CS 70 Discrete Mathematics and Probability Theory
Summer 2021 **DIS 6D**

1 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[Y \geq X \cap X \geq 5]}{\mathbb{P}[Y \geq X]}.$$

$\mathbb{P}[Y \geq X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

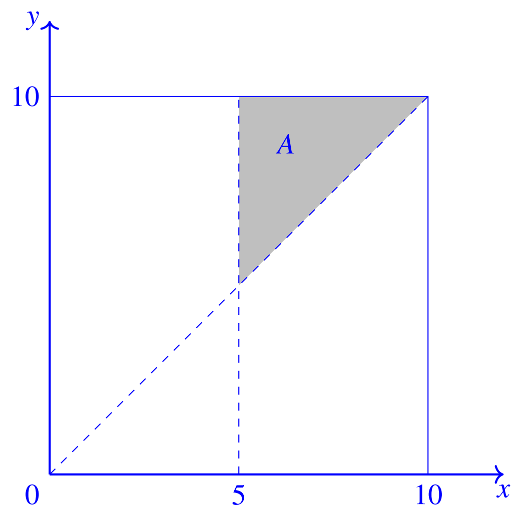

Figure 1: Joint probability density for the spinner.

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

2 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X.

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

Solution:

(a) Since $f(x,y)$ is a probability density function, it must integrate to 1. Then:

$$1 = \int_0^1 \int_0^2 Cxy\,dy\,dx = \int_0^1 2Cx dx = C$$

Therefore, $C = 1$.

(b) To get the marginal distribution of X, we integrate the joint distribution with respect to Y. So:

$$f_X(x) = \int_0^2 f(x,y)dy = \int_0^2 xydy = 2x$$

This is the marginal distribution for $0 \le x \le 1$.

(c) The conditional distribution of Y given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

(d) The conditional distribution of Y given $X = x$ does not depend on x, so they are independent.

Alternatively, you could find the marginal distribution of Y and see it is the same as the conditional distribution of Y:

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 xydx = \frac{y}{2}$$

Notice that since X and Y are independent, $f_X(x)f_Y(y) = xy = f_{X,Y}(x,y)$, i.e. the product of the marginal distributions is the same as the joint distribution.

3 Joint Distributions

(a) Give an example of discrete random variables X and Y with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. You should specify the joint distribution of X and Y.

(b) Give an example of discrete random variables X and Y that (i) are *not independent* and (ii) have the property that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$. Again you should specify the joint distribution of X and Y.

Solution:

(a) Let $P(X = 1) = \frac{1}{2}$, $P(X = -1) = \frac{1}{2}$, and $Y \equiv X$. Then $\mathbb{E}[X] = 1\mathbb{P}[X = 1] + (-1)\mathbb{P}[X = -1] = 0$, and $\mathbb{E}[Y] = \mathbb{E}[X]$. Similarly, since $X = Y$, $\mathbb{E}[XY] = \mathbb{E}[X^2] = 1$ and $\mathbb{E}[X]\mathbb{E}[Y] = 0$.

(b) One example is given by $P(X = -1, Y = \frac{1}{3}) = P(X = 1, Y = \frac{1}{3}) = P(X = 0, Y = -\frac{2}{3}) = \frac{1}{3}$.