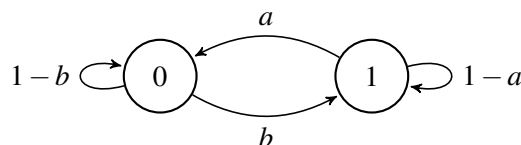


1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i , the chain can transition to any other state j , possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i, i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathcal{X}$. If $d(i) = 1 \forall i \in \mathcal{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



- (a) For what values of a and b is the above Markov chain irreducible? Reducible?
- (b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.
- (c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.
- (d) Construct a transition probability matrix using the above Markov chain.
- (e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

Solution:

- (a) The Markov chain is irreducible if both a and b are non-zero. It is reducible if at least one of a and b is 0.
- (b) We compute $d(0)$ to find that:

$$d(0) = \gcd\{2, 4, 6, \dots\} = 2.$$

This is because if we start at a state X then we can get back to it after taking an even number of steps only (2, 4, 6, 8, etc.), not by taking an odd number of steps (1, 3, 5, 7, etc.). Thus, the chain is periodic with period 2.

(c) We compute $d(0)$ to find that:

$$d(0) = \gcd\{1, 2, 3, \dots\} = 1.$$

Thus, the chain is aperiodic. Notice that the self-loops allow us to stay at the same node, thereby letting us get to any other node in an odd *or* even number of steps.

(d)

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e)

$$\begin{aligned}\pi(0) &= (1-b)\pi(0) + a\pi(1), \\ \pi(1) &= b\pi(0) + (1-a)\pi(1).\end{aligned}$$

One of the equations is redundant. We throw out the second equation and replace it with $\pi(0) + \pi(1) = 1$. This gives the solution

$$\pi = \frac{1}{a+b} \begin{bmatrix} a & b \end{bmatrix}.$$

2 Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p .

- Model this as a Markov chain. What is \mathcal{X} ? Write down the transition matrix.
- What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Solution:

- Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

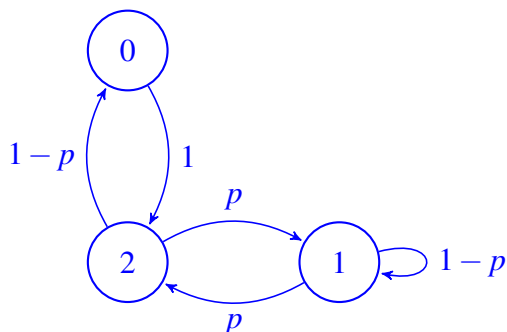
$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p , it rains and Allen brings the umbrella, arriving at state 2. With probability $1 - p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 1] = p, \quad \mathbb{P}[X_{n+1} = 1 \mid X_n = 1] = 1 - p$$

Suppose Allen is in state 2. With probability p , it rains and Allen brings the umbrella, arriving at state 1. With probability $1 - p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\mathbb{P}[X_{n+1} = 1 \mid X_n = 2] = p, \quad \mathbb{P}[X_{n+1} = 0 \mid X_n = 2] = 1 - p$$



We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

- (b) The transition matrices would be expressed as P^2 and P^n . Below we find the stationary distribution.

Observe that the transition matrix has non-zero element in its diagonal, which means the minimum number of steps to transit to state 1 from itself is one. Thus this transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P = \pi$, or $\pi(P - I) = 0$. This yields the balance equations

$$[\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & p \\ 1-p & p & -1 \end{bmatrix} = [0 \quad 0 \quad 0].$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0) + \pi(1) + \pi(2) = 1$.

$$[\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & 1 \\ 1-p & p & 1 \end{bmatrix} = [0 \quad 0 \quad 1]$$

Now solve for the distribution:

$$[\pi(0) \quad \pi(1) \quad \pi(2)] = \frac{1}{3-p} [1-p \quad 1 \quad 1]$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.