Physics 2. Electrical Engineering

Week 10 Dynamic Circuits Analysis 2

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Objectives

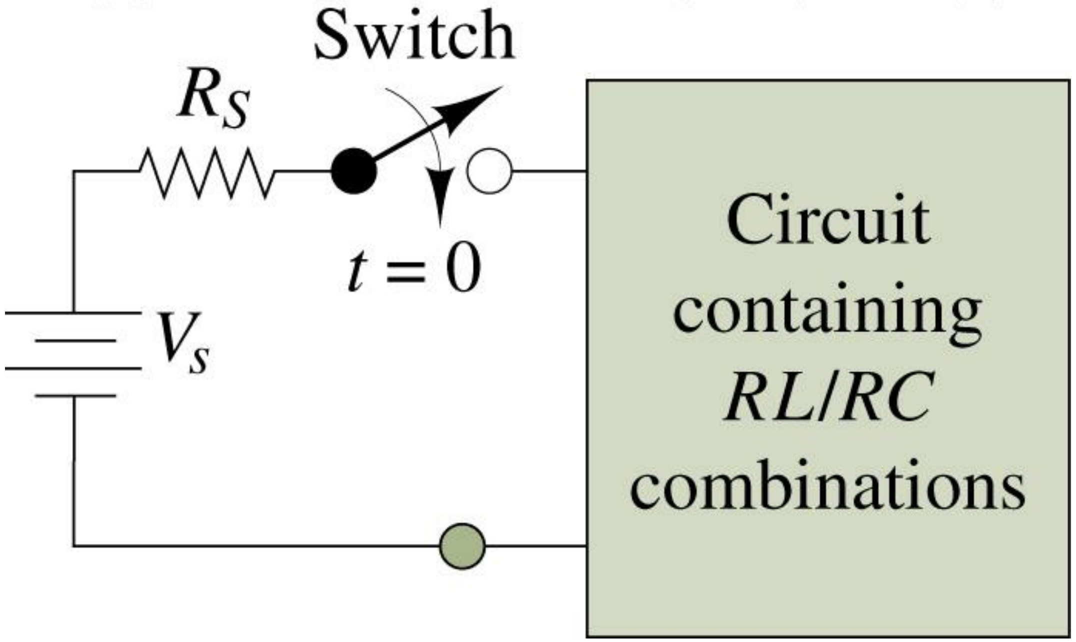
The main objectives of today’s lecture are:

- Review transient response of first-order circuits
- Study the transient response of second-order circuits
- Draw analogies between electrical and mechanical dynamic systems

Transient Response of First-Order Circuits: Review

Introduction to Transient Analysis

The object of transient analysis is to describe the behavior of a voltage or a current during the transition between two distinct steady-state conditions.



Transient Analysis

The previous equation can be rewritten as

$$a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t)$$

$$\frac{a_1}{a_0} \frac{dx(t)}{dt} + x(t) = \frac{b_0}{a_0} f(t)$$

or

$$\tau \frac{dx(t)}{dt} + x(t) = K_S f(t)$$

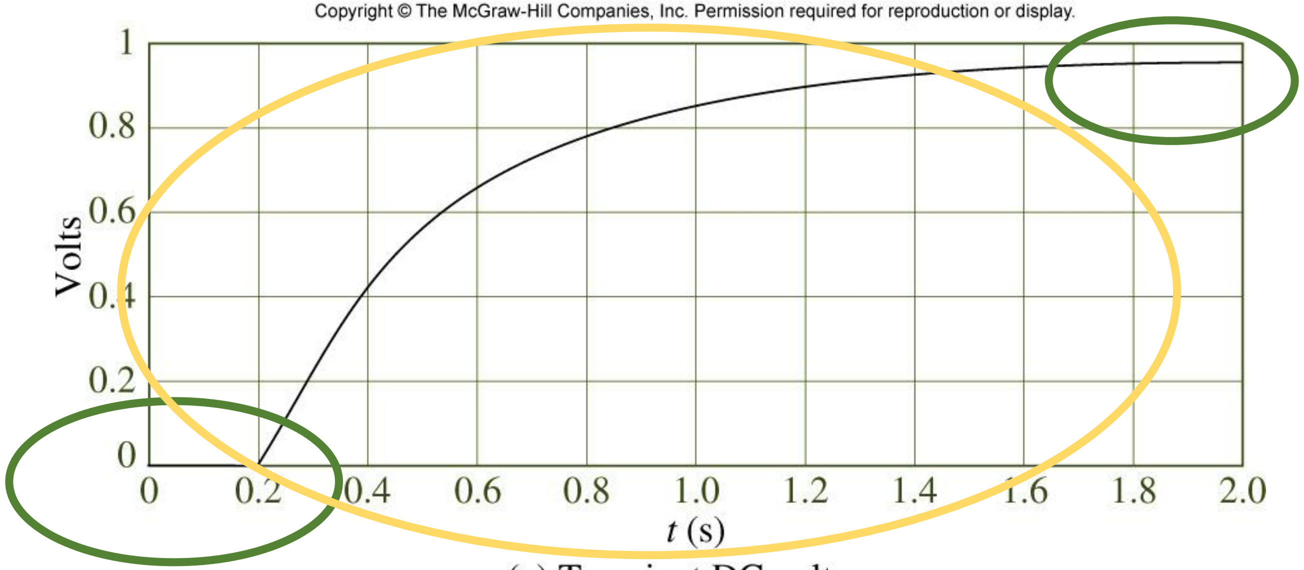
The constants are

- τ - time constant
- K_S – DC gain.

Transient Response (1)

The transient response of a circuit consists of three parts:

1. The steady-state response prior to the transient.
2. The transient response.
3. The steady-state response after the end of the transient.



(a) Transient DC voltage

Transient Response (3)

The steps to find a first-order transient response are:

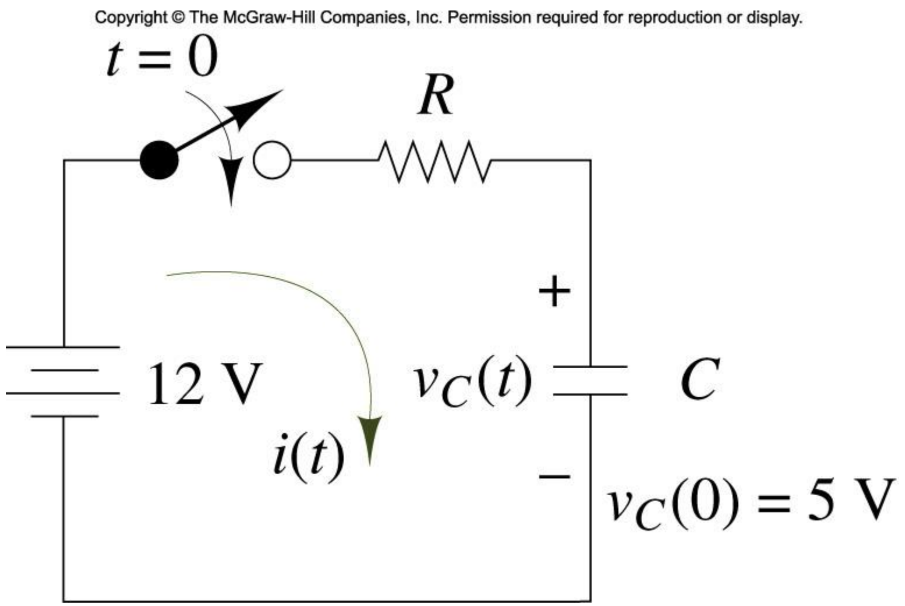
1. Solve for the steady-state response $x(0)$ of the circuit before the switch changes state $(t = 0^-)$ and after the transient has died out $(x(\infty), t \to \infty)$.
2. Identify the initial conditions for the circuit.
3. Write the differential equation of the circuit for $t = 0^+$.
4. Solve for the time constant of the circuit: $\tau = RC$ or $\tau = L/R$.
5. Write the complete solution for the circuit in the form

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}$$

Transient Response: Example (1)

Determine an expression for the capacitor voltage in the circuit shown.

Given: $v_C(t = 0^-)$ = 5 V, v_B = 12 V.



- When the switch has been closed for a long time, the capacitor current becomes zero (capacitor = open circuit). Hence,

$$v_C(\infty) = V_B = 12\text{V}$$

Transient Response: Example (2)

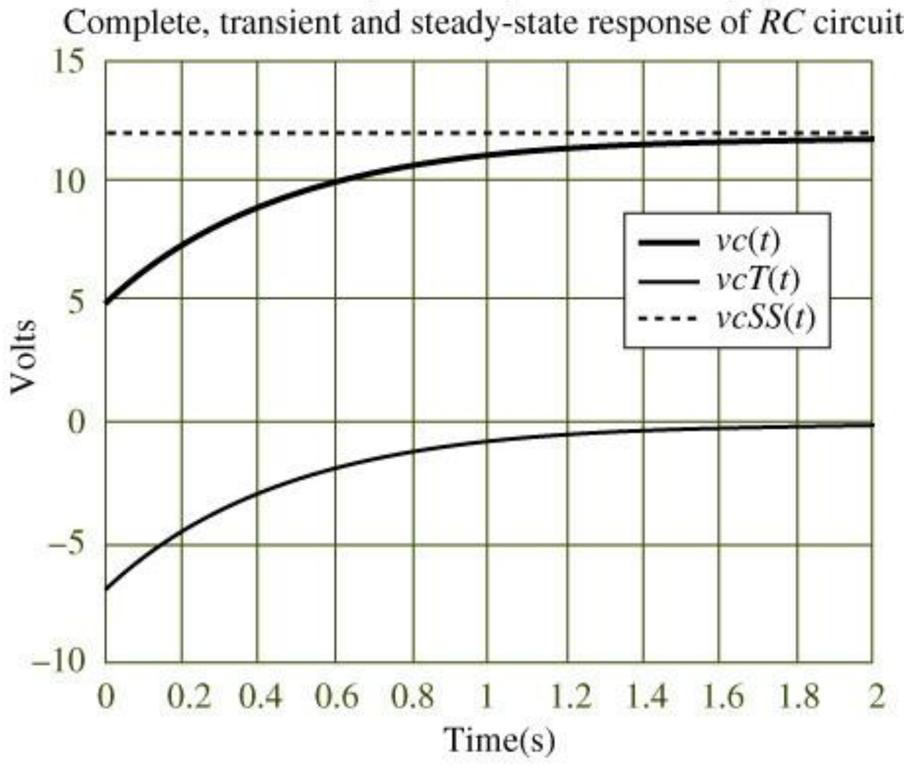
Using KVL, we can write

$$V_B - Ri_C(t) - v_C(t) = V_B - RC\frac{dv_C(t)}{dt} - v_C(t) = 0$$

$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_B, \qquad t \geq 0$$

- Time constant is $\tau = RC$
- Hence,

$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau}$$

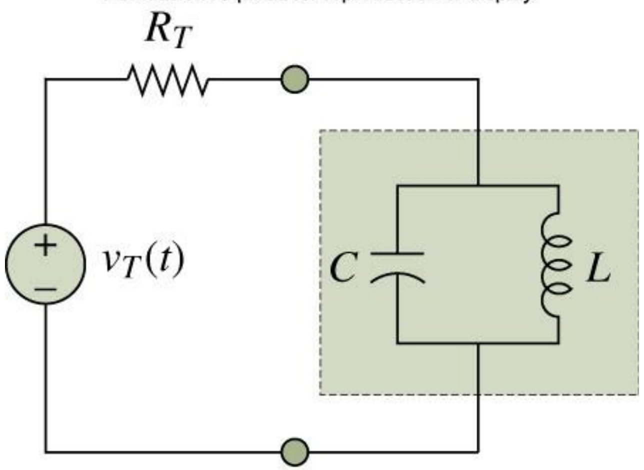


(a)

Transient Response of Second-Order Circuits

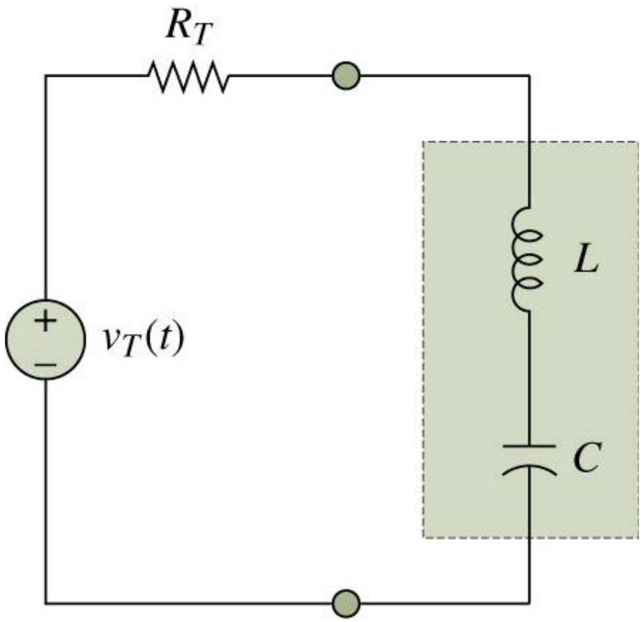
Second-Order Circuits (1)

A simple second-order circuit is a circuit containing R-L-C elements.



Parallel case

(a)



Series case

(b)

Second-Order Circuits: Parallel Case (1)

Let us consider the parallel case first.

- Starting with KVL around the left-hand loop,

$$v_T(t) - R_T i_S(t) - v_C(t) = 0$$

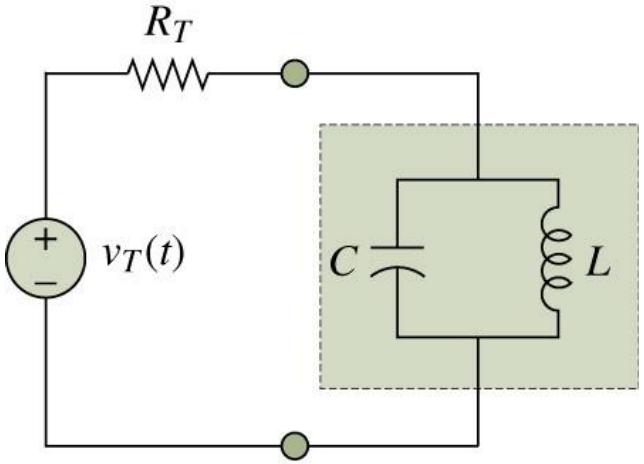
- Then apply KCL to the top node to obtain $i_S(t) - i_C(t) - i_L(t) = 0$

- We know that $v_C(t) = v_L(t)$

- Hence

$$\frac{v_T(t) - v_C(t)}{R_T} - C\frac{dv_C(t)}{dt} - i_L(t) = 0,$$

$$v_C(t) = L\frac{di_L(t)}{dt}$$



Parallel case
(a)

Second-Order Circuits: Parallel Case (2)

Finally, we obtain

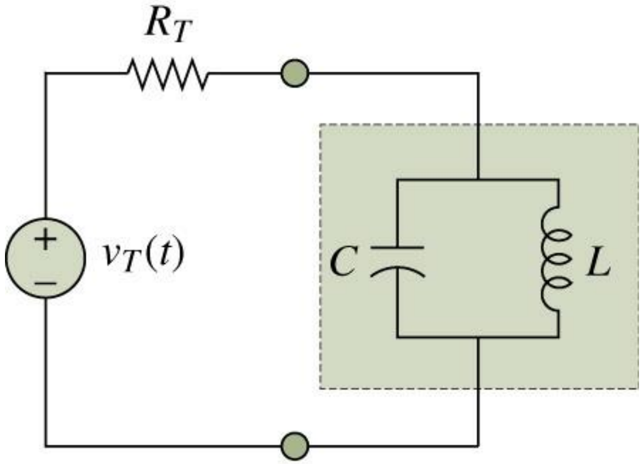
$$\frac{1}{R_T}v_T(t) - \frac{L}{R_T}\frac{di_L(t)}{dt} = LC\frac{d^2 i_L(t)}{dt^2} + i_L(t)$$

or

$$LC\frac{d^2 i_L(t)}{dt^2} + \frac{L}{R_T}\frac{di_L(t)}{dt} + i_L(t) = \frac{1}{R_T}v_T(t)$$

- This equation can be written in the following form:

$$\frac{1}{\omega_n^2}\frac{d^2 x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$$



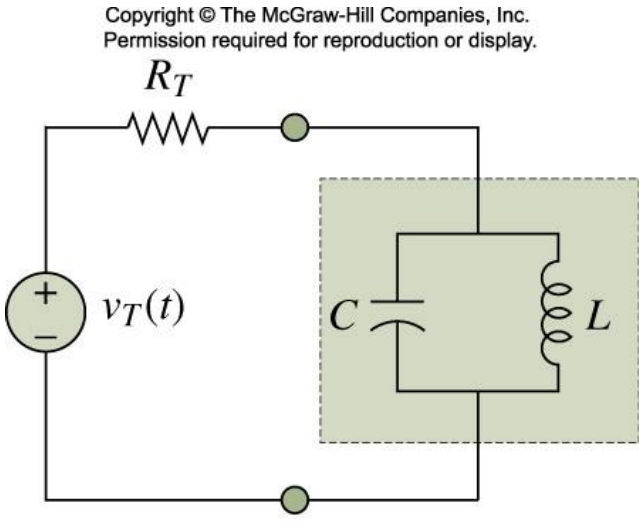
Parallel case

(a)

Second-Order Circuits (2)

The equations of this type are called second-order linear ordinary differential equations

$$\frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$$



Parallel case

(a)

The parameters are:

- ω_n – the natural frequency
- ζ – the damping ratio
- K_S – DC gain.

Second-Order Circuits: Series Case

Now, let us derive the equations for the series RLC circuit.

- Starting with KVL around the loop,

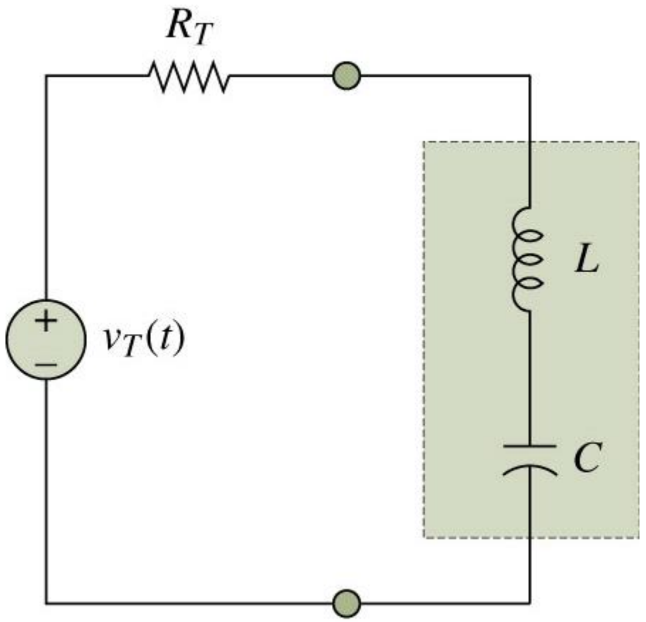
$$v_T - R_T i_L - v_C - v_L = 0$$

- Recalling the equations for capacitor and inductor,

$$v_T - R_T i_L - v_C - L\frac{di_L}{dt} = 0, \quad i_L = i_C = C\frac{dv_C}{dt}$$

- Hence, writing for v_C yields

$$LC\frac{d^2 v_C}{dt^2} + R_T C\frac{dv_C}{dt} + v_C = v_T \quad \rightarrow \frac{1}{\omega_n^2}\frac{d^2 x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$$



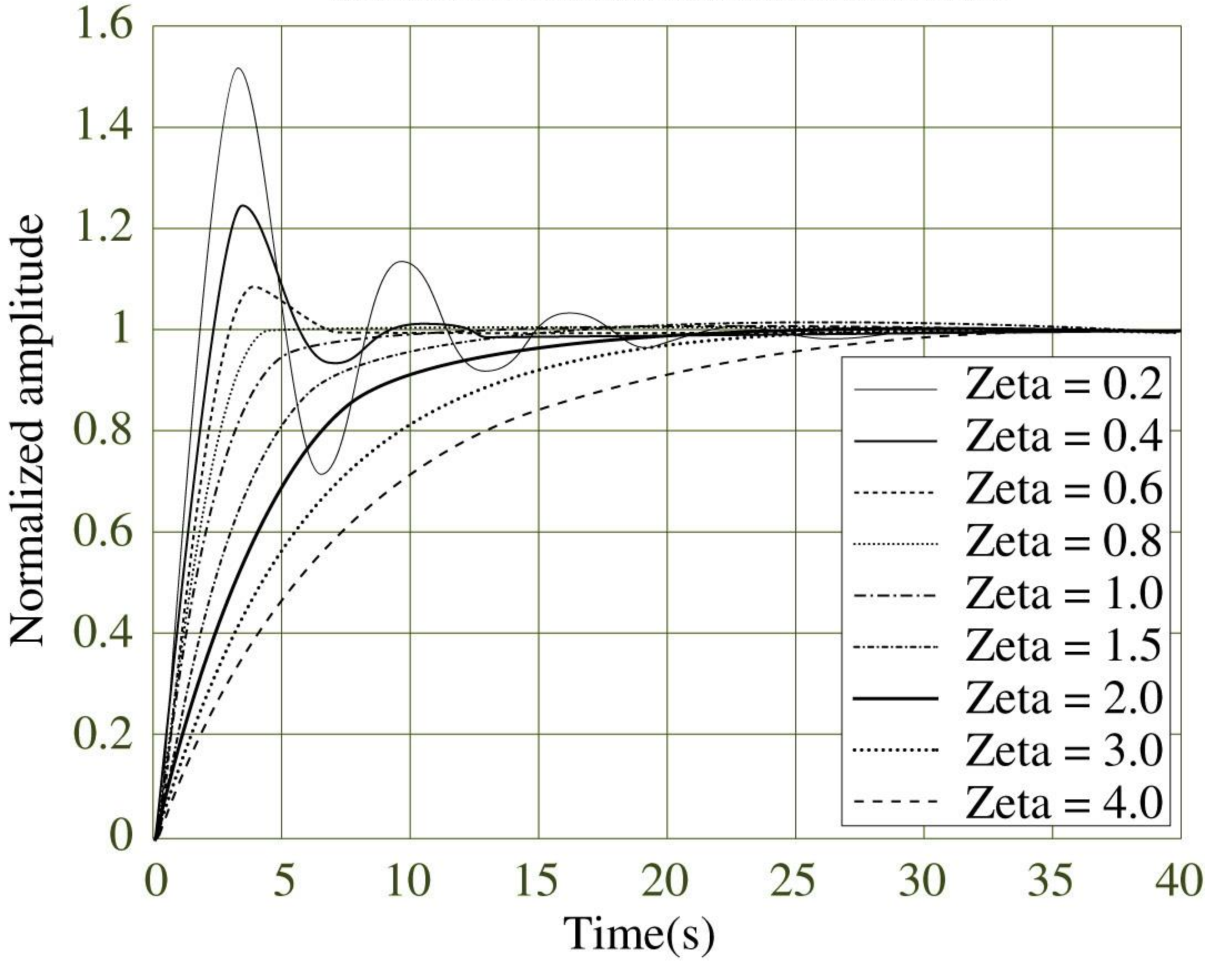
Series case
(b)

Second-Order Circuits (3)

A typical response of switched second-order system is shown here for different values of ζ.

We immediately note three important characteristics of the response:

1. The response asymptotically tends to a constant value.
2. The response oscillates.
3. The oscillations decay (and eventually disappear) as time progresses.



Second-Order Circuits: Natural Response

$$\frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx(t)}{dt}+x(t)=K_Sf(t)$$

The natural response is found by setting the excitation equal to zero.

- Thus, we solve the equation

$$\frac{1}{\omega_n^2}\frac{d^2x_N(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_N(t)}{dt}+x_N(t)=0$$

- The solution is known to be of exponential form: $x_N(t)=\alpha e^{st}$

- Substituting into the initial equation yields

$$\frac{1}{\omega_n^2}s^2\alpha e^{st}+\frac{2\zeta}{\omega_n}s\alpha e^{st}+\alpha e^{st}=0 \quad \rightarrow \frac{s^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}s+1=0$$

Second-Order Circuits: Natural Response

This polynomial in the variable s is called the characteristic polynomial of the differential equation, and it gives rise to two characteristic roots, s_1 and s_2.

$$\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1 = 0$$

$$x_N(t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t}$$

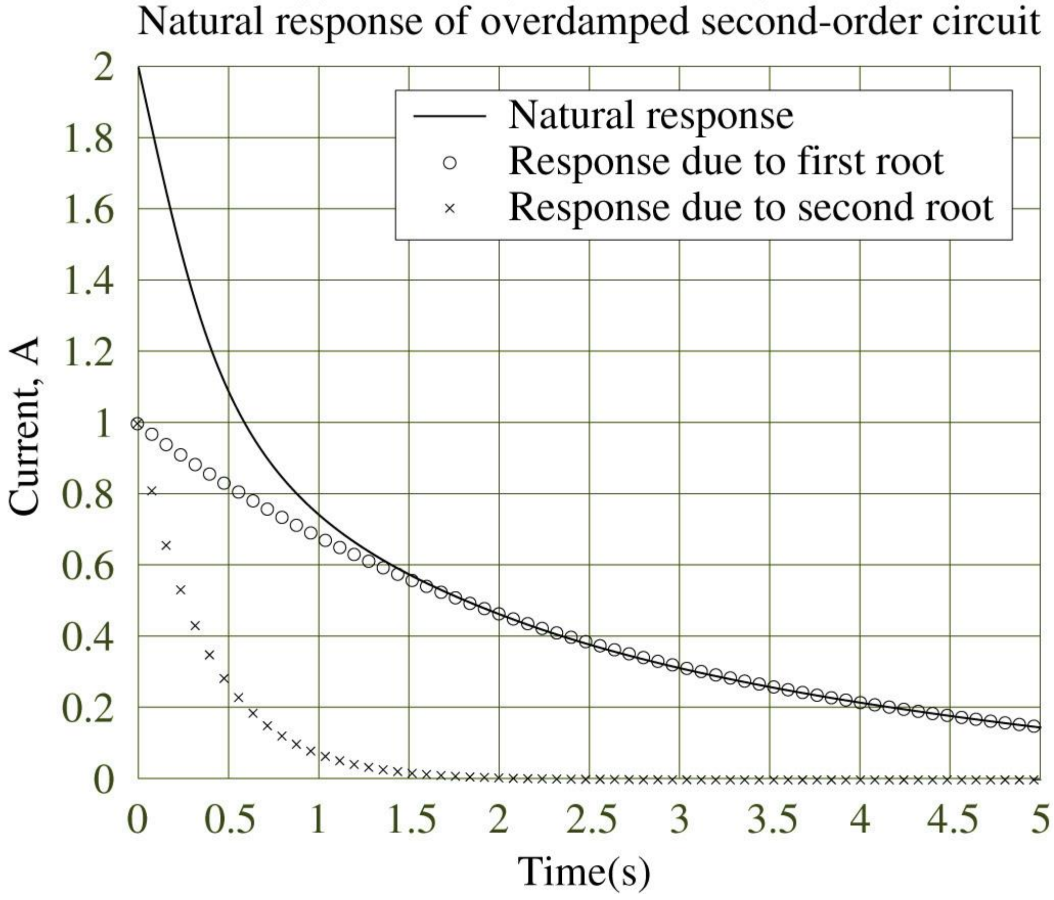
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

For the roots of second-order systems, three cases exist:

1. Real and distinct roots ($\zeta > 1$). This leads to an overdamped response $\quad s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
2. Real and repeated roots ($\zeta = 1$) - critically damped response $\quad s_{1,2} = -\zeta\omega_n = -\omega_n$
3. Complex conjugate roots ($0 < \zeta < 1$) - underdamped response $\quad s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

Second-Order Circuits: Overdamped

1. Real and distinct roots ($\zeta > 1$) - overdamped response



$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

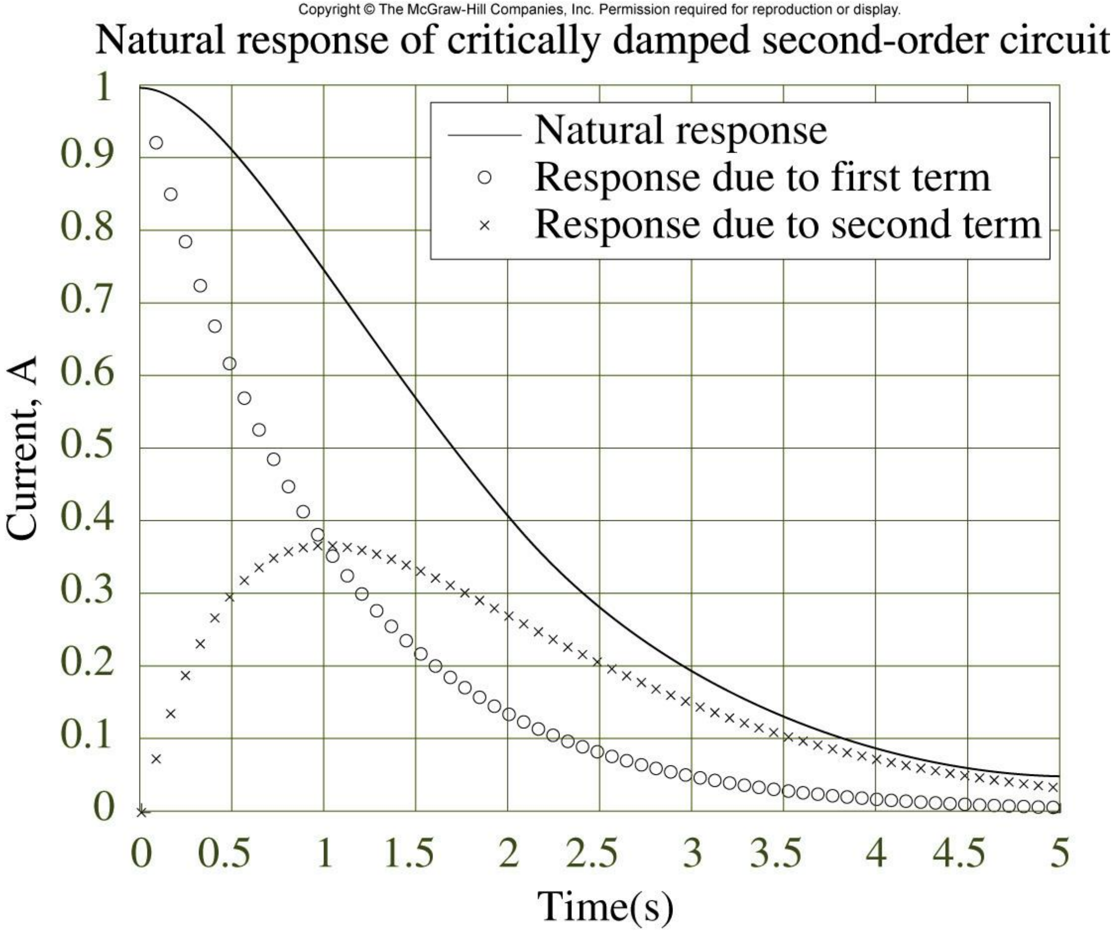
$$x_N(t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t}$$

Second-Order Circuits: Critically Damped

2. Real and repeated roots ($\zeta = 1$) - critically damped response

$$s_{1,2} = -\zeta\omega_n = -\omega_n$$

$$x_N(t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t}$$

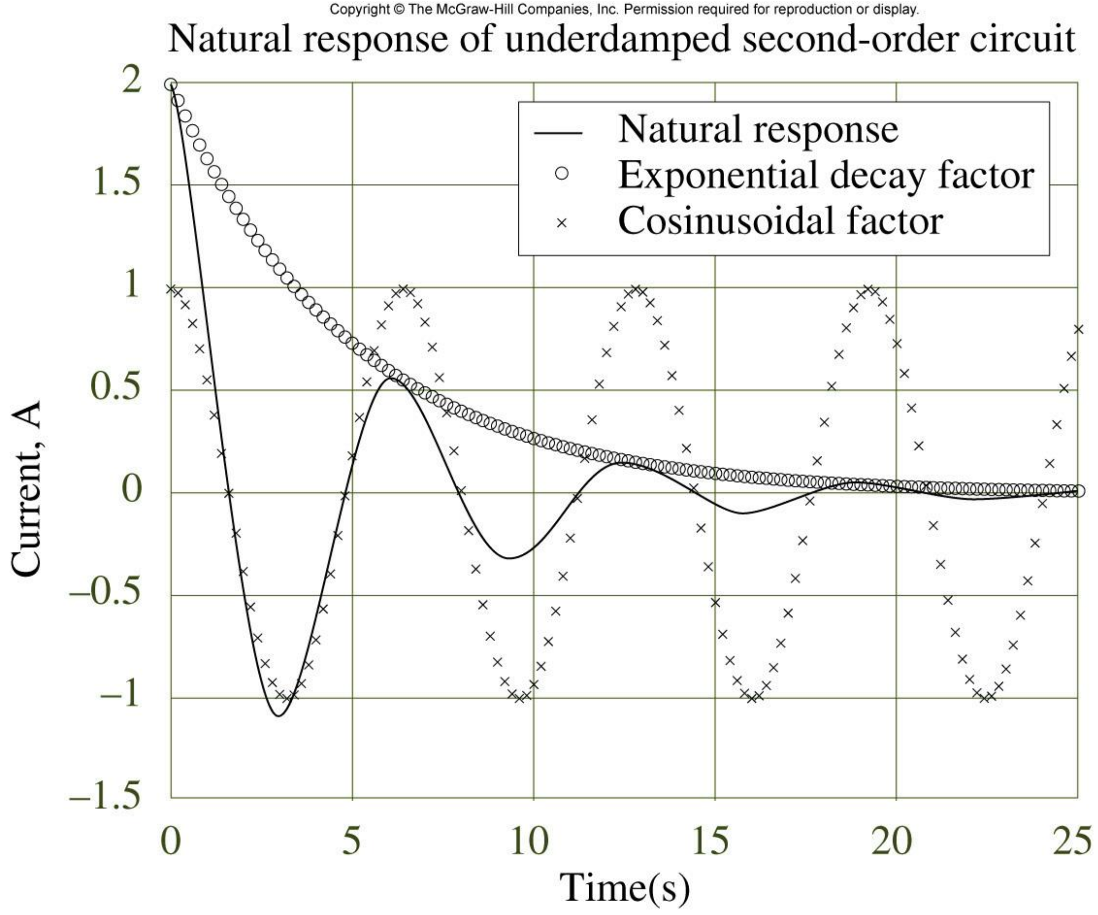


Second-Order Circuits: Underdamped

3. Complex conjugate roots ($0 < \zeta < 1$) - underdamped response

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$x_N(t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t}$$



Second-Order Circuits: Complete Response

The forced response is the solution of the equation

$$\frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx(t)}{dt}+x(t)=K_S f(t)$$

After the transient settles,

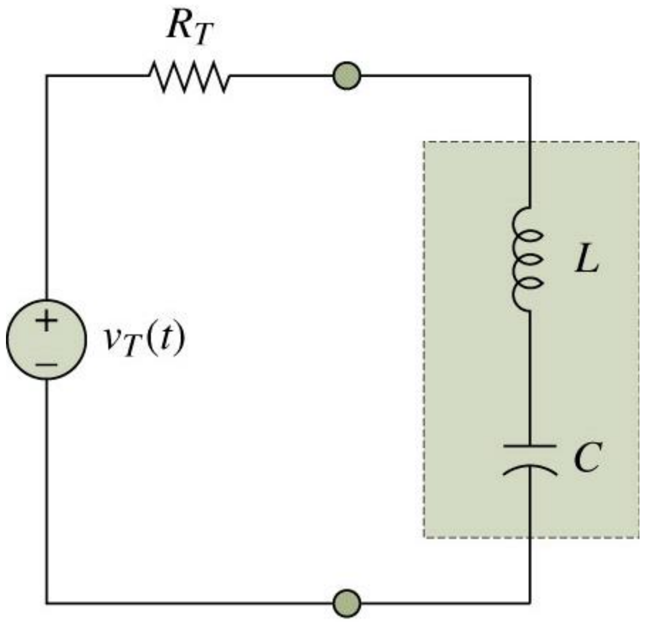
$$x_F(t)=K_S f(t)$$

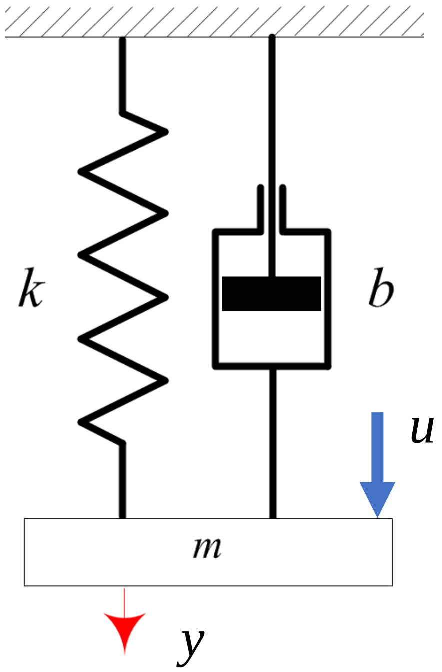
- The complete response is then

$$x(t)=x_N(t)+x_F(t)$$

Second-Order Systems: Analogies (1)

Now, let us compare two systems: electrical and mechanical





$$L\frac{d^2v_C}{dt^2}+R_T\frac{dv_C}{dt}+\frac{1}{C}v_C=\frac{1}{C}v_T$$

$$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=u$$

Second-Order Systems: Analogies (2)

Upon comparing the equations, one can readily identify the analogies between the electrical and mechanical worlds:

$$L\boxed{\frac{d^2 v_C}{dt^2}} + R_T\boxed{\frac{dv_C}{dt}} + \frac{1}{C}\boxed{v_C} = \frac{1}{C}v_T$$

$$m\boxed{\frac{d^2 y}{dt^2}} + b\boxed{\frac{dy}{dt}} + k\boxed{y} = u$$

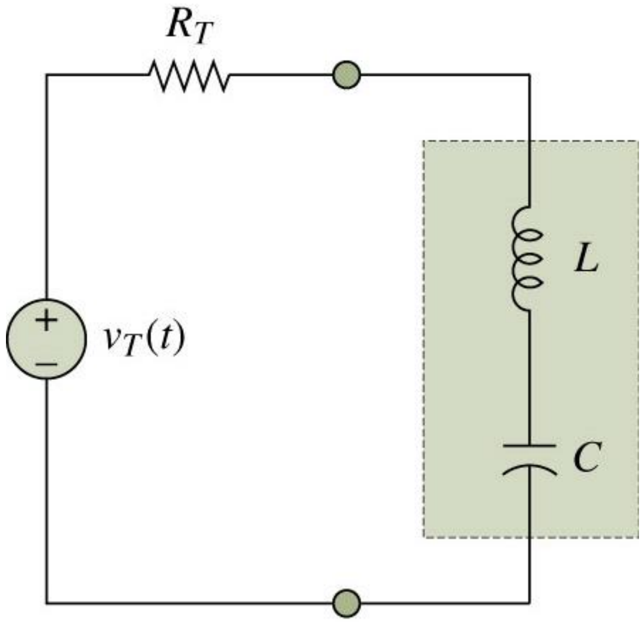
Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping coeff. b
Capacitance C	Compliance $1/k$

Second-Order Systems: Oscillations

Lastly, let us return to an RLC circuit we have started with. Recall the corresponding differential equations:

$$LC\frac{d^2v_C}{dt^2} + R_TC\frac{dv_C}{dt} + v_C = v_T$$

$$\rightarrow \frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_Sf(t)$$



Q: Given that $\zeta = 0$ creates a purely oscillatory response,

- how to turn this circuit into an oscillator?

Thank you for your attention!

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