02/16.

THEORETICAL DISTRIBUTIONS.

i.) Binomial.	i.) Exponential
ii.) Poisson	ii.) Normal
iii.) Geometric	iii.) Uniform.

i.) Binomial distribution:-

A discrete random variable x is said to follow binomial distribution, if its probability mass function is given by

$$P(X=x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \dots n$$

$$q = 1-p.$$

Moment generating function, Mean, Variance of Binomial distribution.

$$(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + nC_3 a^{n-3} b^3 + \dots + b^n.$$

$$M_X(t) = E(e^{tx})$$
$$= \sum e^{tx} \cdot p(x)$$
$$= \sum_{x=0}^{n} e^{tx} \cdot nC_x p^x q^{n-x}$$

$$= nC_0 (pe^t)^0 q^n + nC_1 + (pe^t)^1 q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + nC_n (pe^t)^n$$

$$= q^n + nC_1 (pe^t)^1 q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + (pe^t)^n$$

$$\boxed{M_x(t) = (q + pe^t)^n}$$

Mean

Recall the formula

$$\left[\because \mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{at\ t=0}\right]$$

Now,

$$E(x) = \mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{at\ t=0}$$

$$= \left[\frac{d}{dt}(q+pe^t)^n\right]$$

$$= \left[n(q+pe^t)^{n-1}\right]_{at\ t=0}$$

[1st differentiation]

$$= npe^0 (q+pe^0)^{n-1}$$

$$= np(q+p)^{n-1}$$

$$= np(1)^{n-1}$$

$$\boxed{\begin{array}{l}\text{Mean} = np \\ E(x)\end{array}}$$

Now,

$$E(x^2) = \mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{at\ t=0}$$

$$= \left[\frac{d^2}{dt^2} (q+pe^t)^n\right]_{at\ t=0}$$

$$= \left[\frac{d}{dt} npe^t (q+pe^t)^{n-1}\right]_{at\ t=0}$$ [in the form uv]

$$= np\{e^t(n-1)(q+pe^t)^{n-2}(0+pe^t)+(q+pe^t)^{n-1}e^t\}$$

at $t=0$

$$\left[\because \frac{d}{dx}uv = uv' + vu'\right]$$

$$E(x^2) = np[e^0(n-1)(q+pe^0)^{n-2}(pe^0)+(q+pe^0)^{n-1}e^0]$$

$$= np[(n-1)p+1]$$

$$E(x^2) = np[(np-p+1)]$$

$$E(x^2) = np(np-p+1)$$

$$E(x^2) = n^2p^2 - np^2 + np$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= n^2p^2 - np^2 + np - (n.p)^2$$

$$= np - np^2$$

$$\boxed{var(x) = np(1-p)}$$

and

$$\boxed{variance = npq}$$

Problems under Binomial Distribution :-

1. For a binomial distribution, mean is 6 and Standard deviation is $\sqrt{2}$. Find the first two terms of the distribution.

Soln → Mean = $np = 6$

Standard deviation = $\sqrt{\text{variance}}$

$= \sqrt{npq}$

$= \sqrt{2}$.

$np = 6$ — ①

$npq = 2$ — ②

Put ① in ②

$6q = 2$

$q = 2/6$

$\boxed{q = 1/3}$

$\therefore p = 1 - q = 2/3$

$\boxed{p = 2/3}$

We have $np = 6$

$n(2/3) = 6$

$\boxed{n = 9}$

For a B.D,

$P(x = x) = nC_x \, p^x q^{n-x}$

$x = 0, 1, 2, 3, \dots, n$

Put $(x = 0) = 9C_0 \left(2/3\right)^0 \left(1/3\right)^{9-0}$

$= \left(\frac{1}{3}\right)^9$

$\boxed{\text{Put } x = 1}$

$P(x=1) = 9C_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{9-1}$

$= 18 \times \left(\frac{1}{3}\right)^{9}$

$= 18 \times \left(\frac{1}{3}\right)^{9} = 18 \times \frac{1}{19683}$

$= \frac{18}{19683}$

2) If mean is 4 and variance is 3, find $P(x=1)$ for a binomial distribution.

soln→2→ For a binomial distribution

Mean = 4

variance = 3

$\Rightarrow np = 4$ — ①

$\Rightarrow npq = 3$ — ②

Subsitute ① in ②

$\Rightarrow 4q = 3$

$\Rightarrow \boxed{\therefore q = 3/4}$

$\Rightarrow \therefore p = 1 - 3/4 = 1/4$

$\Rightarrow \boxed{p = 1/4}$

$\Rightarrow n \times \frac{1}{4} = 4$ [considering the Equation 1]

$\Rightarrow \boxed{n = 16}$

$P(x=1) = nC_x p^x q^{n-x}$

$= 16C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{16-1}$

$= 16 \times \frac{1}{4} \times \left(\frac{3}{4}\right)^{15}$

$= 4 \times \left(\frac{3}{4}\right)^{15} = 4 \times \left(\frac{3}{4}\right)^{15}.$

3.) If mean is 2, and the variance is $4/3$, find $P(x=5)$ for the binomial distribution.

Soln→3→ For a binomial distribution.

Mean $= 2$

variance $= 4/3$

$\Rightarrow np = 2$ — ①

$\Rightarrow npq = 4/3$ — ②

subsitute ① in ②

$\Rightarrow 2 \times q = \frac{4}{3}$

$\Rightarrow 2q = \frac{4}{3}$

$\Rightarrow \boxed{\therefore q = \frac{\cancel{4}2}{\cancel{6}3}} \Rightarrow \boxed{q = \frac{2}{3}}$

$\Rightarrow p = 1 - q$

$\Rightarrow \therefore p = 1 - \frac{4}{6} = \frac{\cancel{2}}{\cancel{6}} = \frac{1}{3}$

$\Rightarrow \boxed{\therefore p = \frac{1}{3}}$

$\Rightarrow np = 2$

$\Rightarrow n\frac{1}{3} = 2$

$\Rightarrow \therefore n = 2 \times 3$

$\Rightarrow \boxed{n = 6}$

$P(x=5) = nC_x \, p^x q^{n-x}$

$= 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{6-5}$

$= 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1$

$= \cancel{6} \times \left(\frac{1}{3}\right)^5 \left(\frac{2}{\cancel{3}}\right)^1$

$= 4 \times \left(\frac{1}{3}\right)^5$

$= \frac{4}{243}$

4.) If X is a binomial random variable with $n=6$ satisfying $9(P(X=4)) = P(X=2)$, find p.

In→4→ For a B.D

$$P(X=x) = nC_x p^x q^{n-x},$$

$$x = 0, 1, 2, \dots, n.$$

Given:

$$9\,P(X=4) = P(X=2)$$

$$\Rightarrow 9\left[6C_4 p^4 q^2\right] = 6C_2 p^2 q^4$$

$$\Rightarrow 9 \times \overset{15}{} p^4 q^2 = 15 p^2 q^4$$

$$\Rightarrow 9p^2 = q^2$$

$$\Rightarrow 9p^2 = (1-p)^2$$

$$\Rightarrow 9p^2 = 1 + P - 2P$$

$$\Rightarrow 8p^2 + 2p - 1 = 0$$

$$\Rightarrow p = \frac{-2 \pm \sqrt{4 - 4 \times 8(-1)}}{16}$$

$$\Rightarrow p = \frac{-2 \pm \sqrt{36}}{16}$$

$$\Rightarrow p = \frac{-2 \pm 6}{16}$$

$$\Rightarrow p = 1/4, -1/2$$

P cannot be negative.

$$\therefore p = 1/4$$

5.) Q Four coins are tossed simultaneously. What is the probability of getting
R.V.
a.) exactly 2 heads
b.) atleast 2 heads
c.) atmost 2 heads.

Ans 5→ $n=4$

p = probability of getting head $= 1/2$

$q = 1/2$

For a B.D

$P(X=x) = nC_x p^x q^{n-x}$

$x = 0, 1, 2, 3, \ldots, n$

a.) $P(X=2) = 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$

$= \cancel{6}^{3} \times \frac{1}{\cancel{4}_2} \times \frac{1}{4}$

$= \frac{3}{8}$

$= 0.375.$

b.) $P(X \geq 2) = P(X=2) + P(X=3) + P(X=4)$

$= 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + 4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 + 4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0$

$= 0.375 + 0.25 + 0.0625$

$= 0.6875.$

c.) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= 4C_0 (1/2)^0 (1/2)^4 + 4C_1 (1/2)^1 (1/2)^3 + 4C_2 (1/2)^2 (1/2)^2$

$= 0.0625 + 0.25 + 0.375$

$= 0.6875.$

Aliter:

$$P(x \leq 2) = 1 - P(x > 2)$$
$$= 1 - P((x=3) + P(x=4))$$
$$= 1 - (0.25 + 0.0625)$$
$$= 0.6875$$

6) Fit a Binomial distribution for the following data:

x	0	1	2	3	4	5	Total
frequency f	2	14	20	34	22	8	100

$$P(X=x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

Soln→6→

x	f	fx	Expected frequency $= N \times nC_x p^x q^{n-x}$
0	2	0	$100 \times 5C_0 (0.568)^0 (0.432)^5$ $= 0.0150 \times 100 \simeq 1.5$ Round) approx figure
1	14	14	$100 \times 5C_1 (0.568)^1 (0.432)^4$ $= 9.7868$ $\simeq 10$
2	20	40	25.4638 $\simeq 26$
3	34	102	34.1989 $\simeq 34$
4	22	88	22.4826 $\simeq 22$
5	8	40	5.9121 $\simeq 6$
	N=100	284	

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{284}{100} = 2.84$$

$$\text{Mean} = np = 2.84$$

Here, $\boxed{n = 5}$

$5 \cdot p = 2.84$

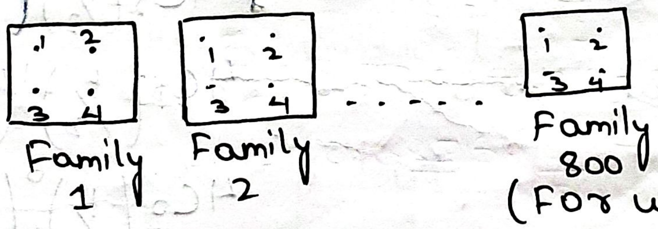
$$\boxed{p = 0.568}$$

$$\boxed{\therefore q = 0.432}$$

x	0	1	2	3	4	5
Observed frequency	2	14	20	34	22	8
Theoretical frequency	2	10	26	34	22	6

7.) Out of 800 families with 4 children each, how many families would be expected to have
a.) 2 boys and 2 girls
b.) atleast 1 boy
c.) atmost 2 girls
d.) children of both sexes.
Assume equal probabilities for boys and girls.

Soln→7) $N = 800, n = 4$



(For understanding)

$P(\text{success})$ = Probability of getting a girl $= \frac{1}{2}$

$q = \frac{1}{2}$

For a binomial distribution,

$$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x},$$
$$x = 0, 1, 2, \ldots n.$$

a.) 2 boys and 2 girls

$P(x=2) = 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$

$= \frac{3}{8}$

$\therefore$ out of 800 families, no. of families having 2 boys and 2 girls $= 800 \times \frac{3}{8} = 300$ families.

b.) at least 1 boy

Boy	Girl
1	3
2	2
3	1
4	0

→ Success.

$P(\text{getting at least 1 boy}) = P(x=3) + P(x=2) + P(x=1) + P(x=0)$

$= 4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + 4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4$

$= 0.9375$

$\therefore$ Out of 800 families, no. of families having at least 1 boy $= 800 \times 0.9375$

$= 750$ families.

c.) atmost 2 girls

≤2

Boy	Girl
2	2
3	1
4	0
	└ success

$P(\text{getting atmost } 2 \text{ girls}) = P(x=2)+P(x=1)+P(x=0)$

$= 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + 4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4$

$= 0.6875$

Out of 800 families, no. of families having atmost 2 girls $= 0.6875 \times 800$
$= 550$ families.

d.) Children of both sides:-

Boy	Girl
1	3
2	2
3	1
	└ success.

P(getting children of both sexes)

$= P(x=3)+P(x=2)+P(x=1)$

$= 4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 +$

$4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3$

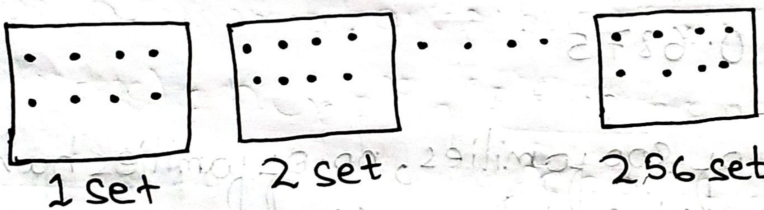
$= 0.875$

∴ out of 800 families, no. of families having children of both sexes $= 800 \times 0.875$

$= 700$ families.

8.) In 256 sets of 8 tosses of a coin, in how many sets one may expect heads and tails in equal numbers?

soln→ 8→



$N = 256$

$n = 8$

p = probability of getting head $= \frac{1}{2}$

$q = \frac{1}{2}$

For a binomial distribution,

$P(x=x) = nC_x p^x q^{n-x}$

$x = 0, 1, 2, \ldots n$

$P(x=4) = 8C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4$

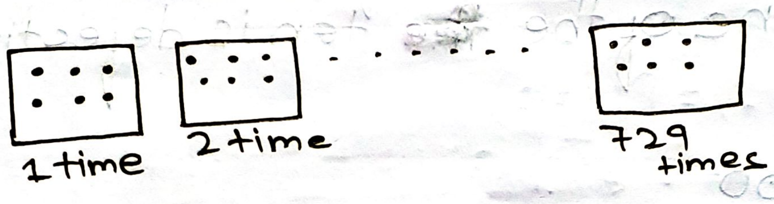
$= 0.2734$

$= 5/32$

In 256 sets, number of sets having heads and tails in equal number $= 256 \times 0.2734$
$= 70$ sets.

9.) Six dice are thrown 729 times. How many times do you expect atleast three dice to show 5 or 6?

Soln→9→



$N = 729$

$n = 6$

p = probability of getting 5 or 6

$= \frac{1}{6} + \frac{1}{6} = \frac{\cancel{2}}{\cancel{6}} = \frac{1}{3}$

$\Rightarrow q = 2/3$

For a B.D,

$P(X=x) = {}^nC_x p^x q^{n-x}$

$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + 6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1 + 6C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0$

$= 0.3196$

$\therefore$ Out of 729 times, number of times atleast 3 dice to show 5 or 6

$= 729 \times 0.3196$

$= 232.9$

$= 233.$

10.) A box contains 100 cell phones, 20 of which are defective. 10 cellphones are selected for inspection. Find the probability that:

i.) atleast 1 is defective
ii.) atmost 3 are defective
iii.) none of the ten is defective.

Soln→10→ $N = 100$

$n = 10$

p = probability of getting a defective $= \frac{20}{100}$

$= \frac{1}{5}$

$q = \frac{4}{5}$

For a binomial distribution,

$P(x = x) = nC_x p^x q^{n-x}$

$x = 0, 1, 2, \ldots, n$

$P(x \geq 1) = P(x=1) + P(x=2) + \ldots P(x=10)$

$= 1 - P(x < 1)$

$= 1 - P(x=0)$

$= 1 - \left[10C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10}\right]$

$= 0.893$

∴ out of 100 cell phones, no. of phones having atleast 1 defective $= 100 \times 0.893$

$= 89.3$

$\simeq 89$ cellphones.

b) Atmost 3 defective

$$P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$

$$= 10C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10} + 10C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^9 + 10C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8$$

$$+ 10C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^7$$

$$= 0.8791.$$

c) none of the ten is defective $= 0.1074.$