CS 70 Fall 2019	Discrete Mathematics and Probability Theory Alistair Sinclair and Yun S. Song	HW 8

Note: *This homework consists of two parts. The first part (questions 1-6) will be graded and will determine your score for this homework. The second part (questions 7-8) will be graded if you submit them, but will not affect your homework score in any way. You are strongly advised to attempt all the questions in the first part. You should attempt the problems in the second part only if you are interested and have time to spare.*

For each problem, justify all your answers unless otherwise specified.

Part 1: Required Problems

1 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\binom{n}{i} = \binom{n}{n-i}$.

(b) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

(c) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$. (Hint: Part (a) might be useful.)

(d) $\sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = 3^n$. (Hint: consider the number of ways of splitting n elements into 3 groups.)

Solution:

(a) Choosing i players out of n to play on a team is the same as choosing $n-i$ players to not play on the team, i.e. $\binom{n}{i} = \binom{n}{n-i}$.

(b) For each i on the LHS, we can think of selecting a team of i members out of a pool of n players, and susequently choosing a captain out of the i team members. The RHS does the same by first choosing the captain out of the n players, and then a subset of the remaining $n-1$ players to constitute the team.

(c) Assume we have n women and n men. Using part (a) we can rewrite the LHS as $\sum_{i=1}^{n} i\binom{n}{i}\binom{n}{n-i}$, which we can interpret as selecting a team of n players by choosing i women and $n-i$ men, and then choosing one of the women to serve as captain. Again, the RHS first chooses a captain, and then selects a remaining $n-1$ players from all remaining men and women to form the team.

(d) We count the number of ways to split n elements into 3 labeled groups by two different methods.

RHS: There are 3 different choices for each element, so 3^n for all of them.

LHS: For every i from 0 to n, choose i elements to go in group A, then for every j from 0 to $n-i$ choose j elements to go in group B, the remaining go in group C. This gives:

$$\sum_{i=0}^{n} \binom{n}{i} \sum_{j=1}^{n-i} \binom{n-i}{j}$$

2 On to Counting

Let us consider two finite sets A and B of cardinalities $|A| = n$ and $|B| = m$, respectively, and ask ourselves how many functions $f : A \to B$ there are that are surjective.

(a) Define F to be the set of all functions $f : A \to B$ (not necessarily surjective). What is the cardinality of F?

(b) For a fixed $b \in B$, define the set $F_b = \{f \in F \ : \ f^{-1}(\{b\}) = \varnothing\}$. What is the cardinality of F_b? How many functions in F_b are surjective? If f is not surjective, is it necessarily contained in $F_{b'}$ for some $b' \in B$?

(c) Use your results from the previous parts to compute the the number of functions from A to B that are surjective.

Solution:

(a) For each $a \in A$, $f(a)$ can be any of the m elements in B, independently of what values f assigns to other elements. Thus, by the first rule of counting we have $|F| = m^n$.

(b) If the pre-image of $\{b\}$ is empty, then no element $a \in A$ can be mapped to b by any $f \in F_b$. That is, for any $a \in A$, $f(a)$ can be any of the $m-1$ choices in $B \setminus \{b\}$, and so again by the first rule of counting, $|F_b| = (m-1)^n$. Moreover, since b isn't in the image of any $f \in F_b$, none of the functions in F_b can be surjective. Conversely, if f is not surjective, then there must exist at least one $b' \in B$ not included in its image, which implies that $f \in F_{b'}$.

(c) Part (b) shows that the set of *non-surjective* functions is, in fact, $F_\times = \bigcup_{b \in B} F_b$, which suggests that counting all surjective functions is really just computing the cardinality of $F \setminus F_\times$. But by (a), we have $|F \setminus F_\times| = |F| - |F_\times| = m^n - |F_\times|$, so all that remains is to find $|F_\times|$, which we will do by using the Inclusion-Exclusion principle:

$$\left| \bigcup_{b \in B} F_b \right| = \sum_{k=1}^{m} (-1)^{k+1} \sum_{S \subset B \,:\, |S| = k} \left| \bigcap_{s \in S} F_s \right| = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} (m-k)^n,$$

where we used the same reasoning as in (b) (but now applied to k elements $b_1,\ldots,b_k \in B$ instead of just one b) to compute $|\bigcap_{s\in S} F_s| = (m-k)^n$ for any subset S of size $|S| = k$ (and using the fact that there are $\binom{m}{k}$ such subsets). Thus, finally we arrive at a total of

$$m^n - \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} (m-k)^n = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

surjective functions from A to B.

3 Flippin' Coins

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes

- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) Since $|\Omega| = 2^3 = 8$ and every outcome has equal probability, $\mathbb{P}[(H,H,T)] = 1/8$.

(f) The event of interest is $E = \{(H,H,T),(H,T,H),(T,H,H)\}$, which has size 3. Whence $\mathbb{P}[E] = 3/8$.

4 Probability Warm-Up

(a) Suppose that we have a bucket of 30 red balls and 70 blue balls. If we pick 20 balls out of the bucket, what is the probability of getting exactly k red balls (assuming $0 \le k \le 20$) if the sampling is done with replacement?

(b) Same as part (a), but the sampling is without replacement.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

Solution:

(a) Let A be the event of getting exactly k red balls. Then treating all balls as distinguishable, we have a total of 100^{20} possibilities to draw a sequence of 20 balls. In order for this sequence to have exactly k red balls, we need to first assign them one of $\binom{20}{k}$ possible locations within the sequence. Once done so, we have 30^k ways of actually choosing the red balls, and 70^{20-k} possibilities for choosing the blue balls. Thus in total we arrive at

$$\mathbb{P}(A) = \frac{\binom{20}{k} \cdot 30^k \cdot 70^{20-k}}{100^{20}} = \binom{20}{k}\left(\frac{3}{10}\right)^k\left(\frac{7}{10}\right)^{20-k}.$$

(b) We note that the size of the sample space is now $\binom{100}{20}$, since we are choosing 20 balls out of a total of 100. To find $|A|$, we need to be able to find out how many ways we can choose k red balls and $20-k$ blue balls. So we have that $|A| = \binom{30}{k}\binom{70}{20-k}$. So

$$\mathbb{P}(A) = \frac{\binom{30}{k}\binom{70}{20-k}}{\binom{100}{20}}.$$

(c) Let B be the event that at least one value is observed more than once. We see that $\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B})$. So we need to find out the probability that the values of the 5 rolls are distinct. We see that $\mathbb{P}(\overline{B})$ simply the number of ways to choose 5 numbers (order matters) divided by the sample space (which is 6^5). So

$$\mathbb{P}(\overline{B}) = \frac{6!}{6^5} = \frac{5!}{6^4}.$$

So,

$$\mathbb{P}(B) = 1 - \frac{5!}{6^4}.$$

5 Monty Halls

For each of the following modified Monty Hall scenarios, decide whether the contestant should switch doors or not. Unless otherwise specified, Monty, as in the original Monty Hall show, reveals a goat behind one door after the contestant has made their first choice.

(a) There are $n > 2$ doors with 1 car and $n-1$ goats.

(b) There are $n > 2$ doors with 1 car and $n-1$ goats, but Monty reveals $n-2$ doors with goats behind them.

(c) There are $n > 2$ doors with $k < n-1$ cars and $n-k$ goats, but Monty reveals $j < n-k$ doors with goats behind them. For what k and j is the relative advantage of switching doors largest? That is, for what values of k and j is the ratio of winning when switching doors to winning when not switching doors largest?

Solution:

Throughout the solution, we will refer to W as the event that the contestant wins, and $\mathbb{P}_S(W)$ and $\mathbb{P}_N(W)$ as the probabilities of this event happening if the contestant is (S)witching or (N)ot switching, respectively.

(a) $\mathbb{P}_N(W) = 1/n$ since only one out of n choices gets us the car. Under the switching strategy two things can happen: Either the first choice hits the car, and so switching (to any of the remaining $n-2$ doors) will inevitably get us the goat, or our first choice picks a goat, leaving one of the remaining $n-2$ cars with the car. Thus

$$\begin{aligned}\mathbb{P}_S(W) &= \left(\underbrace{1}_{\text{first choice = car}} \cdot \underbrace{0}_{\text{second choice = car}} + \underbrace{(n-1)}_{\text{first choice = goat}} \cdot \underbrace{1}_{\text{second choice = car}} \right) \Big/ \underbrace{n(n-2)}_{\text{total \# of choices}} \\ &= \frac{n-1}{n(n-2)} = \frac{1}{n} \cdot \frac{n-1}{n-2}\end{aligned}$$

which is larger than $1/n$ (ever so slightly so the larger n becomes, which demonstrates the intuitive fact that Monty's help gets decreasingly helpful the more doors there are), so switching doors is the better strategy.

(b) $\mathbb{P}_N(W) = 1/n$ remains unchanged. The same approach as in part (a) yields the same enumerator as before. However, since Monty shows us $n-2$ doors, there is only one door to switch to, changing the denominator:

$$\mathbb{P}_S(W) = \left(\underbrace{1}_{\text{first choice = car}} \cdot \underbrace{0}_{\text{second choice = car}} + \underbrace{(n-1)}_{\text{first choice = goat}} \cdot \underbrace{1}_{\text{second choice = car}} \right) \Big/ \underbrace{n \cdot 1}_{\text{total \# of choices}}$$
$$= \frac{n-1}{n} = 1 - \frac{1}{n}$$

so switching is again the better strategy.

(c) Now $\mathbb{P}_N(W) = k/n$ since k doors hide a car. The same reasoning as in part (b) gives us a way to compute the denominator of $\mathbb{P}_S(W)$. However, now the enumerator changes too:

$$\mathbb{P}_S(W) = \left(\underbrace{k}_{\text{first choice = car}} \cdot \underbrace{k-1}_{\text{second choice = car}} + \underbrace{(n-k)}_{\text{first choice = goat}} \cdot \underbrace{k}_{\text{second choice = car}} \right) \Big/ \underbrace{n(n-j-1)}_{\text{total \# of choices}}$$
$$= \frac{k(n-1)}{n(n-j-1)} = \frac{k}{n} \cdot \frac{n-1}{n-j-1}.$$

From here we see that $\mathbb{P}_S(W)/\mathbb{P}_N(W) = \frac{n-1}{n-j-1}$, which is maximal if $j = n-k-1$. In other words, if Monty reveals all but one goat (which he does in the original show where $n = 3, k = 1$ and $j = 1 = n-k-1$), then the contestant can increase their chances of winning by a factor of $\frac{n-1}{k}$ (which is a factor of 2 in the original show). In particular, the largest relative advantage of switching is achieved when $k = 1$.

6 Past Probabilified

For the following experiments, please (i) define an appropriate sample space Ω, (ii) give the probability function $\mathbb{P}$, and (iii) compute $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ for the two given events E_1 and E_2.

(a) Fix a prime $p > 2$, and sample twice with replacement from $\{0, \ldots, p-1\}$, then multiply these two numbers with each other in $(\bmod p)$ space. E_1 = The resulting product is 0, E_2 = The product is $(p-1)/2$.

(b) Sample a random graph on n vertices by including every possible edge with probability $1/2$. E_1 = The graph is complete, E_2 = vertex v_1 has degree d.

(c) Create a random stable marriage instance by having each person's preference list be a random permutation of the opposite gender. Finally, create a random pairing by matching men and women up randomly. E_1 = The resulting pairing is the female-optimal stable pairing, E_2 = All men have distinct favorite women.

Solution:

(a) This is essentially the same as throwing two $\{0,\ldots,p-1\}$-sided dice, so one appropriate sample space is $\Omega = \{(i,j) \,:\, i,j \in \mathrm{GF}(p)\}$. Since there are exactly p^2 such pairs, the probability of sampling each one is $\mathbb{P}[(i,j)] = 1/p^2$.
Now in order for the product $i \cdot j$ to be zero, at least one of them has to be zero. There are exactly $2p-1$ such pairs, and so $\mathbb{P}(E_1) = \frac{2p-1}{p^2}$. For $i \cdot j$ to equal $(p-1)/2$ it doesn't matter what i is as long as $i \neq 0$ and $j \equiv i^{-1}(p-1)/2 \pmod p$. Thus $|E_2| = \left|\left\{(i,j) \,:\, j \equiv i^{-1}(p-1)/2\right\}\right| = p-1$, and whence $\mathbb{P}(E_2) = \frac{p-1}{p^2}$.
Alternative Solution for $\mathbb{P}(E_2)$*:* The previous reasoning showed that $(p-1)/2$ is in no way special, and the probability that $i \cdot j = (p-1)/2$ is the same as $\mathbb{P}(i \cdot j = k)$ for any $k \in \mathrm{GF}(p)$. But $1 = \sum_{k=0}^{p-1} \mathbb{P}(i \cdot j = k) = \mathbb{P}(i \cdot j = 0) + (p-1)\mathbb{P}(i \cdot j = (p-1)/2) = \frac{2p-1}{p^2} + (p-1)\mathbb{P}(i \cdot j = (p-1)/2)$, and so $\mathbb{P}(E_2) = \left(1 - \frac{2p-1}{p^2}\right)/(p-1) = \frac{p-1}{p^2}$ as desired.

(b) Since any n-vertex graph can be sampled, Ω is the set of all graphs on n vertices. As there are $N = 2^{\binom{n}{2}}$ such graphs, the probability of each indivdual one g is $\mathbb{P}(g) = 1/N$ (by the same reasoning that every sequence of fair coin flips is equally likely!).
There is only one complete graph on n vertices, and so $\mathbb{P}(E_1) = 1/N$. For vertex v_1 to have degree d, exactly d of its $n-1$ possible adjacent edges must be present. There are $\binom{n-1}{d}$ choices for such edges, and for any fixed choice, there are $2^{\binom{n}{2}-(n-1)}$ graphs with this choice. So $\mathbb{P}(E_2) = \frac{\binom{n-1}{d} 2^{\binom{n}{2}-(n-1)}}{2^{\binom{n}{2}}} = \binom{n-1}{d}\left(\frac{1}{2}\right)^{n-1}$.

(c) Now there are two random things we need to keep track of: The random preference lists and the random pairing. A person i's preference list can be represented as a permutation σ_i of $\{1,\ldots,n\}$, and the pairing itself is encoded in another permutation ρ of the same set (indicating that man i is paired with woman $\rho(i)$). So $\Omega = \{(\sigma_1,\ldots,\sigma_{2n},\rho) \,:\, \sigma_i, \rho \in S_n\}$, where S_n is the set of permutations of $\{1,\ldots,n\}$. $|\Omega| = (n!)^{2n+1}$, and so $\mathbb{P}(\mathscr{P}) = 1/|\Omega|$ for each $\mathscr{P} \in \Omega$. No matter what $\sigma_1,\ldots,\sigma_{2n}$ are, there is exactly one male-optimal pairing, and so $\mathbb{P}(E_1) = \frac{(n!)^{2n}}{(n!)^{2n+1}} = \frac{1}{n!}$. For E_2, we observe that there are $n!$ possible configurations of all men having distinct favourite women, and that each man has $(n-1)!$ ways of ordering their non-favourite women, so $|E_2| = \underbrace{n!}_{\text{distinct favourites}} \cdot \underbrace{[(n-1)!]^n}_{\text{ordering of non-favourites}} \cdot \underbrace{(n!)^n}_{\text{women's preferences}} \cdot \underbrace{n!}_{\rho}$. Consequently, $\mathbb{P}(E_2) = n!\left(\frac{(n-1)!}{n!}\right)^n = \frac{n!}{n^n}$.

Note: *This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.*

Part 2: Optional Problems

7 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

Solution:

(a) k^p. For each of the p beads, there are k possibilities for its colors. Therefore, by the first counting principle, there are k^p different sequences.

(b) $k^p - k$. You can have k sequences of a beads with only one color.

(c) Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p-1$ other patterns. So the total number of patterns equivalent with some bead sequence is p. Thus, the total number of non-equivalent patterns are $(k^p - k)/p$.

(d) $(k^p - k)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $k^p - k$ has to be divisible by p, i.e., $k^p \equiv k \pmod{p}$, which is Fermat's Little Theorem.

8 Peaceful rooks

A friend of yours, Eithen Quinn, is fascinated by the following problem: placing m rooks on an $n \times n$ chessboard, so that they are in peaceful harmony (i.e. no two threaten each other). Each rook is a chess piece, and two rooks threaten each other if and only if they are in the same row or column. You remind your friend that this is so simple that a baby can accomplish the task. You forget however that babies cannot understand instructions, so when you give the m rooks to your baby niece, she simply puts them on random places on the chessboard. She however, never puts two rooks at the same place on the board.

(a) Assuming your niece picks the places uniformly at random, what is the chance that she places the $(i+1)^{\text{st}}$ rook such that it doesn't threaten any of the first i rooks, given that the first i rooks don't threaten each other?

(b) What is the chance that your niece actually accomplishes the task and does not prove you wrong?

(c) If you were using checker pieces as a replacement for rooks (so that they can be stacked on top of each other), then what would be the probability that your niece's placements result in peace? Assume that two pieces stacked on top of each other threaten each other.

(d) Explain the relationship between your answer to the previous part and the birthday paradox. In particular if we assume that 23 people have a 50% chance of having a repeated birthday (in a 365-day calendar), what is the probability that your niece places 23 stackable pieces in a peaceful position on a 365×365 board?

Solution:

(a) After having placed i rooks in a peaceful position, i of the rows and i of the columns are taken. So for the next rook we have $n-i$ choices for the row and $n-i$ choices for the column in order to remain in a peaceful position. The total number of board cells left is n^2-i. So the chance that the next rook keeps the peace is $\frac{(n-i)^2}{n^2-i}$.

(b) The product over $i=0,\dots,m-1$ gives us the final answer. So the answer is

$$\prod_{i=0}^{m-1}\frac{(n-i)^2}{n^2-i}=\frac{(n!)^2(n^2-m)!}{(n^2)!((n-m)!)^2}$$

(c) The only thing that changes from the previous part is that when placing the i-th piece, we no longer have n^2-i possibilities, but n^2 possibilities. So the answer changes to

$$\prod_{i=0}^{m-1}\frac{(n-i)^2}{n^2}=\frac{(n!)^2}{((n-m)!)^2n^{2m}}$$

(d) l the columns must be different.

All the rows being different is simply the birthday paradox. Similarly all the columns being different is another birthday paradox. So if the probability that m persons have different birthdays in an n-day calendar is p, then the probability that m rooks end up in a peaceful position on an $n\times n$ chessboard is p^2. Of course this can be verified by hand. The answer to the previous part is

$$\frac{(n!)^2}{((n-m)!)^2n^{2m}}=\left(\frac{n!}{(n-m)!n^m}\right)^2$$

The expression inside the parenthesis is the answer to the birthday paradox.

So if the probability p is 0.5 (which roughly happens for $n=365$ and $m=23$), then the probability that rooks end up in a peaceful position is $p^2=0.25$.