# 1 Max of Uniforms

Let $X_1, \ldots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of $n$.

(a) What is the cdf of $X$?

(b) What is the pdf of $X$?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

**Solution:**

(a) $Pr[X \leq x] = x^n$ since in order for $\max(X_1, \ldots X_n) < x$, we must have $X_i < x$ for all $i$. Since they are independent, we can multiply together the probabilities of each of them being less than $x$, which is $x$ itself, as their distributions are uniform.

(b) Taking the derivative of the cdf, we have $f_X(x) = nx^{n-1}$

(c)

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^1 x f_X(x) \\
&= \int_0^1 n x^n dx \\
&= \frac{n}{n+1}
\end{aligned}$$

(d)

$$\mathbb{E}[X^2] = \int_0^1 x^2 f_X(x) = \int_0^1 n x^{n+1} dx = \frac{n}{n+2}$$
$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2}$$

## 2 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius $r$ around the center. Alex's aim follows a uniform distribution over a disk of radius $2r$ around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable $X$ and let the distance of Alex's throw from the center be denoted by the random variable $Y$.

- What's the cumulative distribution function of $X$?
- What's the cumulative distribution function of $Y$?
- What's the probability density function of $X$?
- What's the probability density function of $Y$?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \min\{X,Y\}$?

(d) What's the cumulative distribution function of $V = \max\{X,Y\}$?

(e) What is the expectation of the absolute difference between Michelle's and Alex's distances from the center, that is, what is $\mathbb{E}[|X-Y|]$? [*Hint*: Use parts (c) and (d), together with the continuous version of the tail sum formula, which states that $\mathbb{E}[Z] = \int_0^\infty P(Z \geq z) dz$.]

**Solution:**

(a)
- To get the cumulative distribution function of $X$, we'll consider the ratio of the area where the distance to the center is less than $x$, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}(X \leq x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2}, \qquad x \in [0,r]$$

- Using the same approach as the previous part:

$$\mathbb{P}(Y \leq y) = \frac{\pi y^2}{\pi \cdot 4r^2} = \frac{y^2}{4r^2}, \qquad y \in [0,2r]$$

- We'll take the derivative of the CDF to get the following:

$$f_X(x) = \frac{\mathrm{d}\mathbb{P}(X \leq x)}{\mathrm{d}x} = \frac{2x}{r^2}, \qquad x \in [0,r]$$

- Using the same approach as the previous part:

$$f_Y(y) = \frac{\mathrm{d}\mathbb{P}(Y \leq y)}{\mathrm{d}y} = \frac{y}{2r^2}, \qquad y \in [0,2r]$$

(b) We'll condition on Alex's outcome and then integrate over all the possibilities to get the marginal $\mathbb{P}(X \le Y)$ as following:

$$\mathbb{P}(X \le Y) = \int_0^{2r} \mathbb{P}(X \le Y \mid Y = y) f_Y(y)\,\mathrm{d}y = \int_0^r \frac{y^2}{r^2} \times \frac{y}{2r^2}\,\mathrm{d}y + \int_r^{2r} 1 \times \frac{y}{2r^2}\,\mathrm{d}y$$
$$= \frac{r^4 - 0}{8r^4} + \frac{4r^2 - r^2}{4r^2} = \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$$

Note the range within which $\mathbb{P}(X \le Y) = 1$. This allowed us to separate the integral to simplify our solution. Using this, we can get $\mathbb{P}(Y \le X)$ by the following:

$$\mathbb{P}(Y \le X) = 1 - \mathbb{P}(X \le Y) = \frac{1}{8}$$

A similar approach to the integral above could be used to verify this result.

$$\mathbb{P}(Y \le X) = \int_0^r \mathbb{P}(Y \le X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^r \frac{x^2}{4r^2}\frac{2x}{r^2}\,\mathrm{d}x = \frac{1}{2r^4}\int_0^r x^3\,\mathrm{d}x = \frac{r^4}{8r^4} = \frac{1}{8}$$

(c) Getting the CDF of $U$ relies on the insight that for the minimum of two random variables to be greater than a value, they both need to be greater than that value. Taking the complement of this will give us the CDF of $U$. This allows us to get the following result. For $u \in [0, r]$:

$$\mathbb{P}(U \le u) = 1 - \mathbb{P}(U \ge u) = 1 - \mathbb{P}(X \ge u)\mathbb{P}(Y \ge u) = 1 - \big(1 - \mathbb{P}(X \le u)\big)\big(1 - \mathbb{P}(Y \le u)\big)$$
$$= 1 - \left(1 - \frac{u^2}{r^2}\right)\left(1 - \frac{u^2}{4r^2}\right) = \frac{5u^2}{4r^2} - \frac{u^4}{4r^4}$$

For $u > r$, we get $\mathbb{P}(X > u) = 0$, this makes $\mathbb{P}(U \le u) = 1$.

(d) Getting the CDF of $V$ also relies on a similar insight that for the maximum of two random variables to be smaller than a value, they both need to be smaller than that value. Using this we can get the following result for $v \in [0, r]$:

$$\mathbb{P}(V \le v) = \mathbb{P}(X \le v)\mathbb{P}(Y \le v) = \left(\frac{v^2}{r^2}\right)\left(\frac{v^2}{4r^2}\right) = \frac{v^4}{4r^4}$$

For $v \in [r, 2r]$ we have $\mathbb{P}(X \le v) = 1$, this makes

$$\mathbb{P}(V \le v) = \mathbb{P}(Y \le v) = \frac{v^2}{4r^2}.$$

For $v > 2r$ we have $\mathbb{P}(V \le v) = 1$ since CDFs of both $X$ and $Y$ are 1 in this range.

(e) We can subtract $U$ from $V$ to get this difference. Using the tail-sum formula to calculate the expectation, we can get the following result:

$$\mathbb{E}[|X - Y|] = \mathbb{E}[V - U] = \mathbb{E}[V] - \mathbb{E}[U] = \int_0^{2r} \mathbb{P}(V \ge v)\,\mathrm{d}v - \int_0^r \mathbb{P}(U \ge u)\,\mathrm{d}u$$
$$= \int_0^r \left(1 - \frac{v^4}{4r^4}\right)\mathrm{d}v + \int_r^{2r}\left(1 - \frac{v^2}{4r^2}\right)\mathrm{d}v - \int_0^r \left(1 - \frac{5u^2}{4r^2} + \frac{u^4}{4r^4}\right)\mathrm{d}u$$
$$= \frac{19r}{20} + \frac{5r}{12} - \frac{19r}{30} = \frac{11r}{15}$$

Alternatively, you could derive the density of $U$ and $V$ and use those to calculate the expectation. For $v \in [0, r]$:

$$f_V(v) = \frac{\mathrm{d}\mathbb{P}(V \leq v)}{\mathrm{d}v} = \frac{v^3}{r^4}$$

For $v \in [r, 2r]$:

$$f_V(v) = \frac{\mathrm{d}\mathbb{P}(V \leq v)}{\mathrm{d}v} = \frac{v}{2r^2}$$

Using this we can calculate $\mathbb{E}[V]$ as:

$$\mathbb{E}[V] = \int_0^{2r} v f_V(v)\,\mathrm{d}v = \frac{1}{r^4}\int_0^r v^4\,\mathrm{d}v + \frac{1}{2r^2}\int_r^{2r} v^2\,\mathrm{d}v = \frac{r^5}{5r^4} + \frac{8r^3 - r^3}{6r^2} = \frac{r}{5} + \frac{7r}{6r} = \frac{41r}{30}$$

To calculate $\mathbb{E}[U]$ we will use the following PDF for $u \in [0, r]$:

$$f_U(u) = \frac{\mathrm{d}\mathbb{P}(U \leq u)}{\mathrm{d}u} = \frac{5u}{2r^2} - \frac{u^3}{r^4}$$

We can get the $\mathbb{E}[U]$ by the following:

$$\mathbb{E}[U] = \int_0^r u f_U(u)\,\mathrm{d}u = \int_0^r \left(\frac{5u^2}{2r^2} - \frac{u^4}{r^4}\right)\mathrm{d}u = \frac{5r^3}{6r^2} - \frac{r^5}{5r^4} = \frac{5r}{6} - \frac{r}{5} = \frac{19r}{30}$$

Combining the two results gives us the same result as above:

$$\mathbb{E}[|X - Y|] = \mathbb{E}[V - U] = \mathbb{E}[V] - \mathbb{E}[U] = \frac{41r}{30} - \frac{19r}{30} = \frac{11r}{15}$$