Relationship between linear and far addressing

Let $f(x) = \big((x \gg 4) \ll 16\big) \mid (x \,\&\, 15)$, for $0 \le x < 2^{33}$.

Objective: Prove that $f(x+y) = f(x) + f(y)$ for all x and y in that range.

Proof: Let $X_{31}X_{30} \dots X_2X_1X_0$ be the sequence of the digits of x in order, base 2. Let's see what the operations do to such digits.

$(x \,\&\, 15)$: Since $(15)_{10} = (1111)_2$, it follows that $(x \,\&\, 15)$ is equal to the last four digits of x, that is, in terms of its digits, $\underbrace{0 \dots 0}_{28\text{ zeroes}}\ X_3X_2X_1X_0$.

$(x \gg 4) \ll 16$: By right-shifting the digits of x by 4 places to the right, it follows that
$(x \gg 4) = 0000X_{31}X_{30} \dots X_5X_4$. Then, applying the left-shift of these digits by 16 places to the left, one has that $(x \gg 4) \ll 16 = X_{19}X_{18} \dots X_5X_4\ \underbrace{0 \dots 0}_{16\text{ zeroes}}$.

Then, since $x \mid 0 = x$, it follows that

$f(x) = \big((x \gg 4) \ll 16\big) \mid (x \,\&\, 15) =$

$\left(X_{19}X_{18} \dots X_5X_4\ \underbrace{0 \dots 0}_{12\text{ zeroes}}\ 0000\right) \mid \underbrace{0 \dots 0}_{28\text{ zeroes}}\ X_3X_2X_1X_0$, that is to say,

$f(X_{31}X_{30} \dots X_2X_1X_0) = X_{19}X_{18} \dots X_5X_4\ \underbrace{0 \dots 0}_{12\text{ zeroes}}\ X_3X_2X_1X_0$.

Knowing this, it is possible to compute $f(x) + f(y) = r$ with a little tweak:

if, after the sum, the fourth-to-last digit of r is equal to 0, the operation is completed.
If it is equal to 1 we need to deal with the carry. To achieve this, we need to skip the 12 zeros that preceed the last four digits, which we can do by removing that bit and then re-adding it in the upper bits. We need to xor the result with $1 \ll 4$, in order to remove the carry bit, and then add it back to the result with $1 \ll 16$.

$$\begin{cases} R_4 = 0, \textit{leave the result of the addition as it is} \\ R_4 = 1, \textit{let } r = \big(r \mathbin{\char`^} (1 \ll 4)\big) + (1 \ll 16) \end{cases}$$