)^n\right] = \begin{cases} \frac{8k\ell^2}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

Sub B_n in equation (5)

$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8k\ell^2}{n^3\pi^3} \sin\frac{n\pi x}{\ell} \cos\frac{an\pi t}{\ell}$ //

Q) A tightly stretched string with fixed end points $x=0$ and $x=\ell$, is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(\ell-x)$. Find the displacement.

Solution

The wave equation is $U_{tt} = c^2 U_{xx}$

The ~~soluti~~ suitable sol.

$y(x,t) = (A\cos px + B\sin px)(c\cos apt + D\sin apt)$ — (1)

B.C i) $y(0,t) = 0$ ii) $y(\ell,t) = 0$

iii) ~~$y_t(x,0) = \lambda x(\ell - x)$~~ $y(x,0) = 0$

iii) $y_t(x,0) = \lambda x(\ell - x)$

i) $x=0$ sub (1)

$y(0,t) = A(c\cos apt + D\sin apt)$

$0 = A(c\cos apt + D\sin apt)$

$A = 0$ since $c\cos apt + D\sin apt \neq 0$

Sub in (1)

$y(x,t) = B\sin px\,(c\cos apt + D\sin apt)$ — (2)

ii) $\Rightarrow x = \ell$ sub (2)

$y(\ell,t) = B\sin p\ell\,(c\cos apt + D\sin apt)$

$0 = B\sin p\ell\,(c\cos apt + D\sin apt)$

$\Rightarrow \sin p\ell = 0$ since $B \neq 0$

$\sin p\ell = \sin n\pi$

$p\ell = n\pi$

$\boxed{p = \frac{n\pi}{\ell}}$ Sub in (2)

$y(x,t) = B\sin\frac{n\pi x}{\ell}\left[c\cos\frac{an\pi t}{\ell} + D\sin\frac{an\pi t}{\ell}\right]$ — (3)

iv) $\Rightarrow t = 0$ sub (3)

$y(x,0) = B\sin\frac{n\pi x}{\ell}(c)$

$0 = BC\sin\frac{n\pi x}{\ell}$

$\Rightarrow c = 0$ sub (3)

$y(x,t) = B\sin\frac{n\pi x}{\ell}\, D\sin\frac{an\pi t}{\ell}$

$= BD\sin\frac{n\pi x}{\ell}\sin\frac{an\pi t}{\ell}$

Equation of OA $O(0,0)$ & $A(l, b)$

$x_1\ y_1 \quad x_2\ y_2$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

$$\frac{x-0}{l-0} = \frac{y-0}{b-0}$$

$$\frac{x}{l} b = y$$

$$y = \frac{bx}{l}$$

Equation of AB $A(l, b)$ & $B(2l, 0)$

$x_1\ y_1 \quad x_2\ y_2$

$$\frac{x-l}{2l-l} = \frac{y-b}{0-b}$$

$$\frac{-b(x-l)}{l} = y-b$$

$$y = b - \frac{b}{l}(x-l)$$

$$y = \frac{lb - bx + bl}{l}$$

$$y = \frac{b}{l}(2l-x)$$

iv) $y(x,0) = \begin{cases} \frac{bx}{l}, & 0 < x < l \\ \frac{b(2l-x)}{l}, & l < x < 2l \end{cases}$

Suitable sol

$$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \quad \text{—(1)}$$

i) $\Rightarrow x = 0$ in (1)

$y(0,t) = A(C\cos apt + D\sin apt)$

$0 = A(C\cos apt + D\sin apt)$

$\Rightarrow A = 0$ since $C\cos apt + D\sin apt \neq 0$

(1) $\Rightarrow y(x,t) = B\sin px\,(C\cos apt + D\sin apt)$

ii) $\Rightarrow x = 2l$ in (2)

$y(2l, t) = B\sin p2l\,(C\cos apt + D\sin apt)$

$0 = B\sin 2pl\,(C\cos apt + D\sin apt)$

$\Rightarrow \sin 2pl = 0$ since $B \neq 0$

$2pl = n\pi$

$$\boxed{p = \frac{n\pi}{2l}}$$

(2) $\Rightarrow y(x,t) = B\sin\frac{n\pi x}{2l}\left(C\cos\frac{an\pi t}{2l} + D\sin\frac{an\pi t}{2l}\right)$ —(3)

$$\frac{\partial y}{\partial t}(x,t) = B\sin\frac{n\pi x}{2l}\left[-C\sin\frac{an\pi t}{2l}\left(\frac{an\pi}{2l}\right) + D\cos\frac{an\pi t}{2l}\left(\frac{an\pi}{2l}\right)\right]$$

$$y_t(x,0) = B\sin\frac{n\pi x}{2l}\left[D\frac{an\pi}{2l}\right]$$

$$0 = B\sin\frac{n\pi x}{2l} D\frac{an\pi}{2l}$$

$\Rightarrow D = 0$ sub in (3)

$$y(x,t) = B\sin\frac{n\pi x}{2l}\, C\cos\frac{an\pi t}{2l} \quad \text{—(4)}$$

The most general solution

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}\cos\frac{an\pi t}{2l} \quad \text{—(5)}$$

Sol $u_t = c^2 u_{xx}$ $\quad u_t = 0$

A ——— 30 cm ——— B

$20^\circ C$ $\quad 80^\circ C$

S.S condition prevail

$u_t = 0 \Rightarrow c^2 u_{xx} = 0$

$u_{xx} = 0 \Rightarrow u_x = a$

$u(x) = ax + b \quad - (A)$

At A $\quad x = 0$

$u(0) = a(0) + b$

$\boxed{b = 20}$

At B $\quad x = 30$

$u(30) = a(30) + b$

$80 = 30a + 20$

$60 = 30a$

$\boxed{a = 2}$

Sub 'a' & 'b' in (A)

(A) $\Rightarrow u(x) = 2x + 20, \quad 0 < x < 30$

Boundary condition

i) $u(0,t) = 0$

ii) $u(30,t) = 0$

iii) $u(30, u(x,0) = u(x) = 2x + 20 \quad 0 < x < 30$

Suitable sol

$u(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t} \quad - (1)$

i) $\Rightarrow x = 0$ in (1)

$u(0,t) = A e^{-c^2p^2t}$

$0 = A e^{-c^2p^2t}$

$\Rightarrow A = 0$ since $e^{-c^2p^2t} \neq 0$

$\Rightarrow u(x,t) = B\sin px\, e^{-c^2p^2t} \quad - (2)$

ii) $\Rightarrow x = 30$ in (2)

$u(30,t) = B\sin p30\, e^{-c^2p^2t}$

$0 = B\sin p30\, e^{-c^2p^2t}$

$\Rightarrow \sin p30 = 0$ since $B \neq 0$

$p30 = n\pi$

$\boxed{p = \frac{n\pi}{30}}$ sub in (2)

(2) $\Rightarrow u(x,t) = B\sin\frac{n\pi x}{30}\, e^{-\frac{c^2n^2\pi^2t}{900}} \quad - (3)$

The most general solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}\, e^{-\frac{c^2\pi^2n^2t}{900}} \quad - (4)$$

(iii) $\Rightarrow t = 0$ in (4)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}$$

$$B_n = \frac{2}{30}\int_0^{30} f(x)\sin\frac{n\pi x}{30}\,dx$$

$$= \frac{1}{15}\int_0^{30}(2x+20)\sin\frac{n\pi x}{30}\,dx$$

$u = 2x + 20 \qquad v = \sin\frac{n\pi x}{30}$

$u' = 2 \qquad v_1 = -\frac{\cos n\pi x/30}{n\pi/30}$

$v_2 = -\frac{\sin n\pi x/30}{(n\pi/30)^2}$

$$B_n = \frac{1}{15}\left\{-(2x+20)\frac{\cos n\pi x/30}{n\pi/30} + 2\frac{\sin n\pi x/30}{(n\pi/30)^2}\right\}_0^{30}$$

$$= \frac{1}{15}\left[-(60+20)\frac{\cos n\pi}{n\pi/30} + \frac{2\sin n\pi}{(n\pi/30)^2} + \frac{20}{n\pi/30} - \frac{2\sin 0}{(n\pi/30)^2}\right]$$

$$= \frac{1}{15}\left[-\frac{80(-1)^n \times 30}{n\pi} + \frac{20 \times 30}{n\pi}\right]$$

$$B_n = \frac{30}{15}^{2}\left[\frac{20 - 80(-1)^n}{n\pi}\right]$$

iv) $\Rightarrow t = 0$

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}$$

where $B_n = \frac{\cancel{2}}{\cancel{2l}} \int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\, dx$

$$B_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\, dx$$

$$= \frac{1}{l}\left\{ \int_0^{l} OA \sin\frac{n\pi x}{2l}\, dx + \int_l^{2l} AB \sin\frac{n\pi x}{2l}\, dx \right\}$$

$$B_n = \frac{1}{l}\left\{ \int_0^{l} \frac{bx}{l} \sin\frac{n\pi x}{2l}\, dx + \int_l^{2l} \frac{b(2l-x)}{l} \sin\frac{n\pi x}{l}\, dx \right\}$$

$u = \frac{bx}{l}$ $\quad v = \sin\frac{n\pi x}{2l}$ $\quad u = \frac{b}{l}(2l-x)$

$u' = b/l$ $\quad v_1 = -\frac{\cos(n\pi x/2l)}{n\pi/2l}$ $\quad u' = -\frac{b}{l}$

$v = \sin\frac{n\pi x}{2l}$

$v_2 = -\frac{\sin(n\pi x/2l)}{(n\pi/2l)^2}$ $\quad v_1 = -\frac{\cos(n\pi x/2l)}{n\pi/2l}$

$v_2 = -\frac{\sin(n\pi x/2l)}{(n\pi/2l)^2}$

$$B_n = \frac{1}{l}\left\{ \left(-\frac{bx}{l}\frac{\cos(n\pi x/2l)}{n\pi/2l} + \frac{b}{l}\frac{\sin(n\pi x/2l)}{(n\pi/2l)^2} \right)_0^l + \left(-\frac{b(2l-x)}{l}\frac{\cos(n\pi x/2l)}{n\pi/2l} - \frac{b}{l}\frac{\sin(n\pi x/2l)}{(n\pi/2l)^2} \right)_l^{2l} \right\}$$

$$= \frac{1}{l}\left[-\frac{b\cancel{\cos n\pi/2}}{\cancel{(n\pi/2l)}} + \frac{b}{l}\frac{\sin n\pi/2}{(n\pi/2l)^2} + \frac{b\cancel{\cos n\pi/2}}{\cancel{n\pi/2l}} + \frac{b}{l}\frac{\sin n\pi/2}{(n\pi/2l)^2} \right]$$

$$= \frac{1}{\cancel{l}}\left[\frac{2b}{\cancel{l}} \frac{\sin n\pi/2}{n^2\pi^2} \cdot 4l^2 \right]$$

$$B_n = \frac{8b}{n^2\pi^2} \sin n\pi/2$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2} \sin n\pi/2 \sin\frac{n\pi x}{2l} \cos\frac{an\pi t}{2l} \quad //$$

Steady state condition & zero boundary conditions

1. Heat equation : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Boundary conditions:

i) $u(0,t) = 0, \quad t > 0$

ii) $u(l,t) = 0, \quad t > 0$

iii) $u(x,0) = f(x), \quad 0 < x < l$

Possible solution for heat equation

i) $u(x,t) = (Ae^{px} + Be^{-px})e^{c^2p^2t}$

ii) $u(x,t) = (A\cos px + B\sin px)e^{-c^2p^2t}$

iii) $u(x,t) = Ax + B$

Suitable sol for $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

$u(x,t) = (A\cos px + B\cos px)e^{-c^2p^2t}$

Problems:

1) A rod 30 cm long has its ends A and B kept at 20°C & 80°C respectively until steady state conditions prevail the temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature

$$= 2(20)\left[\frac{1-40(-1)^n}{n\pi}\right]$$

$$B_n = 40\left[\frac{1-40(-1)^n}{n\pi}\right]$$

$\therefore$ (1) $\Rightarrow$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40(1-40(-1)^n)}{n\pi} \sin\frac{n\pi x}{30}\, e^{-\frac{c^2n^2\pi^2 t}{900}} \quad //$$

Q) A metal bar 10 cm long with insulated sides has its ends A and B kept at 20°C & 40°C resp, until steady state conditions prevail. The temp of A is then suddenly raised to 50°C and the same instant that at B is lowered to 10°C. Find the subsequents temp at any point at the bar at anytime.

Sol

$U_t = c^2 U_{xx}$

$U_t = 0 \Rightarrow U_{xx} = 0$

$\Rightarrow U(x) = ax + b, \quad 0 < x < 10$

At A $x = 0$

$u(0) = a(0) + b$

$\boxed{20 = b}$

At B $x = 10$

$u(10) = a(10) + b$

$40 = 10a + 20$

$\boxed{a = 2}$

$\therefore u(x) = 2x + 20, \quad 0 < x < 10$

B.C

i) $u(0,t) = 50$

ii) $u(10,t) = 10$

iii) $u(x,0) = 2x + 10, \quad 0 < x < 10$

Boundary conditions (i) & (ii) are not zero so that we can't find suitable solution.

$\therefore$ we split $u(x,t)$ into two parts.

ie $u(x,t) = U_T(x,t) + U_s(x)$

$\Rightarrow U_T(x,t) = U(x,t) - U_s(x)$ — (1)

To find $U_s(x)$

$50°C = U_s(0)$ A —— B $10°C = U_s(10)$

$U_s(x) = px + q$

$U_s(0) = p(0) + q$

$\boxed{q = 50}$

$U_s(10) = p(10) + q$

$10 = 10p + 50$

$\boxed{p = -4}$

$\therefore U_s(x) = -4x + 50$

(1) $\Rightarrow U_T(x,t) = U(x,t) + 4x - 50$

(i) $\Rightarrow x = 0$

$U_T(0,t) = u(0,t) + 0 - 50$

$= 50 - 50 = 0$

(ii) $\Rightarrow x = 10$

$U_T(10,t) = U(10,t) + 40 - 50$

$= 10 + 40 - 50$

$= 0$

(iii) $\Rightarrow t = 0$

$U_T(x,0) = U(x,0) + 4x - 50$

$= 2x + 10 + 4x - 50$

$U_T(x,0) = 6x - 40, \quad 0 < x < 10$

ie. (I) $U_T(0,t) = 0$

(II) $U_T(10,t) = 0$

(III) $U_T(x,0) = 6x - 40, \quad 0 < x < 10$

Find H.R.S.S for $f(x)=2$ in $(0,\pi)$

Sol
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 2 \sin nx \, dx$$

$$= \frac{4}{\pi}\int_0^{\pi} \sin nx \, dx = \left[-\frac{4 \cos nx}{n\pi}\right]_0^{\pi}$$

$$= \frac{-4}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$b_n = \frac{-4}{n\pi}\left[(-1)^n - 1\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-4}{n\pi}\left[(-1)^n - 1\right] \sin nx \; //$$

4|9|18

& absolut

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}$

Fourier Tra

Let $f(x)$

piecewise

interval

fourier tr

$F[f(x)] =$

Inverse

If $f(x)$

finite

in $(-\infty,$

$f(x) =$

Equati

Properti

i) Linear

$F[$

Suitable solution

$U_T(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t}$ — ②

(I) ⇒ $x = 0$ in ②

$U_T(0,t) = A e^{-c^2p^2t}$

$0 = A e^{-c^2p^2t}$

⇒ $A = 0$ since $e^{-c^2p^2t} \neq 0$

② ⇒

$U_T(x,t) = B\sin px\, e^{-c^2p^2t}$ — ③

I ⇒ $x = 10$ in ③

$U_T(10,t) = B\sin p10\, e^{-c^2p^2t}$

$0 = B\sin p10\, e^{-c^2p^2t}$

$\sin p10 = 0$ since $B \neq 0$

$p10 = n\pi$

$\boxed{p = \frac{n\pi}{10}}$ in ③

③ ⇒

$U_T(x,t) = B\sin\frac{n\pi x}{10}\, e^{-\frac{c^2n^2\pi^2t}{100}}$ — ④

T.M.G.S

$U_T(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}\, e^{-\frac{c^2n^2\pi^2t}{100}}$ — ⑤

(III) ⇒ $t = 0$ in ⑤

$U_T(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}$

$B_n = \frac{2}{10}\int_0^{10} f(x) \sin\frac{n\pi x}{10}\,dx$

$= \frac{1}{5}\int_0^{10} (6x-40)\sin\frac{n\pi x}{10}\,dx$

$u = 6x-40$ $\quad v = \sin n\pi x/10$

$u' = 6$ $\quad v_1 = -\frac{\cos n\pi x/10}{n\pi/10}$

$v_2 = -\frac{\sin n\pi x/10}{(n\pi/10)^2}$

$B_n = \frac{1}{5}\left\{-(6x-40)\frac{\cos n\pi x/10}{n\pi/10} + 6\frac{\sin n\pi x/10}{(n\pi/10)^2}\right\}_0^{10}$

$= \frac{1}{5}\left[-(20)\frac{\cos n\pi}{n\pi/10} + \frac{(-40)}{n\pi/10}\right]$

$= \frac{10^2}{n\pi\, 5}\left[-20(-1)^n - 40\right]$

$= \frac{2(-20)}{n\pi}\left[2 + (-1)^n\right]$

$B_n = -\frac{40}{n\pi}\left(2 + (-1)^n\right)$

∴ ⑤ ⇒

$U_T(x,t) = \sum_{n=1}^{\infty} \frac{-40}{n\pi}(2+(-1)^n)\sin\frac{n\pi x}{10}\, e^{-\frac{c^2n^2\pi^2t}{100}}$ — ④

$u(x,t) = U_T(x,t) + U_S(x)$

$= \sum_{n=1}^{\infty} -\frac{40}{n\pi}(2+(-1)^n)\sin\frac{n\pi x}{10}\, e^{-\frac{c^2n^2\pi^2t}{100}} - 4x + 50$ //