CS 70 Discrete Mathematics and Probability Theory
Summer 2019 James Hulett and Elizabeth Yang **Homework 7**

1 Waiting For the Bus

Edward and Jerry are waiting at the bus stop outside of Soda Hall.

Like many bus systems, buses arrive in periodic intervals. However, the Berkeley bus system is unreliable, so the length of these intervals are random, and follow Exponential distributions.

Edward is waiting for the 51B, which arrives according to an Exponential distribution with parameter λ. That is, if we let the random variable X_i correspond to the difference between the arrival time ith and $i-1$st bus (also known as the inter-arrival time) of the 51B, $X_i \sim \text{Expo}(\lambda)$.

Jerry is waiting for the 79, whose inter-arrival time, follows an Exponential distributions with parameter μ. That is, $Y_i \sim \text{Expo}(\mu)$. Assume that all inter-arrival times are independent.

(a) What is the probability that Jerry's bus arrives before Edward's bus?

(b) After 20 minutes, the 79 arrives, and Jerry rides the bus. However, the 51B still hasn't arrived yet. Let Let D be the additional amount of time Edward needs to wait for the 51B to arrive. What is the distribution of D?

(c) Lavanya isn't picky, so she will wait until either the 51B or the 79 bus arrives. Solve for the distribution of Z, the amount of time Lavanya will wait before catching the bus.

(d) Khalil arrives at the bus stop, but he doesn't feel like riding the bus with Edward. He decides that he will wait for the second arrival of the 51B to ride the bus. Find the distribution of $T = X_1 + X_2$, the amount of time that Khalil will wait to ride the bus. [*Hint*: One way to approach this problem would be to compute the CDF of T and then differentiate the CDF.]

Solution:

(a)

$$\begin{aligned}
\mathbb{P}(Y_i < X_i) &= \int_{t=0}^{\infty} \mathbb{P}(X_i = t \cap Y_i < t) \\
&= \int_{t=0}^{\infty} \mathbb{P}(X_i = t)\mathbb{P}(Y_i < t) \\
&= \int_{t=0}^{\infty} \lambda e^{-\lambda t}(1 - e^{-\mu t}) \\
&= \int_{t=0}^{\infty} \lambda e^{-\lambda t} - \lambda e^{-(\lambda+\mu)t}) \\
&= \int_{t=0}^{\infty} \lambda e^{-\lambda t} - \int_{t=0}^{\infty} \lambda e^{-(\lambda+\mu)t} \\
&= 1 - \frac{\lambda}{\lambda+\mu} \\
&= \frac{\mu}{\mu+\lambda}
\end{aligned}$$

(b) We observe that $\mathbb{P}(D > d) = \mathbb{P}(X > 20 + d | X \geq 20)$. Then, we apply Bayes Rule:

$$\begin{aligned}
\mathbb{P}(X > 20 + d | X \geq 20) &= \frac{\mathbb{P}(X > 20 + d)}{\mathbb{P}(X \geq 20)} \\
&= \frac{e^{\mu(20+d)}}{e^{20\mu}} \\
&= e^{\mu d}
\end{aligned}$$

Thus, the CDF of D is given by $\mathbb{P}(D \leq d) = 1 - \mathbb{P}(D > d) = 1 - e^{\mu d}$. Thus, D is exponentially distributed with parameter μ.

One can also directly apply the memoryless property of the exponential distribution to arrive at this answer.

(c) Lavanya's waiting time is the minimum of the time it takes for the 51B and the time it takes for the 79 to arrive. Thus, $Z = \min(X, Y)$. Following the hint,

$$\begin{aligned}
\mathbb{P}(Z > t) &= \mathbb{P}(X > t \cap Y > t) \\
&= \mathbb{P}(X > t)\mathbb{P}(Y > t) \\
&= (1 - \mathbb{P}(X \leq t))(1 - \mathbb{P}(Y \leq t)) \\
&= (1 - (1 - e^{-\mu t}))(1 - (1 - e^{-\lambda t})) \\
&= e^{-\mu t} e^{-\lambda t} \\
&= e^{-(\mu+\lambda)t}
\end{aligned}$$

It follows that the CDF is Z, $\mathbb{P}(Z \leq t) = 1 - e^{-(\mu+\lambda)t}$. Thus, Z is exponentially distributed with parameter $\mu + \lambda$.

(d) Let $t > 0$. Observe that if $X_1 + X_2 \le t$, then since $X_1, X_2 \ge 0$, it follows that $X_1 \le t$ and $X_2 \le t - X_1$.

$$\begin{aligned}
\mathbb{P}(T \le t) = \mathbb{P}(X_1 \le t, X_2 \le t - X_1) &= \int_0^t \int_0^{t-x_1} \lambda \exp(-\lambda x_1) \lambda \exp(-\lambda x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \lambda^2 \int_0^t \exp(-\lambda x_1) \cdot \frac{1 - \exp(-\lambda(t - x_1))}{\lambda}\,\mathrm{d}x_1 \\
&= \lambda \int_0^t \bigl(\exp(-\lambda x_1) - \exp(-\lambda t)\bigr)\,\mathrm{d}x_1 = \lambda \Bigl(\frac{1 - \exp(-\lambda t)}{\lambda} - t\exp(-\lambda t)\Bigr).
\end{aligned}$$

Upon differentiating the CDF, we have

$$\begin{aligned}
f_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(T \le t) &= \lambda \exp(-\lambda t) - \lambda \exp(-\lambda t) + \lambda^2 t \exp(-\lambda t) \\
&= \lambda^2 t \exp(-\lambda t), \qquad t > 0.
\end{aligned}$$

2 Exponential Practice

Let $X \sim \text{Exponential}(\lambda_X)$ and $Y \sim \text{Exponential}(\lambda_Y)$ be independent, where $\lambda_X, \lambda_Y > 0$. Let $U = \min\{X, Y\}$, $V = \max\{X, Y\}$, and $W = V - U$.

(a) Compute $\mathbb{P}(U > t, X \le Y)$, for $t \ge 0$.

(b) Use the previous part to compute $\mathbb{P}(X \le Y)$. Conclude that the events $\{U > t\}$ and $\{X \le Y\}$ are independent.

(c) Compute $\mathbb{P}(W > t \mid X \le Y)$.

(d) Use the previous part to compute $\mathbb{P}(W > t)$.

(e) Calculate $\mathbb{P}(U > u, W > w)$, for $w > u > 0$. Conclude that U and W are independent. [*Hint*: Think about the approach you used for the previous parts.]

Solution:

(a) One has

$$\begin{aligned}
\mathbb{P}(U > t, X \le Y) = \mathbb{P}(t < X \le Y) &= \int_t^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_t^\infty \int_x^\infty \lambda_X \exp(-\lambda_X x) \lambda_Y \exp(-\lambda_Y y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lambda_X \lambda_Y \int_t^\infty \exp(-\lambda_X x) \cdot \frac{\exp(-\lambda_Y x)}{\lambda_Y}\,\mathrm{d}x = \lambda_X \int_t^\infty \exp\bigl(-(\lambda_X + \lambda_Y)x\bigr)\,\mathrm{d}x \\
&= \frac{\lambda_X}{\lambda_X + \lambda_Y} \exp\bigl(-(\lambda_X + \lambda_Y)t\bigr).
\end{aligned}$$

(b) Take $t = 0$.

$$\mathbb{P}(X \le Y) = \frac{\lambda_X}{\lambda_X + \lambda_Y}.$$

Since X and Y are independent exponentials, $U = \min\{X,Y\} \sim \text{Exponential}(\lambda_X + \lambda_Y)$. So, $\mathbb{P}(U > t) = \exp(-(\lambda_X + \lambda_Y)t)$, and therefore we have $\mathbb{P}(U > t, X \le Y) = \mathbb{P}(X \le Y)\mathbb{P}(U > t)$.

(c) One has

$$\begin{aligned}
\mathbb{P}(W > t, X \le Y) = \mathbb{P}(Y - X > t) &= \int_0^\infty \int_{x+t}^\infty \lambda_X \exp(-\lambda_X x)\lambda_Y \exp(-\lambda_Y y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lambda_X \lambda_Y \int_0^\infty \exp(-\lambda_X x) \cdot \frac{\exp(-\lambda_Y (x+t))}{\lambda_Y}\,\mathrm{d}x \\
&= \lambda_X \exp(-\lambda_Y t) \int_0^\infty \exp\big(-(\lambda_X + \lambda_Y)x\big)\,\mathrm{d}x = \frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-\lambda_Y t).
\end{aligned}$$

So, we see that

$$\mathbb{P}(W > t \mid X \le Y) = \frac{\mathbb{P}(W > t, X \le Y)}{\mathbb{P}(X \le Y)} = \exp(-\lambda_Y t).$$

Alternatively,

$$\begin{aligned}
\mathbb{P}(W > t \mid X \le Y) = \mathbb{P}(Y > X + t \mid X \le Y) &= \int_0^\infty \mathbb{P}(Y > x + t \mid Y \ge x) f_X(x)\,\mathrm{d}x \\
&= \exp(-\lambda_Y t) \int_0^\infty f_X(x)\,\mathrm{d}x = \exp(-\lambda_Y t),
\end{aligned}$$

where we have used the memoryless property of the exponential distribution. Note that in the first line, we are using conditioning:

$$\mathbb{P}(Y > X + t \mid X \le Y) = \int_0^\infty \mathbb{P}(Y > X + t \mid X \le Y, X = x) f_X(x)\,\mathrm{d}x.$$

The probability inside the integral then becomes $\mathbb{P}(Y > x + t \mid Y \ge x, X = x)$, and then one can drop the conditioning on $X = x$ because X and Y are independent.

(d) By switching X and Y in the previous part, we have

$$\mathbb{P}(W > t \mid Y \le X) = \exp(-\lambda_X t).$$

So, we can use the law of total probability to give

$$\begin{aligned}
\mathbb{P}(W > t) &= \mathbb{P}(X \le Y)\mathbb{P}(W > t \mid X \le Y) + \mathbb{P}(Y \le X)\mathbb{P}(W > t \mid Y \le X) \\
&= \frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-\lambda_Y t) + \frac{\lambda_Y}{\lambda_X + \lambda_Y} \exp(-\lambda_X t).
\end{aligned}$$

(e) We calculate

$$\begin{aligned}
\mathbb{P}(U > u, W > w, X \le Y) &= \mathbb{P}(u < X \le X + w < Y) \\
&= \int_u^\infty \int_{x+w}^\infty \lambda_X \exp(-\lambda_X x) \lambda_Y \exp(-\lambda_Y y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lambda_X \lambda_Y \int_u^\infty \exp(-\lambda_X x) \cdot \frac{\exp(-\lambda_Y(x+w))}{\lambda_Y}\,\mathrm{d}x \\
&= \lambda_X \exp(-\lambda_Y w) \int_u^\infty \exp\big(-(\lambda_X + \lambda_Y)x\big)\,\mathrm{d}x \\
&= \frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-\lambda_Y w) \exp\big(-(\lambda_X + \lambda_Y)u\big).
\end{aligned}$$

By switching the roles of X and Y in the above computation, we obtain

$$\mathbb{P}(U > u, W > w, Y \le X) = \frac{\lambda_Y}{\lambda_X + \lambda_Y} \exp(-\lambda_X w) \exp\big(-(\lambda_X + \lambda_Y)u\big).$$

Now, we add the two expressions together to obtain

$$\begin{aligned}
\mathbb{P}(U > u, W > w) &= \Big(\frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-\lambda_Y w) + \frac{\lambda_Y}{\lambda_X + \lambda_Y} \exp(-\lambda_X w)\Big) \exp\big(-(\lambda_X + \lambda_Y)u\big) \\
&= \mathbb{P}(W > w)\mathbb{P}(U > u).
\end{aligned}$$

So, U and W are independent!

3 Normal Darts?

Alex and John are playing a game of darts. Let (X_a, Y_a) and (X_j, Y_j) denote the coordinates of Alex's and John's darts on the board and are distributed in the following way:

1. $X_a, Y_a \sim \mathbb{N}(0,1)$ independently
2. X_j, Y_j are distributed uniformly in a circle of radius 3

The winner of the game is determined by whoever's darts is closer to the center of the board at $(0,0)$. In this question, we will compute the probability that Alex wins the game. We will denote the squared distances of the darts from the center by $r_a = X_a^2 + Y_a^2$ and by $r_j = X_j^2 + Y_j^2$.

(a) What is the distribution of r_a?
Hint: Consider the joint distribution and the following change of variables formula: Suppose we want to integrate the function $f(x,y)$ over the circle $\left(\sqrt{x^2+y^2} \le R\right)$. Then, we have the following change of variables formula:

$$\int_{\sqrt{x^2+y^2} \le R} f(x,y)\,dx\,dy = \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$$

You may find the identity $\sin(\theta)^2 + \cos(\theta)^2 = 1$ useful.

(b) What is the distribution of r_j? (Hint: Try computing the CDF first)

(c) What is the probability that Alex wins the game?

Solution:

(a) Since X_a, X_j are independent, the joint distribution of (X_a, X_j) is given by:

$$f(x,y) = f(x)f(y) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})\frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2}) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)$$

Now, we have:

$$\begin{aligned}\mathbb{P}(r_a \leq t) &= \int_{x^2+y^2\leq t}\frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy = \int_{\sqrt{x^2+y^2}\leq\sqrt{t}}\frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy \\ &= \int_{r=1}^{\sqrt{t}}\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right)rd\theta dr = \int_{r=0}^{\sqrt{t}} r\exp(-r^2/2)dr \\ &= -\exp(-r^2/2)|_{r=0}^{\sqrt{t}} = 1-\exp(-t/2)\end{aligned}$$

(b) Since X_j, Y_j are distributed uniformly in the circle, the cdf of r_j is the area of the inner circle of radius k over the total area of the circle. Therefore, the CDF of the distribution of r_j is given by:

$$\mathbb{P}\left[r_j \leq k\right] = \frac{\pi\cdot k}{\pi\cdot 3^2} = \frac{k}{9}$$

Therefore, we get that the pdf of r_j is given by:

$$f_{r_j}(x) = \frac{1}{9}I\{x \leq 9\}$$

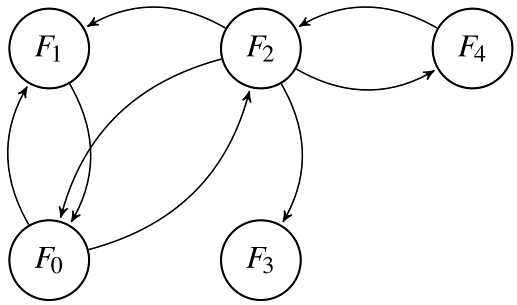
(c) We want to calculate the probability of Alex winning. We integrate over the possible values that John can be away from the board (r_j can be from 0 to 9):

$$\mathbb{P}[\text{Alex Winning}] = \mathbb{P}[r_a \leq r_j] = \int_{t=0}^{9}\mathbb{P}[r_a \leq t]f_{r_j}(t)dt = \int_{t=0}^{9}\frac{1}{9}(1-e^{-\frac{t}{2}}) = \frac{7}{9}+\frac{2}{9}e^{-\frac{9}{2}}$$

4 The Dwinelle Labyrinth

You have decided to take a humanities class this semester, a French class to be specific. Instead of a final exam, your professor has issued a final paper. You must turn in this paper *before* noon to the professor's office on floor 3 in Dwinelle, and it's currently 11:48 a.m.

Let Dwinelle be modeled by the following Markov chain. Instead of rushing to turn it in, we will spend valuable time computing whether or not we *could have* made it. Suppose walking between floors takes 1 minute.



(a) Will you make it in time if you choose a floor to transition to uniformly at random? (If T_i is the number of steps needed to get to F_3 starting from F_i, where $i \in \{0,1,2,3,4\}$, is $\mathbb{E}[T_0] < 12$?)

(b) Will you make it in time, if for every floor, you order all accessible floors and are twice as likely to take higher floors? (If you are considering 1, 2, or 3, you will take each with probabilities $1/7$, $2/7$, $4/7$, respectively.)

Solution:

(a) Write out all of the first-step equations.

$$\begin{aligned}
\mathbb{E}[T_0] &= 1 + \frac{1}{2}\mathbb{E}[T_1] + \frac{1}{2}\mathbb{E}[T_2] \\
\mathbb{E}[T_1] &= 1 + \mathbb{E}[T_0] \\
\mathbb{E}[T_2] &= 1 + \frac{1}{4}\mathbb{E}[T_0] + \frac{1}{4}\mathbb{E}[T_1] + \frac{1}{4}\mathbb{E}[T_3] + \frac{1}{4}\mathbb{E}[T_4] \\
\mathbb{E}[T_3] &= 0 \\
\mathbb{E}[T_4] &= 1 + \mathbb{E}[T_2]
\end{aligned}$$

Let us rewrite these equations, before placing it in matrix form.

$$\begin{aligned}
-1 &= -\mathbb{E}[T_0] + \frac{1}{2}\mathbb{E}[T_1] + \frac{1}{2}\mathbb{E}[T_2] \\
-1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\
-1 &= -\mathbb{E}[T_2] + \frac{1}{4}\mathbb{E}[T_0] + \frac{1}{4}\mathbb{E}[T_1] + \frac{1}{4}\mathbb{E}[T_3] + \frac{1}{4}\mathbb{E}[T_4] \\
0 &= \mathbb{E}[T_3] \\
-1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2]
\end{aligned}$$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/2 & 1/2 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/4 & 1/4 & -1 & 1/4 & 1/4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We can now reduce the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 13 \end{bmatrix}$$

We see that $\mathbb{E}[T_0] = 15$, meaning it will take 15 minutes for us to get to floor 3. Unfortunately, we only have 12 minutes.

(b) Write out all of the first-step equations.

$$\begin{aligned} \mathbb{E}[T_0] &= 1 + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\ \mathbb{E}[T_1] &= 1 + \mathbb{E}[T_0] \\ \mathbb{E}[T_2] &= 1 + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\ \mathbb{E}[T_3] &= 0 \\ \mathbb{E}[T_4] &= 1 + \mathbb{E}[T_2] \end{aligned}$$

Let us rewrite these equations, before placing it in matrix form.

$$\begin{aligned} -1 &= -\mathbb{E}[T_0] + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\ -1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\ -1 &= -\mathbb{E}[T_2] + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\ 0 &= \mathbb{E}[T_3] \\ -1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2] \end{aligned}$$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/3 & 2/3 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/15 & 2/15 & -1 & 4/15 & 8/15 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We row reduce to get the following.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 9.75 \\ 0 & 1 & 0 & 0 & 0 & 10.75 \\ 0 & 0 & 1 & 0 & 0 & 7.75 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8.75 \end{bmatrix}$$

We see that $\mathbb{E}[T_0] = 9.75$, meaning it will take 9.75 minutes for us to get to floor 3. That's fewer than 12 minutes, so if you finished this computation in less than 2 minutes and 15 seconds, you could make it!

5 Faulty Machines

You are trying to use a machine that only works on some days. If on a given day the machine is working, it will break down the next day with probability $0 < b < 1$, and works on the next day $1-b$. If it is not working on a given day, it will work on the next day with probability $0 < r < 1$, and not work on the next day with probability $1-r$. Formulate this process as a Markov chain. As $n \to \infty$, what does the probability that on a given day the machine is working converge to? What properties of the Markov chain allow us to conclude that the probability will actually converge?

Solution: We define the following states $\chi = \{W, B\}$ where W is the state that represents the machine working on a given day, and B is the state that represents the machine broken on a given day. The following are the transition probabilities.

$$\mathbb{P}(W,B) = b; \mathbb{P}(W,W) = 1-b; \mathbb{P}(B,W) = r; \mathbb{P}(B,B) = 1-r$$

We know that the Markov chain is finite and irreducible. Hence, it has a unique invariant distribution π

Furthermore, since the Markov chain has a self-loop of nonzero probability, it is aperiodic.

Hence, for any probability distribution of states at time n, $\pi_n : \lim_{n\to\infty} \pi_n = \pi$

We use the balance equations to find the invariant distribution.

$$\pi = \pi P$$
$$\pi(W) = (1-b)\pi(W) + r\pi(B)$$
$$\pi(B) = b\pi(W) + (1-r)\pi(B)$$
$$\pi(B) + \pi(W) = 1$$
$$\implies \pi(W) = \frac{r}{b+r}, \pi(B) = \frac{b}{b+r}$$

$\implies$ As $n \to \infty$, the probability that on a given day the machine is working is $\pi(W) = \dfrac{r}{b+r}$

6 Three Tails

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Solution:

We can model this problem as a Markov chain with the following states:

- S: Start state, which we are only in before flipping any coins.
- H: We see a head, which means no streak of tails currently exists.
- T: We've seen exactly one tail in a row so far.
- TT: We've seen exactly two tails in a row so far.

- TTT: We've accomplished our goal of seeing three tails in a row and stop flipping.

We can write the first step equations and solve for $\gamma(S)$, only counting heads that we see since we are not looking for the total number of flips. The equations are as follows:

$$\begin{aligned}
\gamma(S) &= .5\gamma(T) + .5\gamma(H) \\
\gamma(H) &= 1 + .5\gamma(H) + .5\gamma(T) \\
\gamma(T) &= .5\gamma(TT) + .5\gamma(H) \\
\gamma(TT) &= .5\gamma(H) + .5\gamma(TTT) \\
\gamma(TTT) &= 0
\end{aligned}$$

From the second equation, we see that

$$.5\gamma(H) = 1 + .5\gamma(T)$$

and can substitute that into equation 3 to get

$$.5\gamma(T) = .5\gamma(TT) + 1.$$

Substituting this into equation 4, we can deduce that $\gamma(TT) = 4$. This allows us to conclude that $\gamma(T) = 6$, $\gamma(H) = 8$, and $\gamma(S) = 7$. On average, we expect to see 7 heads before flipping three tails in a row.