Tutorial 4. Graph theory and electical circuits.

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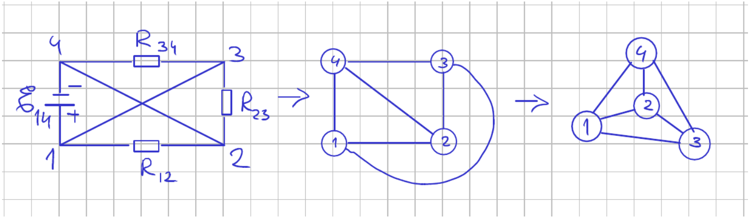
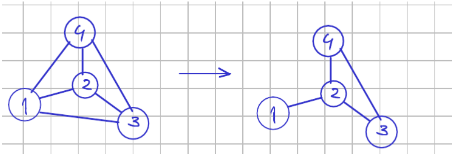
Example 1

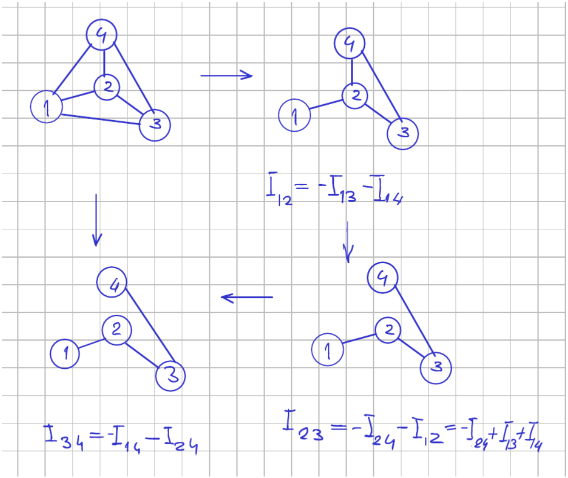


Example 1



$$I_{12} = -I_{13} - I_{14}$$

Example 1

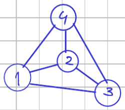


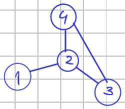
$I_{12} = -I_{13} - I_{14}$

$I_{23} = -I_{24} - I_{12} = -I_{24} + I_{13} + I_{14}$

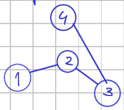
$I_{34} = -I_{14} - I_{24}$

Example 1

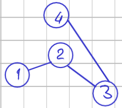




$$I_{12} = -I_{13} - I_{14}$$



$$I_{23} = -I_{24} - I_{12} = -I_{24} + I_{13} + I_{14}$$



$$I_{34} = I_{14} - I_{24}$$

$$I_{12} = -I_{13} - I_{14}$$

$$I_{23} = -I_{24} - I_{12} = -I_{24} + I_{13} + I_{14}$$

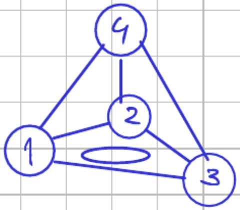
$$I_{34} = I_{14} - I_{24}$$

$$I_{23} + I_{34} = I_{13}$$

$$I_{12} + I_{23} = -I_{24}$$

$$I_{14} = -I_{34} - I_{24}$$

Example 1



$$-R_{23} I_{23} - R_{12} I_{12} = 0;$$

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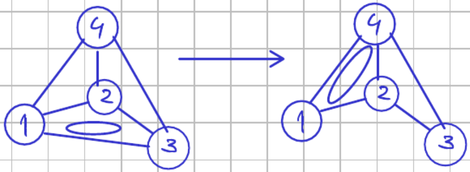
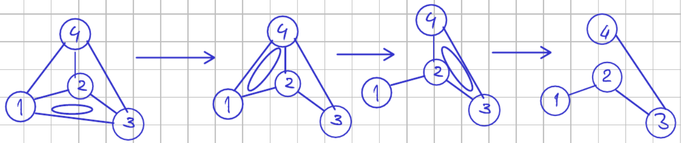


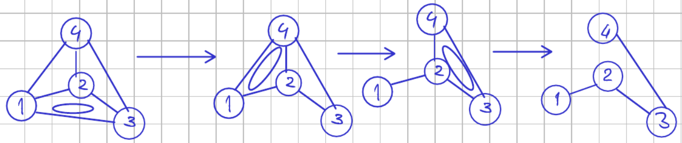
$$-R_{23}I_{23}-R_{12}I_{12}=0;\quad -\mathcal{E}_{14}+R_{12}I_{12}=0;$$

Example 1



$$-R_{23}I_{23}-R_{12}I_{12}=0;\quad -\mathcal{E}_{14}+R_{12}I_{12}=0;\quad R_{23}I_{23}+R_{34}I_{34}=0$$

Example 1

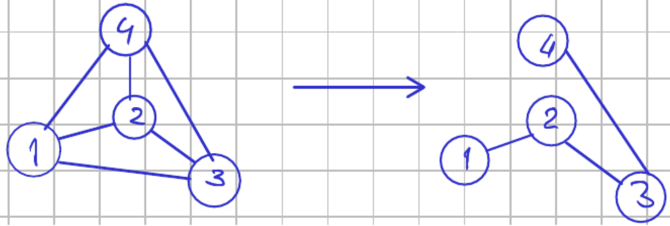


$$-R_{23}I_{23}-R_{12}I_{12}=0;\quad -\mathcal{E}_{14}+R_{12}I_{12}=0;\quad R_{23}I_{23}+R_{34}I_{34}=0$$

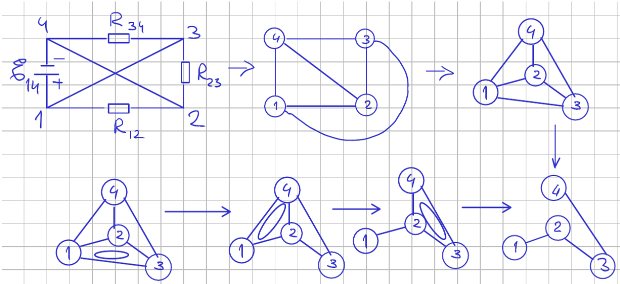
Example 1

$$I_{23}+I_{34}=I_{13}$$
$$I_{12}+I_{23}=-I_{24}$$
$$I_{14}=-I_{34}-I_{24}$$

$$R_{12}I_{12}=\mathcal{E}_{14};$$
$$-R_{23}I_{23}=\mathcal{E}_{14}$$
$$R_{34}I_{34}=\mathcal{E}_{14}$$



Example 1



$$I_{12} = -I_{13} - I_{14}$$

$$I_{23} = -I_{24} - I_{12} = -I_{24} + I_{13} + I_{14}$$

$$I_{34} = -I_{14} - I_{24}$$

$$I_{23} + I_{34} = I_{13}$$

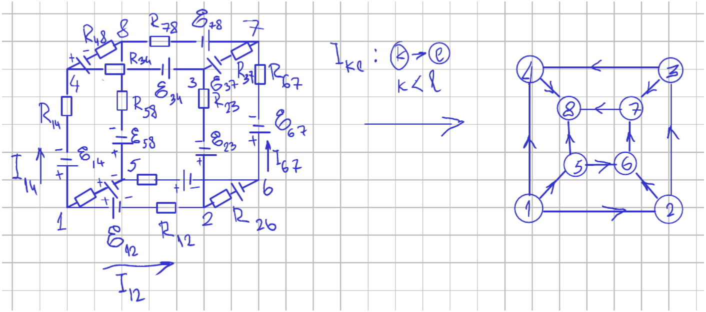
$$I_{12} + I_{23} = -I_{24}$$

$$I_{14} = -I_{34} - I_{24}$$

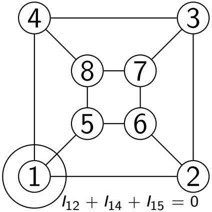
$$-R_{23} I_{23} - R_{12} I_{12} = 0; \quad -\mathcal{E}_{14} + R_{12} I_{12} = 0; \quad R_{23} I_{23} + R_{34} I_{34} = 0$$

$$R_{12} I_{12} = \mathcal{E}_{14}, \quad -R_{23} I_{23} = \mathcal{E}_{14}; \quad R_{34} I_{34} = \mathcal{E}_{14}$$

Example. Cube.



Example. Current into cube edges.



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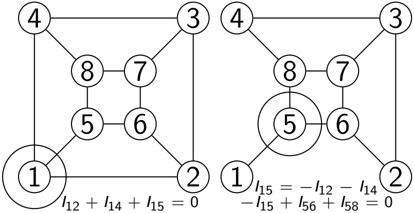

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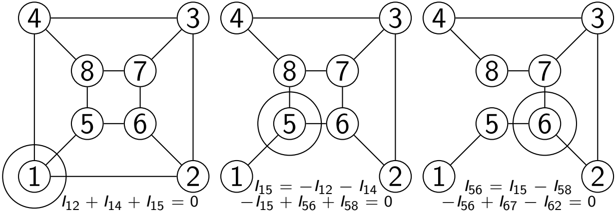
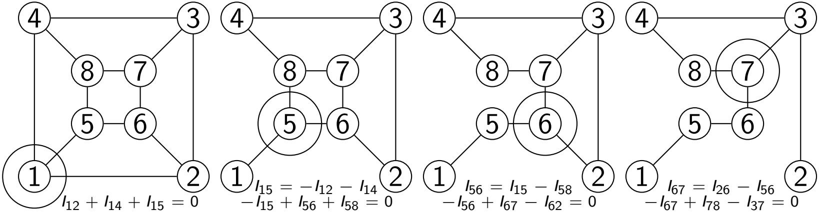
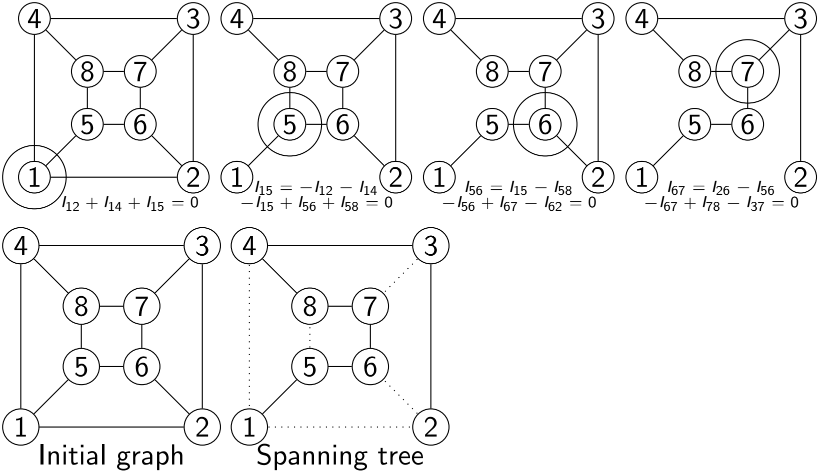


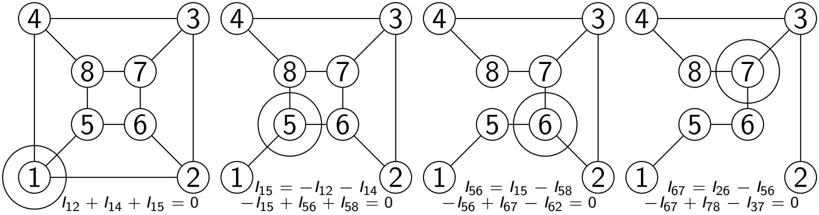
Example. Current into cube edges.

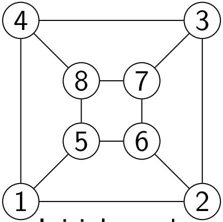


Example. Current into cube edges.

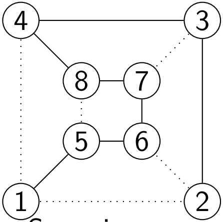


Example. Current into cube edges.





Initial graph

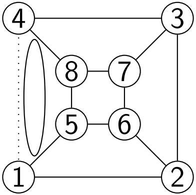


Spanning tree

$$
\begin{gathered}
I_{12},\ I_{14},\ I_{37},\ I_{26},\ I_{58};\\
I_{15} = -I_{12} - I_{14};\\
I_{56} = I_{15} - I_{58};\\
I_{67} = I_{26} + I_{56};\\
I_{78} = I_{37} - I_{67}\\
I_{34} = I_{23} - I_{37};\\
I_{23} = I_{12} - I_{26}.
\end{gathered}
$$

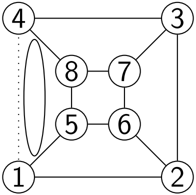
Equations for currents

Example. Voltage into cube edges.

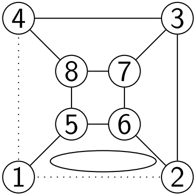


$$R_{15}I_{15} + \mathcal{E}_{15} +$$
$$R_{58}I_{58} + \mathcal{E}_{58} -$$
$$R_{48}I_{48} - \mathcal{E}_{48} -$$
$$R_{14}I_{14} - \mathcal{E}_{14} = 0$$

Example. Voltage into cube edges.

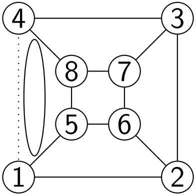


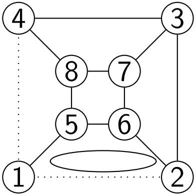
$$\begin{aligned} &R_{15}I_{15} + \mathcal{E}_{15} + \\ &R_{58}I_{58} + \mathcal{E}_{58} - \\ &R_{48}I_{48} - \mathcal{E}_{48} - \\ &R_{14}I_{14} - \mathcal{E}_{14} = 0 \end{aligned}$$

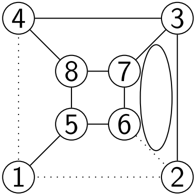


$$\begin{aligned} &-R_{15}I_{15} - \mathcal{E}_{15} - \\ &R_{56}I_{56} - \mathcal{E}_{56} + \\ &R_{26}I_{26} + \mathcal{E}_{26} + \\ &R_{12}I_{12} + \mathcal{E}_{12} = 0 \end{aligned}$$

Example. Voltage into cube edges.





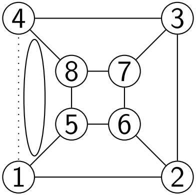


$$\begin{gathered} R_{15} I_{15} + \mathcal{E}_{15} + \\ R_{58} I_{58} + \mathcal{E}_{58} - \\ R_{48} I_{48} - \mathcal{E}_{48} - \\ R_{14} I_{14} - \mathcal{E}_{14} = 0 \end{gathered}$$

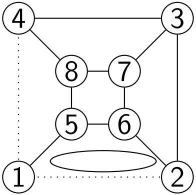
$$\begin{gathered} -R_{15} I_{15} - \mathcal{E}_{15} - \\ R_{56} I_{56} - \mathcal{E}_{56} + \\ R_{26} I_{26} + \mathcal{E}_{26} + \\ R_{12} I_{12} + \mathcal{E}_{12} = 0 \end{gathered}$$

$$\begin{gathered} R_{23} I_{23} + \mathcal{E}_{23} + \\ R_{37} I_{37} + \mathcal{E}_{37} - \\ R_{67} I_{67} - \mathcal{E}_{67} - \\ R_{26} I_{26} - \mathcal{E}_{26} = 0 \end{gathered}$$

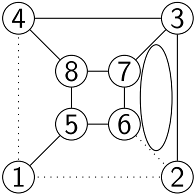
Example. Voltage into cube edges.



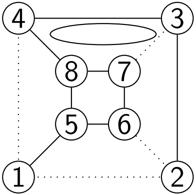
$$\begin{gathered} R_{15}I_{15} + \mathcal{E}_{15} + \\ R_{58}I_{58} + \mathcal{E}_{58} - \\ R_{48}I_{48} - \mathcal{E}_{48} - \\ R_{14}I_{14} - \mathcal{E}_{14} = 0 \end{gathered}$$



$$\begin{gathered} -R_{15}I_{15} - \mathcal{E}_{15} - \\ R_{56}I_{56} - \mathcal{E}_{56} + \\ R_{26}I_{26} + \mathcal{E}_{26} + \\ R_{12}I_{12} + \mathcal{E}_{12} = 0 \end{gathered}$$

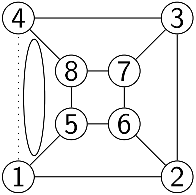


$$\begin{gathered} R_{23}I_{23} + \mathcal{E}_{23} + \\ R_{37}I_{37} + \mathcal{E}_{37} - \\ R_{67}I_{67} - \mathcal{E}_{67} - \\ R_{26}I_{26} - \mathcal{E}_{26} = 0 \end{gathered}$$

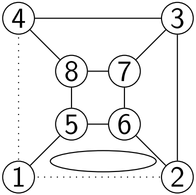


$$\begin{gathered} R_{34}I_{34} + \mathcal{E}_{34} + \\ R_{48}I_{48} + \mathcal{E}_{48} - \\ R_{78}I_{78} - \mathcal{E}_{78} - \\ R_{73}I_{73} - \mathcal{E}_{73} = 0 \end{gathered}$$

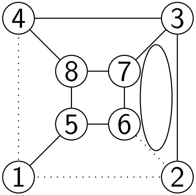
Example. Voltage into cube edges.



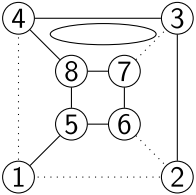
$$\begin{gathered} R_{15} I_{15} + \mathcal{E}_{15} + \\ R_{58} I_{58} + \mathcal{E}_{58} - \\ R_{48} I_{48} - \mathcal{E}_{48} - \\ R_{14} I_{14} - \mathcal{E}_{14} = 0 \end{gathered}$$



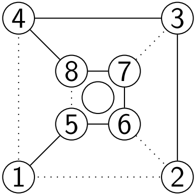
$$\begin{gathered} -R_{15} I_{15} - \mathcal{E}_{15} - \\ R_{56} I_{56} - \mathcal{E}_{56} + \\ R_{26} I_{26} + \mathcal{E}_{26} + \\ R_{12} I_{12} + \mathcal{E}_{12} = 0 \end{gathered}$$



$$\begin{gathered} R_{23} I_{23} + \mathcal{E}_{23} + \\ R_{37} I_{37} + \mathcal{E}_{37} - \\ R_{67} I_{67} - \mathcal{E}_{67} - \\ R_{26} I_{26} - \mathcal{E}_{26} = 0 \end{gathered}$$

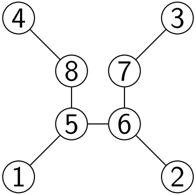


$$\begin{gathered} R_{34} I_{34} + \mathcal{E}_{34} + \\ R_{48} I_{48} + \mathcal{E}_{48} - \\ R_{78} I_{78} - \mathcal{E}_{78} - \\ R_{73} I_{73} - \mathcal{E}_{73} = 0 \end{gathered}$$



$$\begin{gathered} R_{56} I_{56} + \mathcal{E}_{56} + \\ R_{67} I_{67} + \mathcal{E}_{67} + \\ R_{78} I_{78} + \mathcal{E}_{78} - \\ R_{58} I_{58} - \mathcal{E}_{58} = 0 \end{gathered}$$

Example. Spanning trees.



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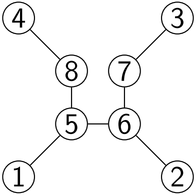
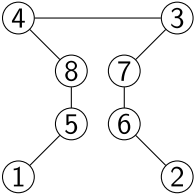
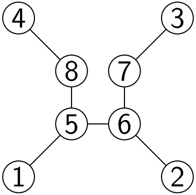
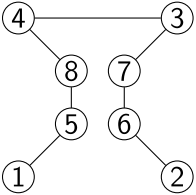
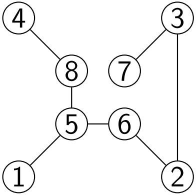




Example. Spanning trees.







Example. System of equations

$$I_{12},\ I_{14},\ I_{37},\ I_{26},\ I_{58};$$

$$I_{15} = -I_{12} - I_{14}; \qquad R_{15}I_{15} + \mathcal{E}_{15} + R_{58}I_{58} + \mathcal{E}_{58} - R_{48}I_{48} - \mathcal{E}_{48} - R_{14}I_{14} - \mathcal{E}_{14} = 0$$

$$I_{56} = I_{15} - I_{58}; \qquad -R_{15}I_{15} - \mathcal{E}_{15} - R_{56}I_{56} - \mathcal{E}_{56} + R_{26}I_{26} + \mathcal{E}_{26} + R_{12}I_{12} + \mathcal{E}_{12} = 0$$

$$I_{67} = I_{26} + I_{56}; \qquad R_{23}I_{23} + \mathcal{E}_{23} + R_{37}I_{37} + \mathcal{E}_{37} - R_{67}I_{67} - \mathcal{E}_{67} - R_{26}I_{26} - \mathcal{E}_{26} = 0$$

$$I_{78} = I_{37} - I_{67} \qquad R_{34}I_{34} + \mathcal{E}_{34} + R_{48}I_{48} + \mathcal{E}_{48} - R_{78}I_{78} - \mathcal{E}_{78} - R_{73}I_{73} - \mathcal{E}_{73} = 0$$

$$I_{34} = I_{23} - I_{37}; \qquad R_{56}I_{56} + \mathcal{E}_{56} + R_{67}I_{67} + \mathcal{E}_{67} + R_{78}I_{78} + \mathcal{E}_{78} - R_{58}I_{58} - \mathcal{E}_{58} = 0$$

$$I_{23} = I_{12} - I_{26}.$$

Kirchhoff's current law

Kirchhoff's voltage law

Circuit parameters and system of equations

In general case the equations look like

$$I_{kl} + I_{km} + I_{kj} = 0,$$

$$\sum_{l=1}^{4} (I_{kl} R_{kl} + \mathcal{E}_{kl}) = 0.$$

The cubic circuit contains following parameters:

- I_{kl} is a current through the edge between node k and node l. The positive direction of the current considers as positive for $k < l$.
- R_{kl} is a resistance of the edge;
- $\mathcal{E}_{kl}$ is a source of the voltage in the edge.

General numbers of the parameters are 12×3.

Circuit parameters and system of equations

The rank of the system which is obtained from the Kirchhoff's current law is equal 7. This means that for given values of current in 5 edges one can obtain the values of current in all 12 edges of the cube.
The Kirchhoff's voltage law allows us to obtain only 5 equations.
In general case the equations are nonlinear because of terms like $I_{kl}R_{kl}$.
Analysis of the nonlinear case looks to difficult in general. One can see the linear case appears if one of the multipliers in $R_{kl}Ikl$ might be obtained independently or known a priori.

Matrix representation of a graph

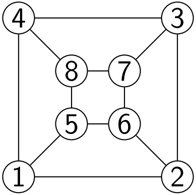
One of formal ways to present a graph passes through set of matrix.

Adjacency matrix defines connections between the nodes of given circuit.

$$A = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ a_{21} & 0 & \dots & \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn-1} & 0 \end{pmatrix}.$$

An element of this matrix a_{ij} declares a number of connections between the i-th and j-th nodes of the circuit.

Example



For the example considered early the adjacency matrix is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Property of the adjacency matrix for circuits

- The dimension of this matrix coincides with squared number of the nodes in the circuit. If n is the number of the nodes, then dimension of the matrix equal to $n \times n$.
- The adjacency matrix has a symmetry with respect to the main diagonal.
- A sum of elements of k-th row or column defines number of connections (edges) for the k-th node.
- For electrical circuit this matrix contains zeros on the main diagonal.

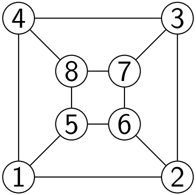
Incident matrix

An incident matrix defines the connections of all edges, say m, between each others.

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & \\ \dots & \dots & \dots & \dots \\ b_{n1} & \dots & b_{nm-1} & b_{nm} \end{pmatrix}.$$

Every element of the matrix $b_{ij} \in \{-1, 0, 1\}$ shows the existence of the connection between i-th vertex and and j-th edge.
The sign of b_{ij} defines the direction of the current through the vertex i into $(+1)$ or away (-1) this vertex.

Example. Incident matrix for the cube



$$B = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Properties of the incident matrix

- Every column contains only two non-zero coefficients.
- A sum of absolute value of coefficients of row equals to sum of the elements of the same row of the adjacency matrix.
- The sum of elements of every column is equal to zero.
- For simplicity let's use as a rule $b_{ij} \geq b_{k,j}$ if $i < k$.

Resistances as an edge's weights

Every lines of the circuit has a resistance which is a positive number or zero.
It is convenient to consider the resistance a weight coefficient for given edge.
Since a value of the resistance is not known then one should consider the list of the resistance as a list of partially unknown parameters R_j, $j \in \{1, \ldots, m\}$.

Sources of voltage and current

A sources of voltage like $\mathcal{E}_k$ and sources of current I_k, where $k \in \{1, \ldots, m\}$.
The sign the value of the source is positive for sources with coincided direction with the direction of corresponding edge.

Kirchhoff's current law

The Kirchhoff's current law allows to formulate the same number of the equations for the current as the number of the nodes of the graph.

$$\sum_{j=k}^{m} b_{k,j} I_k = 0, \quad j \in \{1, \dots, n\}.$$

Kirchhoff's voltage law

Let us define a loop on the graph without repeated passing trough node as a mesh.
Let the mesh be contained a set of edges $M_j = \{E_{k_1,k_2}, \ldots, E_{k_l,k_1}\}$.
The first and final node of this mesh is the node numbered k_1.
The set of all mesh of given graph is denoted $\mathcal{M} = \{M_j\}_{j=1}^{G}$.
The Kirchhoff' voltage law for this mesh looks like an equation:

$$\sum_{k \in M_j} \mathcal{E}_k + R_k I_k = 0, \quad j \in \{1, \ldots, G\}.$$

Properties of the Kirchhoff's laws

The system of the equations

$$\sum_{j=k}^{m} b_{k,j} I_k = 0, \quad j \in \{1, \dots, n\}.,$$

$$\sum_{k \in M_j} \mathcal{E}_k + R_k I_k = 0, \quad j \in \{1, \dots, G\}$$

defines the properties of circuit.
In general case a several parameters of the set $\{\mathcal{E}_j, R_j, I_k\}$, $j \in \{1, \dots, G\}$, $k \in \{1, \dots, n\}$. So the system should be considered as a system for definition of the unknown parameters of the circuit.

Linear case

The system of equation is non-linear with respect to parameters of the circuit because of terms $R_k I_k$ in the Kirchhoff's voltage law. These non-linear terms are linear with respect of unknown data for the following conditions:

- for every number k only one of two parameters I_k and R_k with given index k is unknown;
- if both parameters R_k and I_k are unknown for certain numbers $k \in \{K\}$ but one can find substitutions for every $k \in \{K\}$:

  $$I_k = \sum_J c_{k,j} I_j, \quad j \in \{J\}, \, J \cap \{K\} = \emptyset$$

  and coefficient $c_{k,j}$ can be found from the Kirchhoff's current laws.

Summary

- Applications graph theory to electrical circuits.
- A matrix representations of the graphs.
- Properties of equations for the circuits.