Isfahan University of Technology, Mathemetical Sciences Department

Linear Algebra's Extra Problems #1

February 23, 2021

1. For a set of vectors $S = \{v_1, v_2, \cdots, v_n\}$, prove that $span(S)$ is the intersection of all subspaces that contain S.
2. Determine the dimensions of each of the following vector spaces:
   a) The space of polynomials having degree n or less.
   b) The space $\mathbf{R}^{m\times n}$ of $m \times n$ matrices.
   c) The space of $n \times n$ symmetric matrices.
3. Let $B = \{b_1, b_2, \cdots, b_n\}$ be a basis for a vector space V. Prove that each $v \in V$ can be expressed as a linear combination of the b_i 's

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n \ ,$$

   in only one way—i.e., the coordinates α_i are unique.
4. If V is a vector space over $\mathbb{C}$ of dimension n, prove that V can be regarded as a vector space over $\mathbb{R}$ of dimension $2n$.
5. Let $P_0(X), P_1(X), \cdots, P_n(X)$ be polynomials in $\mathbb{R}_n[X]$ such that, for each i, the degree of $P_i(X)$ is i, prove that $\{P_0(X), P_1(X), \cdots, P_n(X)\}$ is a basis for $\mathbb{R}_n[X]$.
6. Prove that if V and W are three-dimensional subspaces of $\mathbb{R}^5$, then V and W must have a nonzero vector in common.
7. Suppose V is known to have dimension k. Prove that
   a) any k independent vectors in V form a basis;
   b) any k vectors that span V form a basis.

   In other words, if the number of vectors is known to be correct, either of the two properties of a basis implies the other.