Isfahan University of Technology
Department of Mathematical Sciences

Applied Linear Algebra
Assignment #1

Due Date: 17 Esfand 99

1. In each case below, determine whether $v \in \ Span(S)$.
   a) In the vector space $V = \mathbb{R}^3$ (over $\mathbb{F} = \mathbb{R}$), with
   $$v = (-1, 5, 5), \quad S = \{(3, -1, 1), (1, 2, 3)\}$$
   b) In the vector space $V = \mathbb{C}^3$ (over $\mathbb{F} = \mathbb{C}$), with
   $$v = (1, 0, 0), \quad S = \{(i, 1, 0), (-i, 1, 0)\}$$
   c) In the vector space $\mathbb{R}_3[x]$ of polynomials of degree at most 3, with
   $$v = x^3 - 2x^2, \quad S = \{x^3 + 2x + 2, x^2 + x + 3\}$$
2. Let x, y, z be vectors in a vector space V over an arbitrary field $\mathbb{F}$. Prove that:
   a) If $\{x, y, z\}$ is linearly dependent then $\{x, x + y, x + y + z\}$ is also linearly dependent.
   b) If $\{x, x + y, x + y + z\}$ is a basis of V then $\{x, y, z\}$ is a basis of V.
3. For which value(s) of h is the following set of vectors in $\mathbb{R}^3$ linearly dependent?
   $$\left\{ \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix} \right\}$$
4. Find a basis for the space spanned by the set of vectors
   $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ -14 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 10 \end{bmatrix} \right\}$$
5. Consider the vectors $u = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, and $w = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.
   a) Prove that these vectors do not span $\mathbb{R}^3$.

b) Write down a system of equations whose solution set is equal to the span of $\{u, v, w\}$.

6. A matrix A in $\mathbb{R}^{n\times m}$ is called stochastic if the sum of entries in each row is equal to one. Let W be the set of all stochastic matrices in $\mathbb{R}^{n\times m}$.
   a) Prove that W is a subspace of $\mathbb{R}^{n\times m}$.
   b) Find $\dim W$ (with a proof).

7. Determine whether each of the following statements is *True* or *False*. If any item is False, give a counterexample and if it is True prove it.
   a) If $\overrightarrow{v_1}, \overrightarrow{v_2}$ and $\overrightarrow{v_3}$ are in $\mathbb{R}^3$ and $\overrightarrow{v_3}$ is not a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}$, then $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is linearly independent.
   b) If the set of vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\} \subset \mathbb{R}^4$ is linearly independent, then $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is also linearly independent.
   c) The set of nonzero and non-parallel vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\} \subset \mathbb{R}^5$ is always linearly independent.
   d) If W_1 and W_2 are two 2-dimensional subspaces in $\mathbb{R}^4$, then $W_1 \cap W_2$ is a subspace of dimension at least one.

8. Let A be a matrix in $\mathbb{R}^{3\times 3}$ and we define

$$W_A = \{M \in \mathbb{R}^{3\times 3} \mid AM = MA\}.$$

   a) Prove that W_A is a subspace of $\mathbb{R}^{3\times 3}$.

   In each of the following cases, find the dimension of W_A.

   b)
$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

   where a, b, c are distinct real numbers.

   c)
$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix},$$

   where a, b are distinct real numbers.