Control Theory

Tutorial 1
Ordinary Differential Equations
vs
State Space Equations

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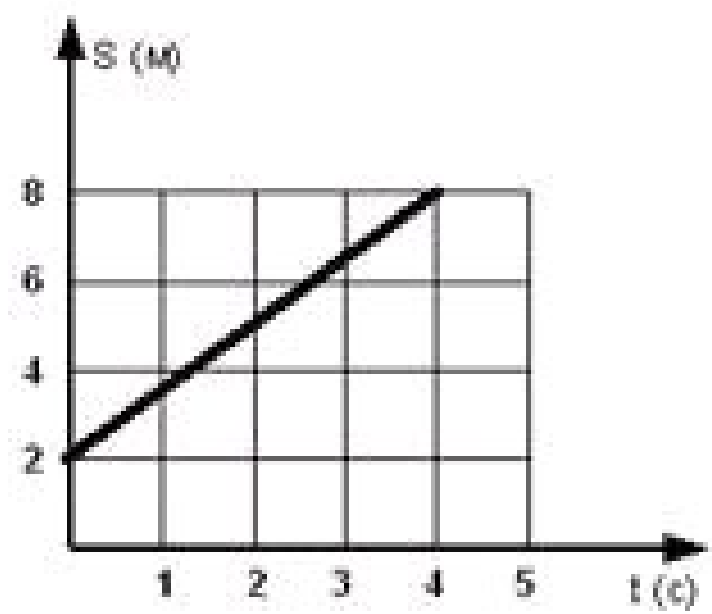
Particle motion

Motion in mechanics means the relative displacement of a body in space with respect to other bodies. This displacement takes place over time.

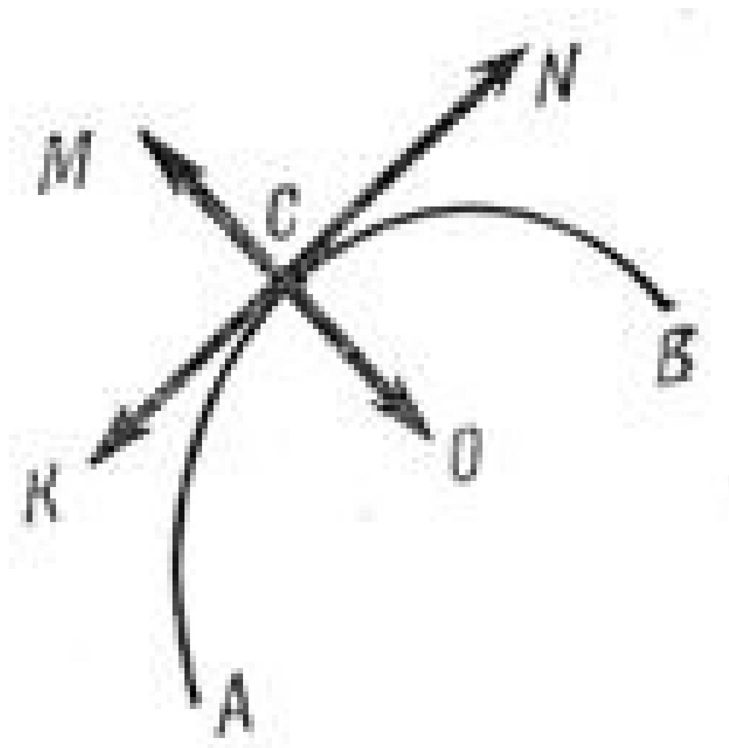
In order to locate a moving particle or body, we assume a coordinate system (reference system, frame of reference).

To describe the motion (the law of motion) of a given body (particle) kinematically means to specify the position of that body (particle) relative to a given frame of reference for any moment of time.

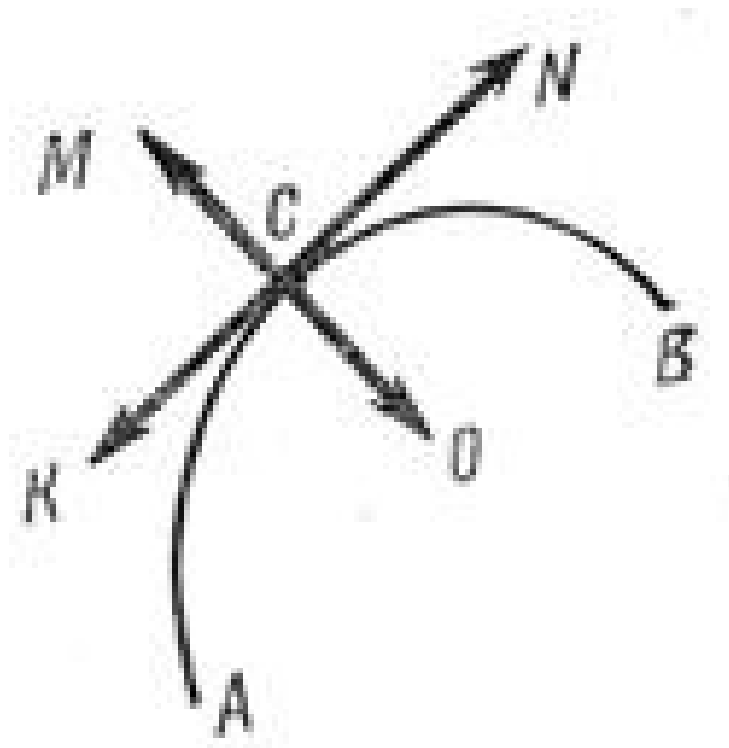
The figure below shows the graph of particle movement on a linear trajectory S(t). Determine the velocity of the particle at the time t=1 s

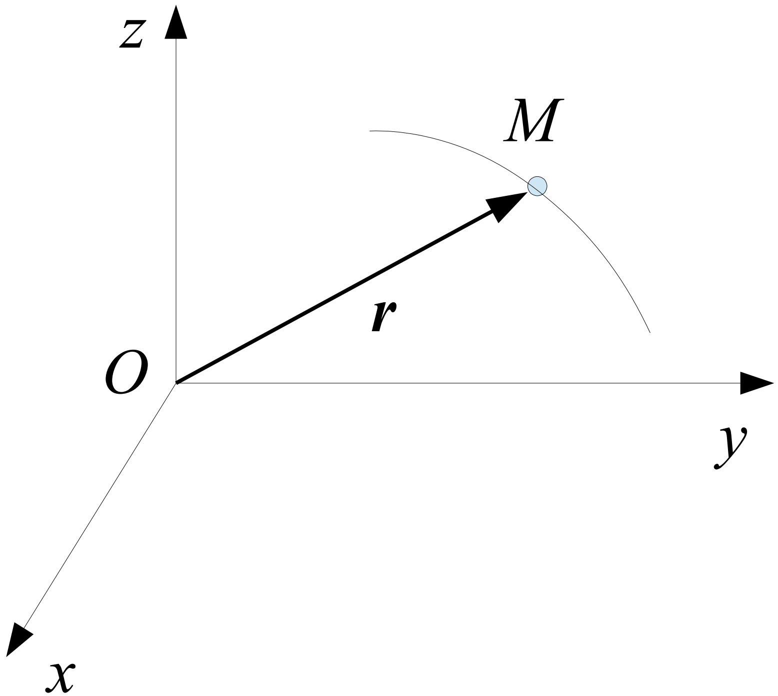


The particle is moving from A to B along the trajectory shown in the figure. Determine the direction of the velocity of point C.



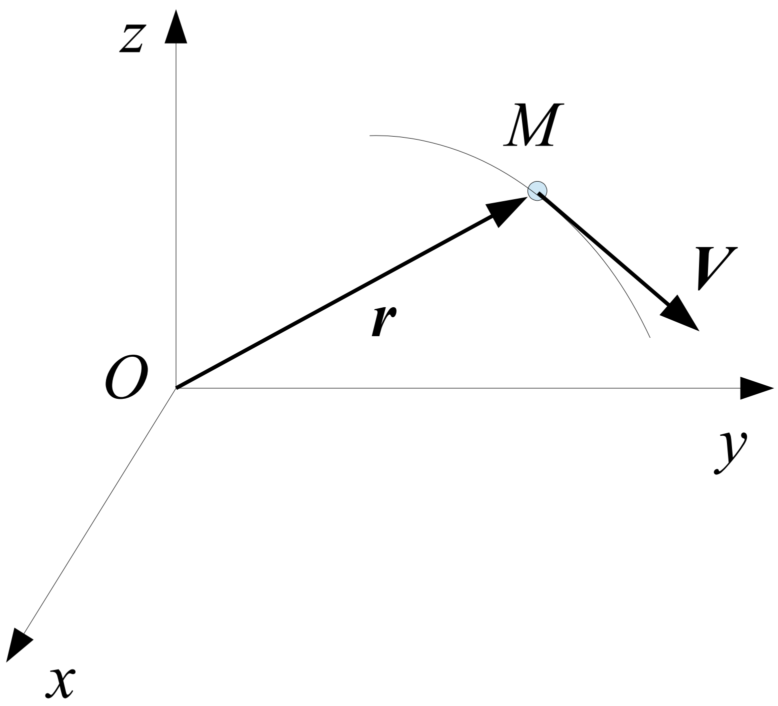
A particle is moving with uniform deceleration from A to B along the ACB trajectory. Determine the direction of the tangential component and normal component of the acceleration of point C.





Coordinates of a Particle

$$\boldsymbol{r} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

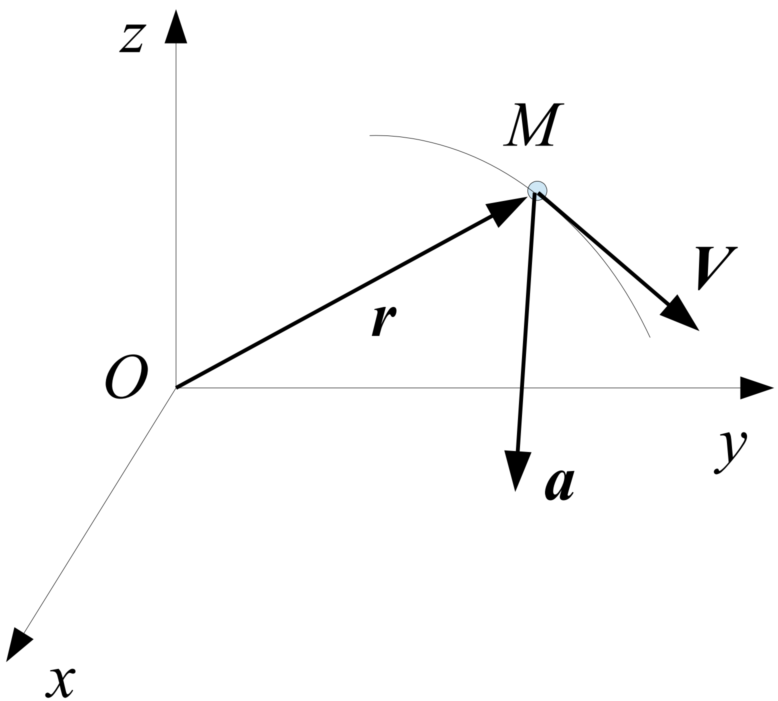


Coordinates of a Particle

$$\boldsymbol{r} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Velocity of a Particle

$$\boldsymbol{V} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}$$



Coordinates of a Particle

$$\boldsymbol{r}=\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Velocity of a Particle

$$\boldsymbol{V}=\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}$$

Acceleration of a Particle

$$\boldsymbol{a}=\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{bmatrix}$$

Main principles of dynamics

All principles of statics maintained valid in dynamics.

Three new principles are added:

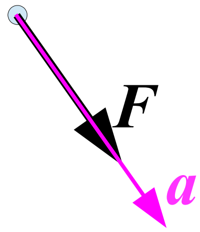
1) Newton's first law — there is a frame of reference, in which a particle either remains at rest or continues to move at a constant velocity, unless acted upon by a force;

2) Newton's second law - in an inertial frame of reference, the force acting on the particle imparts particle acceleration, which is directly proportional to the magnitude of the force and has the same direction;

3) Superposition principle - the action by two or more forces is the sum of the actions that are caused by each force individually

In an inertial frame of reference, the force acting on the particle imparts particle acceleration, which is directly proportional to the magnitude of the force and has the same direction.

$$\boldsymbol{a} \sim \boldsymbol{F}$$

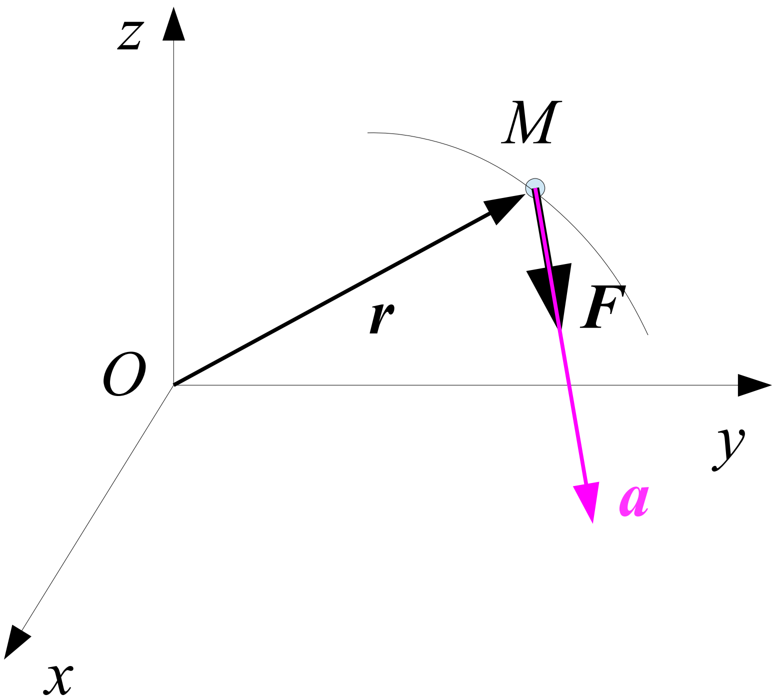


Coefficient of proportionality is called mass.

$$m\boldsymbol{a} = \boldsymbol{F}$$

$$m\boldsymbol{a} = \sum \boldsymbol{F}$$

The first problem of dynamics is to determine the forces acting on a body, knowing the equations of its motion



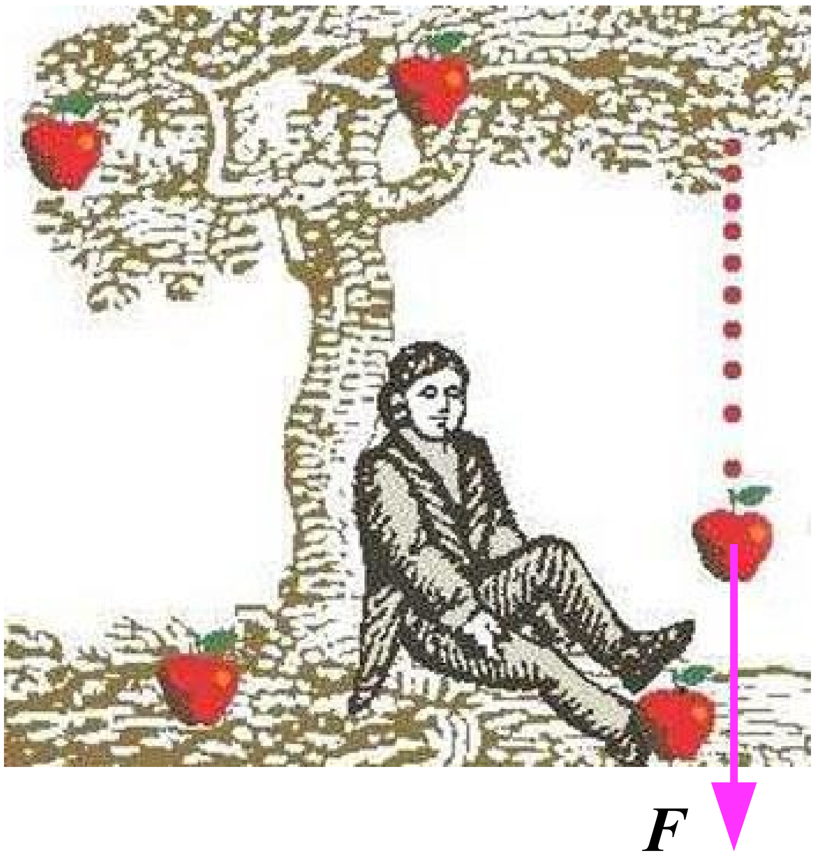
$$m\boldsymbol{a} = \boldsymbol{F}$$

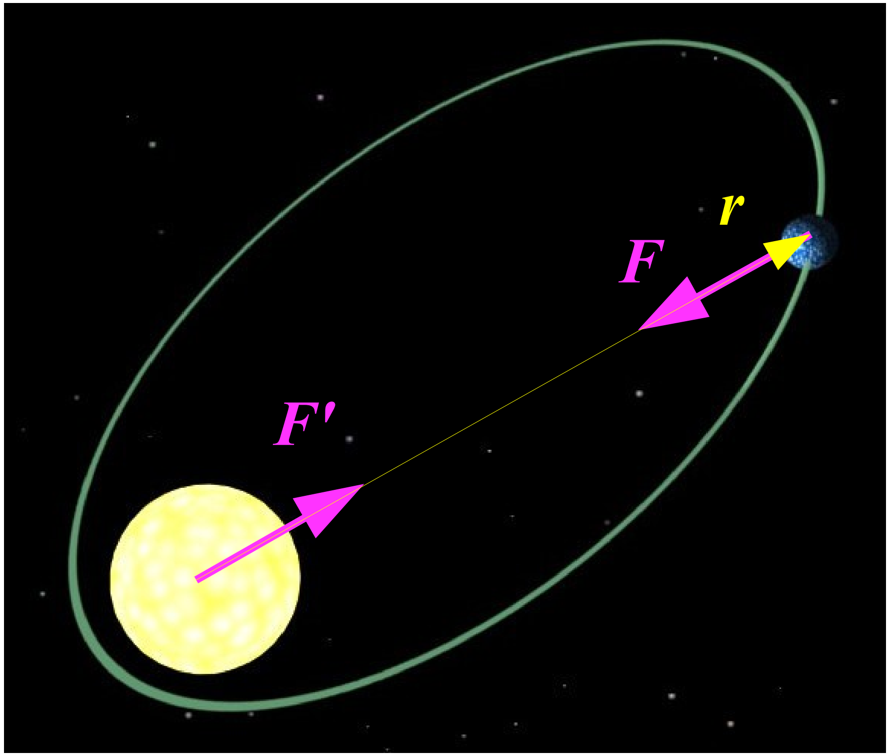
$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}$$

$$F_x = m\ddot{x}$$
$$F_y = m\ddot{y}$$
$$F_z = m\ddot{z}$$

Constant forces and position-dependent forces

Gravitational force

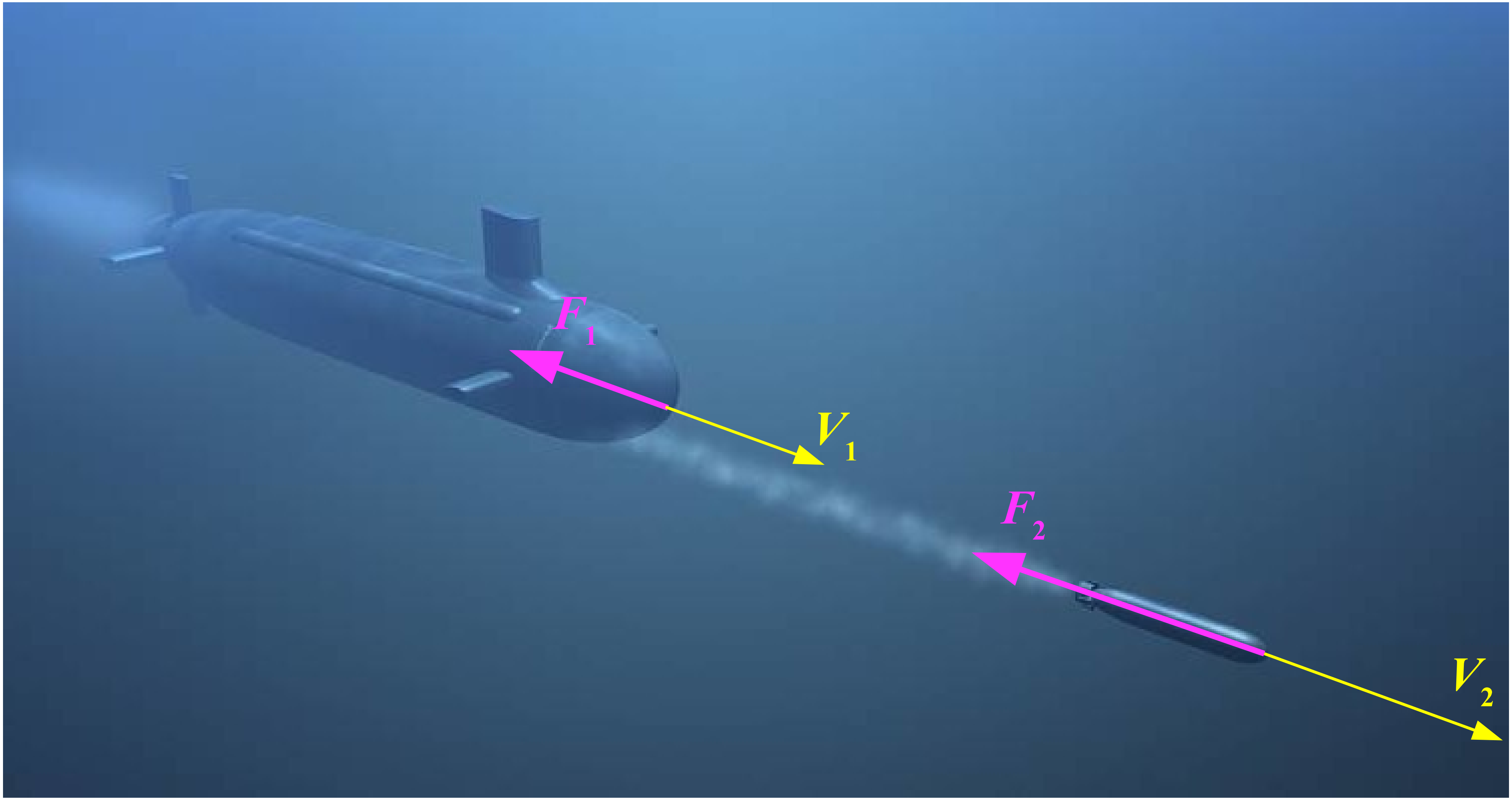




$$\boldsymbol{F} = \boldsymbol{const}$$

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}) = -G\frac{m_1 m_2}{r^3}\boldsymbol{r}$$

Resistance forces

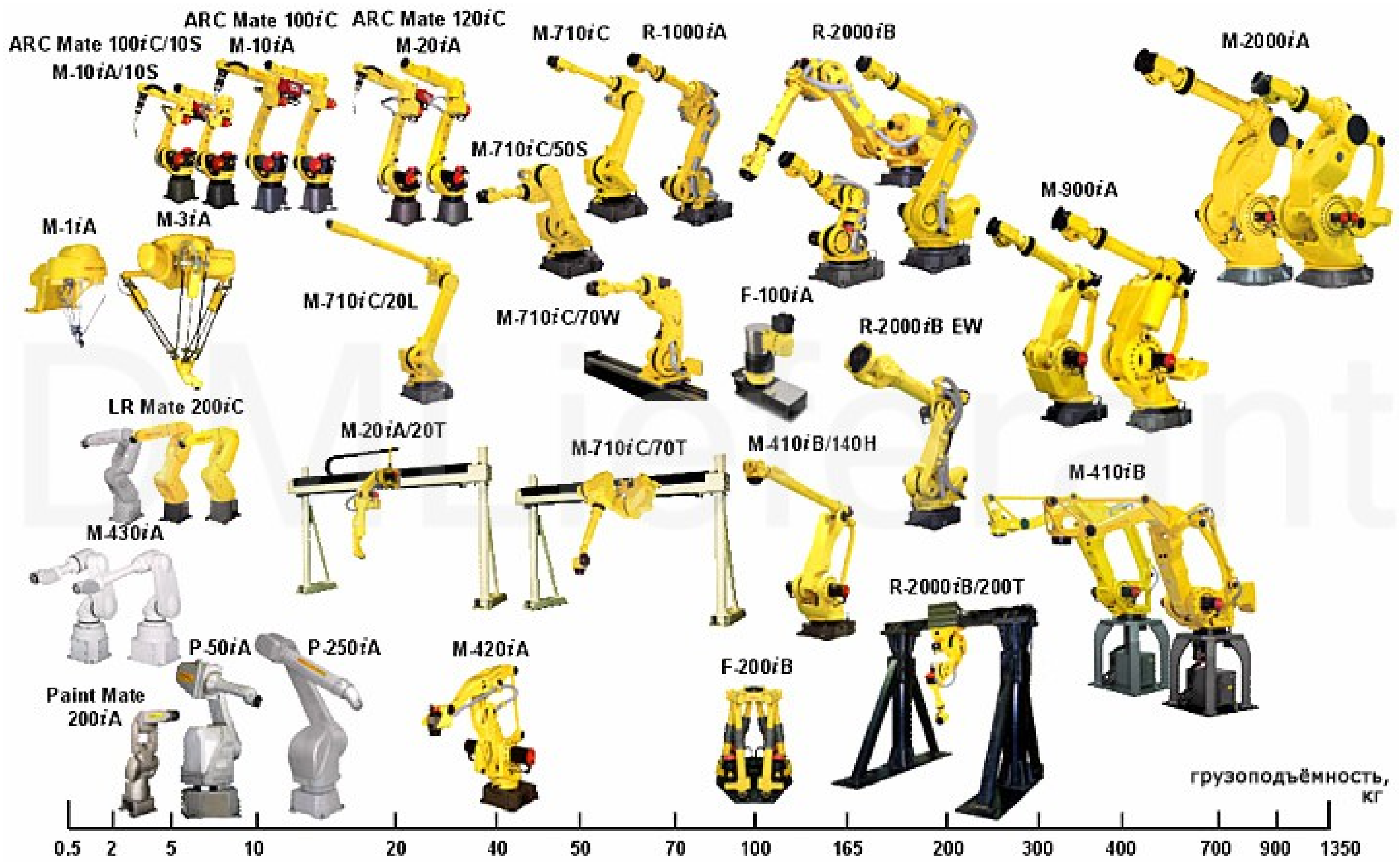


$$F_1 = F_1(V_1) = -\mu V_1$$

$$F_2 = F_2(V_2) = -k V_2 V_2$$

Controlled forces

Time dependent force, acceleration dependent force



$$F = F(t) \qquad F = F(\ddot{r}) \qquad F = F(t, r, \dot{r}, \ddot{r}, \dddot{r}, \ldots)$$

In mechanics we usually use only constant forces, time dependent forces,velocity dependent forces, position-dependent forces

$$\boldsymbol{F} = \boldsymbol{const}$$

$$\boldsymbol{F} = \boldsymbol{F}(t)$$

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$$

$$\boldsymbol{F} = \boldsymbol{F}(\dot{\boldsymbol{r}})$$

$$\boldsymbol{F} = \boldsymbol{F}(t, \boldsymbol{r}, \dot{\boldsymbol{r}})$$

$$F_x = F_x(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$$
$$F_y = F_y(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$$
$$F_z = F_z(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$$

Differential equations

$$\begin{cases} m\ddot{x} = F_x(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ m\ddot{y} = F_y(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) \\ m\ddot{z} = F_z(\boxed{t}, \boxed{x, y, z, \dot{x}, \dot{y}, \dot{z}}) \end{cases}$$

the state of the mechanical system in moment of time *t*

For example:

Initial conditions is the state of the mechanical system in moment of time t_0

$$t_0, \quad V_{x0}, \quad V_{y0}, \quad V_{z0}, \quad x_0, \quad y_0, \quad z_0$$

The second problem of dynamics

Differential equations

$$\begin{cases} m\ddot{x}=F_x(t,x,y,z,\dot{x},\dot{y},\dot{z}) \\ m\ddot{y}=F_y(t,x,y,z,\dot{x},\dot{y},\dot{z}) \\ m\ddot{z}=F_z(t,x,y,z,\dot{x},\dot{y},\dot{z}) \end{cases}$$

$$x=x(t,C_1,C_2,C_3,C_4,C_5,C_6)$$
$$y=y(t,C_1,C_2,C_3,C_4,C_5,C_6)$$
$$z=z(t,C_1,C_2,C_3,C_4,C_5,C_6)$$

$$V_x=\dot{x}=V_x(t,C_1,C_2,C_3,C_4,C_5,C_6)$$
$$V_y=\dot{y}=V_y(t,C_1,C_2,C_3,C_4,C_5,C_6)$$
$$V_z=\dot{z}=V_z(t,C_1,C_2,C_3,C_4,C_5,C_6)$$

Initial conditions

$$t_0,\quad V_{x0},\quad V_{y0},\quad V_{z0},\quad x_0,\quad y_0,\quad z_0$$

$$V_x\big|_{t=t_0}=V_{x0}$$
$$V_y\big|_{t=t_0}=V_{y0}$$
$$V_z\big|_{t=t_0}=V_{z0}$$
$$x\big|_{t=t_0}=x_0$$
$$y\big|_{t=t_0}=y_0$$
$$z\big|_{t=t_0}=z_0$$

$$C_1=C_1(t_0,V_{x0},V_{y0},V_{z0},x_0,y_0,z_0)$$
$$C_2=C_2(t_0,V_{x0},V_{y0},V_{z0},x_0,y_0,z_0)$$
$$C_3=C_3(t_0,V_{x0},V_{y0},V_{z0},x_0,y_0,z_0)$$
$$C_4=C_4(t_0,V_{x0},V_{y0},V_{z0},x_0,y_0,z_0)$$
$$C_5=C_5(t_0,V_{x0},V_{y0},V_{z0},x_0,y_0,z_0)$$
$$C_6=C_6(t_0,V_{x0},V_{y0},V_{z0},x_0,y_0,z_0)$$

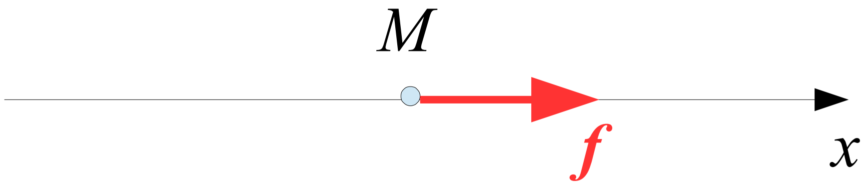
Examples

Example 1

A streetcar starts moving from the station. The resulting traction force is given by the expression f = a - b v, where v is the speed of the streetcar, a, b are given constants. The mass of the streetcar is m. All drag forces are already taken into account in the expression for the resulting traction force.

m = 1e+4 kg, a = 3.1e+4 N, b = 1.6e+3 Ns/m

It is required to make and solve differential equations.



$$m\ddot{x} = a - b\dot{x}$$

$$v = \frac{a}{b}\left(1 - e^{-bt/m}\right)$$

$$x = \frac{a}{b}t + \frac{am}{b^2}\left(e^{-bt/m} - 1\right)$$

$$\dot{v} = \frac{a}{m} - \frac{b}{m}v$$

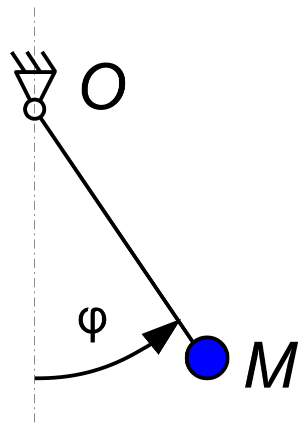
$$\dot{x} = v$$

Example 2

During the test, the streetcar engine power was gradually increased from zero by 1200 N every second. The friction force is constant and equal to 0.02 of the streetcar's weight. The mass of the streetcar is m = 1e+4 kg.

It is required to make and solve differential equations.

Example 3



A mathematical pendulum deviated from its vertical position by the angle φ_0 and began to oscillate.
The pendulum length is l = 1 m.

It is required to make and solve differential equations.

$$\ddot{\varphi} = -\frac{g}{l}\sin\varphi \qquad \ddot{\varphi} \approx -\frac{g}{l}\varphi$$