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تکلیف سری پنجم

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فهرست مطالب

۱ سوال اول

(a) eigenvalues:

$$\det(xI-A)=\det\begin{bmatrix} x-2 & -3 \\ -4 & x-1 \end{bmatrix}=(x-2)(x-1)-12=(x+2)(x-5)\implies \gamma_1=-2,\gamma_2=5$$

(b) find B:

$$-2I-A=\left[\begin{array}{cc|c} -4 & -3 & 0 \\ -4 & -3 & 0 \end{array}\right]\to\left[\begin{array}{cc|c} -4 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]\implies \vec{V}_1=\begin{bmatrix} \frac{-3}{4} \\ 1 \end{bmatrix}$$

$$5I-A=\left[\begin{array}{cc|c} 3 & -3 & 0 \\ -4 & 4 & 0 \end{array}\right]\to\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]\implies \vec{V}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B=\langle\begin{bmatrix} \frac{-3}{4} \\ 1 \end{bmatrix},\begin{bmatrix} 1 \\ 1 \end{bmatrix}\rangle$$

(c) Diagonalize A:

$$D=P^{-1}AP, P=\begin{bmatrix} \frac{-3}{4} & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} \frac{-3}{4} & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]\to\left[\begin{array}{cc|cc} 1 & 0 & \frac{-4}{7} & \frac{4}{7} \\ 0 & 1 & \frac{4}{7} & \frac{3}{7} \end{array}\right]\implies P^{-1}=\begin{bmatrix} \frac{-4}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix}$$

$$\implies D=\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

(d) A^7 :

$$A=PDP^{-1}\implies A^7=PD^7P^{-1}, D^7=\begin{bmatrix} -2^7 & 0 \\ 0 & 5^7 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{-3}{4} & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2^7 & 0 \\ 0 & 5^7 \end{bmatrix}\begin{bmatrix} \frac{-4}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix}$$

$$\approxeq\begin{bmatrix} 44476.53 & 32867.22 \\ 44604.21 & 32739.54 \end{bmatrix}$$

(e) Diagonalize A^{-1} :

$$D_{A^{-1}}=D^{-1}=P^{-1}A^{-1}P=\begin{bmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

۲ سوال دوم

(a) let :

$$e_1=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2=\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_n=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The operation Ae_1 is the first column of A. Ae_2 is the second column. Thus $A(e_1+e_2)$ is the vector that results from the addition of the first and second columns. The vector of all 1's is given by $e_1+e_2+\cdots+e_n$. Applying A to this vector yields :

$$Ae_1+Ae_2+\cdots+Ae_n$$

which is the vector that results from summing all of the columns. this means that $A\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T=\begin{bmatrix} s & s & \cdots & s \end{bmatrix}^T$ and we are done($Av=sI\implies$ s is an eigenvalue of A).

(b) Proof : As has been noted, the all-1s vector is an eigenvector of A with eigenvalue s. Conversely, let $x = (x_1, \cdots, x_n)^T$ be an eigenvector of A with eigenvalue γ. Let x_1 be (an) element of largest absolute value, and by scaling by -1, we may assume that $x_i \geq 0$ without loss of generality. Now the ith component of Ax is γx_i, and its absolute value satisfies :

$$|\gamma| x_i = |(Ax)_i| = |\sum_{j=1}^{n} A_{ij} x_j| \leq \sum_{j=1}^{n} |A_{ij}||x_j| \leq \sum_{j=1}^{n} A_{ij} x_i = x_i \sum_{j=1}^{n} A_{ij} = sx$$

so $|\gamma| \leq s$ ■

۳ سوال چهارم

(a)

$$Cayley-Hamilton: \quad \det(\gamma I - A) = P(\gamma) \xrightarrow{\gamma = A} P(A) = 0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \gamma I - A = \begin{bmatrix} \gamma - a & -b \\ -c & \gamma - d \end{bmatrix}$$

$$\det(\gamma I - A) = (\gamma - a)(\gamma - d) - bc = \gamma^2 - (a+d)\gamma + ad - bc = P(\gamma)$$

$$\det(A) = 0 \implies ad - bc = 0 \implies P(\gamma) = \gamma^2 - (a+d)\gamma$$

$$P(A) = A^2 - (a+d)A = A^2 - trace(A)A = 0 \implies A^2 = trace(A)A \quad \blacksquare$$

(b)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$AB = \begin{bmatrix} ae + bg & * \\ * & cf + dh \end{bmatrix}, BA = \begin{bmatrix} ea + fc & * \\ * & gb + dh \end{bmatrix}$$

$$\implies A = AB - BA = \begin{bmatrix} bg - fc & * \\ * & cf - gb \end{bmatrix}$$

$$\implies trace(A) = bg - fc + cf - gb = 0 \xrightarrow{part(a)} A^2 = 0$$

۴ سوال پنجم

(a)

$$\exists k \quad A^k = O \implies (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = (PD^k P^{-1}) = 0$$

$$\implies D^k = 0 \xrightarrow{D \quad is \quad diagonal} D = 0 \implies A = PDP^{-1} = 0$$

(b) Proof:

(⇒)

Suppose the matrix A is nilpotent. Namely there exists $k \in \mathbb{N}$ such that $A^k = O$. Let λ be an eigenvalue of A and let $\mathbf{x}$ be the eigenvector corresponding to the eigenvalue λ.

Then they satisfy the equality $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying this equality by A on the left, we have

$$A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}.$$

Repeatedly multiplying by A, we obtain that $A^k\mathbf{x} = \lambda^k\mathbf{x}$. (To prove this statement, use mathematical induction.)

Now since $A^k = O$, we get $\lambda^k\mathbf{x} = 0_n$, n-dimensional zero vector.

Since $\mathbf{x}$ is an eigenvector and hence nonzero by definition, we obtain that $\lambda^k = 0$, and hence $\lambda = 0$.

(⇐)

Now we assume that all the eigenvalues of the matrix A are zero.

We prove that A is nilpotent.

There exists an invertible $n \times n$ matrix P such that $P^{-1}AP$ is an upper triangular matrix whose diagonal entries are eigenvalues (A.

(This is always possible. Study a triangularizable matrix or Jordan normal/canonical form.)

Hence we have

$$P^{-1}AP = \begin{bmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Another proof of (⇐) using Cayley-Hamilton theorem

Suppose that all the eigenvalues of the matrix A are zero.

Then the characteristic polynomial of the matrix A is

$$p(t) = \det(A - tI) = \pm t^n.$$

Hence by the Cayley-Hamilton theorem says that

$$p(A) = \pm A^n = O,$$

the zero matrix.

Thus, A is nilpotent.

(c) Proof :

If $A^n = O$, then by definition the matrix A is nilpotent.

On the other hand, suppose A is nilpotent. Then by Part (a), the eigenvalues of A are all zero. Then by the same argument of the proof of part (a) (⇐), we have $A^n = O$.

۵ سوال ششم

$x \neq eigenvalue \implies nul(A - xI) = 0 \implies find\ \gamma\ with\ the\ most\ Geometric\ repetition:$

$$\det(xI - A) = \begin{bmatrix} x-5 & -2 & 1 \\ -2 & x-2 & -2 \\ 1 & -2 & x-5 \end{bmatrix} = x(x-6)^2 \implies x_1 = 0, x_2 = 6$$

$x = 6:$

$$A - 6I = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies we\ have\ 2\ free\ variables \implies nul(A) = 2$$