Physics 2. Electrical Engineering

Week 9 Dynamic Circuits Analysis 1

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Objectives

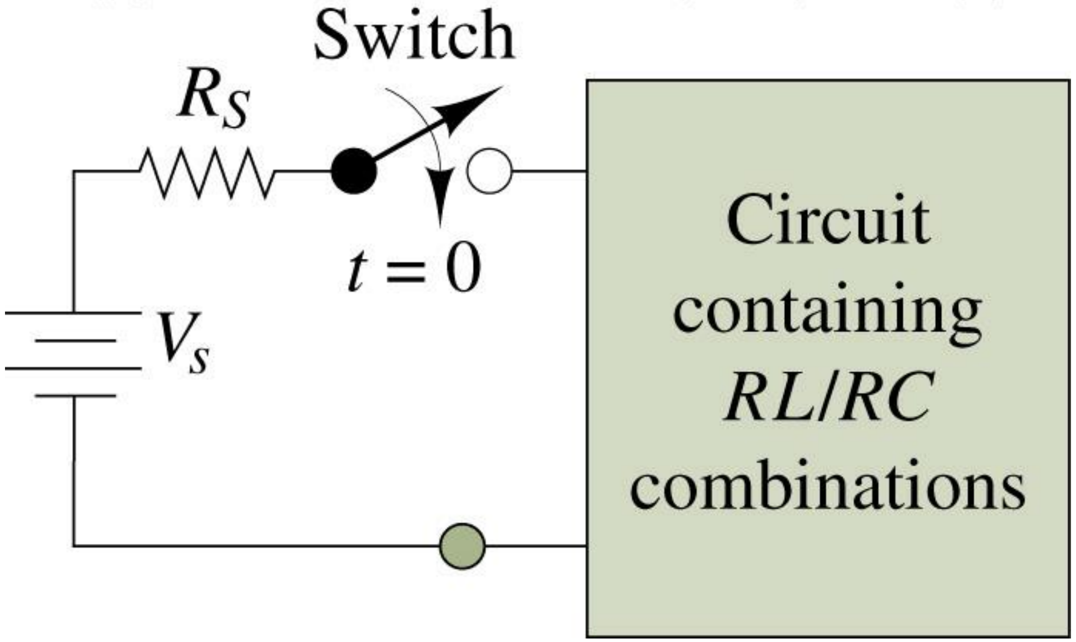
The main objectives of today’s lecture are:

- Become familiar with the transient analysis of dynamic circuits
- Learn to obtain DC steady-state solution of dynamic circuits
- Study the transient response of first-order circuits

Transient Analysis

Introduction to Transient Analysis

The object of transient analysis is to describe the behavior of a voltage or a current during the transition between two distinct steady-state conditions.



Dynamic Circuits

Consider the circuit shown in the figure.

- Applying KCL yields

$$i_R(t) = i_C(t) = \frac{v_S(t) - v_C(t)}{R} = C\frac{dv_C(t)}{dt}$$

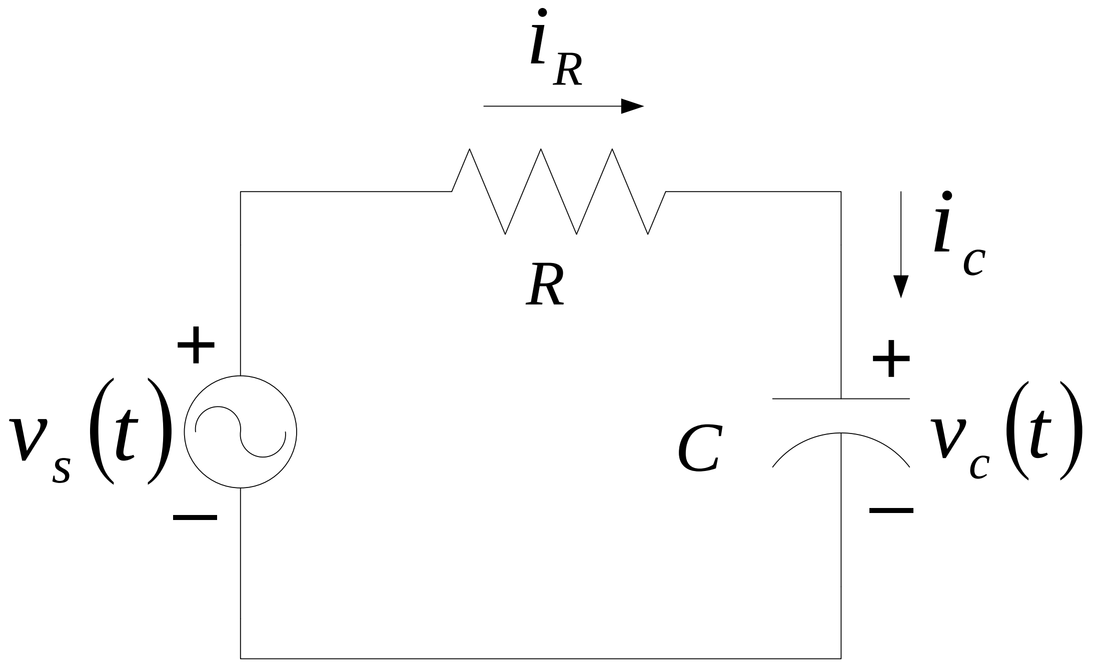
$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_S(t)$$

- Applying KVL yields

$$-v_S(t) + v_R(t) + v_C(t) = 0$$

$$-v_S(t) + Ri_C(t) + \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau = 0$$

$$\frac{di_C(t)}{dt} + \frac{1}{RC}i_C(t) = \frac{1}{R}\frac{dv_S(t)}{dt}$$



Dynamic Circuits: Example 1

Derive the differential equation of the circuit shown below.

- Applying KCL yields

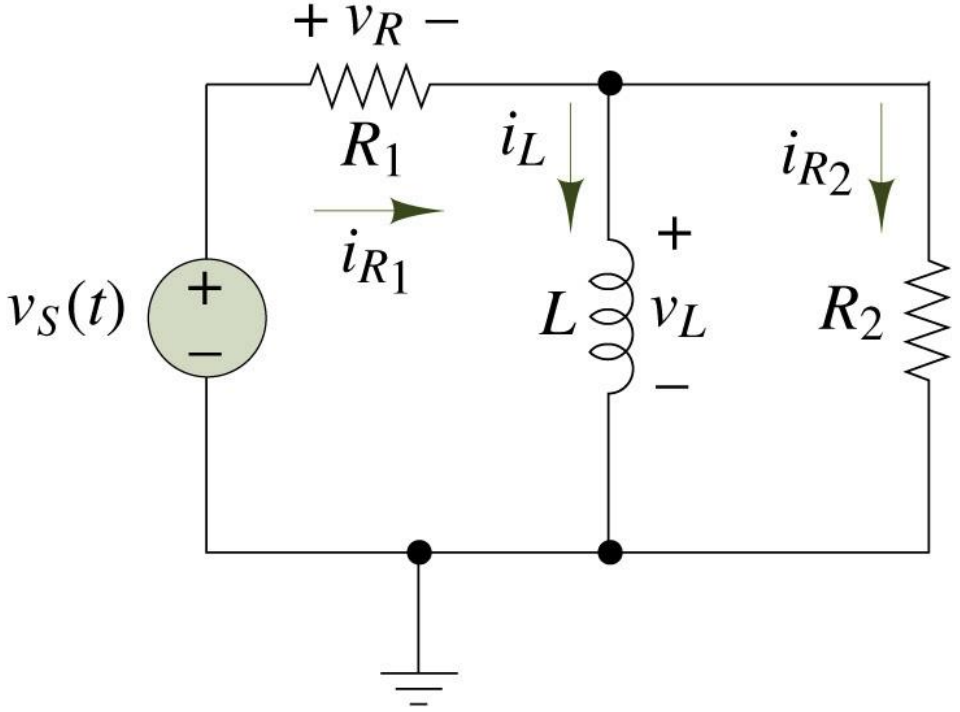
$$i_{R_1} - i_L - i_{R_2} = 0$$

$$\frac{v_s - v_L}{R_1} - i_L - \frac{v_L}{R_2} = 0$$

- Substituting $v_L = L\frac{di_L}{dt}$ yields:

$$\frac{v_s}{R_1} - \frac{L}{R_1}\frac{di_L}{dt} - i_L - \frac{L}{R_2}\frac{di_L}{dt} = 0$$

$$\frac{di_L}{dt} + \frac{R_1 R_2}{L(R_1 + R_2)} i_L = \frac{R_2}{L(R_1 + R_2)} v_s$$



Transient Analysis (1)

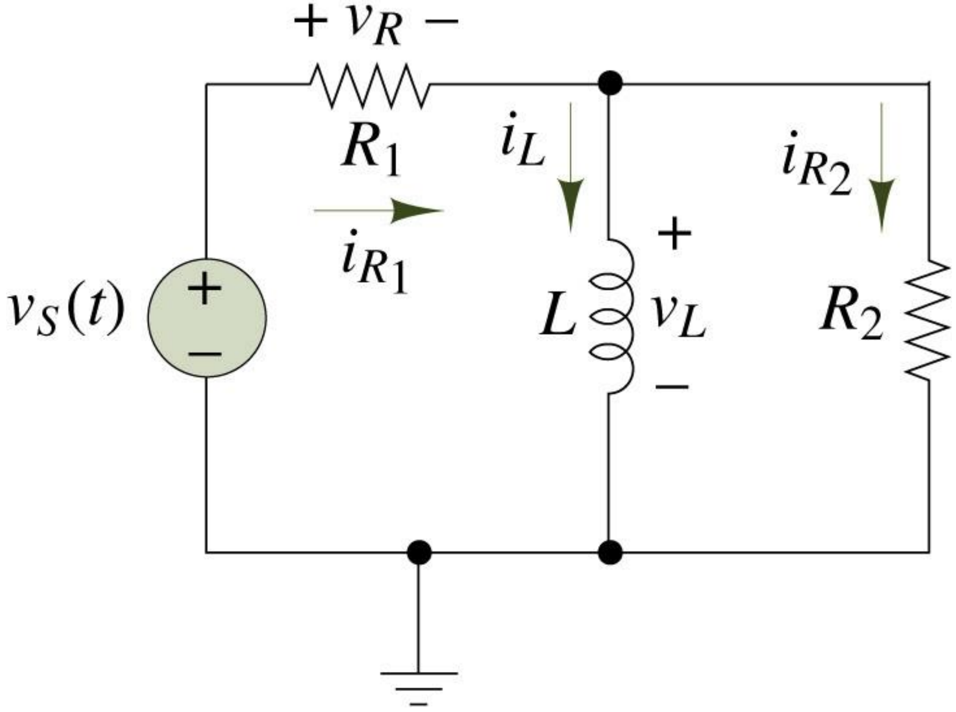
We can generalize the solution from the previous example by writing the differential equation of the following form:

$$\frac{di_L}{dt} + \frac{R_1 R_2}{L(R_1 + R_2)} i_L = \frac{R_2}{L(R_1 + R_2)} v_s$$

$$\downarrow$$

$$a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t)$$

- This is a first-order linear ordinary differential equation.



Transient Analysis (2)

The previous equation can be rewritten as

$$a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t)$$

$$\frac{a_1}{a_0}\frac{dx(t)}{dt} + x(t) = \frac{b_0}{a_0} f(t)$$

or

$$\tau \frac{dx(t)}{dt} + x(t) = K_S f(t)$$

The constants are

- τ - time constant
- K_S – DC gain.

DC Steady-State Solution of Dynamic Circuits

DC Steady State

DC steady state refers to circuits that have been connected to a DC (voltage or current) source for a very long time, such that all voltages and currents in the circuits have become constant.

Recall that, in direct current (DC) steady state,

- all capacitors behave like open circuits and
- all inductors behave like short circuits.

$$i_c(t) = C\frac{dv_C(t)}{dt} \qquad v_L(t) = L\frac{di_L(t)}{dt}$$

$$i_C(t) \to 0 \text{ as } t \to \infty \qquad v_L(t) \to 0 \text{ as } t \to \infty$$

DC Steady State: First Order Systems (1)

Consider the differential equation from Example 1:

$$\frac{di_L(t)}{dt} + \frac{R_1 R_2}{L(R_1 + R_2)} i_L(t) = \frac{R_2}{L(R_1 + R_2)} v_S(t)$$

$$\frac{L(R_1 + R_2)}{R_1 R_2} \frac{di_L(t)}{dt} + i_L(t) = \frac{1}{R_1} v_S(t)$$

or

$$\tau \frac{di_L(t)}{dt} + i_L(t) = K_S v_S(t)$$

$$\frac{di_L(t)}{dt} \to 0 \quad as \quad t \to \infty$$

DC Steady State: First Order Systems (2)

Hence

$$\frac{L(R_1 + R_2)}{R_1 R_2}\frac{di_L(t)}{dt} + i_L(t) = \frac{1}{R_1}v_S(t)$$

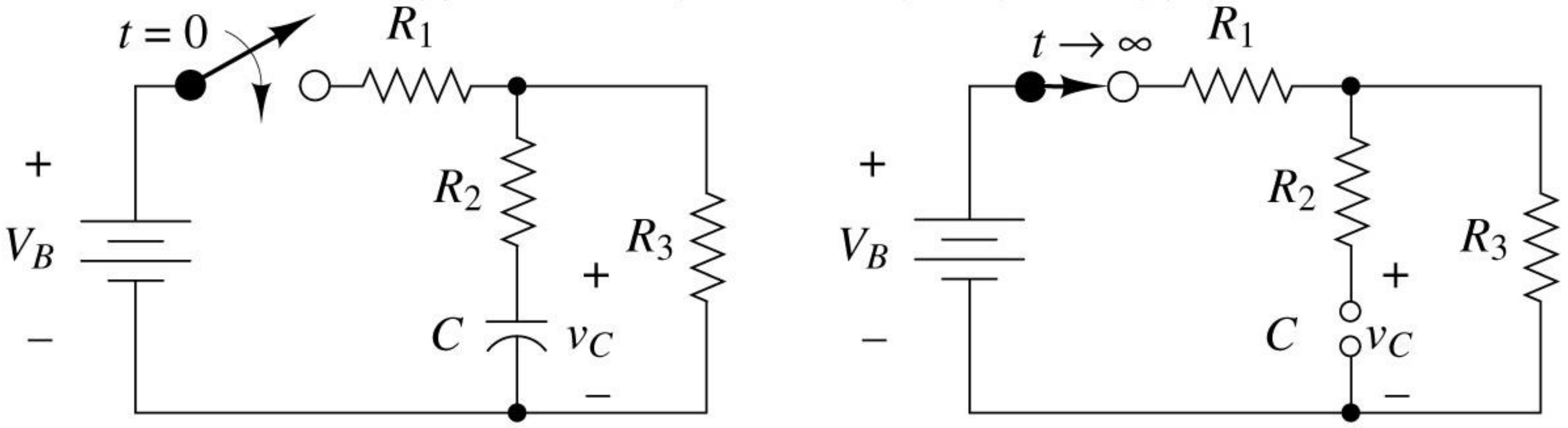
$$\downarrow (\mathrm{t} \to \infty)$$

$$i_L(t) = \frac{1}{R_1}v_S(t) = K_S v_S(t)$$

Note that the steady-state solution is found very easily, and it is determined only by K_S.

Example 2: Initial and Final Conditions

Determine the capacitor voltage in the circuit a long time after the switch has been closed.



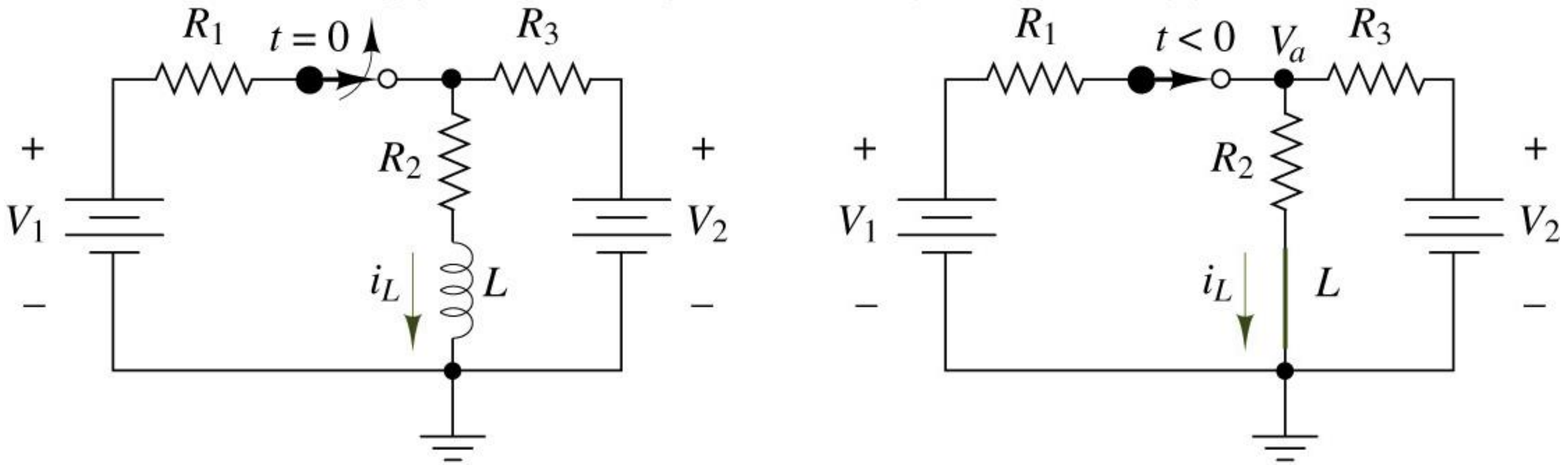
$$V_3(t \to \infty) = V_3(\infty) = \frac{R_3}{R_1 + R_3} V_B$$

The voltage in two parallel branches must be equal:

$$V_3(\infty) = V_C(\infty) + V_2(\infty) \xrightarrow{V_2(\infty)=0} V_3(\infty) = V_C(\infty)$$

Example 3: Initial and Final Conditions

Determine the inductor current in the circuit a just before the switch is opened.



$$\frac{V_1 - V_a}{R_1} - \frac{V_a}{R_2} + \frac{V_2 - V_a}{R_3} = 0$$

Solving yields

$$V_a = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} \left(\frac{V_1}{R_1} + \frac{V_2}{R_3}\right) \quad \rightarrow i_L(t = 0) = i_L(0) = \frac{V_a}{R_2}$$

Transient Response of First-Order Circuits

Transient Response of First-Order Circuits

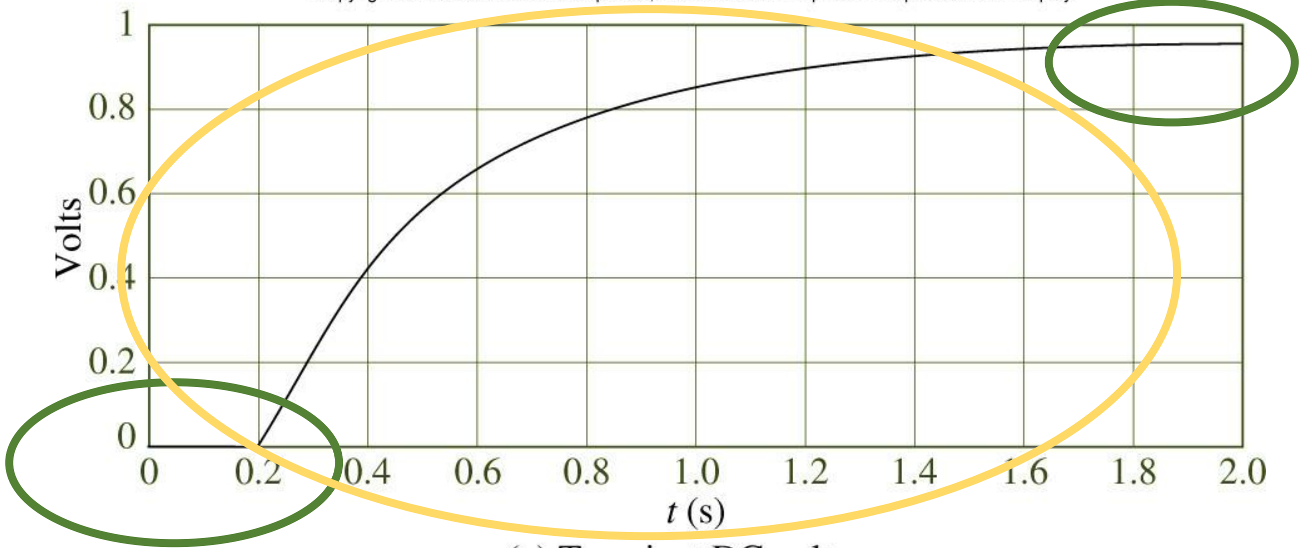
Before proceeding any further, let us outline some properties of any dynamic circuit.

- Initial conditions and final values for the differential equation describing dynamic circuits can be easily obtained, as shown previously.
- Capacitor voltage and inductor current cannot change instantaneously, because instantaneous change of these variables requires infinite power.
- This also means that the values of the inductor current or capacitor voltage immediately before the switch was opened/closed are the same as the values immediately after the switch was opened/closed.

Transient Response (1)

The transient response of a circuit consists of three parts:

1. The steady-state response prior to the transient.
2. The transient response.
3. The steady-state response after the end of the transient.



(a) Transient DC voltage

Transient Response (2)

First-order systems occur very frequently in nature.

- Any circuit containing a single energy storage element (an inductor or a capacitor) and a combination of voltage or current sources and resistors is a first-order circuit.
- The first-order system equation is

$$\frac{a_1}{a_0}\frac{dx(t)}{dt} + x(t) = \frac{b_0}{a_0}f(t)$$

or

$$\tau\frac{dx(t)}{dt} + x(t) = K_S f(t)$$

Transient Response (3)

The steps to find a first-order transient response are:

1. Solve for the steady-state response $x(0)$ of the circuit before the switch changes state $(t = 0^-)$ and after the transient has died out $(x(\infty), t \to \infty)$.
2. Identify the initial conditions for the circuit.
3. Write the differential equation of the circuit for $t = 0^+$.
4. Solve for the time constant of the circuit: $\tau = RC$ or $\tau = L/R$.
5. Write the complete solution for the circuit in the form

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}$$

Transient Response (4)

The solution of a first-order differential equation of the form

$$\tau \frac{dx(t)}{dt} + x(t) = K_S F$$

consists of two parts:

1. The natural response x_N $(F = 0)$
2. The forced response x_F.

The complete response = natural response + forced response

$$x(t) = x_N(t) + x_F(t)$$

Transient Response (5)

To find the natural response, we must solve the equation

$$\tau \frac{dx_N(t)}{dt} + x_N(t) = 0$$

or

$$\frac{dx_N(t)}{dt} = -\frac{x_N(t)}{\tau}$$

- The solution of this equation is known to be of exponential form:

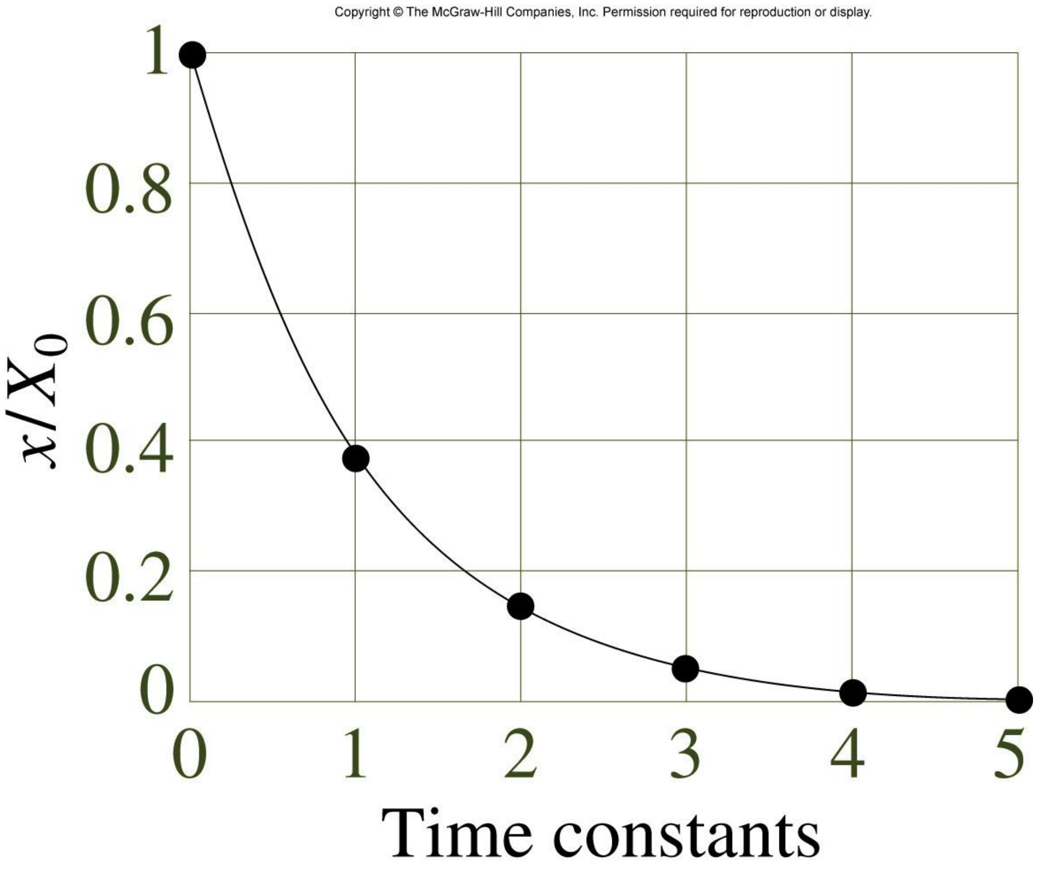
$$x_N(t) = \alpha e^{-t/\tau}$$

Transient Response (6)

Thus, we can create the following table:

$\frac{x(t)}{x(0)}$	$n = \frac{t}{\tau}$
1	0
0.3679	1
0.1353	2
0.0498	3
0.0183	4
0.0067	5

$$x_N(t) = \alpha e^{-t/\tau}$$



Transient Response (7)

The forced response of the system is the solution of the equation

$$\tau \frac{dx_F(t)}{dt} + x_F(t) = K_S F, \quad t \geq 0$$

in which the forcing function F is equal to a constant for $t \geq 0$.

- Since the derivative term becomes zero in response to a constant excitation, forced response is

$$x_F(t) = x_F(\infty) = K_S F, \quad t \geq 0$$

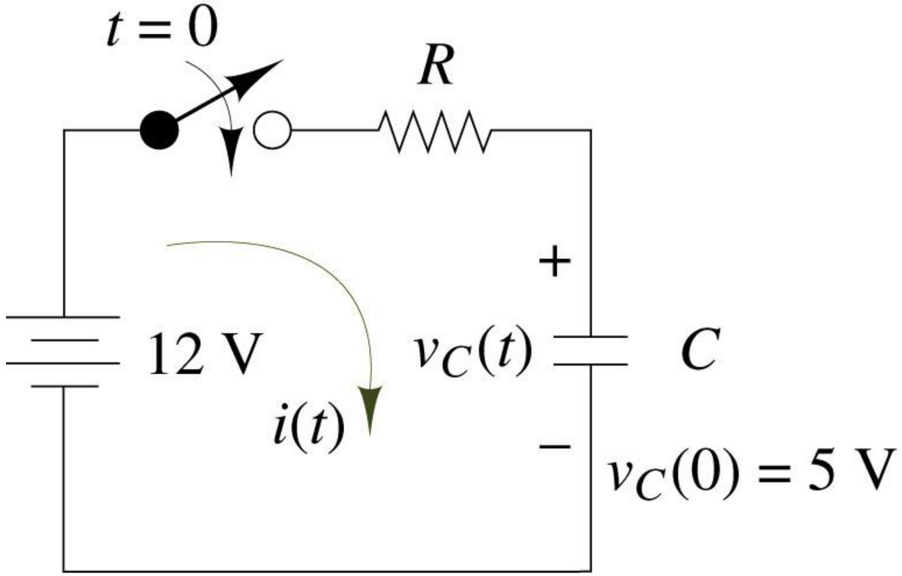
- The complete response is now

$$\begin{aligned} x(t) &= x_N(t) + x_F(t) \\ &= \alpha e^{-t/\tau} + K_S F = \alpha e^{-t/\tau} + x(\infty), \quad t \geq 0 \end{aligned}$$

Transient Response: Example (1)

Determine an expression for the capacitor voltage in the circuit shown.

Given: $v_C(t = 0^-)$ = 5 V, v_B = 12 V.



- When the switch has been closed for a long time, the capacitor current becomes zero (capacitor = open circuit). Hence,

$$v_C(\infty) = V_B = 12\text{V}$$

Transient Response: Example (2)

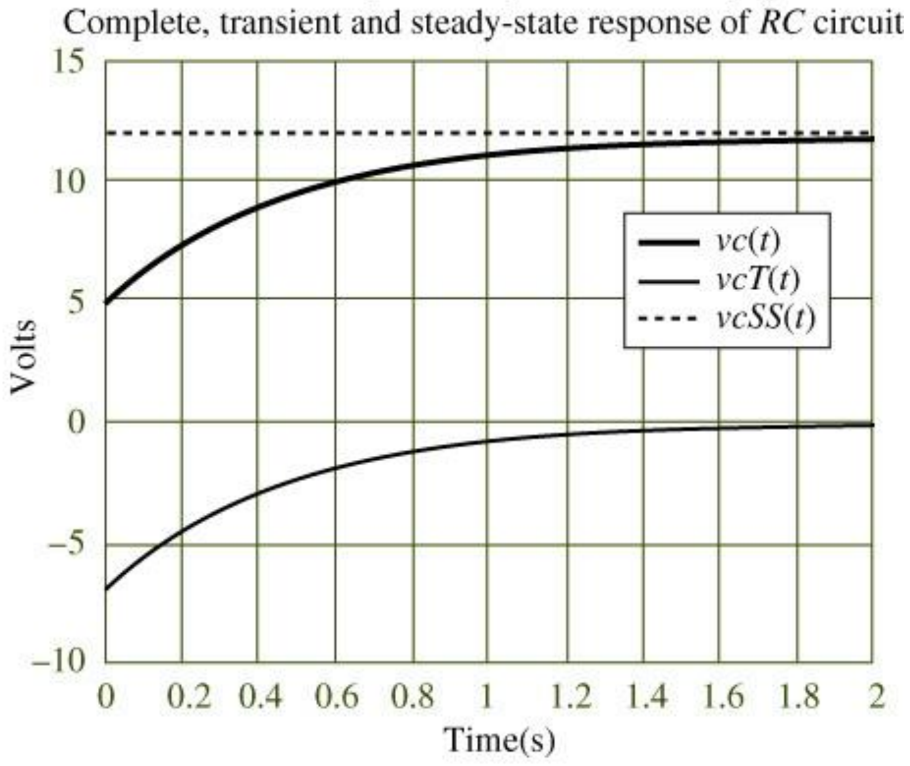
Using KVL, we can write

$$V_B - Ri_C(t) - v_C(t) = V_B - RC\frac{dv_C(t)}{dt} - v_C(t) = 0$$

$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_B, \qquad t \geq 0$$

- Time constant is $\tau = RC$
- Hence,

$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau}$$



(a)

Thank you for your attention!

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