THE GENERALIZED WEIERSTRASS APPROXIMATION THEOREM

by

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1. *Introduction.* Some years ago the writer discovered a generalization of the Weierstrass approximation theorem suggested by an inquiry into certain algebraic properties of the continuous real functions on a topological space [1]. This generalization has since shown itself to be very useful in a variety of similar situations. Interest in it has stimulated several improvements in the proof originally given and has also led to some modifications and extensions of the theorem itself. At the same time many interesting applications to classical problems of analysis have been observed by those working with the generalized approximation theorem. The writer, for instance, has noted a number of such applications in his lectures of 1942-1945, dealing with this and other subjects. Since the proofs thus obtained for several important classical theorems are remarkably simple, there would seem to be some advantage in collecting the relevant material in an expository article where everything could be presented in the light of our most recent knowledge. To offer such an article is our present purpose.

2. *Lattice Formulations of the Generalized Theorem.* The Weierstrass approximation theorem states, of course, that any continuous real function defined on a bounded closed interval of real numbers can be uniformly approximated by polynomials. The generalization with which we shall be concerned here seeks in the first instance to lighten the restrictions imposed on the domain over which the given functions are defined. The difficulty which has to be turned at the very outset in formulating such a generalization is that there are no polynomials on a general domain. It is rather easy, however, to circumvent this difficulty by orienting our inquiry towards the solution of the following question: what functions can be built from the functions of a prescribed family by the application of the algebraic operations (addition, multiplication, and multiplication by real numbers) and of uniform passages to the limit? In the classical case settled by the Weierstrass approximation theorem, the prescribed family consists of just two functions, f_1 and f_2, where $f_1(x) = 1$ and $f_2(x) = x$ for all x in the basic interval. In this, as in other cases which will be noted below, the answer is especially interesting because a very small prescribed family suffices to generate a very much more inclusive family. In his first discussion of the general problem posed above, the author focussed attention on the rôle played in approximation theory by the operations of forming the maximum and the minimum of a pair of functions. The reason why these opera-

tions are technically appropriate to the end in view can be seen even in the classical case of Weierstrass. There it is geometrically evident that a given continuous real function can be uniformly approximated by continuous piecewise linear functions, since to obtain such approximations one has only to inscribe polygons in the graph of the given function; and each piecewise linear function can be obtained from linear functions by means of the operations in question. The approximation of piecewise linear functions by polynomials then becomes the issue. The parts of the author's proof which involve these operations have since been much improved by Kakutani, with the aid of suggestions made by Chevalley, and the results given explicit formulation as a theorem about lattices of continuous functions [2]. Further modifications will be indicated below in the course of our present discussion.

In accordance with the preceding remarks, we shall start with an arbitrary topological space X, the family $\mathfrak{X}$ of all continuous real functions on X, and a prescribed subfamily $\mathfrak{X}_0$ of $\mathfrak{X}$. Our object is to determine the family $\mathfrak{U}(\mathfrak{X}_0)$ of all those functions which can be built from functions in $\mathfrak{X}_0$ by the application of specified algebraic operations and uniform passage to the limit. We shall consider first the case where the specified operations are the lattice operations $\cup$ and $\cap$ defined as follows: $f \cup g = \max(f, g)$ and $f \cap g = \min(f, g)$ are the functions h and k respectively, where $h(x) = \max(f(x), g(x))$ and $k(x) = \min(f(x), g(x))$ for all x in X. Later we shall take up other cases. In general we shall require of X that it be a compact space or even a compact Hausdorff space; but in the course of our preliminary remarks no such restriction will be necessary.

In all the cases we shall consider, $\mathfrak{U}(\mathfrak{X}_0)$ is a part of $\mathfrak{X}$ closed under uniform passage to the limit - in symbols, $\mathfrak{U}(\mathfrak{X}_0) \subset \mathfrak{X}$, $\mathfrak{U}(\mathfrak{U}(\mathfrak{X}_0)) = \mathfrak{U}(\mathfrak{X}_0)$. Let us discuss these statements briefly in the case of the lattice operations. Since $f \cup g$ and $f \cap g$ are continuous whenever f and g are (the mapping of X into the plane given by $x \to (f(x), g(x))$ is continuous and the mappings of the place into the real number system given by $(\xi, \eta) \to \max(\xi, \eta)$ and $(\xi, \eta) \to \min(\xi, \eta)$ respectively are both continuous, so that the composite mappings $x \to \max(f(x), g(x))$ and $x \to \min(f(x), g(x))$ are continuous also) and since the uniform limit of continuous functions is a continuous function, we see that the operations applied in the construction of $\mathfrak{U}(\mathfrak{X}_0)$ work entirely within $\mathfrak{X}$ and hence that $\mathfrak{U}(\mathfrak{X}_0) \subset \mathfrak{X}$. We now observe that $\mathfrak{U}(\mathfrak{X}_0)$ can be constructed in two steps: we first form all the functions obtainable by applying the algebraic operations alone to members of $\mathfrak{X}_0$ and we then form all the functions obtainable from these by uniform passage to the limit. For convenience let us designate the family of functions obtained in the first step by $\mathfrak{U}_1(\mathfrak{X}_0)$ and the family obtained in the second step by $\mathfrak{U}_2(\mathfrak{X}_0)$. It is evident that $\mathfrak{X}_0 \subset \mathfrak{U}_1(\mathfrak{X}_0) \subset \mathfrak{U}_2(\mathfrak{X}_0) \subset \mathfrak{U}(\mathfrak{X})$. We shall show that $\mathfrak{U}_2(\mathfrak{X})$ is closed under the operations allowed and hence that $\mathfrak{U}(\mathfrak{X}) = \mathfrak{U}_2(\mathfrak{X})$. It is then trivial that $\mathfrak{U}(\mathfrak{X})$ is also closed under those operations. It is easy to see that any function f which is a uniform limit of functions f_n in $\mathfrak{U}_2(\mathfrak{X}_0)$ is itself a member of $\mathfrak{U}_2(\mathfrak{X}_0)$: in fact, each f_n can be uniformly approximated by functions in $\mathfrak{U}_1(\mathfrak{X}_0)$ so that, if ϵ is any positive number, f_n and a corresponding function g_n in $\mathfrak{U}_1(\mathfrak{X}_0)$ can be found satisfying the inequalities $|f(x) - f_n(x)| < \epsilon/2$, $|f_n(x) - g_n(x)| < \epsilon/2$, and hence the inequality $|f(x) - g_n(x)| < \epsilon$ for all x in X. It is also fairly easy to see that whenever f and g are in $\mathfrak{U}_2(\mathfrak{X}_0)$ so also are $f\cup g$ and $f\cap g$. For this it is sufficient to observe that, when f and g are uniform limits of the respective sequences f_n and g_n in $\mathfrak{U}_1(\mathfrak{X}_0)$, then $f\cup g$ and $f\cap g$ are uniform limits of the respective sequences $f_n\cup g_n$, and $f_n\cap g_n$ - which are obviously in $\mathfrak{U}_1(\mathfrak{X}_0)$ too. The validity of this observation depends upon the inequalities

$$|\max(\xi,\eta) - \max(\xi',\eta')| \leq |\xi - \xi'| + |\eta - \eta'|,$$

$$|\min(\xi,\eta) - \min(\xi',\eta')| \leq |\xi - \xi'| + |\eta - \eta'|,$$

for which formal proofs based on the equations*

$$\max(\xi,\eta) = 1/2(\xi + \eta + |\xi - \eta|)$$

$$\min(\xi,\eta) = 1/2(\xi + \eta + |\xi - \eta|)$$

are easily given. Using these inequalities and choosing n so that $|f(x) - f_n(x)| < \epsilon/2$, $|g(x) - g_n(x)| < \epsilon/2$ for all x in X, we find directly that $|\max(f(x), g(x)) - \max(f_n(x), g_n(x))| < \epsilon$, $|\min(f(x), g(x)) - \min(f_n(x), g_n(x))| < \epsilon$ for x in X. In case we assume X to be compact, every function in X is automatically bounded. By virtue of this assumption, or by virtue of a direct restriction to the bounded continuous functions on X in the general case, we put ourselves in a position to summarize the preceding remarks in a particularly brief form. In fact, if we restrict $\mathfrak{X}$ to consist of the bounded continuous functions on X and define the distance between two bounded functions f and g to be $\sup\limits_{x\in X} |f(x) - g(x)|$, we thereby make $\mathfrak{X}$ into a complete metric space in which metric convergence is equivalent to uniform convergence. The lattice operations are continuous with respect to this metric. As before, when $\mathfrak{X}_0\subset\mathfrak{X}$ the relations $\mathfrak{X}_0\subset\mathfrak{U}(\mathfrak{X}_0)\subset\mathfrak{X}$, $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{U}(\mathfrak{U}(\mathfrak{X}_0))$ are valid. The first states that the uniform limit of bounded continuous functions is a bounded continuous function, the second that $\mathfrak{U}(\mathfrak{X}_0)$ is metrically and algebraically closed. The proof of the latter fact runs as before; but it can be more briefly stated, as follows: if $\mathfrak{U}_1(\mathfrak{X}_0)$ is the family of all "lattice polynomials" formed from $\mathfrak{X}_0$ and $\mathfrak{U}_2(\mathfrak{X}_0)$ is its metric closure, then $\mathfrak{U}_2(\mathfrak{X}_0)$ is obviously metrically closed and the fact that it is algebraically closed with respect to the lattice operations is a simple, direct consequence of their metric continuity.

We are now ready to determine, in the important case where X is compact, what functions belong to $\mathfrak{U}(\mathfrak{X}_0)$.

Theorem 1: *Let X be a compact space, $\mathfrak{X}$ the family of all continuous (necessarily bounded) real functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the lattice operations and uniform passage to the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathfrak{U}(\mathfrak{X}_0)$ is that, whatever the points x, y in X and whatever the positive number ϵ, there exist a function f_{xy} obtained by applying the lattice operations alone to $\mathfrak{X}_0$ and such that $|f(x) - f_{xy}(x)| < \epsilon$, $|f(y) - f_{xy}(y)| < \epsilon$.*

Proof: The necessity of the stated condition is trivial. It is the sufficiency which requires discussion. Starting with the functions f_{xy} in $\mathfrak{U}_1(\mathfrak{X}_0)$ we shall construct an approximant for f. Let G_y designate the open set $(z;\ f(z) - f_{xy}(z) < \epsilon)$, where x is fixed. By hypothesis x and y are in G_y, so that the union of all the sets G_y is the entire space X. The compactness of X implies the existence of points $y_1, \ldots, y_n$ such that the union of the sets $G_{y_1}, \ldots, G_{y_n}$ is still the entire space X. Setting $g_x = f_{xy_1} \cup \cdots f_{xy_n} = \max\,(f_{xy_1}, \ldots, f_{xy_n})$, we see that for any z in X we have $z \in G_{y_k}$ for a suitable choice of k and hence $g_x(z) \geqq f_{xy_k}(z) > f(z) - \epsilon$. On the other hand the fact that $f_{xy}(x) < f(x) + \epsilon$ implies that $g_x(x) < f(x) + \epsilon$. We can now work in a similar manner with the functions g_x. Let H_x designate the open set $(z;\ g_x(z) < f(z) + \epsilon)$. Evidently x is in H_x, so that the union of all the sets H_x is the entire space X. The compactness of X implies the existence of points $x_1, \ldots, x_m$ such that the union of the sets $H_{x_1}, \ldots, H_{x_m}$ is still the entire space X. Setting $h = g_{x_1} \cap \cdots \cap g_{x_m} = \min\,(g_{x_1}, \ldots, g_{x_m})$, we see that for any z in X we have $z \in H_{x_k}$ for a suitable choice of k and hence $h(z) \leqq g_{x_k}(z) < f(z) + \epsilon$. On the other hand, the fact that $g_x(z) > f(z) - \epsilon$ for all z and all x implies that $h(z) > f(z) - \epsilon$ for all z. Thus we have $|f(z) - h(z)| < \epsilon$ for all z in X. To complete the proof we note that, since only the lattice operations have been used in constructing the functions g_x and h from the functions f_{xy}, these functions are all in $\mathfrak{U}_1(\mathfrak{X}_0)$, as desired.

We may note two simple corollaries, as follows.

Corollary 1: If $\mathfrak{X}_0$ has the property that, whatever the points, x, y, $x \neq y$, in X and whatever the real numbers α and β, there exists a function f_0 in $\mathfrak{X}_0$ for which $f_0(x) = \alpha$ and $f_0(y) = \beta$, then $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$ — in other words, any continuous function on X can be uniformly approximated by lattice polynomials in functions belonging to the prescribed family $\mathfrak{X}_0$.

Corollary 2: If a continuous real function on a compact space X is the limit of a monotonic sequence f_n of continuous functions, then the sequence converges uniformly to f.†

†Professor Andre Weil remarks that the extension to monotonic sets is immediate.

Proof: We take $\mathfrak{X}_0$ to be the totality of functions occurring in the sequence f_n. Then $\mathfrak{U}_1(\mathfrak{X}_0) = \mathfrak{X}_0$ since monotonicity implies that $f_m \cup f_n$ coincides with one of the two functions f_m and f_n while $f_m \cap f_n$ coincides with the other. The assumption that $\lim_{n\to\infty} f_n(x) = f(x)$ for every x now shows that the condition of Theorem 1 is satisfied. Hence f is in $\mathfrak{U}(\mathfrak{X}_0)$; and f is therefore the uniform limit of functions occurring in $\mathfrak{X}_0$. Since $|f(x) - f_n(x)|$ decreases as n increases and since $|f(x) - f_N(x)| < \epsilon$ for all x and a suitable choice of N, we see that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$, as was to be proved.

Theorem 1 tells us that the question, "Can a given function f be approximated in terms of the prescribed family $\mathfrak{X}_0$?" has an answer depending only on the way in which f and $\mathfrak{X}_0$ behave on pairs of points in X. The contraction of a function obtained by suppressing all points of X except the two points x, y of a pair is a function of very simple kind - it is completely described by the ordered pair (α,β) of those real numbers which are its values at x and at y respectively. If $\mathfrak{X}_0(x,y)$ designates the family of functions obtained by contracting every function in $\mathfrak{X}_0$ in this manner, and if $\mathfrak{X}(x,y)$ has a corresponding significance, then everything depends on an examination (for all different pairs x,y) of the question, "Can a given element of $\mathfrak{X}(x,y)$ be approximated in terms of $\mathfrak{X}_0(x,y)$?" This question is that special case of our original problem in which X is a two-element space! When X has just two elements, the approximation problem can be described in slightly different language, as follows. We have to deal with all ordered pairs (α,β) of real numbers - that is, with the cartesian plane. On two such pairs we can perform the operations $\cup$ and $\cap$ defined by the equations

$$(\alpha,\beta)\cup(\gamma,\delta) = (\max(\alpha,\gamma),\ \max(\beta,\delta))$$

$$(\alpha,\beta)\cap(\gamma,\delta) = (\min(\alpha,\gamma),\ \min(\beta,\delta)).$$

Geometrically these operations produce the upper right vertex and lower left vertex respectively of a rectangle with its sides parallel to the coördinate axes and one pair of opposite vertices falling on the points (α,β), (γ,δ). For any given subset S of the plane the problem to be solved is that of finding what points can be generated from it by the above operations and passage to the limit. From what has been said above, it is clear that the points so generated constitute a closed subset S^* of the plane which contains with (α,β) and (γ,δ) the two points described above. It is also clear that this subset is the smallest set enjoying these properties and containing the given subset S. Reverting now to the interpretation of Theorem 1, we see that it can be restated in the following form: if $f \in \mathfrak{X}$, then $f \in \mathfrak{U}(\mathfrak{X}_0)$ if and only if $(f(x),\ f(y)) \in \mathfrak{X}_0(x,y)^*$ for every pair of distinct points

x, y in X. We have not asserted that the conditions corresponding to various pairs x, y are independent of one another, Nor have we asserted that every point (α,β) in $\mathfrak{X}_0(x,y)^*$ can be expressed in the form $\alpha = f(x)$, $\beta = f(y)$ for some f in $\mathfrak{U}(\mathfrak{X}_0)$. Indeed, even in the case where $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ we know only that $\mathfrak{X}_0(x,y)^*$ is the closure of $\mathfrak{X}_0(x,y)$.

It is convenient to express some of the results sketched in the preceding paragraph as a formal theorem. This we do as follows.

Theorem 2: *Let X be a compact space, $\mathfrak{X}$ the family of continuous real functions on X, and $\mathfrak{X}_0$ a subfamily of $\mathfrak{X}$ which is closed under the lattice operations and uniform passage to the limit. Then $\mathfrak{X}_0$ is completely characterized by the system of planar sets $\mathfrak{X}_0(x,y)^* = \mathfrak{X}_0(x,y)^-$.*

Proof: Our hypothesis that $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ shows that $\mathfrak{X}_0(x,y)$ has $\mathfrak{X}_0(x,y)^*$ as its closure, as we remarked above. Let us suppose that $\mathfrak{Y}_0 = \mathfrak{U}(\mathfrak{Y}_0) \subset \mathfrak{X}$ and that $\mathfrak{X}_0(x,y)^* = \mathfrak{Y}_0(x,y)^*$ for all pairs of points x, y in X. Then the conditions for f in $\mathfrak{X}$ to belong to $\mathfrak{X}_0$ are identical to those for it to belong to $\mathfrak{Y}_0$. Hence $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ coincide.

We pass now to the modifications of Theorems 1 and 2 which result when we take into consideration the operations of linear algebra as well as the lattice operations. The newly admitted operations are, more precisely, addition and multiplication by real numbers. In view of the equations (*), which express the lattice operations in terms of the linear operations and the single operation of forming the absolute value, we may take the specified algebraic operations to be simply addition, multiplication by real numbers, and formation of absolute values. The remarks preliminary to Theorem 1 apply, mutatis mutandis, to the present situation. The family $\mathfrak{U}(\mathfrak{X}_0)$ of all functions which can be constructed from $\mathfrak{X}_0 \subset \mathfrak{X}$ by application of the linear lattice operations and uniform passage to the limit is again seen to be obtainable in two steps, the first being algebraic and the second consisting in the adjunction of uniform limits. This family is closed under the operations used to generate it. We now have the following analogue of the results contained in Theorems 1 and 2.

Theorem 3: (*Kakutani* [2]. *Let X be a compact space, $\mathfrak{X}$ the family of all continuous (necessarily bounded) real functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the linear lattice operations and uniform passage to the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathfrak{U}(\mathfrak{X}_0)$ is that f satisfy every linear relation of the form $\alpha g(x) = \beta g(y)$, $\alpha\beta \geqq 0$, which is satisfied by all functions in $\mathfrak{X}_0$. If $\mathfrak{X}_0$ is a closed linear sublattice of $\mathfrak{X}$ - that is, if $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ - then $\mathfrak{X}_0$ is characterized by the system of all the linear relations of this form which are satisfied by every function belonging to it. The linear relations associated with an arbitrary pair of points x,y in X must be equivalent to one of the following distinct types:*

(1) *$g(x) = 0$ and $g(y) = 0$;*
(2) *$g(x) = 0$ and $g(y)$ unrestricted, or vice versa;*
(3) *$g(x) = g(y)$ without restriction on the common value;*
(4) *$g(x) = \lambda g(y)$ or $g(y) = \lambda\, g(x)$ for a unique value λ, $0 < \lambda < 1$.*

Proof: Since $\mathfrak{Y}_0 = \mathfrak{U}(\mathfrak{X}_0)$ is closed under the lattice operations and uniform passage to the limit, Theorem 2 can be applied to $\mathfrak{Y}_0$. However, the fact that $\mathfrak{Y}_0$ is also closed under the linear operations can be expected to produce effective simplifications. Indeed we see that the planar set $\mathfrak{Y}_0(x,y)$, where x and y are arbitrary points in X, must be the entire plane, a straight line passing through the origin, or the one-point set consisting of the origin alone. This appears at once when we observe that if $(\alpha,\beta) \in \mathfrak{Y}_0(x,y)$ then $(\lambda\alpha,\lambda\beta) \in \mathfrak{Y}_0(x,y)$ for every λ, and that if (α,β) and (γ,δ) are in $\mathfrak{Y}_0(x,y)$ then $(\alpha+\gamma,\beta+\delta) \in \mathfrak{Y}_0(x,y)$. Since $\mathfrak{Y}_0(x,y)$ is obviously a closed subset of the plane, we have $\mathfrak{Y}_0(x,y)^* = \mathfrak{Y}_0(x,y)$. When $\mathfrak{Y}_0(x,y)$ is a straight line through the origin we write its equation as $\alpha\xi = \beta\eta$ and observe that $(\beta,\sigma) \in \mathfrak{Y}_0(x,y)$. Since $\mathfrak{Y}_0$ is closed under the operation of forming absolute values, we see that $(|\beta|, |\alpha|) \in \mathfrak{Y}_0(x,y)$. Hence $\alpha|\beta| = |\alpha|\beta$ so that $\alpha\beta|\beta| = |\alpha|\beta^2 \geq 0$ and $\alpha\beta \geq 0$. When $\mathfrak{Y}_0(x,y)$ consists of the origin alone, we have the case enumerated as (1) in the statement of the theorem. When $\mathfrak{Y}_0(x,y)$ is a straight line through the origin we have case (2) if it coincides with one of the coördinate axes, case (3) if it coincides with the bisector of the angle between the positive coördinate axes, and case (4) otherwise. When $\mathfrak{Y}_0(x,y)$ is the entire plane there is no corresponding linear relation, of course. Theorem 2 shows that $\mathfrak{Y}_0$ is characterized by the sets $\mathfrak{Y}_0(x,y) = \mathfrak{Y}_0(x,y)^*$ - in other words, that f in $\mathfrak{X}$ belongs to $\mathfrak{Y}_0 = \mathfrak{U}(\mathfrak{X}_0)$ if and only if $(f(x), f(y)) \in \mathfrak{Y}_0(x,y)$. Since $\mathfrak{X}_0 \subset \mathfrak{Y}_0$, it is clear that the conditions thus imposed on the functions in $\mathfrak{U}(\mathfrak{X}_0)$ are satisfied by the functions in $\mathfrak{X}_0$. On the other hand if all the functions in $\mathfrak{X}_0$ satisfy relations of the kind enumerated in (1) - (4) it is clear that every function in $\mathfrak{U}(\mathfrak{X}_0)$ must do likewise: for the sums, constant multiples, absolute values, and uniform limits of functions which satisfy a condition of any one of these types must satisfy the same condition. Thus the linear relations of the form $\alpha g(x) = \beta g(y)$, $\alpha\beta \geq 0$, satisfied by the functions in $\mathfrak{X}_0$ are identical with those satisfied by the functions in $\mathfrak{U}(\mathfrak{X}_0)$ and serve to characterize the latter family completely.

We may note some simple corollaries to the theorem just proved.

Corollary 1: *In order that $\mathfrak{U}(\mathfrak{X}_0)$ contain a non-vanishing constant function it is necessary and sufficient that the only linear relations of the form $\alpha g(x) = \beta g(y)$, $\alpha\beta > 0$, satisfied by every function in $\mathfrak{X}_0$ be those reducible to the form $g(x) = g(y)$.*

Proof: It is obvious that of conditions (1) - (4) in Theorem 3 only condition (3) can be satisfied by a non-vanishing constant function.

Corollary 2: In order that $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$ *it is sufficient that the functions in* $\mathfrak{X}_0$ *satisfy no linear relation of the form* (1) - (4) *of Theorem 3.*

In order to state a further corollary, we first introduce a convenient definition.

Definition 1: A family of arbitrary functions on a domain X *is said to be a separating family (for that domain) if, whenever x and y are distinct points of* X, *there is some function f in the family with distinct values f(x), f(y) at these points.*

In terms of this definition we have the following result.

Corollary 3: *If X is compact and if* $\mathfrak{X}_0$ *is a separating family for X and contains a non-vanishing constant function, then* $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$.

Proof: Since $\mathfrak{X}_0$ contains a non-vanishing constant function, the only one of conditions (1) - (4) satisfied by every function in $\mathfrak{X}_0$ are those of the form (3). Since $\mathfrak{X}_0$ is a separating family, no linear relation of the form $g(x) = g(y)$, where $x \neq y$, is satisfied by every function in $\mathfrak{X}_0$. Hence Corollary 2 yields the desired result.

Corollary 4: *If* $\mathfrak{X}_0$ *is a separating family, then so is* $\mathfrak{X}$. *If* $\mathfrak{X}$ *is a separating family and* $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$, *then* $\mathfrak{X}_0$ *is also a separating family.*

Proof: The first statement is trivial. The second statement follows at once from the fact that $\mathfrak{X}_0$ is subject to no linear relation of the form $g(x) = g(y)$ which is not also satisfied by every function in $\mathfrak{U}(\mathfrak{X}_0) = \mathfrak{X}$.

It should be remarked that in general the family $\mathfrak{X}$ of all continuous functions on a compact space X need not be a separating family. In case X is a compact Hausdorff space, however, it is well-known that $\mathfrak{X}$ is a separating family: if $x \neq y$, there exists a continuous function f on X such that $f(x) = 0$, $f(y) = 1$.

3. *Linear Ring Formulations of the Generalized Theorem.* We are now ready to discuss the approximation problem when the specified algebraic operations used in the construction of approximants are the linear operations and multiplication. Since the product of two continuous functions is continuous we see that the family $\mathfrak{X}$ of all continuous functions on a topological space $\mathfrak{X}$ is a commutative ring with respect to the two operations of addition and multiplication, and a commutative linear associative algebra or linear ring with respect to the operations of addition, multiplication, and multiplication by real numbers. Hence the formally stated results of this section constitute what may be called the linear-ring formulation of the generalized Weierstrass approximation theorem.

If we now designate by $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions generated from $\mathfrak{X}_0 \subset \mathfrak{X}$ by means of the linear-ring operations and uniform passage of the limit, we have to note a slight modification which must be made in the general statements made in the lattice case. If f and g are uniform limits of the sequences f_n and g_n respectively, the product fg is not in general the uniform limit of the sequence $f_n g_n$ - consider, for example, the case where $f = g$ is a non-bounded function and $f - f_n$ is the constant $1/n$. *We shall therefore suppose that* $\mathfrak{X}$ *consists of all bounded continuous functions on a topological space* X, this boundedness restriction being automatically satisfied when X is compact. By virtue of this restriction we can apply the inequality

$$|fg - f_n g_n| \leqq |f||g - g_n| + |g||f - f_n| + |f - f_n||g - g_n|$$

to show that when f_n and g_n are uniformly convergent sequences in $\mathfrak{X}$ their respective limits f and g are in $\mathfrak{X}$ and that the sequence $f_n g_n$ converges uniformly to the product fg, in $\mathfrak{X}$. When $\mathfrak{X}_0 \subset \mathfrak{X}$ we see as before that $\mathfrak{U}(\mathfrak{X}_0) \subset \mathfrak{X}$, $\mathfrak{U}(\mathfrak{U}(\mathfrak{X}_0)) = \mathfrak{U}(\mathfrak{X}_0)$. It is easy to see that $\mathfrak{U}(\mathfrak{X}_0)$ consists of all those functions, necessarily in $\mathfrak{X}$, which are uniform limits of polynomials in members of $\mathfrak{X}_0$ - in other words, $f \in \mathfrak{X}$ is in $\mathfrak{U}(\mathfrak{X}_0)$ if and only if, whatever the positive number ϵ, there exist functions $f_1, \ldots, f_n$ and a polynomial function $p(\xi_1, \ldots, \xi_n)$ of the real variables $\xi_1, \ldots, \xi_n$ with $p(0, \ldots, 0) = 0$ such that $|f(x) - p(f_1(x), \ldots, f_n(x))| < \epsilon$ for every x in X.

Now in order to prove our principal theorem we shall establish a very special case of the classical Weierstrass approximation theorem, using for this purpose direct and elementary methods which do not depend on any general theory. The result we need is the following proposition.

Theorem 4: *If* ϵ *is any positive number and* $\alpha \leqq \xi \leqq \beta$ *any real interval, then there exists a polynomial* $p(\xi)$ *in the real variable* ξ *with* $p(0) = 0$ *such that* $||\xi| - p(\xi)| < \epsilon$ *for* $\alpha \leq \xi \leq \beta$.

Proof: Unless the point $\xi = 0$ is inside the given interval (α, β), we can obviously take $p(\xi) = \pm\, \xi$. Thus there is no loss of generality in confining our attention to intervals of the form $(-\gamma, \gamma)$ where $\gamma > 0$, since the given interval (α, β) can be included in an interval of this form. Moreover it is obviously sufficient to study the case of the interval $(-1, 1)$ since, if $q(\eta)$, $q(0) = 0$, is a polynomial such that $||\eta| - q(\eta)| < \epsilon/\gamma$ for $-1 \leq \eta \leq 1$, then $p(\xi) = \gamma q(\xi/\gamma)$, $p(0) = 0$, is a polynomial such that $||\xi| - p(\xi)| < \epsilon$ for $-\gamma \leqq \xi \leqq \gamma$. We shall obtain the desired polynomial q for the interval $-1 \leq \eta \leq 1$ as a partial sum of the power series development for $\sqrt{1 - \zeta}$ where $\zeta = 1 - \eta^2$. The validity of the development has to be established directly.

We commence by defining a sequence of constants α_k recursively from the relations

$$\alpha_1 = 1/2, \quad \alpha_k = \frac{1}{2} \sum_{m,\, n \geq 1}^{m+n=k} \alpha_m \alpha_n \quad \text{for } k \geq 2.$$

It is obvious that $\alpha_k > 0$. Putting $\sigma_n = \sum_{k=1}^{k=n} \alpha_k$ we can show inductively that $\sigma_n < 1$. In fact we have $\sigma_1 = \alpha_1 = 1/2 < 1$ and note that $\sigma_n < 1$ implies

$$\sigma_{n+1} = \alpha_1 + \sum_{k=2}^{n+1} \alpha_k = \frac{1}{2} + \frac{1}{2} \sum_{k=2}^{n+1} \sum_{\substack{i+j=k \\ i,j \geq 1}} \alpha_i \alpha_j \leq \frac{1}{2} + \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j$$

$$\leq \frac{1}{2}(1 + \sigma_n^2) < 1.$$

Accordingly the positive term series $\sum_{k=1}^{\infty} \alpha_k$ converges to a sum σ satisfying the inequality $\sigma \leq 1$; and the power series $\sum_{k=1}^{\infty} \alpha_k \zeta^k$ converges uniformly for $|\zeta| \leq 1$ to a continuous function $\sigma(\zeta)$. It is now comparatively easy to identify this function with the function $1 - \sqrt{1 - \zeta}$. To do so we prove that $\sigma(\zeta)(2 - \sigma(\zeta)) = \zeta$. Looking at the partial sums of the power series for $\sigma(\zeta)$, we observe that

$$\left(\sum_{i=1}^{n} \alpha_i \zeta^i\right)\left(2 - \sum_{j=1}^{n} \alpha_j \zeta^j\right) = 2 \sum_{k=1}^{n} \alpha_k \zeta^k - \sum_{i,j=1}^{n} \alpha_i \alpha_j \zeta^{i+j}$$

$$= 2 \sum_{k=1}^{n} \alpha_k \zeta^k - 2 \sum_{k=2}^{n} \alpha_k \zeta^k - \sum_{\substack{i+j \geq n+1 \\ 1 \leq i, j \leq n}} \alpha_i \alpha_j \zeta^{i+j}$$

$$= \zeta - \sum_{\substack{i+j \geq n+1 \\ 1 \leq i, j \leq n}} \alpha_i \alpha_j \zeta^{i+j}$$

in accordance with the definition of the coefficients α_k. The final term here can now be estimated as follows:

$$\left|\sum_{1\leq i,j\leq n}^{i+j\geq n+1} \alpha_i\alpha_j\zeta^{i+j}\right| \leq \sum_{1\leq i,j\leq n}^{i+j\geq n+1} \alpha_i\alpha_j \leq \sum_{k=n+1}^{\infty}\sum_{i,j\geq 1}^{i+j=k} \alpha_i\alpha_j$$

$$\leq 2\sum_{k=n+1}^{\infty} \alpha_k.$$

When n becomes infinite, therefore, this term tends to zero; and passage to the limit in the identity above accordingly yields the relation $\sigma(\zeta)(2 - \sigma(\zeta)) = \zeta$. For each ζ such that $-1 \leq \zeta \leq 1$ we have $\sigma(\zeta) = 1 \pm \sqrt{1 - \zeta}$. Here we decide upon the choice of sign by showing that $\sigma(\zeta) \leq 1$, an inequality incompatible with the upper sign. It is evident that $\sigma(1) = 1$, independently of the choice of sign, and hence that $\sum_{k=1}^{\infty} \alpha_k = \sigma(1) = 1$. Inasmuch as α_k is positive it follows that $\sigma(\zeta) \leq \sigma(|\zeta|) \leq \sigma(1) = 1$, as we intended to show. It is now clear that the power series for $\sqrt{1 - \zeta}$ is given by

$$\sqrt{1 - \zeta} = 1 - \sigma(\zeta) = 1 - \sum_{k=1}^{\infty} \alpha_k\zeta^k = \sum_{k=1}^{\infty} \alpha_k(1 - \zeta^k).$$

Taking η so that $-1 \leq \eta \leq 1$, we have $0 \leq 1 - \eta^2 \leq 1$ and hence

$$|\eta| = \sqrt{\eta^2} = 1 - \sigma(1 - \eta^2) = \sum_{k=1}^{\infty} \alpha_k(1 - (1 - \eta^2)^k),$$

the series being uniformly convergent. The general term of this series is a polynomial in η which vanishes for $\eta = 0$. Hence we can take a suitable one of its partial sums as the required polynomial $q(\eta)$, thus completing our discussion.

We are now ready to give our principal results concerning the generalization of the Weierstrass theorem for the linear-ring operations.

Theorem 5: *Let X be a compact space, $\mathfrak{X}$ the family of all continuous real functions on X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, and $\mathfrak{U}(\mathfrak{X}_0)$ the family of all functions (necessarily continuous) generated from $\mathfrak{X}_0$ by the linear ring operations and uniform passage to* *the limit. Then a necessary and sufficient condition for a function f in $\mathfrak{X}$ to be in $\mathfrak{U}(\mathfrak{X}_0)$ is that f satisfy every linear relation of the form $g(x) = 0$ or $g(x) = g(y)$ which is satisfied by all functions in $\mathfrak{X}_0$. If $\mathfrak{X}_0$ is a closed linear subring of $\mathfrak{X}$ - that is, if $\mathfrak{X}_0 = \mathfrak{U}(\mathfrak{X}_0)$ - then $\mathfrak{X}_0$ is characterized by the system of all the linear relations of this kind which are satisfied by every function belonging to it. In other words, $\mathfrak{X}_0$ is characterized by the partition of X into mutually disjoint closed subsets on each of which every function in $\mathfrak{X}_0$ is constant and by the specification of that one, if any, of these subsets on which every function in $\mathfrak{X}_0$ vanishes.*

Proof: By virtue of Theorem 4, we see that if f is in $\mathfrak{U}(\mathfrak{X}_0)$ then $|f|$ is also in $\mathfrak{U}(\mathfrak{X}_0)$. Indeed, since X is compact, the function f is bounded. Assuming accordingly that $\alpha \le f(x) \le \beta$ for all x, we can find a polynomial $p_n(\xi)$ such that $||\xi| - p_n(\xi)| < 1/n$ for $\alpha \le \xi \le \beta$, while $p_n(0) = 0$. It is clear that $p_n(f)$ is in $\mathfrak{U}(\mathfrak{X}_0)$ and that $||f(x)| - p_n(f(x))| < 1/n$ for all x in X. Hence $|f|$ is the uniform limit of functions - namely, the functions $p_n(f)$ - in $\mathfrak{U}(\mathfrak{X}_0)$. Thus $|f|$ is in $\mathfrak{U}(\mathfrak{X}_0)$, as we wished to prove. By virtue of the formulas (*) connecting the operations $\cup$ and $\cap$ with the operation of forming the absolute value, we now see that whenever f and g are in $\mathfrak{U}(\mathfrak{X}_0)$ then so also are $f \cup g$ and $f \cap g$ - in other words, $\mathfrak{U}(\mathfrak{X}_0)$ is closed under the linear lattice operations, aswell as under the ring operations and uniform passage to the limit. The characterization of closed linear sublattices of $\mathfrak{X}$ given in Theorem 3 applies, naturally, to $\mathfrak{U}(\mathfrak{X}_0)$. It is easy to see that none of the characteristic linear relations can be of the type (4) described there. In fact, if every function in $\mathfrak{U}(\mathfrak{X}_0)$ were to satisfy a linear relation of the form $g(x) = \lambda g(y)$, we would find for every f in $\mathfrak{U}(\mathfrak{X}_0)$ that, f^2 being also in $\mathfrak{U}(\mathfrak{X}_0)$, the relations $f(x) = \lambda f(y)$ $f^2(x) = \lambda f^2(y)$, $\lambda^2 f^2(y) = \lambda f^2(y)$ would hold; and we would conclude that $f(y) = 0$ for every f in $\mathfrak{U}(\mathfrak{X}_0)$ or that $\lambda = 0, 1$. Thus we conclude that f is in $\mathfrak{U}(\mathfrak{X}_0)$ if and only if it satisfies all the linear relations $g(x) = 0$ or $g(x) = g(y)$ satisfied by every function in $\mathfrak{X}_0$. The first characterization of the closed linear subrings of $\mathfrak{X}$ given in the statement of the theorem follows immediately. As to the second characterization, we remark first that the relation $\equiv$ defined by putting $x \equiv y$ if and only if $f(x) = f(y)$ for all f in $\mathfrak{X}_0$ is obviously an equivalence relation *stronger* than the natural equality in X: $x = y$ implies $x = y$; $x \equiv y$ implies $y \equiv x$; $x \equiv y$ and $y \equiv z$ imply $x \equiv z$. Consequently, X is partitioned by this equivalence-relation into mutually disjoint subsets, each a maximal set of mutually equivalent elements. The set of all points y such that $x \equiv y$ is just that partition-class which contains x. Since this set is the intersection or common part of all the sets $X_f = (y;\ f(x) = f(y))$ for the various functions f in $\mathfrak{X}_0$ and since each set X_f is closed by virtue of the continuity of f, we see that the partition-class containing x is closed. If x and y are in distinct partition-classes, then there exists a function f in $\mathfrak{X}_0$ such that $f(x) \neq f(y)$, since otherwise we would have $x \equiv y$ and the two given partition-classes could not be distinct. If a partition-class contains a single point x such that $f(x) = 0$ for every f in $\mathfrak{X}_0$, then all its points obviously have this property. On the other hand, at most one partition-class can contain such a point since, if x and y are points such that $f(x) = 0$, $f(y) = 0$ for every f in $\mathfrak{X}_0$, then $f(x) = f(y)$ for every f in $\mathfrak{X}_0$, $x \equiv y$, and x and y are in the same partition-class. We cannot expect that an arbitrary partition of X into mutually disjoint closed subsets can be derived in the manner just described from some closed linear subring $\mathfrak{X}_0$ of $\mathfrak{X}$. However, partitions obtained from *distinct* closed linear subrings are necessarily *distinct* - except in the case where one subring consists of all the functions in $\mathfrak{X}$ which are constant on each partition-class and the other consists of all those functions which are in the first subring and in addition vanish on one specified partition-class. Thus we see that a closed linear subring is specified by the partition of X into the closed subsets on each of which all its members are constant and the specification of that particular partition-class, if any, on which all its members vanish.

We have at once a pair of useful corollaries.

Corollary 1: *In order that $\mathfrak{U}(\mathfrak{X}_0)$ contain a non-vanishing constant function it is necessary and sufficient that for every x in X there exist some f in $\mathfrak{X}_0$ such that $f(x) \neq 0$.*

Corollary 2: *If $\mathfrak{X}_0$ is a separating family for X, then $\mathfrak{U}(\mathfrak{X}_0)$ either coincides with $\mathfrak{X}$ or is, for a uniquely determined point x_0, the family of all functions f in $\mathfrak{X}$ such that $f(x_0) = 0$. If, conversely, $\mathfrak{X}$ is a separating family for X and $\mathfrak{U}(\mathfrak{X}_0)$ either coincides with $\mathfrak{X}$ or is the family of all those f in $\mathfrak{X}$ which vanish at some fixed point x_0 in X, then $\mathfrak{X}_0$ is a separating family.*

Proof: If $\mathfrak{X}_0$ is a separating family, so also are $\mathfrak{U}(\mathfrak{X}_0)$ and $\mathfrak{X}$. Hence the partition-classes associated with $\mathfrak{U}(\mathfrak{X}_0)$ must each consist of a single point. It follows that $\mathfrak{U}(\mathfrak{X}_0)$ must be as indicated. Conversely, when $\mathfrak{X}$ is a separating family and $\mathfrak{U}(\mathfrak{X}_0)$ is as stated, then $\mathfrak{U}(\mathfrak{X}_0)$ is a separating family. If it were not every f in $\mathfrak{U}(\mathfrak{X}_0)$ vanishes at some point x_0; and there would exist distinct points x and y in X such that $f_0(x) = f_0(y)$ for every f_0 in $\mathfrak{U}(\mathfrak{X}_0)$. Consider now an arbitrary function f in $\mathfrak{X}$. Clearly the function f_0 defined by putting $f_0(z) = f(z) - f(x_0)$ is continuous and vanishes at x_0. Thus f_0 is in $\mathfrak{U}(\mathfrak{X}_0)$, the equation $f_0(x) = f_0(y)$ is verified, and in consequence $f(x) = f(y)$. Thus we find that $f(x) = f(y)$ for every f in $\mathfrak{X}$, against hypothesis. Since $\mathfrak{U}(\mathfrak{X}_0)$ is a separating family, $\mathfrak{X}_0$ must be also. Otherwise, of course, there would exist distinct points x, y in X such that $f_0(x) = f_0(y)$ for every f_0 in $\mathfrak{X}_0$; and then the equation $f(x) = f(y)$ would hold for every f in $\mathfrak{U}(\mathfrak{X}_0)$, contrary to what was just established.

4. *The Characterization of Closed Ideals.* In developing effective general algebraic theories of lattices, linear lattices, and rings, it has been found useful to introduce the concept of an

ideal. Although ideals are independently defined in the different algebraic circumstances mentioned, their theoretical rôles do not differ much from one case to another. Because the results of the preceding sections easily yield characterizations of those ideals in $\mathfrak{X}$ (the family of all continuous real functions on a compact space X) which are closed under uniform passage to the limit, it seems worthwhile to digress from the main line of our discussion long enough to present the very useful facts available in this domain. This we shall now do without further detailed analysis of the concept of an ideal.

When we think of $\mathfrak{X}$ as a lattice - the only algebraic operations taken into consideration being the operations $\cup$ and $\cap$ - we define† an ideal $\mathfrak{X}_0$ to be a non-void subclass of $\mathfrak{X}$ which contains $f \cup g$ together with f and g, and which contains $f \cap g$ together with f. The second condition of this definition is evidently equivalent to the requirement that $\mathfrak{X}_0$ should contain g whenever it contains f and $f(x) \geq g(x)$ for every x. We now have the following characterization of the closed ideals in $\mathfrak{X}$.

Theorem 6: *Let $\mathfrak{X}$ be the lattice of all continuous real functions on a compact space X, $\mathfrak{X}_0$ an arbitrary subfamily of $\mathfrak{X}$, F_0 the extended-real function defined on X through the equation $F_0(x) = \sup_{f \in \mathfrak{X}_0} f(x)$, and $\mathfrak{Y}_0$ the family of all those functions f in $\mathfrak{X}$ such that $f(x) \leq F_0(x)$ for every x in X. When $\mathfrak{X}_0$ is void, $F_0(x) = -\infty$ for every x and $\mathfrak{Y}_0$ is void. Otherwise, $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$; and $\mathfrak{X}_0$ is a closed ideal if and only if $\mathfrak{X}_0 = \mathfrak{Y}_0$. A closed ideal $\mathfrak{X}_0$ is characterized by the associated function F_0.*

Proof: As indicated in the statement of the theorem, we permit $+\infty$ and $-\infty$ to appear as values of F_0, when necessary. When $\mathfrak{X}_0$ is non-void, it is easy to verify that $F_0(x) > -\infty$ for every x and that $\mathfrak{Y}_0$ is non-void and is a closed ideal in $\mathfrak{X}$. For example, if f is in $\mathfrak{Y}_0$ and $g(x) \leq f(x)$ for every x, then obviously $g(x) \leq F_0(x)$ for every x and g is in $\mathfrak{Y}_0$. If $\mathfrak{X}_0$ is a closed ideal, we can show that $\mathfrak{X}_0 = \mathfrak{Y}_0$. To do so we examine the relations between F_0 and the planar sets $\mathfrak{X}_0(x,y)^*$ which characterize $\mathfrak{X}_0$ as a closed sublattice of $\mathfrak{X}$ in accordance with Theorem 2. First of all, it is evident that $\mathfrak{X}_0(x,y)$, and hence also its closure $\mathfrak{X}_0(x,y)^*$, must be contained in the set of points (α,β) such that $\alpha \leq F_0(x)$ and $\beta \leq F_0(y)$. On the other hand, $(F_0(x), F_0(y))$ is a limit point of $\mathfrak{X}_0(x,y)$ and is therefore in $\mathfrak{X}_0(x,y)^*$, as will be verified at once. If $\epsilon < 0$, then there exist functions f and g in $\mathfrak{X}_0$ such that $f(x) > F_0(x) - \epsilon$, $g(y) > F_0(y) - \epsilon$ for any prescribed pair of points x, y in X. The function $h = f \cup g$ is in the ideal $\mathfrak{X}_0$ and satisfies the

†Because of the familiar duality between the two operation $\cup$ and $\cap$, there is also a dual definition in which the roles played here by these operations are interchanged.

relations $h(x) \geq f(x) > F\ (x) - \epsilon$, $h(y) \geq g(y) > F_0(y) - \epsilon$. Thus $(h(x), h(y))$ is a point in $\mathfrak{X}_0(x,y)$ and $|h(x) - F_0(x)| < \epsilon$, $|h(y) - F_0(y)| < \epsilon$, so that $(F_0(x), F_0(y))$ is in $\mathfrak{X}_0(x,y)^*$ as we wished to prove. Now we establish the fact that f is in $\mathfrak{X}_0$ when $f(x) \leq F_0(x)$ for every x. Let f_ϵ be the function in $\mathfrak{X}$ defined by putting $f_\epsilon(x) = f(x) - \epsilon$, $\epsilon > 0$. If x, y are arbitrary points in X, an argument similar to that just carried through shows that there exists a function h in $\mathfrak{X}_0$ satisfying the inequalities $h(x) > f_\epsilon(x) = f(x) - \epsilon$, $h(y) > f_\epsilon(y) = f(y) - \epsilon$. The function $f_{xy} = h \cap f_\epsilon$ then belongs to the ideal $\mathfrak{X}_0$ and has the property that $f(x) - f_{xy}(x) = \epsilon$, $f(y) - f_{xy}(y) = \epsilon$. By Theorem 1 we conclude that f is the uniform limit of functions in the closed ideal $\mathfrak{X}_0$ and hence that f is itself in $\mathfrak{X}_0$. We have now shown that $\mathfrak{Y}_0 \subset \mathfrak{X}_0$. Since $\mathfrak{X}_0 \subset \mathfrak{Y}_0$ by construction, we conclude that $\mathfrak{X}_0 = \mathfrak{Y}_0$ under the present hypothesis. Returning to the case where $\mathfrak{X}_0$ is an arbitrary non-void family, we consider a closed ideal $\mathfrak{X}_1$ containing $\mathfrak{X}_0$. Evidently $\mathfrak{X}_1$ has an associated function F_1 such that $F_1(x) \geq F_0(x)$ for every x. Hence $\mathfrak{X}_1 = \mathfrak{Y}_1 \subset \mathfrak{Y}_0$. Thus $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$. With this the proof of the theorem is complete.

Next we shall consider the case where $\mathfrak{X}$ is treated as a linear lattice - the algebraic operations allowed including the linear operations as well as the two lattice operations. Here an ideal is to be defined as a non-void class closed under the allowed algebraic operations and enjoying the additional property that it contains with f every g such that $|g(x)| \leq |f(x)|$ for all x. Our principal result concerning closed ideals is essentially due to Kakutani [2]; it follows from Theorem 3 much as Theorem 6 follows from Theorem 2.

Theorem 7: *Let $\mathfrak{X}$ be the linear lattice of all the continuous real functions on a compact space X, let $\mathfrak{X}_0$ be an arbitrary non-void subfamily of $\mathfrak{X}$, let X_0 be the closed set of all those points x at which every function f in $\mathfrak{X}_0$ vanishes, and let $\mathfrak{X}_0$ be the family of those functions f in $\mathfrak{X}$ which vanish at every point of X_0. Then $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$; and $\mathfrak{X}_0$ is a closed ideal if and only if $\mathfrak{X}_0 = \mathfrak{Y}_0$. A closed ideal $\mathfrak{X}_0$ is characterized by the associated closed set X_0; in particular, $\mathfrak{X}_0 = \mathfrak{Y}_0 = \mathfrak{X}$ if and only if X_0 is void.*

Proof: It is evident that $\mathfrak{Y}_0$ is a closed ideal containing $\mathfrak{X}_0$. For example, if f is in $\mathfrak{Y}_0$ and $|g(x)| \leq |f(x)|$ for every x, then g vanishes everywhere on X_0 and therefore belongs to $\mathfrak{Y}_0$. If $\mathfrak{X}_0$ is a closed ideal we can show that $\mathfrak{X}_0 = \mathfrak{Y}_0$. To do so we refer to Theorem 3 and consider what linear relations of the form indicated there can be satisfied by every function in $\mathfrak{X}_0$. Obviously the pairs of points x, y which have one or both members in X_0 are of no further interest, as the corresponding linear conditions are those of types (1) and (2), the effect of which has already been taken into account through the introduction of the closed set X_0. Turning to the case where x and y are distinct points not in X_0, we first remark that if we have $f(x) = f(y)$ for every f in $\mathfrak{X}$ then no *effective* restriction is implied by the linear relation corresponding to the pair of points in question. Assuming therefore that g is a function in $\mathfrak{X}$ with $g(x) \neq g(y)$, we may suppose without loss of generality that $g(x) = 1$, $g(y) = 0$ - for we may replace g if necessary by the function h defined through the equation $h(z) = \dfrac{g(z) - g(y)}{g(x) - g(y)}$ for all z in X. Since x is not in X_0 there is a function f in $\mathfrak{X}_0$ such that $f(x) \neq 0$. We may suppose without loss of generality that $f(x) > 1$ for we may replace f if necessary by the function $h = \alpha f$ with a suitable value of α. The function $h = |f| \cap |g|$ is now seen to be in the ideal $\mathfrak{X}_0$ and to satisfy the equations $h(x) = 1$, $h(y) = 0$. Accordingly no linear relation of type (3) or type (4) is satisfied by h. Hence the linear relations which characterize $\mathfrak{X}_0$ as a closed linear sublattice of $\mathfrak{X}$ reduce effectively to those implicit in the statement that every function in $\mathfrak{X}_0$ vanishes throughout X_0. It follows that $\mathfrak{X}_0 = \mathfrak{Y}_0$. Obviously if $\mathfrak{X}_0$ is an arbitrary non-void family and $\mathfrak{X}_1$ is a closed ideal containing $\mathfrak{X}_0$ then the associated closed set X_1 is part of X_0; and $\mathfrak{X}_1 = \mathfrak{Y}_1 \supset \mathfrak{Y}_0$. This completes the proof of the theorem.

Finally we take up the case where $\mathfrak{X}$ is to be regarded as a (linear) ring - the algebraic operations considered being addition and multiplication (and multiplication by real constants). Here an ideal is defined as a non-void subclass of $\mathfrak{X}$ which contains $f + g$ whenever it contains f and g, and which contains fg whenever it contains f. Since multiplication of f by the real number α is equivalent to multiplication of f by the constant function g everywhere equal to α, we see that an ideal automatically contains αf together with f. Our main result reads exactly like Theorem 7, differing from it only in the interpretation which has to be given to the term "ideal."

Theorem 8: *Let $\mathfrak{X}$ be the linear ring of all the continuous real functions on a compact space X, let $\mathfrak{X}_0$ be an arbitrary non-void subfamily of $\mathfrak{X}$, let X_0 be the closed set of all those points x at which every function f in $\mathfrak{X}_0$ vanishes, and let $\mathfrak{Y}_0$ be the family of all those functions f in $\mathfrak{X}$ which vanish at every point of X_0. Then $\mathfrak{Y}_0$ is the smallest closed ideal containing $\mathfrak{X}_0$; and $\mathfrak{X}_0$ is a closed ideal if and only if $\mathfrak{X}_0 = \mathfrak{Y}_0$. A closed ideal $\mathfrak{X}_0$ is characterized by the associated closed set X_0; in particular, $\mathfrak{X}_0 = \mathfrak{Y}_0 = \mathfrak{X}$ if and only if $\mathfrak{X}_0$ is void.*

Proof: It is evident that $\mathfrak{Y}_0$ is a closed ideal containing $\mathfrak{X}_0$. For example, if f is in $\mathfrak{Y}_0$ and g is arbitrary, then fg vanishes throughout X_0 and is therefore a function in $\mathfrak{Y}_0$. If $\mathfrak{X}_0$ is a closed ideal it is a closed linear subring. By using Theorem 5, we can show that $\mathfrak{X}_0 = \mathfrak{Y}_0$. The argument is much like that applied in the discussion of the analogous relation in Theorem 7. Since $\mathfrak{X}_0$ is characterized by the set X_0 and the linear relations of the form $g(x) = g(y)$ (where x and y are outside X_0) which are satisfied by every function in $\mathfrak{X}_0$, we have to eliminate the latter by appealing to the fact that $\mathfrak{X}_0$ is an ideal. If the condition $g(x) = g(y)$ is satisfied for *every* g in $\mathfrak{X}$, then there is no effective condition corresponding to the pair of points x, y. If there is some function g such that $g(x) \neq g(y)$, we may suppose that $g(x) = 1$, $g(y) = 0$. Since f can be found in $\mathfrak{X}_0$ so that $f(x) = 1$, the function $h = fg$ is in $\mathfrak{X}_0$ and $h(x) = 1$, $h(y) = 0$. Hence there is no condition of the form $g(x) = g(y)$ which is satisfied by all functions in $\mathfrak{X}_0$. This establishes the identity of $\mathfrak{X}_0$ and $\mathfrak{Y}_0$. The remainder of the discussion follows exactly the lines laid down in the proof of Theorem 7.

The connection between Theorems 7 and 8 is made plain by the following theorem.

Theorem 9: *If $\mathfrak{X}$ is the family of all continuous real functions on a compact space* X, *then $\mathfrak{X}$ is both a linear lattice and a (linear) ring. A non-void closed subfamily $\mathfrak{X}_0$ of $\mathfrak{X}$ is a linear-lattice ideal if and only if it is a ring-ideal.*

Proof: The result follows immediately from a comparison of Theorems 7 and 8. It would also be possible to give a proof by direct examination of the ideal-properties. Thus, if $\mathfrak{X}_0$ is a closed linear-lattice ideal, we show it to be a ring-ideal as follows. If f is in $\mathfrak{X}_0$ and g in $\mathfrak{X}$, then α can be found so that $|g(x)| \leq \alpha$ for every x - and it therefore follows that the product $h = fg$ satisfies the relations

$$|h(x)| = |f(x)g(x)| \leq |\alpha f(x)| = |(\alpha f)(x)|$$

and hence belongs to $\mathfrak{X}_0$ along with f and αf. On the other hand, if $\mathfrak{X}_0$ is a closed ring-ideal, it is a closed linear sublattice by virtue of Theorem 4. In particular, $\mathfrak{X}_0$ contains $|f|$ together with f. Thus, if f is in $\mathfrak{X}_0$ and g is in $\mathfrak{X}$, the function h_n defined by putting $h_n(x) = |f(x)|g(x)/\left(|f(x)| + \frac{1}{n}\right)$ for all x is in $\mathfrak{X}_0$. If g satisfies the inequality $|g(x)| \leq |f(x)|$ for every x, then

$$|g(x) - h_n(x)| \leq |g(x)|/(n|f(x)| + 1) \leq |f(x)|/(n|f(x)| + 1)$$

$$\leq \frac{1}{n}\left(1 - \frac{1}{n|f(x)| + 1}\right) \leq \frac{1}{n}.$$

Since the sequence h_n thus converges uniformly to g, we see that g is in $\mathfrak{X}_0$ also.

(*to be continued in the next issue*)

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