CS 70 Discrete Mathematics and Probability Theory
Fall 2019 Alistair Sinclair and Yun S. Song **HW 6**

Note: *This homework consists of two parts. The first part (questions 1-3) will be graded and will determine your score for this homework. The second part (questions 4-5) will be graded if you submit them, but will not affect your homework score in any way. You are strongly advised to attempt all the questions in the first part. You should attempt the problems in the second part only if you are interested and have time to spare.*

Part 1: Required Problems

1 Unions and Intersections

Given:

- A is a countable, non-empty set. For all $i \in A$, S_i is an uncountable set.
- B is an uncountable set. For all $i \in B$, Q_i is a countable set.

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $A \cap B$

(b) $A \cup B$

(c) $\bigcup_{i \in A} S_i$

(d) $\bigcap_{i \in A} S_i$

(e) $\bigcup_{i \in B} Q_i$

(f) $\bigcap_{i \in B} Q_i$

Solution:

(a) Always countable. $A \cap B$ is a subset of A, which is countable.

(b) Always uncountable. $A \cup B$ is a superset of B, which is uncountable.

(c) Always uncountable. Let i' be any element of A. $S_{i'}$ is uncountable. Thus, $\bigcup_{i \in A} S_i$, a superset of $S_{i'}$, is uncountable.

(d) Sometimes countable, sometimes uncountable.

Countable: When the S_i are disjoint, the intersection is empty, and thus countable. For example, let $A = \mathbb{N}$, let $S_i = \{i\} \times \mathbb{R} = \{(i, x) \mid x \in \mathbb{R}\}$. Then, $\bigcap_{i \in A} S_i = \varnothing$.

Uncountable: When the S_i are identical, the intersection is uncountable. Let $A = \mathbb{N}$, let $S_i = \mathbb{R}$ for all i. $\bigcap_{i \in A} S_i = \mathbb{R}$ is uncountable.

(e) Sometimes countable, sometimes uncountable.

Countable: Make all the Q_i identical. For example, let $B = \mathbb{R}$, and $Q_i = \mathbb{N}$. Then, $\bigcup_{i \in B} Q_i = \mathbb{N}$ is countable.

Uncountable: Let $B = \mathbb{R}$. Let $Q_i = \{i\}$. Then, $\bigcup_{i \in B} Q_i = \mathbb{R}$ is uncountable.

(f) Always countable. Let b be any element of B. Q_b is countable. Thus, $\bigcap_{i \in B} Q_i$, a subset of Q_b, is also countable.

2 Countability Practice

(a) Do $(0,1)$ and $\mathbb{R}_+ = (0, \infty)$ have the same cardinality? If so, either give an explicit bijection (and prove that it is a bijection) or provide an injection from $(0,1)$ to $(0,\infty)$ and an injection from $(0,\infty)$ to $(0,1)$ (so that by Cantor-Bernstein theorem the two sets will have the same cardinality). If not, then prove that they have different cardinalities.

(b) Is the set of strings over the English alphabet countable? (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.) If so, then provide a method for enumerating the strings. If not, then use a diagonalization argument to show that the set is uncountable.

(c) Consider the previous part, except now the strings are drawn from a countably infinite alphabet $\mathscr{A}$. Does your answer from before change? Make sure to justify your answer.

Solution:

(a) Yes, they have the same cardinality.
Explicit bijection: Consider the bijection $f : (0,1) \to (0,\infty)$ given by

$$f(x) = \frac{1}{x} - 1.$$

We show that f is a bijection by proving separately that it is one-to-one and onto. The function f is one-to-one: suppose that $f(x) = f(y)$. Then,

$$\frac{1}{x} - 1 = \frac{1}{y} - 1,$$
$$\frac{1}{x} = \frac{1}{y},$$
$$x = y.$$

Hence, f is one-to-one.

The function f is onto: take any $y \in (0,\infty)$. Let $x = 1/(1+y)$. Note that $x \in (0,1)$. Then,

$$f(x) = \frac{1}{1/(1+y)} - 1 = 1 + y - 1 = y,$$

so f maps x to y. Hence, f is onto.

We have exhibited a bijection from $(0,1)$ to $(0,\infty)$, so they have the same cardinality. (In fact, they are both uncountable.)

Indirect bijection: The injection from $(0,1)$ to $(0,\infty)$ is trivial; consider the function $f : (0,1) \to (0,\infty)$ given by

$$f(x) = x.$$

It is easy to see that f is injective.

For the other way, consider the function $g : (0,\infty) \to (0,1)$ given by

$$g(x) = \frac{1}{x}.$$

To see that g is injective, suppose $g(x) = g(y)$. Then

$$\frac{1}{x} = \frac{1}{y} \implies x = y.$$

Hence g is injective. Thus we have an injective function from $(0,1)$ to $(0,\infty)$ and an injective function from $(0,\infty)$ to $(0,1)$. By Cantor-Bernstein theorem there exists a bijection from $(0,1)$ to $(0,\infty)$ and hence they have the same cardianlity.

(b) Countable. The English language has a finite alphabet (52 characters if you count only lower-case and upper-case letters, or more if you count special symbols – either way, the alphabet is finite).

We will now enumerate the strings in such a way that each string appears exactly once in the list. We will use the same trick as used in Lecture note 10 to enumerate the elements of $\{0,1\}^*$ We get our bijection by setting $f(n)$ to be the n-th string in the list. List all strings of length 1 in lexicographic order, and then all strings of length 2 in lexicographic order, and then strings of length 3 in lexicographic order, and so forth. Since at each step, there are only finitely many strings of a particular length ℓ, any string of finite length appears in the list. It is also clear that each string appears exactly once in this list.

(c) No, the strings are still countable. Let $\mathscr{A} = \{a_1, a_2, \dots\}$ denote the alphabet. (We are making use of the fact that the alphabet is countably infinite when we assume there is such an enumeration.) We will provide two solutions:

Alternative 1: We will encode the strings into ternary strings. Recall that we used a similar trick in Lecture note 10 to show that the set of all polynomials with natural coefficients is countable. Suppose, for example, we have a string: $S = a_5a_2a_7a_4a_6$. Corresponding to each of the characters in this string, we can write its index as a binary string: $(101, 10, 111, 100, 110)$. Now, we can construct a ternary string where "2" is inserted as a separator between each binary string. Thus we map the string S to a ternary string: 101210211121002110. It is clear that this mapping is injective, since the original string S can be uniquely recovered from this ternary string. Thus we have an injective map to $\{0,1,2\}^*$. From Lecture note 10, we know that the set $\{0,1,2\}^*$ is countable, and hence the set of all strings with finite length over $\mathscr{A}$ is countable.

Alternative 2: We will enumerate all the strings similar to that in part (b), although the enumeration requires a little more finesse. Notice that if we tried to list all strings of length 1, we would be stuck forever, since the alphabet is infinite! On the other hand, if we try to restrict our alphabet and only print out strings containing the first character $a \in \mathscr{A}$, we would also have a similar problem: the list

$$a, aa, aaa, \dots$$

also does not end.

The idea is to restrict *both* the length of the string and the characters we are allowed to use:

(a) List all strings containing only a_1 which are of length at most 1.

(b) List all strings containing only characters in $\{a_1, a_2\}$ which are of length at most 2 and have not been listed before.

(c) List all strings containing only characters in $\{a_1, a_2, a_3\}$ which are of length at most 3 and have not been listed before.

(d) Proceed onwards.

At each step, we have restricted ourselves to a finite alphabet with a finite length, so each step is guaranteed to terminate. To show that the enumeration is complete, consider any string s of length ℓ; since the length is finite, it can contain at most ℓ distinct a_i from the alphabet. Let k denote the largest index of any a_i which appears in s. Then, s will be listed in step $\max(k, \ell)$, so it appears in the enumeration. Further, since we are listing only those strings that have not appeared before, each string appears exactly once in the listing.

3 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

(c) The set of all bijective functions from $\mathbb{N}$ to $\mathbb{N}$.

Solution:

(a) Uncountable: Let us assume the contrary and proceed with a diagonalization argument. If there are countably many such function we can enumerate them as

	0	1	2	3	$\ldots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	$\ldots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$\ldots$
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$\ldots$
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now go along the diagonal and define f such that $f(x) > f_x(x)$ and $f(y) > f(x)$ if $y > x$, which is possible because at step k we only need to find a number $\in \mathbb{N}$ greater than all the $f_j(j)$ for $j \in \{0, \ldots, k\}$. This function differs from each f_i and therefore cannot be on the list, hence the list does not exhaust all non-decreasing functions. As a result, there must be uncountably many such functions.

Alternative Solution: Look at the subset $\mathscr{S}$ of strictly increasing functions. Any such f is uniquely identified by its image which is an infinite subset of $\mathbb{N}$. But the set of infinite subsets of $\mathbb{N}$ is uncountable. This is because the set of all subsets of $\mathbb{N}$ is uncountable, and the set of all finite subsets of $\mathbb{N}$ is countable. So $\mathscr{S}$ is uncountable and hence the set of all non-decreasing functions must be too.

Alternative Solution 2: We can inject the set of infinitely long binary strings into the set of non-decreasing functions as follows. For any infinitely long binary string b, let $f(n)$ be equal to the number of 1's appearing in the first n-digits of b. It is clear that the function f so defined is non-decreasing. Also, since the function f is uniquely defined by the infinitely long binary string, the mapping from binary strings to non-decreasing functions is injective. Since the set of infinite binary strings is uncountable, and we produced an injection from that set to the set of non-decreasing functions, that set must be uncountable as well.

(b) Countable: Let D_n be the subset of non-increasing functions for which $f(0) = n$. Any such function must stop decreasing at some point (because $\mathbb{N}$ has a smallest number), so there can only be finitely many (at most n) points $X_f = \{x_1, \ldots, x_k\}$ at which f decreases. Let y_i be the amount by which f decreases at x_i, then f is fully described by $\{(x_1, y_1), \ldots, (x_k, y_k), (-1, 0), \ldots, (-1, 0)\} \in \mathbb{N}^n = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ (n times), where we padded the k values associated with f with $n-k$ $(-1,0)$s. In Lecture note 10, we have seen that $\mathbb{N} \times \mathbb{N}$ is countable by the spiral method. Using it repeatedly, we get $\mathbb{N}^{(2^l)}$ is countable for all $l \in \mathbb{N}$. This gives us that $\mathbb{N}^n$ is countably for any finite n (because $\mathbb{N}^n \subset \mathbb{N}^{(2^l)}$ where l is such that $2^l \geq n$). Hence D_n is countable. Since each set D_n is countable we can enumerate it. Map an element of D_n to

(n, j) where j is the label of that element produced by the enumeration of D_n. This produces an injective map from $\cup_{n\in\mathbb{N}} D_n$ to $\mathbb{N}\times\mathbb{N}$ and we know that $\mathbb{N}\times\mathbb{N}$ is countable from Lecture note 10 (via spiral method). Now the set of all non-increasing functions is $\cup_{i\in\mathbb{N}} D_n$, and thus countable.

(c) Uncountable: We can inject the set of infinitely long binary strings into the set of bijective functions from $\mathbb{N}$ to $\mathbb{N}$. For any binary string $b = \{b_0, b_1, b_2, \dots\}$, consider the function $f : \mathbb{N} \to \mathbb{N}$ given by

$$\begin{aligned} f(2n) = 2n \text{ and } f(2n+1) = 2n+1 \qquad & \text{if } b_n = 0, \\ f(2n) = 2n+1 \text{ and } f(2n+1) = 2n \qquad & \text{if } b_n = 1. \end{aligned}$$

Note that this is a bijective function. Also, since f is uniquely defined by the binary string b, the mapping from infintely long binary strings to bijective functions is injective. Since the set of infinitely long binary strings is uncountable, and we produced an injection from that set to the set of bijective functions on $\mathbb{N}$, that set must be uncountable as well.

Alternative Solution: We will show that the set of bijections of $\mathbb{N}$ is at least as large as the powerset $\mathscr{P}(\mathbb{N})$ of $\mathbb{N}$, which we know to be uncountable. To do so, we need a little lemma:

Lemma (Shufflability of subsets): If A is a subset of $\mathbb{N}$ and $|A| > 1$, then we can find a bijection $h : A \to A$, so that for all $x \in A$, $h(x) \neq x$. That is, h maps every element to an element *other than itself.*

Proof: If $|A| = 1 < n < \infty$, then we can write $A = \{a_1, \dots, a_n\}$ and define $h(a_i) = a_{i+1 \bmod n}$ and are done. If $|A| = \infty$, then we write $A = \{a_1, a_2, \dots\}$ and define h to swap any two consecutive elements, i.e. $h(a_1) = a_2, h(a_2) = a_1, h(a_3) = a_4, h(a_4) = a_3$, etc.

Now we are in shape to associate with each subset S of $\mathbb{N}$ (ignoring subsets that are of the form $\mathbb{N}\setminus\{x\}$, which we will take care of later), a bijection g_S of $\mathbb{N}$: Namely, let us define g_S so that for all $x \in S$, $g_S(x) = x$, and on $\mathbb{N}\setminus S$, we let g_S be any function h_S from the lemma above. All we need to prove is that g_S and $g_{S'}$ are distinct for distinct S and S'. But if $S \neq S'$, then without loss of generality there exists some $s \in S \setminus S'$. For this s, we have $g_S(s) = s \neq g_{S'}(s)$ and so g_S and $g_{S'}$ must be different. Now, we have constructed an injection that maps the power set $\mathscr{P}(\mathbb{N})$ to a subset of bijective functions on $\mathbb{N}$, except for the special subsets of the form $\mathbb{N}\setminus\{x\}$ for some number x. The reason we excluded these sets is because then we would have to apply the shuffability lemma to the singleton $\{x\}$, which is not possible. Does this break our proof? No! The number of sets that we have ignored is countable, so the *remaining* subset of the power set that we have mapped into bijective functions *is still uncountable*, and thus the set of bijective functions from $\mathbb{N}\to\mathbb{N}$ is uncountable.

Note: *This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.*

Part 2: Optional Problems

4 Counting Shapes

Suppose scaled and shifted copies of a shape S are embedded into the plane $\mathbb{R}^2$. Let $\mathscr{C}$ denote the collection of all these copies. Thus each element in $\mathscr{C}$ determines the scaling and the position of that copy. Suppose further that the embedding is such that no two copies intersect. For example in the case of filled squares, if there is any overlap between two squares, then they intersect. In the case of the (non-filled) square, two copies intersect if and only if their boundaries intersect. Similarly, in the case of the halved-square, if either the boundary or the middle line of one square intersects with either the boundary or the middle line of some other square, then these two squares intersect.

Can $\mathscr{C}$ be uncountable if S is

(a) the filled square: ■ ?

(b) the square: □ ?

(c) the halved square: ⊟ ?

If no uncountable $\mathscr{C}$ exists, prove that all $\mathscr{C}$ must be countable.

Solution:

(a) There can be at most countably many filled squares: We will show that there is an injection from $\mathscr{C}$ to $\mathbb{Q} \times \mathbb{Q}$, which we know to be countable. To do so, we argue that for any filled square $F \in \mathscr{C}$, we can find a point (x_F, y_F) with rational coordinates (i.e. $p_F, q_F \in \mathbb{Q}$) inside F: We know that F has some center (x_F, y_F) and side length $2 \cdot r_F$, and whence $F = [x_F - r_F, x_F + r_F] \times [y_F - r_F, y_F + r_F]$. But any interval must contain a rational number, so in particular there is are rational $p_F \in [x_F - r_F, x_F + r_F]$ and $q_F \in [y_F - r_F, y_F + r_F]$, so that indeed $(p_F, q_F) \in F \cap (\mathbb{Q} \times \mathbb{Q})$.

Now let us define the function f from $\mathscr{C}$ to $\mathbb{Q} \times \mathbb{Q}$ as $f(F) = (p_F, q_F)$ and show that it is injective: If $f(F) = f(F')$, then F and F' share the point $(p_F, q_F) = (p_{F'}, q_{F'})$, so F must be F', for otherwise they would contradict the assumption that any distinct F and F' do not intersect.

(b) We give an example of a $\mathscr{C}$ that is uncountable: Let all squares $Q \in \mathscr{C}$ share the same center (e.g. $(0,0)$), but be of different side lengths $r_Q \in \mathbb{R}$. Then no two squares intersect, and we have $\mathbb{R}$ many of them. But we know that $\mathbb{R}$ is uncountable, and so we have uncountably many squares.

(c) Now $\mathscr{C}$ can be at most countable again. This time, we construct an injection from $\mathscr{C}$ to $(\mathbb{Q} \times \mathbb{Q}) \times (\mathbb{Q} \times \mathbb{Q})$, which we know to be countable. As before, for any $H \in \mathscr{C}$ we can pick a rational point $(p_H, q_H) \in \mathbb{Q} \times \mathbb{Q}$ in the upper half of H and $(s_H, t_H) \in \mathbb{Q} \times \mathbb{Q}$ in the lower half of H. Defining f from $\mathscr{C}$ to $(\mathbb{Q}\times\mathbb{Q})\times(\mathbb{Q}\times\mathbb{Q})$ by $f(H) = ((p_H,q_H),(s_H,t_H))$, we see that f must once more be injective, for if $f(H) = f(H')$, then neither H nor H' can be fully contained in each other's halfs. Therefore, either $H = H'$, or they intersect.

5 Cantor Sums

Show that every real number $x \in [0,1]$ can be written as the average of two numbers in the Cantor set.

Solution:

Let C denote the Cantor set. As seen in Lecture note 10, we have

$$C = \{x \in [0,1] : x \text{ has a ternary representation consisting only of 0's and 2's}\}.$$

Let a be any real number in $[0,1]$. We wish to find $b,c \in C$ such that $a = (b+c)/2$. Let a have a ternary representation $0.a_1a_2a_3\dots$ where $a_i \in \{0,1,2\}$ for all $i \in \{1,2,\dots\}$. This is equivalent to saying,

$$a = \sum_{i=1}^{\infty} a_i 3^{-i}.$$

Thus

$$2a = \sum_{i=1}^{\infty} (2a_i) 3^{-i}.$$

Now depending on a_i select b_i and c_i according to the following table. Note that $2a_i = b_i + c_i$.

a_i	$2a_i$	b_i	c_i
0	0	0	0
1	2	0	2
2	4	2	2

Let $b = \sum_{i=1}^{\infty} b_i 3^{-i}$ and $c = \sum_{i=1}^{\infty} c_i 3^{-i}$. Thus $2a = b + c$, i.e. $a = (b+c)/2$. Also b and c have their ternary representations $0.b_1b_2b_3\dots$ and $0.c_1c_2c_3\dots$ respectively and hence $b,c \in C$.