23/3/17

UNIT- IV

PRINCIPLES OF QUEUEING THEORY

one of the most useful area of appl. of prob. theory is Queue theory

Queuing system ~~function in a following manner~~ In the study of waiting line

Input → | Service | → Output.

The principles in the Queuing system are.

1) The customers and the Server : The customer arrives. ~~He~~ can get service immediately or have to wait in a Queue.

2) The arrival of customer is referred in terms of interarrival time and the service is described by service time.

3) Queue size : It is no. of customers in the Queue.

4) Queue discipline : It is a order in which customers are served (FIFO), LIFO. , priority , Random.
FCFS LCFS

5) System capacity :
It is the maximum no of customer allowed in the system.

6) The basic characteristics of a Queuing system are

1) Arrival pattern
2) Service pattern
3) No of servers
4) System capacity
5) Queue discipline.

Customer behaviour!

1. Balking! A customer who refuses to enter queuing system because the queue is too long.
2. Reneging! A customer who leaves the queue without receiving service because of too much waiting.
3. Jockeying! When there are ||el queues, a customer who jumps from one queue to another with shorter length to reduce waiting time.

State of the system: Transient, steady, explosive

~~Transition state~~ [Kendal's notation for representing queing model (a|b|c) : (d|e)

where

a - arrival time, b = service time

c -> no of servers

d -> Capacity of the system

e -> Queue discipline.

Notation:

$\frac{1}{\lambda}$ = Inter arrival time

λ = Mean Arrival rate [avg. no. of customers entering per unit time]

$\frac{1}{\mu}$ = inter service time

μ = Mean Service rate. [avg. no. of customers served per unit time]

Model NO:-1 : (M|M|1) : (∞|F|FO)

(single server, infinite capacity)

i) $L_s = \frac{\lambda}{\mu-\lambda}$ (Avg. no of customer in the system)

ii) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$ Avg no of customer in the Queue

$L_q \rightarrow$ Length of queue.

iii) Average waiting time of the customer in the system = $W_s = \frac{1}{\mu-\lambda}$

iv) Average waiting time of customer in The queue $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

v) probability of an empty system (or) (idle System)

$$P_0 = 1 - \frac{\lambda}{\mu}.$$

vi) probability of busy system (traffic intensity)

$$\rho = \frac{\lambda}{\mu}.$$

vii) Probability that the waiting time exceeds t.

$$P(W_s > t) = e^{-(\mu-\lambda)t}.$$

viii) Probability that no. of customers in the system exceeds k.

$$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k.$$

ix) steady state property:

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \rho^n P_0.$$

Little's formula :-

It gives the relationships between W_s, W_q, L_s and L_q.

i) $L_s = \lambda W_s$

ii) $L_q = \lambda W_q$

iii) $L_s = L_q + \frac{\lambda}{\mu}$.

iv) $W_q = W_s - 1/\mu$.

PROBLEMS

1) In the Railway yard, goods train arrive at a rate of 30 trains per day. Assume that the inter arrival time follows exponential distribution and the service time is also exponential with an avg. of 36 mins. Calculate the following.

i) The mean queue size.

ii) The prob. that the system size is atleast 10.

iii) If the input of the trains increases for an avg of 33 per day, what will be the change in the above quantity?

Sol

$\lambda = 30/\text{day}$

$= \frac{30}{24 \times 60}$

$= 0.021.$ /min

Service time = 36 mins = $1/\mu$

$\Rightarrow$ Service rate $= \mu = \frac{1}{36} = 0.028$

i) $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(0.021)^2}{0.028(0.028 - 0.0\ldots)}$

$= 2.25$

$\simeq 2$ trains.

ii) $P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k$.

$P(n \geq 10) = \left(\frac{0.021}{0.028}\right)^{10}$

$= 0.0563$

iii) $\lambda = 33/\text{day}$.

$= \frac{33}{24 \times 60} = 0.023$.

$\mu = 0.028$

$L_q = \frac{(0.023)^2}{0.028(0.028 - 0.023)} = 3.7$

$\simeq 4$. trains.

Atleast 10:

$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k$

$P(n \geq 10) = \left(\frac{0.023}{0.028}\right)^{10}$

$= 0.139$ //

2) Customer arrived at One man barber shop. According to poisson process with mean inter arrival time of 20 mins, customer spent an avg of 15 mins in the barber chair. If an hour is used as unit time, then

i) what is the prob. that the customer need not wait for a hair cut? P_0.

ii) what is the expected no of customers in the barber shop and in queue.

iii) How much time can a customer expect to spend in the shop? W_s

iv) find the avg time that the customer spends in the queue.?

v) what is the prob that there will be more than 3 customers in the system.?

Solution:

$\lambda = \frac{1}{20} = 0.05$, $\mu = 1/15 = 0.067$

i) Idle system : $p_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{0.05}{0.067} = 0.253$

ii) $L_s = \frac{\lambda}{\mu - \lambda} = \frac{0.05}{0.067 - 0.05} = 2.941 \simeq 3$ persons.

$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(0.05)^2}{0.067(0.067-0.05)}$

$= 2.194 \simeq 2$ persons.

iii) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{0.067 - 0.05}$

$= 58.82 \simeq 59$ mins.

iv) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{0.05}{0.067(0.067-0.05)}$

$= 43.898$

$\simeq 44$ mins.

v) $P(n \geq k) = p(n>3) = p(n \geq 4)$

$= \left(\frac{\lambda}{\mu}\right)^4 = \left(\frac{0.05}{0.067}\right)^4 = 0.31$

3) Customer arrives at a watch ~~service~~ repair shop. According to a poisson ~~photos~~ process at 1 per every 10 min. and the service time is exponential with mean 8 mins. find

i) The average no of customers in the shop. (L_s)

ii) The avg. time a customer spends in shop (W_s)

iii) find the avg. no. of customers is queue (L_q)

iv) prob. ~~that~~ of the service is idle (P_0)

Sol

$\lambda = 1/10 = 0.1$ $\quad 1/\mu = 8$, $\mu = 1/8 = 0.125$

i) $L_s = \dfrac{\lambda}{\mu - \lambda} = \dfrac{0.1}{0.125 - 0.1} = 4$ persons

ii) $W_s = \dfrac{1}{\mu - \lambda} = \dfrac{1}{0.125 - 0.1} = 40$ mins

iii) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{(0.1)^2}{0.125(0.125 - 0.1)}$

$= 3.2 \simeq 3$ persons

iv) $P_0 = 1 - \dfrac{\lambda}{\mu} = 1 - \dfrac{0.1}{0.125} = \cancel{0.2}$

$= 0.2$ //

4) Customer arrives at an one window drive in bank according to poisson distribution with mean 10/hour. Service time for customer is exponential with mean 5 minutes. The space in front of the window including that for a serviced car can accomodate a max. of 3 cars. Others can wait outside the space.

i) what is the prob that an arriving customer can drive directly to the space in front of the window?

ii) what is the prob. that an arriving customer will have to wait outside the indicated space?

iii) how long an arriving customer expected to wait before being served? Wq

Sol:

$\lambda = 10$ per hour $\quad 1/\mu = 5$

$= 1/6$ per min. $\quad \mu = 1/5$ min.

i) P_0 (idle system)

$= 1 - \lambda/\mu$

$= 1 - \dfrac{1/6}{1/5} = 0.167$

$P_n = \left(\dfrac{\lambda}{\mu}\right)^n \left(1 - \dfrac{\lambda}{\mu}\right)$

$P_1 = \left(\dfrac{\lambda}{\mu}\right)\left(1 - \dfrac{\lambda}{\mu}\right) = \left(\dfrac{1/6}{1/5}\right)\left(1 - \dfrac{1/6}{1/5}\right) = 0.139$

$P_2 = \left(\dfrac{\lambda}{\mu}\right)^2 \left(1 - \dfrac{\lambda}{\mu}\right) = \left(\dfrac{1/6}{1/5}\right)^2 \left(1 - \dfrac{1/6}{1/5}\right) = 0.116$

$P_0 + P_1 + P_2 = \cancel{0.139}\ 0.421$

ii) $P = 1 - 0.421$

$= 0.579.$

iii) $W_q = \dfrac{\lambda}{\mu(\mu - \lambda)} = \dfrac{1/6}{1/5\,(1/5 - 1/6)}$

$= 24.41$ mins. $\equiv 25$ mint.

5) Arrivals at a telephone booth are considered to be a poisson with ~~an~~ avg time of 12 mins b/w 1 arrival and one next. Length of call is ~~dis~~ assumed to be distributed exponentially with mean 4 mins

i) find the avg no of peoples waiting in system (Ls)

ii) what is the prob that a person arriving at the booth hath to wait in the queue?

iii) Also estimate the fraction of one day, when phone will be in use.

iv) what is the prob that, it will take more than 10 mins for a person ~~to~~ wait and complete ~~his~~ hes call? Busy system

v) The Telephone department will install a second booth, when convinced ~~combines~~ that, the arrival would expect to wait atleast 3 mins (on avg).

Sol $\lambda = 1/12$ mins

$\mu = 1/4$ mins

i) $L_s = \dfrac{\lambda}{\mu - \lambda} = \dfrac{1/12}{1/4 - 1/12} = 0.5 \simeq 1$ person.

ii) System busy $P = \lambda/\mu = \dfrac{1/12}{1/4} = 0.333.$

iii) P(Phone in use) i.e, busy system $\Big\} = 0.333$

iv) $P(W_s > t) = e^{-(\mu-\lambda)t}$

$P(W_s > 10) = e^{-(1/4 - 1/12)(10)}$

$= \cancel{0.00}\ 0.1889$

v) The phone will be installed if the expectation of waiting time in the queue will be ≥ 3.

$W_q \geq 3$

$\Rightarrow \dfrac{\lambda}{\mu(\mu-\lambda)} \geq 3$

$\Rightarrow \lambda \geq 3\mu(\mu-\lambda)$

$\Rightarrow \lambda \geq 3 \cdot 1/4 (1/4 - \lambda)$

$\Rightarrow \lambda \geq 3/16 - 3/4\lambda$

$\Rightarrow \lambda + 3/4\lambda \geq 3/16$

$\Rightarrow \dfrac{7\lambda}{4} \geq 3/16$

$\lambda \geq 3/28$

Here, the increase in arrival rate should be atleast $= \frac{3}{28} - \frac{1}{12} \Rightarrow 0.0238$ mins.

to justify a second phone

Model-II $(M|M|1):(K|FIFO)$

(Single Server finite Capacity)

1. P_0 (idle system) $= \begin{cases} \dfrac{1-\lambda/\mu}{1-(\lambda/\mu)^{K+1}}, & \lambda \neq \mu \\ \dfrac{1}{K+1}, & \lambda = \mu \end{cases}$

2. Steady state $P_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^n P_0, & n = 0, 1, 2, \dots K. \\ 0, & n > K \end{cases}$

3. Avg no. of customers in the system.

$$L_s = \begin{cases} \dfrac{\lambda}{\mu-\lambda} - \dfrac{(K+1)(\lambda/\mu)^{K+1}}{1-\left(\dfrac{\lambda}{\mu}\right)^{K+1}}, & \lambda \neq \mu \\ \dfrac{K}{2}, & \lambda = \mu. \end{cases}$$

4. Effective arrival rate.

$$\lambda(eff) = \lambda' = \mu(1-P_0).$$

5. Avg no. customers in the queue

$$L_q = L_s - \frac{\lambda'}{\mu}.$$

6. Avg. time a customer has to spend in the system.

$$W_s = \frac{L_s}{\lambda'}$$

7. Avg time a customer has to spend in the queue.

$$W_q = L_q/\lambda'.$$

8. Probability of customer turned away.

$$P_k = \left(\frac{\lambda}{\mu}\right)^k P_0$$

Problems:

6) An one man barber shop can accomodate a maximum of 5 people at a time (4 waiting and 1 getting a haircut). Customer arrive ~~at a~~ according to a following poisson distribution with mean avg of 5 per hour and service is according to exponential distribution at an avg of 15 mins.

~~λ=5/hour~~

i) what is the % of idle time?

ii) Prob. of a customer turned away.

iii) what is the Expected no. of customers in the queue. (Lq)

iv) what is the Expected time spent in the shop. (Ws)

Sol

$\lambda = 5/\text{hour}$ $\quad \mu = \frac{1}{15}$ $\quad \boxed{k=5} = 4+1$

$= \frac{5}{60} = \frac{1}{12}$ mins

i) $P_0 = \dfrac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \dfrac{1-\frac{1/12}{1/15}}{1-\left(\frac{1/12}{1/15}\right)^{5+1}} = 089$

0.089

ii) $P_k = \left(\frac{\lambda}{\mu}\right)^k P_0$.

$= \left(\frac{\frac{1}{12}}{1/15}\right)^5 \times (0.089) \Rightarrow 0.271$

iii) first find,

$L_s = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)(\lambda/\mu)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$

$\Rightarrow \frac{1/12}{1/15 - 1/12} - \frac{6(1.25)^6}{1-(1.25)^6}$

$= -5 - \frac{22.88}{-2.81} = 3.13$

Then

$\Rightarrow L_q = L_s - \frac{\lambda'}{\mu}$.

$L_q = 3.13 - \left(\frac{0.061}{1/15}\right)$

$= 2.23$

$\simeq 2$ persons.

iv) $W_s = \frac{L_s}{\lambda'}$ where $\lambda' = \mu(1-P_0)$

$= 1/12 (1 - 0.089)$

$= 0.061$.

$= \frac{3.13}{0.061}$

$= 51.48$

$\simeq 51$ mins.

7) At a railway station only 1 train is handled at a time. The Railway yard is sufficient only for two trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an avg rate of 6/hour and the station can handle them on an avg of 12/hour. Assuming poisson arrival and exponential service distribution,

i) find the steady state probabilities of the various no. of trains in the system.

ii) Also find the avg. no. of trains in the system. (Ls)

iii) The avg. waiting time in the system. (Ws)

Sol

$\lambda = 6/\text{hour}$ $\quad \mu = 12/\text{hour}$ $\quad \boxed{k = 3.}$ $(2+1)$

$\lambda = 6$ $\quad \mu = 12$

i) $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad ; n = 1, 2, 3.$

where

$P_0 = \dfrac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{k+1}}$

$= \dfrac{1 - 0.5}{1 - (0.5)^4}$

$= 0.53$

$P_1 = (0.5)(0.53)$

$= 0.266.$

$P_2 = (0.5)^2 (0.53)$

$= 0.132$

$P_3 = (0.5)^3 (0.53)$

$= 0.66$

ii) $L_s = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)(\lambda/\mu)^{k+1}}{1-(\lambda/\mu)^{k+1}}$, $\lambda \neq \mu$

$= \frac{6}{12-6} - \frac{4(0.5)^4}{1-(0.5)^4} = 0.734 \simeq 1$ train

iii) $W_s = \frac{L_s}{\lambda'} = \frac{0.734}{5.604}$ where $\lambda' = \mu(1-P_0) = 12(1-0.53)$
$= 5.604$

$= 0.131$ hrs.

8) Patients arrive at a clinic, according to poisson distribution at 30/hour. The waiting room cannot acomodate more than 14 patients. Examination time ~~per~~ for patient is exponential ~~at~~ with mean rate of 20/hour. ~~find~~

i) find the Effective arrival rate at the clinic.

ii) what is the prob. that an arriving patient will not wait. ? idle.

iii) what is the expected waiting time until a patient is discharged from the clinic ? (hrs)

Sol $\lambda = 30$/hr. $\mu = 20$/hr. $k = 15.(14+1)$
$\lambda = 30$ $= 20.$

i) $\lambda' = \mu(1-P_0)$ where $P_0 = \frac{1-(1.5)}{1-(1.5)^{16}}$

$= 20(1-0.00076)$ $= 0.00076$

$= 19.98$

$\simeq 20$

ii) P_0 (idle system) = 0.00076.

iii) $W_s = \dfrac{L_s}{\lambda'}$

where

$$L_s = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)(\lambda/\mu)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{30}{20-30} - \frac{16 \times (1.5)^{16}}{1 - (1.5)^{16}}$$

$$= 13.024$$

$\therefore$ $W_s = \dfrac{13.024}{19.98} = 0.65$ hrs.