

Note: This homework consists of two parts. The first part (questions 1-5) will be graded and will determine your score for this homework. The second part (questions 6-8) will be graded if you submit them, but will not affect your homework score in any way. You are strongly advised to attempt all the questions in the first part. You should attempt the problems in the second part only if you are interested and have time to spare.

Part 1: Required Problems

1 Induction

Prove the following using induction:

- (a) For all natural numbers n , $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$ is a natural number.
- (b) Let a and b be integers with $a \neq b$. For all natural numbers $n \geq 1$, $(a^n - b^n)$ is divisible by $(a - b)$.
- (c) For all natural numbers n , $(2n)! \leq 2^{2n}(n!)^2$. [Note that $0!$ is defined to be 1.]

Solution:

- (a)
 - Base case (n=0): $P(0)$ asserts that $\frac{0^3}{3} + \frac{0^2}{2} + \frac{0}{6} = 0 \in \mathbb{N}$.
 - Inductive Hypothesis: Assume for arbitrary $k \geq 0$, $P(k)$ is $\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} \in \mathbb{N}$.
 - Inductive Step: Prove the statement for $n = k + 1$: $P(k + 1)$.

$$\begin{aligned}
 \frac{(k+1)^3}{3} + \frac{(k+1)^2}{2} + \frac{k+1}{6} &= \frac{k^3 + 3k^2 + 3k + 1}{3} + \frac{k^2 + 2k + 1}{2} + \frac{k+1}{6} \\
 &= \frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} + \frac{3k^2 + 3k + 1}{3} + \frac{2k + 1}{2} + \frac{1}{6} \\
 &= \underbrace{\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6}}_{\in \mathbb{N}} \quad (\text{Inductive Hypothesis}) \\
 &\quad + \underbrace{k^2 + 2k + 1}_{\in \mathbb{N}} \quad \text{since } k \text{ is a natural number.}
 \end{aligned}$$

Thus, $\frac{(k+1)^3}{3} + \frac{(k+1)^2}{2} + \frac{k+1}{6} \in \mathbb{N}$.

Hence, $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \in \mathbb{N}$ for all $n \geq 0$ by induction.

- (b)
- Base case (n=1): $P(1)$ says that $(a^1 - b^1) = (a - b)$, which is trivially divisible by $(a - b)$.
 - Inductive Hypothesis: For arbitrary $n = k \geq 1$, assume that $P(k)$ is true: $a^k - b^k = q_k(a - b)$, $q_k \in \mathbb{Z}$.
 - Inductive Step: Prove the statement for $n = k + 1$: $P(k + 1)$ gives $a^{k+1} - b^{k+1} = q_{k+1}(a - b)$, $q_{k+1} \in \mathbb{Z}$.

The goal is to express $(a^{k+1} - b^{k+1})$ in terms of $(a^k - b^k)$ and $(a - b)$ (since both of these are divisible by $(a - b)$ which we know their summation is also divisible by $(a - b)$). We can do this as follows:

$$\begin{aligned} a^{k+1} - b^{k+1} &= a(a^k - b^k) + b^k(a - b) \\ &= a \underbrace{(a^k - b^k)}_{\text{divisible by } (a-b)} \quad (\text{Inductive Hypothesis}) \\ &\quad + b^k(a - b) \end{aligned}$$

Alternatively we can write the following:

$$a^{k+1} - b^{k+1} = \frac{1}{2}[(a+b) \underbrace{(a^k - b^k)}_{\text{divisible by } (a-b)} + (a-b)(a^k + b^k)]$$

again each of the terms in the parentheses is divisible by $(a - b)$. For this argument, one should really also verify that both of those terms are even (so that both are integers when divided by 2); but that's easy to see since the parity of the two terms in the two products is equal.

Thus, $a^{k+1} - b^{k+1} = q_{k+1}(a - b)$, $q_{k+1} \in \mathbb{Z}$.

Hence, $(a^n - b^n)$ is divisible by $(a - b)$ for all $n \geq 1$ by induction.

- (c)
- Base case (n=0): $P(0)$ asserts that $(2(0))! = 1 = 2^{(2(0))}(0!)^2$. So we showed the base case is correct.
 - Inductive Hypothesis: For arbitrary $n = k \geq 0$, assume that $P(k)$ is correct which leads to $(2k)! \leq 2^{2k}(k!)^2$.
 - Inductive Step: Prove the statement for $n = k + 1$: i.e., prove that $(2(k+1))! \leq 2^{2(k+1)}((k+1)!)^2$.

1)!^2.

$$\begin{aligned}(2(k+1))! &= (2k)!(2k+2)(2k+1) \\ &\leq 2^{2k}(k!)^2 2(k+1)(2k+1) && \text{(Inductive Hypothesis)} \\ &= 2^{2k+1}(k+1)!k!(2k+1) \\ &\leq 2^{2k+1}(k+1)!k!(2k+2) \\ &\leq 2^{2(k+1)}(k+1)!(k+1)! \\ &= 2^{2(k+1)}((k+1)!)^2.\end{aligned}$$

Thus, $(2(k+1))! \leq 2^{2(k+1)}((k+1)!)^2$.

Hence, $(2n)! \leq 2^{2n}(n!)^2$ holds for all $n \geq 0$ by induction.

2 Make It Stronger

Let $x \geq 1$ be a real number. Use induction to prove that for all positive integers n , all of the entries in the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n$$

are $\leq xn$. (Hint 1: Find a way to strengthen the inductive hypothesis! Hint 2: Try writing out the first few powers.)

Solution: Before starting the proof, writing out the first few powers reveals a telling pattern:

$$\begin{aligned}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^1 &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^2 &= \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^3 &= \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix}\end{aligned}$$

It appears (and we shall soon prove) that the upper left and lower right entries are always 1, the lower left entry is always 0, and the upper right entry is xn . We shall take this to be our inductive hypothesis.

Proof: We prove that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nx \\ 0 & 1 \end{pmatrix}.$$

This claim clearly also proves the original claim in the question, since all elements of this matrix are $\leq xn$ (since $x \geq 1$). Hence, we prove this stronger claim.

- Base case (n=1): $P(1)$ asserts that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. The base case is true.
- Inductive Hypothesis: Assume for arbitrary $k \geq 1$, $P(k)$ is correct: $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & xk \\ 0 & 1 \end{pmatrix}$.
- Inductive Step: Prove the statement for $n = k + 1$,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & xk \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & xk+x \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & x(k+1) \\ 0 & 1 \end{pmatrix}.$$

By the principle of induction, our proposition is therefore true for all $n \geq 1$, so all entries in the matrix will be less than or equal to xn .

3 Strong Induction

Use strong induction to show that for all natural numbers n there exist natural numbers x , y , and z such that $6^n = x^2 + y^2 + z^2$.

Solution: In this induction we need to go two steps back in the induction step, so we verify the validity of the claim for the first two cases. In other words, for proposition

$$P(n) = (\forall n \in \mathbb{N}, \exists x, y, z \in \mathbb{N})(6^n = x^2 + y^2 + z^2),$$

we show that for all natural numbers $k > 1$, $P(k+1)$ follows from $P(k-1)$, and that this requires us to prove both $P(0)$ and $P(1)$ as base cases (else, if we only proved $P(0)$, the induction would only hold for even values of n).

- Base case : $n = 0, n = 1$:
 $n = 0$, we choose $x = 1, y = 0$ and $z = 0$.

$$6^0 = 1 = 1^2 + 0^2 + 0^2 = 1.$$

$P(0)$ is true.

$n = 1$, we choose $x = 2, y = 1$ and $z = 1$.

$$6^1 = 6 = 2^2 + 1^2 + 1^2 = 4 + 1 + 1 = 6.$$

$P(1)$ is true.

- Inductive Hypothesis: For some arbitrary $k-1 > 0$
Assume there exist natural numbers x , y , and z such that: $6^{k-1} = x^2 + y^2 + z^2$.
For this we show that $P(k-1)$ implies $P(k+1)$.

- Inductive Step: We show there exist natural numbers x' , y' , and z' such that: $6^{k+1} = x'^2 + y'^2 + z'^2$
 $P(k+1)$:

$$\begin{aligned} 6^{k+1} &= 6^2 6^{k-1} \\ &= (x^2 + y^2 + z^2) 6^2 \quad (\text{Inductive Hypothesis}) \\ &= (6x)^2 + (6y)^2 + (6z)^2. \end{aligned}$$

We choose $x' = 6x$, $y' = 6y$, and $z' = 6z$.

Thus,

$$6^{k+1} = x'^2 + y'^2 + z'^2.$$

We showed that for all natural numbers $k > 1$, $P(k+1)$ follows from $P(k-1)$ and not from $P(k)$. This requires us to prove both $P(0)$ and $P(1)$ as base cases. If we only proved $P(0)$, the induction would only hold for even values of n .

Hence, we proved by strong induction that for all natural numbers n there exist natural numbers x , y , and z such that $6^n = x^2 + y^2 + z^2$.

4 Long Courtship

- (a) Run the traditional (i.e., male optimal) propose-and-reject algorithm on the following example:

Man	Preference List	Woman	Preference List
1	$A > B > C > D$	A	$2 > 3 > 4 > 1$
2	$B > C > A > D$	B	$3 > 4 > 1 > 2$
3	$C > A > B > D$	C	$4 > 1 > 2 > 3$
4	$A > B > C > D$	D	$1 > 2 > 3 > 4$

State what happens on every day in a table.

- (b) We know from the notes on the stable marriage algorithm (Lemma 4.1) that the propose-and-reject algorithm must terminate after at most n^2 days. This is in fact the maximum number of proposals before the algorithm halts since each man has n women in his list to propose and there are n men who have such a list. Knowing this, prove a sharper bound showing that the algorithm must terminate after at most $n(n-1) + 1$ proposals.
- (c) Is the instance in part (a) a worst-case instance for $n=4$, in the sense that it requires the maximum possible number of proposals?

Solution:

- (a) The stable pairing reached by male propose-and-reject algorithm is $\{(1, D), (2, A), (3, B), (4, C)\}$.

Woman	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9	Day 10
A	1, (4)	(4)	(4)	4, (3)	(3)	(3)	3, (2)	(2)	(2)	(2)
B	(2)	2, (1)	(1)	(1)	1, (4)	(4)	(4)	4, (3)	(3)	(3)
C	(3)	(3)	3, (2)	(2)	(2)	2, (1)	(1)	(1)	1, (4)	(4)
D										(1)

- (b) Explanation 1: Let us prove that there is at most one man who proposes to the last woman in his list.

On the day when a man M proposes to the last woman in his list W , we claim that every other woman must have some man on a string. This is because M proposed to each of these women, and by the Improvement Lemma, once a woman has been proposed to she always has a man on a string. Since there are $n - 1$ other women, they must be paired with all $n - 1$ remaining men. Thus there is only one proposal on this day, and since it is accepted, the algorithm halts.

Therefore, at most one man proposes to his last choice, and thus there are at most $n^2 - (n - 1) = n(n - 1) + 1$ proposals.

Explanation 2:

If we consider a scenario where every man has proposed to $n - 1$ women, then either every woman has received a proposal and thus every woman has accepted a proposal and the algorithm terminates (the algorithm terminates when each man proposes to a different woman). Otherwise there is one woman who has never received a proposal. Since there are n men, the maximum number of possible proposals must be $n(n - 1) + 1$, where 1 is added to account for the last woman to whom a proposal might be extended on the last day.

- (c) In part (a) there were 13 proposals in 10 days which is equal to $4(4 - 1) + 1 = 13$ proposals. Since this is the maximum possible number of proposals before termination, the given example in part (a) is a worst-case instance.

5 The Ranking List

Now that you have practiced the basic algorithm, let's study the stable marriage problem a little bit quantitatively. Here we define the following notation: on day j , let $P_j(M)$ be the rank of the woman that man M proposes to (where the first woman on his list has rank 1 and the last has rank n). Also, let $R_j(W)$ be the total number of men that woman W has rejected up through day $j - 1$ (i.e. not including the proposals on day j). Answer the following questions using the notation above.

- (a) Prove or disprove the following claim: $\sum_M P_j(M) - \sum_W R_j(W)$ is independent of j . If it is true, also give the value of $\sum_M P_j(M) - \sum_W R_j(W)$. The notation, \sum_M and \sum_W , simply means that we are summing over all men and all women.
- (b) Prove or disprove the following claim: one of the **men or women** must be matched to someone who is ranked in the top half of their preference list. You may assume that n is even.

Solution:

- (a) **True.** On day $j = 1$, each man proposes to the first woman on his list so $\sum_M P_1(M) = n$, and no woman rejected any man through day 0, and therefore $\sum_W R_1(W) = 0$. Hence, $\sum_M P_1(M) - \sum_W R_1(W) = n$. In general, each time a woman rejects a man on day $j - 1$, $\sum_W R_j(W)$ increases by exactly 1. Similarly $\sum_M P_j(M)$ increases by exactly 1, since the rejected man proposes to the next woman on his list on day j . Therefore $\sum_M P_j(M) - \sum_W R_j(W)$ stays constant and is independent of j . \square

More formally, we can prove this by induction on j , with $j = 1$ as base case.

Induction Hypothesis: Assume $\sum_M P_j(M) - \sum_W R_j(W) = n$.

Induction Step: The quantity $\sum_W R_{j+1}(W) - \sum_W R_j(W)$ is exactly the number of men rejected by women on day j . But each of the rejected men proposes to the next woman on his list on day $j + 1$, and so this quantity is also equal to $\sum_M P_{j+1}(M) - \sum_M P_j(M)$. Equating the two, we get

$$\sum_W R_{j+1}(W) - \sum_W R_j(W) = \sum_M P_{j+1}(M) - \sum_M P_j(M).$$

Therefore,

$$\sum_M P_{j+1}(M) - \sum_W R_{j+1}(W) = \sum_M P_j(M) - \sum_W R_j(W)$$

and the right hand side is equal to n by the induction hypothesis. \square

- (b) **True.** Assume that no man is matched with a woman in the top half of his preference list. Then each man must have been rejected at least $n/2$ times, for a total of at least $n^2/2$ rejections. This implies that at least one woman must have rejected at least $n/2$ men (because if not, then the total number of rejections must be less than $(n/2) \cdot n$, contradiction). But now, by the improvement lemma, this woman must be matched with a man she likes more than the $n/2$ men she rejected, meaning that the man she is matched with is in the top half of her preference list. \square

Alternative Proof:

Assume towards contradiction that every man and every woman is matched to someone who is ranked in the bottom half of their preference list.

Observe that a man M is matched to someone in the top half of his preference list if and only if $P_m(M) \leq n/2$, where m is the last day of the algorithm. Therefore, if M is matched to someone in the bottom half of his preference list, then $P_m(M) > n/2$, i.e., $P_m(M) \geq n/2 + 1$. Summing over the men gives us $\sum_M P_m(M) \geq n^2/2 + n$. By part (a), it follows that $\sum_W R_m(W) = \sum_M P_m(M) - n \geq n^2/2$.

Observe also that if $R_m(W) \geq n/2$, then by the improvement lemma, W must be matched to someone in the top half of her preference list. Therefore, from our assumption that W is matched to someone in the bottom half of her preference list, we get $R_m(W) < n/2$. Summing over the women gives us $\sum_W R_m(W) < n^2/2$. But this contradicts our earlier result above! \square

Note: This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.

Part 2: Optional Problems

6 Trinomials

Use induction to prove that for all natural numbers n , we have the following expansion:

$$(a+b+c)^n = \sum_{i+j+l=n} \frac{n!}{i!j!l!} a^i b^j c^l, \quad \text{where} \quad 0 \leq i, j, l \leq n, \quad i, j, l \in \mathbb{N}.$$

Solution:

- Base case (n=0): $(a+b+c)^0 = 1 = \sum_{i+j+l=0} \frac{n!}{i!j!l!} a^i b^j c^l = \frac{n!}{0!0!0!} a^0 b^0 c^0$. Thus, the base case ($P(0)$) is correct.
- Inductive Hypothesis: For arbitrary $n = k \geq 0$, assume that $P(k)$ is true: $(a+b+c)^k = \sum_{i+j+l=k} \frac{k!}{i!j!l!} a^i b^j c^l$.
- Inductive Step: Prove the statement for $n = k+1$: $(a+b+c)^{k+1} = \sum_{i+j+l=k+1} \frac{(k+1)!}{i!j!l!} a^i b^j c^l$.

$$\begin{aligned} (a+b+c)^{k+1} &= (a+b+c)^k (a+b+c) \\ &= \left(\sum_{i+j+l=k} \frac{k!}{i!j!l!} a^i b^j c^l \right) (a+b+c) \quad (\text{Inductive Hypothesis}) \\ &= \sum_{i+j+l=k} \frac{k!}{i!j!l!} a^{i+1} b^j c^l + \sum_{i+j+l=k} \frac{k!}{i!j!l!} a^i b^{j+1} c^l + \sum_{i+j+l=k} \frac{k!}{i!j!l!} a^i b^j c^{l+1} \\ &= \sum_{i'+j+l=k+1} \frac{k!}{(i'-1)!j!l!} a^{i'} b^j c^l + \sum_{i+j'+l=k+1} \frac{k!}{i!(j'-1)!l!} a^i b^{j'} c^l \\ &\quad + \sum_{i+j+l'=k+1} \frac{k!}{i!j!(l'-1)!} a^i b^j c^{l'} \end{aligned}$$

Now we can replace i', j', l' with i, j, l .

$$\begin{aligned}
(a+b+c)^{k+1} &= \sum_{i+j+l=k+1} \frac{k!}{(i-1)!j!l!} a^i b^j c^l + \sum_{i+j+l=k} \frac{k!}{i!(j-1)!l!} a^i b^j c^l \\
&\quad + \sum_{i+j+l=k+1} \frac{k!}{i!j!(l-1)!} a^i b^j c^l \\
&= \sum_{i+j+l=k+1} \left(\frac{k!}{(i-1)!j!l!} + \frac{k!}{i!(j-1)!l!} + \frac{k!}{i!j!(l-1)!} \right) a^i b^j c^l \\
&= \sum_{i+j+l=k+1} \frac{k!}{i!j!l!} \underbrace{(i+j+l)}_{=k+1} a^i b^j c^l \\
&= \sum_{i+j+l=k+1} \frac{(k+1)!}{i!j!l!} a^i b^j c^l.
\end{aligned}$$

Thus,

$$(a+b+c)^{k+1} = \sum_{i+j+l=k+1} \frac{(k+1)!}{i!j!l!} a^i b^j c^l$$

Hence, $(a+b+c)^n = \sum_{i+j+l=n} \frac{n!}{i!j!l!} a^i b^j c^l$ holds for all $n \geq 0$ by induction.

7 Airport

Suppose that there are $2n+1$ airports where n is a positive integer. The distances between any two airports are all different. For each airport, there is exactly one airplane departing from it, and heading towards the closest airport. Prove by induction that there is an airport which none of the airplanes are heading towards.

Solution:

For $n=1$, let the 3 airports be A, B, C and let their distance be $|AB|, |AC|, |BC|$. Without loss of generality suppose B, C is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are heading towards airport A .

Now suppose the statement is proven for $n=k$, i.e. when there are $2k+1$ airports. For $n=k+1$, i.e. when there are $2k+3$ airports, the airplanes departing from the closest two airports must be heading towards each other. Removing these two airports reduce the problem to $2k+1$ airports. From the inductive hypothesis, we know that among the $2k+1$ airports remaining, there is an airport with no incoming flights which we call airport Z . When we add back the two airports that we removed, the airplane flights may change; in particular, it is possible that an airplane will now choose to fly to one of these two airports (because the airports that were added may be closer than the airport to which the airplane was previously flying), but observe that none of the airplanes will

choose to fly towards the airport Z. Also, the two airports that were added back will have airplanes flying towards each other, so they too will not fly towards the airport Z. We conclude that the airport Z will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k + 1$. By induction, the claim holds for all $n \geq 3$.

8 A Better Stable Pairing

In this problem we examine a simple way to *merge* two different solutions to a stable marriage problem. Let R, R' be two distinct stable pairings. Define the new pairing $R \wedge R'$ as follows:

For every man m , m 's partner in $R \wedge R'$ is whichever is better (according to m 's preference list) of his partners in R and R' .

Also, we will say that a man/woman *prefers* a pairing R to a pairing R' if he/she prefers his/her partner in R to his/her partner in R' . We will use the following example:

men	preferences	women	preferences
A	1>2>3>4	1	D>C>B>A
B	2>1>4>3	2	C>D>A>B
C	3>4>1>2	3	B>A>D>C
D	4>3>2>1	4	A>B>D>C

- (a) $R = \{(A, 4), (B, 3), (C, 1), (D, 2)\}$ and $R' = \{(A, 3), (B, 4), (C, 2), (D, 1)\}$ are stable pairings for the example given above. Calculate $R \wedge R'$ and show that it is also stable.
- (b) Prove that, for any pairings R, R' , no man prefers R or R' to $R \wedge R'$.
- (c) Prove that, for any stable pairings R, R' where m and w are partners in R but not in R' , one of the following holds:
- m prefers R to R' and w prefers R' to R ; or
 - m prefers R' to R and w prefers R to R' .

[Hint: Let M and W denote the sets of men and women respectively that prefer R to R' , and M' and W' the sets of men and women that prefer R' to R . Note that $|M| + |M'| = |W| + |W'|$. (Why is this?) Show that $|M| \leq |W'|$ and that $|M'| \leq |W|$. Deduce that $|M'| = |W|$ and $|M| = |W'|$. The claim should now follow quite easily.]

(You may assume this result in the next part even if you don't prove it here.)

- (d) Prove an interesting result: for any stable pairings R, R' , (i) $R \wedge R'$ is a pairing [Hint: use the results from (c)], and (ii) it is also stable.

Solution:

- (a) $R \wedge R' = \{(A, 3), (B, 4), (C, 1), (D, 2)\}$. This pairing can be seen to be stable by considering the different combinations of men and women. For instance, A prefers 2 to his current partner 3. However, 2 prefers her current partner D to A . Similarly, A prefers 1 the most, but 1 prefers her current partner C to A . We can prove the stability of this pairing by considering the remaining pairs like this.
- (b) Let m be a man, and let his partners in R and R' be w and w' respectively, and without loss of generality, let $w > w'$ in m 's list. Then his partner in $R \wedge R'$ is w , whom he prefers over w' . However, for m to prefer R or R' over $R \wedge R'$, he must prefer w or w' over w , which is not possible (since $w > w'$ in his list).
- (c) Let M and W denote the sets of men and women respectively that prefer R to R' , and M' and W' the sets of men and women that prefer R' to R . Note that $|M| + |M'| = |W| + |W'|$, since the left-hand side is the number of men who have different partners in the two pairings, and the right-hand side is the number of women who have different partners.

Now, in R there cannot be a pair (m, w) such that $m \in M$ and $w \in W$, since this will be a rogue couple in R' . Hence the partner in R of every man in M must lie in W' , and hence $|M| \leq |W'|$. A similar argument shows that every man in M' must have a partner in R' who lies in W , and hence $|M'| \leq |W|$.

Since $|M| + |M'| = |W| + |W'|$, both these inequalities must actually be tight, and hence we have $|M'| = |W|$ and $|M| = |W'|$. The result is now immediate: if the man m does not partner the woman w in one but not both pairings, then

- either $m \in M$ and $w \in W'$, i.e., m prefers R to R' and w prefers R' to R ,
- or $m \in M'$ and $w \in W$, i.e., m prefers R' to R and w prefers R to R' .

- (d) (i) If $R \wedge R'$ is not a pairing, then it is because two men get the same woman, or two women get the same man. Without loss of generality, assume it is the former case, with $(m, w) \in R$ and $(m', w) \in R'$ causing the problem. Hence m prefers R to R' , and m' prefers R' to R . Using the results of the previous part would imply that w would prefer R' over R , and R over R' respectively, which is a contradiction.

(ii) Now suppose $R \wedge R'$ has a rogue couple (m, w) . Then m strictly prefers w to his partners in both R and R' . Further, w prefers m to her partner in $R \wedge R'$. Let w 's partners in R and R' be m_1 and m_2 . If she is finally matched to m_1 , then (m, w) is a rogue couple in R ; on the other hand, if she is matched to m_2 , then (m, w) is a rogue couple in R' . Since these are the only two choices for w 's partner, we have a contradiction in either case.