CS 70 Discrete Mathematics and Probability Theory
Summer 2021

HW 4

1 Probability Potpourri

Provide brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

(b) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(c) If A and B are disjoint, does that imply they're independent?

Solution:

(a) Start with the right side:

$$\begin{aligned}\mathbb{P}(A) - \mathbb{P}(B) &= [\mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B)] - [\mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)] \\ &= \mathbb{P}(A \setminus B) - \mathbb{P}(B \setminus A) \\ &\leq \mathbb{P}(A \setminus B)\end{aligned}$$

(b) Using total probability rule:

$$\mathbb{P}(D) = \mathbb{P}(D \cap C) + \mathbb{P}(D \cap \overline{C}) = \mathbb{P}(D \mid C) \cdot \mathbb{P}(C) + \mathbb{P}(D \mid \overline{C}) \cdot \mathbb{P}(\overline{C})$$

But we know that $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, so this simplifies to

$$\mathbb{P}(D) = \mathbb{P}(D \mid C) \cdot [\mathbb{P}(C) + \mathbb{P}(\overline{C})] = \mathbb{P}(D \mid C) \cdot 1 = \mathbb{P}(D \mid C)$$

which defines independence.

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\mathbb{P}(A \cap B) = 0$. But these events are not independent: $\mathbb{P}(B \mid A) = 0$, but $\mathbb{P}(B) = 1/6$.

Since disjoint events have $\mathbb{P}(A \cap B) = 0$, we can see that the only time when disjoint A and B are independent is when either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

2 Conditional Practice

(a) Suppose you have 3 bags. Two of them contain a single \$10 bill, and the third contains a single \$5 bill. Suppose you pick one of these bags uniformly at random. You then add a \$5 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag without looking into the bag. Suppose it turns out to be a \$5 bill. If a you draw the remaining bill from the bag, what is the probability that it, too, is a \$5 bill? Show your calculations.

(b) Now suppose that you have a large number of bags, and that each of them contain either a gold or a silver coin (every bag contains exactly one coin). Moreover, these bags are either colored red, blue, or purple (every bag is exactly one of these colors). Half of the bags are red and a third of the bags are blue. Moreover, two thirds of the red bags and one fourth of the blue bags contain gold coins. Lastly, a randomly chosen bag has a $\frac{1}{2}$ probability of containing a silver coin. Suppose that you pick a bag at random and find that it contains a silver coin. What is the probability that the bag you picked was purple?

Solution:

(a) Let A denote the event that the bag originally contained 5 dollars and let B denote the event that the bag originally contained 10 dollars. We are given that $P(A) = 1/3$ and $P(B) = 2/3$.

Now, we add another 5 dollar bill to the bag, and perform an experiment. You draw the first bill, and another student draws the second bill. Let A_1 be the event that the first bill is \$5, let A_2 be the event that the second bill is \$5, let B_1 be the event that the first bill is \$10 and let B_2 be the event that the second bill is \$10. Our sample space for this experiment is $\Omega = \{(A_1,A_2),(A_1,B_2),(B_1,A_2),(B_1,B_2)\}$. Note that $A = A_1 \cap A_2$ and $B = (B_1 \cap A_2) \cup (A_1 \cap B_2)$.

The question asks us to find the conditional probability $P(A_2|A_1)$. We know by the definition of conditional probability that

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}. \tag{1}$$

The only way that both the first and second bill are both 5 dollar bills (the event $A_1 \cap A_2$) is if the bag originally contained a 5 dollar bill. By the problem statement, we know the probability that the bag originally contained a 5 dollar bill is $1/3$. Therefore, $P(A_1 \cap A_2) = \frac{1}{3}$.

We now look at the denominator $P(A_1)$. To calculate the marginal probability of event A_1, we use the law of total probability. This law states that

$$P(A_1) = P(A_1|A) \cdot P(A) + P(A_1|B) \cdot P(B) \tag{2}$$

for two disjoint events A and B.

Let's first look at $P(A_1|A)$. This is the probability that the first bill drawn is a 5 dollar bill, given that the bag initially contained a 5 dollar bill. Since both bills inside the bag are now 5 dollar bills, this probability is just 1! How about the second term, $P(A_1|B)$? This is the probability that the first bill drawn is a 5 dollar bill, given that the bag initially contained a 10 dollar bill. Since the first draw has an equal chance of drawing either bill, this probability is just $1/2$.

Plugging in the appropriate terms in (2), we get $P(A_1) = 1 \cdot \left(\frac{1}{3}\right) + \left(\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) = \frac{2}{3}$. Then plugging into the conditional probability formula (1), we have $P(A_2|A_1) = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \boxed{\mathbf{\frac{1}{2}}}$.

(b) Let R, G, P be the events that the bag we pick is red, green, and purple respectively. Moreover, let G and S denote the events that the bag we pick contains a gold or a silver coin, respectively. Since every bag is exactly one of the stated colors, and since every bag contains either a gold or a silver coin, we have that $\mathbb{P}[R] + \mathbb{P}[B] + \mathbb{P}[P] = \mathbb{P}[G] + \mathbb{P}[S] = 1$. Now, we are given in the problem that $\mathbb{P}[R] = \frac{1}{2}$, $\mathbb{P}[B] = \frac{1}{3}$, and since two-thirds of the red bags and one-fourth of the blue bags contain gold coins, we have that $\mathbb{P}[G \mid R] = \frac{2}{3}$ and $\mathbb{P}[G \mid B] = \frac{1}{4}$. Lastly, we are given $\mathbb{P}[S] = \frac{1}{2}$.

Now, we can see from $\mathbb{P}[R] + \mathbb{P}[B] + \mathbb{P}[P] = \mathbb{P}[G] + \mathbb{P}[S] = 1$ that $\mathbb{P}[P] = \frac{1}{6}$ and $\mathbb{P}[G] = \frac{1}{2}$. Using the Law of Total Probability and Bayes' Theorem, we can expand

$$\begin{aligned}\mathbb{P}[G] &= \mathbb{P}[G \cap R] + \mathbb{P}[G \cap B] + \mathbb{P}[G \cap P] \\ &= \mathbb{P}[R] \cdot \mathbb{P}[G \mid R] + \mathbb{P}[B] \cdot \mathbb{P}[G \mid B] + \mathbb{P}[P] \cdot \mathbb{P}[G \mid P] \\ &= \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{6} \cdot \mathbb{P}[G \mid P].\end{aligned}$$

Now, since $\mathbb{P}[G] = \frac{1}{2}$, we can solve the above to get that $\mathbb{P}[G \mid P] = \frac{1}{2}$. Thus, $\mathbb{P}[S \mid P] = 1 - \mathbb{P}[G \mid P] = \frac{1}{2}$. Finally, we can use Bayes' Theorem again to get that

$$\mathbb{P}[P \mid S] = \frac{\mathbb{P}[S \mid P] \cdot \mathbb{P}[P]}{\mathbb{P}[S]} = \frac{\frac{1}{2} \cdot \frac{1}{6}}{\frac{1}{2}} = \boxed{\frac{1}{6}}.$$

3 Playing Strategically

Bob, Eve and Carol bought new slingshots. Bob is not very accurate, hitting his target with probability $1/3$. Eve is better, hitting her target with probability $2/3$. Carol never misses. They decide to play the following game: They take turns shooting each other. For the game to be fair, Bob starts first, then Eve and finally Carol. Any player who gets shot has to leave the game. The last person standing wins the game. In this problem, we will investigate what Bob's best course of action would be.

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first. (Hint: Let x be the probability Bob wins in a duel against Eve alone, assuming he fires first. If Bob misses his first shot and then Eve misses her first shot, what is the probability Bob wins in terms of x?)

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

(c) Compute the probability of the same events for a duel of Bob against Carol.

(d) Assuming that both Eve and Carol play rationally, conclude that Bob's best course of action is to shoot into the air (i.e., intentionally miss)! (Hint: What happens if Bob misses? What if he doesn't?)

Solution:

(a) Observe that

$$\begin{aligned}\mathbb{P}[E_1] &= \mathbb{P}[\text{Bob hits Eve}] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[\text{Eve misses Bob}]\mathbb{P}[E_1] \\ &= \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3}\mathbb{P}[E_1],\end{aligned}$$

thus, $\mathbb{P}[E_1] = \frac{3}{7}$.

(b) Observe that

$$\begin{aligned}\mathbb{P}[E_2] &= \mathbb{P}[\text{Eve misses Bob}]\left(\mathbb{P}[\text{Bob hits Eve}] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[E_2]\right) \\ &= \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3}\mathbb{P}[E_2]\right),\end{aligned}$$

Thus, $\mathbb{P}[E_2] = \frac{1}{7}$.

(c) The probability of the event E_3 that Bob, going first, survives against Carol is:

$$\begin{aligned}\mathbb{P}[E_3] &= \mathbb{P}[\text{Bob hits Carol}] + \mathbb{P}[\text{Bob misses Carol}]\mathbb{P}[\text{Carol misses Bob}]\mathbb{P}[E_3] \\ &= \frac{1}{3} + \frac{2}{3} \cdot 0 \cdot \mathbb{P}[E_3] \\ &= \frac{1}{3}.\end{aligned}$$

The probability of the event E_4 that Bob, going second, survives against Carol is:

$$\mathbb{P}[E_4] = \mathbb{P}[\text{Carol misses}]\mathbb{P}[E_3] = 0 \cdot \frac{1}{3} = 0.$$

(d) To maximize their chances each player prefers to be left with a weaker opponent. This means that Eve would not shoot at Bob in preference to Carol, and Carol will not shoot at Bob in preference to Eve. Therefore if Bob misses, he will not be shot at until either Eve or Carol lose and he will either be left standing with Eve or Carol, with the first shot in either case.

So Bob is best off not shooting anyone since the advantage he gains by having the first shot exceeds any possible benefit of facing Eve rather than Carol. He should shoot into the air.

4 College Applications

There are n students applying to n colleges. Each college has a ranking over all students (i.e. a permutation) which, for all we know, is completely random and independent of other colleges.

College number i will admit the first k_i students in its ranking. If a student is not admitted to any college, he or she might file a complaint against the board of colleges, and colleges want to avoid that as much as possible.

(a) If for all i, $k_i = 1$ (i.e. if every college only admits the top student on its list), what is the probability that all students will be admitted to at least one college?

(b) What is the probability that a particular student, Alice, does not get admitted to any college? Prove that if the average of all k_i's is at least $2\ln n$, then this probability is at most $1/n^2$. (Hint: use the inequality $1-x \leq e^{-x}$)

(c) Prove that when the average k_i is at least $2\ln n$, then the probability that at least one student does not get admitted to any college is at most $1/n$.

Solution:

(a) If we consider the first choices of all colleges, there are n^n different possibilities, all of which are equally likely because colleges are independently sorting students in a random manner. Out of these we want the possibilities that have all students covered, which is the same as those that have no repeated student (because the number of colleges is the same as the number of students). So we are counting permutations, and we know that there are $n!$ of them. So the probability is $\frac{n!}{n^n}$.

(b) The chance that Alice does not get admitted to college i is $1-\frac{k_i}{n}$. This is because k_i students are admitted out of n total, so the probability that Alice ends up in the top k_i is k_i/n and the probability that she does not is $1-\frac{k_i}{n}$.

Using the fact that each college is independent of one another, the probability that she does not get admitted to any college is just

$$\prod_{i=1}^{n}(1-\frac{k_i}{n})$$

Now using the inequality $1-x \leq e^{-x}$, we get $1-\frac{k_i}{n} \leq e^{-k_i/n}$. Multiplying over all i we get

$$\prod_{i=1}^{n}(1-\frac{k_i}{n}) < \prod_{i=1}^{n} e^{-k_i/n} = e^{-\sum_{i=1}^{n} k_i/n}$$

But $\sum_{i=1}^{n} k_i/n$ is simply the average of all k_i. If this average is at least $2\ln n$, the last expression is at most $e^{-2\ln n}$ which is just $1/n^2$.

(c) If A_i is the event that student i does not get admitted to any college, then $\mathbb{P}[A_i]$ is at most $1/n^2$ by the previous part. Now, we can see that $\cup_{i=1}^n A_i$ is the event that at least one of the students does not get admitted to any college, so by using the union bound we get that

$$\mathbb{P}[\cup_{i=1}^n A_i] \leq \sum_{i=1}^n \mathbb{P}[A_i] \leq \sum_{i=1}^n \frac{1}{n^2} = \frac{1}{n}$$

5 (Un)conditional (In)equalities

Let us consider a sample space $\Omega = \{\omega_1, \ldots, \omega_N\}$ of size $N > 2$ and two probability functions $\mathbb{P}_1$ and $\mathbb{P}_2$ on it. That is, we have two probability spaces: $(\Omega, \mathbb{P}_1)$ and $(\Omega, \mathbb{P}_2)$.

(a) Suppose that for every subset $A \subset \Omega$ of size $|A| = 2$ and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1(\omega \mid A) = \mathbb{P}_2(\omega \mid A)$. Is it necessarily true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$? That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)

(b) Suppose that for every subset $A \subset \Omega$ of size $|A| = k$, where k is some fixed element in $\{2, \ldots, N\}$, and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1(\omega \mid A) = \mathbb{P}_2(\omega \mid A)$. Is it necessarily true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$?

For the following two parts, assume that $\Omega = \left\{(a_1, \ldots, a_k) \mid \sum_{j=1}^k a_j = n\right\}$ is the set of configurations of n balls into k labeled bins, and let $\mathbb{P}_1$ be the probabilities assigned to these configurations by throwing the balls independently one after another into the bins, and let $\mathbb{P}_2$ be the probabilities assigned to these configurations by uniformly sampling one of these configurations.

(c) Let A be the event that all n balls are in exactly one bin.

  (i) What are $\mathbb{P}_1(\omega \mid A)$ and $\mathbb{P}_2(\omega \mid A)$ for any $\omega \in A$?

  (ii) Repeat part (i) for $\omega \in \Omega \setminus A$.

  (iii) Is it true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$?

(d) For the special case of $n = 9$ and $k = 3$, provide two outcomes B and C, so that $\mathbb{P}_1(B) < \mathbb{P}_2(B)$ and $\mathbb{P}_1(C) > \mathbb{P}_2(C)$. Provide justification.

Solution:

(a) Yes, this is indeed true. To see why, let's take the subset $A = \{\omega_i, \omega_j\}$ for some $i, j \in \{1, \ldots, N\}$ and compute: For any $k \in \{1, 2\}$, we have $\mathbb{P}_k(\omega_i \mid A) = \frac{\mathbb{P}_k(\omega_i)}{\mathbb{P}_k(A)}$. Since this expression (by assumption) is the same for $k = 1$ and $k = 2$, we conclude that $\frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_2(\omega_i)} = \frac{\mathbb{P}_1(A)}{\mathbb{P}_2(A)}$. Repeating the reasoning for ω_j, we similarly find that $\frac{\mathbb{P}_1(\omega_j)}{\mathbb{P}_2(\omega_j)} = \frac{\mathbb{P}_1(A)}{\mathbb{P}_2(A)}$, and whence $\frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_1(\omega_j)} = \frac{\mathbb{P}_2(\omega_i)}{\mathbb{P}_2(\omega_j)}$. Since this is true for any $i, j \in \{1, \ldots, N\}$, we can sum over i to get

$$\frac{1}{\mathbb{P}_1(\omega_j)} = \sum_{i=1}^{N} \frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_1(\omega_j)} = \sum_{i=1}^{N} \frac{\mathbb{P}_2(\omega_i)}{\mathbb{P}_2(\omega_j)} = \frac{1}{\mathbb{P}_2(\omega_j)},$$

which shows that $\mathbb{P}_1(\omega_j) = \mathbb{P}_2(\omega_j)$ for all $j \in \{1, \ldots, N\}$.

(b) Yes, it indeed would. There are two ways of verifying this. The first way to convince ourselves that part (b) is true, is to observe that none of the arguments used in part (a) really relied on A having size 2, and so the very same reasoning carries through for A of size k.

The second (more rigorous) one is to observe that if $A' \subset A$ and $\omega \in A'$, then $\mathbb{P}_1(\omega \mid A') = \mathbb{P}_1(\omega \mid A' \cap A) = \frac{\mathbb{P}_1(\omega \mid A)}{\mathbb{P}_1(A' \mid A)} = \frac{\mathbb{P}_2(\omega \mid A)}{\mathbb{P}_2(A' \mid A)} = \mathbb{P}_2(\omega \mid A')$, where the second equality follows from the product rule (Theorem 13.1): $\mathbb{P}_1(A) \cdot \mathbb{P}_1(A' \mid A) \cdot \mathbb{P}_1(\omega \mid A \cap A') = \mathbb{P}_1(\{\omega\} \cap A \cap A') = \mathbb{P}_1(\omega) = \mathbb{P}_1(A)\mathbb{P}_1(\omega \mid A)$. That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ coincide conditional on some event A, they also coincide conditional on any smaller event A'. In particular, if they coincide on all events of size k, they also coincide on all events of size 2, which we have already dealt with in part (a).

(c) There are exactly k outcomes in A (namely, $(n, 0, \ldots, 0), (0, n, 0, \ldots), \ldots, (0, \ldots, 0, n)$; i.e. each bin could be the full one), and all of them are equally likely under either $\mathbb{P}_1$ or $\mathbb{P}_2$. That is, if $\omega \in A$, then $\mathbb{P}_1(\omega) = \left(\frac{1}{k}\right)^n$, and $\mathbb{P}_2(\omega) = \left[\binom{n+k-1}{k-1}\right]^{-1}$. Consequently, for $\omega \in A$,

$$\mathbb{P}_1(\omega \mid A) = \frac{k^{-n}}{k \cdot k^{-n}} = \frac{1}{k} \qquad \mathbb{P}_2(\omega \mid A) = \frac{\binom{n+k-1}{k-1}^{-1}}{k \cdot \binom{n+k-1}{k-1}^{-1}} = \frac{1}{k}.$$

If $\omega \notin A$, then $\mathbb{P}_1(\omega \mid A) = \mathbb{P}_2(\omega \mid A) = 0$, and so $\mathbb{P}_1(\omega \mid A)$ and $\mathbb{P}_2(\omega \mid A)$ coincide for all $\omega \in \Omega$. This, however, does *not* imply that $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same! Indeed, when computing the probability of $\omega \in A$ above, we saw that $\mathbb{P}_1(\omega) \neq \mathbb{P}_2(\omega)$ (remember that the assumption of part (b) was that the conditional probabilities coincide for *all* events of size k, here we have only shown equality conditional on *one* such event).

(d) Intuitively, throwing balls independently one after another makes it much less likely that all balls stack up in one bin as opposed to spreading out more evenly. This suggests taking, e.g., $A = \{\text{all balls land in bin 1}\}$, whose probability we already computed in part (c). That is, to show that $\mathbb{P}_1(A) < \mathbb{P}_2(A)$, we need to show that $k^{-n} < \binom{n+k-1}{k-1}^{-1}$. Plugging in $k = 3$ and $n = 9$, we have

$$k^{-n} = 3^{-9} = 3^{-2} \cdot 3^{-3} \cdot 3^{-4} = \frac{1}{9 \cdot 27 \cdot 3^4} < \frac{1}{5 \cdot 11} = \frac{2}{11 \cdot 10} = \binom{11}{2}^{-1} = \binom{n+k-1}{k-1}^{-1},$$

as desired.

Conversely, the same reasoning suggests that evenly distributed balls are much more likely under $\mathbb{P}_1$ than under $\mathbb{P}_2$. And indeed, letting $B = \{\text{each bin has exactly three balls}\}$, we have

$$\mathbb{P}_1(B) = \binom{9}{3}\binom{6}{3}\binom{3}{3}\left(\frac{1}{3}\right)^9 = \frac{9!}{(3!)^3 \cdot 3^9} = \frac{7!}{3^{10}} > \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{3^6}\cdot\frac{2}{11\cdot 10} = \frac{7}{3}\cdot\frac{6\cdot 5}{3^3}\cdot\frac{4}{3}\mathbb{P}_2(B),$$

and since the last factor is bigger than 1, we have $\mathbb{P}_1(B) > \mathbb{P}_2(B)$ as promised (of course, we could have also just plugged all these powers and factorials into a calculator to get the same result).

6 Cliques in Random Graphs

Consider the graph $G = (V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. Let's call the event that S forms a clique E_S. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \le n^k$. (You might find this useful in part (e))

(e) Prove that the probability that the graph contains a k-clique, for $k \ge 4\log_2 n + 1$, is at most $1/n$.

Solution:

(a) Between every pair of verticles, there is either an edge or not. Since there are two choices for each of the $\binom{n}{2}$ pairs of vertices, the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a fixed set of k vertices to be a k-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) E_{V_1} and E_{V_2} are independent if and only if V_1 and V_2 share at most one vertex: If V_1 and V_2 share at most one vertex, then since edges are added independently of each other, we have

$$\begin{aligned}
\mathbb{P}(E_{V_1} \cap E_{V_2}) &= \mathbb{P}(\text{all edges in } V_1 \text{ and all edges in } V_2 \text{ are present}) \\
&= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2}} \\
&= \mathbb{P}(E_{V_1}) \cdot \mathbb{P}(E_{V_2}).
\end{aligned}$$

Conversely, if V_1 and V_2 share at least two vertices, then their intersection $V_3 = V_1 \cap V_2$ has at least 2 elements, and whence

$$\begin{aligned}
\mathbb{P}\left(E_{V_1} \cap E_{V_2}\right) &= \left(\frac{1}{2}\right)^{\binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} - \binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2} - \binom{|V_3|}{2}} \\
&= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} + \binom{|V_2|}{2} - \binom{|V_3|}{2}} \neq \mathbb{P}\left(E_{V_1}\right) \cdot \mathbb{P}\left(E_{V_2}\right).
\end{aligned}$$

(d) The algebraic solution is an application of the definition of $\binom{n}{k}$:

$$\begin{aligned}
\binom{n}{k} = \frac{n!}{(n-k)!k!} &= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \\
&\leq n \cdot (n-1) \cdots (n-k+1) \\
&\leq n^k
\end{aligned}$$

(e) Let A_S denote the event that S is a k-clique, where $S \subseteq V$ is of size k. Then, the event that the graph contains a k-clique can be described as the union of A_S's over all $S \subseteq V$ of size k. Using the union bound,

$$\mathbb{P}\left[\bigcup_{S \subseteq V, |S| = k} A_S\right] \leq \sum_{S \subseteq V, |S| = k} \mathbb{P}[A_S] = \sum_{S \subseteq V, |S| = k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{k(k-1)/2}} \leq \frac{n^k}{\left(2^{(k-1)/2}\right)^k} \leq \frac{n^k}{\left(2^{(4\log n + 1 - 1)/2}\right)^k} = \frac{n^k}{\left(2^{2\log n}\right)^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \leq \frac{1}{n}.$$

7 Socks

Suppose you have n different pairs of socks (n left socks and n right socks, for $2n$ individual socks total) in your dresser. You take the socks out of the dresser one by one without looking and lay them out in a row on the floor. What is the probability that no two matching socks are next to each other?

Solution: We will use complements and the Principle of Inclusion-Exclusion. We will first find the probability that at least one matching pair of socks are next to each other, and then subtract that from 1 to get the probability we want. Number the pairs of socks 1 up to n. Let A_i be the event that the socks that make up pair i end up next to each other in the row, and let A be the event that at least one matching pair of socks are next to each other. Then we have that

$$A = \bigcup_{i=1}^{n} A_i,$$

and we have from the Principle of Inclusion-Exclusion that

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A_i] - \sum_{1 \le i < j \le n} \mathbb{P}[A_i \cap A_j] + \cdots.$$

Now, we compute these probabilities in a general way. Consider the intersection of k of these sets, $A_{i_1} \cap \cdots \cap A_{i_k}$. This intersection consists of all the outcomes where the socks that make up pair i_1 are next to each other, the socks that make up pair i_2 are next to each other, and so on. Since each outcome is equally likely to occur, we can count the number of outcomes in this set and divide by the total number of outcomes to get the desired probability. Now, each adjacent pair can be in one of two orientations: either the left sock in the pair is to the left of the right sock in the pair, or the opposite. Thus, for each adjacent pair, we have two choices for orientation, giving us a factor of 2^k. Now, we can count the number of arrangements by treating each adjacent pair as a single object, and computing permutations. Since we have k adjacent pairs, this means that we have $2n-k$ objects, hence the number of permutations is $(2n-k)!$. Thus, the total number of outcomes in the intersection of k of the A_i's is $2^k(2n-k)!$. Now, the total number of outcomes is $(2n)!$, so the probability we want is $\frac{2^k(2n-k)!}{(2n)!}$.

Plugging this into the Inclusion-Exclusion expression, we get that

$$\begin{aligned}
\mathbb{P}[A] &= \sum_{i=1}^{n} 2 \cdot \frac{(2n-1)!}{(2n)!} - \sum_{1 \le i < j \le n} 2^2 \cdot \frac{(2n-2)!}{(2n)!} + \cdots \\
&= \binom{n}{1} 2^1 \cdot \frac{(2n-1)!}{(2n)!} - \binom{n}{2} 2^2 \cdot \frac{(2n-2)!}{(2n)!} + \cdots \\
&= \sum_{k=1}^{n} (-1)^k \binom{n}{k} 2^k \cdot \frac{(2n-k)!}{(2n)!},
\end{aligned}$$

hence the desired probability is

$$1 - \mathbb{P}[A] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^k(2n-k)!}{(2n)!}.$$

8 Minesweeper

Minesweeper is a game that takes place on a grid of squares. When you click a square, it reveals either an integer $\in [1,8]$, a mine, or a blank space. If it reveals a mine, you instantly lose. If it reveals a number, that number refers to the number of mines adjacent to that square (including diagonally adjacent). If it reveals a blank space, there were 0 mines adjacent to it.

You are playing on a 8×8 board with 10 mines randomly distributed across the board. In your first move, you click a square near the center of the board (i.e. you click a square that is neither a corner square nor an edge square).

(a) What is the probability that the square reveals

i. a mine?

ii. a blank space?

iii. the number k?

(b) Suppose the first square you clicked revealed the number k. For your next move, you want to minimize the probability of picking a mine. Should you click a square adjacent to your first pick, or a different square? Your answer should depend on the value of k.

(c) Now suppose the first square you clicked revealed the number 1. You then click the square to the right for your next move. What is the probability that this square reveals the number 4?

Solution:

(a) i. There are 10 mines and 64 squares, so the probability of a square being a mine is $\frac{10}{64}$

ii. This is the probability that the picked square and its 8 adjacent squares are not mines. Then, we calculate the probability that all 10 mines are among the other 55 squares. $\frac{\binom{55}{10}}{\binom{64}{10}}$

iii. $\frac{\binom{8}{k}\binom{55}{10-k}}{\binom{64}{10}}$. We choose locations for the k adjacent mines and locations for the remaining $10-k$ mines. The denominator is the total number of possible arrangements of mines.

(b) The probability of picking a mine if you click an adjacent square is $\frac{k}{8}$. The probability of picking a mine if you click a different square is $\frac{10-k}{55}$. You should pick an adjacent square if $\frac{k}{8} \leq \frac{10-k}{55}$. This occurs only when $k = 1$.

(c) The square to the right will share 4 neighbors with the original square. In order to reveal the number 4, one of the mutual neighbors must be a mine. The three new neighbors must also be mines.

The probability that one of the mutual neighbors is a mine is $\frac{1}{2}$. Given that one of the mutual neighbors is a mine, the probability that the three new neighbors are also all mines is $\frac{\binom{52}{6}}{\binom{55}{9}}$ The probability that both these events occur must then be:

$$\frac{1}{2} \times \frac{\binom{52}{6}}{\binom{55}{9}}$$