Physics 2. Electrical Engineering

Week 8.2 Impedance

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Objectives

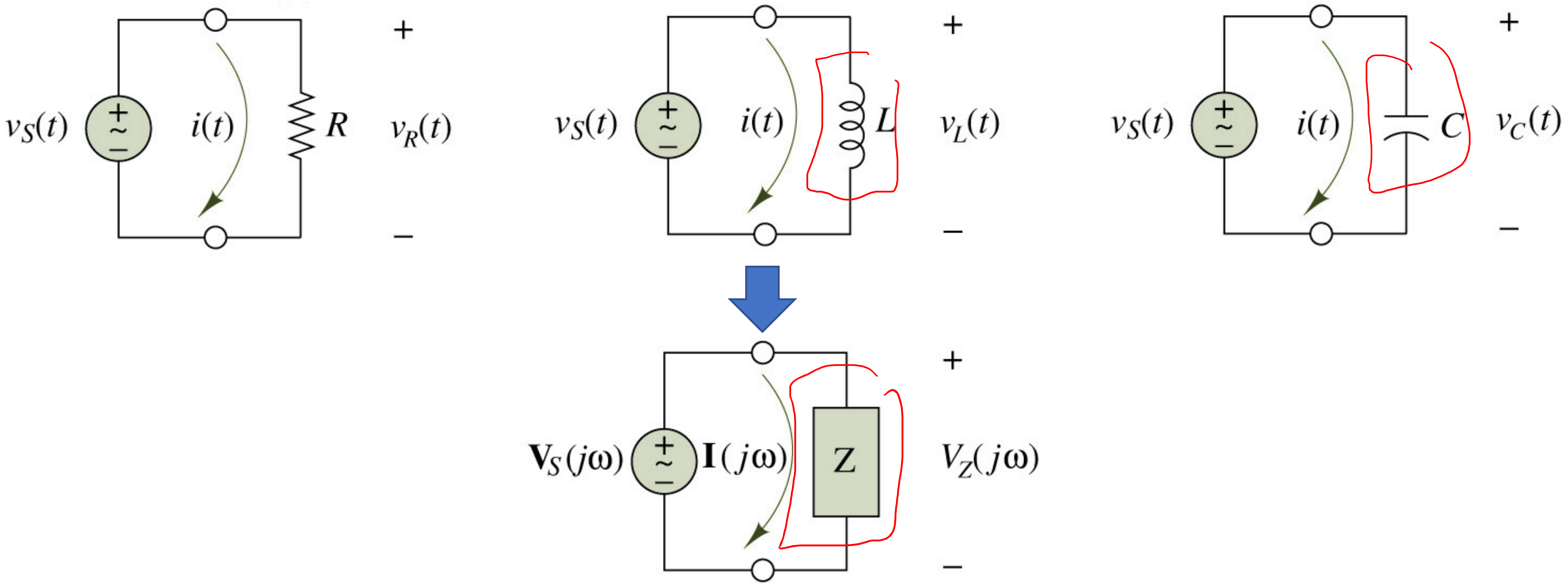
The main objectives of today's lecture are:

- Become familiar with the concept of **impedance**
- Practice finding **impedance of complex circuits**
- Apply phasors and impedance concepts to **analyze dynamic circuits**

Impedance

What is Impedance?

Impedance is a **frequency-dependent resistance**.



AC circuits in phasor/impedance form

Finding Impedance

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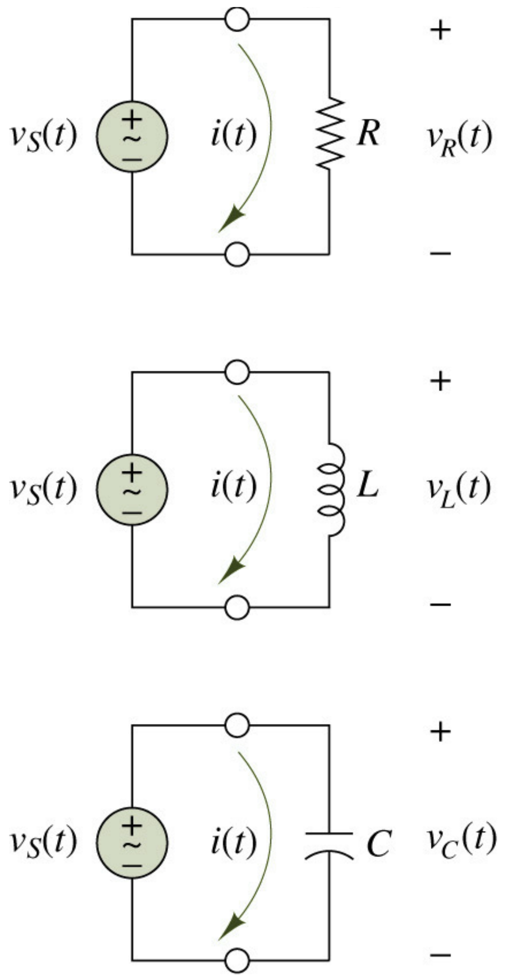
Let the source voltage be defined by

$$v_S(t) = A\cos\omega\, t = Ae^{j0^\circ}$$

Let us examine the frequency-dependent properties of the

- resistor,
- inductor, and
- capacitor.

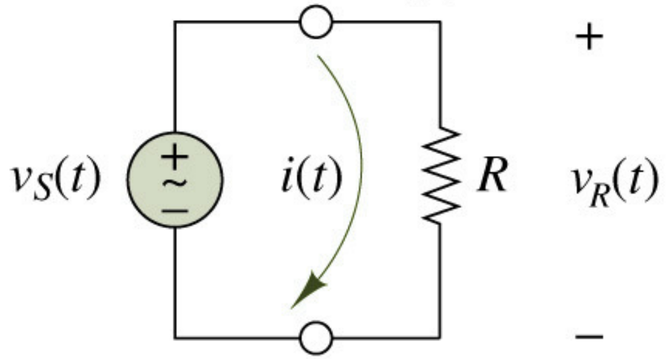
$$Z_R = \frac{V_S(j\omega)}{I_R(j\omega)}$$



AC circuits

Resistor Impedance

$v_S(t) = A\cos\omega t = Ae^{j0°}$



According to Ohm's law, the current flowing through the resistor is

$$i(t) = \frac{v_S(t)}{R} = \frac{A}{R}\cos\omega t$$

- In phasor notation,

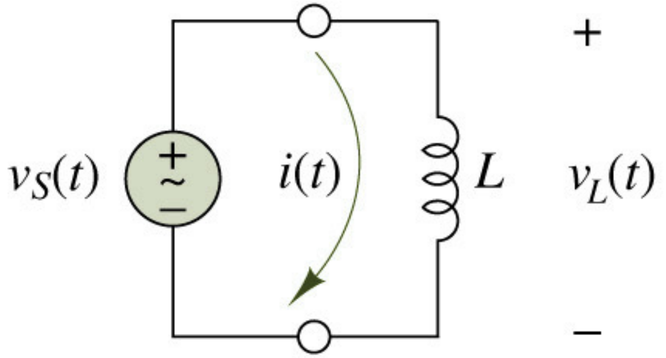
$$V_Z(j\omega) = A\angle 0$$

$$I_Z(j\omega) = \frac{A}{R}\angle 0$$

- Therefore, the impedance Z of a resistor is $Z_R(j\omega) = \frac{V_Z(j\omega)}{I_Z(j\omega)} = R$

Inductor Impedance (1)

$$v_S(t) = A\cos\omega t = Ae^{j0°}$$



Recall the current-voltage relationships for the ideal inductor:

$$i_L(t) = i(t) = \frac{1}{L}\int v_S(t')\,dt'$$

$$i_L(t) = \frac{1}{L}\int A\cos\omega t'\,dt'$$

$$= \frac{A}{\omega L}\sin\omega t = \frac{A}{\omega L}\cos\left(\omega t - \frac{\pi}{2}\right)$$

- In phasor notation,

$$V_Z(j\omega) = A\angle 0$$

$$I(j\omega) = \frac{A}{\omega L}\angle -\frac{\pi}{2}$$

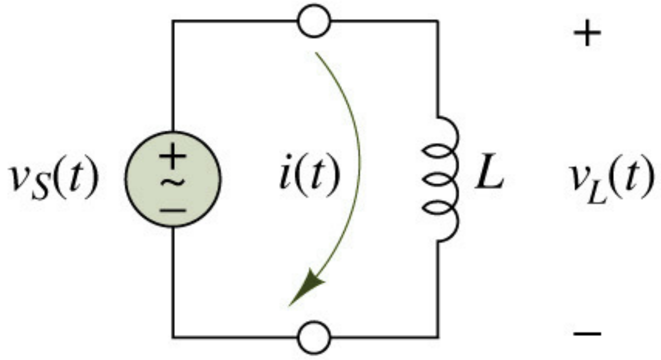
- Thus, the impedance of the inductor is

$$Z_L(j\omega) = \frac{V_Z(j\omega)}{I(j\omega)} = \omega L\angle\frac{\pi}{2} = j\omega L$$

$$\frac{Ae^{j0}}{\frac{A}{\omega L}e^{-j\frac{\pi}{2}}}$$

Inductor Impedance (2)

Hence, the inductor behaves as a complex frequency-dependent resistor, and the magnitude of this complex resistor ωL is proportional to the signal frequency ω.



$$Z_L(j\omega) = \omega L \angle \frac{\pi}{2} = j\omega L$$

This means that

- At **low** frequencies, an inductor acts as a **short circuit**,
- At **high** frequencies, an inductor acts as an **open circuit**.

Capacitor Impedance (1)

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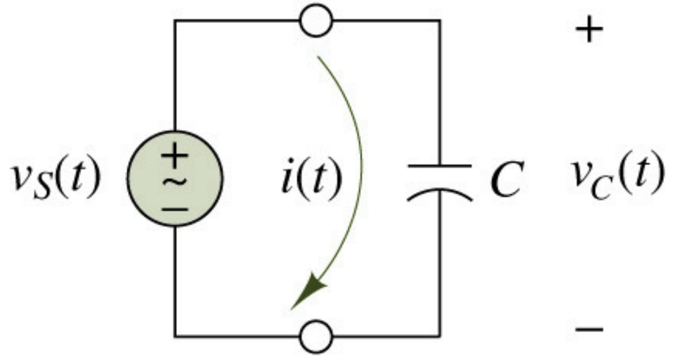
$$v_S(t) = A\cos\omega t = Ae^{j0°}$$



Recall the current-voltage relationships for the ideal capacitor:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$i_C(t) = C\frac{d}{dt}(A\cos\omega t)$$

$$= -CA\omega\sin\omega t$$

$$= \omega CA\cos\left(\omega t + \frac{\pi}{2}\right)$$

- In phasor notation,

$$V_Z(j\omega) = A\angle 0$$

$$I(j\omega) = \omega CA\angle\frac{\pi}{2}$$

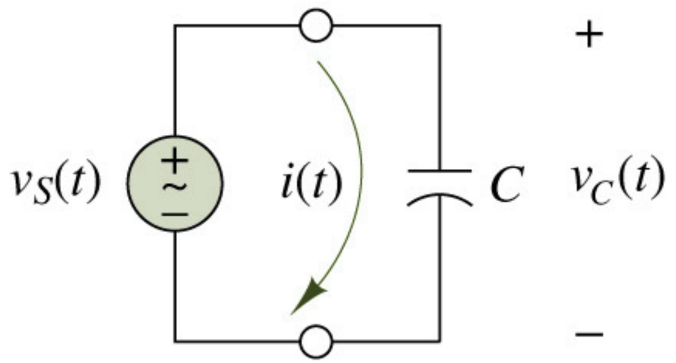
- Thus, the impedance of the capacitor is

$$Z_C(j\omega) = \frac{V_Z(j\omega)}{I(j\omega)} = \frac{1}{\omega C}\angle -\frac{\pi}{2}$$

$$= \frac{-j}{\omega C} = \frac{1}{j\omega C}$$

Capacitor Impedance (2)

Thus, the impedance of a capacitor is also a frequency-dependent complex quantity, with the **impedance of the capacitor** varying as an **inverse function of frequency**.



$$Z_C(j\omega) = \frac{-j}{\omega C} = \frac{1}{j\omega C}$$

Hence,

- At **low** frequencies, a capacitor acts as an **open circuit**,
- At **high** frequencies, a capacitor acts as a **short circuit**.

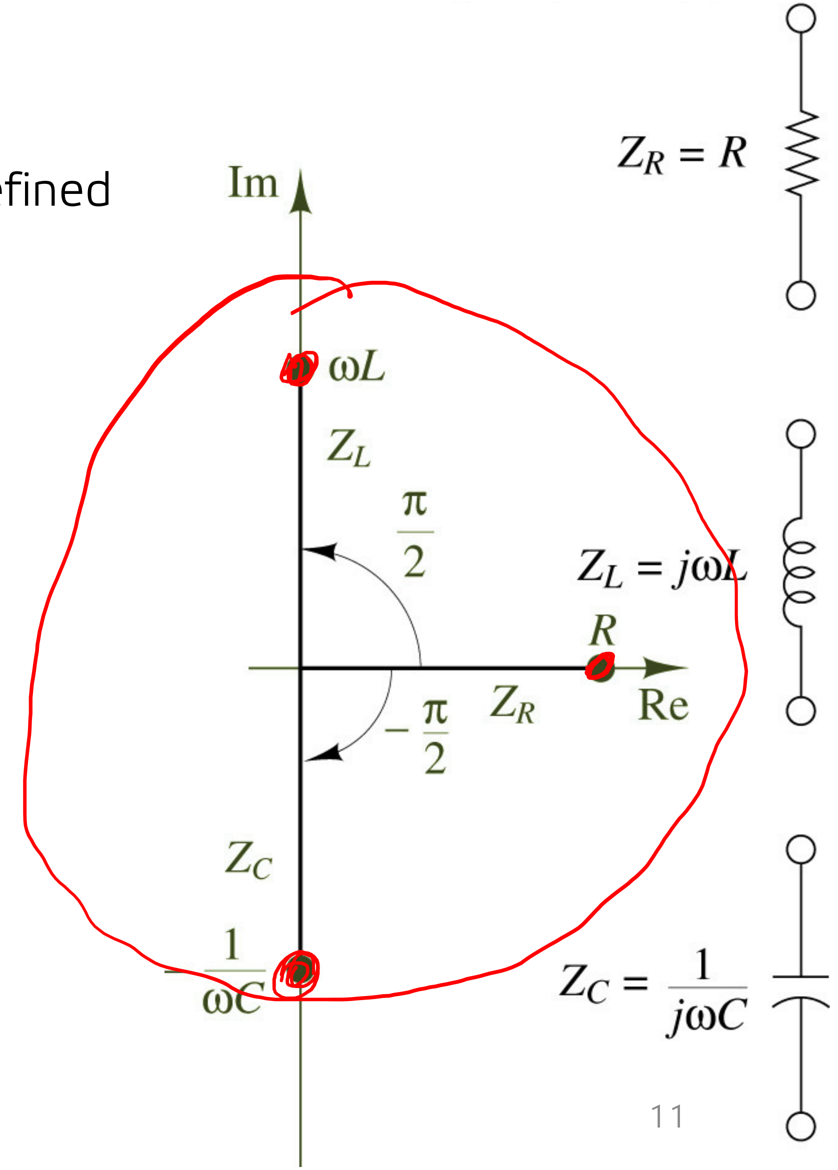
Impedance

In its most general form, the **impedance** of a circuit element is defined as the sum of a real part and an imaginary part:

$$Z(j\omega) = R(j\omega) + jX(j\omega)$$

where the components are

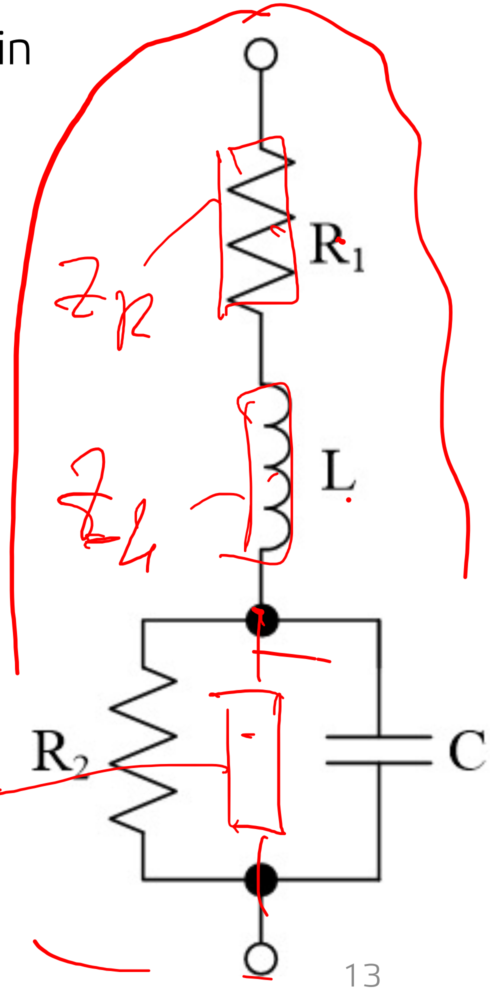
- R: **resistance** and
- X: **reactance**.



Impedance of Complex Circuits

Impedance of Complex Circuits (1)

We can find impedance of rather complex-looking circuits, which may come in extremely handy in their analysis later. Have a look at this example.



Problem: Find the **equivalent impedance** of the circuit shown here.

Known quantities: ω = 10^4 rad/s, R_1 = 100 Ω, L = 10 mH, R_2 = 50 Ω, C = 10 μF.

Just like we did when finding equivalent resistance, we can find Z_{RC} first as

$$Z_{RC} = Z_{R_2} || Z_{C_1} = R_2 \Bigg\| \frac{1}{j\omega C}$$

$$= \frac{R_2(1/j\omega C)}{R_2 + 1/j\omega C} = \frac{R_2}{1 + j\omega C R_2}$$

Z_{RC}

Impedance of Complex Circuits (2)

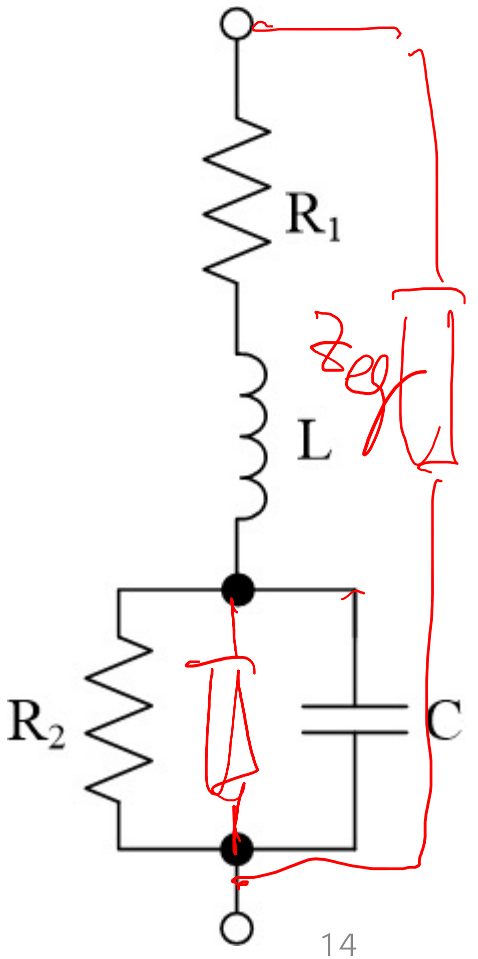
Computation yields $Z_{RC} = \dfrac{50}{1 + j5} = 1.92 - j9.62 = 9.81\angle(-1.3734)\ \Omega$

Next, we determine the equivalent impedance Z_{eq}:

$$Z_{eq} = R_1 + j\omega L + Z_{RC} = 100 + j10^4 \times 10^{-2} + 1.92 - j9.62$$

$$= 101.92 + j90.38 = 136.2\angle 0.723\ \Omega$$

Q: Is this impedance **inductive or capacitive**?

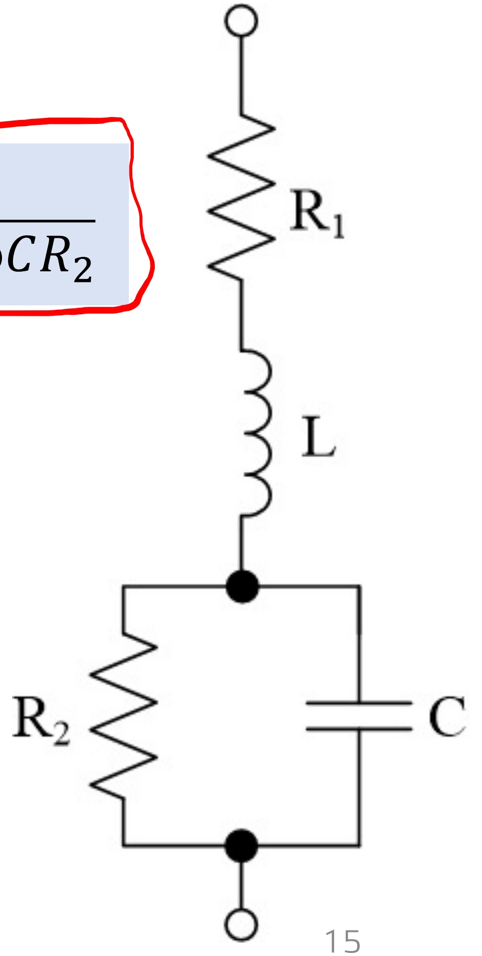


Impedance of Complex Circuits (3)

Let us repeat the previous calculations for

1. ω = 0 rad/s (DC voltage): $Z_{eq} = 150\ \Omega$
2. ω = 1000 rad/s: $Z_{eq} = 140 - j10\ \Omega$
3. ω = 2450 rad/s: $Z_{eq} = 120\ \Omega$

$$Z_{eq} = R_1 + j\omega L + \frac{R_2}{1 + j\omega C R_2}$$

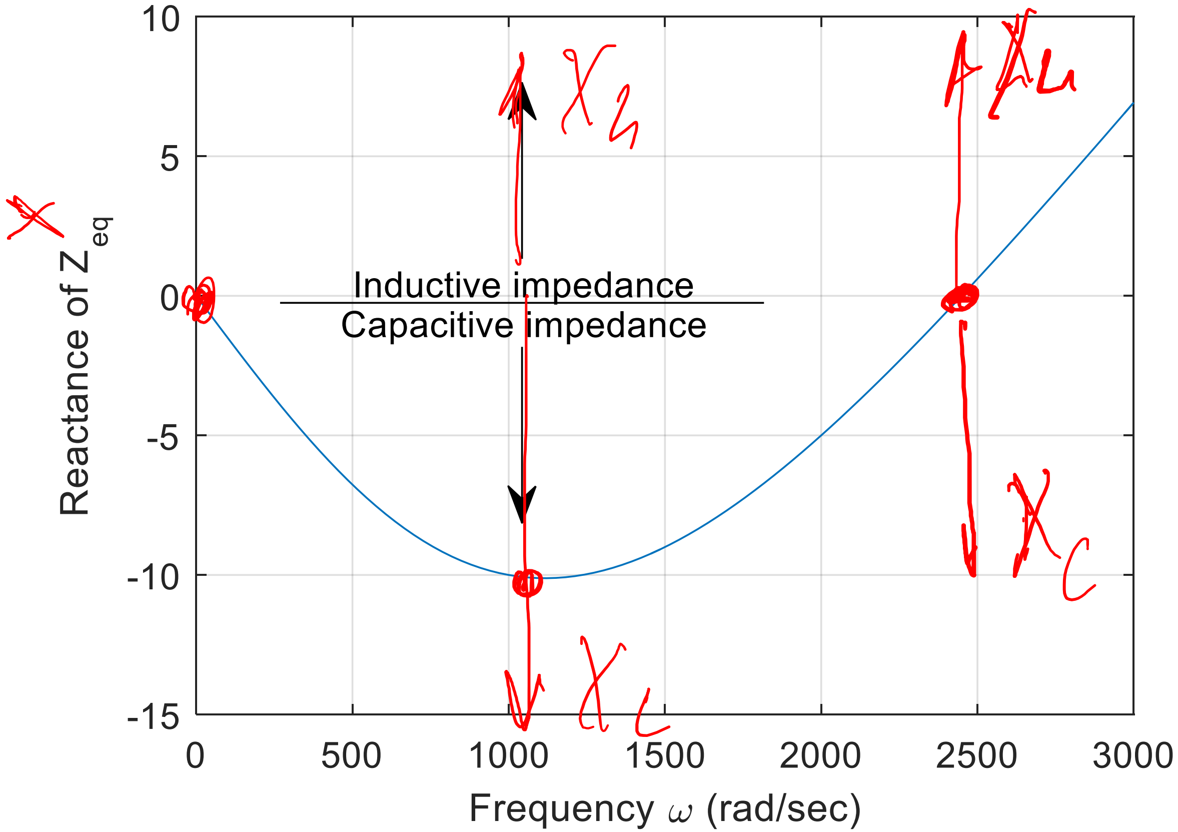


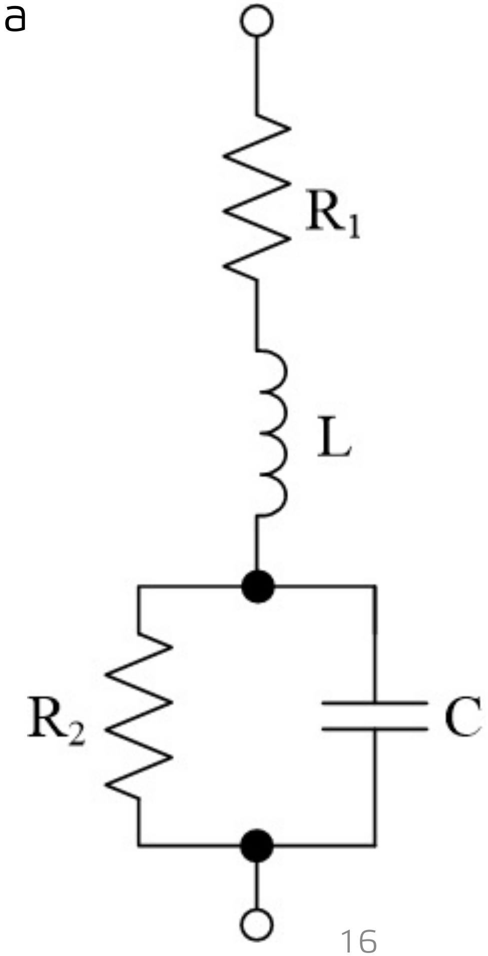
Q1: Are impedance values **inductive or capacitive**?

Q2: What is the **equivalent circuit** for the last case?

Impedance of Complex Circuits (4)

We can conduct a small investigation and plot the reactance of this circuit as a function of frequency as follows:





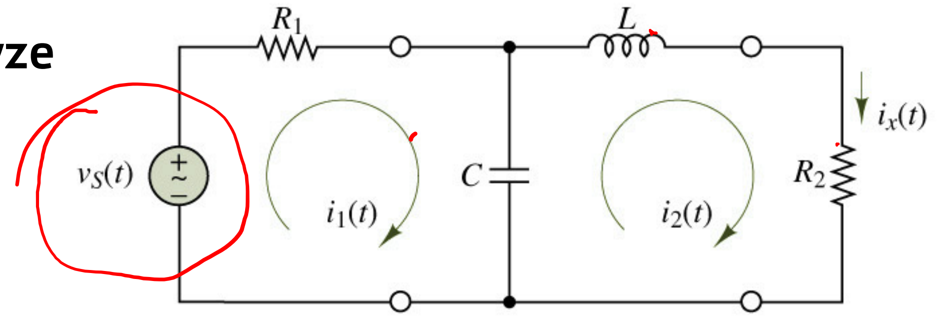
AC Circuit Analysis with Phasors and Impedance

AC Circuit Analysis

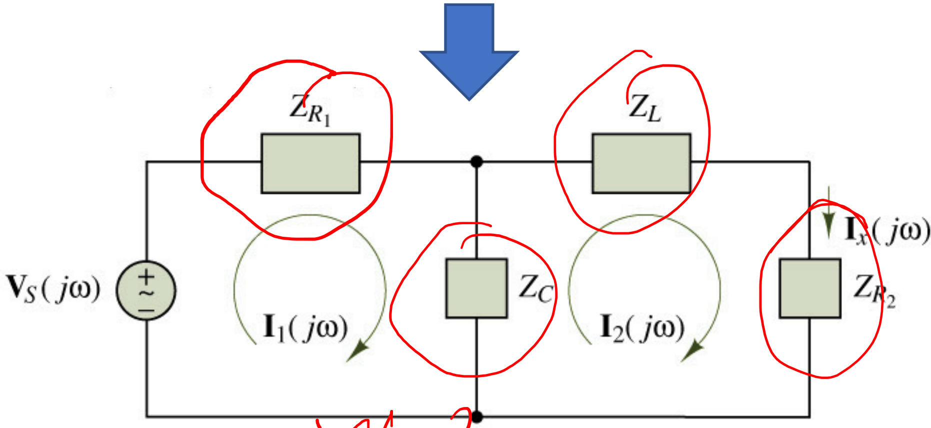
Just like we did so in case of DC circuits, we often need to **analyze AC circuits** with energy storage elements.

One way of doing so is as follows:

1. Represent a sine wave **power source by a phasor**, and each **circuit element by an impedance**.
2. Obtain **solution in the phasor form** by applying the circuit analysis method studied previously (KVL/KCL, node voltage method, network current method, etc.)
3. **Convert the solution** from phasor (frequency) domain into **time domain**.



A sample circuit for AC analysis

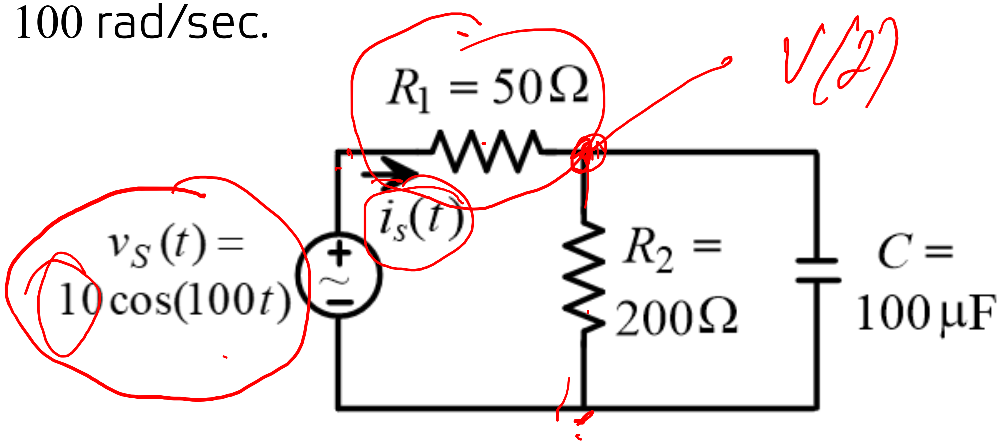


The same circuit in phasor form

$I_1 = I_0 e^{j\theta(\omega)}$ $I_1 = I(\omega) e^{j\theta(\omega)}$

AC Circuit Analysis: Exercise (1)

Given the circuit below, find the source current $i_S(t)$ at $\omega = 100$ rad/sec.



Using the Ohm's law, one can write

$$i_S(t) = \frac{v_S(t) - v(t)}{R_1}$$

Now, let us **convert the voltage source into phasor form** and **find all the impedance values**

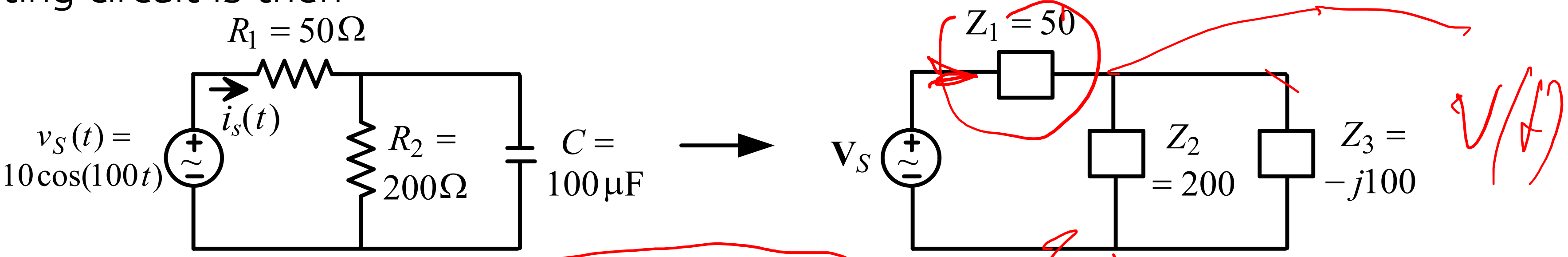
$$v_S(t) = 10\cos\omega\, t, \quad \omega = 377 \quad \text{rad/s}$$

$$V_S(j\omega) = 10\angle 0 \quad \text{V}$$

$$Z_{R_1} = R_1, \quad Z_{R_2} = R_2, \quad Z_C = \frac{1}{j\omega C}.$$

AC Circuit Analysis: Exercise (2)

The resulting circuit is then



Finally, using KCL, we write

$$\frac{V_S - V}{Z_{R_1}} = \frac{V}{Z_{R_2}\|Z_C}$$

Solving yields

$$\frac{V_S}{Z_{R_1}} = V\left(\frac{1}{Z_{R_2}\|Z_C} + \frac{1}{Z_{R_1}}\right) = V\left(\frac{1}{\frac{R_2(1/j\omega C)}{R_2 + (1/j\omega C)}} + \frac{1}{R_1}\right)$$

$$= V\left(\frac{j\omega C R_2 + 1}{R_2} + \frac{1}{R_1}\right) = V\left(\frac{j\omega C R_2 R_1 + R_1 + R_2}{R_1 R_2}\right)$$

AC Circuit Analysis: Exercise (3)

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Substituting all the values yields

$$V = \left(\frac{j\omega C R_2 R_1 + R_1 + R_2}{R_1 R_2}\right)^{-1} \frac{V_S}{R_1} = \left(\frac{R_1 R_2}{j\omega C R_2 R_1 + R_1 + R_2}\right) \frac{V_S}{R_1}$$

$$= \frac{50 \times 200}{j377 \times 10^{-4} \times 50 \times 200 + 50 + 200} \cdot \frac{V_S}{50}$$

$$= 0.4221\angle(-0.9852) V_S = 4.421\angle(-0.9852).$$

Thus,

$$I_S = \frac{V_S - V}{Z_{R_1}} = \frac{10\angle 0 - 4.421\angle(-0.9852)}{50} = 0.1681\angle(0.4537)$$

Thank you for your attention!

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