Laplace Transform and Transfer Functions

Control Theory, Tutorial 3

Spring 2022

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Previously we have seen a simple spring-damper system:

$$m\ddot{x} + \mu\dot{x} + cx = 0 \tag{1}$$

where m, μ, c are mass, damping coefficient and stiffness coefficient, and x is the position of the mass.

We can add external force u, to arrive at the following equation:

$$m\ddot{x} + \mu\dot{x} + cx = u \tag{2}$$

Mechanical systems with inputs (2)

We can use Laplace transform to change $\dot{x}$ to sx, and with that original eq. $m\ddot{x} + \mu\dot{x} + cx = u$ becomes:

$$ms^2x + \mu sx + cx = u \tag{3}$$

$$x = \frac{1}{ms^2 + \mu s + c}u \tag{4}$$

Thus we found the transfer function of the system:

$$W(s) = \frac{1}{ms^2 + \mu s + c} \tag{5}$$

Electric motor current eq.

Dynamics of the current in the windings of an electric motor can be described with the following eq.

$$L\frac{di}{dt} + Ri + C_e\omega = u \tag{6}$$

where L, R, C_e, ω, and u are inductance, resistance, back-EMF (back electromotive force) coefficient, angular velocity of the motor's shaft, and input voltage.

We can transfer this with Laplace transform:

$$Lsi + Ri + C_e\omega = u \tag{7}$$

And we find its transfer function representation:

$$i = \frac{1}{Ls + R}u \tag{8}$$

Notice that the TF from voltage to current is independent from $C_e\omega$.

Electric motor electro-mechanical eq. (1)

Full dynamics equations of a DS motor are

$$\begin{cases} L\frac{di}{dt} + Ri + C_e\omega = u \\ J\frac{d\omega}{dt} = C_\tau i \end{cases} \tag{9}$$

where C_τ and J are torque coef. and apparent inertial coef. of the rotor's shaft.

We can transfer this with Laplace transform:

$$\begin{cases} Lsi + Ri + C_e\omega = u \\ Js\omega = C_\tau i \end{cases} \tag{10}$$

We find transfer function from voltage to angular velocity.

$$\begin{cases} i = \frac{1}{Ls+R}u - \frac{C_e}{Ls+R}\omega \\ Js\omega = C_\tau i \end{cases} \tag{11}$$

$$Js\omega = \frac{C_\tau}{Ls+R}u - \frac{C_\tau C_e}{Ls+R}\omega \tag{12}$$

$$(Ls+R)Js\omega = C_\tau u - C_\tau C_e \omega \tag{13}$$

$$(JLs^2 + JRs + C_\tau C_e)\omega = C_\tau u \tag{14}$$

$$\omega = \frac{C_\tau}{JLs^2 + JRs + C_\tau C_e}u \tag{15}$$

Finally, the transfer function is:

$$W_{\omega,u}(s) = \frac{C_\tau}{JLs^2 + JRs + C_\tau C_e} \tag{16}$$

Other TFs for this system are:

$$W_{\omega,i}(s) = \frac{C_\tau}{Js} \tag{17}$$

$$W_{i,u}(s) = \frac{Js}{JLs^2 + JRs + C_\tau C_e} u \tag{18}$$

Observing eq. $m\ddot{y} + \mu_0\dot{y} + c_0 y = 0$ we can tell that it is stable if (sufficient but not necessary condition) $m > 0$, $\mu > 0$, and $c > 0$ - this follows from the physics of the system.

A more principled approach is to find eigenvalues of the linear system. We start by dividing the equation by m:

$$\ddot{y} + \mu\dot{y} + cy = 0 \tag{19}$$

where $\mu = \mu_0/m$ and $c = c_0/m$. Defining $x_1 = y$ and $x_2 = \dot{y}$, the system can be equivalently represented as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{20}$$

Stability (2)

With linear system $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -c & -\mu\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$, we need to find its eigenvalues. We know that there is a formula for eigenvalues based on trace and determinant:

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \tag{21}$$

where T is trace and D is the determinant.

In our case $T = -\mu$ and $D = c$, and eigenvalues are:

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2} \tag{22}$$

Stability (3)

Lets analyse eigenvalues $\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2}$. We can see that if $\mu \geq 0$ and $c \geq 0$, there are only two scenarios:

1. $\mu^2 - 4c \geq 0$, in which case $\sqrt{\mu^2 - 4c} \leq \mu$, the eigenvalues are purely real and negative.
2. $\mu^2 - 4c < 0$, in which case $\sqrt{\mu^2 - 4c}$ is a purely imaginary number, the eigenvalues are complex with negative real parts.

If $\mu \geq 0$ and $c = 0$, $\lambda_1 = -\mu$, $\lambda_2 = 0$, hence the system is marginally stable.

If $\mu \geq 0$ and $c < 0$, then $\sqrt{\mu^2 - 4c} \geq \mu$, and eigenvalues are purely real and one of them is positive, the system is unstable. If $\mu < 0$ and $c < 0$ at least one of the eigenvalues is still positive.

If $\mu < 0$ and $c \geq 0$, then again there are only two scenarios:

1. $\mu^2 - 4c \geq 0$, in which case $\sqrt{\mu^2 - 4c} \leq \mu$, the eigenvalues are purely real and positive.
2. $\mu^2 - 4c < 0$, in which case $\sqrt{\mu^2 - 4c}$ is a purely imaginary number, the eigenvalues are complex with positive real parts.

Definition

If $\mu \geq 0$ and $c \geq 0$ the system is stable, if $\mu < 0$ or $c < 0$ it is unstable.

Stability (5)

Going back to the eq. $\ddot{y} + \mu\dot{y} + cy = 0$ we can write characteristic eq. for it:

$$k^2 y + \mu k + c = 0 \tag{23}$$

Its roots are given by the formula:

$$k = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2} \tag{24}$$

As we can see, it is exactly the same as the determinant-trace formula.

Stability (6)

Now, lets consider the transfer function:

$$W(s) = \frac{1}{s^2 + \mu s + c} \tag{25}$$

Notice that its determinant is exactly the same as the characteristic polynomial. This illustrates the connection between the stability criteria for transfer function and other representations.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.