Computing AMMs from liquidity pools among multiple independent choices

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April 20, 2024

Abstract

TODO.

1 Introduction

We have the following system of equations, over the variables $\{y_i\}_{i\in[k]}, \{n_i\}_{i\in[k]}$, given values of y'_i

$$\forall i, y'_i = y_i - n_i + \sum_i n_i$$

$$\forall i, p_i(n_i + y_i) = n_i$$

$$\sum p_i = 1$$

$$\forall i, \forall j, y_i n_i = y_j n_j$$

let $N = \sum n_i$, $c = x_i y_i$. These become

$$\forall i, y_i = y'_i + n_i - N$$

$$\forall i, p_i(n_i + y_i) = n_i$$

$$\sum p_i = 1$$

$$\forall i, c = y_i n_i$$

$$N = \sum n_i$$

Plugging the first eqn into the others

$$\forall i, p_i(2n_i + y'_i - N) = n_i$$

$$\sum p_i = 1$$

$$\forall i, c = (y'_i + n_i - N)n_i$$

$$N = \sum n_i$$

And plugging the first into the second

$$\sum \frac{n_i}{(2n_i + y'_i - N)} = 1$$

$$\forall i, c = (y'_i + n_i - N)n_i$$

$$N = \sum n_i$$

Rearranging the second

$$\sum \frac{n_i}{(2n_i + y'_i - N)} = 1$$

$$\forall i, 0 = n_i^2 + n_i(y_i' - N) - c$$
$$N = \sum n_i$$

And applying the quadratic equation (using an assumption that c is positive, the discriminant is greater than B^2, thus we must take the greater root for n_i to be positive.)

$$\sum \frac{n_i}{(2n_i + y_i' - N)} = 1$$
$$\forall i, n_i = \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}$$
$$N = \sum n_i$$

A bit of rearrangement of the first

$$\sum \frac{1}{(2 + \frac{y_i' - N}{n_i})} = 1$$
$$\forall i, n_i = \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}$$
$$N = \sum n_i$$

Now, plugging the second into the others, we have

$$\sum \frac{1}{(2 + \frac{y_i' - N}{\frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}})} = 1$$
$$N = \sum \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}$$

multiplying top and bottom through

$$\sum \frac{\frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}}{(2\frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2} + y_i' - N)} = 1$$
$$N = \sum \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}$$

Simplifying

$$\sum \frac{\frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}}{(N - y_i' + \sqrt{(N - y_i')^2 + 4c} + y_i' - N)} = 1$$
$$N = \sum \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}$$

Simplifying

$$\sum \frac{\frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}}{(\sqrt{(N - y_i')^2 + 4c})} = 1$$
$$N = \sum \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2}$$

Simplifying

$$\sum \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2} \frac{1}{(\sqrt{(N - y_i')^2 + 4c})} = 1$$

$$\sum \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2} = N$$

Simplifying

$$\sum \left(\frac{N - y_i'}{(2\sqrt{(N - y_i')^2 + 4c})} + \frac{1}{2}\right) = 1$$

$$\sum \frac{N - y_i' + \sqrt{(N - y_i')^2 + 4c}}{2} = N$$

1.1 Sufficiency

It is sufficient to solve the above pair of equations, because we can then derive n_i and then y_i and p_i from previously stated formulae.

2 Computing a solution, if it exists.

2.1 Uniqueness

2.1.1 The second equation gives at most one c for any N

This follows because the LHS is monotonically increasing in c, a fact we can use to find this solution, if it exists, by binary search. Call the function $\tilde{c}(N)$

2.1.2 The first equation gives at most one N for any c

This follows because the LHS is monotonically increasing in N, again a fact we can use to find this solution, if it exists, by binary search. Call the function $\tilde{N}(c)$

3 Computing

We know that $0 \leq N \leq \frac{\sum y_i'}{k-1}$ (the latter follows from the sum of the $y_i' = y_i - n_i + \sum_i n_i$ equations). The above maps describe an iteration that we can repeatedly apply: $N \mapsto \tilde{N}(\tilde{c}(N))$. Since N is compact, we can grid-search for a point that is stable under the iteration.

References