Numerical Computing **2023**

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## Solution for Project 5

**Due date:** Friday, 29 December 2023, 11.59 PM

The purpose of this project is to implement the Simplex Method to find the solution of linear programs, involving both the minimisation and the maximisation of the objective function.

## 1. Graphical Solution of Linear Programming Problems [20 points]

Please consider the following two problems:

(1)

$$
\begin{aligned}
\min \quad & z = 4x + y \\
\text{s.t.} \quad & x + 2y \leq 40 \\
& x + y \geq 30 \\
& 2x + 3y \geq 72 \\
& x,\, y \geq 0
\end{aligned}
$$

(2) A tailor plans to sell two types of trousers, with production costs of 25 CHF and 40 CHF, respectively. The former type can be sold for 85 CHF, while the latter for 110 CHF. The tailor estimates a total monthly demand of 265 trousers. Find the number of units of each type of trousers that should be produced in order to maximise the net profit of the tailor, if we assume that the he cannot spend more than 7000 CHF in raw materials.

Start by writing problem (2) as a linear programming problem. Then complete the following tasks:

- Solve the system of inequalities.
- Plot the feasible region identified by the constraints.
- Find the optimal solution and the value of the objective function in that point.

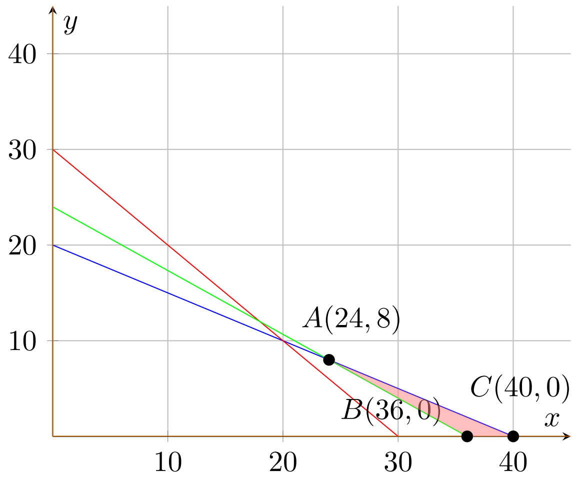
## Problem 1

$$\begin{aligned} \min \quad & z = 4x + y \\ \text{s.t.} \quad & x + 2y \leq 40 \\ & x + y \geq 30 \\ & 2x + 3y \geq 72 \\ & x,\, y \geq 0 \end{aligned}$$

First of all I start solving the systems of inequalities in term of the variable $y$ obtaining the following result:

$$\begin{cases} x + 2y \leq 40 & \rightarrow y \leq \frac{40-x}{2} \\ x + y \geq 30 & \rightarrow y \leq 30 - x \\ 2x + 3y \geq 72 & \rightarrow y \leq \frac{72-2x}{3} \\ x, y \geq 0 \end{cases}$$

Secondly , I represent the feasible area marked by the constraints on the Cartesian plane and I obtain:




Note that I decide to color each straight line with different dyes in order to easily understand the plane: The red line represents the line equation of the constraint $y \leq 30 - x$, the green line is the equation corresponding to the constraint $y \leq \frac{72-2x}{3}$, the blue one is referring to the line equation of the constraint $y \leq \frac{40-x}{2}$ and ,in conclusion, the barely visible orange lines correspond to the non-negativity constraints $x, y \geq 0$

It is possible to notice that there are 3 intersection points that delimited the feasible area : $A(24, 8)$ is the intersection between the inequality $y \leq \frac{72-2x}{3}$ and $y \leq \frac{40-x}{2}$, $B(36, 0)$ is the intersection between the inequality $y \leq \frac{72-2x}{3}$ and $y \leq 0$ and $C(40, 0)$ is the last intersection between the inequality $y \leq \frac{40-x}{2}$ and $y \leq 0$. Although there are different points to be possibly taken as a solution, the Fundamental Theorem of Linear Programming guarantees that there is only one unique solution.

Thirdly given those intersection coordinates, I evaluate the objective function at those vertices and pick the minimum value:

$$\text{at vertex } A(24, 8) : z = 4 * 24 + 8 = 104$$

$$\text{at vertex } B(36, 0) : z = 4 * 36 + 0 = 144$$

$$\text{at vertex } C(40, 0) : z = 4 * 40 + 0 = 160$$

In conclusion it is possible to notice that the point $A(24, 8)$ is the coordinates that minimize the objective function $z$ .Therefore, $x = 24$ and $y = 8$ is the optimal solution for the linear programming problem.

## Problem 2

First of all , to find a solution to the tailor plan problem it is needed to formulate the equivalent linear programming problem. The two type of trousers that tailor wants to sell can be identify in the linear programming problem as the variable $x$ for the former type and $y$ for the latter one. Since tailor's objective is to maximise profit, the objective function will be dictated by the sum of the profits of the two types of trousers, from the initial text it can be notice that : for the first type of trousers the production costs are 25CHF and the possible selling price can be 85CHF , thus, the profit that can be achieved from this type of trousers can be calculate as $85 - 25 = 60$CHF; instead for the latter type of pants the production and selling values are respectively to 40CHF and 110CHF, thereby the profit is equal to compute $110 - 40 = 70$CHF. Then given those information , it is possible to transcribe the objective function as:

$$z = 60x + 70y$$

Now the constraints have to be defined :
"*The tailor estimates a total monthly demand of 265 trousers*" means that the monthly request is not going to exceed 265 pants and it can be formulate that $x + y \leq 265$.
"*if we assume that the he cannot spend more than 7000 CHF in raw materials*" can be interpreted as an upper bound for the total trousers production cost and this expression can be formulate as $25x + 40y \leq 7000$.
Moreover, the problem formulation needs the non-negativity constraints $x, y \geq 0$ to guarantee positive values.

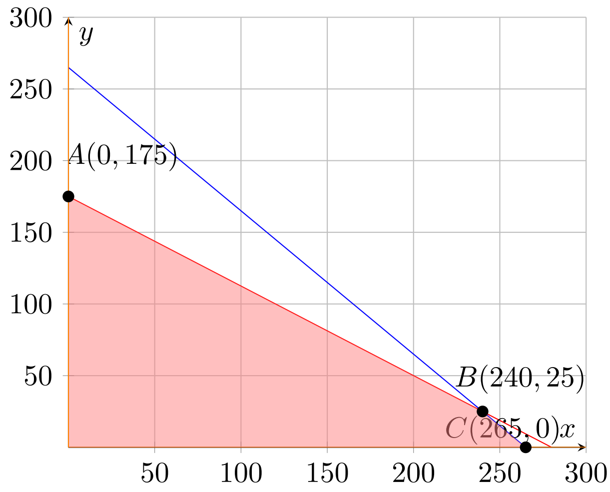
Finally , the linear programming problem can be expressed as:

$$\begin{aligned} \max \quad & z = 60x + 70y \\ \text{s.t.} \quad & x + y \leq 265 \\ & 25x + 40y \leq 7000 \\ & x,\, y \geq 0 \end{aligned}$$

First of all I start solving the systems of inequalities in term of the variable $y$ obtaining the following result:

$$\begin{cases} x + y \leq 265 & \rightarrow y \leq 265 - x \\ 25x_1 + 40x_2 \leq 7000 & \rightarrow y \leq \frac{7000-25x}{40} \\ x, y \geq 0 \end{cases}$$

Secondly , I represent the feasible area marked by the constraints on the Cartesian plane and I obtain:




Note that I decide to color each straight line with different dyes in order to easily understand the plane: The red line represents the line equation of the constraint $y \leq \frac{7000-25x}{40}$, the blue line is the equation corresponding to the constraint $y \leq 265 - x$ and ,in conclusion, the barely visible orange lines correspond to the non-negativity constraints $x, y \geq 0$.

It is possible to notice that there are 3 intersection points that delimited the feasible area : $A(0, 175)$ is the intersection between the inequality $y \leq \frac{7000-25x}{40}$ and the y-axis $x = 0$, $B(240, 25)$ is the intersection between the inequality $y \leq \frac{7000-25x}{40}$ and $y \leq 265 - x$ and $C(265, 0)$ is the last intersection between the inequality $y \leq 265 - x$ and $y \leq 0$.

Thirdly given those intersection coordinates, I evaluate the objective function at those vertices and pick the maximum value:

$$\text{at vertix } A(0, 175) : z = 60 * 0 + 70 * 175 = 12250$$

$$\text{at vertix } B(240, 25) : z = 60 * 240 + 70 * 25 = 16450$$

$$\text{at vertix } C(265, 0) : z = 60 * 265 + 70 * 0 = 10600$$

In conclusion it is possible to notice that the point $B(240, 25)$ is the coordinates that maximize the objective function $z$ . Therefore, $x = 240$ and $y = 25$ is the optimal solution for maximizing the net profit of the tailor.

## 2. Implementation of the Simplex Method [30 points]

In this first part of the assignment, you are required to complete 2 functions which are part of a dummy implementation of the simplex method. Specifically you have to complete the TODOs in:

- *standardise.m*, which writes a maximisation or minimisation input problem in standard form;
- *simplexSolve.m*, which solves a maximisation or minimisation problem using the simplex method.

You are given also some already-implemented functions to help you in your task: *simplex.m* is a wrapper which calls all the functions necessary to find a solution to the linear program; *auxiliary.m* solves the auxiliary problem to find a feasible starting basic solution of the linear program; *printSol.m* is a function which prints the optimal solution found by the simplex algorithm. Finally, *testSimplex.m* presents a series of 6 problems to check if your implementation is correct, before moving to the next part of the assignment. Additional details to aid you in your implementation can be found in the comments inside the code.

My files implementation correctly passed all the test contained in `testSimplex.m` document and my implementations are available inside project folder in the respective named matlab files

## 3. Applications to Real-Life Example: Cargo Aircraft [25 points]

In this second part of the assignment, you are required to use the simplex method implementation to solve a real-life problem taken from economics (constrained profit maximisation).

A cargo aircraft has 4 compartments (indicated simply as $S_1, \ldots, S_4$) used to store the goods to be transported. Details about the weight capacity and storage capacity of the different compartments can be inferred from the data reported in the following table:

| Compartment | Weight Capacity ($t$) | Storage Capacity ($m^3$) |
|---|---|---|
| $S_1$ | 18 | 11930 |
| $S_2$ | 32 | 22552 |
| $S_3$ | 25 | 11209 |
| $S_4$ | 17 | 5870 |

The following four cargos are available for shipment during the next flight:

| Cargo | Weight ($t$) | Volume ($m^3/t$) | Profit (CHF/$t$) |
|---|---|---|---|
| $C_1$ | 16 | 320 | 135 |
| $C_2$ | 32 | 510 | 200 |
| $C_3$ | 40 | 630 | 410 |
| $C_4$ | 28 | 125 | 520 |

Any proportion of the four cargos can be accepted, and the profit obtained for each cargo is increased by 10% if it is put in $S_2$, by 20% if it is put in $S_3$ and by 30% if it is put in $S_4$, due to the better storage conditions. The objective of this problem is to determine which amount of the different cargos will be transported and how to allocate it among the different compartments, while maximising the profit of the owner of the cargo plane. Specifically you have to:

1. Formulate the problem above as a linear program: what is the objective function? What are the constraints? Write down all equations, with comments explaining what you are doing.

   The linear programming formulation for the problem starts with the definition of the variable $x$ to handle various states of the goods on an aircraft:
   $x_{ij}$ refers to the quantity loaded in the cargo $i$ that is allocated to compartment $j$ .
   To give an idea , $x_{1,2}$ refers to the quantity loaded in the cargo $C_1$ which is allocated to the compartment $S_2$.

   Having said that, it is required to formulate the objective function in a way that "*Any proportion of the four cargos can be accepted, and the profit obtained for each cargo is increased by 10% if it is put in S2, by 20% if it is put in S3 and by 30% if it is put in S4, due to the better storage conditions. The objective of this problem is to determine which amount of the different cargos will be transported and how to allocate it among the different compartments, while maximising the profit of the owner of the cargo plane*" which can be express as :

$$
\begin{aligned}
\max z = {} & (135 * 1 * x_{11}) + (135 * 1.1 * x_{12}) \\
& + (135 * 1.2 * x_{13}) + (135 * 1.3 * x_{14}) \\
& + (200 * 1 * x_{21}) + (200 * 1.1 * x_{22}) \\
& + (200 * 1.2 * x_{23}) + (200 * 1.3 * x_{24}) \\
& + (410 * 1 * x_{31}) + (410 * 1.1 * x_{32}) \\
& + (410 * 1.2 * x_{33}) + (410 * 1.3 * x_{34}) \\
& + (520 * 1 * x_{41}) + (520 * 1.1 * x_{42}) \\
& + (520 * 1.2 * x_{43}) + (520 * 1.3 * x_{44})
\end{aligned}
$$

Then I define the constraints for the objective function by analyzing the information contained in both tables:
I formulate the next constraints in the following order:

1) Compartments weight capacity

$$\begin{aligned}
\textbf{s.t.} \quad & \\
& x_{11} + x_{21} + x_{31} + x_{45} \leq 18 \\
& x_{12} + x_{22} + x_{32} + x_{46} \leq 32 \\
& x_{13} + x_{23} + x_{33} + x_{47} \leq 25 \\
& x_{14} + x_{24} + x_{34} + x_{48} \leq 17
\end{aligned}$$

2) Volume capacity of compartments

$$\begin{aligned}
& 320 * x_{11} + 510 * x_{21} + 630 * x_{31} + 125 * x_{45} \leq 11930 \\
& 320 * x_{12} + 510 * x_{22} + 630 * x_{32} + 125 * x_{46} \leq 22552 \\
& 320 * x_{13} + 510 * x_{23} + 630 * x_{33} + 125 * x_{47} \leq 11209 \\
& 320 * x_{14} + 510 * x_{24} + 630 * x_{34} + 125 * x_{48} \leq 5870
\end{aligned}$$

3) Volume availability of each cargo

$$\begin{aligned}
& x_{11} + x_{12} + x_{13} + x_{14} \leq 16 \\
& x_{21} + x_{22} + x_{23} + x_{24} \leq 32 \\
& x_{31} + x_{32} + x_{33} + x_{34} \leq 40 \\
& x_{41} + x_{42} + x_{43} + x_{44} \leq 28
\end{aligned}$$

Then , I also add the non-negativity constraints which are necessary to avoid negative values for the weights :

$$\begin{aligned}
& x_{11}, x_{12}, x_{13}, x_{14} \geq 0 \\
& x_{21}, x_{22} x_{23}, x_{24} \geq 0 \\
& x_{31}, x_{32} x_{33}, x_{34} \geq 0 \\
& x_{41}, x_{42} x_{43}, x_{44} \geq 0
\end{aligned}$$

2. Create a script *exercise2.m* which uses the simplex method implemented in the previous exercise to solve the problem. What is the optimal solution? Visualise it graphically and briefly comment the results obtained (are you surprised of this outcome on the basis of your data?).

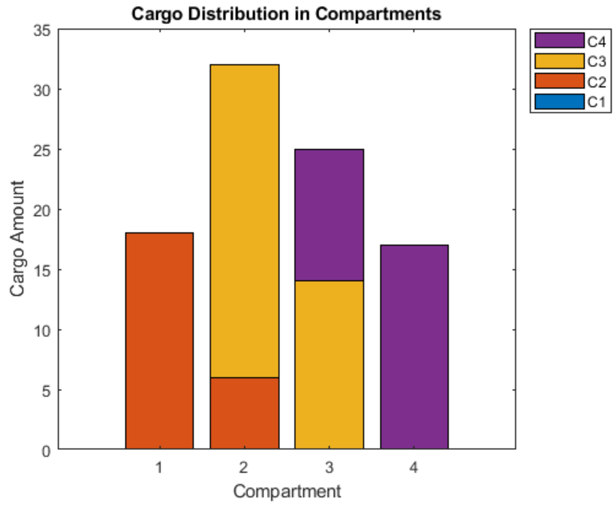
   The file `exercise2.m` is available inside the project folder included of comments about code. The optimal solution and the variables associated to that solution that I obtain from the computation of algorithm are :

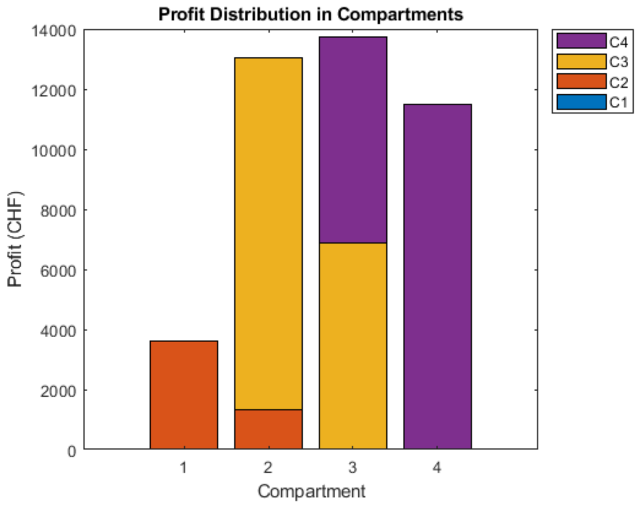
```
Variables and optimal solution:

x5 = 18.000000        % x2,1
x6 = 6.000000         % x2,2
x10 = 26.000000       % x3,2
x11 = 14.000000       % x3,3
x15 = 11.000000       % x4,3
x16 = 17.000000       % x4,4
s5 = 2750.000000
s6 = 3112.000000
s7 = 1014.000000
s8 = 3745.000000
s9 = 16.000000
s10 = 8.000000

Optimal value of z = 41890
```

For the graphical representation of the problem optimal solution I decided to use two bar plots one for showing the cargo distributions and one for showing the profit distribution inside each compartment.I think that it is best way to correctly show the distribution of how each compartment was used for the various loads in order to maximize the final profit, this graphical interpretion of data makes easily understand how in practice the optimal solution can be reached.




Examining the various plots it is possible to notice that the results reflect a strategic allocation of cargoes in order to maximize the final profit. Especially ,as it is possible to notice from the plot , the cargo $C_4$, which is also the most profitable load, is distributed in the compartment $S_3$ and $S_4$ so as to completely maximize the profit for the next flight given that the profit of the cargo is increase by 30%, if added to $S_3$, and 40%, if added to $S_4$ .

## 4. Cycling and Degeneracy [10 points]

Consider now the following simple problem:

$$\begin{aligned} \max\ & z = 3x_1 + 4x_2, \\ \text{s.t.}\ & 4x_1 + 3x_2 \leq 12 \\ & 4x_1 + x_2 \leq 8 \\ & 4x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0. \end{aligned}$$

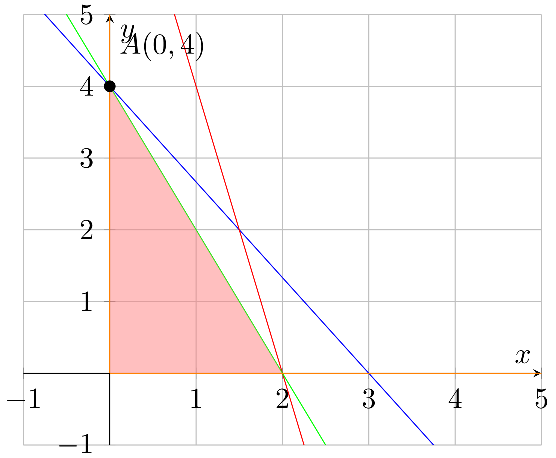
1. Create a script *exercise3.m* which uses the simplex method implemented above to solve this problem. Do you achieve convergence within the maximum number of iterations (given by the maximum number of possible basic solutions)? Do you notice any strange behaviour? (*hint:* check, e.g., the indices of the entering and departing variables).

   Once I run the `exercise3.m` script, I notice the error "*Incorrect loop, more iterations than the number of basic solutions*" which it means that the convergence has not been reached before the maximum number of possible basic solutions

   I also notice that during the computation of each iteration the simplex algorithm calculate repeatedly the same solution without reach the convergence. For example I notice that at the iteration `nIter`=0 the variables assume $x_B = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ , $x_D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,$c_B = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}$ and $c_D = \begin{bmatrix} 4 & 0 \end{bmatrix}$. Then in the subsequent iteration `nIter`=1 the variables are equal to $x_B = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ $x_D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,$c_B = \begin{bmatrix} 0 & 3 & 4 \end{bmatrix}$ and $c_D = \begin{bmatrix} 0 & 0 \end{bmatrix}$. These results are continuously repeated in the following iterations until the error is not thrown and I believe that this cycling behaviour without finding the optimal solution is due to the value assume by the variable $x_B$. This conclusion is given by the fact that $x_B$ is a *degenerate* solution because its third component is equal to 0 and this assumption can generate some implications during the finding solutions procedures of the algorithm consequently generating this cycling behavior

2. Look at the number of constraints and at the number of unknowns: what can you notice about the underlying system of equations? Represent them graphically and try to use this information to explain the behaviour of your solver in the previous point.

   I can notice that the constraints are really similar between each other and looking at the plot corresponding to the system it is possible to notice that they are redundant. Probably the cycling behaviour effect is caused by the presence of those redundant constraints which do not influence the final feasible area.




Note that I decide to color each straight line with different dyes in order to easily understand the plane: The red line represents the line equation of the constraint $4x_1 + x_2 \leq 8$, the blue line is the equation corresponding to the constraint $4x_1 + 3x_2 \leq 12$,the green line is the equation corresponding to the constraint $4x_1 + 2x_2 \leq 8$,and ,in conclusion, the barely visible orange lines correspond to the non-negativity constraints $x_1, x_2 \geq 0$.

## 5. Quality of the Code & Report [15 points]

Each project sums up to 100 points, out of which 15 points are dedicated to the general quality of your written report and of the code submitted. Your report should be a coherent document. If there are theoretical questions, explain and justify your answers. If you made a particular choice in your implementation that might be out of the ordinary, please explain it in the report. The code you submit should be self-contained and executable. It should include the set-up for all the results that you obtained and listed in your report. Furthermore, it should be readable and include comments explaining the more complicated steps.

## 6. References


1. ”Linear programming and the Simplex method” slides of the lesson of 13th December 2023 available on iCorsi platform

2. Ron Larson,”Elementary Linear Algebra” chapter 9 about linear programming and the simplex method