Physics 2. Electrical Engineering

Week 8.1 Phasors

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Objectives

The main objectives of today's lecture are:

- Become familiar with the **forced response in dynamic circuits**
- Review **Euler's identity**
- Learn the concept of **phasors**

Inductors and Capacitors

Inductors vs Capacitors

Let us compare the I-V relationships of inductors and capacitors.

Inductors

$$v_L(t) = L\frac{di_L}{dt}$$

$$i_L(t) = \frac{1}{L}\int_0^t v_L(t')dt' + i_0$$

Capacitors

$$i(t) = C\frac{dv(t)}{dt}$$

$$v_C(t) = \frac{1}{C}\int_0^t i_C(t')dt' + v_0$$

Dynamic Circuits

Dynamic Circuits

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Consider the circuit in the figure below.

- Applying **KCL** yields

$$i_R(t) = i_C(t) = \frac{v_S(t) - v_C(t)}{R} = C\frac{dv_C(t)}{dt}$$

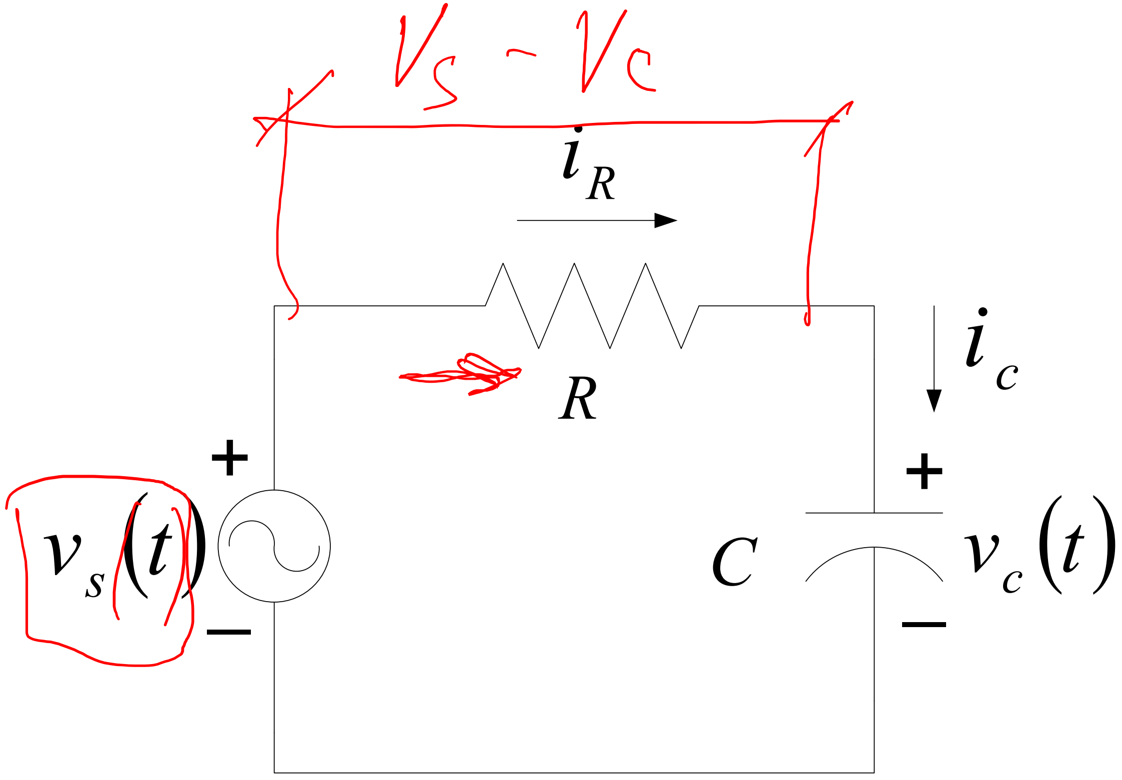
1 $$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_S(t)$$

- Applying **KVL** yields

$$-v_S(t) + v_R(t) + v_C(t) = 0$$

$$-v_S(t) + Ri_C(t) + \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\,d\tau = 0$$

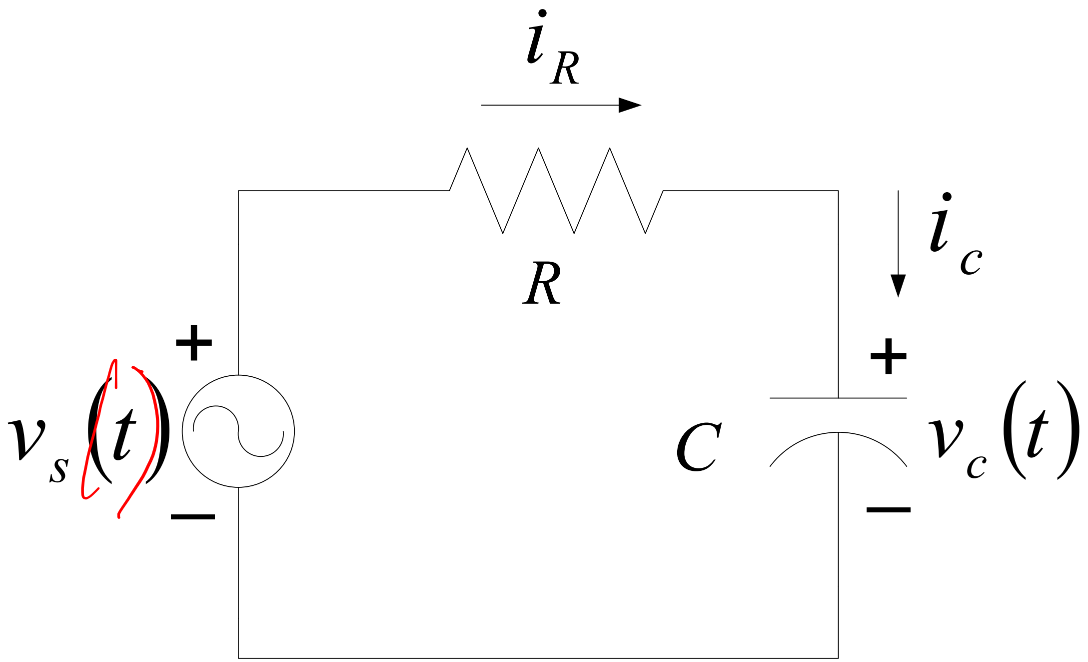
2 $$\frac{di_C(t)}{dt} + \frac{1}{RC}i_C(t) = \frac{1}{R}\frac{dv_S(t)}{dt}$$



Forced Response (1)

Consider the same circuit, where now the source produces a **sinusoidal voltage**:

- In sinusoidally excited linear circuit, all branch voltages and currents are **sinusoids at the same frequency** as the excitation signal.
- The **amplitudes** of these voltages and currents are a **scaled version** of the excitation amplitude.
- The voltages and currents may be **shifted in phase** with respect to the excitation signal.



$$v_s(t) = V\cos\omega t$$

$$v_R = V_R \cos(\omega t + \phi_R)$$

Forced Response (2)

Now, let us return to one of the original differential equations we obtained above (for example, the one derived with the KCL method):

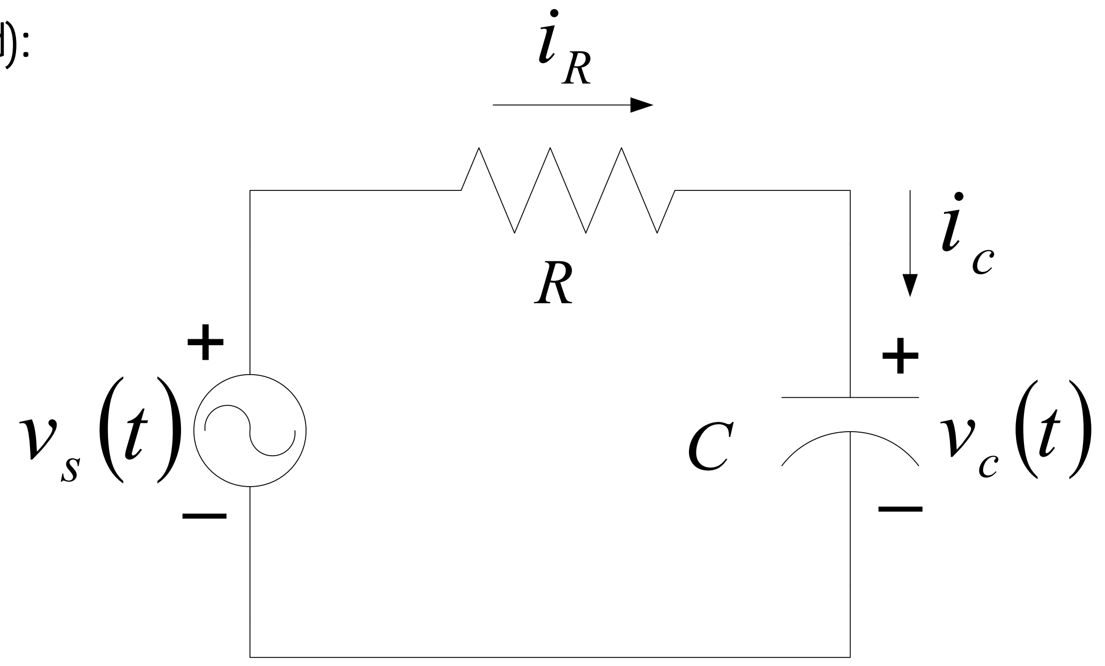
$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}V\cos\omega t$$

Since the output voltage (v_C) waveform can be of

- **different magnitude and shifted by phase**, but
- will have the **same frequency**,

we can rewrite the equation above as follows:

$$v_C(t) = C\cos(\omega t + \varphi) = A\sin\omega t + B\cos\omega t$$



$$v_s(t) = V\cos\omega t$$

$V_c = V_c \cos(\omega t + \varphi_c)$

Forced Response (3)

$$\frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = \frac{1}{RC} V \cos \omega\, t$$

$$v_C(t) = A \sin \omega\, t + B \cos \omega\, t$$

Substituting one equation into the other yields

$$A\omega \cos \omega\, t - B\omega \sin \omega\, t + \frac{1}{RC}(A \sin \omega\, t + B \cos \omega\, t) = \frac{1}{RC} V \cos \omega\, t$$

If the coefficients of like terms are grouped, the following equation is obtained:

$$\left(\frac{A}{RC} - B\omega\right) \sin \omega\, t + \left(A\omega + \frac{B}{RC} - \frac{V}{RC}\right) \cos \omega\, t = 0$$

Forced Response (4)

$$v_C(t) = A\sin\omega t + B\cos\omega t$$

$$\left(\frac{A}{RC} - B\omega\right)\sin\omega t + \left(A\omega + \frac{B}{RC} - \frac{V}{RC}\right)\cos\omega t = 0$$

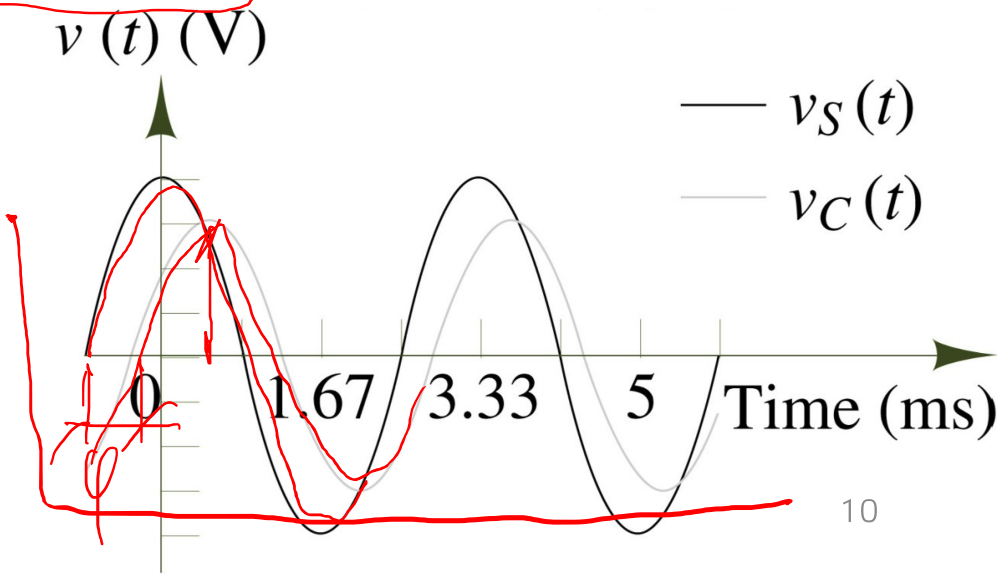
0 0

The coefficients of sine and cosine components must both be identically zero in order for equation above to hold. Thus,

$$A = \frac{V\omega RC}{1+\omega^2(RC)^2}, \qquad B = \frac{V}{1+\omega^2(RC)^2}$$

- Hence, the solution is

$$v_C(t) = \frac{V\omega RC}{1+\omega^2(RC)^2}\sin\omega t + \frac{V}{1+\omega^2(RC)^2}\cos\omega t$$



Forced Response: Conclusion

These observations indicate that **three parameters uniquely define a sinusoid**:

- *frequency,*
- *amplitude,* and
- *phase*.

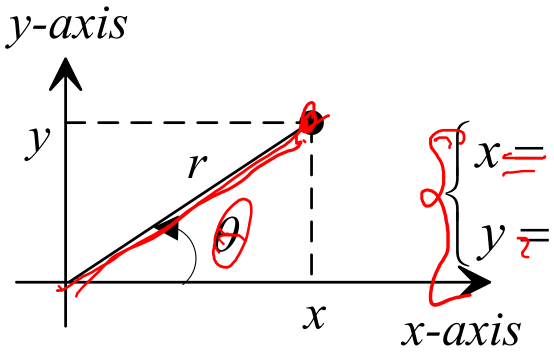
But if this is the case, is it necessary to carry the “excess luggage,” that is, the sinusoidal functions? Might it be possible to simply keep track of the three parameters just mentioned?

Fortunately, the answers to these two questions are **no** and **yes**, respectively, as discussed in the next part of today’s lecture.

Phasors

Cartesian vs Polar Coordinates

As you might recall, a position of a point on a plane can be defined by either **cartesian and polar coordinates** interchangeably as



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) + 2n\pi \quad (n = 0, \pm 1, \cdots) \end{cases}$$

- Assuming that the coordinate along the x-axis is a real part of a complex number $(z = x + jy)$ while that along the y-axis is its imaginary counterpart, one can write the following equation:

$$z = x + jy = r(\cos\theta + j\sin\theta)$$

$j = \sqrt{-1}$

Euler's Identity (1)

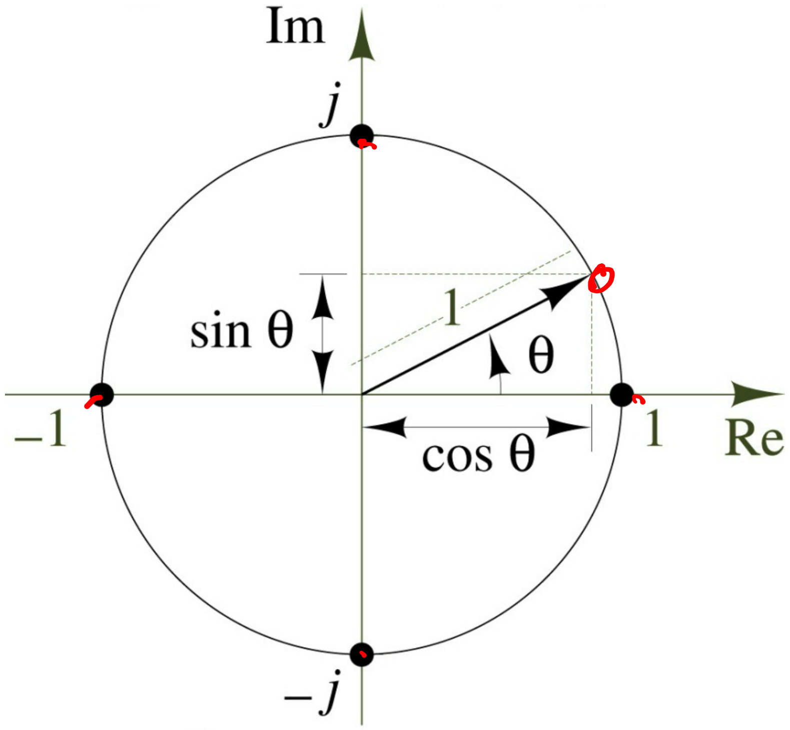
Named after the Swiss mathematician Leonhard Euler, **Euler's identity** forms the basis of **phasor notation**. Simply stated,

- the identity defines the complex exponential $e^{j\theta}$ as a point in the complex plane, which may be represented by real and imaginary components:

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}, \qquad \boxed{\left|e^{j\theta}\right| = |\cos\theta + j\sin\theta| = 1}$$

$$\cos\theta = 1 + \frac{x^2}{2!} \cdot - \qquad \sqrt{\cos^2\theta + \sin^2\theta}$$

$$\sin = x + \frac{x^3}{3!} + \ldots$$



$e^{j\theta} = \cos\theta + j\sin\theta$

Euler's Identity (2)

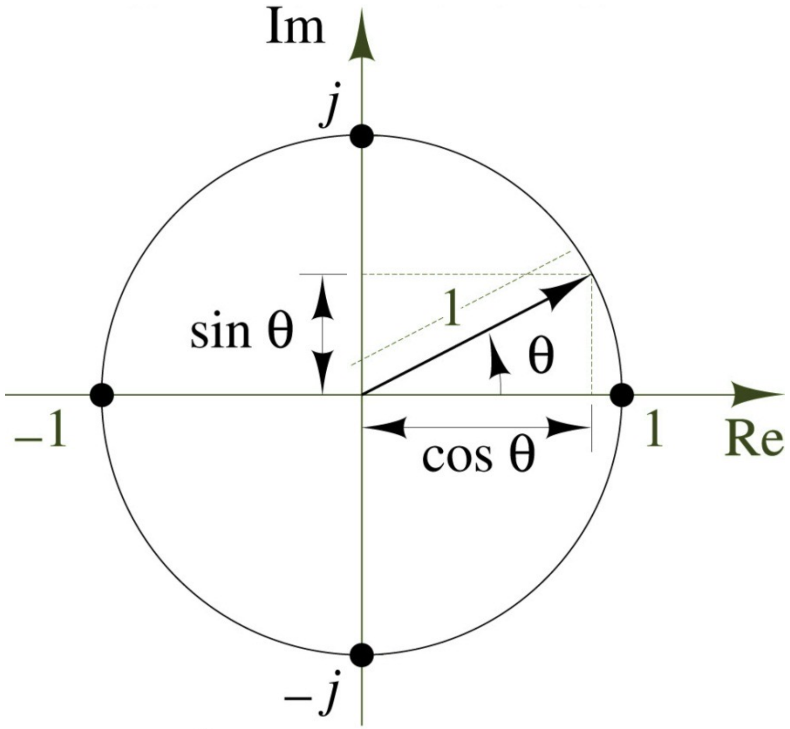
Writing Euler's identity corresponds to **equating the polar form** of a complex number to its **rectangular form**.

- For example, consider a vector of length A making an angle θ with the real axis.

The following equation illustrates the relationship between the rectangular and polar forms:

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta = A\angle\theta$$

In effect, Euler's identity is simply a trigonometric relationship in the complex plane.



$e^{j\theta} = \cos\theta + j\sin\theta$

Phasors (1)

To see how complex numbers can be used to represent sinusoidal signals, let us rewrite the expression for a **generalized sinusoid considering Euler’s equation**:

$$A\cos(\omega t+\theta)=\mathrm{Re}[\underline{Ae^{j(\omega t+\theta)}}]$$

$A\sin(\omega t+\theta) = \mathrm{Im}[Ae^{j(\omega t+\theta)}]$

- This is true because, upon expanding the right-hand side, one arrives to the following:

$$\mathrm{Re}[Ae^{j(\omega t+\theta)}]$$
$$=\underline{\mathrm{Re}[A\cos(\omega t+\theta)+jA\sin(\omega t+\theta))]}$$
$$=\underline{A\cos(\omega t+\theta)}$$

Phasors (2)

$$A\cos(\omega t + \theta) = \mathrm{Re}[Ae^{j(\omega t+\theta)}]$$

Thus,

- it is possible to **express a generalized sinusoid** as the **real part** of a complex vector whose argument, or angle, is given by $\omega t + \theta$ and
- whose length, or **magnitude**, **is equal** to the peak **amplitude** of the sinusoid.

The complex phasor corresponding to the sinusoidal signal $A\cos(\omega t + \theta)$ is therefore defined to be the complex number $Ae^{j\theta}$:

$Ae^{j\omega t}e^{j\theta}$

$$Ae^{j\theta} = A\angle\theta = \text{complex phasor notation for } A\cos(\omega t + \theta)$$

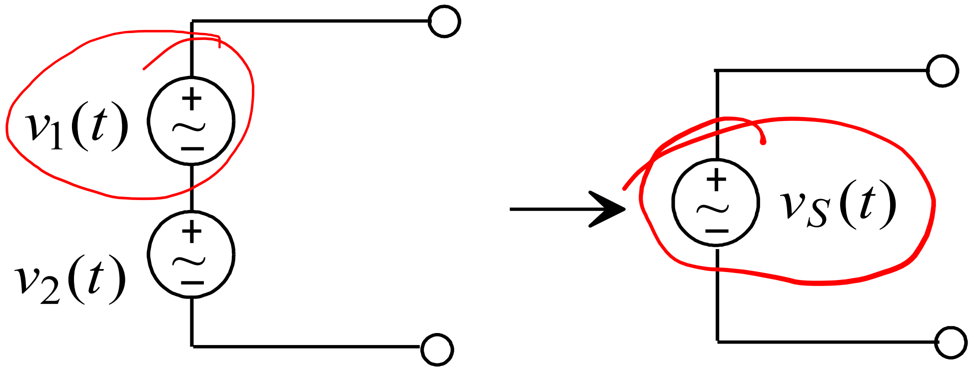
It is important to explicitly point out that **this is a *definition***.

- Phasor notation can be further simplified (**omitting the frequency**) using the following:

$$A\cos(\omega t + \theta) = \mathrm{Re}[Ae^{j(\omega t+\theta)}] = \mathrm{Re}[Ae^{j\theta}e^{j\omega t}]$$

Phasors: Example (1)

Compute the phasor voltage resulting from the series connection of two sinusoidal voltage sources.



$$\begin{cases} v_1(t) = 15\cos(377t + \frac{\pi}{4}) \\ v_2(t) = 15\cos(377t + \frac{\pi}{12}) \end{cases}$$

$= \varphi_1$

$= \varphi_2$

Phasors: Example (2)

Writing the given voltages in phasor notation yields:

$15\cos 45^\circ + j\,15\sin 45^\circ$

$$\begin{cases} v_1(t) = 15\cos(377t + \frac{\pi}{4}) \\ v_2(t) = 15\cos(377t + \frac{\pi}{12}) \end{cases} \Rightarrow \begin{cases} \mathbf{V}_1(j\omega) = 15e^{j45^\circ} = 10.61 + j10.61 \\ \mathbf{V}_2(j\omega) = 15e^{j15^\circ} = 14.49 + j3.88 \end{cases}$$

Now, adding the two voltages yields:

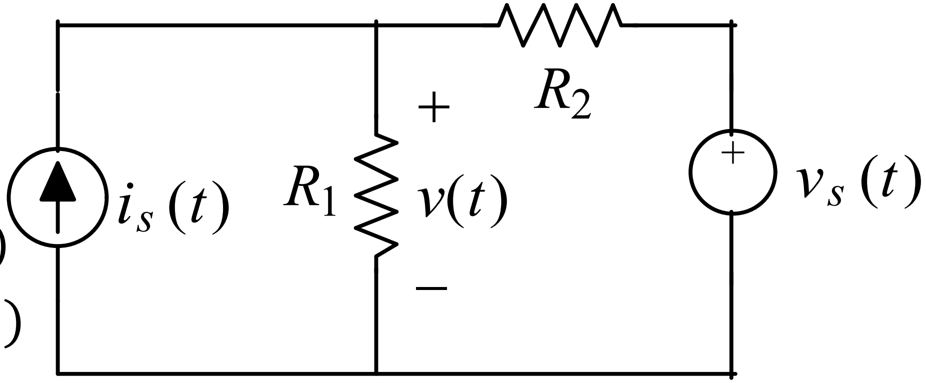
$$\mathbf{V}_s(j\omega) = \mathbf{V}_1(j\omega) + \mathbf{V}_2(j\omega) = 25.10 + j14.49 = 28.98e^{\frac{j\pi}{6}} = 28.98\angle\frac{\pi}{6} \quad \mathbf{V}$$

$A = \sqrt{25{,}10^2 + 14{,}49^2}$ $\quad \theta = \tan^{-1}\left(\frac{14{,}49}{25{,}10}\right)$

$$v_s(t) = 28.98\cos(377t + \frac{\pi}{6}) \quad \text{V}$$

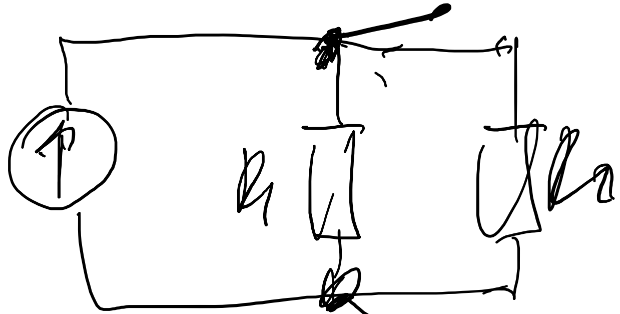
Superposition of AC Signals (1)

Assume we would like to analyze a circuit with two power signals of sinusoidal waveforms of different frequencies.

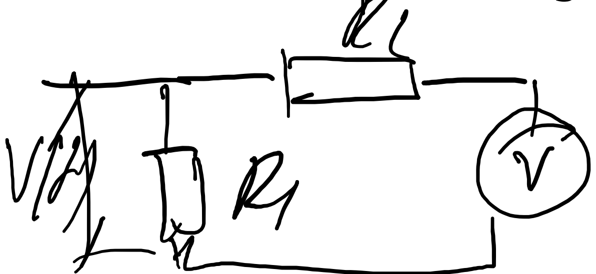


$$\begin{cases} i_S(t) = 0.5\cos(2\pi \cdot 100t) \\ v_S(t) = 20\cos(2\pi \cdot 1000t) \end{cases} \Rightarrow \begin{cases} I_S(j\omega_1) = 0.5e^{j0^\circ}(\omega_1 = 2\pi \cdot 100) \\ V_S(j\omega_2) = 20e^{j0^\circ}(\omega_2 = 2\pi \cdot 1000) \end{cases}$$

- Assume the resistance values are: $R_1 = 150\ \Omega, R_2 = 50\ \Omega$



Let us use the **principle of superposition** and study the effect of each source on the voltage $v(t)$ separately.



Superposition of AC Signals (2)

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Short-circuiting the voltage source yields:

$$v_S(t) = 0 \Rightarrow \mathbf{V}_{i_S}(j\omega_1) = \frac{R_2}{R_1 + R_2} R_1 \mathbf{I}_S(j\omega_1) = \frac{50 \cdot 150}{150 + 50} \cdot 0.5e^{j0^\circ} = 18.75e^{j0^\circ}$$

- **Open-circuiting** the current source yields:

$$i_s(t) = 0 \Rightarrow \mathbf{V}_{v_S}(j\omega_2) = \frac{R_1}{R_1 + R_2} \mathbf{V}_S(j\omega_2) = \frac{150}{150 + 50} \cdot 20e^{j0^\circ} = 15e^{j0^\circ}$$

- Adding the two,

$$\mathbf{V}(j\omega) = \mathbf{V}_{i_S}(j\omega_1) + \mathbf{V}_{v_S}(j\omega_2) \quad \Rightarrow v(t) = 18.75\cos(2\pi \cdot 100t) + 15.0\cos(2\pi \cdot 1000t)$$

$$\mathbf{V} \neq 18.75e^{j0^\circ} + 15.0e^{j0^\circ} = 33.75e^{j0^\circ}$$

Superposition of AC Signals (3)

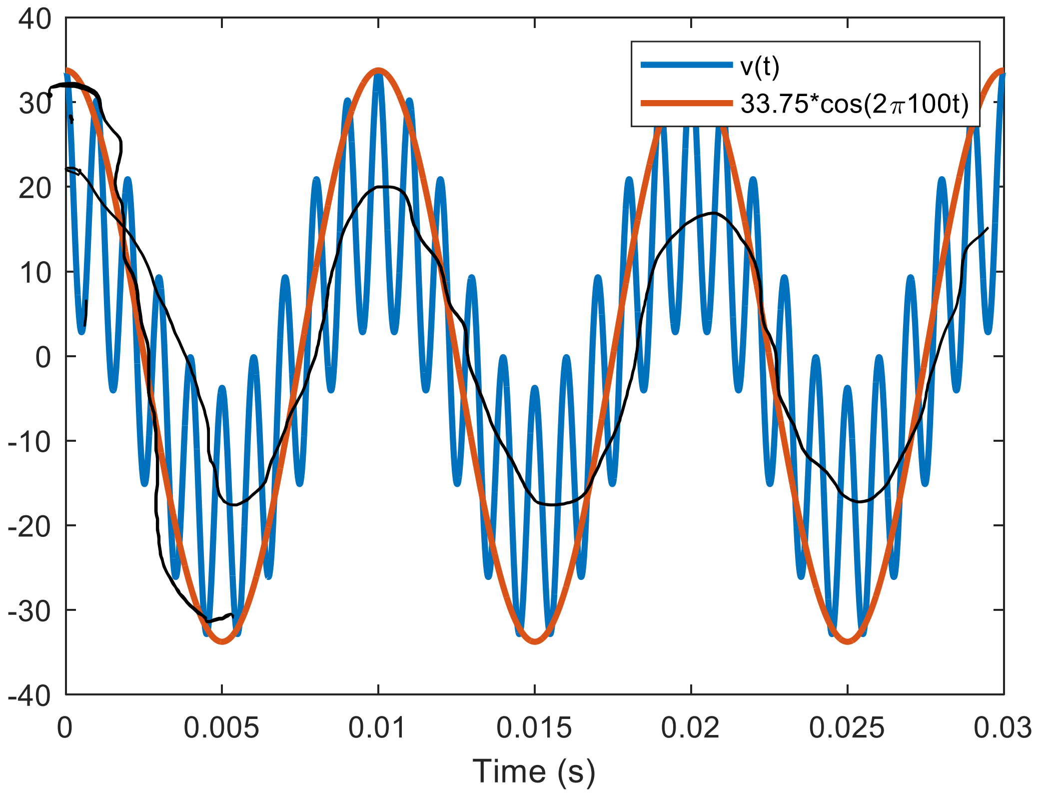
Thus, if you plot

$$v(t) = 18.75\cos(2\pi \cdot 100t) + 15.0\cos(2\pi \cdot 1000t)$$

and

$$v(t) = 33.75\cos(2\pi \times 100t)$$

you may see how actually **different these waveforms are**:



Thank you for your attention!

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