CS 70 Discrete Mathematics and Probability Theory
Summer 2021 DIS 05D

1 Warm-up

For each of the following parts, you may leave your answer as an expression.

(a) You throw darts at a board until you hit the center area. Assume that the throws are i.i.d. and the probability of hitting the center area is $p = 0.17$. What is the probability that you hit the center on your eighth throw?

(b) Let $X \sim \text{Geometric}(0.2)$. Calculate the expectation and variance of X.

(c) Suppose the accidents occurring weekly on a particular stretch of a highway is Poisson distributed with average number of accidents equal to 3 cars per week. Calculate the probability that there is at least one accident this week.

(d) Consider an experiment that consists of counting the number of α particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on average, 3.2 such α-particles are given off per second, what is a good approximation to the probability that no more that 2 α-particles will appear in a second?

Solution:

(a) $(0.17)(1-0.17)^7$

Let N denote the random variable that you hit the center on your X-th turn. Then $X \sim \text{Geometric}(0.17)$ and hence,

$$\mathbb{P}(X = 8) = (0.17)(1-0.17)^7 \approx 0.0461.$$

(b) $\mathbb{E}(X) = 5$ **and** $\text{Var}(X) = 20$

This follows from $\mathbb{E}(X) = 1/p$ and $\text{Var}(X) = (1-p)/(p^2)$ for $X \sim \text{Geometric}(p)$ as seen in lecture.

(c) $1 - e^{-3}$

Let X denote the number of accidents occurring on the stretch of highway in question during this week. We have $X \sim \text{Poisson}(3)$ and hence,

$$\begin{aligned} \mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0), \\ &= 1 - e^{-3}\frac{3^0}{0!} \\ &= 1 - e^{-3} \approx 0.9502. \end{aligned}$$

(d) $e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2}e^{-3.2}$

We model the number of α-particles given off during the second considered as a Poisson random variable with parameter $\lambda = 3.2$. Hence,

$$\mathbb{P}(X \leq 2) = e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2}e^{-3.2} = 0.382.$$

2 Coupon Collector Variance

It's that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{Var}(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}(X)$. *[Hint: Try to break this problem down using indicators as with the coupon collector's problem. Are the indicators independent?]*

Solution:

Note that this is the coupon collector's problem, but now we have to find the variance. Let X_i be the number of visits we need to make before we have collected the ith unique Monopoly card actually obtained, given that we have already collected $i-1$ unique Monopoly cards. Then $X = \sum_{i=1}^{n} X_i$ and each X_i is geometrically distributed with $p = (n-i+1)/n$. Moreover, the indicators themselves are independent, since each time you collect a new card, you are starting from a clean slate.

$$\begin{aligned}
\operatorname{Var}(X) &= \sum_{i=1}^{n} \operatorname{Var}(X_i) && \text{(as the } X_i \text{ are independent)} \\
&= \sum_{i=1}^{n} \frac{1-(n-i+1)/n}{[(n-i+1)/n]^2} && \text{(variance of a geometric r.v. is } (1-p)/p^2\text{)} \\
&= \sum_{j=1}^{n} \frac{1-j/n}{(j/n)^2} && \text{(by noticing that } n-i+1 \text{ takes on all values from 1 to } n\text{)} \\
&= \sum_{j=1}^{n} \frac{n(n-j)}{j^2} \\
&= \sum_{j=1}^{n} \frac{n^2}{j^2} - \sum_{j=1}^{n} \frac{n}{j} \\
&= n^2 \left(\sum_{j=1}^{n} \frac{1}{j^2} \right) - \mathbb{E}(X) && \text{(using the coupon collector problem expected value).}
\end{aligned}$$