CS 70	Discrete Mathematics and Probability Theory	
Summer 2019	James Hulett and Elizabeth Yang	HW 1

1 Propositional Logic Language

For each of the following sentences, use the notation introduced in class to convert the sentence into propositional logic. Then write the statement's negation in propositional logic.

(a) (5 Points) The cube of a negative integer is negative.

(b) (5 Points) There are no integer solutions to the equation $x^2 - y^2 = 10$.

(c) (5 Points) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.

(d) (5 Points) For any two distinct real numbers, we can find a rational number in between them.

Solution:

(a) We can rephrase the sentence as "if $n < 0$, then $n^3 < 0$", which can be written as

$$(\forall n \in \mathbb{Z})((n < 0) \implies (n^3 < 0))$$

or equivalently as

$$(\forall n \in \mathbb{Z})((n \geq 0) \vee (n^3 < 0)).$$

The latter is easier to negate, and its negation is given by

$$(\exists n \in \mathbb{Z})((n < 0) \wedge (n^3 \geq 0))$$

(b) The sentence is

$$(\forall x, y \in \mathbb{Z})(x^2 - y^2 \neq 10).$$

The negation is

$$(\exists x, y \in \mathbb{Z})(x^2 - y^2 = 10)$$

(c) Let $p(x) = x^3 + x + 1$. The sentence can be read "there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x". Or,

$$(\exists x \in \mathbb{R})\left((p(x) = 0) \wedge ((\forall y \in \mathbb{R})(p(y) = 0) \implies (x = y))\right).$$

Its negation is given by

$$(\forall x \in \mathbb{R})\left((p(x) \neq 0) \vee ((\exists y \in \mathbb{R})(p(y) = 0) \wedge (x \neq y))\right).$$

This can be equivalently expressed as

$$(\forall x \in \mathbb{R})((p(x) = 0) \implies ((\exists y \in \mathbb{R})(p(y) = 0) \wedge (x \neq y))).$$

(d) The sentence can be read "if x and y are distinct real numbers, then there is a rational number z between x and y." Or,

$$(\forall x, y \in \mathbb{R})((x \neq y) \implies ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x))).$$

Equivalently,

$$(\forall x, y \in \mathbb{R})(x = y) \vee ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x)).$$

Note that $x < z < y$ is mathematical shorthand for $(x < z) \wedge (z < y)$, so the above statement is equivalent to

$$(\forall x, y \in \mathbb{R})(x = y) \vee ((\exists z \in \mathbb{Q})((x < z) \wedge (z < y)) \vee ((y < z) \wedge (z < x))).$$

Then the negation is

$$(\exists x, y \in \mathbb{R})(x \neq y) \wedge ((\forall z \in \mathbb{Q})((z \leq x) \vee (z \geq y)) \wedge ((y \geq z) \vee (x \leq z))).$$

2 Miscellaneous Logic

(a) Let the statement, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\ G(x,y)$, be true for predicate $G(x,y)$.

For each of the following statements, decide if the statement is certainly true, certainly false, or possibly true, and justify your solution. (If possibly true, provide a specific example where the statement is false and a specific example where the statement is true.)

(i) (2 Points) $G(3,4)$

(ii) (2 Points) $(\forall x \in \mathbb{R})\ G(x,3)$

(iii) (2 Points) $\exists y\ G(3,y)$

(iv) (2 Points) $\forall y\ \neg G(3,y)$

(v) (2 Points) $\exists x\ G(x,4)$

(b) (5 Point) Give an expression using terms involving $\vee, \wedge$ and $\neg$ which is true if and only if exactly one of X, Y, and Z is true.

Solution:

(a) (i) **Possibly true.**

The statement only guarantees there exists some y such that $G(3,y)$ is true, not that $G(3,4)$ in particular is true, so this is possibly but not necessarily true.

Choose $G(x,y)$ to be always true and statement is true.

Choose $G(x,y)$ to be $x > y$ to be false.

(ii) **Possibly true.**

In the same vein as the previous part, we are guaranteed the existence of *a* y for each x, not that $G(x,3)$ is necessarily true.

Choose $G(x,y)$ to be always true and statement is true.

Choose $G(x,y)$ to be $x > y$ to be false.

(iii) **True.**
The original statement is that for every x, there is a y where $G(x,y)$ is true which implies that for $x=3$, there is a y where $G(x,y)$ is true.

(iv) **False.**
This is the negation of the statement above.

(v) **Possibly true.**
This is similar to part b. We don't have information about $G(x,4)$ specifically - only that there exists a y for x such that $G(x,y)$ is true.
Choose $G(x,y)$ to be always true and statement is true.
Choose $G(x,y)$ to be $y \neq 4$.

(b) $(X \land \neg Y \land \neg Z) \lor (\neg X \land Y \land \neg Z) \lor (\neg X \land \neg Y \land Z)$
There are 3 cases in which exactly one of X, Y, and Z are true, and the cases are joined with "or"s because we only require one case to occur. Within each case, we fix one of X, Y, and Z to be true and the other two to be false. Since the three cases are mutually exclusive, at least one of them being true is equivalent to exactly one of them being true.

3 Logical Equivalence?

Decide whether each of the following is true or false and justify your answer:

(a) (5 Points) $\forall x \, \big(P(x) \land Q(x)\big) \equiv \forall x \, P(x) \land \forall x \, Q(x)$

(b) (5 Points) $\forall x \, \big(P(x) \lor Q(x)\big) \equiv \forall x \, P(x) \lor \forall x \, Q(x)$

(c) (5 Points) $\exists x \, \big(P(x) \lor Q(x)\big) \equiv \exists x \, P(x) \lor \exists x \, Q(x)$

(d) (5 Points) $\exists x \, \big(P(x) \land Q(x)\big) \equiv \exists x \, P(x) \land \exists x \, Q(x)$

Solution:

(a) **True**.

Assume that the LHS is true. Then we know for an arbitrary x $P(x) \land Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the RHS is true. Now assume the RHS. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. Thus the LHS is true.

(b) **False**. If $P(1)$ is true, $Q(1)$ is false, $P(2)$ is false and $Q(2)$ is true, the left-hand side will be true, but the right-hand side will be false.

(c) **True**

Assuming that the LHS is true, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the RHS is true. To prove the other direction, assume the LHS is false. Then there does not exists an x for which $P(x) \vee Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. Therefore the RHS is false.

(d) **False**. If $P(1)$ is true and $P(x)$ is false for all other x, and $Q(2)$ is true and $Q(x)$ is false for all other x, the right hand side would be true. However, there would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true.

4 Fermat's Contradiction

(5 Points) Prove that $2^{1/n}$ is not rational for any integer $n \geq 3$. (*Hint*: Use Fermat's Last Theorem. It states that there exists no positive integers a, b, c s.t. $a^n + b^n = c^n$ for $n \geq 3$.)

Solution:

If not, then there exists an integer $n \geq 3$ such that $2^{1/n} = \frac{p}{q}$ where p, q are positive integers. Thus, $2q^n = p^n$, and this implies

$$q^n + q^n = p^n,$$

which is a contradiction to the Fermat's Last Theorem.

5 Prove or Disprove

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Note 2) you used.

(a) (4 Points) For all natural numbers n, if n is odd then $n^2 + 3n$ is even.

(b) (4 Points) For all real numbers a, b, if $a + b \geq 20$ then $a \geq 17$ or $b \geq 3$.

(c) (4 Points) For all real numbers r, if r is irrational then $r + 1$ is irrational.

(d) (4 Points) For all natural numbers n, $10n^3 > n!$.

(e) (4 Points) For all natural numbers a where a^5 is odd, then a is odd.

Solution:

1. True/False: For all natural numbers n, if n is odd then $n^2 + 3n$ is even.

   True.
   Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 3n$, we get $(2k+1)^2 + 3 \times (2k+1)$. Simplifying the expression yields $4k^2 + 10k + 4$. This can be rewritten as $2 \times (2k^2 + 5k + 2)$. Since $2k^2 + 5k + 2$ is a natural number, by the definition of even numbers, $n^2 + 3n$ is even. ■

2. True/False: For all real numbers a, b, if $a+b \geq 20$ then $a \geq 17$ or $b \geq 3$.

   True.
   Proof: We will use a proof by contraposition. Suppose that $a < 17$ and $b < 3$ (note that this is equivalent to $\neg(a \geq 17 \vee b \geq 3)$). Since $a < 17$ and $b < 3$, $a+b < 20$ (note that $a+b < 20$ is equivalent to $\neg(a+b \geq 20)$). Thus, if $a+b \geq 20$, then $a \geq 17$ or $b \geq 3$ (or both, as "or" is not "exclusive or" in this case). By contraposition, for all real numbers a, b, if $a+b \geq 20$ then $a \geq 17$ or $b \geq 3$. ■

3. True/False: For all real numbers r, if r is irrational then $r+1$ is irrational.

   True.
   Proof: We will use a proof by contraposition. Assume that $r+1$ is rational. Since $r+1$ is rational, it can be written in the form a/b where a and b are integers. Then r can be written as $(a-b)/b$. By the definition of rational numbers, r is a rational number, since both $a-b$ and b are integers. By contraposition, if r is irrational, then $r+1$ is irrational. ■

4. True/False: For all natural numbers n, $10n^3 > n!$.

   False.
   Proof: We will use proof by counterexample. Let $n = 10$. $10 \times 10^3 = 10{,}000$. $(10!) = 3{,}628{,}800$. Since $10n^3 < n!$, the claim is false. ■

5. True/False: For all natural numbers a where a^5 is odd, then a is odd.

   True.
   Proof: This will be proof by contraposition. The contrapositive is "If a is even, then a^5 is even." Let a be even. By the definition of even, $a = 2k$. Then $a^5 = (2k)^5 = 2(16k^5)$, which implies a^5 even. By contraposition, for all natural numbers a where a^5 is odd, then a is odd.
   ■

6 Divisibility Induction

Prove the following statements using induction.

(a) (5 Points) For all $n \in \mathbb{N}$ with $n \geq 1$, the number $n^3 - n$ is divisible by 3.

(b) (5 Points) For all $n \in \mathbb{N}$ with $n \geq 1$, the number $5^n - 4n - 1$ is divisible by 16.

(c) (5 Points) You need to send in an envelope with n cents of postage on it. However, you only have 3 cent and 7 cent postage stamps. You do not want to go over n cents of postage. Prove that you can make exactly n cents of postage with 3 cent and 7 cent stamps as long as $n \geq 12$.

Solution:

1. Base Case: $n = 1$. $1^3 - 1 = 0$, 0 is divisible by 3.

Assume that for $n \leq k$, where $k \geq 1$, $k^3 - k$ is divisible by 3.
Now consider $n = k+1$. We want to show that $(k+1)^3 - (k+1)$ is also divisible by 3.

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 2k = (k^3 - k) + 3k^2 + 3k$$

By the Inductive Hypothesis, we know the part in parentheses is divisible by 3 and the rest has a factor of 3, so the whole term is divisible by 3.

2. Base Case: $n = 1$. $5^1 - 4(1) - 1 = 0$, and 0 is divisible by 16.

   Assume that for $n \leq k$, where $k \geq 1$, $5^k - 4k - 1$ is divisible by 16.
   Now consider $n = k+1$. We want to show that $5^{k+1} - 4(k+1) - 1$ is divisible by 16.

   $$5^{k+1} - 4(k+1) - 1 = 5 \cdot 5^k - 4k - 4 - 1 = 4 \cdot 5^k + (5^k - 4k - 1) - 4$$

   We know the part in the parentheses is divisible by 16 by the inductive hypothesis. We can use induction again to prove that the remaining part is also divisible by 16.

   New Proposition: $4 \cdot 5^n - 4$ is divisible by 16 for $n \geq 1$ and $n \in \mathbb{N}$.

   Base Case: $n = 1$. $4 \cdot 5^1 - 4 = 16$, which is divisible by 16.

   Assume that for $n \leq k$, where $k \geq 1$, $4 \cdot 5^k - 4$ is divisible by 16.
   Now consider $n = k+1$. We want to show that $4 \cdot 5^{k+1} - 4$ is also divisible by 16.

   $$4 \cdot 5^{k+1} - 4 = 20 \cdot 5^k - 4 = 16 \cdot 5^k + (4 \cdot 5^k - 4)$$

   The expression in parentheses is divisible by 16 by the inductive hypothesis. The rest is also divisible by 16 because 16 is a factor.

3. Base Case: $n = 12$. We can form postage of 12 cents using four 3-cent stamps.

   Inductive Step: For an arbitrary k cents, $\exists a, b \in \mathbb{N}$ s.t. $3a + 7b = k$. Now, consider $k+1$:

   $$k + 1 = 3a + 7b + 1$$

   Case 1: $a \geq 2$, decrement a by 2 and increment b by 1 to make $(k+1)$ cents with the stamps.

   Case 2: $a \leq 1$. If $(k+1) \geq 12$, this means that $b \geq 2$ as well. (Otherwise, if both $a \leq 1$ and $b \leq 1$, then $k \leq 10$, a contradiction.) Now, we can decrement b by 2 and increment a by 5 to make $(k+1)$ cents.

   An alternate solution is to use strong induction and consider 3 different base cases.

   - $n = 12$: We use four 3-cent stamps.
   - $n = 13$: We use two 3-cent stamps and one 7-cent stamp.
   - $n = 14$: We use two 7-cent stamps.

   Now, the inductive hypothesis is that for $12 \leq n \leq k$, we can form n cents with 3-cent and 7-cent stamps.
   Now consider $n = k+1$. We may assume $n \geq 15$, since we already considered $n = 12, 13, 14$ for the base cases. If we commit to using a 3-cent stamp, we are left to cover $(n-3)$ cents

with 3-cent and 7-cent stamps. We are now done because via our inductive hypothesis, we are able to make $(n-3)$ cents with the 3-cent and 7-cent stamps.

Note that we needed four base cases here because only when $n \geq 15$ was it true that we could form $(n-3)$ cents with the 3-cent and 7-cent stamps. For instance, if we did not handle $n = 14$ as a base case, we would need a way to make $n-3 = 11$ cents with 3-cent and 7-cent stamps, which is impossible.

7 A Coin Game

(10 Points) Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

Solution:

We can prove this by strong induction on n.

Base Case: If $n = 1$, you start with a stack of one coin, so the game immediately terminates. Your total score is zero–and indeed, $\frac{n(n-1)}{2} = \frac{1 \cdot 0}{2} = 0$.

Inductive Step: Suppose that if you start with i coins (for i between 1 and n inclusive), your score will be $\frac{i(i-1)}{2}$ no matter what strategy you employ. Now suppose you start with $n+1$ coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks s_1 and s_2 (so $s_1 + s_2 = n+1$ and $s_1, s_2 \geq 1$). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get $s_1 s_2$ points from the first split. From the inductive hypothesis (which we can apply because s_1 and s_2 are between 1 and n), we know that the total number of points you get from future splits of stack 1 is $\frac{s_1(s_1-1)}{2}$ and similarly that the total number of points you get from future splits of stack 2 is $\frac{s_2(s_2-1)}{2}$, regardless of what strategy you employ in splitting them. Thus, the total number of points we score is

$$\begin{aligned}
s_1s_2 + \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1)+2s_1s_2+s_2(s_2-1)}{2} \\
&= \frac{\big(s_1(s_1-1)+s_1s_2\big)+\big(s_2(s_2-1)+s_1s_2\big)}{2} \\
&= \frac{s_1(s_1+s_2-1)+s_2(s_1+s_2-1)}{2} \\
&= \frac{(s_1+s_2)(s_1+s_2-1)}{2}
\end{aligned}$$

Since $s_1+s_2 = n+1$, this works out to $\frac{(n+1)(n+1-1)}{2}$, which is what we wanted to show your total number of points came out to. This completes our proof by induction.

8 Leaves in a Tree

A *leaf* in a tree is a vertex with degree 1.

(a) (10 Points) Consider a tree with $n \geq 3$ vertices. What is the largest possible number of leaves the tree could have? Prove that this maximum m is possible to achieve, and further that there cannot exist a tree with more than m leaves.

(b) (10 Points) Prove that every tree on $n \geq 2$ vertices must have at least two leaves.

Solution:

(a) We claim the maximum number of leaves is $n-1$. This is achieved when there is one vertex that is connected to all other vertices (this is called the *star graph*).

We now show that a tree on $n \geq 3$ vertices cannot have n leaves. Suppose the contrary that there is a tree on $n \geq 3$ vertices such that all its n vertices are leaves. Pick an arbitrary vertex x, and let y be its unique neighbor. Since x and y both have degree 1, the vertices x, y form a connected component separate from the rest of the tree, contradicting the fact that a tree is connected.

(b) We give a proof by contradiction. Consider the longest path $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}$ between two vertices $x = v_0$ and $y = v_k$ in the tree (here the length of a path is how many edges it uses, and if there are multiple longest paths then we just pick one of them). We claim that x and y must be leaves. Suppose the contrary that x is not a leaf, so it has degree at least two. This means x is adjacent to another vertex z different from v_1. Observe that z cannot appear in the path from x to y that we are considering, for otherwise there would be a cycle in the tree. Therefore, we can add the edge $\{z, x\}$ to our path to obtain a longer path in the tree, contradicting our earlier choice of the longest path. Thus, we conclude that x is a leaf. By the same argument, we conclude y is also a leaf.

The case when a tree has only two leaves is called the *path graph*, which is the graph on $V = \{1,2,\ldots,n\}$ with edges $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$.

Alternate Solution: We know that $\sum_{v\in V}\deg(v) = 2|E|$. Since our graph is a tree, we have $\sum_{v\in V}\deg(v) = 2(|V|-1)$. We can expand the sum of the degrees as $n_1 + 2n_2 + 3n_3 + \ldots + dn_d$, where n_i is the number of vertices with degree i, and d is the maximum degree in the graph. If there are fewer than two leaves, either $n_1 = 0$ or $n_1 = 1$.

If $n_1 = 0$, we can lower bound $\sum_{v\in V}\deg(v)$ as follows:

$$\sum_{v\in V}\deg(v) = 2n_2 + 3n_3 + 4n_4 + \ldots + dn_d \geq 2n_2 + 2n_3 + 2n_4 \ldots 2n_d \geq 2(n_2 + n_3 + n_4 + \ldots + n_d)$$

The sum $(n_2 + n_3 + n_4 + \ldots + n_d)$ is equal to $|V| - n_1 = |V|$. Plugging this back into our inequalities above, we have $\sum_{v\in V}\deg(v) \geq 2(|V|)$. However, this contradicts the fact that we must have $\sum_{v\in V}\deg(v) = 2(|V|-1)$.

If $n_1 = 1$, we can use a similar lower bound:

$$\sum_{v\in V}\deg(v) > 2n_2 + 3n_3 + 4n_4 + \ldots + dn_d \geq 2n_2 + 2n_3 + 2n_4 \ldots 2n_d \geq 2(n_2 + n_3 + n_4 + \ldots + n_d)$$

Instead of having equality like in the $n_1 = 0$ case, note that we have a *strict* inequality. Now, we know the sum $(n_2 + n_3 + n_4 + \ldots + n_d)$ is equal to $|V| - n_1 = |V| - 1$. Plugging this back into our inequalities above, we have $\sum_{v\in V}\deg(v) > 2(|V|-1)$ with strict inequality, which contradicts the fact that $\sum_{v\in V}\deg(v) = 2(|V|-1)$.