HOMEWORK 7

SOLUTIONS

Problem 1. For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

(b) $V = P_1(\mathbb{R})$; $T(a + bx) = (6a - 6b) + (12a - 11b)x$; $\beta = \{\, 3 + 4x, 2 + 3x \,\}$

(d) $V = P_2(\mathbb{R})$; $T(a + bx + cx^2) = (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2$;
$\beta = \{\, x - x^2, -1 + x^2, -1 - x + x^2 \,\}$

(f) $V = M_{2\times 2}(\mathbb{R})$; $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$;
$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

Solution. We know that β consists of eigenvectors of T if and only if $[T]_\beta$ is diagonal.

(b) $[T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$; yes

(d) $[T]_\beta = \begin{pmatrix} 4 & 2 & 0 \\ 4 & 0 & 3 \\ -4 & -2 & -3 \end{pmatrix}$; no

(f) $[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; yes

□

Problem 2. For each of the following matrices $A \in M_{n\times n}(F)$, (i) determine all of the eigenvalues of A; (ii) for each eigenvalue λ of A, find the set of eigenvectors corresponding to λ; (iii) if possible, find a basis for F^n consisting of eigenvectors of A; (iv) if successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbb{R}$

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = \mathbb{C}$

Solution. The eigenvalues satisfy $\det(A - \lambda I) = 0$, and for each eigenvalue λ, the eigenvectors are the non-zero elements of $\ker(A - \lambda I)$. If there is a basis of eigenvectors, then we can take Q to have columns given by this basis and D to be the diagonal matrix whose diagonal entries are the eigenvalues (in order).

(a) (i) $\lambda = 1, 2, 3$

(ii) E_1 has basis $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$; E_2 has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$; E_3 has basis $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(iii) Taking these three vectors together gives a basis for $\mathbb{R}^3$

(iv) $Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(b) (i) $\lambda = -1, 1$

(ii) E_{-1} has basis $\left\{ \begin{pmatrix} -1+i \\ 2 \end{pmatrix} \right\}$; E_1 has basis $\left\{ \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right\}$

(iii) Taking these two vectors together gives a basis for $\mathbb{C}^2$

(iv) $Q = \begin{pmatrix} -1+i & 1+i \\ 2 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\square$

Problem 3. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

(e) $V = P_2(\mathbb{R})$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$

(g) $V = P_3(\mathbb{R})$ and $T(f(x)) = xf'(x) + f''(x) - f(2)$

Solution. We start with the standard basis ε on V and compute $[T]_\varepsilon$. We then compute the eigenvalues of $[T]_\varepsilon$, which are the same as the eigenvalues of T, and corresponding eigenvectors of $[T]_\varepsilon$ are the ε-coordinate vectors of the eigenvectors of T.

(e) $\lambda = 0, 2, 4$; $\beta = \left\{ -3 + x, -3 - 13x + 4x^2, 1 + x \right\}$

(g) $\lambda = -1, 1, 2, 3$; $\beta = \left\{ 1, -1 + x, -2 + 3x^2, -7 + 6x + 2x^3 \right\}$

□

Problem 4. A *scalar matrix* is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all of the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalisable matrix having only one eigenvalue is a scalar matrix.

(c) Using part (b), prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalisable.

Solution. (a) If A is similar to λI, then for some invertible matrix S, we have $A = S^{-1}(\lambda I)S = \lambda I$.

(b) If the $n \times n$ matrix A is diagonalisable with eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicity), then A is similar to $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. If A has only one distinct eigenvalue λ, then A is similar to $\operatorname{diag}(\lambda, \dots, \lambda) = \lambda I$, so $A = \lambda I$.

(c) The matrix given has only the eigenvalue 1, but is not a scalar matrix, so not diagonalisable.

□

Problem 5. For any square matrix A, prove that A and A^{t} have the same characteristic polynomial (and hence the same eigenvalues).

Solution. We have

$$p_{A^{\mathrm{t}}}(\lambda) = \det(A^{\mathrm{t}} - \lambda I) = \det((A - \lambda I)^{\mathrm{t}}) = \det(A - \lambda I) = p_A(\lambda).$$

□

Problem 6. (a) Let T be a linear operator on a vector space V, and let $\mathbf{x}$ be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that $\mathbf{x}$ is an eigenvector of T^m corresponding to the eigenvalue λ^m.

(b) State and prove the analogous result for matrices.

Solution. (a) We have $\mathbf{x} \neq \mathbf{0}$ by assumption and

$$T^m(\mathbf{x}) = T^{m-1}(\lambda\mathbf{x}) = \lambda T^{m-1}(\mathbf{x}) = \lambda^2 T^{m-2}(\mathbf{x}) = \cdots = \lambda^m \mathbf{x}.$$

(b) Let A be an $n \times n$ matrix and let X be a non-zero eigenvector of A corresponding to the eigenvalue λ. Then X is a non-zero eigenvector of A^m corresponding to the eigenvalue λ^m, as

$$A^m X = A^{m-1}(\lambda X) = \lambda A^{m-1} X = \lambda^2 A^{m-2} X = \cdots = \lambda^m X.$$

□

Problem 7. Let T be a linear operator on $M_{2\times 2}(\mathbb{R})$ defined by $T(A) = A^{\mathrm{t}}$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) For $n = 2$, find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix for $n \geq 2$.

Solution. (a) Note that $T^2(A) = (A^{\mathrm{t}})^{\mathrm{t}} = A$ for all A, so $T^2 = I$. If λ is an eigenvalue for T, then λ^2 is an eigenvalue for $T^2 = I$, so $\lambda^2 = 1$ and $\lambda = \pm 1$. Both values are attained, say for example

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(b) If $T(A) = A$, so that A is an eigenvector for the eigenvalue 1, then $A^{\mathrm{t}} = A$, so A is symmetric. If $T(A) = -A$, so that A is an eigenvector for the eigenvalue -1, then $A^{\mathrm{t}} = -A$, so A is skew-symmetric.

(c) $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

(d) We form β by concatenating bases for the spaces of symmetric matrices (Example 19 in Section 1.6) and skew-symmetric matrices (Homework 3 Problem 5).

□

Problem 8. Let $A, B \in M_{n\times n}(\mathbb{C})$.

(a) Prove that if B is invertible, then there exists a scalar c such that $A + cB$ is *not* invertible.

(b) Find non-zero 2×2 matrices A and B such that A and $A + cB$ are invertible for all $c \in \mathbb{C}$.

Solution. (a) Let

$$f(c) = \det(A + cB) = \det(AB^{-1} + cI)\det B;$$

the values of c for which $A + cB$ is not invertible are precisely the ones for which $f(c) = 0$. Since B is invertible, $\det B \neq 0$, while by inspection, $\det(AB^{-1} + cI)$ is a polynomial in c of leading term c^n. Hence $f(c)$ is a polynomial of degree $n \geq 1$, which necessarily has a root.

(b) If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

then A is invertible and

$$\det(A + cB) = \begin{vmatrix} 1+c & c \\ -c & 1-c \end{vmatrix} = 1,$$

so $A + cB$ is invertible for all c.

□

Problem 9. Let $A = (A_{ij})$ be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that $f(t) = (A_{1,1} - t)(A_{2,2} - t)\cdots(A_{n,n} - t) + q(t)$, where $q(t)$ is a polynomial of degree at most $n - 2$.

(b) Using part (a), show that $\operatorname{tr} A = (-1)^{n-1}a_{n-1}$.

Solution. (a) This result can be proved by induction. Alternatively, from the expansion

$$f(t) = \sum_{\sigma \text{ permutations}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} [A - tI]_{i,\sigma(i)},$$

we obtain a factor involving t whenever $\sigma(i) = i$. When σ is the identity permutation, we obtain the first term of $f(t)$ as shown. Any other permutation moves at least two indices, so we obtain at most $n - 2$ factors involving a t, hence a polynomial of degree at most $n - 2$. The sum over all such non-identity permutations gives us the polynomial $q(t)$, still of degree at most $n - 2$.

(b) Expanding the product,

$$\begin{aligned} f(t) &= (-t)^n + (-t)^{n-1}(A_{1,1} + A_{2,2} + \cdots + A_{n,n}) + (\text{terms of degree at most } n-2) + q(t) \\ &= (-1)^n t^n + (-1)^{n-1}\operatorname{tr}(A)t^{n-1} + r(t) \end{aligned}$$

for $r(t)$ of degree at most $n - 2$. Hence $a_{n-1} = (-1)^{n-1}\operatorname{tr} A$, so $\operatorname{tr} A = (-1)^{n-1}a_{n-1}$.

□