

1 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

- (a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?
- (b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?
- (c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

Solution:

- (a) Let $X \sim \text{Exponential}(1/50)$ be the time until the bulb is broken. For an exponential random variable with parameter λ , the density function is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$. So in this case $\lambda = 1/50$. Thus we can integrate the density to find the probability that the lightbulb broke in the first 30 days:

$$\mathbb{P}[X < 30] = \int_0^{30} \left(\frac{1}{50} \cdot e^{-x/50} \right) dx = 1 - e^{-30/50} = 1 - e^{-3/5} \approx 0.451.$$

- (b) The new bulb's waiting time Y is i.i.d. with the old bulb's. So the answer is

$$\mathbb{P}[Y > 30] = 1 - \mathbb{P}[Y < 30] = 1 - (1 - e^{-3/5}) = e^{-3/5} \approx 0.549.$$

- (c) The bulb is memoryless, so the probability it will last 60 days given that it has lasted 30 days, is just the probability it will last 30 days:

$$\mathbb{P}[X > 60 \mid X > 30] = \mathbb{P}[X - 30 > 30 \mid X > 30] = \mathbb{P}[X > 30] = e^{-3/5} \approx 0.549.$$

2 Darts Again

Edward and Khalil are playing darts.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $1/2$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}(X < Y)$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X , $\mathbb{P}(X < x)$.]

Solution: We are given that $Y \sim \text{Exponential}(1/2)$. We now find the distribution of X by solving for the CDF of X , $\mathbb{P}(X < x)$. To get this, we'll consider the ratio of the area where the distance to the center is less than x , compared to the entire available area. This gives us the following expression:

$$\begin{aligned}\mathbb{P}(X < x) &= \frac{\pi x^2}{\pi 10^2} \\ &= \frac{x^2}{100}\end{aligned}$$

Differentiating gives us the PDF of X , which is given by $f_X(x) = \frac{x}{50}$. Now, we solve for $\mathbb{P}(X < Y)$:

$$\begin{aligned}\mathbb{P}(X < Y) &= \int_{x=0}^{10} \mathbb{P}(X = x) \mathbb{P}(Y > x) dx \\ &= \int_{x=0}^{10} \frac{x}{50} e^{-0.5x} dx\end{aligned}$$

Evaluating this integral gives us $\mathbb{P}(X < Y) \approx 0.0767$.