Contents

Chapter 1

几何

1.1 几何公式

- 三角形
  - 半周长

$$P = \frac{a+b+c}{2}$$

  - 面积

$$S = \frac{aH_a}{2} = \frac{ab\sin C}{2}$$
$$= \sqrt{P(P-a)(P-b)(P-c)}$$

  - 中线

$$M_a = \frac{\sqrt{2(b^2+c^2)-a^2}}{2}$$
$$= \frac{\sqrt{b^2+c^2+2bc\cos A}}{2}$$

  - 角平分线

$$T_a = \frac{\sqrt{bc((b+c)^2-a^2)}}{b+c}$$
$$= 2\frac{bc\cos\frac{A}{2}}{b+c}$$

  - 高线

$$H_a = b\sin C = c\sin B$$
$$= \sqrt{b^2 - (\frac{a^2+b^2-c^2}{2a})^2}$$

  - 内切圆半径

$$r = \frac{S}{P} = a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{B+C}{2}}$$
$$= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$
$$= P\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$$
$$= \sqrt{\frac{(P-a)(P-b)(P-c)}{P}}$$

  - 外接圆半径

$$R = \frac{abc}{4S}$$
$$= \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

- 四边形
  D_1,D_2 为对角线长，M 为对角线中点连线，A 为对角线夹角

$$a^2+b^2+c^2+d^2 = D_1^2+D_2^2+4M^2$$
$$S = D_1D_2\frac{\sin A}{2}$$

  以下对圆的内接四边形，P 为半周长

$$ac+bd = D_1D_2$$
$$S = \sqrt{(P-a)(P-b)(P-c)(P-d)}$$

- 正 n 边形
  R 为外接圆半径，r 为内切圆半径
  - 中心角

$$A = \frac{2\pi}{n}$$

  - 内角

$$C = \frac{(n-2)\pi}{n}$$

  - 边长

$$a = 2\sqrt{R^2-r^2}$$
$$= 2R\sin\frac{A}{2} = 2r\tan\frac{A}{2}$$

  - 面积

$$S = \frac{nar}{2} = nr^2\tan\frac{A}{2}$$
$$= \frac{nR^2\sin A}{2} = \frac{na^2}{4\tan\frac{A}{2}}$$

- 圆
  - 弧长
    $$l = rA$$
  - 弦长
    $$a = 2\sqrt{2hr - h^2} = 2r\sin\frac{A}{2}$$
  - 弓形高
    $$\begin{aligned} h &= r - \sqrt{r^2 - \frac{a^2}{4}} \\ &= r(1-\cos\frac{A}{2}) = a\frac{\tan\frac{A}{4}}{2} \end{aligned}$$
  - 扇形面积
    $$S_1 = \frac{rl}{2} = \frac{r^2A}{2}$$
  - 弓形面积
    $$S_2 = \frac{rl - a(r-h)}{2} = \frac{r^2(A - \sin A)}{2}$$
- 棱柱
  A 为底面积，h 为高，l 为棱长，p 为直截面周长
  - 体积
    $$V = Ah$$
  - 侧面积
    $$S = lp$$
  - 全面积
    $$T = S + 2A$$
- 棱锥
  A 为底面积，h 为高
  - 体积
    $$V = \frac{Ah}{3}$$

  以下对正棱锥，l 为斜高，p 为底面周长
  - 侧面积
    $$S = \frac{lp}{2}$$
  - 全面积
    $$T = S + A$$
- 棱台
  A_1，A_2 为上下底面积，h 为高
  - 体积
    $$V = \frac{(A_1 + A_2 + \sqrt{A_1A_2})h}{3}$$

  以下为正棱台，p_1，p_2 为上下底面周长，l 为斜高
  - 侧面积
    $$S = \frac{(p_1 + p_2)l}{2}$$
  - 全面积
    $$T = S + A_1 + A_2$$
- 圆柱
  - 侧面积
    $$S = 2\pi rh$$
  - 全面积
    $$T = 2\pi r(h + r)$$
  - 体积
    $$V = \pi r^2 h$$
- 圆锥
  - 母线
    $$l = \sqrt{h^2 + r^2}$$
  - 侧面积
    $$S = \pi rl$$
  - 全面积
    $$T = \pi r(l + r)$$
  - 体积
    $$V = \frac{\pi r^2 h}{3}$$
- 圆台
  - 母线
    $$l = \sqrt{h^2 + (r_1 - r_2)^2}$$
  - 侧面积
    $$S = \pi(r_1 + r_2)l$$
  - 全面积
    $$T = \pi r_1(l + r_1) + \pi r_2(l + r_2)$$
  - 体积
    $$V = \frac{\pi(r_1^2 + r_2^2 + r_1r_2)h}{3}$$
- 球

– 全面积

$$T = 4\pi r^2$$

– 体积

$$V = \frac{4\pi r^3}{3}$$

- 球台

– 侧面积

$$S = 2\pi r h$$

– 全面积

$$T = \pi(2rh + r_1^2 + r_2^2)$$

– 体积

$$V = \frac{\pi h(3(r_1^2 + r_2^2) + h^2)}{6}$$

- 球扇形
  h 为球冠高，r_0 为球冠底面半径

– 全面积

$$T = \pi r(2h + r_0)$$

– 体积

$$V = \frac{2\pi r^2 h}{3}$$

1.2 注意

1. 注意舍入方式 (0.5 的舍入方向)；防止输出 -0
2. 几何题注意多测试不对称数据；误差限缺省使用 1e-8
3. 整数几何注意 xmult 和 dmult 是否会出界；符点几何注意 eps 的使用
4. 避免使用斜率；注意除数是否会为 0
5. 公式一定要化简后再代入
6. 判断同一个 2π 域内两角度差 (beta) 应该是
   abs(a1-a2) < beta || abs(a1-a2) > pi + pi - beta
   判断相等时将 beta 换成 eps
7. 需要的话尽量使用 atan2，注意：
   - atan2(0,0) = 0
   - atan2(1,0) = pi/2
   - atan2(-1,0) = -pi/2
   - atan2(0,1) = 0
   - atan2(0,-1) = pi
8. cross product = $|u||v|\sin\alpha$
   dot product = $|u||v|\cos\alpha$
9. $(P_1 - P_0) \times (P_2 - P0)$ 结果的意义：
   - 正：$< P_0, P_1 >$ 在 $< P_0, P_2 >$ 顺时针 $(0, \pi)$ 内
   - 负：$< P_0, P_1 >$ 在 $< P_0, P_2 >$ 逆时针 $(0, \pi)$ 内
   - 0 ：$< P_0, P_1 >$，$< P_0, P_2 >$ 共线，夹角为 0 或 π

1.3 geo(猛犸也钻地)

```
// 计算几何 By 猛犸也钻地 @ 2012.08.21

/* 命名约定 //
   圆：圆心在 u，一般情况下半径 r 大于等于 0
   直线：经过点 u 和 v 的直线，u 不重合于 v
   射线：起点在 u，途经点 v，u 不重合于 v
   线段：起点在 u，终点在 v，u 不重合于 v
   散点集：点的可空集合
   多边形：至少有三个点，沿多边形的边依次排列，边不重合，图形不自交
   凸多边形：各内角均小于 180 度的多边形
   平面：由不共线的三点 uvw 所表示
```

```
// 所有函数都会默认传入的参数已满足上面的命名约定 */

# include <vector>
# include <cmath>
# include <utility>
# include <algorithm>
using namespace std;
using namespace rel_ops;

// typedef long long NUM;
typedef double NUM;
const NUM EPS = 1e-12, MAGIC = 2.71828e18;
// 因为有相对误差判断，所以 EPS 不要设得太宽

inline NUM sqr(NUM a) {
    return a*a;
}
inline NUM cmp(NUM a, NUM b) {
    // return a-b; // 坐标为浮点数时，使用下面那个
    return fabs(a-b)>=EPS+fabs(a)*EPS?a-b:0;
}

//------------------------------------------------------------------------//

struct VEC {
    NUM x,y;
} NOVEC = {MAGIC,MAGIC};
struct RAY {
    VEC u,v;
} NORAY = {NOVEC,NOVEC};
struct CIR {
    VEC u;
    NUM r;
} NOCIR = {NOVEC,MAGIC};

inline NUM sqr(const VEC &a) {
    return sqr(a.x)+sqr(a.y);
}
inline double abs(const VEC &a) {
    return sqrt(sqr(a));
}
inline NUM cmp(const VEC &a, const VEC &b) {
    NUM at=cmp(a.x,b.x);
    return !at?cmp(a.y,b.y):at;
}

inline VEC operator +(const VEC &a, const VEC &b) {
    return (VEC) {
        a.x+b.x,a.y+b.y
    };
}
inline VEC operator -(const VEC &a, const VEC &b) {
    return (VEC) {
        a.x-b.x,a.y-b.y
    };
```

```
}
inline NUM operator *(const VEC &a, const VEC &b) {
    return a.x*b.y-a.y*b.x;
}
inline NUM operator %(const VEC &a, const VEC &b) {
    return a.x*b.x+a.y*b.y;
}
inline VEC operator -(const VEC &a) {
    return (VEC) {
        -a.x,-a.y
    };
}
inline VEC operator ~(const VEC &a) {
    return (VEC) {
        -a.y,+a.x
    };
}
inline VEC operator *(NUM u, const VEC &a) {
    return (VEC) {
        u *a.x,u *a.y
    };
}
inline VEC operator *(const VEC &a, NUM u) {
    return (VEC) {
        a.x *u,a.y *u
    };
}
inline VEC operator /(const VEC &a, NUM u) {
    return (VEC) {
        a.x/u,a.y/u
    };
}
inline VEC operator /(const VEC &a, const VEC &b) {
    return a%b/sqr(b)*b;
}
inline bool operator ==(const VEC &a, const VEC &b) {
    return !cmp(a,b);
}
inline bool operator <(const VEC &a, const VEC &b) {
    return cmp(a,b)<0;
}

//  返回值        cmp_side                cmp_axis
//  ==  0       a 和 b 相互平行     /    a 和 b 相互垂直
//  <= -EPS     a 在 b 的左手侧     /    a 和 b 朝向相反（内角大于 90 度）
//  >= +EPS     a 在 b 的右手侧     /    a 和 b 朝向相同（内角小于 90 度）
NUM cmp_side(const VEC &a, const VEC &b) {
    return cmp(a.x*b.y,+a.y*b.x);
}
NUM cmp_axis(const VEC &a, const VEC &b) {
    return cmp(a.x*b.x,-a.y*b.y);
}

// 求向量 a 长度缩放至 u 单位后的新向量, a 不能是零向量
// 求向量 a 绕坐标原点 o, 逆时针转 u 度后的新向量
```

```
VEC resize(const VEC &a, NUM u) {
    return u/abs(a)*a;
}
VEC rotate(const VEC &a, NUM u) {
    return (VEC) {
        cos(u)*a.x-sin(u)*a.y,sin(u)*a.x+cos(u)*a.y
    };
}

struct PolarVEC { // 极角序比较，使用此函子排序后呈逆时针顺序
    bool operator ()(const VEC &a, const VEC &b) {
        NUM at=cmp_side(a,b);
        return at?at>0:cmp(sqr(a),sqr(b))<0;
    } // 两向量在比较时，靠右手侧的优先，同向时短的优先
};

// 点在直线上的投影（到直线的最近点）
// 点在圆周上的投影（到圆周的最近点）
VEC project(const VEC &p, const RAY &l) {
    return (p-l.u)/(l.v-l.u)+l.u;
}
VEC project(const VEC &p, const CIR &c) {
    if(!cmp(p,c.u)) return NOVEC;
    return resize(p-c.u,c.r)+c.u;
}

// 求凸多边形上，朝某个方向看过去的最远点的编号，复杂度 O(logn)
// 如果有多解，则返回相对于观测向量，在凸多边形上相对顺序更靠前的点
int apoapsis(const VEC &v, const vector<VEC>& u) {
    if(!cmp((VEC) {
    0,0
},v)) return -1;
    int l=0,r=u.size()-1;
    NUM s=cmp_axis(u[r]-u[0],v);
    NUM t=cmp_axis(u[1]-u[0],v);
    if(s<=0 && t<=0) return !s?r:0;
    while(l<r) {
        int m=(l+r)/2,e=cmp_axis(u[m]-u[0],v);
        if((e>=0 && e<cmp_axis(u[m+1]-u[0],v))
                || (e< 0 && t<0)) l=m+1;
        else r=m;
    }
    return r;
}

// 求两直线的交点
// 求直线与圆的交点，交线段的方向与原先直线相同
// 求两圆相交的交点，交线段的方向为圆心 a 到 b 连线方向逆指针转 90 度
// 求直线与凸多边形的交点，交线段的方向与原先直线相同，复杂度 O(logn)
VEC intersect(const RAY &a, const RAY &b) {
    VEC s=a.u-a.v,t=b.u-b.v;
    NUM at=cmp_side(s,t);
    if(!at) return NOVEC;
```

```
    return a.u+(b.u-a.u)*t/at*s;
}
RAY intersect(const RAY &l, const CIR &c) {
    VEC s=l.u+(c.u-l.u)/(l.v-l.u);
    NUM at=cmp(c.r*c.r,sqr(s-c.u));
    if(at<0) return NORAY;
    VEC t=resize(l.v-l.u,sqrt(at));
    return (RAY) {
        s-t,s+t
    };
}
RAY intersect(const CIR &a, const CIR &b) {
    NUM l=sqr(b.u-a.u);
    NUM w=(1+(a.r*a.r-b.r*b.r)/l)*0.5;
    NUM e=cmp(a.r*a.r/l,w*w);
    if(e<0) return NORAY;
    VEC t=sqrt(e)*~(b.u-a.u);
    VEC s=a.u+w*(b.u-a.u);
    return (RAY) {
        s-t,s+t
    };
}
RAY intersect(RAY l, const vector<VEC>& u) {
    int n=u.size(),p,q,lo,hi;
    VEC o=l.v-l.u;
    if(cmp_side(u[1]-u[0],u[2]-u[0])<0) o=-o;
    NUM pt=cmp_side(o,u[p=apoapsis(~ o,u)]-l.u);
    NUM qt=cmp_side(o,u[q=apoapsis(~-o,u)]-l.u);
    if(pt*qt>0) return NORAY;
    for(; p<n+n; o=-o) { // 只执行两次，分别计算 (p,q] 和 (q,p] 段和直线的交点
        lo=p,hi=q+=n;
        swap(p+=n,q);
        while(lo<hi) {
            int at=(lo+hi+1)/2;
            if(cmp_side(o,u[at%n]-l.u)>=0) lo=at;
            else hi=at-1;
        }
        if(!cmp_side(o,u[lo%n]-l.u)) l.u=u[lo%n];
        else l.u=intersect((RAY) {
            u[lo%n],u[(lo+1)%n]
        },l);
        swap(l.u,l.v);
    }
    return l;
}

// 判断三点是否共线
// 判断点在直线上的投影点，是否在线段上
// 判断点和线的位置关系，在外侧为 0，在直线上为 1 或 2(在线段上时为 2)
// 判断点和圆的位置关系，在外侧为 0，内部为 1，边上为 2
// 判断点和任意简单多边形的位置关系，在外侧为 0，内部为 1，边上为 2
// 快速地判断点和凸多边形的位置关系，在外侧为 0，内部为 1，边上为 2
// 判断两条线的位置关系，斜相交为 0，垂直为 1，平行为 2，重合为 3
bool collinear(const VEC &a, const VEC &b, const VEC &c) {
    return !cmp_side(a-b,b-c);
```

```
}
bool seg_range(const VEC &p, const RAY &l) {
    return cmp_axis(p-l.u,p-l.v)<=0;
}
int relation(const VEC &p, const RAY &l) {
    if(cmp_side(p-l.u,p-l.v)) return 0;
    return cmp_axis(p-l.u,p-l.v)>0?1:2;
}
int relation(const VEC &p, const CIR &c) {
    NUM at=cmp(sqr(c.r),sqr(c.u-p));
    return at?at<0?0:1:2;
}
int relation(const VEC &p, const vector<VEC>& u) {
    int n=u.size(),ret=0;
    for(int i=0; i<n; i++) {
        VEC s=u[i]-p,t=u[(i+1)%n]-p;
        if(t<s) swap(s,t);
        if(!cmp_side(s,t) && cmp_axis(s,t)<=0) return 2;
        if(cmp(s.x+p.x,p.x)<=0 && cmp(t.x+p.x,p.x)>0
                && cmp_side(s,t)>0) ret^=1;
    }
    return ret;
}
int relation_convex(const VEC &p, const vector<VEC>& u) {
    int n=u.size(),l=0,r=n-1,o=cmp_side(u[1]-u[0],u[r]-u[0])<0?-1:1;
    if(relation(p,(RAY) {
    u[0],u[1]
    })==2
    || relation(p,(RAY) {
        u[0],u[r]
    })==2) return 2;
    while(l<r) {
        int m=(l+r+1)/2;
        if(cmp_side(p-u[0],u[m]-u[0])*o<=0) l=m;
        else r=m-1;
    }
    if(!r || r==n-1) return 0;
    NUM at=cmp_side(p-u[r],u[r+1]-u[r])*o;
    return at?at<0:2;
}
int relation(const RAY &a, const RAY &b) {
    NUM at=cmp_side(a.u-a.v,b.u-b.v);
    return at?!cmp_axis(a.u-a.v,b.u-b.v):!cmp_side(a.u-b.u,a.u-b.v)+2;
}

// 由 ax+by+c=0 构造直线
// 由直径上的两点构造一个圆
// 由三角形的顶点构造外接圆
RAY make_line(NUM a, NUM b, NUM c) {
    if(!cmp(a,0) && !cmp(b,0)) return NORAY;
    else if(!cmp(a,0)) return (RAY) { {0,-c/b}, {1,-c/b}
    };
    else if(!cmp(b,0)) return (RAY) { {-c/a,0}, {-c/a,1}
    };
    return (RAY) { {0,-c/b}, {-c/a,0}
    };
}
```

```
CIR make_circle(const VEC &a, const VEC &b) {
    return (CIR) {
        (a+b)/2,abs(a-b)/2
    };
}
CIR make_circle(const VEC &a, const VEC &b, const VEC &c) {
    if(!cmp_side(a-b,a-c)) return NOCIR;
    NUM x=(c-b)%(a-c),y=(c-b)*(a-b);
    VEC m=(x/y*~(a-b)+a+b)/2;
    return (CIR) {
        m,abs(a-m)
    };
}

// 求三点的内切圆
// 求点到圆的两个切点，返回的切点分别在点到圆心连线方向的左侧和右侧
// 求两圆的两条公切线，切线段的方向与圆心 a 到 b 连线方向相同
//      默认是外公切线，若将其中的一个圆半径设为负数，则求出的是内公切线
CIR tangent_circle(const VEC &a, const VEC &b, const VEC &c) {
    if(!cmp_side(a-b,a-c)) return NOCIR;
    NUM x=abs(b-c),y=abs(c-a),z=abs(a-b);
    VEC m=(a*x+b*y+c*z)/(x+y+z);
    return (CIR) {
        m,fabs((m-a)*(a-b)*1.0/z)
    };
}
RAY tangent(const VEC &p, const CIR &c) {
    NUM l=sqr(p-c.u),e=cmp(l,c.r*c.r);
    if(e<0) return NORAY;
    NUM x=c.r/sqrt(l),y=sqrt(e/l);
    VEC s=resize(p-c.u,1),t=~s;
    RAY lr= {c.u+c.r *x *s-c.r *y *t,
             c.u+c.r *x *s+c.r *y *t
            };
    return lr;
}
pair<RAY,RAY> tangent(const CIR &a, const CIR &b) {
    NUM o=a.r-b.r,l=sqr(b.u-a.u),e=cmp(l,o*o);
    if(e<0) return make_pair(NORAY,NORAY);
    NUM x=o/sqrt(l),y=sqrt(e/l);
    VEC s=resize(b.u-a.u,1),t=~s;
    RAY ll= {a.u+a.r *x *s+a.r *y *t,
             b.u+b.r *x *s+b.r *y *t
            };
    RAY rr= {a.u+a.r *x *s-a.r *y *t,
             b.u+b.r *x *s-b.r *y *t
            };
    return make_pair(ll,rr);
}

// 由散点集构造一个最小覆盖圆，期望复杂度 O(n)
CIR min_covering_circle(vector<VEC> u) {
    random_shuffle(u.begin(),u.end());
    int n=u.size(),i,j,k,z=1%n;
    CIR ret;
    for(ret=make_circle(u[0],u[z]),i=2; i<n; i++) if(!relation(u[i],ret))
```

```
            for(ret=make_circle(u[0],u[i]),j=1; j<i; j++) if(!relation(u[j],ret))
                    for(ret=make_circle(u[i],u[j]),k=0; k<j; k++) if(!relation(u[k],ret)\
)
                            ret=make_circle(u[i],u[j],u[k]);
    return ret;
}

// 求散点集的二维凸包，并按逆时针顺序排列
// 若传入的点集不足以构成凸多边形，则返回的点集是退化后的点或线段
vector<VEC> convex_hull(vector<VEC> u) {
    sort(u.begin(),u.end()); // 这两行是排序 + 去重，如果数据已经有保证
    u.erase(unique(u.begin(),u.end()),u.end()); // 则可省略相应的操作
    if(u.size()<3) return u;
    vector<VEC> c;
    for(size_t i=0,o=1,m=1; ~i; i+=o) {
        while(c.size()>m) {
            VEC a=c.back()-c[c.size()-2];
            VEC b=c.back()-u[i];
            if(cmp_side(a,b)<0) break; // 改成<=0 则保留共线点
            c.pop_back();
        }
        c.push_back(u[i]);
        if(i+1==u.size()) m=c.size(),o=-1; // 条件成立时切换至上凸壳
    }
    c.pop_back();
    return c;
}

//------------------------------------------------------------------//

struct TOR {
    NUM x,y,z;
} NOTOR = {MAGIC,MAGIC,MAGIC};
struct SIG {
    TOR u,v;
} NOSIG = {NOTOR,NOTOR};
struct PLN {
    TOR u,v,w;
} NOPLN = {NOTOR,NOTOR,NOTOR};

inline NUM sqr(const TOR &a) {
    return sqr(a.x)+sqr(a.y)+sqr(a.z);
}
inline double abs(const TOR &a) {
    return sqrt(sqr(a));
}
inline NUM cmp(const TOR &a, const TOR &b) {
    NUM at=cmp(a.x,b.x);
    if(!at) at=cmp(a.y,b.y);
    return !at?cmp(a.z,b.z):at;
}

inline TOR operator +(const TOR &a, const TOR &b) {
    return (TOR) {
        a.x+b.x,a.y+b.y,a.z+b.z
```

```
    };
}
inline TOR operator -(const TOR &a, const TOR &b) {
    return (TOR) {
        a.x-b.x,a.y-b.y,a.z-b.z
    };
}
inline TOR operator *(const TOR &a, const TOR &b) {
    return (TOR) {
        a.y *b.z-a.z *b.y,a.z *b.x-a.x *b.z,a.x *b.y-a.y *b.x
    };
}
inline NUM operator %(const TOR &a, const TOR &b) {
    return a.x*b.x+a.y*b.y+a.z*b.z;
}
inline TOR operator -(const TOR &a) {
    return (TOR) {
        -a.x,-a.y,-a.z
    };
}
inline TOR operator *(NUM u, const TOR &a) {
    return (TOR) {
        u *a.x,u *a.y,u *a.z
    };
}
inline TOR operator *(const TOR &a, NUM u) {
    return (TOR) {
        a.x *u,a.y *u,a.z *u
    };
}
inline TOR operator /(const TOR &a, NUM u) {
    return (TOR) {
        a.x/u,a.y/u,a.z/u
    };
}
inline TOR operator /(const TOR &a, const TOR &b) {
    return a%b/sqr(b)*b;
}
inline bool operator ==(const TOR &a, const TOR &b) {
    return !cmp(a,b);
}
inline bool operator <(const TOR &a, const TOR &b) {
    return cmp(a,b)<0;
}

// 下面两个函数类似于它们的二维版本, 但 cmp_side 只能用于判定向量是否平行
int cmp_side(const TOR &a, const TOR &b) {
    return cmp(a.y*b.z,a.z*b.y)
            || cmp(a.z*b.x,a.x*b.z)
            || cmp(a.x*b.y,a.y*b.x);
}
NUM cmp_axis(const TOR &a, const TOR &b) {
    NUM x=a.x*b.x,y=a.y*b.y,z=a.z*b.z;
    if((x<0)==(y<0)) return cmp(x+y,-z);
    if((x<0)==(z<0)) return cmp(x+z,-y);
    // 注释掉上面两行可以大幅提升程序速度, 但有极小概率出现精度问题
    return cmp(y+z,-x);
```

```
}

// 求平面 c 的法向量
// 求向量 a 长度缩放至 u 单位后的新向量, a 不能是零向量
// 求向量 a 绕转轴向量 o, 逆时针转 u 度后的新向量
// 求点 p 在平面 c 上的投影坐标, 以 uv 为 x 轴的基, uw 为 y 轴的基, u 为原点
inline TOR normal(const PLN &c) {
    return (c.v-c.u)*(c.w-c.u);
}
TOR resize(const TOR &a, NUM u) {
    return u/abs(a)*a;
}
TOR rotate(const TOR &a, NUM u, const TOR &o) {
    return a*cos(u)+resize(o,1)*a*sin(u);
}
VEC shadow(const TOR &p, const PLN &c) {
    TOR a=c.v-c.u,b=c.w-c.u;
    TOR r=(c.u-p)/(a*b)+p;
    return (VEC) {
        r%a/sqr(a),r%b/sqr(b)
    };
}

// 点在直线上的投影 (到直线的最近点)
// 点在平面上的投影 (到平面的最近点)
TOR project(const TOR &p, const SIG &l) {
    return (p-l.u)/(l.v-l.u)+l.u;
}
TOR project(const TOR &p, const PLN &c) {
    return (c.u-p)/normal(c)+p;
}

// 求两直线的交点
// 求两平面的交线
// 求直线与平面的交点
TOR intersect(const SIG &a, const SIG &b) {
    TOR s=b.u-b.v,p=s*(b.u-a.u);
    TOR t=a.u-a.v,q=s*t;
    if(cmp_axis(p,t) || !cmp_side(s,t)) return NOTOR;
    NUM at=cmp_axis(p,q);
    return a.u+(at?at<0?-1:1:0)*sqrt(sqr(p)/sqr(q))*t;
}
TOR intersect(const SIG &l, const PLN &c) {
    TOR at=l.v-l.u,o=normal(c);
    if(!cmp_axis(o,at)) return NOTOR;
    return l.u+(c.u-l.u)%o/(at%o)*at;
}
SIG intersect(const PLN &a, const PLN &b) {
    TOR o=normal(a);
    SIG s= {b.u,b.v},t= {b.u,b.w},r= {b.v,b.w};
    s.u=intersect(cmp_axis(s.u-s.v,o)?s:r,a);
    t.u=intersect(cmp_axis(t.u-t.v,o)?t:r,a);
    return (SIG) {
        s.u,t.u
```

```
    };
}

// 判断四点是否共面
// 判断三点是否共线
// 判断点在直线上的投影点，是否在线段上
// 判断点和线的位置关系，在外侧为 0，在直线上为 1 或 2(在线段上时为 2)
// 判断点和面的位置关系，在面内为 0，在正方向为 1，在负方向为-1
// 判断两条线的位置关系，其他情况下为 0，垂直为 1，平行为 2，重合为 3
// 判断线和面的位置关系，斜相交为 0，垂直为 1，平行为 2，线在面内为 3
// 判断两平面的位置关系，斜相交为 0，垂直为 1，平行为 2，重合为 3
bool coplanar(const TOR &a, const TOR &b, const TOR &c, const TOR &d) {
    return !cmp_axis(a-b,(a-c)*(a-d));
}
bool collinear(const TOR &a, const TOR &b, const TOR &c) {
    return !cmp_side(a-b,b-c);
}
bool seg_range(const TOR &p, const SIG &l) {
    return cmp_axis(p-l.u,p-l.v)<=0;
}
int relation(const TOR &p, const SIG &l) {
    if(cmp_side(p-l.u,p-l.v)) return 0;
    return cmp_axis(p-l.u,p-l.v)>0?1:2;
}
int relation(const TOR &p, const PLN &c) {
    NUM at=cmp_axis(p-c.u,normal(c));
    return at?at<0?-1:1:0;
}
int relation(const SIG &a, const SIG &b) { // 注意，异面垂直也算垂直
    NUM at=cmp_side(a.u-a.v,b.u-b.v);
    return at?!cmp_axis(a.u-a.v,b.u-b.v):!cmp_side(a.u-b.u,a.u-b.v)+2;
}
int relation(const SIG &l, const PLN &c) {
    TOR o=normal(c),e=l.v-l.u;
    return cmp_axis(e,o)?!cmp_side(e,o):!cmp_axis(c.u-l.u,o)+2;
}
int relation(const PLN &a, const PLN &b) {
    TOR p=normal(a),q=normal(b);
    return cmp_side(p,q)?!cmp_axis(p,q):!cmp_axis(a.u-b.u,p)+2;
}

// 由 ax+by+cz+d=0 构造平面
PLN make_plane(NUM a, NUM b, NUM c, NUM d) {
    if(cmp(a,0)) return (PLN) { {-d/a,0,0}, {(-b-d)/a,1,0}, {(-c-d)/a,0,1}
    };
    if(cmp(b,0)) return (PLN) { {0,-d/b,0}, {1,(-a-d)/b,0}, {0,(-c-d)/b,1}
    };
    if(cmp(c,0)) return (PLN) { {0,0,-d/c}, {1,0,(-a-d)/c}, {0,1,(-b-d)/c}
    };
    return NOPLN;
}

// 求散点集的三维凸包，返回每个三角面的顶点编号，复杂度 O(nlogn)
```

```
// 从凸包外看，face 中的顶点按逆时针顺序排列，与 edge 中的邻面一一对应
// 传入的点集不能含有重点，若返回值为空集，则说明所有点共面
// NUM 的类型为 long long 时，坐标的范围不要超过 10^6，建议使用浮点类型
struct TPL {
    int u[3];
};
vector<TPL> convex_hull(const vector<TOR>& p) {
    vector<TPL> face,edge;
    static vector<int> F[100005],G[100005*7]; // 注意设置最大结点数
    int n=p.size(),i,j,k;
    if(n<=3) return face;
    vector<int> u(n),v(4),at(n),go(n),by(n);
    for(i=0; i<n; i++) u[i]=i;
    random_shuffle(u.begin(),u.end());
    TOR a=p[u[0]]-p[u[1]],b;
    for(i=2; i<n; i++) if(cmp_side(a,b=p[u[0]]-p[u[i]])) break;
    for(j=i; j<n; j++) if(cmp_axis(a*b,p[u[0]]-p[u[j]])) break;
    if(i>=n || j>=n) return face;
    swap(u[i],u[2]),swap(u[j],u[3]);
    b=p[u[0]]+p[u[1]]+p[u[2]]+p[u[3]];
    for(i=0; i<4; i++) {
        a=(p[u[i]]-p[u[j=(i+1)%4]])*(p[u[i]]-p[u[k=(i+2)%4]]);
        if(cmp_axis(p[u[i]]*4-b,a)<0) swap(j,k),a=-a;
        face.push_back((TPL) {{
                u[i],u[j],u[k]
            }
        });
        edge.push_back((TPL) {{
                (k+1)%4,(i+1)%4,(j+1)%4
            }
        });
        for(j=4; j<n; j++) if(cmp_axis(p[u[j]]-p[u[i]],a)>0)
                F[j].push_back(i),G[i].push_back(j);
    }
    for(i=4; i<n; F[i++].clear()) {
        int x=n,m=F[i].size(),c=v.size();
        for(j=0; j<m; j++) v[F[i][j]]++;
        for(j=0; j<m; j++) if(v[F[i][j]]>0) {
                v[F[i][j]]=-1234567890;
                for(k=0; k<3; k++) {
                    if(v[edge[F[i][j]].u[k]]) continue;
                    at[x=face[F[i][j]].u[k]]=F[i][j];
                    go[x]=k;
                }
            }
        if(x==n) continue;
        for(j=x,k=-1; k!=x; j=k) {
            k=face[at[j]].u[(go[j]+1)%3];
            a=(p[j]-p[k])*(p[j]-p[u[i]]);
            int t=v.size(),w=edge[at[j]].u[go[j]];
            v.push_back(0);
            face.push_back((TPL) {{
                    j,k,u[i]
                }
            });
            edge.push_back((TPL) {{
```

```
                    w,t+1,t-1
                }
            });
            *find(edge[w].u,edge[w].u+3,at[j])=t;
            vector<int>::const_iterator o,z;
            z=set_union(G[at[j]].begin(),G[at[j]].end(),
                        G[w].begin(),G[w].end(),by.begin());
            for(o=by.begin(); o!=z; ++o)
                if(*o>i && cmp_axis(p[u[*o]]-p[u[i]],a)>0)
                    F[*o].push_back(t),G[t].push_back(*o);
        }
        edge[edge.back().u[1]=c].u[2]=edge.size()-1;
    }
    int m=v.size();
    for(i=j=0; i<m; G[i++].clear())
        if(!v[i]) face[j]=face[i],edge[j++]=edge[i];
    face.erase(face.begin()+j,face.end());
    edge.erase(edge.begin()+j,edge.end());
    return face;
}
```

1.4 geo

```
# include <cmath>
# include <algorithm>
using namespace std;
const int MAXN = 1000;
const double eps = 1e-8, PI = atan2(0, -1);
inline double sqr(double x) {
    return x * x;
}
inline bool zero(double x) {
    return (x > 0 ? x : -x) < eps;
}
inline int sgn(double x) {
    return (x > eps ? 1 : (x + eps < 0 ? -1 : 0));
}
struct point {
    double x, y;
    point(double x, double y):x(x), y(y) {}
    point() {}
    bool operator == (const point &a) const {
        return sgn(x - a.x) == 0 && sgn(y - a.y) == 0;
    }
    bool operator != (const point &a) const {
        return sgn(x - a.x) != 0 || sgn(y - a.y) != 0;
    }
    bool operator < (const point &a) const {
        return sgn(x - a.x) < 0 || sgn(x - a.x) == 0 && sgn(y - a.y) < 0;
    }
    point operator + (const point &a) const {
        return point(x + a.x, y + a.y);
    }
    point operator - (const point &a) const {
        return point(x - a.x, y - a.y);
    }
```

```
    point operator * (const double &a) const {
        return point(x * a, y * a);
    }
    point operator / (const double &a) const {
        return point(x / a, y / a);
    }
    double operator * (const point &a) const {
        return x * a.y - y * a.x;     //xmult
    }
    double operator ^ (const point &a) const {
        return x * a.x + y * a.y;     //dmult
    }
    double length() const {
        return sqrt(sqr(x) + sqr(y));
    }
    point trunc(double a) const {
        return (*this) * (a / length());
    }
    point rotate(double ang) const {
        point p(sin(ang), cos(ang));
        return point((*this) * p, (*this) ^ p);
    }
    point rotate(const point &a) const {
        point p(-a.y, a.x);
        p = p.trunc(1.0);
        return point((*this) * p, (*this) ^ p);
    }
};
bool isConvex(int n, const point *p) {
    int i, s[3] = {1, 1, 1};
    for(i = 0; i < n && /*s[1] && */s[0] | s[2]; i++)
        s[sgn((p[(i + 1) % n] - p[i]) * (p[(i + 2) % n] - p[i])) + 1] = 0;
    return /*s[1] && */s[0] | s[2];
} //去掉注释即不允许相邻边共线
bool insideConvex(const point &q, int n, const point *p) {
    int i, s[3] = {1, 1, 1};
    for(i = 0; i < n && /*s[1] && */s[0] | s[2]; i++)
        s[sgn((p[(i + 1) % n] - p[i]) * (q - p[i])) + 1] = 0;
    return /*s[1] && */s[0] | s[2];
} //去掉注释即严格在形内
inline bool dotsInline(const point &p1, const point &p2, const point &p3) {
    return zero((p1 - p3) * (p2 - p3));
} //三点共线
inline int decideSide(const point &p1, const point &p2, const point &l1, const point &l2\
) {
    return sgn((l1 - l2) * (p1 - l2)) * sgn((l1 - l2) * (p2 - l2));
} //点 p1 和 p2, 直线 l1-l2,-1 表示在异侧,0 表示在线上,1 表示同侧
inline bool dotOnlineIn(const point &p, const point &l1, const point &l2) {
    return zero((p - l2) * (l1 - l2)) && (l1.x - p.x) * (l2.x - p.x) < eps && (l1.y - p.\
y) * (l2.y - p.y) < eps;
} //判点是否在线段及其端点上
inline bool parallel(const point &u1, const point &u2, const point &v1, const point &v2)\
 {
```

```
    return zero((u1 - u2) * (v1 - v2));
} //判直线平行
inline bool perpendicular(const point &u1, const point &u2, const point &v1, const point\
 &v2) {
    return zero((u1 - u2) ^ (v1 - v2));
} //判直线垂直
inline bool intersectIn(const point &u1, const point &u2, const point &v1, const point &\
v2) {
    if(!dotsInline(u1, u2, v1) || !dotsInline(u1, u2, v2))
        return decideSide(u1, u2, v1, v2) != 1 && decideSide(v1, v2, u1, u2) != 1;
    else
        return dotOnlineIn(u1, v1, v2) || dotOnlineIn(u2, v1, v2) || dotOnlineIn(v1, u1,\
 u2) || dotOnlineIn(v2, u1, u2);
} //判两线段相交，包括端点和部分重合
inline bool intersectEx(const point &u1, const point &u2, const point &v1, const point &\
v2) {
    return decideSide(u1, u2, v1, v2) < 0 && decideSide(v1, v2, u1, u2) < 0;
} //判两线段相交，不包括端点和部分重合
inline bool insidePolygon(const point &q, int n, const point *p, bool onEdge = true) {
    if(dotOnlineIn(q, p[n - 1], p[0])) return onEdge;
    for(int i = 0; i + 1 < n; i++) if(dotOnlineIn(q, p[i], p[i + 1])) return onEdge;
 # define getq(i) Q[(sgn(p[i].x-q.x)>0)<<1|sgn(p[i].y-q.y)>0]
 # define difq(a,b,i,j) (a==b?0:(a==((b+1)&3)?1:(a==((b+3)&3)?-1:(sgn((p[i]-q)*(p[j]-q))<<\
 1))))
    int Q[4] = {2, 1, 3, 0}, oq = getq(n-1), nq = getq(0), qua = difq(nq, oq, n - 1, 0);
    oq = nq;
    for(int i = 1; i < n; i++) {
        nq = getq(i);
        qua += difq(nq, oq, i - 1, i);
        oq = nq;
    }
    return qua != 0; //象限环顾法，较好
     /*point q1; int i = 0, cnt = 0; const double OFFSET = 1e6;//坐标上限
     for(q1 = point(rand() + OFFSET, rand() + OFFSET);i < n;) for(i = cnt = 0;i < n && !d\
otsInline(q, q1, p[i]);i++) cnt += intersectEx(q, q1, p[i], p[(i + 1) % n]);
     return cnt & 1;*/ //考验 rp 的射线法
} //判点在任意多边形内
inline point intersection(const point &u1, const point &u2, const point &v1, const point\
 &v2) {
    return u1 + (u2 - u1) * (((u1 - v1) * (v1 - v2)) / ((u1 - u2) * (v1 - v2)));
} //求两直线交点，须预判是否平行
inline point ptoline(const point &p, const point &l1, const point &l2) {
    point t = p;
    t.x += l1.y - l2.y;
    t.y += l2.x - l1.x;
    return intersection(p, t, l1, l2);
} //点到直线的最近点，注意 l1 不能等于 l2
inline double disptoline(const point &p, const point &l1, const point &l2) {
    return fabs((p - l2) * (l1 - l2)) / (l1 - l2).length();
} //点到直线距离，注意 l1 不能等于 l2
inline point ptoseg(const point &p, const point &l1, const point &l2) {
```

```
    point t = p;
    t.x += l1.y - l2.y;
    t.y += l2.x - l1.x;
    if(sgn((l1 - p) * (t - p)) * sgn((l2 - p) * (t - p)) > 0)
        return (p - l1).length() < (p - l2).length() ? l1 : l2;
    else
        return intersection(p, t, l1, l2);
} //点到线段的最近点, 注意 l1 不能等于 l2
inline double disptoseg(const point &p, const point &l1, const point &l2) {
    point t = point(l1.y - l2.y, l2.x - l1.x);
    if(sgn((l1 - p) * t) * sgn((l2 - p) * t) > 0)
        return min((p - l1).length(), (p - l2).length());
    else
        return disptoline(p, l1, l2);
} //点到线段距离, 注意 l1 不能等于 l2
double fermentpoint(int m, point p[]) {
    point u(0, 0), v;
    double step = 0, nowbest = 0, now;
    for(int i = 0; i < m; ++i) u = u + p[i];
    u = u / m;
    for(int i = 0; i < m; ++i) nowbest += (u - p[i]).length();
    for(step = u.x + u.y; step > 1e-10; step *= 0.97) //对结果有影响, 注意调整
        for(int i = -1; i <= 1; i++)
            for(int j = -1; j <= 1; j++) {
                v = u + point(i, j) * step;
                now = 0;
                for(int i = 0; i < m; ++i) now += (v - p[i]).length();
                if(now < nowbest) {
                    nowbest = now;
                    u = v;
                }
            }
    return nowbest;
} //模拟退火求费马点
void polygonCut(int &n, point *p, const point &l1, const point &l2, const point &side) {
    int m = 0, i;
    point pp[MAXN]; //尽量定义成全局变量
    for(i = 0; i < n; i++) {
        if(decideSide(p[i], side, l1, l2) == 1) pp[m++] = p[i];
        if(decideSide(p[i], p[(i + 1) % n], l1, l2) < 1 && !(zero((p[i] - l2) * (l1 - l2\
)) && zero((p[(i + 1) % n] - l2) * (l1 - l2))))
            pp[m++] = intersection(p[i], p[(i + 1) % n], l1, l2);
    }
    for(n = i = 0; i < m; i++)
        if(!i || !zero(pp[i].x - pp[i - 1].x) || !zero(pp[i].y - pp[i-1].y)) p[n++] = pp\
[i];
    if(zero(p[n - 1].x - p[0].x) && zero(p[n - 1].y - p[0].y)) n--;
    if(n < 3) n = 0;
} //将多边形沿 l1,l2 确定的直线在 side 侧切割, 保证 l1,l2,side 不共线
inline double Seg_area(const point &p1, const point &p2, const point &p0, double R) {
    point tmp = (p0 - p1).rotate(p2 - p1);
    double d = -tmp.y, h1 = -tmp.x, h2 = h1 + (p2 - p1).length();
    if(d >= R || d <= -R) return R * R * (atan2(d, h1) - atan2(d, h2));
    double dh = sqrt(R * R - d * d);
    if(h2 < -dh || dh < h1) return R * R * (atan2(d, h1) - atan2(d, h2));
```

```
    double ret = 0;
    if(h1 < -dh) ret += atan2(d, h1) - atan2(d, -dh);
    if(h2 > dh) ret += atan2(d, dh) - atan2(d, h2);
    return ret * R * R + d * (min(h2, dh) - max(h1, -dh));
} //圆与线段交的有向面积
int graham(int n, point *p, point *ch, bool comEdge = false) {
    if(n < 3) {
        for(int i = 0; i < n; i++) ch[i] = p[i];
        return n;
    }
    const double e1 = comEdge ? eps : -eps;
    int i, j, k;
    sort(p, p + n);
    ch[0] = p[0];
    ch[1] = p[1];
    for(i = j = 2; i < n; ch[j++] = p[i++]) while(j > 1 && (ch[j - 2] - ch[j - 1]) * (p[\
i] - ch[j - 1]) > e1) j--;
    ch[k = j++] = p[n - 2];
    for(i = n - 3; i > 0; ch[j++] = p[i--]) while(j > k && (ch[j - 2] - ch[j - 1]) * (p[\
i] - ch[j - 1]) > e1) j--;
    while (j > k && (ch[j - 2] - ch[j - 1]) * (ch[0] - ch[j - 1]) > e1) j--;
    return j;
} //求凸包,p 会被打乱顺序,ch 为逆时针,comEdge 为 true 时保留共线点, 重点会导致不稳定
```

1.5 geo3d

```
# include <cmath>
# include <algorithm>
using namespace std;
const int MAXN = 1000;
const double eps = 1e-8;
const double PI = atan2(0.0, -1.0);
inline double sqr(double x) {
    return x * x;
}
inline bool zero(double x) {
    return (x > 0 ? x : -x) < eps;
}
inline int sgn(double x) {
    return (x > eps ? 1 : (x + eps < 0 ? -1 : 0));
}
struct point3 {
    double x, y, z;
    point3(double x, double y, double z):x(x), y(y), z(z) {}
    point3() {}
    bool operator == (const point3 &a) const {
        return sgn(x - a.x) == 0 && sgn(y - a.y) == 0 && sgn(z - a.z) == 0;
    }
    bool operator != (const point3 &a) const {
        return sgn(x - a.x) != 0 || sgn(y - a.y) != 0 || sgn(z - a.z) != 0;
    }
    bool operator < (const point3 &a) const {
        return sgn(x - a.x) < 0 || sgn(x - a.x) == 0 && sgn(y - a.y) < 0 || sgn(x - a.x)\
 == 0 && sgn(y - a.y) == 0 && sgn(z - a.z) < 0;
    }
```

```
    point3 operator + (const point3 &a) const {
        return point3(x + a.x, y + a.y, z + a.z);
    }
    point3 operator - (const point3 &a) const {
        return point3(x - a.x, y - a.y, z - a.z);
    }
    point3 operator * (const double &a) const {
        return point3(x * a, y * a, z * a);
    }
    point3 operator / (const double &a) const {
        return point3(x / a, y / a, z / a);
    }
    point3 operator * (const point3 &a) const {
        return point3(y * a.z - z * a.y, z * a.x - x * a.z, x * a.y - y * a.x);     //xmu\
lt
    }
    double operator ^ (const point3 &a) const {
        return x * a.x + y * a.y + z * a.z;     //dmult
    }
    double sqrlen() const {
        return sqr(x) + sqr(y) + sqr(z);
    }
    double length() const {
        return sqrt(sqrlen());
    }
    point3 trunc(double a) const {
        return (*this) * (a / length());
    }
    point3 rotate(const point3 &a, const point3 &b, const point3 &c) const {
        return point3(a ^ (*this), b ^ (*this), c ^ (*this));     //abc 正交且模为 1
    }
};
inline point3 pvec(const point3 &a, const point3 &b, const point3 &c) {
    return (a - b) * (b - c);
} //平面法向量
inline bool dotsInline(const point3 &a, const point3 &b, const point3 &c) {
    return zero(((a - b) * (b - c)).length());
} //判三点共线
inline bool dotsOnplane(const point3 &a, const point3 &b, const point3 &c, const point3 \
&d) {
    return zero(pvec(a, b, c) ^ (d - a));
} //判四点共面
inline bool dotOnlineIn(const point3 &p, const point3 &l1, const point3 &l2) {
    return zero(((p - l1) * (p - l2)).length()) && (l1.x - p.x) * (l2.x - p.x) < eps && \
(l1.y - p.y) * (l2.y - p.y) < eps && (l1.z - p.z) * (l2.z - p.z) < eps;
} //判点是否在线段上，包括端点和共线
inline bool dotInplaneIn(const point3 &p, const point3 &a, const point3 &b, const point3\
 &c) {
    return zero(((a - b) * (a - c)).length() - ((p - a) * (p - b)).length() - ((p - b) *\
 (p - c)).length() - ((p - c) * (p - a)).length());
} //判点是否在空间三角形上，包括边界，须保证 abc 不共线
inline int decideSide(const point3 &p1, const point3 &p2, const point3 &l1, const point3\
 &l2) {
    return sgn(((l1 - l2) * (p1 - l2)) ^ ((l1 - l2) * (p2 - l2)));
```

```
}//点 p1 和 p2, 直线 l1-l2,-1 表示在异侧,0 表示在线上,1 表示同侧, 须保证所有点共面
inline int decideSide(const point3 &p1, const point3 &p2, const point3 &a, const point3 \
&b, const point3 &c) {
    return sgn((pvec(a, b, c) ^ (p1 - a)) * (pvec(a, b, c) ^ (p2 - a)));
}//点 p1 和 p2, 平面 abc,-1 表示在异侧,0 表示在面上,1 表示同侧
inline bool parallel(const point3 &u1, const point3 &u2, const point3 &v1, const point3 \
&v2) {
    return zero(((u1 - u2) * (v1 - v2)).length());
}//判两直线平行
inline bool parallel(const point3 &a, const point3 &b, const point3 &c, const point3 &d,\
 const point3 &e, const point3 &f) {
    return zero((pvec(a, b, c) * pvec(d, e, f)).length());
}//判两平面平行
inline bool parallel(const point3 &l1, const point3 &l2, const point3 &a, const point3 &\
b, const point3 &c) {
    return zero((l1 - l2) ^ pvec(a, b, c));
}//判直线与平面平行
inline bool perpendicular(const point3 &u1, const point3 &u2, const point3 &v1, const po\
int3 &v2) {
    return zero((u1 - u2) ^ (v1 - v2));
}//判两直线垂直
inline bool perpendicular(const point3 &a, const point3 &b, const point3 &c, const point\
3 &d, const point3 &e, const point3 &f) {
    return zero(pvec(a, b, c) ^ pvec(d, e, f));
}//判两平面垂直
inline bool perpendicular(const point3 &l1, const point3 &l2, const point3 &a, const poi\
nt3 &b, const point3 &c) {
    return zero(((l1 - l2) * pvec(a, b, c)).length());
}//判直线与平面垂直
inline bool intersectIn(const point3 &u1, const point3 &u2, const point3 &v1, const poin\
t3 &v2) {
    if(!dotsOnplane(u1, u2, v1, v2)) return false;
    if(!dotsInline(u1, u2, v1) || !dotsInline(u1, u2, v2))
        return decideSide(u1, u2, v1, v2) < 1 && decideSide(v1, v2, u1, u2) < 1;
    return dotOnlineIn(u1, v1, v2) || dotOnlineIn(u2, v1, v2) || dotOnlineIn(v1, u1, u2)\
 || dotOnlineIn(v2, u1, u2);
}//判两线段相交, 包括端点和部分重合
inline bool intersectEx(const point3 &u1, const point3 &u2, const point3 &v1, const poin\
t3 &v2) {
    return dotsOnplane(u1, u2, v1, v2) && decideSide(u1, u2, v1, v2) < 0 && decideSide(v\
1, v2, u1, u2) < 0;
}//判两线段相交, 不包括端点和部分重合
inline bool intersect(const point3 &l1, const point3 &l2, const point3 &a, const point3 \
&b, const point3 &c, bool edge = true) {
    return decideSide(l1, l2, a, b, c) < edge && decideSide(a, b, l1, l2, c) < edge && d\
ecideSide(b, c, l1, l2, a) < edge && decideSide(c, a, l1, l2, b) < edge;
}//判线段与空间三角形相交,edge 表示是否包括交于边界和部分包含
point3 intersection(const point3 &u1, const point3 &u2, const point3 &v1, const point3 &\
v2) {
    point3 p0 = (u1 - v1) * (v1 - v2), p1 = (u1 - u2) * (v1 - v2);
    return u1 + (u2 - u1) * (sgn(p0 ^ p1) * sqrt(p0.sqrlen() / p1.sqrlen()));
}//计算两直线交点, 须预判直线是否共面和平行
```

```
point3 intersection(const point3 &l1, const point3 &l2, const point3 &a, const point3 &b\
, const point3 &c) {
    point3 temp = pvec(a, b, c);
    return l1 + (l2 - l1) * ((temp ^ (a - l1)) / (temp ^ (l2 - l1)));
} //计算直线与平面交点，须预判是否平行，并保证三点不共线
void intersection(const point3 &a, const point3 &b, const point3 &c, const point3 &d, co\
nst point3 &e, const point3 &f, point3 &p1, point3 &p2) {
    p1 = parallel(d, e, a, b, c) ? intersection(e, f, a, b, c) : intersection(d, e, a, b\
, c);
    p2 = parallel(f, d, a, b, c) ? intersection(e, f, a, b, c) : intersection(f, d, a, b\
, c);
} //计算两平面交线，注意事先判断是否平行，并保证三点不共线,p-q 为交线
inline double disptoline(const point3 &p, const point3 &l1, const point3 &l2) {
    return sqrt(((p - l1) * (l2 - l1)).sqrlen() / (l1 - l2).sqrlen());
} //点到直线距离
inline point3 ptoline(const point3 &p, const point3 &l1, const point3 &l2) {
    point3 temp = l2 - l1;
    return l1 + temp * ((p - l1) ^ temp) / (temp ^ temp);
} //点到直线最近点
inline double disptoplane(const point3 &p, const point3 &a, const point3 &b, const point\
3 &c) {
    point3 temp = pvec(a, b, c);
    return fabs(temp ^ (p - a)) / temp.length();
} //点到平面距离
inline point3 ptoplane(const point3 &p, const point3 &a, const point3 &b, const point3 &\
c) {
    return intersection(p, p + pvec(a, b, c), a, b, c);
} //点到平面最近点
inline double dislinetoline(const point3 &u1, const point3 &u2, const point3 &v1, const \
point3 &v2) {
    point3 temp = (u1 - u2) * (v1 - v2);
    return fabs((u1 - v1) ^ temp) / temp.length();
} //直线到直线距离
void linetoline(const point3 &u1, const point3 &u2, const point3 &v1, const point3 &v2, \
point3 &p1, point3 &p2) {
    point3 ab = u2 - u1, cd = v2 - v1, ac = v1 - u1;
    p2 = v1 + cd * (((ab ^ cd) * (ac ^ ab) - (ab ^ ab) * (ac ^ cd)) / ((ab ^ ab) * (cd ^\
 cd) - sqr(ab ^ cd)));
    p1 = ptoline(p2, u1, u2);
} //直线到直线的最近点对,p1 在 u 上,p2 在 v 上，须保证直线不平行
inline double angleCos(const point3 &u1, const point3 &u2, const point3 &v1, const point\
3 &v2) {
    return ((u1 - u2) ^ (v1 - v2)) / sqrt((u1 - u2).sqrlen() * (v1 - v2).sqrlen());
} //两直线夹角 cos 值
inline double angleCos(const point3 &a, const point3 &b, const point3 &c, const point3 &\
d, const point3 &e, const point3 &f) {
    point3 p1 = pvec(a, b, c), p2 = pvec(d, e, f);
    return (p1 ^ p2) / sqrt(p1.sqrlen() * p2.sqrlen());
} //两平面夹角 cos 值
inline double angleSin(const point3 &l1, const point3 &l2, const point3 &a, const point3\
 &b, const point3 &c) {
    point3 temp = pvec(a, b, c);
    return ((l1 - l2) ^ temp) / sqrt((l1 - l2).sqrlen() * temp.sqrlen());
```

```
}//直线平面夹角 sin 值
double angle(double lng1, double lat1, double lng2, double lat2) {
    double dlng = fabs(lng1 - lng2) * PI / 180;
    while(dlng >= PI + PI) dlng -= PI + PI;
    if(dlng > PI) dlng = PI + PI - dlng;
    lat1 *= PI / 180;
    lat2 *= PI / 180;
    return acos(cos(lat1) * cos(lat2) * cos(dlng) + sin(lat1) * sin(lat2));
}//计算大圆劣弧圆心角,lat(-90, 90) 表示纬度,lng 表示经度
```

1.6 三维几何

```
//三维几何函数库
# include <cmath>

const double EPS = 1e-8;

struct Point3D {
    double x, y, z;
};

struct Line3D {
    Point3D a, b;
};

struct Plane {
    Point3D a, b, c;
};

struct PlaneF {
    // ax + by + cz + d = 0
    double a, b, c, d;
};

inline bool zero(double x) {
    return (x > 0 ? x : -x) < EPS;
}

//平方
inline double sqr(double d) {
    return d * d;
}

//计算 cross product U x V
inline Point3D xmult(const Point3D &u, const Point3D &v) {
    Point3D ret;
    ret.x = u.y * v.z - v.y * u.z;
    ret.y = u.z * v.x - u.x * v.z;
    ret.z = u.x * v.y - u.y * v.x;
    return ret;
}

//计算 dot product U . V
inline double dmult(const Point3D &u, const Point3D &v) {
```

```
    return u.x * v.x + u.y * v.y + u.z * v.z;
}

//矢量差 U - V
inline Point3D subt(const Point3D &u, const Point3D &v) {
    Point3D ret;
    ret.x = u.x - v.x;
    ret.y = u.y - v.y;
    ret.z = u.z - v.z;
    return ret;
}

//取平面法向量
inline Point3D pvec(const Plane &s) {
    return xmult(subt(s.a, s.b), subt(s.b, s.c));
}
inline Point3D pvec(const Point3D &s1, const Point3D &s2, const Point3D &s3) {
    return xmult(subt(s1, s2), subt(s2, s3));
}
inline Point3D pvec(const PlaneF &p) {
    Point3D ret;
    ret.x = p.a;
    ret.y = p.b;
    ret.z = p.c;
    return ret;
}

//两点距离
inline double dis(const Point3D &p1, const Point3D &p2) {
    return sqrt((p1.x - p2.x)*(p1.x - p2.x) + (p1.y - p2.y)*(p1.y - p2.y) + (p1.z - p2.z\
)*(p1.z - p2.z));
}

//向量大小
inline double vlen(const Point3D &p) {
    return sqrt(p.x*p.x + p.y*p.y + p.z*p.z);
}

//向量大小的平方
inline double sqrlen(const Point3D &p) {
    return (p.x*p.x + p.y*p.y + p.z*p.z);
}

//判三点共线
bool dotsInline(const Point3D &p1, const Point3D &p2, const Point3D &p3) {
    return sqrlen(xmult(subt(p1, p2), subt(p2, p3))) < EPS;
}

//判四点共面
bool dotsOnplane(const Point3D &a, const Point3D &b, const Point3D &c, const Point3D &d)\
 {
    return zero(dmult(pvec(a, b, c), subt(d, a)));
}

```

```
//判点是否在线段上，包括端点和共线
bool dotOnlineIn(const Point3D &p, const Line3D &l) {
    return zero(sqrlen(xmult(subt(p, l.a), subt(p, l.b)))) && (l.a.x - p.x) * (l.b.x - p\
.x) < EPS && (l.a.y - p.y) * (l.b.y - p.y) < EPS && (l.a.z - p.z) * (l.b.z - p.z) < EPS;
}
bool dotOnlineIn(const Point3D &p, const Point3D &l1, const Point3D &l2) {
    return zero(sqrlen(xmult(subt(p, l1), subt(p, l2)))) && (l1.x - p.x) * (l2.x - p.x) \
< EPS && (l1.y - p.y) * (l2.y - p.y) < EPS && (l1.z - p.z) * (l2.z - p.z) < EPS;
}

//判点是否在线段上，不包括端点
bool dotOnlineEx(const Point3D &p, const Line3D &l) {
    return dotOnlineIn(p, l) && (!zero(p.x - l.a.x) || !zero(p.y - l.a.y) || !zero(p.z -\
 l.a.z)) && (!zero(p.x - l.b.x) || !zero(p.y - l.b.y) || !zero(p.z - l.b.z));
}
bool dotOnlineEx(const Point3D &p, const Point3D &l1, const Point3D &l2) {
    return dotOnlineIn(p, l1, l2) && (!zero(p.x - l1.x) || !zero(p.y - l1.y) || !zero(p.\
z - l1.z)) && (!zero(p.x - l2.x) || !zero(p.y - l2.y) || !zero(p.z - l2.z));
}

//判点是否在空间三角形上，包括边界，三点共线无意义
bool dotInplaneIn(const Point3D &p, const Plane &s) {
    return zero(vlen(xmult(subt(s.a, s.b), subt(s.a, s.c))) - vlen(xmult(subt(p, s.a), s\
ubt(p, s.b))) - vlen(xmult(subt(p, s.b), subt(p, s.c))) - vlen(xmult(subt(p, s.c), subt(\
p, s.a))));
}
bool dotInplaneIn(const Point3D &p, const Point3D &s1, const Point3D &s2, const Point3D \
&s3) {
    return zero(vlen(xmult(subt(s1, s2), subt(s1, s3))) - vlen(xmult(subt(p, s1), subt(p\
, s2))) - vlen(xmult(subt(p, s2), subt(p, s3))) - vlen(xmult(subt(p, s3), subt(p, s1))))\
;
}

//判点是否在空间三角形上，不包括边界，三点共线无意义
bool dotInplaneEx(const Point3D &p, const Plane &s) {
    return dotInplaneIn(p, s) && sqrlen(xmult(subt(p, s.a), subt(p, s.b))) > EPS && sqrl\
en(xmult(subt(p, s.b), subt(p, s.c))) > EPS && sqrlen(xmult(subt(p, s.c), subt(p, s.a)))\
 > EPS;
}
bool dotInplaneEx(const Point3D &p, const Point3D &s1, const Point3D &s2, const Point3D \
&s3) {
    return dotInplaneIn(p, s1, s2, s3) && sqrlen(xmult(subt(p, s1), subt(p, s2))) > EPS \
&& sqrlen(xmult(subt(p, s2), subt(p, s3))) > EPS && sqrlen(xmult(subt(p, s3), subt(p, s1\
))) > EPS;
}

//判两点在线段同侧，点在线段上返回 0，不共面无意义
bool sameSide(const Point3D &p1, const Point3D &p2, const Line3D &l) {
    return dmult(xmult(subt(l.a, l.b), subt(p1, l.b)), xmult(subt(l.a, l.b), subt(p2, l.\
b))) > EPS;
}
bool sameSide(const Point3D &p1, const Point3D &p2, const Point3D &l1, const Point3D &l2\
) {
    return dmult(xmult(subt(l1, l2), subt(p1, l2)), xmult(subt(l1, l2), subt(p2, l2))) >\
 EPS;
}
```

```cpp

//判两点在线段异侧，点在线段上返回 0，不共面无意义
bool oppositeSide(const Point3D &p1, const Point3D &p2, const Line3D &l) {
    return dmult(xmult(subt(l.a, l.b), subt(p1, l.b)), xmult(subt(l.a, l.b), subt(p2, l.\
b))) < -EPS;
}
bool oppositeSide(const Point3D &p1, const Point3D &p2, const Point3D &l1, const Point3D\
 &l2) {
    return dmult(xmult(subt(l1, l2), subt(p1, l2)), xmult(subt(l1, l2), subt(p2, l2))) <\
 -EPS;
}

//判两点在平面同侧，点在平面上返回 0
bool sameSide(const Point3D &p1, const Point3D &p2, const Plane &s) {
    return dmult(pvec(s), subt(p1, s.a)) * dmult(pvec(s), subt(p2, s.a)) > EPS;
}
bool sameSide(const Point3D &p1, const Point3D &p2, const Point3D &s1, const Point3D &s2\
, const Point3D &s3) {
    return dmult(pvec(s1, s2, s3), subt(p1, s1)) * dmult(pvec(s1, s2, s3), subt(p2, s1))\
 > EPS;
}
bool sameSide(const Point3D &p1, const Point3D &p2, const PlaneF &s) {
    return (s.a * p1.x + s.b * p1.y + s.c * p1.z + s.d) * (s.a * p2.x + s.b * p2.y + s.c\
 * p2.z + s.d) > EPS;
}

//判两点在平面异侧，点在平面上返回 0
bool oppositeSide(const Point3D &p1, const Point3D &p2, const Plane &s) {
    return dmult(pvec(s), subt(p1, s.a)) * dmult(pvec(s), subt(p2, s.a)) < -EPS;
}
bool oppositeSide(const Point3D &p1, const Point3D &p2, const Point3D &s1, const Point3D\
 &s2, const Point3D &s3) {
    return dmult(pvec(s1, s2, s3), subt(p1, s1)) * dmult(pvec(s1, s2, s3), subt(p2, s1))\
 < -EPS;
}
bool oppositeSide(const Point3D &p1, const Point3D &p2, const PlaneF &s) {
    return (s.a*p1.x+s.b*p1.y+s.c*p1.z+s.d) * (s.a*p2.x+s.b*p2.y+s.c*p2.z+s.d) < -EPS;
}

//判两直线平行
bool parallel(const Line3D &u, const Line3D &v) {
    return sqrlen(xmult(subt(u.a, u.b), subt(v.a, v.b))) < EPS;
}
bool parallel(const Point3D &u1, const Point3D &u2, const Point3D &v1, const Point3D &v2\
) {
    return sqrlen(xmult(subt(u1, u2), subt(v1, v2))) < EPS;
}

//判两平面平行
bool parallel(const Plane &u, const Plane &v) {
    return sqrlen(xmult(pvec(u), pvec(v))) < EPS;
}
bool parallel(const Point3D &u1, const Point3D &u2, const Point3D &u3, const Point3D &v1\
, const Point3D &v2, const Point3D &v3) {
    return sqrlen(xmult(pvec(u1, u2, u3), pvec(v1, v2, v3))) < EPS;
```

```
}
bool parallel(const PlaneF &u, const PlaneF &v) {
    return sqrlen(xmult(pvec(u), pvec(v))) < EPS;
}

//判直线与平面平行
bool parallel(const Line3D &l, const Plane &s) {
    return zero(dmult(subt(l.a, l.b), pvec(s)));
}
bool parallel(const Point3D &l1, const Point3D &l2, const Point3D &s1, const Point3D &s2\
, const Point3D &s3) {
    return zero(dmult(subt(l1, l2), pvec(s1, s2, s3)));
}
bool parallel(const Line3D &l, const PlaneF &s) {
    return zero(dmult(subt(l.a, l.b), pvec(s)));
}

//判两直线垂直
bool perpendicular(const Line3D &u, const Line3D &v) {
    return zero(dmult(subt(u.a, u.b), subt(v.a, v.b)));
}
bool perpendicular(const Point3D &u1, const Point3D &u2, const Point3D &v1, const Point3\
D &v2) {
    return zero(dmult(subt(u1, u2), subt(v1, v2)));
}

//判两平面垂直
bool perpendicular(const Plane &u, const Plane &v) {
    return zero(dmult(pvec(u), pvec(v)));
}
bool perpendicular(const Point3D &u1, const Point3D &u2, const Point3D &u3, const Point3\
D &v1, const Point3D &v2, const Point3D &v3) {
    return zero(dmult(pvec(u1, u2, u3), pvec(v1, v2, v3)));
}
bool perpendicular(const PlaneF &u, const PlaneF &v) {
    return zero(dmult(pvec(u), pvec(v)));
}

//判直线与平面垂直
bool perpendicular(const Line3D &l, const Plane &s) {
    return sqrlen(xmult(subt(l.a, l.b), pvec(s))) < EPS;
}
bool perpendicular(const Point3D &l1, const Point3D &l2, const Point3D &s1, const Point3\
D &s2, const Point3D &s3) {
    return sqrlen(xmult(subt(l1, l2), pvec(s1, s2, s3))) < EPS;
}
bool perpendicular(const Line3D &l, const PlaneF &s) {
    return sqrlen(xmult(subt(l.a, l.b), pvec(s))) < EPS;
}

//判两线段相交，包括端点和部分重合
bool intersectIn(const Line3D &u, const Line3D &v) {
    if (!dotsOnplane(u.a, u.b, v.a, v.b)) {
        return 0;
    } else if (!dotsInline(u.a, u.b, v.a) || !dotsInline(u.a, u.b, v.b)) {
        return !sameSide(u.a, u.b, v) && !sameSide(v.a, v.b, u);
```

```
    } else {
        return dotOnlineIn(u.a, v) || dotOnlineIn(u.b, v) || dotOnlineIn(v.a, u) || dotO\
nlineIn(v.b, u);
    }
}
bool intersectIn(const Point3D &u1, const Point3D &u2, const Point3D &v1, const Point3D \
&v2) {
    if (!dotsOnplane(u1, u2, v1, v2)) {
        return 0;
    } else if (!dotsInline(u1, u2, v1) || !dotsInline(u1, u2, v2)) {
        return !sameSide(u1, u2, v1, v2) && !sameSide(v1, v2, u1, u2);
    } else {
        return dotOnlineIn(u1, v1, v2) || dotOnlineIn(u2, v1, v2) || dotOnlineIn(v1, u1,\
 u2) || dotOnlineIn(v2, u1, u2);
    }
}

//判两线段相交，不包括端点和部分重合
bool intersectEx(const Line3D &u, const Line3D &v) {
    return dotsOnplane(u.a, u.b, v.a, v.b) && oppositeSide(u.a, u.b, v) && oppositeSide(\
v.a, v.b, u);
}
bool intersectEx(const Point3D &u1, const Point3D &u2, const Point3D &v1, const Point3D \
&v2) {
    return dotsOnplane(u1, u2, v1, v2) && oppositeSide(u1, u2, v1, v2) && oppositeSide(v\
1, v2, u1, u2);
}

//判线段与空间三角形相交，包括交于边界和（部分）包含
bool intersectIn(const Line3D &l, const Plane &s) {
    return !sameSide(l.a, l.b, s) && !sameSide(s.a, s.b, l.a, l.b, s.c) && !sameSide(s.b\
, s.c, l.a, l.b, s.a) && !sameSide(s.c, s.a, l.a, l.b, s.b);
}
bool intersectIn(const Point3D &l1, const Point3D &l2, const Point3D &s1, const Point3D \
&s2, const Point3D &s3) {
    return !sameSide(l1, l2, s1, s2, s3) && !sameSide(s1, s2, l1, l2, s3) && !sameSide(s\
2, s3, l1, l2, s1) && !sameSide(s3, s1, l1, l2, s2);
}

//判线段与空间三角形相交，不包括交于边界和（部分）包含
bool intersectEx(const Line3D &l, const Plane &s) {
    return oppositeSide(l.a, l.b, s) && oppositeSide(s.a, s.b, l.a, l.b, s.c) && opposit\
eSide(s.b, s.c, l.a, l.b, s.a) && oppositeSide(s.c, s.a, l.a, l.b, s.b);
}
bool intersectEx(const Point3D &l1, const Point3D &l2, const Point3D &s1, const Point3D \
&s2, const Point3D &s3) {
    return oppositeSide(l1, l2, s1, s2, s3) && oppositeSide(s1, s2, l1, l2, s3) && oppos\
iteSide(s2, s3, l1, l2, s1) && oppositeSide(s3, s1, l1, l2, s2);
}

//计算两直线交点，注意事先判断直线是否共面和平行！
//线段交点请另外判线段相交（同时还是要判断是否平行！）
# include <algorithm>

using namespace std;
```

```
Point3D intersection(Point3D u1, Point3D u2, Point3D v1, Point3D v2) {
    double dxu = u2.x - u1.x;
    double dyu = u2.y - u1.y;
    double dzu = u2.z - u1.z;
    double dxv = v2.x - v1.x;
    double dyv = v2.y - v1.y;
    double dzv = v2.z - v1.z;
    double t;
    if (!zero(dxu * dyv - dyu * dxv)) {
        t = (dyv * (v1.x - u1.x) + dxv * (u1.y - v1.y)) / (dxu * dyv - dyu * dxv);
    } else if (!zero(dxu * dzv - dzu * dxv)) {
        t = (dzv * (v1.x - u1.x) + dxv * (u1.z - v1.z)) / (dxu * dzv - dzu * dxv);
    } else {
        t = (dzv * (v1.y - u1.y) + dyv * (u1.z - v1.z)) / (dyu * dzv - dzu * dyv);
    }
    Point3D ret;
    ret.x = u1.x + dxu * t;
    ret.y = u1.y + dyu * t;
    ret.z = u1.z + dzu * t;
    return ret;
}

//计算直线与平面交点，注意事先判断是否平行，并保证三点不共线！
//线段和空间三角形交点请另外判断
Point3D intersection(const Line3D &l, const Plane &s) {
    Point3D ret = pvec(s);
    double t = (ret.x * (s.a.x - l.a.x) + ret.y * (s.a.y - l.a.y) + ret.z * (s.a.z - l.a\
.z)) / (ret.x * (l.b.x - l.a.x) + ret.y * (l.b.y - l.a.y) + ret.z * (l.b.z - l.a.z));
    ret.x = l.a.x + (l.b.x - l.a.x) * t;
    ret.y = l.a.y + (l.b.y - l.a.y) * t;
    ret.z = l.a.z + (l.b.z - l.a.z) * t;
    return ret;
}
Point3D intersection(const Point3D &l1, const Point3D &l2, const Point3D &s1, const Poin\
t3D &s2, const Point3D &s3) {
    Point3D ret = pvec(s1, s2, s3);
    double t = (ret.x * (s1.x - l1.x) + ret.y * (s1.y - l1.y) + ret.z * (s1.z - l1.z)) /\
 (ret.x * (l2.x - l1.x) + ret.y * (l2.y - l1.y) + ret.z * (l2.z - l1.z));
    ret.x = l1.x + (l2.x - l1.x) * t;
    ret.y = l1.y + (l2.y - l1.y) * t;
    ret.z = l1.z + (l2.z - l1.z) * t;
    return ret;
}
Point3D intersection(const Line3D &l, const PlaneF &s) {
    Point3D ret = subt(l.b, l.a);
    double t = -(dmult(pvec(s), l.a) + s.d) / (dmult(pvec(s), ret));
    ret.x = ret.x * t + l.a.x;
    ret.y = ret.y * t + l.a.y;
    ret.z = ret.z * t + l.a.z;
    return ret;
}

//计算两平面交线，注意事先判断是否平行，并保证三点不共线！
Line3D intersection(const Plane &u, const Plane &v) {
    Line3D ret;
    ret.a = parallel(v.a, v.b, u.a, u.b, u.c) ? intersection(v.b, v.c, u.a, u.b, u.c) : \
```

```
intersection(v.a, v.b, u.a, u.b, u.c);
    ret.b = parallel(v.c, v.a, u.a, u.b, u.c) ? intersection(v.b, v.c, u.a, u.b, u.c) : \
intersection(v.c, v.a, u.a, u.b, u.c);
    return ret;
}
Line3D intersection(const Point3D &u1, const Point3D &u2, const Point3D &u3, const Point\
3D &v1, const Point3D &v2, const Point3D &v3) {
    Line3D ret;
    ret.a = parallel(v1, v2, u1, u2, u3) ? intersection(v2, v3, u1, u2, u3) : intersecti\
on(v1, v2, u1, u2, u3);
    ret.b = parallel(v3, v1, u1, u2, u3) ? intersection(v2, v3, u1, u2, u3) : intersecti\
on(v3, v1, u1, u2, u3);
    return ret;
}

//点到直线距离
double disptoline(const Point3D &p, const Line3D &l) {
    return vlen(xmult(subt(p, l.a), subt(l.b, l.a))) / dis(l.a, l.b);
}
double disptoline(const Point3D &p, const Point3D &l1, const Point3D &l2) {
    return vlen(xmult(subt(p, l1), subt(l2, l1))) / dis(l1, l2);
}

//点到直线最近点
Point3D ptoline(const Point3D &p, const Line3D &l) {
    Point3D ab = subt(l.b, l.a);
    double t = - dmult(subt(p, l.a), ab) / sqrlen(ab);
    ab.x *= t;
    ab.y *= t;
    ab.z *= t;
    return subt(l.a, ab);
}

//点到平面距离
double disptoplane(const Point3D &p, const Plane &s) {
    return fabs(dmult(pvec(s), subt(p, s.a))) / vlen(pvec(s));
}
double disptoplane(const Point3D &p, const Point3D &s1, const Point3D &s2, const Point3D\
 &s3) {
    return fabs(dmult(pvec(s1, s2, s3), subt(p, s1))) / vlen(pvec(s1, s2, s3));
}
double disptoplane(const Point3D &p, const PlaneF &s) {
    return fabs((dmult(pvec(s), p)+s.d) / vlen(pvec(s)));
}

//点到平面最近点
Point3D ptoplane(const Point3D &p, const PlaneF &s) {
    Line3D l;
    l.a = p;
    l.b = pvec(s);
    l.b.x += p.x;
    l.b.y += p.y;
    l.b.z += p.z;
    return intersection(l, s);
}

```

```
//直线到直线距离
double dislinetoline(const Line3D &u, const Line3D &v) {
    Point3D n = xmult(subt(u.a, u.b), subt(v.a, v.b));
    return fabs(dmult(subt(u.a, v.a), n)) / vlen(n);
}
double dislinetoline(const Point3D &u1, const Point3D &u2, const Point3D &v1, const Poin\
t3D &v2) {
    Point3D n = xmult(subt(u1, u2), subt(v1, v2));
    return fabs(dmult(subt(u1, v1), n)) / vlen(n);
}

//直线到直线的最近点对
//p1 在 u 上，p2 在 v 上，p1 到 p2 是 uv 之间的最近距离
//注意，保证两直线不平行
void linetoline(const Line3D &u, const Line3D &v, Point3D &p1, Point3D &p2) {
    Point3D ab = subt(u.b, u.a), cd = subt(v.b, v.a), ac = subt(v.a, u.a);
    double r = (dmult(ab, cd) * dmult(ac, ab) - sqrlen(ab) * dmult(ac, cd)) / (sqrlen(ab\
)*sqrlen(cd) - sqr(dmult(ab, cd)));
    p2.x = v.a.x + r * cd.x;
    p2.y = v.a.y + r * cd.y;
    p2.z = v.a.z + r * cd.z;
    p1 = ptoline(p2, u);
}

//两直线夹角 cos 值
double angleCos(const Line3D &u, const Line3D &v) {
    return dmult(subt(u.a, u.b), subt(v.a, v.b)) / vlen(subt(u.a, u.b)) / vlen(subt(v.a,\
 v.b));
}
double angleCos(const Point3D &u1, const Point3D &u2, const Point3D &v1, const Point3D &\
v2) {
    return dmult(subt(u1, u2), subt(v1, v2)) / vlen(subt(u1, u2)) / vlen(subt(v1, v2));
}

//两平面夹角 cos 值
double angleCos(const Plane &u, const Plane &v) {
    return dmult(pvec(u), pvec(v)) / vlen(pvec(u)) / vlen(pvec(v));
}
double angleCos(const Point3D &u1, const Point3D &u2, const Point3D &u3, const Point3D &\
v1, const Point3D &v2, const Point3D &v3) {
    return dmult(pvec(u1, u2, u3), pvec(v1, v2, v3)) / vlen(pvec(u1, u2, u3)) / vlen(pve\
c(v1, v2, v3));
}
double angleCos(const PlaneF &u, const PlaneF &v) {
    return dmult(pvec(u), pvec(v)) / (vlen(pvec(u)) * vlen(pvec(v)));
}

//直线平面夹角 sin 值
double angleSin(const Line3D &l, const Plane &s) {
    return dmult(subt(l.a, l.b), pvec(s)) / vlen(subt(l.a, l.b)) / vlen(pvec(s));
}
double angleSin(const Point3D &l1, const Point3D &l2, const Point3D &s1, const Point3D &\
s2, const Point3D &s3) {
    return dmult(subt(l1, l2), pvec(s1, s2, s3)) / vlen(subt(l1, l2)) / vlen(pvec(s1, s2\
, s3));
```

```
}
double angleSin(Line3D l, const PlaneF &s) {
    return dmult(subt(l.a, l.b), pvec(s)) / (vlen(subt(l.a, l.b)) * vlen(pvec(s)));
}

// 平面方程形式转化 Plane -> PlaneF
PlaneF planeToPlaneF(const Plane &p) {
    PlaneF ret;
    Point3D m = xmult(subt(p.b, p.a), subt(p.c, p.a));
    ret.a = m.x;
    ret.b = m.y;
    ret.c = m.z;
    ret.d = -m.x * p.a.x - m.y * p.a.y - m.z * p.a.z;
    return ret;
}
```

1.7 三角形

```
# include <cmath>

struct Point {
    double x, y;
};
struct Line {
    Point a, b;
};

inline double dis(const Point &p1, const Point &p2) {
    return sqrt((p1.x - p2.x) * (p1.x - p2.x) + (p1.y - p2.y) * (p1.y - p2.y));
}

Point intersection(const Line &u, const Line &v) {
    Point ret = u.a;
    double t = ((u.a.x - v.a.x) * (v.a.y - v.b.y) - (u.a.y - v.a.y) * (v.a.x - v.b.x)) /\
 ((u.a.x - u.b.x) * (v.a.y - v.b.y) - (u.a.y - u.b.y) * (v.a.x - v.b.x));
    ret.x += (u.b.x - u.a.x) * t;
    ret.y += (u.b.y - u.a.y) * t;
    return ret;
}

//外心
Point circumcenter(const Point &a, const Point &b, const Point &c) {
    Line u, v;
    u.a.x = (a.x + b.x) / 2;
    u.a.y = (a.y + b.y) / 2;
    u.b.x = u.a.x - a.y + b.y;
    u.b.y = u.a.y + a.x - b.x;
    v.a.x = (a.x + c.x) / 2;
    v.a.y = (a.y + c.y) / 2;
    v.b.x = v.a.x - a.y + c.y;
    v.b.y = v.a.y + a.x - c.x;
    return intersection(u, v);
}

//内心
```

```
Point incenter(const Point &a, const Point &b, const Point &c) {
    Line u, v;
    double m, n;
    u.a = a;
    m = atan2(b.y - a.y, b.x - a.x);
    n = atan2(c.y - a.y, c.x - a.x);
    u.b.x = u.a.x + cos((m + n) / 2);
    u.b.y = u.a.y + sin((m + n) / 2);
    v.a = b;
    m = atan2(a.y - b.y, a.x - b.x);
    n = atan2(c.y - b.y, c.x - b.x);
    v.b.x = v.a.x + cos((m + n) / 2);
    v.b.y = v.a.y + sin((m + n) / 2);
    return intersection(u, v);
}

//垂心
Point perpencenter(const Point &a, const Point &b, const Point &c) {
    Line u, v;
    u.a = c;
    u.b.x = u.a.x - a.y + b.y;
    u.b.y = u.a.y + a.x - b.x;
    v.a = b;
    v.b.x = v.a.x - a.y + c.y;
    v.b.y = v.a.y + a.x - c.x;
    return intersection(u, v);
}

//重心
//到三角形三顶点距离的平方和最小的点
//三角形内到三边距离之积最大的点
Point barycenter(const Point &a, const Point &b, const Point &c) {
    Line u, v;
    u.a.x = (a.x + b.x) / 2;
    u.a.y = (a.y + b.y) / 2;
    u.b = c;
    v.a.x = (a.x + c.x) / 2;
    v.a.y = (a.y + c.y) / 2;
    v.b = b;
    return intersection(u, v);
}
```

1.8 圆

```
# include <cmath>

const double EPS = 1e-8;

struct Point {
    double x, y;
};

inline double xmult(const Point &p1, const Point &p2, const Point &p0) {
    return (p1.x - p0.x) * (p2.y - p0.y) - (p2.x - p0.x) * (p1.y - p0.y);
}
```

```
inline double dis(const Point &p1, const Point &p2) {
    return sqrt((p1.x - p2.x) * (p1.x - p2.x) + (p1.y - p2.y) * (p1.y - p2.y));
}

double disptoline(const Point &p, const Point &l1, const Point &l2) {
    return fabs(xmult(p, l1, l2)) / dis(l1, l2);
}

Point intersection(const Point &u1, const Point &u2, const Point &v1, const Point &v2) {
    Point ret = u1;
    double t = ((u1.x - v1.x) * (v1.y - v2.y) - (u1.y - v1.y) * (v1.x - v2.x)) / ((u1.x \
- u2.x) * (v1.y - v2.y) - (u1.y - u2.y) * (v1.x - v2.x));
    ret.x += (u2.x - u1.x) * t;
    ret.y += (u2.y - u1.y) * t;
    return ret;
}

//判直线和圆相交，包括相切
int intersectLineCircle(const Point &c, double r, const Point &l1, const Point &l2) {
    return disptoline(c, l1, l2) < r + EPS;
}

//判线段和圆相交，包括端点和相切
int intersectSegCircle(const Point &c, double r, const Point &l1, const Point &l2) {
    double t1 = dis(c, l1) - r, t2 = dis(c, l2) - r;
    Point t = c;
    if (t1 < EPS || t2 < EPS) {
        return t1 > -EPS || t2 > -EPS;
    }
    t.x += l1.y - l2.y;
    t.y += l2.x - l1.x;
    return xmult(l1, c, t) * xmult(l2, c, t) < EPS && disptoline(c, l1, l2) - r < EPS;
}

//判圆和圆相交，包括相切
int intersectCircleCircle(const Point &c1, double r1, const Point &c2, double r2) {
    return dis(c1, c2) < r1 + r2 + EPS && dis(c1, c2) > fabs(r1 - r2) - EPS;
}

//计算圆上到点 p 最近点，如 p 与圆心重合，返回 p 本身
Point dotToCircle(const Point &c, double r, const Point &p) {
    Point u, v;
    if (dis(p, c)<EPS) {
        return p;
    }
    u.x = c.x + r * fabs(c.x - p.x) / dis(c, p);
    u.y = c.y + r * fabs(c.y - p.y) / dis(c, p) * ((c.x - p.x) * (c.y - p.y) < 0 ? -1 : \
1);
    v.x = c.x - r * fabs(c.x - p.x) / dis(c, p);
    v.y = c.y - r * fabs(c.y - p.y) / dis(c, p) * ((c.x - p.x) * (c.y - p.y) < 0 ? -1 : \
1);
    return dis(u, p) < dis(v, p) ? u : v;
}
```

```
//计算直线与圆的交点，保证直线与圆有交点
//计算线段与圆的交点可用这个函数后判点是否在线段上
void intersectionLineCircle(const Point &c, double r, const Point &l1, const Point &l2, \
Point &p1, Point &p2) {
    Point p = c;
    p.x += l1.y - l2.y;
    p.y += l2.x - l1.x;
    p = intersection(p, c, l1, l2);
    double t = sqrt(r * r - dis(p, c) * dis(p, c)) / dis(l1, l2);
    p1.x = p.x + (l2.x - l1.x) * t;
    p1.y = p.y + (l2.y - l1.y) * t;
    p2.x = p.x - (l2.x - l1.x) * t;
    p2.y = p.y - (l2.y - l1.y) * t;
}

//计算圆与圆的交点，保证圆与圆有交点，圆心不重合
void intersectionCircleCircle(const Point &c1, double r1, const Point &c2, double r2, Po\
int &p1, Point &p2) {
    Point u, v;
    double t = (1 + (r1 * r1 - r2 * r2) / dis(c1, c2) / dis(c1, c2)) / 2;
    u.x = c1.x + (c2.x - c1.x) * t;
    u.y = c1.y + (c2.y - c1.y) * t;
    v.x = u.x + c1.y - c2.y;
    v.y = u.y - c1.x + c2.x;
    intersectionLineCircle(c1, r1, u, v, p1, p2);
}
```

1.9 球面

```
# include <cmath>

const double PI = acos(-1.0);

//计算圆心角 lat 表示纬度， - 90 <= w <= 90, lng 表示经度
//返回两点所在大圆劣弧对应圆心角, 0 <= angle <= PI
double angle(double lng1, double lat1, double lng2, double lat2) {
    double dlng = fabs(lng1 - lng2) * PI / 180;
    while (dlng >= PI + PI) {
        dlng -= PI + PI;
    }
    if (dlng > PI) {
        dlng = PI + PI - dlng;
    }
    lat1 *= PI / 180;
    lat2 *= PI / 180;
    return acos(cos(lat1) * cos(lat2) * cos(dlng) + sin(lat1) * sin(lat2));
}

//计算距离，r 为球半径
double lineDist(double r, double lng1, double lat1, double lng2, double lat2) {
    double dlng = fabs(lng1 - lng2) * PI / 180;
    while (dlng >= PI + PI) {
        dlng -= PI + PI;
    }
```

```
    if (dlng > PI) {
        dlng = PI + PI - dlng;
    }
    lat1 *= PI / 180;
    lat2 *= PI / 180;
    return r * sqrt(2 - 2 * (cos(lat1) * cos(lat2) * cos(dlng) + sin(lat1) * sin(lat2)))\
;
}

//计算球面距离, r 为球半径
inline double sphereDist(double r, double lng1, double lat1, double lng2, double lat2) {
    return r * angle(lng1, lat1, lng2, lat2);
}
```

1.10 网格 (pick)

```
# include <cstdlib>

struct Point {
    int x, y;
};

int gcd(int a, int b) {
    return b ? gcd(b, a % b) : a;
}

//多边形上的网格点个数
int gridOnedge(int n, const Point *p) {
    int i, ret = 0;
    for (i = 0; i < n; i++) {
        ret += gcd(abs(p[i].x - p[(i + 1) % n].x), abs(p[i].y - p[(i + 1) % n].y));
    }
    return ret;
}

//多边形内的网格点个数
int gridInside(int n, const Point *p) {
    int i, ret = 0;
    for (i = 0; i < n; i++) {
        ret += p[(i + 1) % n].y * (p[i].x - p[(i + 2) % n].x);
    }
    return (abs(ret) - gridOnedge(n, p)) / 2 + 1;
}
```

Chapter 2

组合

2.1 组合公式

```
1. C(m,n)=C(m,m-n)
2. C(m,n)=C(m-1,n)+C(m-1,n-1)

错排 (derangement)
D(n) = n!(1 - 1/1! + 1/2! - 1/3! + ... + (-1)^n/n!)
     = (n-1)(D(n-2) + D(n-1))

Q(n) = D(n) + D(n-1)

Catalan numbers:
Ca(n)=C(2n-2,n-1)/n

K-dimensional Catalan numbers:
A(n) = 0! * 1! * .. * (k - 1)! * (k * n)! / (n! * (n + 1)! * .. * (n + k - 1)!)

2-d:
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900
3-d:
1, 5, 42, 462, 6006, 87516, 1385670, 23371634, 414315330
4-d:
1, 14, 462, 24024, 1662804, 140229804, 13672405890, 1489877926680

求和公式,k = 1..n
1. sum( k ) = n(n+1)/2
2. sum( 2k-1 ) = n^2
3. sum( k^2 ) = n(n+1)(2n+1)/6
4. sum( (2k-1)^2 ) = n(4n^2-1)/3
5. sum( k^3 ) = (n(n+1)/2)^2
6. sum( (2k-1)^3 ) = n^2(2n^2-1)
7. sum( k^4 ) = n(n+1)(2n+1)(3n^2+3n-1)/30
8. sum( k^5 ) = n^2(n+1)^2(2n^2+2n-1)/12
9. sum( k(k+1) ) = n(n+1)(n+2)/3
10. sum( k(k+1)(k+2) ) = n(n+1)(n+2)(n+3)/4
12. sum( k(k+1)(k+2)(k+3) ) = n(n+1)(n+2)(n+3)(n+4)/5
```

2.2 字典序全排列

```
//字典序全排列与序号的转换
int perm2Num(int n, const int *p) {
    int ret = 0, k = 1;
    for (int i = n - 2; i >= 0; k *= n - (i--)) {
        for (int j = i + 1; j < n; j++) {
            if (p[j] < p[i]) {
                ret += k;
            }
        }
    }
    return ret;
}

void num2Perm(int n, int *p, int t) {
    for (int i = n - 1; i >= 0; i--) {
        p[i] = t % (n - i);
        t /= n - i;
    }
    for (int i = n - 1; i; i--) {
        for (int j = i - 1; j >= 0; j--) {
            if (p[j] <= p[i]) {
                p[i]++;
            }
        }
    }
}
```

2.3 字典序组合

```
//字典序组合与序号的转换
//comb 为组合数 C(n, m), 必要时换成大数, 注意处理 C(n, m) = 0 | n < m
int comb(int n, int m) {
    int ret = 1;
    m = m < (n - m) ? m :(n - m);
    for (int i = n - m + 1; i <= n; ret *= (i++));
    for (int i = 1; i <= m; ret /= (i++));
    return m < 0 ? 0 : ret;
}

int comb2Num(int n, int m, const int *c) {
    int ret = comb(n, m);
    for (int i = 0; i < m; i++) {
        ret -= comb(n - c[i], m - i);
    }
    return ret;
}

void num2Comb(int n, int m, int *c, int t) {
    int j = 1, k;
    for (int i = 0; i < m; c[i++] = j++) {
        for (; t > (k = comb(n - j, m - i - 1)); t -= k, j++);
    }
}
```

2.4 排列组合生成

```
//genPerm 产生字典序排列 P(n, m)
//genComb 产生字典序组合 C(n, m)
//genPermSwap 产生相邻位对换全排列 P(n, n)
//产生元素用 1..n 表示
//dummy 为产生后调用的函数, 传入 a[] 和 n, a[0]..a[n-1] 为一次产生的结果
const int MAXN = 100;

void dummy(const int *a, int n) {
    //...
}

void genPermRc(int *a, int n, int m, int k, int *temp, bool *tag) {
    if (k == m) {
        dummy(temp, m);
    } else {
        for (int i = 0; i < n; i++) {
            if (!tag[i]) {
                temp[k] = a[i], tag[i] = true;
                genPermRc(a, n, m, k + 1, temp, tag);
                tag[i] = 0;
            }
        }
    }
}

void genPerm(int n, int m) {
    int a[MAXN], temp[MAXN];
    bool tag[MAXN] = {false};
    for (int i = 0; i < n; i++) {
        a[i] = i + 1;
    }
    genPermRc(a, n, m, 0, temp, tag);
}

void genCombRc(int *a, int s, int e, int m, int &count, int *temp) {
    if (m == 0) {
        dummy(temp, count);
    } else {
        for (int i = s; i <= e - m + 1; i++) {
            temp[count++] = a[i];
            genCombRc(a, i + 1, e, m - 1, count, temp);
            count--;
        }
    }
}

void genComb(int n, int m) {
    int a[MAXN], temp[MAXN], count = 0;
    for (int i = 0; i < n; i++) {
        a[i] = i + 1;
    }
    genCombRc(a, 0, n-1, m, count, temp);
}
```

```
void genPermSwapRc(int *a, int n, int k, int *pos, int *dir) {
    int p1, p2, t;
    if (k == n) {
        dummy(a, n);
    } else {
        genPermSwapRc(a, n, k + 1, pos, dir);
        for (int i = 0; i < k; i++) {
            p2 = (p1 = pos[k]) + dir[k];
            t = a[p1];
            a[p1] = a[p2];
            a[p2] = t;
            pos[a[p1] - 1] = p1;
            pos[a[p2] - 1] = p2;
            genPermSwapRc(a, n, k + 1, pos, dir);
        }
        dir[k] = -dir[k];
    }
}

void genPermSwap(int n) {
    int a[MAXN], pos[MAXN], dir[MAXN];
    for (int i = 0; i < n; i++) {
        a[i] = i + 1;
        pos[i] = i;
        dir[i] = -1;
    }
    genPermSwapRc(a, n, 0, pos, dir);
}
```

2.5 生成 gray 码

```
//生成 reflected gray code
//每次调用 gray 取得下一个码
//000...000 是第一个码,100...000 是最后一个码
void gray(int n, int *code) {
    int t = 0;
    for (int i = 0; i < n; t += code[i++]);
    if (t & 1) {
        for (n--; !code[n]; n--);
    }
    code[n - 1] = 1 - code[n - 1];
}
```

2.6 置换 (polya)

```
//求置换的循环节, polya 原理
//perm[0..n-1] 为 0..n-1 的一个置换 (排列)
//返回置换最小周期, num 返回循环节个数
const int MAXN = 1000;

int gcd(int a, int b) {
```

```
    return b ? gcd(b, a % b) : a;
}

int polya(int *perm, int n, int &num) {
    int i, j, p, v[MAXN] = {0}, ret = 1;
    for (num = i = 0; i < n; i++) {
        if (!v[i]) {
            num++;
            p = i;
            for (j=0; !v[p = perm[p]]; j++) {
                v[p] = 1;
            }
            ret *= j / gcd(ret, j);
        }
    }
    return ret;
}
```

Chapter 3

数论

3.1 整除规则

```
最后 n 位可以被 2^n 整除, 则原数被 2^n 整除
各位数和可以被 3,9 整除, 则原数被 3,9 整除
最后 n 位可以被 5^n 整除, 则原数被 5^n 整除

对于其他的小素数有通用的方法:
删除最低位 (设为 d), 剩下的数减去 y*d 得到的新数被 x 整除, 则原数可以被 x 整除.(此过程可以重复直到 \
数小到可以直接判)
x  y
7  2
11 1
13 9
17 5
19 17
23 16
29 26
31 3
37 11
41 4
43 30
47 14
```

3.2 分解质因数

```
//分解质因数
//primeFactor() 传入 n, 返回不同质因数的个数
//f 存放质因数, nf 存放对应质因数的个数
//先调用 initPrime(), 其中第二个 initPrime() 更快

# include <cmath>

const int PSIZE = 100000;

int plist[PSIZE], pcount = 0;

bool prime(int n) {
```

```
    if ((n != 2 && !(n % 2)) || (n != 3 && !(n % 3)) || (n != 5 && !(n % 5)) || (n != 7 \
&& !(n % 7))) {
        return false;
    }
    for (int i = 0; plist[i] * plist[i] <= n; ++i) {
        if (n % plist[i] == 0) {
            return false;
        }
    }
    return n > 1;
}

void initPrime() {
    plist[pcount++] = 2;
    for (int i = 3; i < 100000; i += 2) {
        if (prime(i)) {
            plist[pcount++] = i;
        }
    }
}

int primeFactor(int n, int *f, int *nf) {
    int cnt = 0;
    int n2 = (int) sqrt((double) n);
    for (int i = 0; n > 1 && plist[i] <= n2; ++i) {
        if (n % plist[i] == 0) {
            for (nf[cnt] = 0; n % plist[i] == 0; n /= plist[i]) {
                ++nf[cnt];
            }
            f[cnt++] = plist[i];
        }
    }
    if (n > 1) {
        nf[cnt] = 1;
        f[cnt++] = n;
    }
    return cnt;
}

//产生 MAXN 以内的所有素数
//note:2863311530 就是 10101010101010101010101010101010
//给所有 2 的倍数赋初值
# include <cmath>

const int MAXN = 100000000;
unsigned int plist[6000000], pcount;
unsigned int isPrime[(MAXN >> 5) + 1];

# define setbitzero(a) (isPrime[(a) >> 5] &= (~(1 << ((a) & 31))))
# define setbitone(a) (isPrime[(a) >> 5] |= (1 << ((a) & 31)))
# define ISPRIME(a) (isPrime[(a) >> 5] & (1 << ((a) & 31)))

void initPrime() {
    int i, j, m;
    int t = (MAXN >> 5) + 1;
```

```
    for (i = 0; i < t; ++i) {
        isPrime[i] = 2863311530;
    }
    plist[0] = 2;
    setbitone(2);
    setbitzero(1);
    m = (int) sqrt((double) MAXN);
    pcount = 1;
    for (i = 3; i <= m; i += 2) {
        if (ISPRIME(i)) {
            plist[pcount++] = i;
            for (j = i << 1; j <= MAXN; j += i) {
                setbitzero(j);
            }
        }
    }
    if (!(i & 1)) {
        ++i;
    }
    for (; i <= MAXN; i += 2) {
        if (ISPRIME(i)) {
            plist[pcount++] = i;
        }
    }
}

// O(n) 筛素数表
// 返回素数个数
// 素数保存在 plist 里面
// minp 存放最小的素因子
# include <cstring>

const int MAXN = 1000000;

int minp[MAXN + 1], plist[MAXN + 1];

int prime(int n = MAXN) {
    int num = 0;
    memset(minp, 0, sizeof(minp));
    for (int i = 2; i <= n; i++) {
        if (!minp[i]) plist[num++] = i, minp[i] = i;
        for (int j = 0; j < num && i * plist[j] <= n; j++) {
            minp[i * plist[j]] = plist[j];
            if (i % plist[j] == 0) break;
        }
    }
    return num;
}

// 先调用 prime 初始化
// 然后就可以调用 factorize 分解质因素
// 结果存在 p 里面，返回质因数个数
// 要保证 n >= 2
int factorize(int n, int *p) {
```

```
    int num = 0;
    while (n != 1) {
        p[num++] = minp[n];
        n /= minp[n];
    }
    return num;
}
```

3.3 同余方程合并

```
/*
    同余方程合并
    x % m[1] = c[1]
    x % m[2] = c[2]
    ...
    x % m[n] = c[n]
    注意下标从 1 开始
    返回最小解 x
    返回值为-1 表示无解
    可以处理 m[i] 不互素的情况,c[i]>=m[i] 将认为无解
*/
# define N 105
typedef long long lld;
lld exgcd(lld a,lld b,lld &x,lld &y) {
    if (!b) {
        x=1;
        y=0;
        return a;
    } else {
        lld res=exgcd(b,a%b,x,y);
        lld t=x;
        x=y;
        y=t-(a/b)*y;
        return res;
    }
}

inline lld mod(lld x,lld y) {
    lld res=x%y;
    if (res<0) res+=y;
    return res;
}

lld solve(int n,lld m[],lld c[]) {
    for (int i=1; i<=n; i++)
        if (c[i]>=m[i])
            return -1;
    lld ans=c[1],LCM=m[1],x,y,g,mm;
    for (int i=2; i<=n; i++) {
        g=exgcd(LCM,m[i],x,y);
        if ((c[i]-ans)%g) return -1;
        ans=mod( ans + LCM * mod( (c[i]-ans)/g*x, m[i]/g )  , LCM/g*m[i]);
        LCM=LCM/g*m[i];
```

```
    }
    return mod(ans,LCM);
}
```

3.4 模线性方程 (组)

```
//扩展 Euclid 求解 gcd(a,b)=ax+by
int extGcd(int a, int b, int &x, int &y) {
    int t, ret;
    if (!b) {
        x = 1;
        y = 0;
        return a;
    }
    ret = extGcd(b, a % b, x, y);
    t = x;
    x = y;
    y = t - a / b * y;
    return ret;
}

//计算 m^a, O(loga), 本身没什么用, 注意这个按位处理的方法 :-P
int exponent(int m, int a) {
    int ret = 1;
    for (; a; a >>= 1, m *= m) {
        if (a & 1) {
            ret *= m;
        }
    }
    return ret;
}

//计算幂取模 a^b mod n, O(logb)
int modularExponent(int a, int b, int n) {
    //a^b mod n
    int ret = 1;
    for (; b; b >>= 1, a = (int) ((long long) a * a % n)) {
        if (b & 1) {
            ret = (int) ((long long) ret * a % n);
        }
    }
    return ret;
}

//求解模线性方程 ax=b (mod n)
//返回解的个数, 解保存在 sol[] 中
//要求 n>0, 解的范围 0..n-1
int modularLinear(int a, int b, int n, int *sol) {
    int d, e, x, y, i;
    d = extGcd(a, n, x, y);
    if (b % d) {
        return 0;
    }
```

```
    e = (x *(b / d) % n + n) % n;
    for (i = 0; i < d; i++) {
        sol[i] = (e + i *(n / d)) % n;
    }
    return d;
}

//求解模线性方程组 (中国余数定理)
//  x = b[0] (mod w[0])
//  x = b[1] (mod w[1])
//  ...
//  x = b[k-1] (mod w[k-1])
//要求 w[i]>0,w[i] 与 w[j] 互质, 解的范围 1..n,n=w[0]*w[1]*...*w[k-1]
int modularLinearSystem(int b[], int w[], int k) {
    int d, x, y, a = 0, m, n = 1, i;
    for (i = 0; i < k; i++) {
        n *= w[i];
    }
    for (i = 0; i < k; i++) {
        m = n / w[i];
        d = extGcd(w[i], m, x, y);
        a = (a + y * m * b[i]) % n;
    }
    return (a + n) % n;
}
```

3.5 欧拉函数

```
int gcd(int a, int b) {
    return b ? gcd(b, a % b): a;
}

inline int lcm(int a, int b) {
    return a / gcd(a, b) * b;
}

//求 1..n-1 中与 n 互质的数的个数
int eular(int n) {
    int ret = 1, i;
    for (i = 2; i *i <= n; i++) {
        if (n % i == 0) {
            n /= i;
            ret *= i - 1;
            while (n % i == 0) {
                n /= i;
                ret *= i;
            }
        }
    }
    if (n > 1) {
        ret *= n - 1;
    }
    return ret;
}
```

```
// O(n) 求 1 到 n 的欧拉函数，顺便筛出一个素数表
// 素数保存在 plist 里面，euler 存放欧拉函数值
// 返回素数个数
# include <cstring>

const int MAXN = 1000000;

int euler[MAXN + 1], plist[MAXN + 1];

int doEuler(int n = MAXN) {
    int num = 0;
    memset(euler, 0, sizeof(euler));
    euler[1] = 1;
    for (int i = 2; i <= n; i++) {
        if (!euler[i]) plist[num++] = i, euler[i] = i - 1;
        for (int j = 0; j < num && i * plist[j] <= n; j++) {
            if (i % plist[j] == 0)
                euler[i * plist[j]] = euler[i] * plist[j];
            else
                euler[i * plist[j]] = euler[i] * (plist[j] - 1);
            if (i % plist[j] == 0) break;
        }
    }
    return num;
}
```

3.6 离散对数

```
/*
    离散对数
    传入 a,b,p, 返回 a^x % p = b 的最小解 x, 若无解返回-1.
    可以处理 p 不为素数的情况
    代码中把 b>=p 的情况判为无解
    算法复杂度 O( p^0.5*lg (p^0.5) )
*/
int gcd(int a,int b) {
    return b?gcd(b,a%b):a;
}

int pow_mod(int a,long long b,int p) {
    int r=1;
    if (p==1) return 0;
    for(; b; b>>=1,a=(long long)a*a%p)
        if (b&1) r=(long long)r*a%p;
    return r;
}

int phi(int n) {
    int i,m=n;
    for(i=2; n/i>=i; i++) if (n%i==0) {
            m=m/i*(i-1);
```

```
            while(n%i==0) n/=i;
        }
    if (n>1) m=m/n*(n-1);
    return m;
}

map <int,int> Mp;

int logorithm(int a,int b,int p) {
    if (b>=p) return -1;
    Mp.clear();
    int r=0,d,i,j,t1,t2,m=(int)ceil(sqrt(p)+1e-9),m1;
    if (p==1) return 0;
    for(i=0,t1=1; i<32; i++) {
        if (t1==b) return i;
        t1=(long long)t1*a%p;
    }
    for(t1=d=1; gcd((long long)t1*a%p,p)!=d; r++) {
        t1=(long long)t1*a%p;
        d=gcd(t1,p);
    }
    if (b%d!=0) return -1;
    if (t1==b) return r;
    Mp[t1]=0;
    int pre=t1;
    for(i=1; i<m; i++) {
        int tmp=1LL*pre*a%p;
        if (!Mp.count(tmp)) Mp[tmp]=i;
        pre=tmp;
        if (tmp==b) return r+i;
    }
    m1=phi(p/d);
    m1=pow_mod(a,m1-m%m1,p);
    for(i=1; i<m; i++) {
        b=(long long)b*m1%p;
        map <int,int>::iterator it=Mp.find(b);
        if (it!=Mp.end()) return i*m+it->second+r;
    }
    return -1;
}
```

3.7 素数表

```
//用素数表判定素数，先调用 initPrime
int plist[10000], pcount = 0;

bool prime(int n) {
    int i;
    if ((n != 2 && !(n % 2)) || (n != 3 && !(n % 3)) || (n != 5 && !(n % 5)) || (n != 7 \
&& !(n % 7))) {
        return false;
    }
    for (i = 0; plist[i] * plist[i] <= n; i++) {
        if (!(n % plist[i])) {
            return false;
        }
```

```
    }
    return n > 1;
}

void initPrime() {
    plist[pcount++] = 2;
    for (int i = 3; i < 50000; i += 2) {
        if (prime(i)) {
            plist[pcount++] = i;
        }
    }
}
```

3.8 素数随机判定 (miller rabin)

```
//miller rabin
//判断自然数 n 是否为素数
//time 越高失败概率越低，一般取 10 到 50
# include <cstdlib>

//a^b mod n
int modularExponent(int a, int b, int n) {
    int ret = 1;
    for (; b; b >>= 1, a = (int) ((long long) a * a % n)) {
        if (b & 1) {
            ret = (int) ((long long) ret * a % n);
        }
    }
    return ret;
}

// Carmicheal number: 561, 41041, 825265, 321197185,
// 5394826801, 232250619601, 9746347772161, 1436697831295441, 60977817398996785,
// 7156857700403137441, 1791562810662585767521, 87674969936234821377601
bool millerRabin(int n, int time = 10) {
    if (n == 1 || (n != 2 && !(n % 2)) || (n != 3 && !(n % 3)) || (n != 5 && !(n % 5)) |\
| (n != 7 && !(n % 7))) {
        return false;
    }
    while (time--) {
        if (modularExponent(((rand() & 0x7fff << 16) + rand() & 0x7fff + rand() & 0x7fff\
) % (n - 1) + 1, n - 1, n) != 1) {
            return false;
        }
    }
    return true;
}
```

3.9 线性相关

```
//判线性相关（正交化）
//传入 m 个 n 维向量
```

```
# include <cmath>
const int MAXN = 100;
const double EPS = 1e-10;

bool linearDependent(int m, int n, double vec[][MAXN]) {
    double ort[MAXN][MAXN], e;
    int i, j, k;
    if (m > n) {
        return true;
    }
    for (i = 0; i < m; i++) {
        for (j = 0; j < n; j++) {
            ort[i][j] = vec[i][j];
        }
        for (k = 0; k < i; k++) {
            for (e = j = 0; j < n; j++) {
                e += ort[i][j] * ort[k][j];
            }
            for (j = 0; j < n; j++) {
                ort[i][j] -= e * ort[k][j];
            }
            for (e = j = 0; j < n; j++) {
                e += ort[i][j] * ort[i][j];
            }
            if (fabs(e = sqrt(e)) < EPS) {
                return 1;
            }
            for (j = 0; j < n; j++) {
                ort[i][j] /= e;
            }
        }
    }
    return false;
}
```

3.10 线性规划

```
/* 求 c1 x1 + c2 x2 + ... + cn xn 在 x1, x2, ..., xn >= 0 时的最大值
   n 表示未知数的数量, m 表示约束的数量。
   每个限制条件为:
       Ai1 x1 + Ai2 x2 + ... + Ain xn <= bi (1 <= i <= m)
   目标函数为:
       maximize(c1 x1 + c2 x2 + ... + cn xn)
使用方法:
   1. 首先调用 LP.init(n, m) 函数
   2. 对于每一个约束条件, 可以使用 constraint 函数进行输入, 或者直接写系数至 LP.A
       以及 LP.b(其中 LP.A[n+i][j] 记录系数 Aij, LP.b[n+i] 记录 bi, 1<=i<=m, 1<=j<=n)
   3. 对于目标函数, 可以使用 coefficient 函数进行输入, 或者直接写入至 LP.c[1..n]
   4. 调用 simplex() 函数 (或 simplex(true) 函数, LP.b[1..n] 分别记录 x1..xn 的取值)
    如果返回值为 true, 则 LP.v 记录目标函数最大值
    如果返回值为 false, 则 LP.failtype 记录失败类型, 1 表示无解, 2 表示解无穷大
```

```
    5. 注意：使用完毕要重复使用的时候需要 delete 式子或者自行修改 init 函数清空所有数组
*/
# include <cstring>
# include <iostream>
# include <cmath>
using namespace std;
const int MAXS = 3000; //sum of n and m
const double eps = 1e-10, oo = 1e100; //infinite
inline bool equ(double a, double b) {
    return (a < b) ? (a + eps >= b) : (a <= b + eps);
}
class LinearProgramming {
    double tA[MAXS+1][MAXS+1], tb[MAXS+1], tc[MAXS+1];
public:
    double A[MAXS+1][MAXS+1], b[MAXS+1], c[MAXS+1], v;
    int N[MAXS+2], B[MAXS+2], n, m, cnt, failtype;

    void init(int _n, int _m) {
        n = _n;
        m = _m;
        cnt = 1;
        failtype = 0;
        memset(c, 0, sizeof(double) * (n + m + 1));
    }
    void constraint(double _A[], double _b) { //_A * x <= b
        memcpy(A[n + cnt] + 1, _A, sizeof(double) * n);
        b[n + cnt++] = _b;
    }
    void coefficient(double _c[]) {
        memcpy(c + 1, _c, sizeof(double) * n);
    }
    void pivot(int l, int e) {
        int i, j;
        tb[e] = b[l] / A[l][e];
        tA[e][l] = 1 / A[l][e];
        for(i = 1; i <= N[0]; i++) if(N[i] != e) tA[e][N[i]] = A[l][N[i]] / A[l][e];
        for(i = 1; i <= B[0]; i++) {
            tb[B[i]] = b[B[i]] - A[B[i]][e] * tb[e];
            tA[B[i]][l] = -A[B[i]][e] * tA[e][l];
            for(j = 1; j <= N[0]; j++) if(N[j] != e) tA[B[i]][N[j]] = A[B[i]][N[j]] - tA\
[e][N[j]] * A[B[i]][e];
        } //for(i)
        v += tb[e] * c[e];
        tc[l] = -tA[e][l] * c[e];
        for(i = 1; i <= N[0]; i++) if(N[i] != e) tc[N[i]] = c[N[i]] - tA[e][N[i]] * c[e]\
;
        for(i = 1; i <= N[0]; i++) if(N[i] == e) N[i] = l;
        for(i = 1; i <= B[0]; i++) if(B[i] == l) B[i] = e;
        for(i = 1; i <= B[0]; i++) {
            for(j = 1; j <= N[0]; j++) A[B[i]][N[j]] = tA[B[i]][N[j]];
            b[B[i]] = tb[B[i]];
        }
        for(i = 1; i <= N[0]; i++) c[N[i]] = tc[N[i]];
    }
    bool opt() { //false stands for unbounded
```

```
        while (true) {
            int l, e, tl, te;
            double maxUp = -1;
            for(int ie = 1; ie <= N[0]; ie++) {
                if(c[te = N[ie]] <= eps) continue;
                double d = oo;
                tl = MAXS + 1;
                for(int i = 1; i <= B[0]; i++)
                    if(A[B[i]][te] > eps) {
                        double temp = b[B[i]] / A[B[i]][te];
                        if(equ(d, oo) || temp < d || equ(temp, d) && B[i] < tl) {
                            d = temp;
                            tl = B[i];
                        }
                    }
                if(tl == MAXS + 1) return false;
                if(d * c[te] > maxUp) {
                    maxUp = d * c[te];
                    l = tl;
                    e = te;
                }
            } //for(ie)
            if(equ(maxUp, -1)) break;
            pivot(l, e);
        }
        return true;
    }
    void delete0() {
        int p, i;
        for(p = 1; p <= B[0]; p++) if(B[p] == 0) break;
        if(p <= B[0]) {
            for(i = 1; i <= N[0]; i++)
                if(A[0][N[i]] > eps || A[0][N[i]] < -eps) break;
            pivot(0, N[i]);
        }
        for(p = 1; p <= N[0]; p++) if(N[p] == 0) break;
        for(i = p; i < N[0]; i++) N[i] = N[i+1];
        N[0]--;
    }
    bool initialize() { //false stands for infeasible
        int i;
        N[0] = n;
        B[0] = m;
        v = 0;
        for(i = 1; i <= n; i++) N[i] = i;
        for(i = 1; i <= m; i++) B[i] = n + i;
        int l = B[1];
        for(i = 2; i <= m; i++) if(b[B[i]] < b[l]) l = B[i];
        if(b[l] >= 0) return true;
        double origC[MAXS+1];
        memcpy(origC, c, sizeof(double) * (n + m + 1));
        N[++N[0]] = 0;
        for(i = 1; i <= B[0]; i++) A[B[i]][0] = -1;
        memset(c, 0, sizeof(double) * (n + m + 1));
        c[0] = -1;
        pivot(l, 0);
        opt();
```

```
        if(v < -eps) return false;
        delete0();
        memcpy(c, origC, sizeof(double) * (n + m + 1));
        bool inB[MAXS+1];
        memset(inB, false, sizeof(bool) * (n + m + 1));
        for(i = 1; i <= B[0]; i++) inB[B[i]] = true;
        for(i = 1; i <= n + m; i++)
            if(inB[i] && c[i] != 0) {
                v += c[i] * b[i];
                for(int j = 1; j <= N[0]; j++) c[N[j]] -= A[i][N[j]] * c[i];
                c[i] = 0;
            }
        return true;
    }
    bool simplex(bool calcEachAns = false) {
        if(!initialize()) {
            failtype = 1;
            return false;
        }
        if(!opt()) {
            failtype = 2;
            return false;
        }
        if(!calcEachAns) return true;
        bool inN[MAXS+1];
        memset(inN, false, sizeof(bool) * (n + m + 1));
        for(int i = 1; i <= N[0]; i++) inN[N[i]] = true;
        for(int i = 1; i <= n; i++) if(inN[i]) b[i] = 0;
        return true;
    }
} LP;
```

3.11 阶乘最后非零位

```
//求阶乘最后非零位, 复杂度 O(nlogn)
//返回该位,n 以字符串方式传入
# include <cstring>
const int MAXN = 10000;

int lastdigit(char *buf) {
    const int mod[20] =  {
        1, 1, 2, 6, 4, 2, 2, 4, 2, 8, 4, 4, 8, 4, 6, 8, 8, 6, 8, 2
    };
    int len = strlen(buf), a[MAXN], i, c, ret = 1;
    if (len == 1) {
        return mod[buf[0] - '0'];
    }
    for (i = 0; i < len; i++) {
        a[i] = buf[len - 1 - i] - '0';
    }
    for (; len; len -= !a[len - 1]) {
        ret = ret * mod[a[1] % 2 * 10 + a[0]] % 5;
        for (c = 0, i = len - 1; i >= 0; i--) {
            c = c * 10 + a[i];
            a[i] = c / 5;
```

```
            c %= 5;
        }
    }
    return ret + ret % 2 * 5;
}
```

Chapter 4

数值

4.1 FFT

```
//用傅立叶变换可在 O(nlogn) 求卷积
//传入长度为 len(保证为 2 的幂) 的数组 a,inv 为 1 和-1 时分别做正反 DFT 变换
//常数较大
# include <complex>
void FFT(complex<double> *a,int len,int inv) { //eps=1e-12
    for (int i=0, n1=0, n2=0; i < len; ++i, n2 ^= (len/(i&-i)>>1), n1^=(i&-i))
        if (n1 > n2)
            swap(a[n1], a[n2]);
    for(int m = 1; m <= len >> 1; m <<= 1) {
        complex<double> w0(cos(PI / m), sin(PI / (inv * m))), w = 1, t;
        for(int k = 0; k < len; k += (m << 1), w = 1)
            for(int j = 0; j < m; ++j, w *= w0) {
                t=w*a[k+j+m];
                a[k+j+m]=a[k+j]-t;
                a[k+j]+=t;
            }
    }
    if(inv == -1)
        for(int i = 0; i < len; ++i)
            a[i] /= len;
}
```

4.2 周期性方程 (追赶法)

```
/*  追赶法解周期性方程, 复杂度 o(n)
    周期性方程定义: | b0 c0 ...                 a0   |           =  x0
                     | a1 b1 c1 ...                   |           =  x1
                     |          ...                   | *   X     =  ...
                     |      ...         an-2 bn-2 cn-2 |           =  xn-2
                     | cn-1                 an-1 bn-1 |           =  xn-1
    输入: a[],b[],c[],x[],n
    输出: 求解结果 X 在 x[] 中
*/
```

```
void run(double a[], double b[], double c[], double x[], int n) {
    c[0] /= b[0];
    a[0] /= b[0];
    x[0] /= b[0];
    for (int i = 1; i < n - 1; i++) {
        double temp = b[i] - a[i] * c[i - 1];
        c[i] /= temp;
        x[i] = (x[i] - a[i] * x[i - 1]) / temp;
        a[i] = -a[i] * a[i - 1] / temp;
    }
    a[n - 2] = -a[n - 2] - c[n - 2];
    for (int i = n - 3; i >= 0; i--) {
        a[i] = -a[i] - c[i] * a[i + 1];
        x[i] -= c[i] * x[i + 1];
    }
    x[n - 1] -= (c[n - 1] * x[0] + a[n - 1] * x[n - 2]);
    x[n - 1] /= (c[n - 1] * a[0] + a[n - 1] * a[n - 2] + b[n - 1]);
    for (int i = n - 2; i >= 0; i--) {
        x[i] += a[i] * x[n - 1];
    }
}
```

4.3 多项式求根 (牛顿法)

```
/* 牛顿法解多项式的根
    输入: 多项式系数 c[], 多项式度数 n, 求在 [a,b] 间的根
    输出: 根
    要求保证 [a,b] 间有根
*/
# include <cmath>
# include <cstdlib>

double f(int m, double c[], double x) {
    int i;
    double p = c[m];
    for (i = m; i > 0; i--) {
        p = p * x + c[i - 1];
    }
    return p;
}

int newton(double x0, double *r, double c[], double cp[], int n, double a, double b, dou\
ble eps) {
    const int MAX_ITERATION = 1000;
    int i = 1;
    double x1, x2, fp, eps2 = eps / 10.0;
    x1 = x0;
    while (i < MAX_ITERATION) {
        x2 = f(n, c, x1);
        fp = f(n - 1, cp, x1);
        if ((fabs(fp) < 0.000000001) && (fabs(x2) > 1.0)) {
            return 0;
        }
```

```
        x2 = x1 - x2 / fp;
        if (fabs(x1 - x2) < eps2) {
            if (x2 < a || x2 > b) {
                return 0;
            }
            *r = x2;
            return 1;
        }
        x1 = x2;
        i++;
    }
    return 0;
}

double polynomialRoot(double c[], int n, double a, double b, double eps) {
    double *cp;
    int i;
    double root;

    cp = (double *) calloc(n, sizeof(double));
    for (i = n - 1; i >= 0; i--) {
        cp[i] = (i + 1) * c[i + 1];
    }
    if (a > b) {
        root = a;
        a = b;
        b = root;
    }
    if ((!newton(a, &root, c, cp, n, a, b, eps)) && (!newton(b, &root, c, cp, n, a, b, e\
ps))) {
        newton((a + b) * 0.5, &root, c, cp, n, a, b, eps);
    }
    free(cp);
    if (fabs(root) < eps) {
        return fabs(root);
    } else {
        return root;
    }
}
```

4.4 定积分计算 (Romberg)

```
/*  Romberg 求定积分
    输入：积分区间 [a,b]，被积函数 f(x,y,z)
    输出：积分结果

    f(x,y,z) 示例：
    double f0( double x, double l, double t )
    {
        return sqrt(1.0+l*l*t*t*cos(t*x)*cos(t*x));
    }
*/
# include <cmath>
```

```
double romberg(double a, double b, double(*f)(double x, double y, double z), double eps,\
 double l, double t) {
    const int MAXN = 1000;
    int i, j, temp2, min;
    double h, R[2][MAXN], temp4;

    for (i = 0; i < MAXN; i++) {
        R[0][i] = 0.0;
        R[1][i] = 0.0;
    }
    h = b - a;
    min = (int)(log(h * 10.0) / log(2.0)); //h should be at most 0.1
    R[0][0] = ((*f)(a, l, t) + (*f)(b, l, t)) * h * 0.50;
    i = 1;
    temp2 = 1;
    while (i < MAXN) {
        i++;
        R[1][0] = 0.0;
        for (j = 1; j <= temp2; j++) {
            R[1][0] += (*f)(a + h *((double)j - 0.50), l, t);
        }
        R[1][0] = (R[0][0] + h * R[1][0]) * 0.50;
        temp4 = 4.0;
        for (j = 1; j < i; j++) {
            R[1][j] = R[1][j - 1] + (R[1][j - 1] - R[0][j - 1]) / (temp4 - 1.0);
            temp4 *= 4.0;
        }
        if ((fabs(R[1][i - 1] - R[0][i - 2]) < eps) && (i > min)) {
            return R[1][i - 1];
        }
        h *= 0.50;
        temp2 *= 2;
        for (j = 0; j < i; j++) {
            R[0][j] = R[1][j];
        }
    }
    return R[1][MAXN - 1];
}

double integral(double a, double b, double(*f)(double x, double y, double z), double eps\
, double l, double t) {
    const double PI = 3.1415926535897932;
    int n;
    double R, p, res;

    n = (int)(floor)(b * t * 0.50 / PI);
    p = 2.0 * PI / t;
    res = b - (double)n * p;
    if (n) {
        R = romberg(a, p, f, eps / (double)n, l, t);
    }
    R = R *(double)n + romberg(0.0, res, f, eps, l, t);

    return R / 100.0;
}
```

```
// 其实不妨先考虑用复化 Simpson 公式
// S = h / 6 * [f(A) + 4 * Σf(Xk+1/2) + 2 * Σf(Xk) + f(B)]; k = 0..n-1
```

4.5 定积分计算 (变步长 simpson)

```
//变步长辛普森积分
double func(double x) {
    return ...; //根据题目需要实现
}
double Simpson_VariStep(double x1,double xh,double eps) {
    int subs=1,n=1,i;
    double result,x,p,width=xh-x1,t1=width*(func(x1)+func(xh))/2.0,t2;
    double s1=t1,s2=s1+2.0*eps;
    while(subs) {
        for (p=0.0,i=0; i<=n-1; ++i) {
            x=x1+(i+0.5)*width;
            p=p+func(x);
        }
        t2=(t1+width*p)/2.0;
        s2=(t2*4-t1)/3.0;
        result=s2;
        subs=(fabs(s2-s1)>=eps);
        t1=t2;
        s1=s2;
        n+=n;
        width=width/2.0;
    }
    return s2;
}
```

4.6 线性方程组 (gauss)

```
# include <cmath>

const int MAXN = 100;
const double EPS = 1e-10;

//列主元 gauss 消去求解 a[][]x[]=b[]
//返回是否有唯一解, 若有解在 b[] 中
bool gaussCpivot(int n, double a[][MAXN], double b[]) {
    int i, j, k, row;
    double maxp, t;
    for (k = 0; k < n; k++) {
        for (maxp = 0, i = k; i < n; i++) {
            if (fabs(a[i][k]) > fabs(maxp)) {
                maxp = a[row = i][k];
            }
        }
        if (fabs(maxp) < EPS) {
            return false;
        }
        if (row != k) {
```

```
            for (j = k; j < n; j++) {
                swap(a[k][j], a[row][j]);
            }
            swap(b[k], b[row]);
        }
        for (j = k + 1; j < n; j++) {
            a[k][j] /= maxp;
            for (i = k + 1; i < n; i++) {
                a[i][j] -= a[i][k] * a[k][j];
            }
        }
        b[k] /= maxp;
        for (i = k + 1; i < n; i++) {
            b[i] -= b[k] * a[i][k];
        }
    }
    for (i = n - 1; i >= 0; i--) {
        for (j = i + 1; j < n; j++) {
            b[i] -= a[i][j] * b[j];
        }
    }
    return true;
}

//全主元 gauss 消去解 a[][]x[]=b[]
//返回是否有唯一解, 若有解在 b[] 中
bool gaussTpivot(int n, double a[][MAXN], double b[]) {
    int i, j, k, row, col, index[MAXN];
    double maxp, t;
    for (i = 0; i < n; i++) {
        index[i] = i;
    }
    for (k = 0; k < n; k++) {
        for (maxp = 0, i = k; i < n; i++) {
            for (j = k; j < n; j++) {
                if (fabs(a[i][j]) > fabs(maxp)) {
                    maxp = a[row = i][col = j];
                }
            }
        }
        if (fabs(maxp) < EPS) {
            return false;
        }
        if (col != k) {
            for (i = 0; i < n; i++) {
                swap(a[i][col], a[i][k]);
            }
            swap(index[col],  index[k]);
        }
        if (row != k) {
            for (j = k; j < n; j++) {
                swap(a[k][j], a[row][j]);
            }
            swap(b[k], b[row]);
        }
        for (j = k + 1; j < n; j++) {
            a[k][j] /= maxp;
```

```
            for (i = k + 1; i < n; i++) {
                a[i][j] -= a[i][k] * a[k][j];
            }
        }
        b[k] /= maxp;
        for (i = k + 1; i < n; i++) {
            b[i] -= b[k] * a[i][k];
        }
    }
    for (i = n - 1; i >= 0; i--) {
        for (j = i + 1; j < n; j++) {
            b[i] -= a[i][j] * b[j];
        }
    }
    for (k = 0; k < n; k++) {
        a[0][index[k]] = b[k];
    }
    for (k = 0; k < n; k++) {
        b[k] = a[0][k];
    }
    return true;
}
```

Chapter 5

结构

5.1 ST 表

```
// RMQ
// MAXL = ceil(lg(MAXN))
// 根据需要重写以下函数

# define BIN(i) (1 << (i))
# define HLF(i) (BIN(i) >> 1)
# define PRE(i) ((i) > 0 ? (i) - 1 : 0)

// 为了省时间可以把 lg(x) 做成表, 可以参考位运算加速
// 注意 lg(1 << i) = i - 1; lg((1 << i) + 1) = i;
const double eps = 1e-8;
const double ln2 = log(2.0);
inline int lg2(double x) {
    return (int)floor(fabs(log(x) / ln2 - eps));
}
```

```
/************ 一维 ************/

// MAXL = min{(1 << MAXL) >= MAXN};

template<int MAXL, class T = int, int MAXN = 1 << MAXL>
struct RMQ {
    T e[MAXN];
    int rmq[MAXL][MAXN];

    // 重写 cmp, 比较两个下标, 返回较“小”下标
    int cmp(int l, int r) {
        return e[l] <= e[r] ? l : r;
    }

    // 请直接对 e 赋值后调用
    void init(const int n) {
        for (int i = 0; i < n; i++)
            rmq[0][i] = i;
        for (int i = 0; BIN(i + 1) <= n; i++)
            for (int j = 0; j <= n - BIN(i + 1); j++)
                rmq[i + 1][j] = cmp(rmq[i][j], rmq[i][j + BIN(i)]);
    }

    // [l, r) (l < r)
    int index(int l, int r) {
        int b = lg2(r - l);
        return cmp(rmq[b][l], rmq[b][r - (1 << b)]);
    }
    T value(int l, int r) {
        return e[index(l, r)];
    }
};


/************ 二维 ************/

// 如果 MLE 就把 int 改成 short
typedef pair<int, int> IndexType;

// MAXR = min{(1 << MAXR) >= MAXM};
// MAXC = min((1 << MAXC) >= MAXN);

template<int MAXR, int MAXC, class T = int, int MAXM = 1 << MAXR, int MAXN = 1 << MAXC>
struct RMQ2 {
    T e[MAXM][MAXN];
    IndexType rmq[MAXR][MAXC][MAXM][MAXN];

    IndexType cmp(const IndexType &lhs, const IndexType &rhs) {
        return e[lhs.first][lhs.second] <= e[rhs.first][rhs.second] ? lhs : rhs;
    }

    void init(int m, int n) {
        for (int x = 0; x < m; x++)
```

```
            for (int y = 0; y < n; y++)
                rmq[0][0][x][y] = make_pair(x, y);
        for (int i = 0, ii; ii = PRE(i), BIN(i) <= m; i++)
            for (int j = 0, jj; jj = PRE(j), BIN(j) <= n; j++)
                for (int x = 0, xx; xx = HLF(i), x <= m - BIN(i); x++)
                    for (int y = 0, yy; yy = HLF(j), y <= n - BIN(j); y++)
                        rmq[i][j][x][y] = cmp(
                                                    cmp(rmq[ii][jj][x]     [y], rmq[ii][jj][x]\
     [y + yy]),
                                                    cmp(rmq[ii][jj][x + xx][y], rmq[ii][jj][x \
+ xx][y + yy])
                                                );
    }

    IndexType index(int x1, int y1, int x2, int y2) {
        int xx = lg2(x2 - x1), yy = lg2(y2 - y1);
        return cmp(
                   cmp(rmq[xx][yy][x1]            [y1], rmq[xx][yy][x1]            [y2 -\
 (1 << yy)]),
                   cmp(rmq[xx][yy][x2 - (1 << xx)][y1], rmq[xx][yy][x2 - (1 << xx)][y2 -\
 (1 << yy)])
               );
    }

    T value(int x1, int y1, int x2, int y2) {
        IndexType i = index(x1, y1, x2, y2);
        return e[i.first][i.second];
    }
};

/*
可以在开头定义一个全局变量，代替用浮点函数的 lg2

template<int MAXN>
struct LG2
{
    int lg2[MAXN + 1];

    LG2()
    {
        lg2[0] = -1;
        for (int i = 1; i <= MAXN; i++) {
            lg2[i] = lg2[i - 1] + ((i & (i - 1)) == 0);
        }
    }

    int operator()(int x) const { return lg2[x]; }
};

LG2<65536> lg2;
 */
```

5.2 Splay

```
#include <algorithm>
using namespace std;

/*
每次先调用一下 Splay::init()
不同的题目注意修改 newNode, pushdown, update 三个函数就足够了
Splay 命名空间里的函数都是对树的操作，对结点的操作都归类到 Node 里。
空间吃紧时可修改 erase 函数，增加内存池管理
*/
const int N = 130005;
struct Node {
    int key, size;
    bool rvs;
    Node *f, *ch[2];
    void set(int c, Node *x);    //设当前结点的左 (0)/右 (1) 儿子为 x
    void fix();                  //令两个儿子的父亲指针指向自己，主要为了写起来方便，意义 \
明确
    void pushdown();             //标记下传
    void update();               //从儿子处更新自己的信息
    void rotate();               //向上旋转
    void Splay(Node *);          //把当前 Node 旋转到参数传入的 Node 下面。Node 默认为 null，直 \
接调用 Splay() 则旋转到根
} statePool[N], *null;          //本模板统一用 null 代替 NULL
void Node::set(int c, Node *x) {
    ch[c] = x;
    x->f = this;
}
void Node::fix() {
    ch[0]->f = this;
    ch[1]->f = this;
}
void Node::pushdown() {
    if (this == null) return;
    if (rvs) {
        ch[0]->rvs ^= 1;
        ch[1]->rvs ^= 1;
        rvs = 0;
        swap(ch[0], ch[1]);
    }
}
void Node::update() {
    if (this == null) return;
    size = ch[0]->size + ch[1]->size + 1;
}
void Node::rotate() {
    Node *x = f;
    bool o = f->ch[0] == this;
    x->set(!o, ch[o]);
    x->f->set(x->f->ch[1] == x, this);
    set(o, x);
```

```
    x->update();
    update();
}
void Node::Splay(Node *goal = null) {
    pushdown();
    for (; f != goal;) {
        f->f->pushdown();
        f->pushdown();
        pushdown();
        if (f->f == goal) {
            rotate();
        } else if ((f->f->ch[0] == f) ^ (f->ch[0] == this)) {
            rotate();
            rotate();
        } else {
            f->rotate();
            rotate();
        }
    }
}
namespace Splay {
int poolCnt;
Node *newNode() {
    Node *p = &statePool[poolCnt++];
    p->f = p->ch[0] = p->ch[1] = null;
    p->size = 1;
    p->rvs = 0;
    return p;
}
//该命名空间里的函数必须先调用 init()
void init() {
    poolCnt = 0;
    null = newNode();
    null->size = 0;
}

//用 a[l..r] 的值建立一棵完全平衡的 Splay tree。返回树根。
template <class T> Node *build(int l, int r, T a[]) {
    if (l > r) return null;
    Node *p = newNode();
    int mid = (l + r) / 2;
    p->key = a[mid];
    if (l < r) {
        p->ch[0] = build(l, mid - 1, a);
        p->ch[1] = build(mid + 1, r, a);
        p->update();
        p->fix();
    }
    return p;
}

//返回树中第 i 个元素，若没有其它操作，请记得 select 后进行 Splay 操作以保证均摊复杂度。
Node *select(Node *root, int i) {
    for (Node *p = root;;) {
        p->pushdown();
        if (p->ch[0]->size + 1 == i) {
```

```
            return p;
        } else if (p->ch[0]->size >= i) {
            p = p->ch[0];
        } else {
            i -= p->ch[0]->size + 1;
            p = p->ch[1];
        }
    }
}

//返回结点 p 在树中的排名
int rank(Node *p) {
    p->Splay();
    return p->ch[0]->size + 1;
}

// 返回 >= a 的结点，若没有则返回 null，若之后没有其它操作，最好进行 Splay 操作以保证均摊复 \
杂度。
// 返回 null 时可以这样保证复杂度： select(root, root->size)->Splay();
template <class T> Node *lower_bound(Node *root, T a) {
    Node *ret = null;
    for (Node *p = root; p != null; ) {
        p->pushdown();
        if (a < p->key) {
            p = p->ch[1];
        } else {
            ret = p;
            p = p->ch[0];
        }
    }
    return ret;
}

//传入两树根，将之合并 (可以处理 null)。p 中结点均在 q 中结点的左边。
Node *merge(Node *p, Node *q) {
    p->f = q->f = null;
    if (p == null) return q;
    if (q == null) return p;
    q = select(q, 1);
    q->Splay();
    q->set(0, p);
    q->update();
    return q;
}

//当 p 为根，且 q 为 p 的右儿子时，翻转从 p 到 q 的所有结点，返回树根
Node *reverse(Node *p, Node *q) {
    swap(p->ch[0], q->ch[0]);
    p->ch[1] = q->ch[1];
    q->ch[1] = p;
    q->f = null;
    p->fix();
    q->fix();
    p->update();
    q->update();
```

```
    p->ch[0]->rvs ^= 1;
    return q;
}

//翻转第 l 个元素到第 r 个元素，返回树根，下标从 1 开始。
Node *reverse(Node *root, int l, int r) {
    if (l >= r) return root;
    Node *p = select(root, l);
    p->Splay();
    Node *q = select(p, r);
    q->Splay(p);
    return reverse(p, q);
}

//在 p 的前一位插入 q，返回树根
Node *insert(Node *p, Node *q) {
    p->Splay();
    if (p->ch[0] == null) {
        p->set(0, q);
    } else {
        Node *t = select(p, p->ch[0]->size);
        t->Splay(p);
        t->set(1, q);
        t->update();
    }
    p->update();
    return p;
}

//在第 i 个元素前插入结点 q，返回树根
Node *insert(Node *root, Node *q, int i) {
    if (i > root->size) {
        Node *p = select(root, root->size);
        p->Splay();
        p->set(1, q);
        p->update();
        return p;
    } else {
        Node *p = select(root, i);
        return insert(p, q);
    }
}

//删除以 p 为根的子树
Node *erase(Node *p) {
    if (p->f != null) {
        Node *q = p->f;
        q->pushdown();
        q->set(q->ch[1] == p, null);
        q->update();
        q->Splay();
        return q;
    } else {
        return null;
    }
}
```

```
//删除第 l 个到第 r 个结点, 返回树根
Node *erase(Node *root, int l, int r) {
    if (l > r) return root;
    if (l == r) {
        Node *p = select(root, l);
        p->Splay();
        return merge(p->ch[0], p->ch[1]);
    } else {
        Node *p = select(root, l);
        p->Splay();
        Node *q = select(p, r);
        q->Splay(p);
        return merge(p->ch[0], q->ch[1]);
    }
}

//切开结点 p 与其左儿子之间的边, 返回左子树的根以及原树的根。
pair <Node *, Node *> split(Node *p) {
    Node *q = p->ch[0];
    p->ch[0] = null;
    q->f = null;
    p->update();
    p->Splay();
    return make_pair(q, p);
}
}
```

5.3 划分树

```
// 划分树 by yxdb
// 功能: 求任意一段区间第 K 小元素, 也可求中位数. 仅能处理静态数组, 可处理含有相同元素的情 \
况.
// 对于 N 个元素的数组, 空间复杂度为 O(NLogN), 初始化时间复杂度为 O(NLogN), 每次查询时间复杂 \
度为 O(LogN).
// 查询前先调用 build(n, d) 初始化, d 数组 [0, n) 为需要处理的数据.
// query(l, r, k) 返回数组中 [l, r] 的第 k 小元素 (k 为 1 时即为求最小值).
template <class T, int N>
class PartitionTree {
    pair<T,int> a[N];
    int p[25][N], L[25][N], R[25][N];
    bool isL[25][N];
    void build(int d, int l, int r) {
        if (l+1==r) return;
        int m=(l+r)>>1;
        for (int i=l, j=l, k=m; i<r; ++i) {
            L[d][i]=j, R[d][i]=k;
            p[d+1][(isL[d][i]=p[d][i]<m)?j++:k++]=p[d][i];
        }
        build(d+1, l, m);
        build(d+1, m, r);
    }
public:
```

```
    void build(int n, T *d) {
        for (int i=0; i<n; ++i) a[i]=make_pair(d[i],i);
        sort(a, a+n);
        for (int i=0; i<n; ++i) p[0][a[i].second]=i;
        build(0, 0, n);
    }
    T query(int l, int r, int k) {
        int d, cnt;
        for (d=0; l<r; ++d) {
            cnt=L[d][r]-L[d][l]+isL[d][r];
            if (cnt>=k)  l=L[d][l], r=L[d][r]-!isL[d][r];
            else k-=cnt, l=R[d][l], r=R[d][r]- isL[d][r];
        }
        return a[p[d][l]].first;
        // 若改为 return a[p[d][l]].second; 可求第 k 小元素的位置.
    }
};
```

5.4 子阵和

```
//求 sum{a[0..m-1][0..n-1]}
//维护和查询复杂度均为 O(logm*logn)
//用于动态求子阵和, 数组内容保存在 Sum.a[][] 中
//可以改成其他数据类型
# include <cstring>

const int MAXN = 10000;

inline int lowbit(int x) {
    return (x & -x);
}


template<class elemType>
class Sum {
    elemType a[MAXN][MAXN], c[MAXN][MAXN], ret;
    int m, n, t;
    void init(int i, int j) {
        memset(a, 0, sizeof(a));
        memset(c, 0, sizeof(c));
        m=i;
        n=j;
    }

    void update(int i, int j, elemType v) {
        for (v -= a[i][j], a[i++][j++] += v, t = j; i <= m; i += lowbit(i)) {
            for (j = t; j <= n; j += lowbit(j)) {
                c[i - 1][j - 1] += v;
            }
        }
    }

    elemType query(int i, int j) {
        for (ret = 0, t = j; i; i ^= lowbit(i)) {
```

```
            for (j = t; j; j ^= lowbit(j)) {
                ret += c[i - 1][j - 1];
            }
        }
        return ret;
    }
};
```

5.5 左偏树

```
// 左偏树是可以高效做合并操作的堆

# include <algorithm>     // swap
# include <functional>    // less

template<typename T = int, typename Pred = less<T> >
struct LeftistTree {
    struct node_type {
        T v;
        int d;
        node_type *l, *r;
        node_type(T v, int d) {
            this->v = v;
            this->d = d;
            l = NULL;
            r = NULL;
        }
        ~node_type() {
            delete l;
            delete r;
        }
    };
private:
    node_type *root;
    // 比较
    static Pred pr;
    // 核心操作，将以 l 和 r 为根的左偏树合并，返回新的根节点，复杂度 O(lgn)
    static node_type *merge(node_type *l, node_type *r) {
        if (l == NULL) {
            return r;
        }
        if (r == NULL) {
            return l;
        }

        if (pr(r->v, l->v)) {
            swap(l, r);
        }
        l->r = merge(l->r, r);
        if (l->r != NULL && (l->l == NULL || l->r->d > l->l->d)) {
            swap(l->l, l->r);
        }
        if (l->r == NULL) {
            l->d = 0;
```

```
        } else {
            l->d = l->r->d + 1;
        }

        return l;
    }
public:
    LeftistTree() {
        root = NULL;
    }
    ~LeftistTree() {
        delete root;
    }
    // 合并操作将让被合并的树变为空
    void merge(LeftistTree &t) {
        root = merge(root, t.root);
        t.root = NULL;
    }
    void push(T v) {
        root = merge(root, new node_type(v, 0));
    }
    void pop() {
        node_type *l = root->l, *r = root->r;

        root->l = NULL;
        root->r = NULL;
        delete root;
        root = merge(l, r);
    }
    T front() {
        return root->v;
    }
};
```

5.6 并查集

```
//带路径压缩的并查集，动态维护和查询等价类
//维护和查询复杂度略大于 O(1)
//集合元素取值 0 ~ MAXN-1,find(k) 返回所在的并查集根的编号
//init 默认不等价
const int MAXN = 1000000;
struct Dset {
    int p[MAXN];
    void init(int N) {
        for (int i = 0; i < N; ++i)
            p[i] = i;
    }
    int find(int k) {
        return p[k] == k ? k : (p[k] = find(p[k]));
    }
    void setFriend(int i, int j) {
        p[find(i)] = p[find(j)];
    }
    bool isFriend(int i, int j) {
```

```
        return find(i) == find(j);
    }
};
```

5.7 扩展并查集

```
//扩展并查集 by asmn
//已知许多组 v[i]^v[j]=k, 询问给某一对 v[i]^v[j] 的值
//setDiff(i,j,k) 输入 v[i]^v[j]=k 的关系
//返回值若为 false 表示输入的关系与之前有矛盾, 不予处理
//query(i,j,ans) 查询 v[i]^v[j] 的结果 ans
//返回若值为 false 表示结果无法由已有关系推得
# include <cstdio>
# include <algorithm>
using namespace std;
const int MAXN = 100000;
struct T {
    int prv, dif;
    T() {}
    T(int prv, int dif):prv(prv),dif(dif) {}
    T operator ^=(const T a) {
        return T(prv = a.prv, dif ^= a.dif);
    }
};
struct Dset {
    T p[MAXN];
    void init(int n) {
        for (int i = 0; i < n; ++i) {
            p[i] = T(i, 0);
        }
    }
    T find(int k) {
        return p[k].prv == k ? p[k] : (p[k] ^= find(p[k].prv));
    }
    // set v[i] ^ v[j] = k
    bool setDiff(int i, int j, int k) {
        T ti = find(i), tj = find(j);
        tj.dif ^= ti.dif ^ k;
        p[ti.prv] ^= tj;
        if (p[tj.prv].dif) {
            p[tj.prv].dif = 0;
            return false;
        } else {
            return true;
        }
    }
    //query ans = v[i] ^ v[j]
    bool query(int i, int j, int &ans) {
        ans = find(i).dif ^ find(j).dif;
        return (find(i).prv == find(j).prv);
    }
};
```

```
int main() {
    int i, j, ans;
    char op;
    Dset A;
    A.init(100);
    while (scanf(" %c %d%d", &op, &i, &j) != EOF) {
        switch (op) {
        case 'f':
            if (A.setDiff(i, j, 0)) {
                puts("OK");
            } else {
                puts("NO");
            }
            break;
        case 'e':
            if (A.setDiff(i, j, 1)) {
                puts("OK");
            } else {
                puts("NO");
            }
            break;
        case 'q':
            if (A.query(i, j, ans)) {
                printf("%s\n", ans ? "e" : "f");
            } else {
                puts("I dont know");
            }

        }
    }
}
```

5.8 树状数组

```
// 树状数组 By 猛犸也钻地 @ 2011.11.24

/* 使用提示 //
   单点修改/区间查询:
       使 P 位置增加 V           modify(P,V);
       查询 [L,R] 区间的和      getsum(R)-getsum(L-1);
   区间修改/单点查询:
       使 [L,R] 区间各增加 V     modify(L,V),modify(R+1,-V);
       查询 P 位置的值          getsum(P);
   区间修改/区间查询:
       使 [0,R] 区间各增加 V     A.modify(R,R*V),B.modify(R,-V);
       查询 [0,R] 区间的和      A.getsum(R)+B.getsum(R)*R;
   多维树状数组和单维的类似，比如修改操作是这样的：
       for(int i=x+BIAS;i<SIZE;i++)
           for(int j=y+BIAS;j<SIZE;j++) u[i][j]+=v;
   模板里的 BIAS 用来避免下溢出，比如查询 L-1 时，可能会取到 0 或-1 的位置
```

```
// 因此 SIZE 通常设为最大结点数 +2*BIAS, 而 BIAS 通常设为 5 或 10 */

# include <cstring>
using namespace std;

class BITree {
public:
    static const int SIZE = 100010, BIAS = 5;
    long long u[SIZE];
    void clear() {
        memset(u,0,sizeof(u));
    }
    void modify(int x, long long v) {
        for(x+=BIAS; x<SIZE; x+=x&-x) u[x]+=v;
    }
    long long getsum(int x) {
        long long s=0;
        for(x+=BIAS; x; x-=x&-x) s+=u[x];
        return s;
    }
};
```

5.9 树链剖分

```
// 树链剖分 By 猛犸也钻地 @ 2012.02.10

# include <vector>
# include <cstring>
using namespace std;

class TreeDiv {
public:
    static const int SIZE = 100005; // SIZE 为最大结点数 +1
    int sz[SIZE],lv[SIZE]; // sz[] 为子树的总结点数, lv[] 为结点的深度
    int rt[SIZE],fa[SIZE]; // rt[] 为结点所在的树根, fa[] 为结点的父亲
    // seg[] 为结点所在的链的编号, idx[] 为结点在链上的位置
    // low[] 为链首结点, top[] 为链尾结点的父亲, len[] 为链上结点的个数
    int cnt,seg[SIZE],idx[SIZE],low[SIZE],top[SIZE],len[SIZE];
    // 遍历某条链的方法: for(int x=low[u];x!=top[u];x=fa[x])
    // 传入结点个数 n 及各结点的出边 e[], 对树或森林进行剖分, 返回链数 cnt
    int init(int n, const vector<int> e[]) {
        memset(len,cnt=0,sizeof(len));
        memset(lv,0,sizeof(lv));
        for(int i=0; i<n; i++) if(!lv[i]) {
                segment(rt[i]=fa[i]=i,e);
                top[seg[i]]=fa[i]=SIZE-1; // 指向虚根
            }
        return cnt;
    }
    // 求 x 和 y 的最近公共祖先, 不在同一棵树上则返回-1, 复杂度 O(logN)
    int lca(int x, int y) {
```

```
        if(rt[x]!=rt[y]) return -1;
        while(seg[x]!=seg[y]) {
            int p=top[seg[x]],q=top[seg[y]];
            if(lv[p]>lv[q]) x=p;
            else y=q;
        }
        return lv[x]<lv[y]?x:y;
    }
private:
    void segment(int x, const vector<int> e[]) {
        int y,t=-1;
        sz[x]=1;
        rt[x]=rt[fa[x]];
        lv[x]=lv[fa[x]]+1;
        for(size_t i=0; i<e[x].size(); i++) if(e[x][i]!=fa[x]) {
                fa[y=e[x][i]]=x;
                segment(y,e);
                sz[x]+=sz[y];
                if(t<0 || sz[y]>sz[t]) t=y;
            }
        seg[x]=~t?seg[t]:cnt;
        idx[x]=~t?idx[t]+1:0;
        if(t<0) low[cnt++]=x;
        len[seg[x]]++;
        top[seg[x]]=fa[x];
    }
};
```

5.10 矩形并

```
//线段树求矩形并得面积和周长
//要保证传入的 x1 < x2, y1 < y2
# include<algorithm>
# include <cstring>
# define MAXN 40000
using namespace std;
typedef long long LL;
const int K = 1;
int tot, X[MAXN * 2], CX;
pair<pair<int, int>, pair<int,int> > E[MAXN * 2];

//矩形类
struct Rect {
    int x1, y1, x2, y2;
} R[MAXN];

struct SegTree {
    int L, R, Lson, Rson, cover, length[K + 1];
    void build(int, int);
    void add(int, int, int);
} T, A[MAXN * 4];

void SegTree::build(int l, int r) {
    L = l;
```

```
    R = r;
    cover = 0;
    memset(length, 0, sizeof(length));
    length[0] = X[R + 1] - X[L];
    if (L == R) return ;
    int mid = (L + R) >> 1;
    A[Lson = tot++].build(l, mid);
    A[Rson = tot++].build(mid + 1, r);
}
//线段树的填删线段，只能删除之前加入过的线段
void SegTree::add(int v, int l, int r) {
    memset(length, 0, sizeof(length));
    if (l <= L && R <= r) {
        cover += v;
        if (L == R) {
            length[min(cover, K)] += X[R + 1] - X[L];
        } else {
            for (int i = 0; i <= K; ++i) {
                length[min(i + cover, K)] += A[Lson].length[i] + A[Rson].length[i];
            }
        }
        return ;
    }
    int mid = (L + R) >> 1;
    if (l <= mid) A[Lson].add(v, l, r);
    if (r > mid) A[Rson].add(v, l, r);
    for (int i = 0; i <= K; ++i) {
        length[min(i + cover, K)] += A[Lson].length[i] + A[Rson].length[i];
    }
}

//此子函数用于把坐标离散化至 X 数组，建立线段树，并把边的事件用 y 坐标排序
void discrete(Rect *R, int N) {
    int i, tx1, tx2;
    for (CX = i = 0; i < N; ++i) {
        X[CX++] = R[i].x1;
        X[CX++] = R[i].x2;
    }
    sort(X, X + CX);
    CX = unique(X, X + CX) - X;
    T.build(tot = 0, CX - 2);
    for (i = 0; i < N; ++i) {
        tx1 = lower_bound(X, X + CX, R[i].x1) - X;;
        tx2 = lower_bound(X, X + CX, R[i].x2) - X;;
        E[i].second = E[i + N].second = make_pair(tx1, tx2);
        E[i].first = make_pair(R[i].y1, -1);
        E[i + N].first = make_pair(R[i].y2, 1);
    }
    sort(E, E + (N << 1));
}

//求矩形并得周长，传入储存矩形的数组 R 和矩形个数 N
int perimeter(Rect *R, int N) {
    int ret = 0, i, k, prv;
    for (k = 0; k < 2; ++k) {
        discrete(R, N);
```

```cpp
        for (prv = i = 0; i < (N << 1); ++i) {
            T.add(-E[i].first.second, E[i].second.first, E[i].second.second - 1);
            ret += abs(T.length[K] - prv);
            prv = T.length[K];
        }
        for (i = 0; i < N; ++i) {
            swap(R[i].x1, R[i].y1);
            swap(R[i].x2, R[i].y2);
        }
    }
    return ret;
}

//求矩形并的面积，传入储存矩形的数组 R 和矩形个数 N
LL area(Rect *R, int N) {
    int i, k, prv;
    LL ret = 0;
    discrete(R, N);
    prv = E[0].first.first;
    for(i = 0; i < (N << 1); ++i) {
        ret += (LL)T.length[K] * (E[i].first.first - prv);
        prv = E[i].first.first;
        T.add(-E[i].first.second, E[i].second.first, E[i].second.second - 1);
    }
    return ret;
}
int main() {}
```

5.11 线段树

```cpp
//线段树
//可以处理加入边和删除边不同的情况
//incSet 和 decSeg 用于加入边
//segLen 求长度
//t 传根节点 (一律为 1)
//L0, r0 传树的节点范围 (一律为 1..t)
//L, r 传线段 (端点)
const int MAXN = 10000;
class SegTree {
    int n, cnt[MAXN], len[MAXN];

    SegTree(int t) : n(t) {
        for (int i = 1; i <= t; i++) {
            cnt[i] = len[i] = 0;
        }
    };

    void update(int t, int L, int r);
    void incSet(int t, int L0, int r0, int L, int r);
    void decSeg(int t, int L0, int r0, int L, int r);
    int segLen(int t, int L0, int r0, int L, int r);
};

```

```
inline int length(int L, int r) {
    return r - L;
}

void SegTree::update(int t, int L, int r) {
    if (cnt[t] || r - L == 1) {
        len[t] = length(L, r);
    } else {
        len[t] = len[t + t] + len[t + t + 1];
    }
}

void SegTree::incSet(int t, int L0, int r0, int L, int r) {
    if (L0 == L && r0 == r) {
        cnt[t]++;
    } else {
        int m0 = (L0 + r0) >> 1;
        if (L < m0) {
            incSet(t + t, L0, m0, L, m0 < r ? m0 : r);
        }
        if (r > m0) {
            incSet(t + t + 1, m0, r0, m0 > L ? m0 : L, r);
        }
        if (cnt[t + t] && cnt[t + t + 1]) {
            cnt[t + t]--;
            update(t + t, L0, m0);
            cnt[t + t + 1]--;
            update(t + t + 1, m0, r0);
            cnt[t]++;
        }
    }
    update(t, L0, r0);
}

void SegTree::decSeg(int t, int L0, int r0, int L, int r) {
    if (L0 == L && r0 == r) {
        cnt[t]--;
    } else if (cnt[t]) {
        cnt[t]--;
        if (L > L0) {
            incSet(t, L0, r0, L0, L);
        }
        if (r < r0) {
            incSet(t, L0, r0, r, r0);
        }
    } else {
        int m0 = (L0 + r0) >> 1;
        if (L < m0) {
            decSeg(t + t, L0, m0, L, m0 < r ? m0 : r);
        }
        if (r > m0) {
            decSeg(t + t + 1, m0, r0, m0 > L ? m0 : L, r);
        }
    }
    update(t, L0, r0);
}

int SegTree::segLen(int t, int L0, int r0, int L, int r) {
```

```cpp
    if (cnt[t] || (L0 == L && r0 == r)) {
        return len[t];
    } else {
        int m0 = (L0 + r0) >> 1, ret = 0;
        if (L < m0) {
            ret += segLen(t + t, L0, m0, L, m0 < r ? m0 : r);
        }
        if (r > m0) {
            ret += segLen(t + t + 1, m0, r0, m0 > L ? m0 : L, r);
        }
        return ret;
    }
}
```

5.12 线段树扩展

```cpp
//线段树扩展
//可以计算长度和线段数
//可以处理加入边和删除边不同的情况
//incSeg 和 decSeg 用于加入边
//segLen 求长度, setCut 求线段数
//t 传根节点 (一律为 1)
//L0, r0 传树的节点范围 (一律为 1..t)
//L, r 传线段 (端点)
const int MAXN = 10000;
class SegTree {
public:
    int n, cnt[MAXN], len[MAXN], cut[MAXN], bl[MAXN], br[MAXN];

    SegTree(int t) : n(t) {
        for (int i = 1; i <= t; i++) {
            cnt[i] = len[i] = cut[i] = bl[i] = br[i] = 0;
        }
    };

    void update(int t, int L, int r);
    void incSeg(int t, int L0, int r0, int L, int r);
    void decSeg(int t, int L0, int r0, int L, int r);
    int segLen(int t, int L0, int r0, int L, int r);
    int setCut(int t, int L0, int r0, int L, int r);
};

inline int length(int L, int r) {
    return r - L;
}

void SegTree::update(int t, int L, int r) {
    if (cnt[t] || r - L == 1) {
        len[t] = length(L, r);
        cut[t] = bl[t] = br[t] = 1;
    } else {
        len[t] = len[t + t] + len[t + t + 1];
        cut[t] = cut[t + t] + cut[t + t + 1];
```

```
        if (br[t + t] && bl[t + t + 1]) {
            cut[t]--;
        }
        bl[t] = bl[t + t];
        br[t] = br[t + t + 1];
    }
}

void SegTree::incSeg(int t, int L0, int r0, int L, int r) {
    if (L0 == L && r0 == r) {
        cnt[t]++;
    } else {
        int m0 = (L0 + r0) >> 1;
        if (L < m0) {
            incSeg(t + t, L0, m0, L, m0 < r ? m0 : r);
        }
        if (r > m0) {
            incSeg(t + t + 1, m0, r0, m0 > L ? m0 : L, r);
        }
        if (cnt[t + t] && cnt[t + t + 1]) {
            cnt[t + t]--;
            update(t + t, L0, m0);
            cnt[t + t + 1]--;
            update(t + t + 1, m0, r0);
            cnt[t]++;
        }
    }
    update(t, L0, r0);
}

void SegTree::decSeg(int t, int L0, int r0, int L, int r) {
    if (L0 == L && r0 == r) {
        cnt[t]--;
    } else if (cnt[t]) {
        cnt[t]--;
        if (L > L0) {
            incSeg(t, L0, r0, L0, L);
        }
        if (r < r0) {
            incSeg(t, L0, r0, r, r0);
        }
    } else {
        int m0 = (L0 + r0) >> 1;
        if (L < m0) {
            decSeg(t + t, L0, m0, L, m0 < r ? m0 : r);
        }
        if (r > m0) {
            decSeg(t + t + 1, m0, r0, m0 > L ? m0 : L, r);
        }
    }
    update(t, L0, r0);
}

int SegTree::segLen(int t, int L0, int r0, int L, int r) {
    if (cnt[t] || (L0 == L && r0 == r)) {
        return len[t];
    } else {
        int m0 = (L0 + r0) >> 1, ret=0;
```

```
        if (L < m0) {
            ret += segLen(t + t, L0, m0, L, m0 < r ? m0 : r);
        }
        if (r > m0) {
            ret += segLen(t + t + 1, m0, r0, m0 > L ? m0 : L, r);
        }
        return ret;
    }
}

int SegTree::setCut(int t, int L0, int r0, int L, int r) {
    if (cnt[t]) {
        return 1;
    }
    if (L0 == L && r0 == r) {
        return cut[t];
    } else {
        int m0 = (L0 + r0) >> 1, ret = 0;
        if (L < m0) {
            ret += setCut(t + t, L0, m0, L, m0 < r ? m0 : r);
        }
        if (r > m0) {
            ret += setCut(t + t + 1, m0, r0, m0 > L ? m0 : L, r);
        }
        if (L < m0 && r > m0 && br[t + t] && bl[t + t + 1]) {
            ret--;
        }
        return ret;
    }
}
```

Chapter 6

字符串

6.1 Trie 图 (dd engi)

```
/*
    Trie 图，即用于多串匹配的字符串自动机。
    对于字符串 S 中是否存在模式串 s_1,s_2,..,s_n 的匹配的问题，
    可以用 O((|s_1|+|s_2|+...+|s_n|)|Σ|) 的时间预处理，O(|S|) 的时间回答询问。
    若深入理解，也可根据具体情况扩展之，以解决很多字符串有关的题。
    注意事项：
        1. 字符的类型、字符集的大小可改。
```

```
        2. ELEMENT_MAX 一般可设为 |s_1|+|s_2|+...+|s_n| 的最大可能值。
        3. 传递的字符串中每个字符应在 [0,SIGMA) 的区间内。
    使用方法:
        1. 处理每个测试点前调用 init() 初始化。
        2. 用 insert 插入 s_1,s_2,...
        3. 调用 build_graph 建立 trie 图。
        4. 调用 match 查询 s 中是否存在匹配。
*/
# include <cstring>
# include <queue>
# include <cstdio>
using namespace std;
typedef struct node *trie;
const int SIGMA = 26;
const int ELEMENT_MAX = 50000;
int tot;
struct node {
    bool match;
    trie pre, child[SIGMA];
} T[ELEMENT_MAX];

void trie_init() {
    tot = 1;
    memset(T, 0, sizeof(T));
}
void insert(char *s, int n) {
    trie t = T;
    for (int i = 0; i < n; ++i) {
        int c = s[i] - 'a';
        if (!t->child[c]) {
            t->child[c] = &T[tot++];
        }
        t = t->child[c];
    }
    t->match = true;
}
void build_graph() {
    trie t = T;
    queue <trie> Q;
    for (int i = 0; i < SIGMA; ++i) {
        if (t->child[i]) {
            t->child[i]->pre = t;
            Q.push(t->child[i]);
        } else {
            t->child[i] = t;
        }
    }
    while (!Q.empty()) {
        t = Q.front();
        Q.pop();
        t->match |= t->pre->match;
        for (int i = 0; i < SIGMA; ++i) {
            if (t->child[i]) {
                t->child[i]->pre = t->pre->child[i];
```

```
                Q.push(t->child[i]);
            } else {
                t->child[i] = t->pre->child[i];
            }
        }
    }
}
bool match(char *s, int n) {
    trie t = T;
    for (int i = 0; i < n; ++i) {
        int c = s[i] - 'a';
        t = t->child[c];
    }
    return t->match;
}
//示例程序
int main() {
    trie_init();
    insert("abcd", 4);
    insert("bc", 2);
    build_graph();

    printf("%s\n", match("abc", 3) ? "Yes" : "No"); //output: Yes
}
```

6.2 Trie 图 (猛犸也钻地)

```
// 确定性 AC 自动机 (Trie 图) By 猛犸也钻地 @ 2011.11.24

# include <cstring>
# include <algorithm>
using namespace std;

class TrieGraph {
public:
    static const int SIZE = 100005; // 最大结点总数，约为模板串长度之和
    static const int LEAF = 26;     // 每个结点下的叶子数量
    // next[] 指向了含有相同后缀但更短的一个字符串，n 为当前存在的结点总数
    int next[SIZE],e[SIZE][LEAF],n; // e[][] 为结点的各个叶子的编号
    int data[SIZE]; // data[] 一般用位标记维护当前的串匹配上了哪些模式串
    void init() {
        static int ok=0;
        if(!ok++) n=SIZE;
        fill_n(next,n,0);
        fill_n(data,n,0);
        memset(e,-1,n*sizeof(e[0]));
        n=1;
    }
    void insert(const char *s, int idx = 0) {
        int x=0;
        for(int i=0; s[i]; i++) {
            int c=s[i]-'a'; // 根据题目的字符集修改这里的映射方式
```

```
            x=~e[x][c]?e[x][c]:e[x][c]=n++;
        }
        data[x]|=1<<idx;
    }
    void make() {
        static int q[SIZE],m;
        next[0]=m=0;
        for(int c=0; c<LEAF; c++)
            if(~e[0][c]) next[q[m++]=e[0][c]]=0;
            else e[0][c]=0;
        for(int i=0; i<m; i++) {
            int x=q[i];
            data[x]|=data[next[x]]; // 求 next[] 路径上的前缀和
            for(int c=0; c<LEAF; c++) {
                int t=e[next[x]][c];
                if(~e[x][c]) next[q[m++]=e[x][c]]=t;
                else e[x][c]=t;
            }
        }
    }
};
```

6.3 后缀数组 ($\log^2 n$)

```
// 后缀数组 (倍增算法) By 猛犸也钻地 @ 2012.02.02

#include <algorithm>
using namespace std;

class SuffixArray {
public:
    // 数组下标和后缀顺序都从 0 开始标号，注意排序后最小的串为空串
    // sa[i] 表示第 i 小的后缀从字符串的第 sa[i] 个位置开始
    // rk[i] 表示字符串第 i 个位置开始的后缀是第 rk[i] 小的字符串
    // ht[i] 表示 sa[i] 和 sa[i-1] 所代表的字符串的公共前缀的长度
    static const int SIZE = 100005;
    int sa[SIZE],rk[SIZE],ht[SIZE],tmp[SIZE*2],len;
    struct Compare {
        int &len,* tmp;
        bool operator ()(int x, int y) const {
            if(tmp[x]!=tmp[y]) return tmp[x]<tmp[y];
            return tmp[x+len]<tmp[y+len];
        }
    };
    // 对字符串 s[0..n-1] 做后缀排序，s[n] 必须为'\0'
    // 排序后的 sa[0] 是后缀"\0"，sa[1..n] 为一般后缀
    // 对于 int 类型的串，同样需保证有且仅有 s[n] 的值为 0
    void gao(const char *s, int n) {
        Compare cmp= {len,tmp};
        for(int i=0; i<=n; i++) rk[sa[i]=i]=s[i];
        for(len=1; rk[sa[n]]!=n; len<<=1) {
            copy(rk,rk+n+1,tmp);
```

```
            sort(sa,sa+n+1,cmp);
            for(int i=rk[*sa]=0; i<n; i++)
                rk[sa[i+1]]=rk[sa[i]]+cmp(sa[i],sa[i+1]);
        }
        for(int i=len=ht[*sa]=0; i<n; i++) {
            if(len) --len;
            while(s[i+len]==s[sa[rk[i]-1]+len]) len++;
            ht[rk[i]]=len;
        }
    }
};
```

6.4 后缀数组 (线性)

```
//Author t__nt
//Linear Suffix Array

//Function suffixArray is copy from
//Linear Work Suffix Array Construction
//Written by Juha K.arkk.ainen, Peter Sandersy AND Stefan Burkhardt

//Function Height is written by myself

//Count the number of the same substring in two given string
//the pairs of the same substrings are not shorter than N

# include<cstdio>
# include<cstring>

inline bool leq(int a1, int a2, int b1, int b2) { // lexicographic order
    return(a1 < b1 || a1 == b1 && a2 <= b2);
} // for pairs

inline bool leq(int a1, int a2, int a3, int b1, int b2, int b3) {
    return(a1 < b1 || a1 == b1 && leq(a2,a3, b2,b3));
} // and triples
// stably sort a[0..n-1] to b[0..n-1] with keys in 0..K from r

void radixPass(int *a, int *b, int *r, int n, int K) { // count occurrences
    int *c = new int[K + 1]; // counter array
    for (int i = 0; i <= K; i++) c[i] = 0; // reset counters
    for (int i = 0; i < n; i++) c[r[a[i]]]++; // count occurrences
    for (int i = 0, sum = 0; i <= K; i++) { // exclusive prefix sums
        int t = c[i];
        c[i] = sum;
        sum += t;
    }
    for (int i = 0; i < n; i++) b[c[r[a[i]]]++] = a[i]; // sort
    delete [] c;
}
```

```
// find the suffix array SA of T[0..n-1] in {1..K}^n
// require T[n]=T[n+1]=T[n+2]=0, n>=2
void suffixArray(int *T, int *SA, int n, int K) {
    int n0=(n+2)/3, n1=(n+1)/3, n2=n/3, n02=n0+n2;
    int *R = new int[n02 + 3];
    R[n02]= R[n02+1]= R[n02+2]=0;
    int *SA12 = new int[n02 + 3];
    SA12[n02]=SA12[n02+1]=SA12[n02+2]=0;
    int *R0 = new int[n0];
    int *SA0 = new int[n0];
    //******* Step 0: Construct sample ********
    // generate positions of mod 1 and mod 2 suffixes
    // the "+(n0-n1)" adds a dummy mod 1 suffix if n%3 == 1
    for (int i=0, j=0; i < n+(n0-n1); i++) if (i%3 != 0) R[j++] = i;
    //******* Step 1: Sort sample suffixes ********
    // lsb radix sort the mod 1 and mod 2 triples
    radixPass(R , SA12, T+2, n02, K);
    radixPass(SA12, R , T+1, n02, K);
    radixPass(R , SA12, T , n02, K);
    // find lexicographic names of triples and
    // write them to correct places in R
    int name = 0, c0 = -1, c1 = -1, c2 = -1;
    for (int i = 0; i < n02; i++) {
        if (T[SA12[i]] != c0 || T[SA12[i]+1] != c1 || T[SA12[i]+2] != c2) {
            name++;
            c0 = T[SA12[i]];
            c1 = T[SA12[i]+1];
            c2 = T[SA12[i]+2];
        }
        if (SA12[i] % 3 == 1) {
            R[SA12[i]/3] = name;     // write to R1
        } else {
            R[SA12[i]/3 + n0] = name;     // write to R2
        }
    }
    // recurse if names are not yet unique
    if (name < n02) {
        suffixArray(R, SA12, n02, name);
        // store unique names in R using the suffix array
        for (int i = 0; i < n02; i++) R[SA12[i]] = i + 1;
    } else // generate the suffix array of R directly
        for (int i = 0; i < n02; i++) SA12[R[i] - 1] = i;
    //******* Step 2: Sort nonsample suffixes ********
    // stably sort the mod 0 suffixes from SA12 by their first character
    for (int i=0, j=0; i < n02; i++) if (SA12[i] < n0) R0[j++] = 3*SA12[i];
    radixPass(R0, SA0, T, n0, K);
    //******* Step 3: Merge ********
    // merge sorted SA0 suffixes and sorted SA12 suffixes
    for (int p=0, t=n0-n1, k=0; k < n; k++) {
# define GetI() (SA12[t] < n0 ? SA12[t] * 3 + 1 : (SA12[t] - n0) * 3 + 2)
```

```
        int i = GetI(); // pos of current offset 12 suffix
        int j = SA0[p]; // pos of current offset 0 suffix
        if (SA12[t] < n0 ? // different compares for mod 1 and mod 2 suffixes
                leq(T[i], R[SA12[t] + n0], T[j], R[j/3]) :
                leq(T[i],T[i+1],R[SA12[t]-n0+1], T[j],T[j+1],R[j/3+n0])) { // suffix fro\
m SA12 is smaller
            SA[k] = i;
            t++;
            if (t == n02) // done --- only SA0 suffixes left
                for (k++; p < n0; p++, k++) SA[k] = SA0[p];
        } else { // suffix from SA0 is smaller
            SA[k] = j;
            p++;
            if (p == n0) // done --- only SA12 suffixes left
                for (k++; t < n02; t++, k++) SA[k] = GetI();
        }
    }
    delete [] R;
    delete [] SA12;
    delete [] SA0;
    delete [] R0;
}

// LCP(i,j)=min{hgt[k]|i+1<=k<=j}
// hgt[k]=LCP(i-1,i)
// O(4n)
void Height(int *T,int *SA,int *height,int n) {
    int *rank=new int[n+3];
    int *h=new int[n+3];
    for(int i=0; i<n; i++) {
        rank[SA[i]]=i;
    }
    for(int i=0; i<n; i++) {
        if(rank[i]==0) {
            h[i]=0;
        } else {
            if(i==0 || h[i-1]<=1) {
                int t1=i,t2=SA[rank[i]-1],cc=0;
                for(int j=0; t1+j<n && t2+j<n; j++) {
                    if(T[t1+j]==T[t2+j]) {
                        cc++;
                    } else {
                        break;
                    }
                }
                h[i]=cc;
            } else {
                int t1=i,t2=SA[rank[i]-1],cc=h[i-1]-1;
                for(int j=h[i-1]-1; t1+j<n && t2+j<n; j++) {
                    if(T[t1+j]==T[t2+j]) {
                        cc++;
                    } else {
                        break;
                    }
```

```
                }
                h[i]=cc;
            }
        }
    }
    for(int i=0; i<n; i++) height[i]=h[SA[i]];
    delete [] rank;
    delete [] h;
}
```

6.5 后缀自动机

```
/*
这是求解多串 LCS 的例程
使用时先调用 init(), 再依次在线调用 add 就可以构建出对应串的后缀自动机 (注意这是个在线过程)
每次 add 至多加入两个点
大致来说建成的后缀自动机有这么几个性质
1、对于任意一个给定串的后缀, 在上面按顺序转移, 最终一定会转移到 last。所以任意一个子串在 \
上面跑, 不可能遇到 null 边
2、既然满足 1, 那么显然跑到某一个状态以后, 后续可接收的串的集合必然一致。
3、val 表示的是能从 root 转移到这个状态的最长串的长度
4、而能 root 转移到本状态的串的长度实际是在这个区间内 [this->fa->val + 1, this->val]
5、this 能接收的字符串的集合是 fa 对应的结点能接收的字符串的集合的子集。
6、不会存在状态 p, this 可接收的字符串集合是 p 可以接收的字符串集合的子集, 而且 p 对应集合的势 \
比 fa 的要小
*/

template <class T> void checkmin(T &t,T x) {
    if (x < t) t = x;
}
template <class T> void checkmax(T &t,T x) {
    if (x > t) t = x;
}
# define foreach(it,v) for (__typeof((v).begin()) it = (v).begin();it != (v).end();it++)
const int N = 250005;

struct Node {
    Node *ch[26], *fa;
    int val;
    int len[10];
    Node():
        val(0), fa(NULL) {
        memset(ch, 0, sizeof(ch));
        memset(len, 0, sizeof(len));
    }
} pool[N * 2 + 5], *last, *root;
vector <Node *> vec[N];

namespace SAM {
int cnt;

```

```
void init() {
    if (cnt)
        for (int i = 0; i < cnt; i++)
            pool[i] = Node();
    cnt = 1;
    root = &pool[0];
    last = root;
}

void add(int c) {
    Node *p = last, *np = &pool[cnt++];
    last = np;
    np->val = p->val + 1;
    for  (; p && !p->ch[c]; p = p->fa)
        p->ch[c] = np;
    if (!p) {
        np->fa = root;
    } else {
        Node *q = p->ch[c];
        if (p->val + 1 == q->val) {
            np->fa = q;
        } else {
            Node *nq = &pool[cnt++];
            *nq = *q;
            nq->val = p->val + 1;
            q->fa = nq;
            np->fa = nq;
            for (; p && p->ch[c] == q; p = p->fa)
                p->ch[c] = nq;
        }
    }
}
}

int m, n;
char S[N], T[N];

int main() {
    SAM::init();
    scanf("%s", S);
    m = strlen(S);
    for (int i = 0; i < m; i++)
        SAM::add(S[i] - 'a');
    int k;
    for (k = 0; scanf("%s", T) != EOF; k++) {
        Node *p = root;
        int cnt = 0;
        n = strlen(T);
        for (int i = 0; i < n; i++) {
            int c = T[i] - 'a';
            if (p->ch[c]) {
                cnt++;
                p = p->ch[c];
            } else {
                for (; p && !p->ch[c]; p = p->fa);
                if (!p) {
                    p = root;
                    cnt = 0;
```

```
                } else {
                    cnt = p->val + 1;
                    p = p->ch[c];
                }
            }
            checkmax(p->len[k], cnt);
        }
    }
    for (int i = 0; i < SAM::cnt; i++) {
        vec[pool[i].val].push_back(&pool[i]);
    }
    int ans = 0;
    for (int i = m; i >= 0; i--) {
        foreach (it, vec[i]) {
            Node *p = *it;
            int now = p->val;
            for (int j = 0; j < k; j++) {
                checkmin(now, p->len[j]);
                if (p->fa) {
                    checkmax(p->fa->len[j], p->len[j]);
                }
            }
            checkmax(ans, now);
        }
    }
    printf("%d\n", ans);
}
```

6.6 字符串最小表示

```
/*
    求字符串的最小表示
    输入：字符串
    返回：字符串最小表示的首字母位置 (0...size-1)
*/
# include <vector>
using namespace std;

template <class T>
int minString(const vector<T>  & str) {
    int i, j, k;
    vector<T> ss(str.size() << 1);
    for (i = 0; i<str.size(); i++) {
        ss[i] = ss[i + str.size()] = str[i];
    }
    for (i = k = 0, j = 1; k<str.size() && i<str.size() && j<str.size();) {
        for (k = 0; k<str.size() && ss[i + k] == ss[j + k]; k++);
        if (k<str.size()) {
            if (ss[i + k]> ss[j + k]) {
                i += k + 1;
            } else {
                j += k + 1;
            }
            if (i == j) {
                j++;
```

```
            }
        }
    }
    return i < j ? i : j;
}
```

6.7 最长回文子串

```
// 最长回文子串 (Manacher) By 猛犸也钻地 @ 2011.11.24

# include <vector>
# include <algorithm>
using namespace std;

// 传入字符串 s 和长度 n, 返回最长回文子串的直径, 复杂度 O(n)
int manacher(const char *s, int n) {
    int u=0,v=0,ret=0;
    vector<int> r(n+n-1);   // r[i] 表示以 i/2 为圆心的最长回文子串的半径
    for(int i=0; i<n+n-1; i++) {  // 比如字符串 abaaba, 看作 a.b.a.a.b.a
        int x=max(min((u-i)/2,r[v-i]),0); // 对应的半径就是 10201310201
        int p=i/2-x,q=(i+1)/2+x;
        while(~p && q<n && s[p]==s[q]) p--,q++,x++;
        if(u<q+q) u=q+q,v=i+i;
        ret=max(ret,x+x-1+i%2);
        r[i]=x;
    }
    return ret; // 需要每个位置的半径的话可以返回 r
}
```

6.8 模式匹配 (kmp)

```
//模式匹配,kmp 算法, 复杂度 O(m+n)
//返回匹配位置,-1 表示匹配失败, 传入匹配串和模式串和长度
//可更改元素类型, 更换匹配函数
const int MAXN = 10000;
# define _match(a, b) ((a) == (b))

template <class elemType>
int patMatch(int ls, const elemType *str, int lp, const elemType *pat) {
    int fail[MAXN] = {-1}, i = 0, j;
    for (j = 1; j < lp; j++) {
        for (i = fail[j - 1]; i >= 0 && !_match(pat[i + 1], pat[j]); i = fail[i]);
        fail[j] = (_match(pat[i + 1], pat[j]) ? i + 1 : - 1);
    }
    for (i = j = 0; i < ls && j < lp; i++) {
        if (_match(str[i], pat[j])) {
            j++;
        } else if (j) {
            j = fail[j - 1] + 1;
            i--;
```

```
        }
    }
    return j == lp ? (i - lp) : -1;
}

// 统计次数

# define MAXN 10000
# define _match(a,b) ((a)==(b))
typedef char elem_t;

int pat_match(int ls,elem_t *str,int lp,elem_t *pat) {
    int ret = 0;
    int fail[MAXN]= {-1},i=0,j;
    for (j=1; j<lp; j++) {
        for (i=fail[j-1]; i>=0&&!_match(pat[i+1],pat[j]); i=fail[i]);
        fail[j]=(_match(pat[i+1],pat[j])?i+1:-1);
    }
    for (i=j=0; i<ls; i++) {
        if (_match(str[i],pat[j])) {
            j++;
            if (j == lp) {
                ++ret;
                j=fail[j-1]+1;
            }
        } else if (j)
            j=fail[j-1]+1,i--;
    }
    return ret;
}


// 扩展 KMP, 复杂度 O(m+n)
// 传入匹配串 str 和模式串 pat 及长度, 返回 A[i] 值与 B[i] 值
// A[i] 表示 pat[i..m-1] 与 pat[0..m-1] 的最长公共前缀的长度
// B[i] 表示 str[i..n-1] 与 pat[0..m-1] 的最长公共前缀的长度

void extKMP(int n, const char str[], int m, const char pat[], int A[], int B[]) {
    A[0] = m;
    int ind = 0, k = 1;
    while (ind + 1 < m && pat[ind + 1] == pat[ind]) ind++;
    A[1] = ind;
    for (int i = 2; i < m; i++) {
        if (i <= k + A[k] - 1 && A[i - k] + i < k + A[k]) {
            A[i] = A[i - k];
        } else {
            ind = max(0, k + A[k] - i);
            while (ind + i < m && pat[ind + i] == pat[ind]) ind++;
            A[i] = ind, k = i;
        }
    }
    ind = 0, k = 0;
    while (ind < n && str[ind] == pat[ind]) ind++;
    B[0] = ind;
    for (int i = 1; i < n; i++) {
```

```
        if (i <= k + B[k] - 1 && A[i - k] + i < k + B[k]) {
            B[i] = A[i - k];
        } else {
            ind = max(0, k + B[k] - i);
            while (ind + i < n && ind < m && str[ind + i] == pat[ind]) ind++;
            B[i] = ind, k = i;
        }
    }
}
```

Chapter 7

图论

7.1 网络流

7.1.1 最大流 (dinic d)

```
/*
    最大流 Dinic 算法 by dd_engi
    1. 算法被封装成了一个 struct。
    2.struct 需要在全局变量中声明或者 new 出来，千万不要声明成栈上的局部变量。
    3. 每次使用前，dinic.init(S,T) 给定源与汇的编号，然后用 dinic.add_edge(x,y,w) 添加每条有 \
容量的边。
    4. 调用 dinic.flow() 进行计算，返回最大流的值；每条边的流量有储存在 edge.f 里。
    5. 同一个 struct 可以处理多组数据，但每次都要先 init。
    6. 不需要知道总点数，点的编号可以不连续，但是所有的编号都需要在 [0,MAXN) 之间。
    7. 可处理多重边。
    8.dinic.cut() 是一个附送的功能，调用 flow() 后，可用它求出最小割中的 T 集。返回 T 集的大小 \
，元素保存在传入的数组中。
*/
# include <cstdio>
# include <cstring>
# include <climits>
using namespace std;

const int MAXN=22000,MAXM=440000;
struct Dinic {
    struct edge {
        int x,y; //两个顶点
        int c; //容量
```

```
        int f; //当前流量
        edge *next,*back; //下一条边, 反向边
        edge(int x,int y,int c,edge *next):x(x),y(y),c(c),f(0),next(next),back(0) {}
        void *operator new(size_t, void *p) {
            return p;
        }
    } *E[MAXN],*data; //E[i] 保存顶点 i 的边表
    char storage[2*MAXM *sizeof(edge)];
    int S,T; //源、汇

    int Q[MAXN]; //DFS 用到的 queue
    int D[MAXN]; //距离标号, -1 表示不可达
    void DFS() {
        memset(D,-1,sizeof(D));
        int head=0,tail=0;
        Q[tail++]=S;
        D[S]=0;
        for(;;) {
            int i=Q[head++];
            for(edge *e=E[i]; e; e=e->next) {
                if(e->c==0)continue;
                int j=e->y;
                if(D[j]==-1) {
                    D[j]=D[i]+1;
                    Q[tail++]=j;
                    if(j==T)return;
                }
            }
            if(head==tail)break;
        }
    }
    edge *cur[MAXN]; //当前弧
    edge *path[MAXN]; //当前找到的增广路
    int flow() {
        int res=0; //结果, 即总流量
        int path_n; //path 的大小
        for(;;) {
            DFS();
            if(D[T]==-1)break;
            memcpy(cur,E,sizeof(E));
            path_n=0;
            int i=S;
            for(;;) {
                if(i==T) { //已找到一条增广路, 增广之
                    int mink=0;
                    int delta=INT_MAX;
                    for(int k=0; k<path_n; ++k) {
                        if(path[k]->c < delta) {
                            delta = path[k]->c;
                            mink=k;
                        }
                    }
                    for(int k=0; k<path_n; ++k) {
```

```
                        path[k]->c -= delta;
                        path[k]->back->c += delta;
                    }
                    path_n=mink; //回退
                    i=path[path_n]->x;
                    res+=delta;
                }
                edge *e;
                for(e=cur[i]; e; e=e->next) {
                    if(e->c==0)continue;
                    int j=e->y;
                    if(D[i]+1==D[j])break; //找到一条弧，加到路径里
                }
                cur[i]=e; //当前弧结构，访问过的不能增广的弧不会再访问
                if(e) {
                    path[path_n++]=e;
                    i=e->y;
                } else { //该节点已没有任何可增广的弧，从图中删去，回退一步
                    D[i]=-1;
                    if(path_n==0)break;
                    path_n--;
                    i=path[path_n]->x;
                }
            }
        }
        return res;
    }
    int cut(int *s) {
        int rst=0;
        for(int i=0; i<MAXN; ++i)
            if(D[i]==-1&&E[i])
                s[rst++]=i;
        return rst;
    }
    void init(int _S,int _T) {
        S=_S,T=_T;
        data=(edge *)storage;
        memset(E,0,sizeof(E));
    }
    void add_edge(int x,int y,int w) { //加进一条 x 至 y 容量为 w 的边，需要保证 0<=x,y<MAXN, 0\
<w<=INT_MAX
        E[x]=new((void *)data++) edge(x,y,w,E[x]);
        E[y]=new((void *)data++) edge(y,x,0,E[y]);
        E[x]->back = E[y];
        E[y]->back = E[x];
    }
};


/**** 用来 AC POJ3469 的示范用法 ****/
Dinic dinic;
int main() {
    int N,M;
    while(2==scanf("%d%d",&N,&M)) {
        int rst=0;
```

```
        int S=0,T=N+1;
        dinic.init(S,T);
        for(int i=1; i<=N; ++i) {
            int a,b;
            scanf("%d%d",&a,&b);
            dinic.add_edge(S,i,a);
            dinic.add_edge(i,T,b);
        }
        for(int i=0; i<M; ++i) {
            int x,y,w;
            scanf("%d%d%d",&x,&y,&w);
            dinic.add_edge(x,y,w);
            dinic.add_edge(y,x,w);
        }
        rst=dinic.flow();
        printf("%d\n",rst);
    }
}
```

7.1.2 最高标号先流推进

```
//邻接表形式, 邻接阵接口, 复杂度 O(V^3)
//返回最大流量,f 返回每条边的流量, 返回最大流量
//传入网络节点数 n, 容量 mat, 源点 s, 汇点 t
const int MAXN = 210;
const int INF = 1000000000;

int maxFlow(int n, const int mat[][MAXN], int s, int t, int f[][MAXN]) {
    int g[MAXN][MAXN], cur[MAXN], h[MAXN], e[MAXN], q[MAXN], l[MAXN * 2][MAXN], i, j, k,\
 head, tail, ret, checked, p, o;
    for (i = 0; i < n; i++) {
        h[i] =  - 1;
        cur[i] = 1;
        for (e[i] = g[i][0] = j = 0; j < n; f[i][j++] = 0) {
            if (mat[i][j] || mat[j][i]) {
                g[i][++g[i][0]] = j;
            }
        }
    }
    for (i = 0; i < 2 * n; i++) {
        l[i][0] = 0;
    }
    for (l[h[q[head = 0] = t] = 0][++l[0][0]] = t, tail = 1; head < tail; head++) {
        for (i = 1; i <= g[j = q[head]][0]; i++) {
            if (h[k = g[j][i]] < 0) {
                h[k] = h[j] + 1;
                q[tail++] = k;
                if (k != s) {
                    l[h[k]][++l[h[k]][0]] = k;
                }
            }
        }
    }
    for (h[s] = n, i = 1; i <= g[s][0]; i++) {
        j = g[s][i];
        f[j][s] = -(f[s][j] = e[j] = mat[s][j]);
```

```
    }
    for (i = n; i; i--) {
        for (checked = 0, j = l[i][0]; j;) {
            if ((k = l[i][j]) == s) {
                j--;
            } else if (!e[k]) {      //Full
                if (checked) {       //Update
                    if (i && l[i][0] == 1) {
                        for (p = h[k] + 1; p < n; l[n][0] += l[p][0], l[p++][0] = 0) {
                            for (o = 1; o <= l[p][0]; o++) {
                                l[n][++l[n][0]] = l[p][o];
                                h[l[p][o]] = n;
                            }
                        }
                    }
                    l[h[k]][++l[h[k]][0]] = k;
                    l[i][j] = l[i][l[i][0]--];
                    i = h[k];
                    break;
                } else {
                    j--;
                }
            } else if (cur[k] > g[k][0]) {  //Relabel
                for (checked = p = 1, o = INF; p <= g[k][0]; p++) {
                    if (mat[k][g[k][p]] > f[k][g[k][p]] && h[g[k][p]] < o) {
                        o = h[g[k][p]];
                    }
                }
                h[k] = o + 1, cur[k] = 1;
            } else if ((o = mat[k][p = g[k][cur[k]]] - f[k][p]) && h[k] == h[p] + 1) {
                o = o < e[k] ? o : e[k];
                f[p][k] = -(f[k][p] += o);
                e[k] -= o;
                e[p] += o;
            } else {
                cur[k]++;
            }
        }
    }
    for (ret = 0, i = 1; i <= g[s][0]; i++) {
        ret += f[s][g[s][i]];
    }
    return ret;
}
```

7.1.3 网络流 (全功能)

```
// 网络流 By 猛犸也钻地 @ 2012.02.10

# include <cstring>
# include <queue>
# include <algorithm>
using namespace std;

class Network {
```

```
public:
    static const int SIZE = 1005;          // 最大点数
    static const int INF = 1000000007;     // 流量的极大值
    typedef int VAL;     // 费用的类型
    typedef struct ARC {
        int t,c;
        VAL w;
        ARC *o;
    } *PTR;
    ARC arc[200005];     // 最大边数，注意一次普通加边操作需要占用两条边
    PTR now[SIZE],e[SIZE];              // now[] 为当前弧，e[] 为出边链表
    int cnt[SIZE],l[SIZE],r[SIZE],edge; // cnt[] 为层总数，l[] 为层次标号
    VAL sum;    // sum 为当前流网络下的费用
    // 传入源点 S 和汇点 T，返回流量，处理费用流时把下面改成 spfa_johnson
    int flow(int S, int T) {
        return improved_sap(S,T,INF);
    }
    ARC &REV(PTR x) {
        return arc[(x-arc)^1];    // 取反向边
    }
    void clear() {
        memset(e,edge=sum=0,sizeof(e));    // 清空边表
    }
    // 加入一条 x 到 y 的有向边，容量为 c，费用为 w
    PTR add_edge(int x, int y, int c, VAL w = 0) {
        e[x]=&(arc[edge++]=(ARC) {
            y,c,+w,e[x]
        });
        e[y]=&(arc[edge++]=(ARC) {
            x,0,-w,e[y]
        });
        return e[x];
    }
    // 加入一条 x 到 y 的无向边，容量为 c，费用为 0
    PTR add_edge_simple(int x, int y, int c) {
        e[x]=&(arc[edge++]=(ARC) {
            y,c,0,e[x]
        });
        e[y]=&(arc[edge++]=(ARC) {
            x,c,0,e[y]
        });
        return e[x];
    }
    // 加入一条 x 到 y 的有向边，下界为 lo，上界为 hi，费用为 w
    // 超级源在 SIZE-2，超级汇在 SIZE-1，注意给这两个点预留空间
    PTR add_edge_bounded(int x, int y, int lo, int hi, VAL w = 0) {
        add_edge(SIZE-2,y,lo,w);
        add_edge(x,SIZE-1,lo,0);
        return add_edge(x,y,hi-lo,w);
    }
    // 对 S 至 T 且出弧为 now[] 的增广路进行松弛，返回被阻塞的结点
```

```
    int aug(int S, int T, int &can) {
        int x,z=T,use=can;
        for(x=S; x!=T; x=now[x]->t) if(use>now[x]->c) use=now[z=x]->c;
        for(x=S; x!=T; x=now[x]->t) {
            now[x]->c-=use;
            REV(now[x]).c+=use;
            sum+=use*now[x]->w;
        }
        can-=use;
        return z;
    }
    // 无权值最短路增广算法，用在无费用的网络流上，返回流量
    int improved_sap(int S, int T, int can) {
        if(S==T) return can; // can 为本次增广的流量上限
        int in=can,x,m;
        memcpy(now,e,sizeof(now));
        memset(cnt,0,sizeof(cnt));
        memset(l,127,sizeof(l));
        for(int i=m=l[r[0]=T]=0; i<=m; i++) {
            cnt[l[x=r[i]]]++;
            for(PTR u=e[x]; u; u=u->o)
                if(l[u->t]>=INF && REV(u).c) l[r[++m]=u->t]=l[x]+1;
        }
        for(x=r[S]=S; l[S]<INF; x=r[x]) {
JMP:
            for(PTR &u=now[x]; u; u=u->o) if(l[u->t]<l[x] && u->c) {
                    r[u->t]=x;
                    x=u->t==T?aug(S,T,can):u->t;
                    if(x==T) return in;
                    else goto JMP;
                }
            if(!--cnt[l[x]]) break;
            else l[x]=INF;
            for(PTR u=e[x]; u; u=u->o)
                if(l[u->t]+1<l[x] && u->c) now[x]=u,l[x]=l[u->t]+1;
            if(l[x]<INF) cnt[l[x]]++;
        }
        return in-can;
    }
    // 连续最短路增广算法，可以处理不含负费用圈的费用流，返回流量
    int spfa_johnson(int S, int T, int can) {
        if(S==T) return can; // can 为本次增广的流量上限
        int in=can,x,m;
        VAL phi[SIZE],len[SIZE],MAXW=1000000007; // 费用极大值
        fill_n(phi,SIZE,MAXW);
        memset(l,0,sizeof(l));
        phi[r[0]=T]=0;
        for(int i=m=0; i<=m; i++) { // 从汇点出发反向 SPFA
            l[x=r[i%SIZE]]=0;
            for(PTR u=e[x]; u; u=u->o) { // 下面这行如果是浮点比较要加 EPS
                if(phi[x]+REV(u).w>=phi[u->t] || !REV(u).c) continue;
                phi[u->t]=phi[x]+REV(u).w;
                if(!l[u->t]) l[r[++m%SIZE]=u->t]=1;
            }
        }
```

```
        do {
            typedef pair<VAL,int> TPL;
            priority_queue<TPL> q;
            fill_n(len,SIZE,MAXW);
            memset(l,0,sizeof(l));
            q.push(TPL(len[T]=0,T));
            while(q.size()) {
                x=q.top().second;
                q.pop();
                if(!l[x]) l[x]=1;
                else continue;
                for(PTR u=e[x]; u; u=u->o) {
                    if(!REV(u).c || l[u->t]) continue;
                    VAL at=len[x]+phi[x]+REV(u).w-phi[u->t];
                    if(at>=len[u->t]) continue; // 如果是浮点比较要加 EPS
                    len[u->t]=at;
                    now[u->t]=&REV(u);
                    q.push(TPL(-at,u->t));
                }
            }
            for(x=0; x<SIZE; x++) phi[x]+=len[x];
        } while(phi[S]<MAXW && aug(S,T,can)!=T);
        // 使用 phi[S]<MAXW 求最小费用最大流，使用 phi[S]<0 求最小费用流
        return in-can;
    }
    // 判断无源汇上下界可行流是否存在
    // 加入边 (T,S)=INF 可处理带源汇的情况，此时反向弧 S->T 的 c 即为流量
    bool feasible_bounded() {
        flow(SIZE-2,SIZE-1);
        for(PTR u=e[SIZE-2]; u; u=u->o) if(u->c) return false;
        return true;
    }
    // 有源汇上下界最大/最小流，返回-1 表示不存在可行流，否则返回流量
    int max_flow_bounded(int S, int T) {
        add_edge(T,S,INF);
        bool ok=feasible_bounded();
        int ret=e[S]->c;
        edge-=2,e[S]=e[S]->o,e[T]=e[T]->o;
        return ok?ret+flow(S,T):-1;
    }
    int min_flow_bounded(int S, int T) {
        flow(SIZE-2,SIZE-1);
        add_edge(T,S,INF);
        bool ok=feasible_bounded();
        int ret=e[S]->c;
        edge-=2,e[S]=e[S]->o,e[T]=e[T]->o;
        return ok?ret:-1;
    }
    // 将所有带下界的边合并回原图中
    void merge_bounded() {
        for(PTR u=e[SIZE-1]; u; u=u->o) u->c=REV(u).c=0;
        for(PTR u=e[SIZE-2]; u; u=u->o) {
            (u+4)->c+=u->c;
            (u+5)->c+=REV(u).c;
            u->c=REV(u).c=0;
```

```
        }
    }
};
```

7.2 匹配

7.2.1 一般图匹配 (Edmonds 邻接阵形式)

```
//一般图匹配，Edmonds 带花树算法，时间复杂度 O(n^4)，一般达不到这么大
//输入：图的点数 N，邻接阵 mat（全局），非 0 值表示有边，点编号为 0,1,...,N-1
//输出：一般图的最大匹配数 *2，match 数组存匹配，未匹配顶点记为-1
# include <cstdio>
# include <cstring>
# include <algorithm>
# include <vector>
# include <queue>
using namespace std;
const int MAXN = 255;
bool mat[MAXN][MAXN], vis[MAXN], mk[MAXN];
int prv[MAXN], base[MAXN], match[MAXN];
int go(int i, int root) {
    int ret = -1;
    while(true) {
        mk[i] ^= 1;
        if (!mk[i] && ret < 0) {
            ret = i;
        }
        if (i == root) break;
        mk[base[match[i]]] ^= 1;
        i = base[prv[match[i]]];
    }
    return ret;
}
void reset(int i, int newbase) {
    while (base[i] != newbase) {
        i = prv[match[i]];
        if (base[i] != newbase)
            prv[i] = match[i];
    }
}
int findpath(int root, int N) {
    memset(vis, 0, sizeof(vis));
    memset(prv, 0xff, sizeof(prv));
    for (int i = 0; i < N; ++i)
        base[i] = i;
    queue <int> Q;
    Q.push(root);
    vis[root] = true;
    while (!Q.empty()) {
        int i = Q.front();
        Q.pop();
        for (int j = 0; j < N; ++j) {
            if (mat[i][j] && (base[i] != base[j]) && (match[i] != j)) {
                if ((j == root) || ((match[j] >= 0) && (prv[match[j]] >= 0))) {
```

```
                    memset(mk, 0, sizeof(mk));
                    go(base[i], root);
                    int newbase = go(base[j], root);
                    reset(i, newbase);
                    reset(j, newbase);
                    if (base[i] != newbase)     prv[i] = j;
                    if (base[j] != newbase)     prv[j] = i;
                    for (int k = 0; k < N; ++k) {
                        if (mk[base[k]]) {
                            base[k] = newbase;
                            if (!vis[k]) {
                                Q.push(k);
                                vis[k] = true;
                            }
                        }
                    }
                } else if (prv[j] < 0) {
                    prv[j] = i;
                    if (match[j] >= 0) {
                        Q.push(match[j]);
                        vis[match[j]] = true;
                    } else {
                        return j;
                    }
                }
            }
        }
    }
    return -1;
}
int Edmonds(int N) {
    memset(match, 0xff, sizeof(match));
    for (int i = 0; i < N; ++i) {
        if (match[i] < 0) {
            int j = findpath(i, N);
            while (j >= 0) {
                match[j] = prv[j];
                swap(j, match[match[j]]);
            }
        }
    }
    int ret = 0;
    for (int i = 0; i < N; ++i)
        ret += (match[i] >= 0);
    return ret;
}
int main() {
    //freopen("in.txt", "r", stdin);
    //freopen("out.txt", "w", stdout);
    int i, j, N;
    scanf("%d", &N);
    while (scanf("%d%d", &i, &j) != EOF) {
        mat[i - 1][j - 1] = mat[j - 1][i - 1] = true;
    }
    printf("%d\n", Edmonds(N));
    for (i = 0; i < N; ++i) {
        if (match[i] > i) {
```

```
            printf("%d %d\n", i + 1, match[i] + 1);
        }
    }
}
```

7.2.2 二分图最佳匹配 (kuhn munkras 邻接阵形式)

```
//二分图最佳匹配,kuhn munkras 算法, 邻接阵形式, 复杂度 O(m*m*n)
//返回最佳匹配值, 传入二分图大小 m,n 和邻接阵 mat, 表示权值
//match1,match2 返回一个最佳匹配, 未匹配顶点 match 值为-1
//一定注意 m<=n, 否则循环无法终止
//最小权匹配可将权值取相反数
# include <cstring>
const int MAXN = 310;
const int INF = 1000000000;
# define _clr(x) memset(x, 0xff, sizeof(int) * n)

int kuhnMunkras(int m, int n, int mat[][MAXN], int *match1, int *match2) {
    int s[MAXN + 1], t[MAXN], l1[MAXN], l2[MAXN], p, q, ret = 0, i, j, k;
    for (i = 0; i < m; i++) {
        l1[i] =  - INF;
        for (j = 0; j < n; j++) {
            l1[i] = mat[i][j] > l1[i] ? mat[i][j]: l1[i];
        }
    }
    for (i = 0; i < n; l2[i++] = 0);
    _clr(match1);
    _clr(match2);
    for (i = 0; i < m; i++) {
        _clr(t);
        for (s[p = q = 0] = i; p <= q && match1[i] < 0; p++) {
            k = s[p];
            for (j = 0; j < n && match1[i] < 0; j++) {
                if (l1[k] + l2[j] == mat[k][j] && t[j] < 0) {
                    s[++q] = match2[j];
                    t[j] = k;
                    if (s[q] < 0) {
                        for (p = j; p >= 0; j = p) {
                            match2[j] = k = t[j];
                            p = match1[k];
                            match1[k] = j;
                        }
                    }
                }
            }
        }
        if (match1[i] < 0) {
            i--;
            p = INF;
            for (k = 0; k <= q; k++) {
                for (j = 0; j < n; j++) {
                    if (t[j] < 0 && l1[s[k]] + l2[j] - mat[s[k]][j] < p) {
                        p = l1[s[k]] + l2[j] - mat[s[k]][j];
                    }
```

```
                    }
                }
                for (j = 0; j < n; j++) {
                    l2[j] += t[j] < 0 ? 0 : p;
                }
                for (k = 0; k <= q; k++) {
                    l1[s[k]] -= p;
                }
            }
        }
    for (i = 0; i < m; i++) {
        ret += mat[i][match1[i]];
    }
    return ret;
}
```

7.2.3 二分图最大匹配 (hopcroft kart 邻接表形式)

```
// Hopcroft_Karp matching algorithm, 图的大小为 n 和 m, 返回最大匹配数
// vector 存储, 复杂度 O(sqrt(V)*E)
// match1 和 match2 为最大匹配, 未匹配节点 match1 为-1, match2 为 n
// 每次使用之前将边信息存入 E

# include <cstring>
# include <vector>
using namespace std;

const int MAXN = 50005;
const int MAXM = 200005;

int n, m;
int match1[MAXN], match2[MAXN];
int Q[MAXN], D1[MAXN], D2[MAXN];
vector <int> E[MAXN];

inline bool bfs() {
    int s = 0, t = 0, u, v;
    memset(D1, -1, sizeof(D1));
    memset(D2, -1, sizeof(D2));
    for (int i = 0; i < n; i++)
        if (match1[i] == -1)
            Q[t++] = i, D1[i] = 0;
    while (s != t)
        if ((u = Q[s++]) != n)
            for (int i = 0; i < (int)E[u].size(); i++)
                if (D2[v = E[u][i]] == -1) {
                    D2[v] = D1[u] + 1;
                    if (D1[match2[v]] == -1)
                        D1[Q[t++] = match2[v]] = D2[v] + 1;
                }
    return D1[n] != -1;
}

bool dfs(int u) {
    for (int i = 0, v; i < (int)E[u].size(); i++)
```

```
        if (D2[v = E[u][i]] == D1[u] + 1 && (D2[v] = -1) && (match2[v] == n || dfs(match\
2[v]))) {
            match1[u] = v;
            match2[v] = u;
            D1[u] = -1;
            return true;
        }
    D1[u] = -1;
    return false;
}

inline int hopcroft_karp() {
    memset(match1, -1, sizeof(match1));
    for (int i = 0; i < m; i++)
        match2[i] = n;
    int ret = 0;
    for (int i = 0; i < n; i++)
        for (int j = 0, u; j < E[i].size(); j++)
            if (match2[u = E[i][j]] == n) {
                match1[i] = u;
                match2[u] = i;
                ret++;
                break;
            }
    while (bfs())
        for (int i = 0; i < n; i++)
            if (match1[i] == -1 && dfs(i))
                ret++;
    return ret;
}
```

7.2.4 二分图最大匹配 (hungary bfs 邻接阵形式)

```
//二分图最大匹配,hungary 算法, 邻接阵形式, 复杂度 O(n*n*m)
//返回最大匹配数, 传入二分图大小 n,m 和邻接阵 mat, 非零元素表示有边
//match1,match2 返回一个最大匹配, 未匹配顶点 match 值为-1
# include <cstring>
const int MAXN = 310;
# define _clr(x) memset(x, 0xff, sizeof(int) * MAXN)

int hungary(int n, int m, const bool mat[][MAXN], int *match1, int *match2) {
    int s[MAXN + 1], t[MAXN], p, q, ret = 0, i, j, k;
    _clr(match1);
    _clr(match2);
    for (i = 0; i < n; ret += (match1[i++] >= 0)) {
        _clr(t);
        for (s[p = q = 0] = i; p <= q && match1[i] < 0; p++) {
            k = s[p];
            for (j = 0; j < m && match1[i] < 0; j++) {
                if (mat[k][j] && t[j] < 0) {
                    s[++q] = match2[j];
                    t[j] = k;
                    if (s[q] < 0) {
                        for (p = j; p >= 0; j = p) {
                            match2[j] = k = t[j];
```

```
                    p = match1[k];
                    match1[k] = j;
                }
            }
        }
    }
  }
 }
 return ret;
}
```

7.2.5 二分图最大匹配 (hungary dfs 邻接阵形式)

```
//二分图最大匹配,hungary dfs 算法, 邻接阵形式, 复杂度 O(n*n*m)
//返回最大匹配数, 传入二分图大小 n,m 和邻接阵 mat, 非零元素表示有边
//match1,match2 返回一个最大匹配, 未匹配顶点 match 值为-1
# define _clr(x) memset(x, 0xff, sizeof(int) * MAXN)
const int MAXN = 200;
int mk[MAXN], match1[MAXN], match2[MAXN], mat[MAXN][MAXN], n, m;
int path(int i) {
    for (int j = 0; j < m; j++)
        if (mat[i][j] && !(mk[j]++) && (match2[j] < 0 || path(match2[j]))) {
            match1[i] = j;
            match2[j] = i;
            return 1;
        }
    return 0;
}
int hungary() {
    int res(0);
    _clr(match1);
    _clr(match2);
    for (int i = 0; i < n; i++)
        if (match1[i] < 0) {
            memset(mk, 0, sizeof(mk));
            res += path(i);
        }
    return res;
}
```

7.3 生成树

7.3.1 多源最小树形图 (邻接阵形式)

```
//多源最小树形图,edmonds 算法, 邻接阵形式, 复杂度 O(n^3)
//返回最小生成树的长度, 构造失败返回负值
//传入图的大小 n 和邻接阵 mat, 不相邻点边权 inf
//可更改边权的类型,pre[] 返回树的构造, 用父结点表示
//传入时 pre[] 数组清零, 用-1 标出可能的源点
# include <string.h>
# define MAXN 120
```

```
# define inf 1000000000
typedef int elem_t;
elem_t edmonds(int n,elem_t mat[][MAXN*2],int *pre) {
    elem_t ret=0;
    int c[MAXN*2][MAXN*2],l[MAXN*2],p[MAXN*2],m=n,t,i,j,k;
    for (i=0; i<n; l[i]=i,i++);
    do {
        memset(c,0,sizeof(c)),memset(p,0xff,sizeof(p));
        for (t=m,i=0; i<m; c[i][i]=1,i++);
        for (i=0; i<t; i++)
            if (l[i]==i&&pre[i]!=-1) {
                for (j=0; j<m; j++)
                    if
                    (l[j]==j&&i!=j&&mat[j][i]<inf&&(p[i]==-1||mat[j][i]<mat[p[i]][i]))
                        p[i]=j;
                if ((pre[i]=p[i])==-1)
                    return -1;
                if (c[i][p[i]]) {
                    for (j=0; j<=m; mat[j][m]=mat[m][j]=inf,j++);
                    for (k=i; l[k]!=m; l[k]=m,k=p[k])
                        for (j=0; j<m; j++)
                            if (l[j]==j) {
                                if (mat[j][k]-mat[p[k]][k]<mat[j][m])
                                    mat[j][m]=mat[j][k]-mat[p[k]][k];
                                if (mat[k][j]<mat[m][j])
                                    mat[m][j]=mat[k][j];
                            }
                    c[m][m]=1,l[m]=m,m++;
                }
                for (j=0; j<m; j++)
                    if (c[i][j])
                        for (k=p[i]; k!=-1&&l[k]==k; c[k][j]=1,k=p[k]);
            }
    } while (t<m);
    for (; m-->n; pre[k]=pre[m])
        for (i=0; i<m; i++)
            if (l[i]==m) {
                for (j=0; j<m; j++)
                    if (pre[j]==m&&mat[i][j]==mat[m][j])
                        pre[j]=i;
                if (mat[pre[m]][m]==mat[pre[m]][i]-mat[pre[i]][i])
                    k=i;
            }
    for (i=0; i<n; i++)
        if (pre[i]!=-1)
            ret+=mat[pre[i]][i];
    return ret;
}
```

7.3.2 最小生成树 (kruskal 邻接表形式)

```
//无向图最小生成树,kruskal 算法, 邻接表形式, 复杂度 O(mlogm)
//返回最小生成树的长度, 传入图的大小 n 和邻接表 list
//可更改边权的类型,edge[][2] 返回树的构造, 用边集表示
//如果图不连通, 则对各连通分支构造最小生成树, 返回总长度
```

```
# include <cstring>
const int MAXN = 200;
const int INF = 1000000000;

# define _run(x) for(; p[t = x]; x = p[x], p[t] = (p[x] ? p[x] : x))
# define _run_both _run(i); _run(j)

class DSet {
public:
    int p[MAXN], t;
    void init() {
        memset(p, 0, sizeof(p));
    }
    void setFriend(int i, int j) {
        _run_both;
        p[i] = (i == j ? 0 : j);
    }
    bool isFriend(int i, int j) {
        _run_both;
        return i == j && i;
    }
};

typedef double elemType;

struct Edge {
    int from, to;
    elemType len;
    Edge *next;
};

struct HeapNode {
    int a, b;
    elemType len;
};

# define _cp(a,b) ((a).len < (b).len)

class MinHeap {
public:
    HeapNode h[MAXN *MAXN];
    int n, p, c;
    void init() {
        n = 0;
    }
    void ins(HeapNode e) {
        for (p = ++n; p > 1 && _cp(e, h[p >> 1]); h[p] = h[p >> 1], p >>= 1);
        h[p] = e;
    }
    bool del(HeapNode &e) {
        if (!n) {
            return false;
        }
        e = h[p = 1];
        for (c = 2; c < n && _cp(h[c += (c < n - 1 && _cp(h[c + 1], h[c]))], h[n]); c <<\
= 1) {
```

```
            h[p] = h[c];
            p = c;
        }
        h[p] = h[n--];
        return true;
    }
};

elemType kruskal(int n, const Edge *list[], int edge[][2]) {
    DSet u;
    MinHeap h;
    const Edge *t;
    HeapNode e;
    elemType ret = 0;
    int i, m = 0;
    u.init(), h.init();
    for (i = 0; i < n; i++) {
        for (t = list[i]; t; t = t->next) {
            if (i < t->to) {
                e.a = i;
                e.b = t->to;
                e.len = t->len;
                h.ins(e);
            }
        }
    }
    while (m < n - 1 && h.del(e)) {
        if (!u.isFriend(e.a + 1, e.b + 1)) {
            edge[m][0] = e.a;
            edge[m][1] = e.b;
            ret += e.len;
            u.setFriend(e.a + 1, e.b + 1);
        }
    }
    return ret;
}
```

7.3.3 最小生成树 (prim+priority queue 邻接阵形式)

```
// 不是太苛刻的情况下推荐使用
// 复杂度为 O(|E|+|V|lg|V|), 但常数较大, 而且复杂度不是很严格
// 边权非负!

# define Rec pair<T, int>

template<class T>
T Prim(int n, vector<pair<int, T> > e[], T mind[], int *pre = NULL) {
    int s = 0;
    priority_queue<Rec, vector<Rec>, greater<Rec> > q;
    vector<bool> mark(n, false);

    // pre 为 NULL 则不做记录
    if (pre != NULL) {
        fill(pre, pre + n, -1);
        pre[s] = s;
```

```
    }
    // mind 初始化部分注意修改
    fill(mind, mind + n, numeric_limits<T>::max());
    mind[s] = T();
    q.push(make_pair(T(), s));

    T ret = T();

    while (!q.empty()) {
        s = q.top().second;
        if (!mark[s]) {
            mark[s] = true;
            ret += q.top().first;
            q.pop();
            for (typename vector<pair<int, T> >::const_iterator i = e[s].begin(); i != e\
[s].end(); ++i) {
                if (!mark[i->first] && mind[i->first] > i->second) {
                    mind[i->first] = i->second;
                    if (pre != NULL) {
                        pre[i->first] = s;
                    }
                    q.push(make_pair(mind[i->first], i->first));
                }
            }
        } else {
            q.pop();
        }
    }

    return ret;
}
```

7.3.4 最小生成树 (prim 邻接阵形式)

```
//无向图最小生成树,prim 算法, 邻接阵形式, 复杂度 O(n^2)
//返回最小生成树的长度, 传入图的大小 n 和邻接阵 mat, 不相邻点边权 INF
//可更改边权的类型,pre[] 返回树的构造, 用父结点表示, 根节点 (第一个)pre 值为-1
//必须保证图的连通的!
const int MAXN = 200;
const int INF = 1000000000;

template <class elemType>
elemType prim(int n, const elemType mat[][MAXN], int *pre) {
    elemType mind[MAXN], ret = 0;
    int v[MAXN], i, j, k;
    for (i = 0; i<n; i++) {
        mind[i] = INF;
        v[i] = 0;
        pre[i] = -1;
    }
    for (mind[j = 0] = 0; j<n; j++) {
        for (k = -1, i = 0; i<n; i++) {
            if (!v[i] && (k == -1 || mind[i]<mind[k])) {
                k = i;
            }
```

```
        }
        v[k] = 1;
        ret += mind[k];
        for (i = 0; i < n; i++) {
            if (!v[i] && mat[k][i] < mind[i]) {
                mind[i] = mat[pre[i] = k][i];
            }
        }
    }
    return ret;
}
```

7.3.5 次最小生成树

```
// 次小生成树, 复杂度 O(n^2)
// 传入邻接阵 mat, 不存在边权 inf
// 返回次小生成树长度和树的构造 pre[]
// 如返回 inf 则不存在次小生成树
// 必须保证图的连通
const int maxn = 100;
const int inf = 1000000000;
typedef int elem_t;

elem_t prim(int n, elem_t mat[][maxn], int *pre) {
    elem_t min[maxn], ret = 0;
    int v[maxn], i, j, k;

    for (i = 0; i < n; ++i)
        min[i] = inf, v[i] = 0, pre[i] = -1;
    for (min[j = 0] = 0; j < n; ++j) {
        for (k = -1, i = 0; i < n; ++i) if (!v[i] && (k == -1 || min[i] < min[k]))
                k = i;
        for (v[k] = 1, ret += min[k], i = 0; i < n; ++i)
            if (!v[i] && mat[k][i] < min[i])
                min[i] = mat[pre[i] = k][i];
    }

    return ret;
}

elem_t sbmst(int n, elem_t mat[][maxn], int *pre) {
    elem_t min = inf, t, ret = prim(n, mat, pre);
    int i, j, ti, tj;
    for (i = 0; i < n; ++i) for (j = 0; j < n && pre[i] != -1; ++j) if (i != j && pre[i]\
 != j && pre[j] != i)
                if (mat[j][i] < inf && (t = mat[j][i] - mat[pre[i]][i]) < min)
                    min = t, ti = i, tj = j;
    pre[ti] = tj;
    return ret + min;
}
```

7.4 连通性

7.4.1 无向图关键点、关键边和块

```
//无向图关键点、关键边和双连通块（由关键点分割得到的块）
//复杂度 O(m)，边存在 E 中，要保证传入时的 E 没有重边
//调用 ArtEdge_ArtVertex_Components(总点数)
//结果关键点储存在 keyV 中
//结果关键边储存在 keyE 中
# include <cstring>
# include <vector>
using namespace std;
# define MP(i,j) make_pair(i, j)
# define MAXN 10000
int dfn[MAXN], low[MAXN], st[MAXN], top;
vector <pair<int, int> > keyE;
vector <int> keyV, E[MAXN];

void tarjan(int now, int cnt) {
    int part = (cnt > 1);
    st[top++] = now;
    dfn[now] = low[now] = cnt;
    for (int ii = E[now].size() - 1; ii >= 0; --ii) {
        int i = E[now][ii];
        if (!dfn[i]) {
            tarjan(i, cnt + 1);
            low[now] = min(low[now], low[i]);
            if (low[i] > dfn[now])          // 这两行用于求取关键边
                keyE.push_back(MP(now, i));

            if (low[i] >= dfn[now]) {
                if(++part == 2)             // 以下两行求取关键点
                    keyV.push_back(now);

                vector <int> A;             //以下四行求取双联通块
                A.push_back(now);
                for (st[top] = 0; st[top] != i; A.push_back(st[--top]));
                dummy(A); //每次 dummy 调用的 A 中包含一个联通块
            }
        } else if (dfn[i] != dfn[now] - 1)
            low[now] = min(low[now], dfn[i]);
    }
    //此时 part 值等于将此点去掉之后它原先所在的联通块分成的块数
}
void ArtEdge_ArtVertex_Components(int N) {
    memset(dfn, 0, sizeof(dfn));
    memset(low, 0, sizeof(low));
    keyE.clear();
    keyV.clear();
    for (int i = top = 0; i < N; ++i)
        if (!dfn[i])
            tarjan(i, 1);
```

```
}
```

7.4.2 有向图强连通分量

```
//有向图强连通分量, 复杂度 O(m)
//边存在 E 中, 调用 Components(总点数)
//返回 id 数组储存每个点所在的强连通分量编号 (从 0 开始)
//最后 num 表示一共有几个强连通分量
# include <cstring>
# include <vector>
using namespace std;
# define MP(i,j) make_pair(i, j)
# define MAXN 10000
int dfn[MAXN], low[MAXN], id[MAXN], num, st[MAXN], top, in[MAXN], tot;
vector <int> E[MAXN];

void tarjan(int now) {
    in[st[top++] = now] = true;
    dfn[now] = low[now] = ++tot;
    int i;
    for (int ii = E[now].size() - 1; ii >= 0; --ii) {
        i = E[now][ii];
        if (!dfn[i]) {
            tarjan(i);
            low[now] = min(low[now], low[i]);
        } else if (in[i])
            low[now] = min(low[now], dfn[i]);
    }
    if (dfn[now] == low[now]) {
        do {
            i = st[--top];
            in[i] = false;
            id[i] = num;
        } while (i != now);
        ++num;
    }
}
void Components(int N) {
    memset(dfn, 0, sizeof(dfn));
    memset(low, 0, sizeof(low));
    memset(in, 0, sizeof(in));
    memset(id, 0xff, sizeof(id));
    for (int i = top = num = tot = 0; i < N; ++i)
        if (!dfn[i])
            tarjan(i);
}
```

7.5 NP 搜索

7.5.1 带权最大团

```c
/* wclique.c exact algorithm for finding one maximum-weight
   clique in an arbitrary graph,
   10.2.2000, Patric R. J. Ostergard,
   patric.ostergard@hut.fi */

/* compile: gcc wclique.c -o wclique -O2 */

/* usage: wclique infile */

/* infile format: see http://www.tcs.hut.fi/~pat/wclique.html */

# include <stdio.h>
# include <sys / times.h>
# include <sys / types.h>

# define INT_SIZE (8*sizeof(int))
# define TRUE 1
# define FALSE 0
# define MAX_VERTEX 2000  /* maximum number of vertices */
# define MAX_WEIGHT 1000000  /* maximum weight of vertex */
# define is_edge(a,b) (bit[a][b / INT_SIZE]&(mask[b%INT_SIZE]))

int Vnbr,Enbr;           /* number of vertices/edges */
int clique[MAX_VERTEX];  /* table for pruning */
int bit[MAX_VERTEX][MAX_VERTEX/INT_SIZE+1];
int wt[MAX_VERTEX];

int pos[MAX_VERTEX];     /* reordering function */
int set[MAX_VERTEX];     /* current clique */
int rec[MAX_VERTEX];     /* best clique so far */
int record;      /* weight of best clique */
int rec_level;           /* # of vertices in best clique */

unsigned mask[INT_SIZE];
void graph();            /* reads graph */

struct tms bf;
int timer1;
double timer11;

main (argc,argv)
int argc;
char *argv[];
{
    int i,j,k,p;
```

```c
    int min_wt,max_nwt,wth;
    int new[MAX_VERTEX],used[MAX_VERTEX];
    int nwt[MAX_VERTEX];
    int count;
    FILE *infile;

    /* read input */
    if(argc < 2) {
        printf("Usage: wclique infile\n");
        exit(1);
    }
    if((infile=fopen(argv[1],"r"))==NULL)
        fileerror();

    /* initialize mask */
    mask[0] = 1;
    for(i=1; i<INT_SIZE; i++)
        mask[i] = mask[i-1]<<1;

    /* read graph */
    graph(infile);

    /* "start clock" */
    times(&bf);
    timer1 = bf.tms_utime;

    /* order vertices */
    for(i=0; i<Vnbr; i++) {
        nwt[i] = 0;
        for(j=0; j<Vnbr; j++)
            if (is_edge(i,j)) nwt[i] += wt[j];
    }
    for(i=0; i<Vnbr; i++)
        used[i] = FALSE;
    count = 0;
    do {
        min_wt = MAX_WEIGHT+1;
        max_nwt = -1;
        for(i=Vnbr-1; i>=0; i--)
            if((!used[i])&&(wt[i]<min_wt))
                min_wt = wt[i];
        for(i=Vnbr-1; i>=0; i--) {
            if(used[i]||(wt[i]>min_wt)) continue;
            if(nwt[i]>max_nwt) {
                max_nwt = nwt[i];
                p = i;
            }
        }
        pos[count++] = p;
        used[p] = TRUE;
        for(j=0; j<Vnbr; j++)
            if((!used[j])&&(j!=p)&&(is_edge(p,j)))
                nwt[j] -= wt[p];
    } while(count<Vnbr);

    /* main routine */
```

```
    record = 0;
    wth = 0;
    for(i=0; i<Vnbr; i++) {
        wth += wt[pos[i]];
        sub(i,pos,0,0,wth);
        clique[pos[i]] = record;
        times(&bf);
        timer11 = (bf.tms_utime - timer1)/100.0;
        printf("level = %3d(%d) best = %2d time = %8.2f\n",i+1,Vnbr,record,timer11);
    }
    printf("Record: ");
    for(i=0; i<rec_level; i++)
        printf ("%d ",rec[i]);
    printf ("\n");
}

int sub(ct,table,level,weight,l_weight)
int ct,level,weight,l_weight;
int *table;
{
    register int i,j,k;
    int best;
    int curr_weight,left_weight;
    int newtable[MAX_VERTEX];
    int *p1,*p2;

    if(ct<=0) { /* 0 or 1 elements left; include these */
        if(ct==0) {
            set[level++] = table[0];
            weight += l_weight;
        }
        if(weight>record) {
            record = weight;
            rec_level = level;
            for (i=0; i<level; i++) rec[i] = set[i];
        }
        return 0;
    }
    for(i=ct; i>=0; i--) {
        if((level==0)&&(i<ct)) return 0;
        k = table[i];
        if((level>0)&&(clique[k]<=(record-weight))) return 0;  /* prune */
        set[level] = k;
        curr_weight = weight+wt[k];
        l_weight -= wt[k];
        if(l_weight<=(record-curr_weight)) return 0; /* prune */
        p1 = newtable;
        p2 = table;
        left_weight = 0;
        while (p2<table+i) {
            j = *p2++;
            if(is_edge(j,k)) {
                *p1++ = j;
                left_weight += wt[j];
            }
        }
        if(left_weight<=(record-curr_weight)) continue;
```

```
            sub(p1-newtable-1,newtable,level+1,curr_weight,left_weight);
        }
        return 0;
}

void graph(fp)
FILE *fp;
{
        register int i,j,k;
        int weight,degree,entry;

        if(!fscanf(fp,"%d %d\n",&Vnbr,&Enbr))
            fileerror();
        for(i=0; i<Vnbr; i++)     /* empty graph table */
            for(j=0; j<Vnbr/INT_SIZE+1; j++)
                bit[i][j] = 0;
        for(i=0; i<Vnbr; i++) {
            if(!fscanf(fp,"%d %d",&weight,&degree))
                fileerror();
            wt[i] = weight;
            for(j=0; j<degree; j++) {
                if(!fscanf(fp,"%d",&entry))
                    fileerror();
                bit[i][entry/INT_SIZE] |= mask[entry%INT_SIZE]; /* record edge */
            }
        }
        fclose(fp);
}

int fileerror() {
        printf("Error in graph file\n");
        exit();
}
```

7.5.2 最大团 (n 小于 64)(faster)

```
/**
 * WishingBone's ACM/ICPC Routine Library
 *
 * maximum clique solver
 */

// 不知道怎么用……

# include <vector>

using std::vector;

// clique solver calculates both size and consitution of maximum clique
// uses bit operation to accelerate searching
// graph size limit is 63, the graph should be undirected
// can optimize to calculate on each component, and sort on vertex degrees
// can be used to solve maximum independent set
```

```cpp
class clique {
public:
    static const long long ONE = 1;
    static const long long MASK = (1 << 21) - 1;
    char *bits;
    int n, size, cmax[63];
    long long mask[63], cons;
    // initiate lookup table
    clique() {
        bits = new char[1 << 21];
        bits[0] = 0;
        for (int i = 1; i < 1 << 21; ++i) bits[i] = bits[i >> 1] + (i & 1);
    }
    ~clique() {
        delete bits;
    }
    // search routine
    bool search(int step, int size, long long more, long long con);
    // solve maximum clique and return size
    int sizeClique(vector<vector<int> >& mat);
    // solve maximum clique and return constitution
    vector<int> consClique(vector<vector<int> >& mat);
};

// search routine
// step is node id, size is current solution, more is available mask, cons is
// constitution mask
bool clique::search(int step, int size, long long more, long long cons) {
    if (step >= n) {
        // a new solution reached
        this->size = size;
        this->cons = cons;
        return true;
    }
    long long now = ONE << step;
    if ((now & more) > 0) {
        long long next = more & mask[step];
        if (size + bits[next & MASK] + bits[(next >> 21) & MASK] + bits[next >>
                42] >= this->size
                && size + cmax[step] > this->size) {
            // the current node is in the clique
            if (search(step + 1, size + 1, next, cons | now)) return true;
        }
    }
    long long next = more & ~now;
    if (size + bits[next & MASK] + bits[(next >> 21) & MASK] + bits[next >> 42]
            > this->size) {
        // the current node is not in the clique
        if (search(step + 1, size, next, cons)) return true;
    }
    return false;
}

// solve maximum clique and return size
```

```
int clique::sizeClique(vector<vector<int> >& mat) {
    n = mat.size();
    // generate mask vectors
    for (int i = 0; i < n; ++i) {
        mask[i] = 0;
        for (int j = 0; j < n; ++j) if (mat[i][j] > 0) mask[i] |= ONE << j;
    }
    size = 0;
    for (int i = n - 1; i >= 0; --i) {
        search(i + 1, 1, mask[i], ONE << i);
        cmax[i] = size;
    }
    return size;
}

// solve maximum clique and return constitution
// calls sizeClique and restore cons
vector<int> clique::consClique(vector<vector<int> >& mat) {
    sizeClique(mat);
    vector<int> ret;
    for (int i = 0; i < n; ++i) if ((cons & (ONE << i)) > 0) ret.push_back(i);
    return ret;
}
```

7.5.3 最大团

```
const int maxn = 50;

void clique(int n, int mat[][maxn], int num, int U[], int size, int C[], int &_max, int \
ok) {
    int i, j, k, tmp[maxn];
    if (num == 0) {
        if (size > _max) {
            ok = 1;
            _max = size;
        }
        return;
    }
    for (i = 0; i < num && !ok; ++i) {
        if (size + num  - i <= _max) return;
        if (size + C[U[i]] <= _max) return;
        for (k = 0, j = i + 1; j < num; ++j) if (mat[U[i]][U[j]])
                tmp[k++] = U[j];
        clique(n, mat, k, tmp, size + 1, C, _max, ok);
    }
}

int max_clique(int n, int mat[][maxn]) {
    int i, j, k, U[maxn], C[maxn], _max;
    for (_max = 0, i = n - 1; i >= 0; --i) {
        for (k = 0, j = i + 1; j < n; ++j) if (mat[i][j])
                U[k++] = j;
        clique(n, mat, k, U, 1, C, _max, 0);
        C[i] = _max;
    }
    return _max;
```

```
}
```

7.6 应用

7.6.1 前序表转化

```
//将用边表示的树转化为前序表示的树
//传入节点数 n 和邻接表 list[], 邻接表必须是双向的, 会在函数中释放
//pre[] 返回前序表,map[] 返回前序表中的节点到原来节点的映射
const int MAXN = 10000;
struct Node {
    int to;
    Node *next;
};

void prenode(int n, Node *list[], int *pre, int *map, int *v, int now, int last, int &id\
) {
    Node *t;
    int p = id++;
    for (v[map[p] = now] = 1, pre[p] = last; list[now];) {
        t = list[now];
        list[now] = t->next;
        if (!v[t->to]) {
            prenode(n, list, pre, map, v, t->to, p, id);
        }
    }
}

void makepre(int n, Node *list[], int *pre, int *map) {
    int v[MAXN], id = 0, i;
    for (i = 0; i < n; v[i++] = 0);
    prenode(n, list, pre, map, v, 0,  - 1, id);
}
```

7.6.2 拓扑排序 (邻接阵形式)

```
//拓扑排序, 邻接阵形式, 复杂度 O(n^2)
//如果无法完成排序, 返回 0, 否则返回 1,ret 返回有序点列
//传入图的大小 n 和邻接阵 mat, 不相邻点边权 0
const int MAXN = 100;

bool toposort(int n, int mat[][MAXN], int *ret) {
    int d[MAXN], i, j, k;
    for (i = 0; i < n; i++) {
        for (d[i] = j = 0; j < n; d[i] += mat[j++][i]);
    }
    for (k = 0; k < n; ret[k++] = i) {
        for (i = 0; d[i] && i < n; i++);
        if (i == n) {
            return false;
        }
        for (d[i] =  - 1, j = 0; j < n; j++) {
            d[j] -= mat[i][j];
```

```
        }
    }
    return true;
}
```

7.6.3 无向图全局最小割

```
// 无向图全局最小割 (Stoer-Wagner) By 猛犸也钻地 @ 2012.08.22

# include <vector>
# include <algorithm>
using namespace std;

class StoerWagner {
public:
    typedef int VAL; // 权值的类型
    static const int SIZE = 505;          // 最大结点个数
    static const VAL INF = 1000000007;    // 最大权值之和
    VAL sum,e[SIZE][SIZE];
    // 传入结点个数 n 及权值矩阵 a[][], 返回无向图全局最小割的边权之和 sum
    // 对于矩阵 a[][] 中不存在的边, 权值设为 0
    int gao(int n, const VAL a[SIZE][SIZE]) {
        vector<int> v,idx(n);
        for(int i=0; i<n; i++) copy(a[i],a[i]+n,e[idx[i]=i]);
        for(int i=0; i<n; i++) e[i][i]=0;
        for(sum=INF; idx.size()>=2; n=idx.size()) {
            vector<VAL> s(n);
            for(int i=0; i<n; i++) v.push_back(i);
            int p=0,t=0;
            while(v.size()) {
                int m=v.size(),x=-1;
                for(int i=0; i<m; i++) if(x<0 || s[x]<s[i]) x=i;
                for(int i=0; i<m; i++) s[i]+=e[idx[v[x]]][idx[v[i]]];
                v.erase(v.begin()+x);
                s.erase(s.begin()+x);
                swap(p=v[x],t);
            }
            VAL now=0;
            for(int i=0; i<n; i++) if(i!=t) now+=e[idx[t]][idx[i]];
            for(int i=0; i<n; i++) {
                e[idx[i]][idx[p]]+=e[idx[i]][idx[t]];
                e[idx[p]][idx[i]]+=e[idx[t]][idx[i]];
            }
            idx.erase(idx.begin()+t);
            sum=min(sum,now);
        }
        return sum;
    }
};
```

7.6.4 无向图最小环

```
// 无向图最小环 (Floyd) By 猛犸也钻地 @ 2012.09.13

# include <vector>
# include <cstring>
using namespace std;

class Floyd {
public:
    typedef int VAL; // 权值的类型
    static const int SIZE = 105;
    vector<int> path;
    VAL len[SIZE][SIZE],ans;
    int src[SIZE][SIZE];
    // 传入结点个数 n 及权值矩阵 a[][], 返回最小环的长度 ans, 方案记在 path 中
    // 对于矩阵 a[][] 中不存在的边, 权值设为 1e9+7 或 0x7F7F7F7F 之类的极大值
    VAL gao(int n, const VAL a[SIZE][SIZE]) {
        ans=1e9+7; // 若最后的返回值大于等于 1e9, 则不存在最小环
        memset(src,-1,sizeof(src));
        memcpy(len, a,sizeof(len));
        for(int k=0; k<n; k++) {
            for(int i=0; i<k; i++) for(int j=i+1; j<k; j++) {
                    VAL tmp=a[k][i]+a[j][k];
                    if(len[i][j]>=ans-tmp) continue;
                    path.clear();
                    getpath(i,j);
                    path.push_back(k);
                    path.push_back(i);
                    ans=tmp+len[i][j];
                }
            for(int i=0; i<n; i++) for(int j=0; j<n; j++) {
                    VAL tmp=len[i][k]+len[k][j];
                    if(tmp>=len[i][j]) continue;
                    len[i][j]=tmp;
                    src[i][j]=k;
                }
        }
        return ans;
    }
private:
    void getpath(int i, int j) {
        int k=src[i][j];
        if(~k) {
            getpath(i,k);
            getpath(k,j);
        } else {
            path.push_back(j);
        }
    }
};
```

7.6.5 最佳边割集

```
//最佳边割集
const int MAXN = 100;
const int INF = 1000000000;

int maxFlow(int n, int mat[][MAXN], int source, int sink) {
    int v[MAXN], c[MAXN], p[MAXN], ret = 0, i, j;
    while (true) {
        for (i = 0; i < n; i++) {
            v[i] = c[i] = 0;
        }
        for (c[source] = INF;;) {
            for (j = -1, i = 0; i < n; i++) {
                if (!v[i] && c[i] && (j == -1 || c[i] > c[j])) {
                    j = i;
                }
            }
            if (j < 0) {
                return ret;
            }
            if (j == sink) {
                break;
            }
            for (v[j] = 1, i = 0; i < n; i++) {
                if (mat[j][i] > c[i] && c[j] > c[i]) {
                    c[i] = mat[j][i] < c[j] ? mat[j][i]: c[j];
                    p[i] = j;
                }
            }
        }
        for (ret += j = c[i = sink]; i != source; i = p[i]) {
            mat[p[i]][i] -= j;
            mat[i][p[i]] += j;
        }
    }
}

int bestEdgeCut(int n, int mat[][MAXN], int source, int sink, int set[][2], int &mincost\
) {
    int m0[MAXN][MAXN], m[MAXN][MAXN], i, j, k, l, ret = 0, last;
    if (source == sink) {
        return -1;
    }
    for (i = 0; i < n; i++) {
        for (j = 0; j < n; j++) {
            m0[i][j] = mat[i][j];
        }
    }
    for (i = 0; i < n; i++) {
        for (j = 0; j < n; j++) {
            m[i][j] = m0[i][j];
        }
    }
    mincost = last = maxFlow(n, m, source, sink);
    for (k = 0; k < n && last; k++) {
        for (l = 0; l < n && last; l++) {
            if (m0[k][l]) {
```

```
                for (i = 0; i < n + n; i++) {
                    for (j = 0; j < n + n; j++) {
                        m[i][j] = m0[i][j];
                    }
                }
                m[k][l] = 0;
                if (maxFlow(n, m, source, sink) == last - mat[k][l]) {
                    set[ret][0] = k;
                    set[ret++][1] = l;
                    m0[k][l] = 0;
                    last -= mat[k][l];
                }
            }
        }
    }
    return ret;
}
```

7.6.6 最佳顶点割集

```
//最佳顶点割集
const int MAXN = 100;
const int INF = 1000000000;

int maxFlow(int n, int mat[][MAXN], int source, int sink) {
    int v[MAXN], c[MAXN], p[MAXN], ret = 0, i, j;
    while (true) {
        for (i = 0; i < n; i++) {
            v[i] = c[i] = 0;
        }
        for (c[source] = INF;;) {
            for (j = -1, i = 0; i < n; i++) {
                if (!v[i] && c[i] && (j == -1 || c[i] > c[j])) {
                    j = i;
                }
            }
            if (j < 0) {
                return ret;
            }
            if (j == sink) {
                break;
            }
            for (v[j] = 1, i = 0; i < n; i++) {
                if (mat[j][i] > c[i] && c[j] > c[i]) {
                    c[i] = mat[j][i] < c[j] ? mat[j][i] : c[j];
                    p[i] = j;
                }
            }
        }
        for (ret += j = c[i = sink]; i != source; i = p[i]) {
            mat[p[i]][i] -= j;
            mat[i][p[i]] += j;
        }
    }
}

int bestVertexCut(int n, int mat[][MAXN], int *cost, int source, int sink, int *set, int\
```

```
&mincost) {
    int m0[MAXN][MAXN], m[MAXN][MAXN], i, j, k, ret = 0, last;
    if (source == sink || mat[source][sink]) {
        return -1;
    }
    for (i = 0; i < n + n; i++) {
        for (j = 0; j < n + n; j++) {
            m0[i][j] = 0;
        }
    }
    for (i = 0; i < n; i++) {
        for (j = 0; j < n; j++) {
            if (mat[i][j]) {
                m0[i][n + j] = INF;
            }
        }
    }
    for (i = 0; i < n; i++) {
        m0[n + i][i] = cost[i];
    }
    for (i = 0; i < n + n; i++) {
        for (j = 0; j < n + n; j++) {
            m[i][j] = m0[i][j];
        }
    }
    mincost = last = maxFlow(n + n, m, source, n + sink);
    for (k = 0; k < n && last; k++) {
        if (k != source && k != sink) {
            for (i = 0; i < n + n; i++) {
                for (j = 0; j < n + n; j++) {
                    m[i][j] = m0[i][j];
                }
            }
            m[n + k][k] = 0;
            if (maxFlow(n + n, m, source, n + sink) == last - cost[k]) {
                set[ret++] = k;
                m0[n + k][k] = 0;
                last -= cost[k];
            }
        }
    }
    return ret;
}
```

7.6.7 最小路径覆盖

```
//最小路径覆盖,O(n^3)
//求解最小的路径覆盖图中所有点, 有向图无向图均适用
//注意此问题等价二分图最大匹配, 可以用邻接表或正向表减小复杂度
//返回最小路径条数,pre 返回前指针 (起点-1),next 返回后指针 (终点-1)
#include <cstring>
const int MAXN = 310;
#define _clr(x) memset(x, 0xff, sizeof(int) * n)

int hungary(int n, const bool mat[][MAXN], int *match1, int *match2) {
```

```
    int s[MAXN], t[MAXN], p, q, ret = 0, i, j, k;
    _clr(match1);
    _clr(match2);
    for (i = 0; i < n; ret += (match1[i++] >= 0)) {
        _clr(t);
        for (s[p = q = 0] = i; p <= q && match1[i] < 0; p++) {
            for (k = s[p], j = 0; j < n && match1[i] < 0; j++) {
                if (mat[k][j] && t[j] < 0) {
                    s[++q] = match2[j];
                    t[j] = k;
                    if (s[q] < 0) {
                        for (p = j; p >= 0; j = p) {
                            match2[j] = k = t[j];
                            p = match1[k];
                            match1[k] = j;
                        }
                    }
                }
            }
        }
    }
    return ret;
}

inline int pathCover(int n, const bool mat[][MAXN], int *pre, int *next) {
    return n - hungary(n, mat, next, pre);
}
```

7.6.8 最小边割集

```
//最小边割集
const int MAXN = 100;
const int INF = 1000000000;

int maxFlow(int n, int mat[][MAXN], int source, int sink) {
    int v[MAXN], c[MAXN], p[MAXN], ret = 0, i, j;
    while (true) {
        for (i = 0; i < n; i++) {
            v[i] = c[i] = 0;
        }
        for (c[source] = INF;;) {
            for (j = -1, i = 0; i < n; i++) {
                if (!v[i] && c[i] && (j == -1 || c[i] > c[j])) {
                    j = i;
                }
            }
            if (j < 0) {
                return ret;
            }
            if (j == sink) {
                break;
            }
            for (v[j] = 1, i = 0; i < n; i++) {
                if (mat[j][i] > c[i] && c[j] > c[i]) {
                    c[i] = mat[j][i] < c[j] ? mat[j][i] : c[j];
                    p[i] = j;
                }
```

```
            }
        }
        for (ret += j = c[i = sink]; i != source; i = p[i]) {
            mat[p[i]][i] -= j;
            mat[i][p[i]] += j;
        }
    }
}

int minEdgeCut(int n, int mat[][MAXN], int source, int sink, int set[][2]) {
    int m0[MAXN][MAXN], m[MAXN][MAXN], i, j, k, l, ret = 0, last;
    if (source == sink) {
        return  - 1;
    }
    for (i = 0; i < n; i++) {
        for (j = 0; j < n; j++) {
            m0[i][j] = (mat[i][j] != 0);
        }
    }
    for (i = 0; i < n; i++) {
        for (j = 0; j < n; j++) {
            m[i][j] = m0[i][j];
        }
    }
    last = maxFlow(n, m, source, sink);
    for (k = 0; k < n && last; k++) {
        for (l = 0; l < n && last; l++) {
            if (m0[k][l]) {
                for (i = 0; i < n + n; i++) {
                    for (j = 0; j < n + n; j++) {
                        m[i][j] = m0[i][j];
                    }
                }
                m[k][l] = 0;
                if (maxFlow(n, m, source, sink) < last) {
                    set[ret][0] = k;
                    set[ret++][1] = l;
                    m0[k][l] = 0;
                    last--;
                }
            }
        }
    }
    return ret;
}
```

7.6.9 最小顶点割集

```
//最小顶点割集
const int MAXN = 100;
const int INF = 1000000000;

int maxFlow(int n, int mat[][MAXN], int source, int sink) {
    int v[MAXN], c[MAXN], p[MAXN], ret = 0, i, j;
    while (true) {
        for (i = 0; i < n; i++) {
            v[i] = c[i] = 0;
```

```
        }
        for (c[source] = INF;;) {
            for (j = -1, i = 0; i < n; i++) {
                if (!v[i] && c[i] && (j == -1 || c[i] > c[j])) {
                    j = i;
                }
            }
            if (j < 0) {
                return ret;
            }
            if (j == sink) {
                break;
            }
            for (v[j] = 1, i = 0; i < n; i++) {
                if (mat[j][i] > c[i] && c[j] > c[i]) {
                    c[i] = mat[j][i] < c[j] ? mat[j][i] : c[j];
                    p[i] = j;
                }
            }
        }
        for (ret += j = c[i = sink]; i != source; i = p[i]) {
            mat[p[i]][i] -= j;
            mat[i][p[i]] += j;
        }
    }
}

int minVertexCut(int n, int mat[][MAXN], int source, int sink, int *set) {
    int m0[MAXN][MAXN], m[MAXN][MAXN], i, j, k, ret = 0, last;
    if (source == sink || mat[source][sink]) {
        return  - 1;
    }
    for (i = 0; i < n + n; i++) {
        for (j = 0; j < n + n; j++) {
            m0[i][j] = 0;
        }
    }
    for (i = 0; i < n; i++) {
        for (j = 0; j < n; j++) {
            if (mat[i][j]) {
                m0[i][n + j] = INF;
            }
        }
    }
    for (i = 0; i < n; i++) {
        m0[n + i][i] = 1;
    }
    for (i = 0; i < n + n; i++) {
        for (j = 0; j < n + n; j++) {
            m[i][j] = m0[i][j];
        }
    }
    last = maxFlow(n + n, m, source, n + sink);
    for (k = 0; k < n && last; k++) {
        if (k != source && k != sink) {
            for (i = 0; i < n + n; i++) {
                for (j = 0; j < n + n; j++) {
                    m[i][j] = m0[i][j];
```

```
                }
            }
            m[n + k][k] = 0;
            if (maxFlow(n + n, m, source, n + sink) < last) {
                set[ret++] = k;
                m0[n + k][k] = 0;
                last--;
            }
        }
    }
    return ret;
}
```

7.6.10 树的优化算法

```
const int MAXN = 1000;

//最大顶点独立集
int maxNodeIndependent(int n, int *pre, int *set) {
    int c[MAXN], i, ret = 0;
    for (i = 0; i < n; i++) {
        c[i] = set[i] = 0;
    }
    for (i = n - 1; i >= 0; i--) {
        if (!c[i]) {
            set[i] = 1;
            if (pre[i] != -1) {
                c[pre[i]] = 1;
            }
            ret++;
        }
    }
    return ret;
}

//最大边独立集
int maxEdgeIndependent(int n, int *pre, int *set) {
    int c[MAXN], i, ret = 0;
    for (i = 0; i < n; i++) {
        c[i] = set[i] = 0;
    }
    for (i = n - 1; i >= 0; i--) {
        if (!c[i] && pre[i] != -1 && !c[pre[i]]) {
            set[i] = 1;
            c[pre[i]] = 1;
            ret++;
        }
    }
    return ret;
}

//最小顶点覆盖集
int minNodeCover(int n, int *pre, int *set) {
    int c[MAXN], i, ret = 0;
    for (i = 0; i < n; i++) {
        c[i] = set[i] = 0;
```

```
    }
    for (i = n - 1; i >= 0; i--) {
        if (!c[i] && pre[i] != -1 && !c[pre[i]]) {
            set[i] = 1;
            c[pre[i]] = 1;
            ret++;
        }
    }
    return ret;
}

//最小顶点支配集
int minNodeDominant(int n, int *pre, int *set) {
    int c[MAXN], i, ret = 0;
    for (i = 0; i < n; i++) {
        c[i] = set[i] = 0;
    }
    for (i = n - 1; i >= 0; i--) {
        if (!c[i] && (pre[i] == -1 || !set[pre[i]])) {
            if (pre[i] !=  - 1) {
                set[pre[i]] = 1;
                c[pre[i]] = 1;
                if (pre[pre[i]] != -1) {
                    c[pre[pre[i]]] = 1;
                }
            } else {
                set[i] = 1;
            }
            ret++;
        }
    }
    return ret;
}
```

7.6.11 欧拉回路 (邻接阵形式)

```
//求欧拉回路或欧拉路, 邻接阵形式, 复杂度 O(n^2)
//返回路径长度,path 返回路径 (有向图时得到的是反向路径)
//传入图的大小 n 和邻接阵 mat, 不相邻点边权 0
//可以有自环与重边, 分为无向图和有向图

const int MAXN = 100;

void findPathU(int n, int mat[][MAXN], int now, int &step, int *path) {
    int i;
    for (i = n - 1; i >= 0; i--) {
        while (mat[now][i]) {
            mat[now][i]--;
            mat[i][now]--;
            findPathU(n, mat, i, step, path);
        }
    }
    path[step++] = now;
}

```

```
void findPathD(int n, int mat[][MAXN], int now, int &step, int *path) {
    int i;
    for (i = n - 1; i >= 0; i--) {
        while (mat[now][i]) {
            mat[now][i]--;
            findPathD(n, mat, i, step, path);
        }
    }
    path[step++] = now;
}

int euclidPath(int n, int mat[][MAXN], int start, int *path) {
    int ret = 0;
    findPathU(n, mat, start, ret, path);
//  findPathD(n, mat, start, ret, path);
    return ret;
}
```

Chapter 8

应用

8.1 简单位操作

<table>
<tr><th>功能</th><th>示例</th><th>位运算</th><th>备注</th></tr>
<tr><td>把右数第 k 位置 0</td><td></td><td>x & (~(1 << k))</td><td>k 从 0 开始计数</td></tr>
<tr><td>把右数第 k 位置 1</td><td></td><td>x | (1 << k)</td><td>k 从 0 开始计数</td></tr>
<tr><td>把右数第 k 位取反</td><td></td><td>x ^ (1 << k)</td><td>k 从 0 开始计数</td></tr>
<tr><td>得到右数第 k 位值</td><td></td><td>(x >> k) & 1</td><td>k 从 0 开始计数</td></tr>
<tr><td>得到末尾 k 位</td><td></td><td>x & ((1 << k) - 1)</td><td></td></tr>
<tr><td>把最右边的 1 置 0</td><td>01011000 -> 01010000</td><td>x & (x - 1)</td><td>把结果跟 0 作比较可以得出 x 是否为 2 的幂</td></tr>
<tr><td>得到最右边的 1 的掩码</td><td>01011000 -> 00001000</td><td>x & (-x)</td><td></td></tr>
<tr><td>把最右边的 0 置 1</td><td>01011000 -> 01011001</td><td>x | (x + 1)</td><td></td></tr>
<tr><td>得到最右边的 0 的掩码</td><td>10100111 -> 00001000</td><td>(~x) & (x + 1)</td><td></td></tr>
<tr><td>把末尾连续 0 串置 1</td><td>01011000 -> 01011111</td><td>x | (x - 1)</td><td>如果 x 为 0 则结果为全 1</td></tr>
<tr><td>得到末尾连续 0 串的掩码</td><td>01011000 -> 00000111</td><td>(~x) & (x - 1)</td><td>或者使用 ~(x | (-x)) 和 (x & (-x)) - 1</td></tr>
<tr><td>得到最右边 1 及其右边连续 0 串的掩码</td><td>01011000 -> 00001111</td><td>x ^ (x - 1)</td><td>如果 x 为 0 则结果为全 1</td></tr>
<tr><td>把最右边的连续 1 串置 0</td><td>01011000 -> 01000000</td><td>((x | (x - 1)) + 1) & x</td><td>把结果跟 0 作比较可以得出 x 是否为 2^j-2^k (j ≥ k ≥ 0)</td></tr>
<tr><td>得到最右边的连续 1 串的掩码</td><td>01011000 -> 00011000</td><td>(((x | (x - 1)) + 1) & x) ^ x</td><td></td></tr>
<tr><td>下舍入到 2k 倍数</td><td></td><td>x & ((-1) << k)</td><td></td></tr>
<tr><td>上舍入到 2k 倍数</td><td></td><td>t = (1 << k) - 1; x = (x + t) & (~t)</td><td></td></tr>
</table>

GCC 内建函数, 接受 unsigned int, 返回 int:

- `__builtin_ffs`
  二进制末尾 (最低位开始) 第一个 1 的 index (从 1 开始), x 是 0 时返回 0
- `__builtin_clz`
  二进制开头 (最高位开始) 连续的 0 的个数, x 不能是 0
- `__builtin_ctz`
  二进制末尾 (最低位开始) 连续的 0 的个数, x 不能是 0
- `__builtin_popcount`
  二进制 1 的个数
- `__builtin_parity`
  奇偶校验, `__builtin_popcount` % 2

8.2 Joseph

```
// 编号 1 to n
// 每次第 d 个人出局
// 返回剩下的人的编号
int joseph(int n, int d) {
    int ret = 1;
    for (int i = 2; i <= n; i++)
        ret = (ret + d - 1) % i + 1;
    return ret;
}

//Joseph 问题模拟
//传入 n,m, r 返回出环的序列
//时间复杂度 O(nlogn)
# include <cstring>

const int MAXN = 32768;

void josephus(int n, int m, int r[]) {
    int d[MAXN * 2], i, j, nbase, p, t;
    for (nbase = 1; nbase < n; nbase <<= 1);
    memset(d, 0, sizeof(d));
    for (i = 0; i < n; i++) {
        d[nbase + i] = 1;
    }
    for (i = nbase - 1; i; i--) {
        d[i] = d[i << 1] + d[i << 1 | 1];
    }
    for (j = i = 0; i < n; i++) {
        for (j = t = (j - 1+m) % (n - i), p = 1; p < nbase;) {
            d[p]--;
            if (t < d[p << 1]) {
                p = p << 1;
            } else {
                t -= d[p << 1];
                p = p << 1 | 1;
            }
        }
```

```
        r[i] = p - nbase;
        d[p]--;
    }
}
```

8.3 位操作

```
// 遍历一个掩码的所有子集掩码，不包括 0 和其自身
// 传入表示超集的掩码
void iterateSubset(int mask) {
    for(int sub = (mask - 1) & mask; sub > 0; sub = (sub - 1) & mask) {
        int incsub = ~sub & mask; // 递增顺序的子集
        // gogogo
    }
}

// 下一个包含同样数量二进制 1 的掩码
// 传入掩码，掩码不能为 0
unsigned snoob(unsigned mask) {
    unsigned smallest, ripple, ones;
    // x = xxx0 1111 0000
    smallest = mask & -mask;          //     0000 0001 0000
    ripple = mask + smallest;         //     xxx1 0000 0000
    ones = mask ^ ripple;             //     0001 1111 0000
    ones = (ones >> 2) / smallest;    //     0000 0000 0111
    return ripple | ones;             //     xxx1 0000 0111
}

// 遍历 {0, 1, ··· , n-1} 的所有 k 元子集
// 传入 n 和 k,k 不为 0
void iterateSubset(int n, int k) {
    int s = (1 << k) - 1;
    while (!(s & 1 << n)) {
        // gogogo
        s = snoob(s);
    }
}

// 求一个 32 位整数二进制 1 的位数
// 初始化函数先调用一次
int ones[256];

void initOnes() {
    for (int i = 1; i < 256; ++i)
        ones[i] = ones[i & (i - 1)] + 1;
}

int countOnes(int n) {
    return ones[n & 255] + ones[(n >> 8) & 255] + ones[(n >> 16) & 255] + ones[(n >> 24)\
 & 255];
```

```
}

// 求 32 位整数二进制 1 的位数，不需要初始化 (from Beautiful Code 2007 Ch. 10)
int countOnes(unsigned x) {
    x = x - ((x >> 1) & 0x55555555);
    x = (x & 0x33333333) + ((x >> 2) & 0x33333333);
    x = (x + (x >> 4)) & 0x0F0F0F0F;
    x = x + (x >> 8);
    x = x + (x >> 16);
    return x & 0x0000003F;
}


// 求一个 32 位整数二进制 1 的位数的奇偶性
// 偶数返回 0，奇数返回 1
int parityOnes(unsigned n) {
    n ^= n >> 1;
    n ^= n >> 2;
    n ^= n >> 4;
    n ^= n >> 8;
    n ^= n >> 16;
    return n & 1; // n 的第 i 位是原数第 i 位到最左侧位的奇偶性
}

// 对一个 32 位整数进行位反转
unsigned revBits(unsigned n) {
    n = (n & 0x55555555) << 1 | (n >> 1) & 0x55555555;
    n = (n & 0x33333333) << 2 | (n >> 2) & 0x33333333;
    n = (n & 0x0F0F0F0F) << 4 | (n >> 4) & 0x0F0F0F0F;
    n = (n << 24) | ((n & 0xFF00) << 8) | ((n >> 8) & 0xFF00) | (n >> 24);
    return n;
}

// 对 n 个数根据二进制掩码求部分和，类似的可以用于求 gcd，求二进制 1 的个数等等
void partSum(int n, int a[], int s[]) {
    s[0] = 0;
    for (int i = 0; i < n; i++) {
        s[1 << i] = a[i];
    }
    for (int i = 1; i < (1 << n); i++) {
        s[i] = s[i&(i-1)] + s[i^(i&(i-1))];
    }
}
```

8.4 布尔母函数

```
//布尔母函数
//判 m[] 个价值为 w[] 的货币能否构成 value
//适合 m[] 较大 w[] 较小的情况
//返回布尔量
//传入货币种数 n，个数 m[]，价值 w[] 和目标值 value
const int MAXV = 100000;
```

```

bool genfunc(int n, const int *m, const int *w, int value) {
    int i, j, k, c;
    char r[MAXV];
    for (r[0] = i = 1; i <= value; r[i++] = 0);
    for (i = 0; i < n; i++) {
        for (j = 0; j < w[i]; j++) {
            c = m[i] * r[k = j];
            while ((k += w[i]) <= value) {
                if (r[k]) {
                    c = m[i];
                } else if (c) {
                    r[k] = 1;
                    c--;
                }
            }
            if (r[value]) {
                return true;
            }
        }
    }
    return false;
}
```

8.5 最大子阵和

```
//求最大子阵和, 复杂度 O(n^3)
//传入阵的大小 m,n 和内容 mat[][]
//返回最大子阵和, 重载返回子阵位置 (maxsum=list[s1][s2]+...+list[e1][e2])
//可更改元素类型
const int MAXN = 100;

template <class elemType>
elemType maxsum(int m, int n, const elemType mat[][MAXN]) {
    elemType matsum[MAXN][MAXN + 1], ret, sum;
    int i, j, k;
    for (i = 0; i < m; i++) {
        for (matsum[i][j = 0] = 0; j < n; j++) {
            matsum[i][j + 1] = matsum[i][j] + mat[i][j];
        }
    }
    for (ret = mat[0][j = 0]; j < n; j++) {
        for (k = j; k < n; k++) {
            for (sum = 0, i = 0; i < m; i++) {
                sum = (sum > 0 ? sum : 0) + matsum[i][k + 1] - matsum[i][j], ret = (sum \
> ret ? sum : ret);
            }
        }
    }
    return ret;
}

template <class elemType>
elemType maxsum(int m, int n, const elemType mat[][MAXN], int &s1, int &s2, int &e1, int\
 &e2) {
```

```
    elemType matsum[MAXN][MAXN + 1], ret, sum;
    int i, j, k, s;
    for (i = 0; i < m; i++) {
        for (matsum[i][j = 0] = 0; j < n; j++) {
            matsum[i][j + 1] = matsum[i][j] + mat[i][j];
        }
    }
    for (ret = mat[s1 = e1 = 0][s2 = e2 = j = 0]; j < n; j++) {
        for (k = j; k < n; k++) {
            for (sum = 0, s = i = 0; i < m; i++, s = (sum > 0 ? s : i)) {
                if ((sum = (sum > 0 ? sum : 0) + matsum[i][k + 1] - matsum[i][j]) > ret)\
 {
                    ret = sum;
                    s1 = s;
                    s2 = i;
                    e1 = j;
                    e2 = k;
                }
            }
        }
    }
    return ret;
}
```

8.6 最长公共单调子序列

```
/**
 * 最长公共递增子序列, 时间复杂度 O(n^2 * logn), 空间 O(n^2)
 * n 为 a 的大小, m 为 b 的大小
 * 结果在 ans 中
 * "define _cp(a,b) ((a)<(b))" 求解最长严格递增序列
 */
const int MAXN = 1000;
# define _cp(a,b) ((a) < (b))

typedef int elemType;

elemType DP[MAXN][MAXN];
int num[MAXN], p[1 << 20];
int lis(int n, const elemType *a, int m, const elemType *b, elemType *ans) {
    int i, j, l, r, k;
    DP[0][0] = 0;
    num[0] = (b[0] == a[0]);
    for (i = 1; i < m; i++) {
        num[i] = (b[i] == a[0]) || num[i - 1];
        DP[i][0] = 0;
    }
    for (i = 1; i < n; i++) {
        if (b[0] == a[i] && !num[0]) {
            num[0] = 1;
            DP[0][0] = i << 10;
        }
        for (j = 1; j < m; j++) {
            for (k = ((l = 0) + (r = num[j - 1] - 1)) >> 1; l <= r; k = (l + r) >> 1) {
```

```
                if (_cp(a[DP[j - 1][k] >> 10], a[i])) {
                    l = k + 1;
                } else {
                    r = k - 1;
                }
            }
            if (l < num[j - 1] && i == (DP[j - 1][l] >> 10)) {
                if (l >= num[j]) {
                    DP[j][num[j]++] = DP[j - 1][l];
                } else {
                    DP[j][l] = _cp(a[DP[j][l] >> 10], a[i]) ? DP[j][l] : DP[j - 1][l];
                }
            }
            if (b[j] == a[i]) {
                for (k = ((l = 0) + (r = num[j] - 1)) >> 1; l <= r; k = (l + r) >> 1) {
                    if (_cp(a[DP[j][k] >> 10], a[i])) {
                        l = k + 1;
                    } else {
                        r = k - 1;
                    }
                }
                DP[j][l] = (i << 10) + j;
                num[j] += (l >= num[j]);
                p[DP[j][l]] = l ? DP[j][l - 1]:  - 1;
            }
        }
    }

    for (k = DP[m - 1][i = num[m - 1] - 1]; i >= 0; k = p[k]) {
        ans[i--] = a[k >> 10];
    }
    return num[m - 1];
}
```

8.7 最长子序列

```
//最长单调子序列, 复杂度 O(nlogn)
//注意最小序列覆盖和最长序列的对应关系, 例如
//"define _cp(a,b) ((a)>(b))" 求解最长严格递减序列, 则
//"define _cp(a,b) (!((a)>(b)))" 求解最小严格递减序列覆盖
//可更改元素类型和比较函数
const int MAXN = 10000;
# define _cp(a,b) ((a)>(b))

template <class elemType>
int subseq(int n, const elemType *a) {
    int b[MAXN + 1], i, l, r, m, ret = 0;
    for (i = 0; i < n; b[l] = i++, ret += (l> ret)) {
        for (m = ((l = 1) + (r = ret)) >> 1; l <= r; m = (l + r) >> 1) {
            if (_cp(a[b[m]], a[i])) {
                l = m + 1;
            } else {
                r = m - 1;
            }
```

```
        }
    }
    return ret;
}

template <class elemType>
int subseq(int n, const elemType *a, elemType *ans) {
    int b[MAXN + 1], p[MAXN], i, l, r, m, ret = 0;
    for (i = 0; i < n; p[b[l] = i++] = b[l - 1], ret += (l> ret)) {
        for (m = ((l = 1) + (r = ret)) >> 1; l <= r; m = (l + r) >> 1) {
            if (_cp(a[b[m]], a[i])) {
                l = m + 1;
            } else {
                r = m - 1;
            }
        }
    }
    for (m = b[i = ret]; i; ans[--i] = a[m], m = p[m]);
    return ret;
}
```

8.8 行列式求模

```
// @author Navi
// 高斯消元法行列式求模。复杂度 O(n^3logn)。
// n 为行列式大小, 计算 |mat| % m
const int MAXN = 200;
typedef long long LL;
int detMod(int n, int m, int mat[][MAXN]) {
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            mat[i][j] %= m;
    LL ret = 1;
    for (int i = 0; i < n; i++) {
        for (int j = i + 1; j < n; j++)
            while (mat[j][i] != 0) {
                LL t = mat[i][i] / mat[j][i];
                for (int k = i; k < n; k++) {
                    mat[i][k] = (mat[i][k] - mat[j][k] * t) % m;
                    int s = mat[i][k];
                    mat[i][k] = mat[j][k];
                    mat[j][k] = s;
                }
                ret = -ret;
            }
        if (mat[i][i] == 0)
            return 0;
        ret = ret * mat[i][i] % m;
    }
    if (ret < 0)
        ret += m;
    return (int)ret;
}

// @author Navi
```

```
// 高斯消元法行列式求模。复杂度 O(n^3 + n^2logn)。
// n 为行列式大小，计算 |mat| % m
// 速度只比 O(n^3logn) 的快一些，推荐用另外那个。
const int MAXN = 200;
typedef long long LL;
int detMod(int n, int m, int mat[][MAXN]) {
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            mat[i][j] %= m;
    LL ret = 1;
    for (int i = 0; i < n; i++) {
        for (int j = i + 1; j < n; j++) {
            LL x1 = 1, y1 = 0, x2 = 0, y2 = 1, p = i, q = j;
            if (mat[i][i] < 0) {
                x1 = -1;
                mat[i][i] = -mat[i][i];
                ret = -ret;
            }
            if (mat[j][i] < 0) {
                y2 = -1;
                mat[j][i] = -mat[j][i];
                ret = -ret;
            }
            while (mat[i][i] != 0 && mat[j][i] != 0) {
                if (mat[i][i] <= mat[j][i]) {
                    int t = mat[j][i] / mat[i][i];
                    mat[j][i] -= mat[i][i] * t;
                    x2 -= x1 * t;
                    y2 -= y1 * t;
                } else {
                    int t = mat[i][i] / mat[j][i];
                    mat[i][i] -= mat[j][i] * t;
                    x1 -= x2 * t;
                    y1 -= y2 * t;
                }
            }
            x1 %= m;
            y1 %= m;
            x2 %= m;
            y2 %= m;
            if (mat[i][i] == 0 && mat[j][i] != 0) {
                ret = -ret;
                p = j;
                q = i;
                mat[i][i] = mat[j][i];
                mat[j][i] = 0;
            }
            for (int k = i + 1; k < n; k++) {
                int s = mat[i][k], t = mat[j][k];
                mat[p][k] = (s * x1 + t * y1) % m;
                mat[q][k] = (s * x2 + t * y2) % m;
            }
        }
        if (mat[i][i] == 0)
            return 0;
        ret = ret * mat[i][i] % m;
    }
```

```
    if (ret < 0)
        ret += m;
    return (int)ret;
}
```

8.9 逆序对数

```
//序列逆序对数，复杂度 O(nlogn)
//传入序列长度和内容，返回逆序对数
//可更改元素类型和比较函数
#include <cstring>
const int MAXN = 1000000;
#define _cp(a,b) ((a) <= (b))

typedef int elemType;
elemType tmp[MAXN];

long long inv(int n, elemType *a) {
    int left = n >> 1, r = n - left, i, j;
    long long ret = (r > 1 ? (inv(left, a) + inv(r, a + left)) : 0);
    for (i = j = 0; i <= left; tmp[i + j] = a[i], i++) {
        for (ret += j; j < r && (i == left || !_cp(a[i], a[left + j])); tmp[i + j] = a[l\
eft + j], j++);
    }
    memcpy(a, tmp, sizeof(elemType) * n);
    return ret;
}
```

Chapter 9

其他

9.1 分数

```cpp
# include <cmath>

struct Frac {
    int num, den;
};

int gcd(int a, int b) {
    int t;
    if (a < 0) {
        a = -a;
    }
    if (b < 0) {
        b = -b;
    }
    if (!b) {
        return a;
    }
    while (t = a % b) {
        a = b;
        b = t;
    }
    return b;
}

void simplify(Frac &f) {
    int t;
    if (t = gcd(f.num, f.den)) {
        f.num /= t;
        f.den /= t;
    } else {
        f.den = 1;
    }
}

Frac f(int n, int d, int s = 1) {
    Frac ret;
    if (d < 0) {
        ret.num = -n;
        ret.den = -d;
    } else {
```

```
        ret.num = n;
        ret.den = d;
    }
    if (s) {
        simplify(ret);
    }
    return ret;
}

Frac convert(double x) {
    Frac ret;
    for (ret.den = 1; fabs(x - int(x)) > 1e-10; ret.den *= 10, x *= 10);
    ret.num = (int)x;
    simplify(ret);
    return ret;
}

int fraqcmp(Frac a, Frac b) {
    int g1 = gcd(a.den, b.den), g2 = gcd(a.num, b.num);
    if (!g1 || !g2) {
        return 0;
    }
    return b.den / g1 *(a.num / g2) - a.den / g1 *(b.num / g2);
}

Frac add(Frac a, Frac b) {
    int g1 = gcd(a.den, b.den), g2, t;
    if (!g1) {
        return f(1, 0, 0);
    }
    t = b.den / g1 * a.num + a.den / g1 * b.num;
    g2 = gcd(g1, t);
    return f(t / g2, a.den / g1 *(b.den / g2), 0);
}

Frac sub(Frac a, Frac b) {
    return add(a, f(-b.num, b.den, 0));
}

Frac mul(Frac a, Frac b) {
    int t1 = gcd(a.den, b.num), t2 = gcd(a.num, b.den);
    if (!t1 || !t2) {
        return f(1, 1, 0);
    }
    return f(a.num / t2 *(b.num / t1), a.den / t1 *(b.den / t2), 0);
}

Frac div(Frac a, Frac b) {
    return mul(a, f(b.den, b.num, 0));
}
```

9.2 日期

```
int days[12] = {31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31};

class Date {
```

```cpp
public:
    //判闰年
    inline static bool isLeap(int year) {
        return (year % 4 == 0 && year % 100 != 0) || year % 400 == 0;
    }

    int year, month, day;

    //判合法性
    inline bool isLegal() const {
        if (month <= 0 || month > 12) {
            return false;
        }
        if (month == 2) {
            return day > 0 && day <= 28 + isLeap(year);
        }
        return day > 0 && day <= days[month - 1];
    }

    //比较日期大小
    inline int compareTo(const Date &other) const {
        if (year != other.year) {
            return year - other.year;
        }
        if (month != other.month) {
            return month - other.month;
        }
        return day - other.day;
    }

    //返回指定日期是星期几  0 (Sunday) ... 6 (Saturday)
    inline int toWeekday() const {
        int tm = month >= 3 ? (month - 2) : (month + 10);
        int ty = month >= 3 ? year : (year - 1);
        return (ty + ty / 4 - ty / 100 + ty / 400 + (int)(2.6 * tm - 0.2) + day) % 7;
    }

    //日期转天数偏移
    inline int toInt() const {
        int ret = year * 365 + (year - 1) / 4 - (year - 1) / 100 + (year - 1) / 400;
        days[1] += isLeap(year);
        for (int i = 0; i < month - 1; ret += days[i++]);
        days[1] = 28;
        return ret + day;
    }

    //天数偏移转日期
    inline void fromInt(int a) {
        year = a / 146097 * 400;
        for (a %= 146097; a >= 365 + isLeap(year); a -= 365 + isLeap(year), year++);
        days[1] += isLeap(year);
        for (month = 1; a >= days[month - 1]; a -= days[month - 1], month++);
        days[1] = 28;
        day = a + 1;
    }
```

```
};
```

9.3 矩阵

```
# include <cmath>
const int MAXN = 100;

# define zero(x) (fabs(x) < 1e-10)

struct mat {
    int n, m;
    double data[MAXN][MAXN];
};

bool mul(mat &c, const mat &a, const mat &b) {
    int i, j, k;
    if (a.m != b.n) {
        return false;
    }
    c.n = a.n;
    c.m = b.m;
    for (i = 0; i < c.n; i++) {
        for (j = 0; j < c.m; j++) {
            for (c.data[i][j] = k = 0; k < a.m; k++) {
                c.data[i][j] += a.data[i][k] * b.data[k][j];
            }
        }
    }
    return true;
}

bool inv(mat &a) {
    int i, j, k, is[MAXN], js[MAXN];
    double t;
    if (a.n != a.m) {
        return false;
    }
    for (k = 0; k < a.n; k++) {
        for (t = 0, i = k; i < a.n; i++) {
            for (j = k; j < a.n; j++) {
                if (fabs(a.data[i][j]) > t) {
                    t = fabs(a.data[is[k] = i][js[k] = j]);
                }
            }
        }
        if (zero(t)) {
            return false;
        }
        if (is[k] != k) {
            for (j = 0; j < a.n; j++) {
                t = a.data[k][j];
                a.data[k][j] = a.data[is[k]][j];
                a.data[is[k]][j] = t;
            }
        }
        if (js[k] != k) {
```

```
            for (i = 0; i < a.n; i++) {
                t = a.data[i][k];
                a.data[i][k] = a.data[i][js[k]];
                a.data[i][js[k]] = t;
            }
        }
        a.data[k][k] = 1 / a.data[k][k];
        for (j = 0; j < a.n; j++) {
            if (j != k) {
                a.data[k][j] *= a.data[k][k];
            }
        }
        for (i = 0; i < a.n; i++) {
            if (i != k) {
                for (j = 0; j < a.n; j++) {
                    if (j != k) {
                        a.data[i][j] -= a.data[i][k] * a.data[k][j];
                    }
                }
            }
        }
        for (i = 0; i < a.n; i++) {
            if (i != k) {
                a.data[i][k] *= -a.data[k][k];
            }
        }
    }
    for (k = a.n - 1; k >= 0; k--) {
        for (j = 0; j < a.n; j++) {
            if (js[k] != k) {
                t = a.data[k][j];
                a.data[k][j] = a.data[js[k]][j];
                a.data[js[k]][j] = t;
            }
        }
        for (i = 0; i < a.n; i++) {
            if (is[k] != k) {
                t = a.data[i][k];
                a.data[i][k] = a.data[i][is[k]];
                a.data[i][is[k]] = t;
            }
        }
    }
    return true;
}

double det(const mat &a) {
    int i, j, k, sign = 0;
    double b[MAXN][MAXN], ret = 1, t;
    if (a.n != a.m) {
        return 0;
    }
    for (i = 0; i < a.n; i++) {
        for (j = 0; j < a.m; j++) {
            b[i][j] = a.data[i][j];
        }
    }
    for (i = 0; i < a.n; i++) {
```

```
        if (zero(b[i][i])) {
            for (j = i + 1; j < a.n; j++) {
                if (!zero(b[j][i])) {
                    break;
                }
            }
            if (j == a.n) {
                return 0;
            }
            for (k = i; k < a.n; k++) {
                t = b[i][k], b[i][k] = b[j][k], b[j][k] = t;
            }
            sign++;
        }
        ret *= b[i][i];
        for (k = i + 1; k < a.n; k++) {
            b[i][k] /= b[i][i];
        }
        for (j = i + 1; j < a.n; j++) {
            for (k = i + 1; k < a.n; k++) {
                b[j][k] -= b[j][i] * b[i][k];
            }
        }
    }
    if (sign & 1) {
        ret = -ret;
    }
    return ret;
}
```

Chapter 10

附录

10.1 算法描述

10.1.1 弦图与区间图

```
// 弦图与区间图 By 猛犸也钻地 @ 2012.09.13

/* 相关定义 //
1. 子图：原图点集的子集 + 原图边集的子集
2. 诱导子图：原图点集的子集 + 这些点在原图中所连出的边集
3. 团：原图的一个子图，且是完全图
4. 极大团：此团不是其他团的子集
5. 最大团：点数最多的团 -> 团数
6. 最小染色：用最少的颜色给点染色使相邻点颜色不同 -> 色数
7. 最大独立集：原图点集的子集，任意两点在原图中没有边相连
8. 最小团覆盖：用最少个数的团覆盖所有的点
   推论 -> 团数<= 色数，最大独立集数<= 最小团覆盖数
9. 弦：连接环中不相邻的两个点的边
10. 弦图：图中任意长度大于 3 的环都至少有 1 个弦
   推论 -> 弦图的每一个诱导子图一定是弦图
           弦图的任一个诱导子图不同构于 Cn(n>3)
11. 单纯点：记 N(v) 为点 v 相邻点的集合，若 N(v)+{v} 是一个团，则 v 为单纯点
   引理 -> 任何一个弦图都至少有一个单纯点
           不是完全图的弦图至少有两个不相邻的单纯
// 重点内容 //
12. 完美消除序列：点的序列 v1,v2,..,vn，满足 vi 在 {vi,vi+1,..,vn} 中是单纯点
   定理 -> 一个无向图是弦图，当且仅当它有一个完美消除序列
   构造算法 -> 令 cnt[i] 为第 i 个点与多少个已标记的点相邻，初值全为零
               每次选择一个 cnt[i] 最大的结点并打上标记
               标记顺序的逆序则为完美消除序列
   判定算法 -> 对于每个 vi，其出边为 vi1,vi2,..,vik
               然后判断 vi1 与 vi2,vi3,..,vik 是否都相邻
               若存在不相邻的情况，则说明不是完美消除序列
13. 弦图各类算法：
   最小染色算法 -> 按照完美消除序列，从后向前，依次染上可以染的最小颜色
   最大独立集算法 -> 按照完美消除序列，从前向后，能选则选
   最小团覆盖算法 -> 求出最大独立集，记为 {p1,p2,..,pk}
                     则 {N(p1)+{p1},N(p2)+{p2},..,N(pk)+{pk}} 即为解
16. 区间图：坐标轴上的一些区间看作点，任意两个交集非空的区间之间有边
   定理 -> 区间图一定是弦图 */
```

10.1.2 生成树

```
// 生成树相关的一些问题 By 猛犸也钻地 @ 2012.02.24

/* 度限制生成树 //
Q: 求一个最小生成树，其中 V0 连接的边不能超过 K 个或只能刚好 K 个
1. 去掉所有和 V0 连接的边，对每个连通分量求最小生成树
2. 如果除去点 V0 外共有 T 个连通分量，且 T>K，无解
3. 于是现在有一个最小 T 度生成树，然后用 dp[V] 计算出该点到 V0 的路径上
   权值最大的边是多少，再枚举和 V0 连接的没有使用过的边，找出一条边
   使得用那条边替换已有的边，增加的权值最小，不停替换直到 V0 出度为 K */

/* 次小生成树 //
Q: 求一个次小生成树，要求权值之和必须大于等于或严格大于其最小生成树
1. 求最小生成树
2. 找一个根然后 dp，求出每个点往上走 2^L 能到达的祖先是谁，以及
   这段路径上的最大边和次大边（如果仅要求大于等于的话就不需要次大边）
3. 枚举没有使用过的边，利用上面得到的信息，在 O(logN) 时间内对每条边
   计算出其能够替换的已有的最大和次大边，然后找出最佳替换方式 */

/* 斯坦纳树 //
Q: 求一个包含指定的 K 个特殊点的最小生成树，其他点不一定在树中
1. 用 dp[mask][x] 记录树根在点 x，mask 所对应的特殊点集在树中的最小权值之和
2. 将 dp[][] 初始化为正无穷，只有 dp[1<<i][Ai] 被初始化为 0，Ai 为第 i 个特殊点
3. 先求出所有点对间最短路，然后枚举 mask，依次做两种转移：
3.1. 枚举 x 和 mask 的子集 sub，合并两棵子树
     dp[mask][x]=min(dp[mask][x],dp[sub][x]+dp[mask^sub][x])
3.2. 枚举 x 和 y，计算结点从 y 移动到 x 的花费
     dp[mask][x]=min(dp[mask][x],dp[mask][y]+minDistance(y,x))
     在上面的转移中，也可以把这些点同时放到队列里，用 spfa 更新最短路 */

/* 生成树计数 //
Q: 给定一个无权的无向图 G，求生成树的个数
1. 令矩阵 D[][] 为度数矩阵，其中 D[i][i] 为结点 i 的度，其他位置的值为 0
2. 令矩阵 A[][] 为邻接矩阵，当结点 i 和 j 之间有 x 条边时，D[i][j]=D[j][i]=x
3. 令矩阵 C=D-A，矩阵 C' 为矩阵 C 抽去第 k 行和第 k 列后的一个 n-1 阶的子矩阵
   其中 k 可以任意设定，构造完 C' 后，生成树的个数即为 C' 行列式的值 */
```

10.2 应用

10.2.1 N 皇后构造解

```
//N 皇后构造解,n>=4

void even1(int n, int *p) {
    int i;
    for (i = 1; i <= n / 2; i++) {
        p[i - 1] = 2 * i;
    }
    for (i = n / 2 + 1; i <= n; i++) {
        p[i - 1] = 2 * i - n - 1;
    }
}

void even2(int n, int *p) {
    int i;
```

```
    for (i = 1; i <= n / 2; i++) {
        p[i - 1] = (2 * i + n / 2 - 3) % n + 1;
    }
    for (i = n / 2 + 1; i <= n; i++) {
        p[i - 1] = n - (2 *(n - i + 1) + n / 2 - 3) % n;
    }
}

void generate(int, int *);

void odd(int n, int *p) {
    generate(n - 1, p);
    p[n - 1] = n;
}

void generate(int n, int *p) {
    if (n & 1) {
        odd(n, p);
    } else if (n % 6 != 2) {
        even1(n, p);
    } else {
        even2(n, p);
    }
}
```

10.2.2 大数 (整数类封装)

```
// 不推荐使用, 最好用 Java

# include <iostream>
# include <cstring>
using namespace std;

# define DIGIT   4
# define DEPTH   10000
# define MAX     100
typedef int bignum_t[MAX+1];

int read(bignum_t a,istream &is=cin) {
    char buf[MAX*DIGIT+1],ch;
    int i,j;
    memset((void *)a,0,sizeof(bignum_t));
    if (!(is>>buf)) return 0;
    for (a[0]=strlen(buf),i=a[0]/2-1; i>=0; i--)
        ch=buf[i],buf[i]=buf[a[0]-1-i],buf[a[0]-1-i]=ch;
    for (a[0]=(a[0]+DIGIT-1)/DIGIT,j=strlen(buf); j<a[0]*DIGIT; buf[j++]='0');
    for (i=1; i<=a[0]; i++)
        for (a[i]=0,j=0; j<DIGIT; j++)
            a[i]=a[i]*10+buf[i*DIGIT-1-j]-'0';
    for (; !a[a[0]]&&a[0]>1; a[0]--);
    return 1;
}

void write(const bignum_t a,ostream &os=cout) {
    int i,j;
```

```
    for (os<<a[i=a[0]],i--; i; i--)
        for (j=DEPTH/10; j; j/=10)
            os<<a[i]/j%10;
}

int comp(const bignum_t a,const bignum_t b) {
    int i;
    if (a[0]!=b[0])
        return a[0]-b[0];
    for (i=a[0]; i; i--)
        if (a[i]!=b[i])
            return a[i]-b[i];
    return 0;
}

int comp(const bignum_t a,const int b) {
    int c[12]= {1};
    for (c[1]=b; c[c[0]]>=DEPTH; c[c[0]+1]=c[c[0]]/DEPTH,c[c[0]]%=DEPTH,c[0]++);
    return comp(a,c);
}

int comp(const bignum_t a,const int c,const int d,const bignum_t b) {
    int i,t=0,O=-DEPTH*2;
    if (b[0]-a[0]<d&&c)
        return 1;
    for (i=b[0]; i>d; i--) {
        t=t*DEPTH+a[i-d]*c-b[i];
        if (t>0) return 1;
        if (t<O) return 0;
    }
    for (i=d; i; i--) {
        t=t*DEPTH-b[i];
        if (t>0) return 1;
        if (t<O) return 0;
    }
    return t>0;
}

void add(bignum_t a,const bignum_t b) {
    int i;
    for (i=1; i<=b[0]; i++)
        if ((a[i]+=b[i])>=DEPTH)
            a[i]-=DEPTH,a[i+1]++;
    if (b[0]>=a[0])
        a[0]=b[0];
    else
        for (; a[i]>=DEPTH&&i<a[0]; a[i]-=DEPTH,i++,a[i]++);
    a[0]+=(a[a[0]+1]>0);
}

void add(bignum_t a,const int b) {
    int i=1;
    for (a[1]+=b; a[i]>=DEPTH&&i<a[0]; a[i+1]+=a[i]/DEPTH,a[i]%=DEPTH,i++);
    for (; a[a[0]]>=DEPTH; a[a[0]+1]=a[a[0]]/DEPTH,a[a[0]]%=DEPTH,a[0]++);
}

void sub(bignum_t a,const bignum_t b) {
    int i;
```

```
    for (i=1; i<=b[0]; i++)
        if ((a[i]-=b[i])<0)
            a[i+1]--,a[i]+=DEPTH;
    for (; a[i]<0; a[i]+=DEPTH,i++,a[i]--);
    for (; !a[a[0]]&&a[0]>1; a[0]--);
}

void sub(bignum_t a,const int b) {
    int i=1;
    for (a[1]-=b; a[i]<0; a[i+1]+=(a[i]-DEPTH+1)/DEPTH,a[i]-=(a[i]-DEPTH+1)/DEPTH*DEPTH,\
i++);
    for (; !a[a[0]]&&a[0]>1; a[0]--);
}

void sub(bignum_t a,const bignum_t b,const int c,const int d) {
    int i,O=b[0]+d;
    for (i=1+d; i<=O; i++)
        if ((a[i]-=b[i-d]*c)<0)
            a[i+1]+=(a[i]-DEPTH+1)/DEPTH,a[i]-=(a[i]-DEPTH+1)/DEPTH*DEPTH;
    for (; a[i]<0; a[i+1]+=(a[i]-DEPTH+1)/DEPTH,a[i]-=(a[i]-DEPTH+1)/DEPTH*DEPTH,i++);
    for (; !a[a[0]]&&a[0]>1; a[0]--);
}

void mul(bignum_t c,const bignum_t a,const bignum_t b) {
    int i,j;
    memset((void *)c,0,sizeof(bignum_t));
    for (c[0]=a[0]+b[0]-1,i=1; i<=a[0]; i++)
        for (j=1; j<=b[0]; j++)
            if ((c[i+j-1]+=a[i]*b[j])>=DEPTH)
                c[i+j]+=c[i+j-1]/DEPTH,c[i+j-1]%=DEPTH;
    for (c[0]+=(c[c[0]+1]>0); !c[c[0]]&&c[0]>1; c[0]--);
}

void mul(bignum_t a,const int b) {
    int i;
    for (a[1]*=b,i=2; i<=a[0]; i++) {
        a[i]*=b;
        if (a[i-1]>=DEPTH)
            a[i]+=a[i-1]/DEPTH,a[i-1]%=DEPTH;
    }
    for (; a[a[0]]>=DEPTH; a[a[0]+1]=a[a[0]]/DEPTH,a[a[0]]%=DEPTH,a[0]++);
    for (; !a[a[0]]&&a[0]>1; a[0]--);
}

void mul(bignum_t b,const bignum_t a,const int c,const int d) {
    int i;
    memset((void *)b,0,sizeof(bignum_t));
    for (b[0]=a[0]+d,i=d+1; i<=b[0]; i++)
        if ((b[i]+=a[i-d]*c)>=DEPTH)
            b[i+1]+=b[i]/DEPTH,b[i]%=DEPTH;
    for (; b[b[0]+1]; b[0]++,b[b[0]+1]=b[b[0]]/DEPTH,b[b[0]]%=DEPTH);
    for (; !b[b[0]]&&b[0]>1; b[0]--);
}

void div(bignum_t c,bignum_t a,const bignum_t b) {
    int h,l,m,i;
    memset((void *)c,0,sizeof(bignum_t));
    c[0]=(b[0]<a[0]+1)?(a[0]-b[0]+2):1;
```

```
    for (i=c[0]; i; sub(a,b,c[i]=m,i-1),i--)
        for (h=DEPTH-1,l=0,m=(h+l+1)>>1; h>l; m=(h+l+1)>>1)
            if (comp(b,m,i-1,a)) h=m-1;
            else l=m;
    for (; !c[c[0]]&&c[0]>1; c[0]--);
    c[0]=c[0]>1?c[0]:1;
}

void div(bignum_t a,const int b,int &c) {
    int i;
    for (c=0,i=a[0]; i; c=c*DEPTH+a[i],a[i]=c/b,c%=b,i--);
    for (; !a[a[0]]&&a[0]>1; a[0]--);
}

void sqrt(bignum_t b,bignum_t a) {
    int h,l,m,i;
    memset((void *)b,0,sizeof(bignum_t));
    for (i=b[0]=(a[0]+1)>>1; i; sub(a,b,m,i-1),b[i]+=m,i--)
        for (h=DEPTH-1,l=0,b[i]=m=(h+l+1)>>1; h>l; b[i]=m=(h+l+1)>>1)
            if (comp(b,m,i-1,a)) h=m-1;
            else l=m;
    for (; !b[b[0]]&&b[0]>1; b[0]--);
    for (i=1; i<=b[0]; b[i++]>>=1);
}

int length(const bignum_t a) {
    int t,ret;
    for (ret=(a[0]-1)*DIGIT,t=a[a[0]]; t; t/=10,ret++);
    return ret>0?ret:1;
}

int digit(const bignum_t a,const int b) {
    int i,ret;
    for (ret=a[(b-1)/DIGIT+1],i=(b-1)%DIGIT; i; ret/=10,i--);
    return ret%10;
}

int zeronum(const bignum_t a) {
    int ret,t;
    for (ret=0; !a[ret+1]; ret++);
    for (t=a[ret+1],ret*=DIGIT; !(t%10); t/=10,ret++);
    return ret;
}

void comp(int *a,const int l,const int h,const int d) {
    int i,j,t;
    for (i=l; i<=h; i++)
        for (t=i,j=2; t>1; j++)
            while (!(t%j))
                a[j]+=d,t/=j;
}

void convert(int *a,const int h,bignum_t b) {
    int i,j,t=1;
    memset(b,0,sizeof(bignum_t));
    for (b[0]=b[1]=1,i=2; i<=h; i++)
        if (a[i])
            for (j=a[i]; j; t*=i,j--)
```

```
                if (t*i>DEPTH)
                    mul(b,t),t=1;
    mul(b,t);
}

void combination(bignum_t a,int m,int n) {
    int *t=new int[m+1];
    memset((void *)t,0,sizeof(int)*(m+1));
    comp(t,n+1,m,1);
    comp(t,2,m-n,-1);
    convert(t,m,a);
    delete []t;
}

void permutation(bignum_t a,int m,int n) {
    int i,t=1;
    memset(a,0,sizeof(bignum_t));
    a[0]=a[1]=1;
    for (i=m-n+1; i<=m; t*=i++)
        if (t*i>DEPTH)
            mul(a,t),t=1;
    mul(a,t);
}

# define SGN(x) ((x)>0?1:((x)<0?-1:0))
# define ABS(x) ((x)>0?(x):-(x))

int read(bignum_t a,int &sgn,istream &is=cin) {
    char str[MAX*DIGIT+2],ch,*buf;
    int i,j;
    memset((void *)a,0,sizeof(bignum_t));
    if (!(is>>str)) return 0;
    buf=str,sgn=1;
    if (*buf=='-') sgn=-1,buf++;
    for (a[0]=strlen(buf),i=a[0]/2-1; i>=0; i--)
        ch=buf[i],buf[i]=buf[a[0]-1-i],buf[a[0]-1-i]=ch;
    for (a[0]=(a[0]+DIGIT-1)/DIGIT,j=strlen(buf); j<a[0]*DIGIT; buf[j++]='0');
    for (i=1; i<=a[0]; i++)
        for (a[i]=0,j=0; j<DIGIT; j++)
            a[i]=a[i]*10+buf[i*DIGIT-1-j]-'0';
    for (; !a[a[0]]&&a[0]>1; a[0]--);
    if (a[0]==1&&!a[1]) sgn=0;
    return 1;
}

struct bignum {
    bignum_t num;
    int sgn;
public:
    inline bignum() {
        memset(num,0,sizeof(bignum_t));
        num[0]=1;
        sgn=0;
    }
    inline int operator!() {
        return num[0]==1&&!num[1];
    }
```

```
    inline bignum &operator=(const bignum &a) {
        memcpy(num,a.num,sizeof(bignum_t));
        sgn=a.sgn;
        return *this;
    }
    inline bignum &operator=(const int a) {
        memset(num,0,sizeof(bignum_t));
        num[0]=1;
        sgn=SGN(a);
        add(num,sgn*a);
        return *this;
    };
    inline bignum &operator+=(const bignum &a) {
        if(sgn==a.sgn)add(num,a.num);
        else if(sgn&&a.sgn) {
            int ret=comp(num,a.num);
            if(ret>0)sub(num,a.num);
            else if(ret<0) {
                bignum_t t;
                memcpy(t,num,sizeof(bignum_t));
                memcpy(num,a.num,sizeof(bignum_t));
                sub(num,t);
                sgn=a.sgn;
            } else memset(num,0,sizeof(bignum_t)),num[0]=1,sgn=0;
        } else if(!sgn)memcpy(num,a.num,sizeof(bignum_t)),sgn=a.sgn;
        return *this;
    }
    inline bignum &operator+=(const int a) {
        if(sgn*a>0)add(num,ABS(a));
        else if(sgn&&a) {
            int ret=comp(num,ABS(a));
            if(ret>0)sub(num,ABS(a));
            else if(ret<0) {
                bignum_t t;
                memcpy(t,num,sizeof(bignum_t));
                memset(num,0,sizeof(bignum_t));
                num[0]=1;
                add(num,ABS(a));
                sgn=-sgn;
                sub(num,t);
            } else memset(num,0,sizeof(bignum_t)),num[0]=1,sgn=0;
        } else if(!sgn)sgn=SGN(a),add(num,ABS(a));
        return *this;
    }
    inline bignum operator+(const bignum &a) {
        bignum ret;
        memcpy(ret.num,num,sizeof(bignum_t));
        ret.sgn=sgn;
        ret+=a;
        return ret;
    }
    inline bignum operator+(const int a) {
        bignum ret;
        memcpy(ret.num,num,sizeof(bignum_t));
        ret.sgn=sgn;
        ret+=a;
        return ret;
    }
```

```
    inline bignum &operator-=(const bignum &a) {
        if(sgn*a.sgn<0)add(num,a.num);
        else if(sgn&&a.sgn) {
            int ret=comp(num,a.num);
            if(ret>0)sub(num,a.num);
            else if(ret<0) {
                bignum_t t;
                memcpy(t,num,sizeof(bignum_t));
                memcpy(num,a.num,sizeof(bignum_t));
                sub(num,t);
                sgn=-sgn;
            } else memset(num,0,sizeof(bignum_t)),num[0]=1,sgn=0;
        } else if(!sgn)add(num,a.num),sgn=-a.sgn;
        return *this;
    }
    inline bignum &operator-=(const int a) {
        if(sgn*a<0)add(num,ABS(a));
        else if(sgn&&a) {
            int ret=comp(num,ABS(a));
            if(ret>0)sub(num,ABS(a));
            else if(ret<0) {
                bignum_t t;
                memcpy(t,num,sizeof(bignum_t));
                memset(num,0,sizeof(bignum_t));
                num[0]=1;
                add(num,ABS(a));
                sub(num,t);
                sgn=-sgn;
            } else memset(num,0,sizeof(bignum_t)),num[0]=1,sgn=0;
        } else if(!sgn)sgn=-SGN(a),add(num,ABS(a));
        return *this;
    }
    inline bignum operator-(const bignum &a) {
        bignum ret;
        memcpy(ret.num,num,sizeof(bignum_t));
        ret.sgn=sgn;
        ret-=a;
        return ret;
    }
    inline bignum operator-(const int a) {
        bignum ret;
        memcpy(ret.num,num,sizeof(bignum_t));
        ret.sgn=sgn;
        ret-=a;
        return ret;
    }
    inline bignum &operator*=(const bignum &a) {
        bignum_t t;
        mul(t,num,a.num);
        memcpy(num,t,sizeof(bignum_t));
        sgn*=a.sgn;
        return *this;
    }
    inline bignum &operator*=(const int a) {
        mul(num,ABS(a));
        sgn*=SGN(a);
        return *this;
    }
```

```
    inline bignum operator*(const bignum &a) {
        bignum ret;
        mul(ret.num,num,a.num);
        ret.sgn=sgn*a.sgn;
        return ret;
    }
    inline bignum operator*(const int a) {
        bignum ret;
        memcpy(ret.num,num,sizeof(bignum_t));
        mul(ret.num,ABS(a));
        ret.sgn=sgn*SGN(a);
        return ret;
    }
    inline bignum &operator/=(const bignum &a) {
        bignum_t t;
        div(t,num,a.num);
        memcpy(num,t,sizeof(bignum_t));
        sgn=(num[0]==1&&!num[1])?0:sgn*a.sgn;
        return *this;
    }
    inline bignum &operator/=(const int a) {
        int t;
        div(num,ABS(a),t);
        sgn=(num[0]==1&&!num[1])?0:sgn*SGN(a);
        return *this;
    }
    inline bignum operator/(const bignum &a) {
        bignum ret;
        bignum_t t;
        memcpy(t,num,sizeof(bignum_t));
        div(ret.num,t,a.num);
        ret.sgn=(ret.num[0]==1&&!ret.num[1])?0:sgn*a.sgn;
        return ret;
    }
    inline bignum operator/(const int a) {
        bignum ret;
        int t;
        memcpy(ret.num,num,sizeof(bignum_t));
        div(ret.num,ABS(a),t);
        ret.sgn=(ret.num[0]==1&&!ret.num[1])?0:sgn*SGN(a);
        return ret;
    }
    inline bignum &operator%=(const bignum &a) {
        bignum_t t;
        div(t,num,a.num);
        if (num[0]==1&&!num[1])sgn=0;
        return *this;
    }
    inline int operator%=(const int a) {
        int t;
        div(num,ABS(a),t);
        memset(num,0,sizeof(bignum_t));
        num[0]=1;
        add(num,t);
        return t;
    }
    inline bignum operator%(const bignum &a) {
        bignum ret;
```

```
        bignum_t t;
        memcpy(ret.num,num,sizeof(bignum_t));
        div(t,ret.num,a.num);
        ret.sgn=(ret.num[0]==1&&!ret.num[1])?0:sgn;
        return ret;
    }
    inline int operator%(const int a) {
        bignum ret;
        int t;
        memcpy(ret.num,num,sizeof(bignum_t));
        div(ret.num,ABS(a),t);
        memset(ret.num,0,sizeof(bignum_t));
        ret.num[0]=1;
        add(ret.num,t);
        return t;
    }
    inline bignum &operator++() {
        *this+=1;
        return *this;
    }
    inline bignum &operator--() {
        *this-=1;
        return *this;
    };
    inline int operator>(const bignum &a) {
        return sgn>0?(a.sgn>0?comp(num,a.num)>0:1):(sgn<0?(a.sgn<0?comp(num,a.num)<0:0):\
a.sgn<0);
    }
    inline int operator>(const int a) {
        return sgn>0?(a>0?comp(num,a)>0:1):(sgn<0?(a<0?comp(num,-a)<0:0):a<0);
    }
    inline int operator>=(const bignum &a) {
        return sgn>0?(a.sgn>0?comp(num,a.num)>=0:1):(sgn<0?(a.sgn<0?comp(num,a.num)<=0:0\
):a.sgn<=0);
    }
    inline int operator>=(const int a) {
        return sgn>0?(a>0?comp(num,a)>=0:1):(sgn<0?(a<0?comp(num,-a)<=0:0):a<=0);
    }
    inline int operator<(const bignum &a) {
        return sgn<0?(a.sgn<0?comp(num,a.num)>0:1):(sgn>0?(a.sgn>0?comp(num,a.num)<0:0):\
a.sgn>0);
    }
    inline int operator<(const int a) {
        return sgn<0?(a<0?comp(num,-a)>0:1):(sgn>0?(a>0?comp(num,a)<0:0):a>0);
    }
    inline int operator<=(const bignum &a) {
        return sgn<0?(a.sgn<0?comp(num,a.num)>=0:1):(sgn>0?(a.sgn>0?comp(num,a.num)<=0:0\
):a.sgn>=0);
    }
    inline int operator<=(const int a) {
        return sgn<0?(a<0?comp(num,-a)>=0:1):(sgn>0?(a>0?comp(num,a)<=0:0):a>=0);
    }
    inline int operator==(const bignum &a) {
        return (sgn==a.sgn)?!comp(num,a.num):0;
    }
    inline int operator==(const int a) {
        return (sgn *a>=0)?!comp(num,ABS(a)):0;
    }
```

```
    inline int operator!=(const bignum &a) {
        return (sgn==a.sgn)?comp(num,a.num):1;
    }
    inline int operator!=(const int a) {
        return (sgn *a>=0)?comp(num,ABS(a)):1;
    }
    inline int operator[](const int a) {
        return digit(num,a);
    }
    friend inline istream &operator>>(istream &is,bignum &a) {
        read(a.num,a.sgn,is);
        return is;
    }
    friend inline ostream &operator<<(ostream &os,const bignum &a) {
        if(a.sgn<0)os<<'-';
        write(a.num,os);
        return os;
    }
    friend inline bignum sqrt(const bignum &a) {
        bignum ret;
        bignum_t t;
        memcpy(t,a.num,sizeof(bignum_t));
        sqrt(ret.num,t);
        ret.sgn=ret.num[0]!=1||ret.num[1];
        return ret;
    }
    friend inline bignum sqrt(const bignum &a,bignum &b) {
        bignum ret;
        memcpy(b.num,a.num,sizeof(bignum_t));
        sqrt(ret.num,b.num);
        ret.sgn=ret.num[0]!=1||ret.num[1];
        b.sgn=b.num[0]!=1||ret.num[1];
        return ret;
    }
    inline int length() {
        return ::length(num);
    }
    inline int zeronum() {
        return ::zeronum(num);
    }
    inline bignum C(const int m,const int n) {
        combination(num,m,n);
        sgn=1;
        return *this;
    }
    inline bignum P(const int m,const int n) {
        permutation(num,m,n);
        sgn=1;
        return *this;
    }
};
```

10.2.3 幻方构造

```
//幻方构造 (L!=2)
const int MAXN = 100;

```

```
void dllb(int L, int si, int sj, int sn, int d[][MAXN]) {
    int n, i = 0, j = L / 2;
    for (n = 1; n <= L * L; n++) {
        d[i + si][j + sj] = n + sn;
        if (n % L) {
            i = (i) ? (i - 1): (L - 1);
            j = (j == L - 1) ? 0 : (j + 1);
        } else {
            i = (i == L - 1) ? 0 : (i + 1);
        }
    }
}

void magicOdd(int L, int d[][MAXN]) {
    dllb(L, 0, 0, 0, d);
}

void magic4k(int L, int d[][MAXN]) {
    int i, j;
    for (i = 0; i < L; i++) {
        for (j = 0; j < L; j++) {
            d[i][j] = ((i % 4 == 0 || i % 4 == 3) && (j % 4 == 0 || j % 4 == 3) || (i % \
4 == 1 || i % 4 == 2) && (j % 4 == 1 || j % 4 == 2)) ? (L * L - (i * L + j)) : (i * L + \
j + 1);
        }
    }
}

void magicOther(int L, int d[][MAXN]) {
    int i, j, t;
    dllb(L / 2, 0, 0, 0, d);
    dllb(L / 2, L / 2, L / 2, L * L / 4, d);
    dllb(L / 2, 0, L / 2, L * L / 2, d);
    dllb(L / 2, L / 2, 0, L * L / 4 * 3, d);
    for (i = 0; i < L / 2; i++) {
        for (j = 0; j < L / 4; j++) {
            if (i != L / 4 || j) {
                t = d[i][j];
                d[i][j] = d[i + L / 2][j];
                d[i + L / 2][j] = t;
            }
        }
    }
    t = d[L / 4][L / 4];
    d[L / 4][L / 4] = d[L / 4+L / 2][L / 4];
    d[L / 4+L / 2][L / 4] = t;
    for (i = 0; i < L / 2; i++) {
        for (j = L - L / 4+1; j < L; j++) {
            t = d[i][j];
            d[i][j] = d[i + L / 2][j];
            d[i + L / 2][j] = t;
        }
    }
}

void generate(int L, int d[][MAXN]) {
    if (L % 2) {
        magicOdd(L, d);
```

```
    } else if (L % 4 == 0) {
        magic4k(L, d);
    } else {
        magicOther(L, d);
    }
}
```

10.2.4 最大子串匹配

```
//最大子串匹配，复杂度 O(mn)
//返回最大匹配值，传入两个串和串的长度，重载返回一个最大匹配
//注意做字符串匹配是串末的'\0' 没有置!
//可更改元素类型，更换匹配函数和匹配价值函数
# include <cstring>
# include <algorithm>
using namespace std;

const int MAXN = 100;

# define _match(a, b) ((a) == (b))
# define _value(a, b) 1

template <class elemType>
int strMatch(int m, const elemType *a, int n, const elemType *b) {
    int match[MAXN + 1][MAXN + 1], i, j;
    memset(match, 0, sizeof(match));
    for (i = 0; i < m; i++) {
        for (j = 0; j < n; j++) {
            match[i + 1][j + 1] = max(max(match[i][j + 1], match[i + 1][j]), (match[i][j\
] + _value(a[i], b[i])) * _match(a[i], b[j]));
        }
    }
    return match[m][n];
}

template <class elemType>
int strMatch(int m, const elemType *a, int n, const elemType *b, elemType *ret) {
    int match[MAXN + 1][MAXN + 1], last[MAXN + 1][MAXN + 1], i, j, t;
    memset(match, 0, sizeof(match));
    for (i = 0; i < m; i++) {
        for (j = 0; j < n; j++) {
            match[i + 1][j + 1] = (match[i][j + 1] > match[i + 1][j] ? match[i][j + 1] :\
 match[i + 1][j]);
            last[i + 1][j + 1] = (match[i][j + 1] > match[i + 1][j] ? 3 : 1);
            if ((t = (match[i][j] + _value(a[i], b[i])) * _match(a[i], b[j]))> match[i +\
 1][j + 1]) {
                match[i + 1][j + 1] = t;
                last[i + 1][j + 1] = 2;
            }
        }
    }
    for (; match[i][j]; i -= (last[t = i][j] > 1), j -= (last[t][j] < 3)) {
        ret[match[i][j] - 1] = (last[i][j] < 3 ? a[i - 1]: b[j - 1]);
    }
```

```
    return match[m][n];
}
```

10.2.5 最大子段和

```
//求最大子段和，复杂度 O(n)
//传入串长 n 和内容 list[]
//返回最大子段和，重载返回子段位置 (maxsum=list[start]+...+list[end])
//可更改元素类型
template <class elemType>
elemType maxsum(int n, const elemType *list) {
    elemType ret, sum = 0;
    int i;
    for (ret = list[i = 0]; i < n; i++) {
        sum = (sum > 0 ? sum : 0) + list[i];
        ret = (sum > ret ? sum : ret);
    }
    return ret;
}

template <class elemType>
elemType maxsum(int n, const elemType *list, int &start, int &end) {
    elemType ret, sum = 0;
    int s, i;
    for (ret = list[start = end = s = i = 0]; i < n; i++, s = (sum > 0 ? s : i)) {
        if ((sum = (sum > 0 ? sum : 0) + list[i])> ret) {
            ret = sum;
            start = s;
            end = i;
        }
    }
    return ret;
}
```

10.2.6 第 k 元素

```
//一般可用 STL 的 kth_element()

//取第 k 个元素,k=0..n-1
//平均复杂度 O(n)
//注意 a[] 中的顺序被改变
# define _cp(a, b) ((a) < (b))

template <class elemType>
elemType kthElement(int n, const elemType *a, int k) {
    elemType t, key;
    int left = 0, r = n - 1, i, j;
    while (left < r) {
        for (key = a[((i = left - 1) + (j = r + 1)) >> 1]; i < j;) {
            for (j--; _cp(key, a[j]); j--);
            for (i++; _cp(a[i], key); i++);
            if (i<j) {
                t = a[i];
```

```
                a[i] = a[j];
                a[j] = t;
            }
        }
        if (k > j) {
            left = j + 1;
        } else {
            r = j;
        }
    }
    return a[k];
}
```

10.2.7 骰子

```
//Author: t__nt
//骰子的基本操作
//展开后对应的标号
//     3
//1 2 6 5
//     4
# include<cstdio>
//通过上表面和前面, 得出右侧面
const int getFace[7][7]=//[upward][forward]
    //0 1 2 3 4 5 6
{   {0,0,0,0,0,0,0}     //0
    ,{0,0,3,5,2,4,0}    //1
    ,{0,4,0,1,6,0,3}    //2
    ,{0,2,6,0,0,1,5}    //3
    ,{0,5,1,0,0,6,2}    //4
    ,{0,3,0,6,1,0,4}    //5
    ,{0,0,4,2,5,3,0}
};  //6
//对面的对应标号
const int oppo[7]= {0,6,5,4,3,2,1};

class Dice {
public:
    int up,forward,right;

    Dice() {}

    Dice(int a,int b):up(a),forward(b) {
        right=getFace[a][b];
    }

    Dice(int a,int b,int c):up(a),forward(b),right(c) {}

    void goLeft() {
        int a=up,b=forward,c=right;
        up=c;
        forward=b;
```

```
            rigth=oppo[a];
        }

        void goRight() {
            int a=up,b=forward,c=right;
            up=oppo[c];
            forward=b;
            rigth=a;
        }

        void goUp() {
            int a=up,b=forward,c=right;
            up=b;
            forward=oppo[a];
            rigth=c;
        }

        void goDown(Pos &tmppos,Pos &pos) {
            int a=up,b=forward,c=right;
            up=oppo[b];
            forward=a;
            rigth=c;
        }

};
```

Chapter 11

Cheat Sheet

11.1 Theoretical Computer Science

Theoretical Computer Science Cheat Sheet

Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)\ \forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0\ \forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n}\frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

Identities

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\displaystyle\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\displaystyle\left[{n \atop 1}\right] = (n-1)!$, **15.** $\displaystyle\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[{n \atop n}\right] = 1$, **17.** $\displaystyle\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\displaystyle\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\displaystyle\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\displaystyle\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\displaystyle\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\displaystyle\left\langle{0 \atop k}\right\rangle = \begin{cases}1 & \text{if } k = 0,\\ 0 & \text{otherwise}\end{cases}$ **26.** $\displaystyle\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \sum_{k=0}^{n} \begin{bmatrix} k \\ m \end{bmatrix} n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix},$

39. $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^{n} \left\langle\!\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

40. $\begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_k \binom{n}{k} \begin{Bmatrix} k+1 \\ m+1 \end{Bmatrix} (-1)^{n-k},$

41. $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k},$

42. $\begin{Bmatrix} m+n+1 \\ m \end{Bmatrix} = \sum_{k=0}^{m} k \begin{Bmatrix} n+k \\ k \end{Bmatrix},$

43. $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^{m} k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix},$

44. $\binom{n}{m} = \sum_k \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},$

45. $(n-m)! \binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{m-k}, \quad \text{for } n \ge m,$

46. $\begin{Bmatrix} n \\ n-m \end{Bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix},$

47. $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{Bmatrix} m+k \\ k \end{Bmatrix},$

48. $\begin{Bmatrix} n \\ \ell+m \end{Bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{Bmatrix} k \\ \ell \end{Bmatrix} \begin{Bmatrix} n-k \\ m \end{Bmatrix} \binom{n}{k},$

49. $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i / 2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$1\big(T(n) - 3T(n/2) = n\big)$$
$$3\big(T(n/2) - 3T(n/4) = n/2\big)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$3^{\log_2 n - 1}\big(T(2) - 3T(1) = 2\big)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$\begin{aligned} n \sum_{i=0}^{m-1} c^i &= n \left(\frac{c^m - 1}{c - 1} \right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find

$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \ge 0} g_{i+1} x^i = \sum_{i \ge 0} 2g_i x^i + \sum_{i \ge 0} x^i.$$

We choose $G(x) = \sum_{i \ge 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \ge 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) \\ &= x \left(2 \sum_{i \ge 0} 2^i x^i - \sum_{i \ge 0} x^i \right) \\ &= \sum_{i \ge 0} (2^{i+1} - 1) x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520},\ldots$$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$

$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{n} k\binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x)\Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$

$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$

$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1}\sum_{i_i < \cdots < i_k}\Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$

$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

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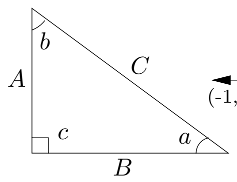
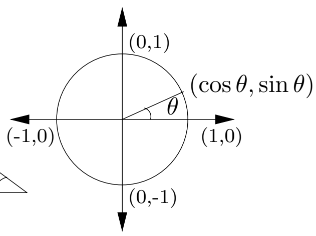
Trigonometry





Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

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Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ - ceg - fha - ibd. \end{matrix}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

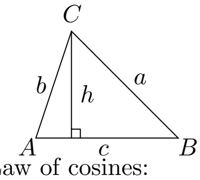
Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a + b + c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:

$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

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Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

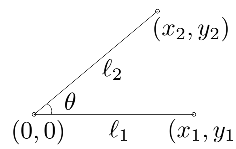
Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$
$$\left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p},$$
$$\lim_{p \to \infty} \left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

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π

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:

$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:

$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$

$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$

$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, 2. $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, 3. $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, 5. $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, 6. $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, 8. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, 10. $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, 12. $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \sec u\dfrac{du}{dx}$, 14. $\dfrac{d(\csc u)}{dx} = -\cot u \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, 16. $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$, 18. $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, 20. $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, 22. $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

23. $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, 24. $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

25. $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$, 26. $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u\dfrac{du}{dx}$,

27. $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, 28. $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

29. $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, 30. $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

31. $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, 32. $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx$, 2. $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, 4. $\displaystyle\int \frac{1}{x}dx = \ln x$, 5. $\displaystyle\int e^x\,dx = e^x$,

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, 7. $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle\int \sin x\,dx = -\cos x$, 9. $\displaystyle\int \cos x\,dx = \sin x$,

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, 11. $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, 13. $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle\int \arcsin\tfrac{x}{a}dx = \arcsin\tfrac{x}{a} + \sqrt{a^2-x^2}, \quad a > 0$,

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Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$

16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$

17. $\int \sin^2(ax) dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax) dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\, dx = \tan x,$

20. $\int \csc^2 x\, dx = -\cot x,$

21. $\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx,$

22. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx,$

23. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx, \quad n \neq 1,$

24. $\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx, \quad n \neq 1,$

25. $\int \sec^n x\, dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx, \quad n \neq 1,$

26. $\int \csc^n x\, dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x\, dx, \quad n \neq 1,$

27. $\int \sinh x\, dx = \cosh x,$

28. $\int \cosh x\, dx = \sinh x,$

29. $\int \tanh x\, dx = \ln|\cosh x|,$

30. $\int \coth x\, dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x\, dx = \arctan \sinh x,$

32. $\int \operatorname{csch} x\, dx = \ln\left|\tanh \frac{x}{2}\right|,$

33. $\int \sinh^2 x\, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\, dx = \tanh x,$

36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$

37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln|a^2 - x^2|,$

38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$

39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0,$

40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$

41. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$

44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right|,$

45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}},$

46. $\int \sqrt{a^2 \pm x^2}\, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln\left|x + \sqrt{a^2 \pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0,$

48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln\left|\frac{x}{a + bx}\right|,$

49. $\int x\sqrt{a + bx}\, dx = \frac{2(3bx - 2a)(a + bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a + bx}}{x} dx = 2\sqrt{a + bx} + a \int \frac{1}{x\sqrt{a + bx}} dx,$

51. $\int \frac{x}{\sqrt{a + bx}} dx = \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right|, \quad a > 0,$

52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

53. $\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}(a^2 - x^2)^{3/2},$

54. $\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

56. $\int \frac{x\, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$

57. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a,} \quad a > 0,$

58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln\left|\frac{a + \sqrt{a^2 + x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$

60. $\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln\left|\frac{x}{a + \sqrt{a^2 + x^2}}\right|,$

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Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \tfrac{1}{a}\arccos\tfrac{a}{|x|}, \quad a>0,$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac,\\ \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac,\end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0,\\ \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0,\end{cases}$$

68. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0,\\ \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0,\end{cases}$$

71. $$\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$$

72. $$\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax)+\tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$$

73. $$\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax)-\tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$$

74. $$\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a}-\tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$$

75. $$\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1}-\frac{1}{(n+1)^2}\right),$$

76. $$\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$$

$$\begin{array}{lclcl}
x^1 = & x^{\underline{1}} & = & x^{\overline{1}} \\
x^2 = & x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{2}}-x^{\overline{1}} \\
x^3 = & x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}} \\
x^4 = & x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}} \\
x^5 = & x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}} & = & x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}
\end{array}$$

$$\begin{array}{lclc}
x^{\overline{1}} = & x^1 & x^{\underline{1}} = & x^1 \\
x^{\overline{2}} = & x^2+x^1 & x^{\underline{2}} = & x^2-x^1 \\
x^{\overline{3}} = & x^3+3x^2+2x^1 & x^{\underline{3}} = & x^3-3x^2+2x^1 \\
x^{\overline{4}} = & x^4+6x^3+11x^2+6x^1 & x^{\underline{4}} = & x^4-6x^3+11x^2-6x^1 \\
x^{\overline{5}} = & x^5+10x^4+35x^3+50x^2+24x^1 & x^{\underline{5}} = & x^5-10x^4+35x^3-50x^2+24x^1
\end{array}$$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\, f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu) = c\Delta u$, $\quad \Delta(u+v) = \Delta u+\Delta v$,

$\Delta(uv) = u\Delta v + \mathrm{E}\, v\Delta u$,

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}}$,

$\Delta(H_x) = x^{\underline{-1}}$, $\quad \Delta(2^x) = 2^x$,

$\Delta(c^x) = (c-1)c^x$, $\quad \Delta\binom{x}{m} = \binom{x}{m-1}$.

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x$,

$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x$,

$\sum u\Delta v\,\delta x = uv - \sum \mathrm{E}\, v\Delta u\,\delta x$,

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}$, $\quad \sum x^{\underline{-1}}\,\delta x = H_x$,

$\sum c^x\,\delta x = \frac{c^x}{c-1}$, $\quad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}$.

Falling Factorial Powers:

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$$
$$x^{\underline{0}} = 1,$$
$$x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$$
$$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$$

Rising Factorial Powers:

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$$
$$x^{\overline{0}} = 1,$$
$$x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$$
$$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$$

Conversion:

$$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$$
$$= 1/(x+1)^{\overline{-n}},$$
$$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$$
$$= 1/(x-1)^{\underline{-n}},$$
$$x^n = \sum_{k=1}^n \left\{{n \atop k}\right\} x^{\underline{k}} = \sum_{k=1}^n \left\{{n \atop k}\right\}(-1)^{n-k}x^{\overline{k}},$$
$$x^{\underline{n}} = \sum_{k=1}^n \left[{n \atop k}\right](-1)^{n-k}x^k,$$
$$x^{\overline{n}} = \sum_{k=1}^n \left[{n \atop k}\right]x^k.$$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} i x^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$$

$$\ln\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i,$$

$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i} \right]x^i,$$

$$(e^x-1)^n = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n} \right]\frac{n!x^i}{i!},$$

$$x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p\frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

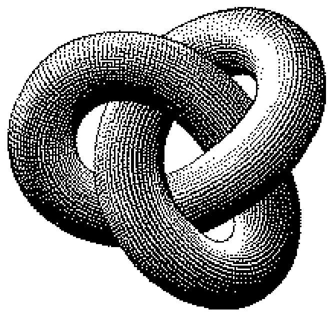
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^ii!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A\neq 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Fibonacci Numbers

$1,1,2,3,5,8,13,21,34,55,89,\ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \tfrac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)