Note: *This homework consists of two parts. The first part (questions 1-4) will be graded and will determine your score for this homework. The second part (questions 5-6) will be graded if you submit them, but will not affect your homework score in any way. You are strongly advised to attempt all the questions in the first part. You should attempt the problems in the second part only if you are interested and have time to spare.*

For each problem, justify all your answers unless otherwise specified.

Part 1: Required Problems

1 Counting Boot Camp

Let's get some practice with counting! For this problem, you do not need to show work that justifies your answers. You may also leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many sequences of 15 coin-flips are there that contain exactly 4 heads?

(b) How many ways are there to arrange n heads and k tails into a sequence?

(c) How many 99-coin-flip sequences are there that contain more heads than tails?

(d) A poker hand contains 5 cards from a standard 52-card deck. The order of the cards in a poker hand is irrelevant. How many different 5-card poker hands are there?

(e) How many different 5-card poker hands are there that contain no aces?

(f) How many different 5-card poker hands are there that contain all four aces?

(g) How many different 5-card poker hands are there that contain exactly 3 spades?

(h) An anagram of HALLOWEEN is any re-ordering of the letters of HALLOWEEN, i.e., any string made up of the letters H, A, L, L, O, W, E, E, N in any order. The anagram does not have to be an English word.
How many different anagrams of HALLOWEEN are there?

(i) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a non-negative integer?

(j) How many solutions does $y_0 + y_1 = n$ have, if each y must be a positive integer?

(k) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a positive integer?

Solution:

(a) This is just the number of ways to choose 4 positions out of 15 positions to place the heads, and so is $\binom{15}{4}$.

(b) $\binom{n+k}{k}$. There are $n+k$ total coin flips in the sequence, and we need to pick k of them to be tails (the rest will be heads).

(c) **Answer 1:** There are $\binom{99}{k}$ sequences with k heads and $99-k$ tails. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such sequences is $\sum_{k=50}^{99} \binom{99}{k}$.
This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99} \binom{99}{k} = \sum_{k=50}^{99} \binom{99}{99-k} = \sum_{l=0}^{49} \binom{99}{l}$$

by substituting $l = 99-k$. Now $\sum_{k=50}^{99} \binom{99}{k} + \sum_{l=0}^{49} \binom{99}{l} = \sum_{m=0}^{99} \binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99} \binom{99}{k} = (1/2) \cdot 2^{99} = 2^{98}$.

Answer 2: Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-flip sequence can have the same number of heads and tails. Let A be the set of 99-flip sequences with more heads than tails, and B be the set of 99-flip sequences with more tails than heads. Now take any 99-coin-flip sequence x with more heads than tails i.e. $x \in A$. If all the flips of x are reversed, then you get a sequence y with more tails than heads, and so $y \in B$. This operation of reversals creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every length-99 sequence is either in A or in B, and since there are 2^{99} such sequences, we get $|A| = |B| = (1/2) \cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(d) We have to choose 5 cards out of 52 cards, so this is just $\binom{52}{5}$.

(e) We now have to choose 5 cards out of 48 non-ace cards. So this is $\binom{48}{5}$.

(f) We now require the four aces to be present. So we have to choose the remaining 1 card in our hand from the 48 non-ace cards, so there are 48 ways to do this.

(g) We need our hand to contain 3 out of the 13 spade cards, and 2 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{3}\binom{39}{2}$ ways to make up the hand.

(h) In this 9 letter word, the letters L and E are each repeated 2 times while the other letters appear once. Hence, the number 9! overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of 2! for the number of ways of permuting the 2 L's among themselves and another factor of 2! for the number of ways of permuting the 2 E's among themselves). Hence, there are $9!/(2!)^2$ different anagrams.

(i) $\binom{n+k}{k}$. We can imagine this as a sequence of n ones and k plus signs: y_0 is the number of ones before the first plus, y_1 is the number of ones between the first and second plus, etc. We can now count the number of sequences using the “balls and bins” method (also known as “stars and bars”).

(j) $n-1$. We can just enumerate the solutions here. y_0 can take values $1,2,\dots,n-1$ and this uniquely fixes the value of y_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(k) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2 That's Numberwang!

Congratulations! You've earned a spot on the game show "Numberwang".

(a) How many permutations of NUMBERWANG contain "GAME" as a substring? How about as a subsequence (meaning the letters of "GAME" have to appear in that order, but not necessarily next to each other)?

(b) In round 1 of Numberwang, each player chooses an ordered sequence of 5 digits. A valid sequence must have the property that it is non-increasing when read from left to right. For example, 99621 is a valid sequence, but 43212 is not. How many choices of valid sequences are there? (**Hint:** Relate the problem to balls and bins.)

(c) To win round 2 of Numberwang, a contestant must choose five nonnegative integers x_0, x_1, x_2, x_3, x_4 such that $x_0 + x_1 + x_2 + x_3 + x_4 = 100$, and $x_i \equiv i \pmod 5$. How many ways are there to pick a winning set of integers?

Solution:

(a) If we need "GAME" to be a substring, then this is equivalent to finding the number of ways to arrange 7 items in a row ("GAME", "N", "U", "B", "R", "W", and "N"), where two of the items are the same (the two "N"s). This amounts to $7!/2! = 2520$ ways.

If we need "GAME" to be a subsequence, let's first choose which 4 positions contain the letters of GAME; then, that uniquely determines the positions of G, A, M, E, since they have to be in that order. There are $\binom{10}{4}$ ways to do this, and $6!/2!$ ways to choose the positions of the remaining six letters (two of which are the same), so there are a total of $\binom{10}{4}\frac{6!}{2!} = 75600$ valid subsequences.

(b) This is actually a “balls and bins” (or “stars and bars”) problem in disguise! We have five digits (“balls”), and 10 “bins” describing the values of the digits from 0 to 9: one bin for nines, one bin for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. So we just need to count how many ways there are to distribute the digits into the 10 bins: so our answer is $\binom{10+5-1}{10-1} = \binom{14}{9}$.

(c) Let $x_i = 5y_i + i$ for nonnegative integers y_i (we do this because of the modulo conditions). Then, the equation we must satisfy becomes $(5y_0+0)+(5y_1+1)+(5y_2+2)+(5y_3+3)+(5y_4+4) = 100$, which simplifies down to $y_0+y_1+y_2+y_3+y_4 = 18$ for nonnegative integers y_i. This is a standard balls and bins (or stars and bars) problem, with 18 balls and 5 bins, so our answer is $\binom{22}{4}$.

Why is $y_0+y_1+y_2+y_3+y_4 = 18$ a balls and bins problem? y_i is the number of balls in bin i, and 18 is the number of balls in total. Here are a few possible arrangements, with $\star$s representing balls:

$$\underbrace{\star\star\star\star\star\star}_{y_0=6} \mid \underbrace{\quad}_{y_1=0} \mid \underbrace{\star\star\star\star\star\star\star}_{y_2=7} \mid \underbrace{\star\star\star\star\star}_{y_3=5} \mid \underbrace{\quad}_{y_4=0}$$

$$\underbrace{\star\star}_{y_0=2} \mid \underbrace{\star\star\star\star}_{y_1=4} \mid \underbrace{\star\star\star\star}_{y_2=4} \mid \underbrace{\star\star\star\star\star\star}_{y_3=6} \mid \underbrace{\star\star}_{y_4=2}$$

Notice that we arrange the balls into <u>5 bins</u> using <u>4 dividers</u>. The total number of ways we can arrange 18 balls and 4 dividers side-by-side in a line is $\binom{22}{4}$. Why?

- 22 is the total number of objects in the line, 18 balls + 4 dividers.
- 4 is the number of dividers to be placed in the line.

"22 choose 4", or $\binom{22}{4}$, is the number of ways we can position 4 dividers on a line <u>with</u> 18 indistinguishable balls.

3 Shipping Crates

A widget factory has four loading docks for storing crates of ready-to-ship widgets. Any time a loading dock contains at least 5 crates, a truck picks up 5 crates from that dock and ships them away (e.g., if 6 crates are sent to a loading dock, the truck removes 5, leaving 1 leftover crate still in the dock).

Suppose the factory produces 8 indistinguishable crates of widgets and sends each crate to one of the four loading docks. After all the shipping has been done, how many possible configurations of leftover crates in loading docks are there? (We consider two configurations to be the same if, for every loading dock, the two configurations have the same number of leftover crates in that dock.)

Solution:

Consider a scenario in which crates do not get removed from loading docks. With "balls and bins" (or "stars and bars") we see there are $\binom{11}{3}$ ways to distribute the crates before shipping.

Now, let's consider what happens when crates get shipped.

Case 1: No docks received ≥ 5 crates. Then the configuration does not change after shipping.

Case 2: One dock received ≥ 5 crates. After shipping, 5 crates were removed, leaving 3 leftover crates in the docks.

In Case 2, there are 4 ways to choose which dock has ≥ 5 crates, and once we put 5 crates into that dock, we are left to distribute 3 leftover crates among 4 docks in $\binom{6}{3}$ ways. So, there are $4\binom{6}{3}$ ways to arrive at one of these configurations with only 3 leftover crates, but there are only $\binom{6}{3}$ different ways the 3 leftover crates can be distributed after the shipping is done.

In Case 1, we can count all of the ways to distribute the crates before shipping, such that no dock ever recieves more than 4 crates. We already saw that there are $4\binom{6}{3}$ ways to send out the crates such that some dock *does* receive ≥ 5 crates, and $\binom{11}{3}$ total ways to distribute the crates before shipping, so there are $\binom{11}{3} - 4\binom{6}{3}$ ways to send out the crates such that no crates get removed. These configurations don't change after shipping, so Case 1 contributes $\binom{11}{3} - 4\binom{6}{3}$ different configurations after shipping.

In total, summing up the total configurations (after shipping) from Cases 1 and 2, we have

$$\binom{11}{3} - 4\binom{6}{3} + \binom{6}{3} = \binom{11}{3} - 3\binom{6}{3} = 105.$$

4 Picking Teams

(a) The 25 students in CS 70 have a strange way of picking a study group. Each person is given a distinct number 1 through 25 based on random chance, and participants may form a group as long of the sum of their numbers is **at most** 162. (The TAs are afraid of the number 162 for some reason...)

How many different study groups are possible? That is, how many different subsets of students are there, which are allowable under the ranking rule? (**Hint:** $162 = [(\sum_{i=1}^{25} i) - 1]/2)$)

(b) UC Berkeley is forming a new, n-person robotics team. There are $2n$ interested students: n mechanical engineers and n programmers. (Assume that no student is both a mechanical engineer and a programmer.)

Find the number of distinct n-person teams.

(c) Suppose that all robotics teams must also name a team captain who is a mechanical engineer. Find the number of ways to pick an n-person team with a mechanical engineer as the captain.

Solution:

(a) Note that the sum of the first 25 positive integers is 325. We can construct a bijection between sets of numbers that have sum ≤ 162 and sum greater than 162, namely, for every set of numbers with sum ≤ 162, the numbers not chosen form a set with sum greater than 162. Therefore, exactly half of all subsets of the 25 students have numbers that sum to ≤ 162, or $(1/2)(2^{25}) = 2^{24}$ subsets.

(b) **Method 1:** $\binom{2n}{n}$ This is simply the number of ways you can pick n students to be on the team.
Method 2: $\sum_{k=0}^{n}\binom{n}{k}^2$

We begin by noticing that $\binom{n}{k}^2 = \binom{n}{k}\cdot\binom{n}{n-k}$. This product gives us the number of ways of picking k mechanical engineers and $n-k$ programmers to be on the team. We add up all the products involving anywhere from 0 mechanical engineers all the way to n. This gives us the total number of ways to pick a team of n students.

(c) Take the same scenario as part (a) except now you have to choose a team captain who is also a mechanical engineer.

Method 1: $n\binom{2n-1}{n-1}$
This is the number of ways of picking the team captain multiplied with the number of ways of picking the rest of the team from the remaining students. The product gives the total number of teams with a mechanical engineer team captain.

Method 2: $\sum_{k=1}^{n} k\binom{n}{k}^2$

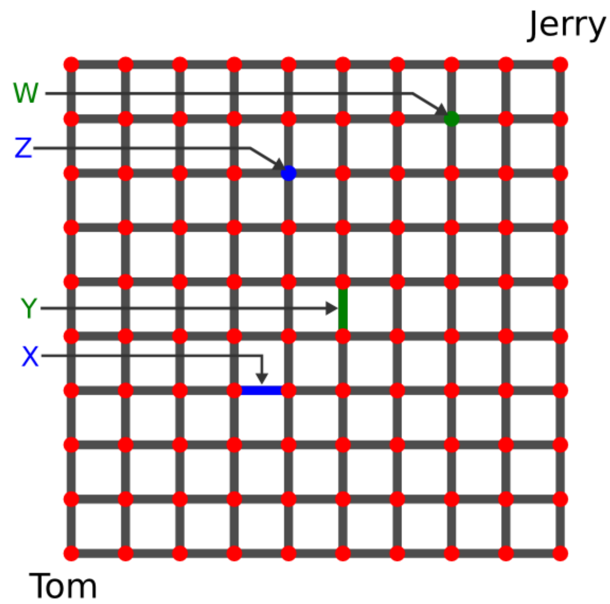
We begin similarly by noticing that $k\cdot\binom{n}{k}^2 = k\cdot\binom{n}{k}\cdot\binom{n}{n-k}$. Here as before we are picking k mechanical engineers and $n-k$ programmers to be on the team. However amongst the k mechanical engineers on the team, we choose one of them to be the team leader. We add up all the products involving anywhere from 1 mechanical engineers all the way to n. This gives us the total number of ways to pick a team of n students with a team captain who is a mechanical engineer.

Note: *This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.*

Part 2: Optional Problems

5 Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.



(a) How many such shortest paths exist?

(b) How many shortest paths pass through the edge labeled X?

(c) The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

(d) How many shortest paths pass through the vertex labeled Z? The vertex labeled W? Both the vertices Z and W? Neither vertex Z nor vertex W?

Solution:

(a) Each row in the maze has 9 edges, and so does each column. Any shortest path that Tom can take to Jerry will have exactly 9 horizontal edges going right (let's call these "H" edges) and 9 vertical edges going up (let's call these "V" edges).

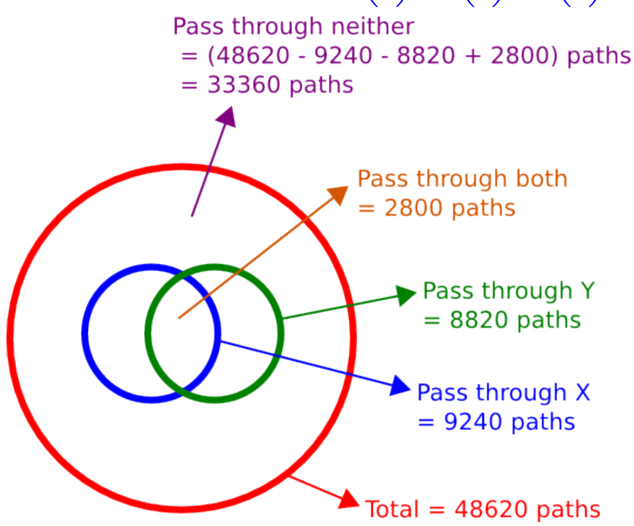
Observe also that every shortest path from Tom to Jerry can be described by a unique sequence consisting of 9 "H"s and 9 "V"s. For example, one such path is HHHHHHHHHVVVVVVVVV (which represents the path that goes all the way to the right, and then all the way to the top). Conversely, every such sequence of exactly 9 "H"s and 9 "V"s corresponds to a unique shortest path from Tom to Jerry.

Therefore, the number of shortest paths is exactly the same as the number of ways of arranging 9 "H"s and 9 "V"s in a sequence, which is $\binom{18}{9} = 48620$.

(b) For a shortest path to pass through the edge X, it has to first get to the left vertex of X. So the first portion of the path has to start at the origin, and end at the left vertex of X. Using the same logic as above, there are exactly $\binom{6}{3} = 20$ ways to complete this "first leg" of the path (consisting of 3 "H" edges and 3 "V" edges). Having chosen one of these 20 ways, the path has to then go from the right vertex of X to the top right corner of the maze (the "second leg"). This second leg will consist of 5 "H" edges and 6 "V" edges, and using the same logic, there are exactly $\binom{11}{5} = 462$ possibilities. Therefore, the total number of shortest paths that pass through the edge X is $20 \times 462 = 9240$.

(c) Using similar logic, any shortest path that passes through Y has to consist of 2 legs, the first leg going from the origin to the bottom vertex of Y, and the second leg going from the top vertex of Y to the top right corner of the maze. The first leg will consist of exactly 5 "H"s and 4 "V"s, while the second leg will consist of exactly 4 "H"s and 4 "V"s. So the total number of such shortest paths will be $\binom{9}{5} \times \binom{8}{4} = 8820$.

By a similar argument, let's try to figure out how many paths will pass through both X and Y. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 3 "H"s and 3 "V"s (going from the origin to the left edge of X), the second leg consisting of 1 "H" and 1 "V" (going from the right vertex of X to the bottom vertex of Y), and the third leg consisting of 4 "H"s and 4 "V"s (going from the top vertex of Y to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{6}{3} \times \binom{2}{1} \times \binom{8}{4} = 2800$.



Finally, we know that there are 48620 shortest paths in all, of which 9240 pass through X, 8820 pass through Y, and 2800 pass through both. So the number of paths that pass through neither is 33360 (see the figure above for an intuitive explanation).

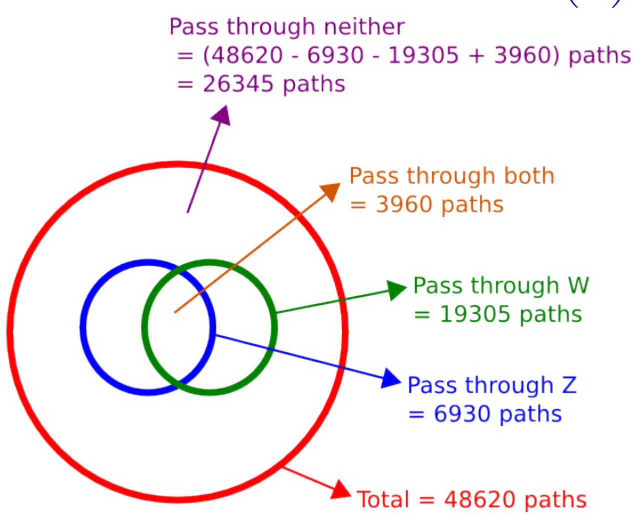
(d) This part is very similar in spirit to the previous one, except that in this case, each leg of the path we consider begins exactly where the previous leg ended, and *not* to the right or to the top of where the previous leg ended.

For concreteness, let's find out how many shortest paths pass through vertex Z. Observe that for a shortest path to pass through Z, it has to first get to Z. So the first portion of the path has to start at the origin, and end at Z. Using the same logic as above, there are exactly $\binom{11}{4} = 330$ ways to complete this "first leg" of the path (consisting of 4 "H" edges and 7 "V" edges).

Having chosen one of these 330 ways, the path has to then go from Z to the top right corner of the maze. This second leg will consist of 5 "H" edges and 2 "V" edges, and so there are exactly $\binom{7}{2} = 21$ possibilities. Therefore, the total number of shortest paths that pass through the vertex Z is $330 \times 21 = 6930$.

Using similar logic, any shortest path that passes through W has to consist of 2 legs, the first leg going from the origin to W, and the second leg going from W to the top right corner of the maze. The first leg will consist of exactly 7 "H"s and 8 "V"s, while the second leg will consist of exactly 2 "H"s and 1 "V". So the total number of such shortest paths will be $\binom{15}{7} \times \binom{3}{1} = 19305$.

By a similar argument, let's try to figure out how many paths will pass through both Z and W. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 4 "H"s and 7 "V"s (going from the origin to Z), the second leg consisting of 3 "H"s and 1 "V" (going from Z to W), and the third leg consisting of 2 "H"s and 1 "V" (going from W to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{11}{4} \times \binom{4}{1} \times \binom{3}{1} = 3960$.



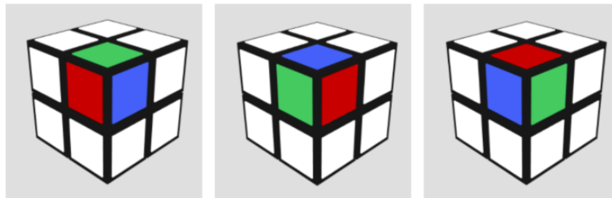
Finally, we know that there are 48620 shortest paths in all, of which 6930 pass through Z, 19305 pass through W, and 3960 pass through both. So the number of paths that pass through neither is 26345 (see the figure above for an intuitive explanation).

6 Rubik's Cube Scrambles

We wish to count the number of ways to scramble a $2 \times 2 \times 2$ Rubik's Cube, and take a quick look at the $3 \times 3 \times 3$ cube. Leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) The $2 \times 2 \times 2$ Rubik's Cube is assembled from 8 "corner pieces" arranged in a $2 \times 2 \times 2$ cube. How many ways can we assign all the corner pieces a position?

(b) Each corner piece has three distinct colors on it, and so can also be oriented three different ways once it is assigned a position (see figure below). How many ways can we *assemble* (assign each piece a position and orientation) a $2 \times 2 \times 2$ Rubik's Cube?



Three orientations of a corner piece

(c) The previous part assumed we can take apart pieces and assemble them as we wish. But certain configurations are unreachable if we restrict ourselves to just turning the sides of the cube. What this means for us is that if the orientations of 7 out of 8 of the corner pieces are determined, there is only 1 valid orientation for the eighth piece. Given this, how many ways are there to *scramble* (as opposed to *assemble*) a $2 \times 2 \times 2$ Rubik's Cube?

(d) We decide to treat scrambles that differ only in overall positioning (in other words, the entire cube is flipped upside-down or rotated but otherwise unaltered) as the same scramble. Then we overcounted in the previous part! How does this new condition change your answer to the previous part?

(e) Now consider the $3 \times 3 \times 3$ Rubik's Cube. In addition to 8 corner pieces, we now have 12 "edge" pieces, each of which can take 2 orientations. How many ways can we *assemble* a $3 \times 3 \times 3$ Rubik's Cube?

Solution:

(a) $8!$ ways. This is just the number of permutations of 8 objects.

(b) $8! \cdot 3^8$. Each of eight pieces has 3 possible orientations, to add onto the $8!$ permutations from part (a).

(c) $8! \cdot 3^7$. We divide the previous part by three. We can still choose the orientation of 7 of 8 corner pieces, but the eighth is then fixed.

(d) For each scramble, there are $6 \cdot 4 = 24$ ways to position/rotate it. We can see this by first choosing one of 6 faces to be on "top", then choosing one of four rotations with that particular face on top. So now we have $\frac{1}{24} \cdot 8! \cdot 3^7$.

(e) $8! \cdot 3^8 \cdot 12! \cdot 2^{12}$. We use the same procedure on the edge pieces as we do the corner pieces.