Simple Linear Regression: Estimation

EC 320: Introduction to Econometrics

Winter 2022

Prologue

Last Time

We considered a simple linear regression of Y_i on X_i:

$$Y_i = \beta_1 + \beta_2 X_i + u_i.$$

- β_1 and β_2 are **population parameters** that describe the *"true"* relationship between X_i and Y_i.

- **Problem:** We don't know the population parameters. The best we can do is to estimate them.

Last Time

We derived the OLS estimator by picking estimates that minimize $\sum_{i=1}^{n} \hat{u}_i^2$.

- **Intercept:**

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}.$$

- **Slope:**

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}.$$

We used these formulas to obtain estimates of the parameters β_1 and β_2 in a regression of Y_i on X_i.

Last Time

With the OLS estimates of the population parameters, we constructed a regression line:

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i.$$

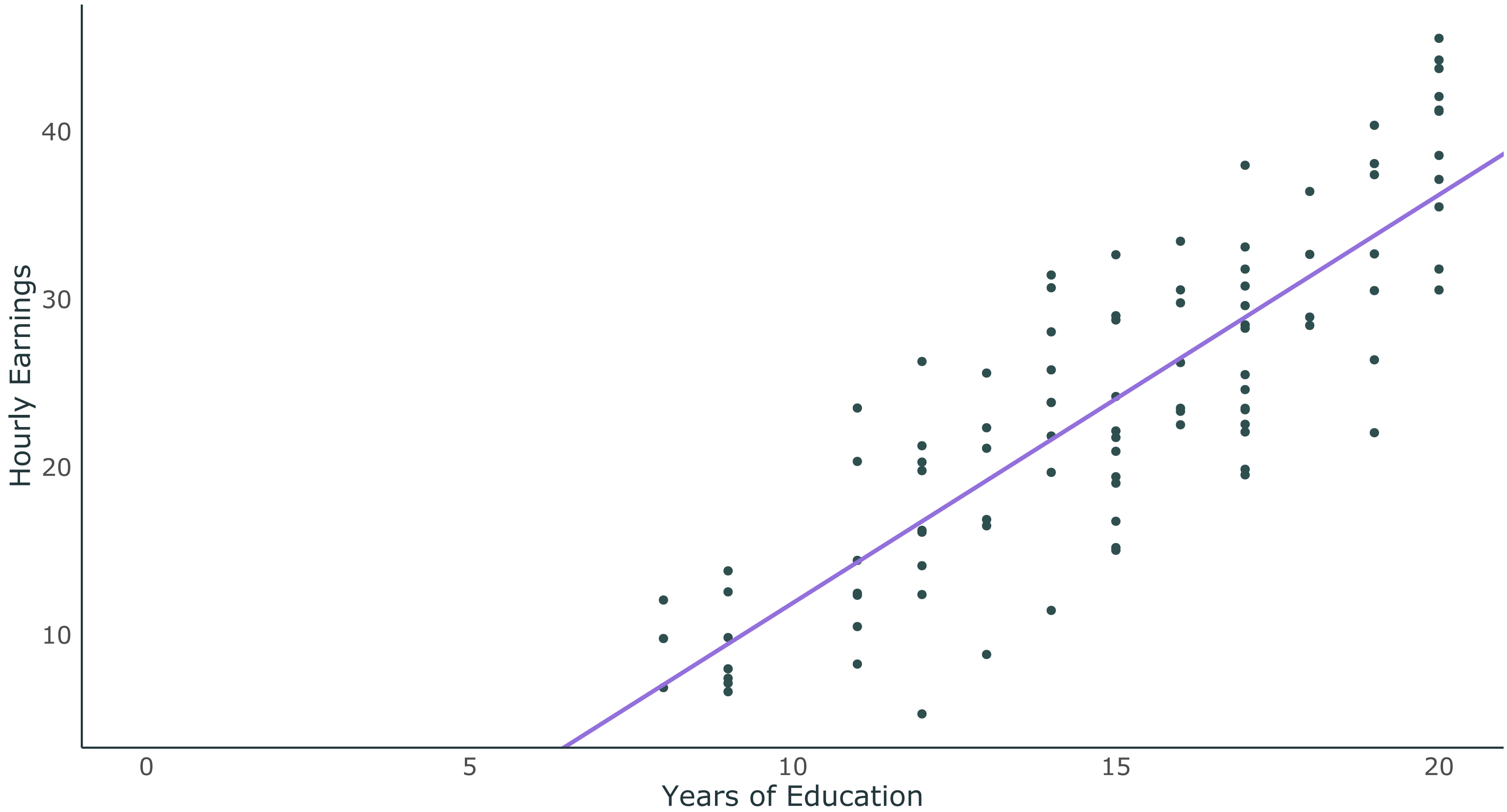
- $\hat{Y}_i$ are predicted or **fitted** values of Y_i.
- You can think of $\hat{Y}_i$ as an estimate of the average value of Y_i given a particular of X_i.

OLS still produces prediction errors: $\hat{u}_i = Y_i - \hat{Y}_i$.

- Put differently, there is a part of Y_i we can explain and a part we cannot: $Y_i = \hat{Y}_i + \hat{u}_i$.

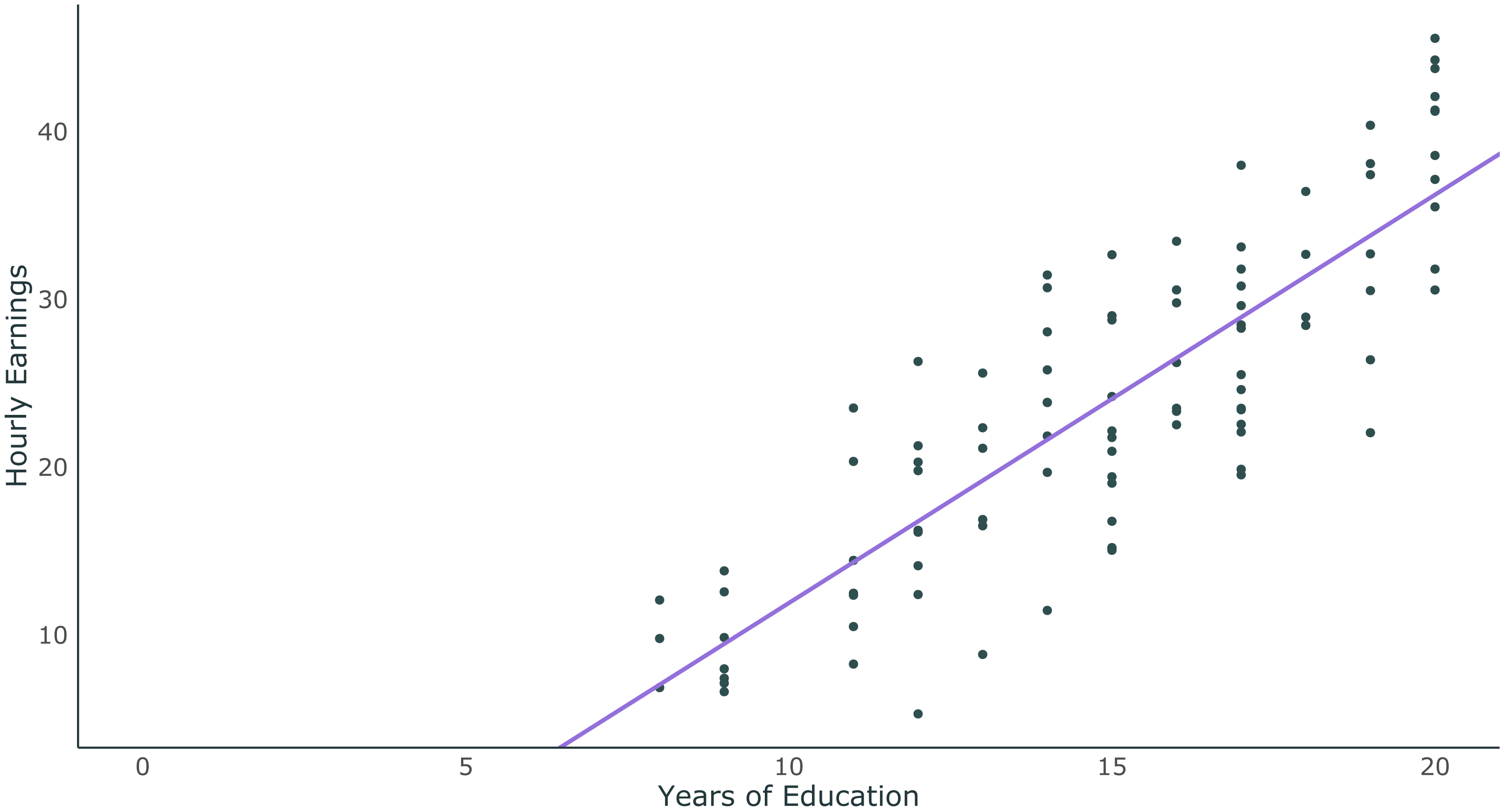
Review

What is the equation for the regression model estimated below?



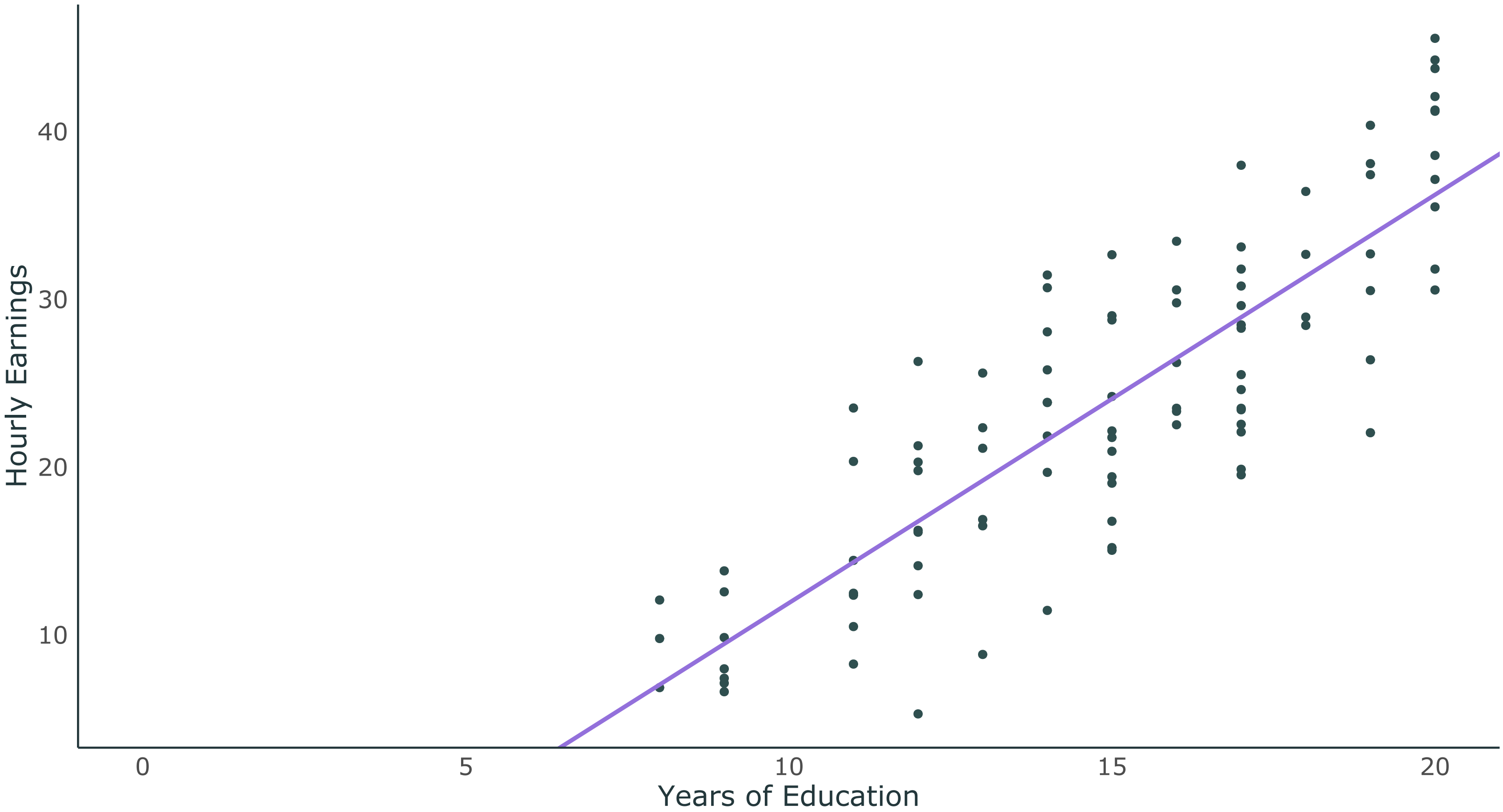
Review

The estimated **intercept** is -12.44. What does this tell us?



Review

The estimated **slope** is 2.43. How do we interpret it?



Today

Agenda

1. Highlight important properties of OLS.

2. Discuss goodness of fit: how well does one variable explain another?

3. Units of measurement.

OLS Properties

OLS Properties

The way we selected OLS estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ gives us three important properties:

1. Residuals sum to zero: $\sum_{i=1}^{n} \hat{u}_i = 0$.
2. The sample covariance between the independent variable and the residuals is zero: $\sum_{i=1}^{n} X_i \hat{u}_i = 0$.
3. The point $(\bar{X}, \bar{Y})$ is always on the regression line.

OLS Residuals

Residuals sum to zero: $\sum_{i=1}^{n} \hat{\boldsymbol{u}}_i = \mathbf{0}$.

- By extension, the sample mean of the residuals are zero.
- You will prove this in Problem Set 3.

OLS Residuals

The sample covariance between the independent variable and the residuals is zero: $\sum_{i=1}^{n} X_i \hat{u}_i = 0$.

- You will prove a version of this in Problem Set 3.

OLS Regression Line

The point $(\bar{X}, \bar{Y})$ is always on the regression line.

- Start with the regression line: $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$.
- $\hat{Y}_i = \bar{Y} - \hat{\beta}_2 \bar{X} + \hat{\beta}_2 X_i$.
- Plug $\bar{X}$ into X_i:

$$\begin{aligned} \hat{Y}_i &= \bar{Y} - \hat{\beta}_2 \bar{X} + \hat{\beta}_2 \bar{X} \\ &= \bar{Y}. \end{aligned}$$

Goodness of Fit

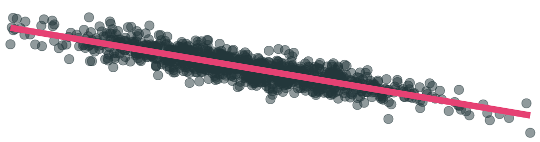
Goodness of Fit

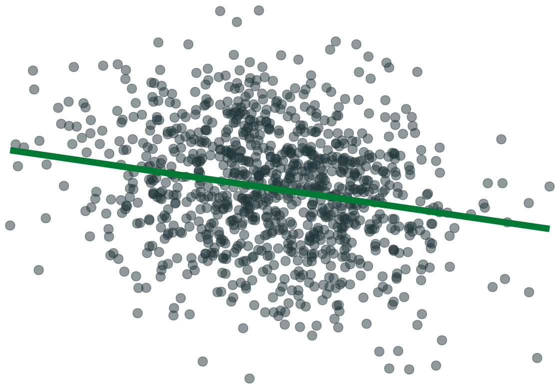
Regression 1 *vs.* Regression 2

- Same slope.

- Same intercept.

Q: Which fitted regression line *"explains"*[*] the data better?





[*] *Explains = fits.*

Goodness of Fit

Regression 1 *vs.* Regression 2

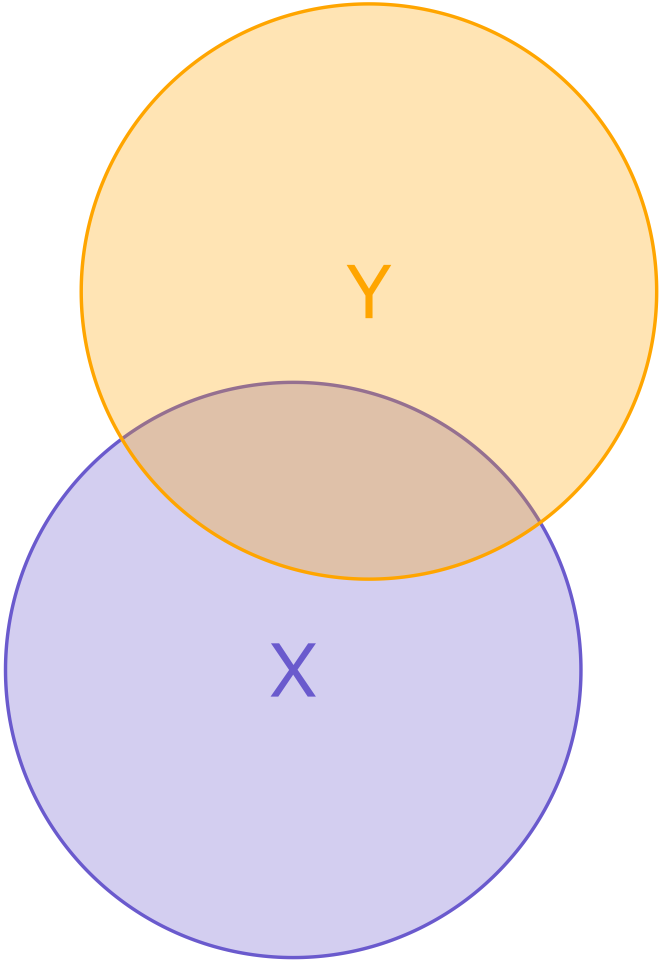
The **coefficient of determination** R^2 is the fraction of the variation in Y_i *"explained"* by X_i in a linear regression.

- $R^2 = 1 \implies X_i$ explains *all* of the variation in Y_i.
- $R^2 = 0 \implies X_i$ explains *none* of the variation in Y_i.

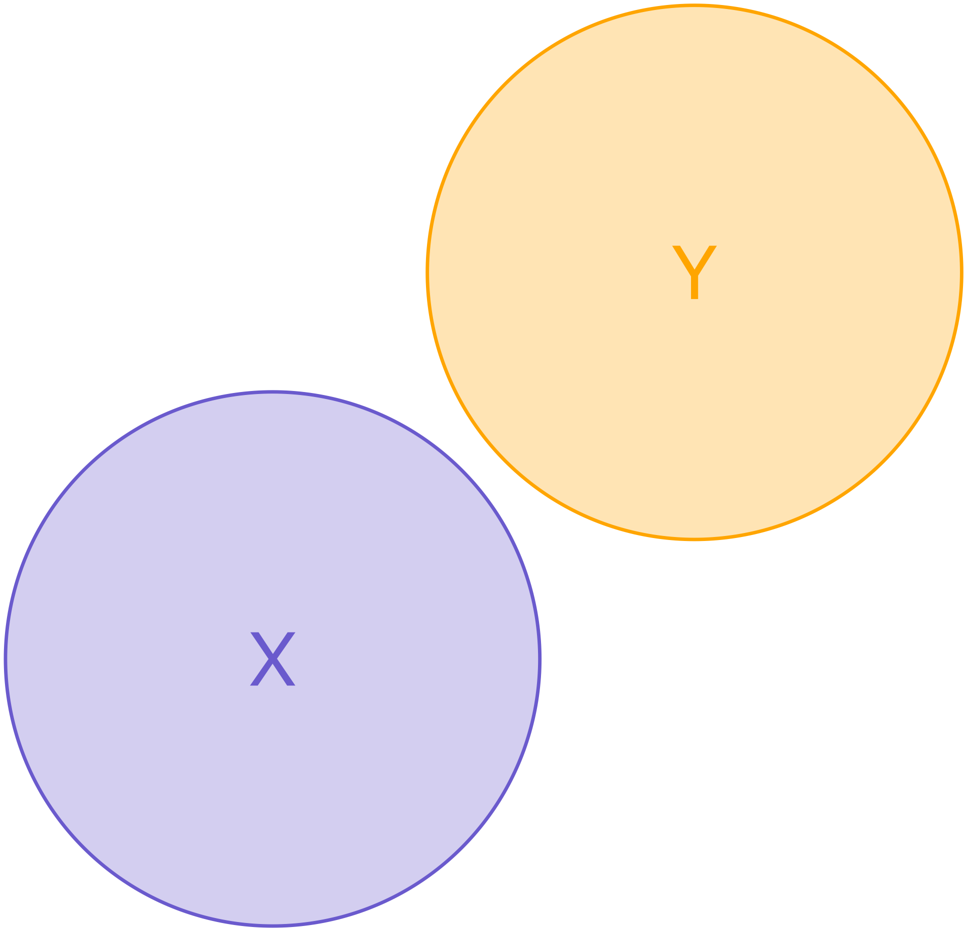
R^2 = 0.76

R^2 = 0.04

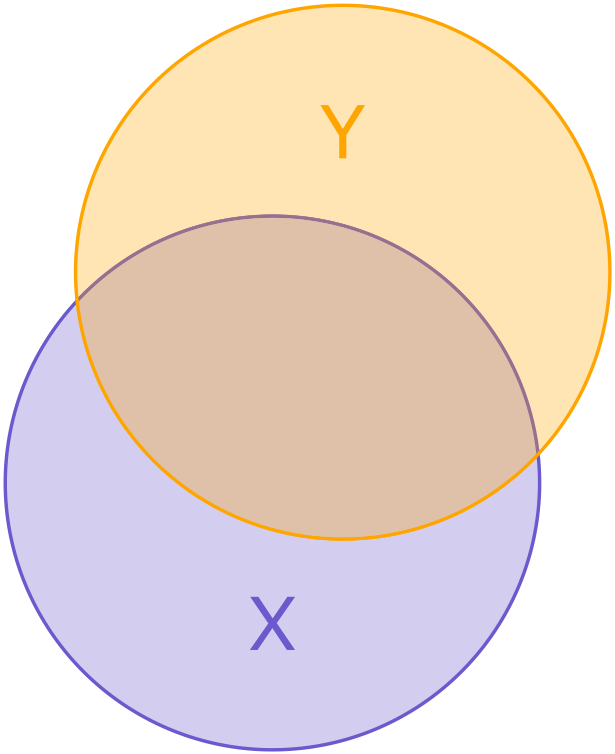
Goodness of Fit



Goodness of Fit



Goodness of Fit



Explained and Unexplained Variation

Residuals remind us that there are parts of Y_i we can't explain.

$$Y_i = \hat{Y}_i + \hat{u}_i$$

- Sum the above, divide by n, and use the fact that OLS residuals sum to zero to get $\bar{\hat{u}} = 0 \implies \bar{Y} = \bar{\hat{Y}}$.

Total Sum of Squares (TSS) measures variation in Y_i:

$$\text{TSS} \equiv \sum_{i=1}^{n} (Y_i - \bar{Y})^2.$$

- We will decompose this variation into explained and unexplained parts.

Explained and Unexplained Variation

Explained Sum of Squares (ESS) measures the variation in $\hat{Y}_i$:

$$\text{ESS} \equiv \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2.$$

Residual Sum of Squares (RSS) measures the variation in $\hat{u}_i$:

$$\text{RSS} \equiv \sum_{i=1}^{n} \hat{u}_i^2.$$

Goal: Show that $\text{TSS} = \text{ESS} + \text{RSS}$.

Step 1: Plug $Y_i = \hat{Y}_i + \hat{u}_i$ into TSS.

$$
\begin{aligned}
&\text{TSS} \\
&\quad = \sum_{i=1}^n (Y_i - \bar{Y})^2 \\
&\quad = \sum_{i=1}^n ([\hat{Y}_i + \hat{u}_i] - [\bar{\hat{Y}} + \bar{\hat{u}}])^2
\end{aligned}
$$

Step 2: Recall that $\bar{\hat{u}} = 0$ and $\bar{Y} = \bar{\hat{Y}}$.

$$
\begin{aligned}
&\text{TSS} \\
&\quad = \sum_{i=1}^n \left([\hat{Y}_i - \bar{Y}] + \hat{u}_i \right)^2 \\
&\quad = \sum_{i=1}^n \left([\hat{Y}_i - \bar{Y}] + \hat{u}_i \right) \left([\hat{Y}_i - \bar{Y}] + \hat{u}_i \right) \\
&\quad = \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2 + \sum_{i=1}^n \hat{u}_i^2 + 2 \sum_{i=1}^n \left((\hat{Y}_i - \bar{Y}) \hat{u}_i \right)
\end{aligned}
$$

Step 3: Notice **ESS** and **RSS**.

$$
\begin{aligned}
&\text{TSS} \\
&\quad = \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2 + \sum_{i=1}^n \hat{u}_i^2 + 2 \sum_{i=1}^n \left((\hat{Y}_i - \bar{Y}) \hat{u}_i \right) \\
&\quad = \text{ESS} + \text{RSS} + 2 \sum_{i=1}^n \left((\hat{Y}_i - \bar{Y}) \hat{u}_i \right)
\end{aligned}
$$

Step 4: Simplify.

$$
\begin{aligned}
&\mathrm{TSS} \\
&\quad = \mathrm{ESS} + \mathrm{RSS} + 2 \textstyle\sum_{i=1}^{n} \left((\hat{Y}_i - \bar{Y}) \hat{u}_i \right) \\
&\quad = \mathrm{ESS} + \mathrm{RSS} + 2 \textstyle\sum_{i=1}^{n} \hat{Y}_i \hat{u}_i - 2\bar{Y} \textstyle\sum_{i=1}^{n} \hat{u}_i
\end{aligned}
$$

Step 5: Shut down the last two terms. Notice that

$$
\begin{aligned}
&\textstyle\sum_{i=1}^{n} \hat{Y}_i \hat{u}_i \\
&\quad = \textstyle\sum_{i=1}^{n} (\hat{\beta}_1 + \hat{\beta}_2 X_i) \hat{u}_i \\
&\quad = \hat{\beta}_1 \textstyle\sum_{i=1}^{n} \hat{u}_i + \hat{\beta}_2 \textstyle\sum_{i=1}^{n} X_i \hat{u}_i \\
&\quad = 0
\end{aligned}
$$

Goodness of Fit

Calculating R^2

- $R^2 = \frac{\text{ESS}}{\text{TSS}}$.
- $R^2 = 1 - \frac{\text{RSS}}{\text{TSS}}$.

R^2 is related to the correlation between the actual values of Y and the fitted values of Y.

- Can show that $R^2 = (r_{Y,\hat{Y}})^2$.

Goodness of Fit

So what?

In the social sciences, low R^2 values are common.

Low R^2 doesn't mean that an estimated regression is useless.

- In a randomized control trial, R^2 is usually less than 0.1.

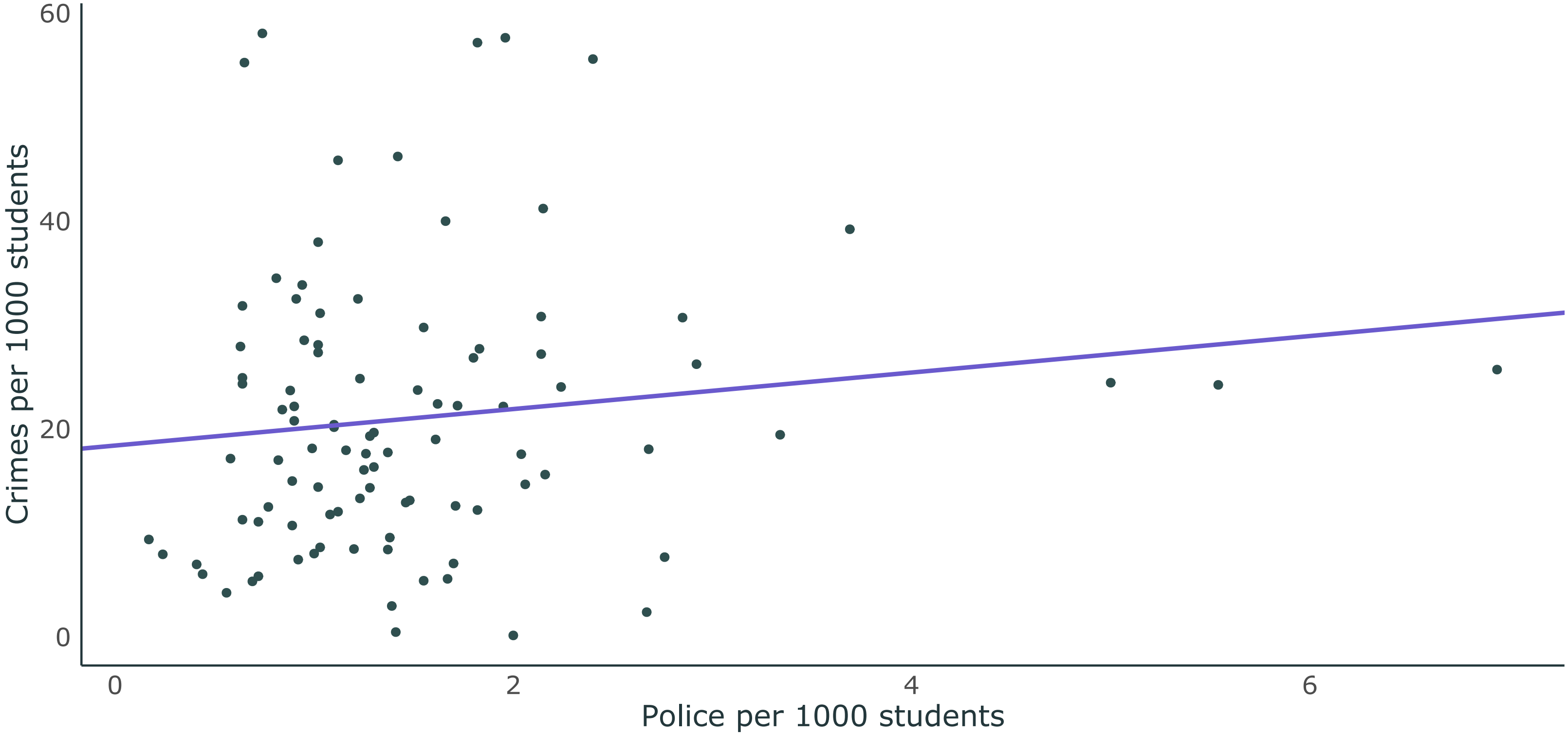
High R^2 doesn't necessarily mean you have a *"good"* regression.

- Worries about selection bias and omitted variables still apply.

Units of Measurement

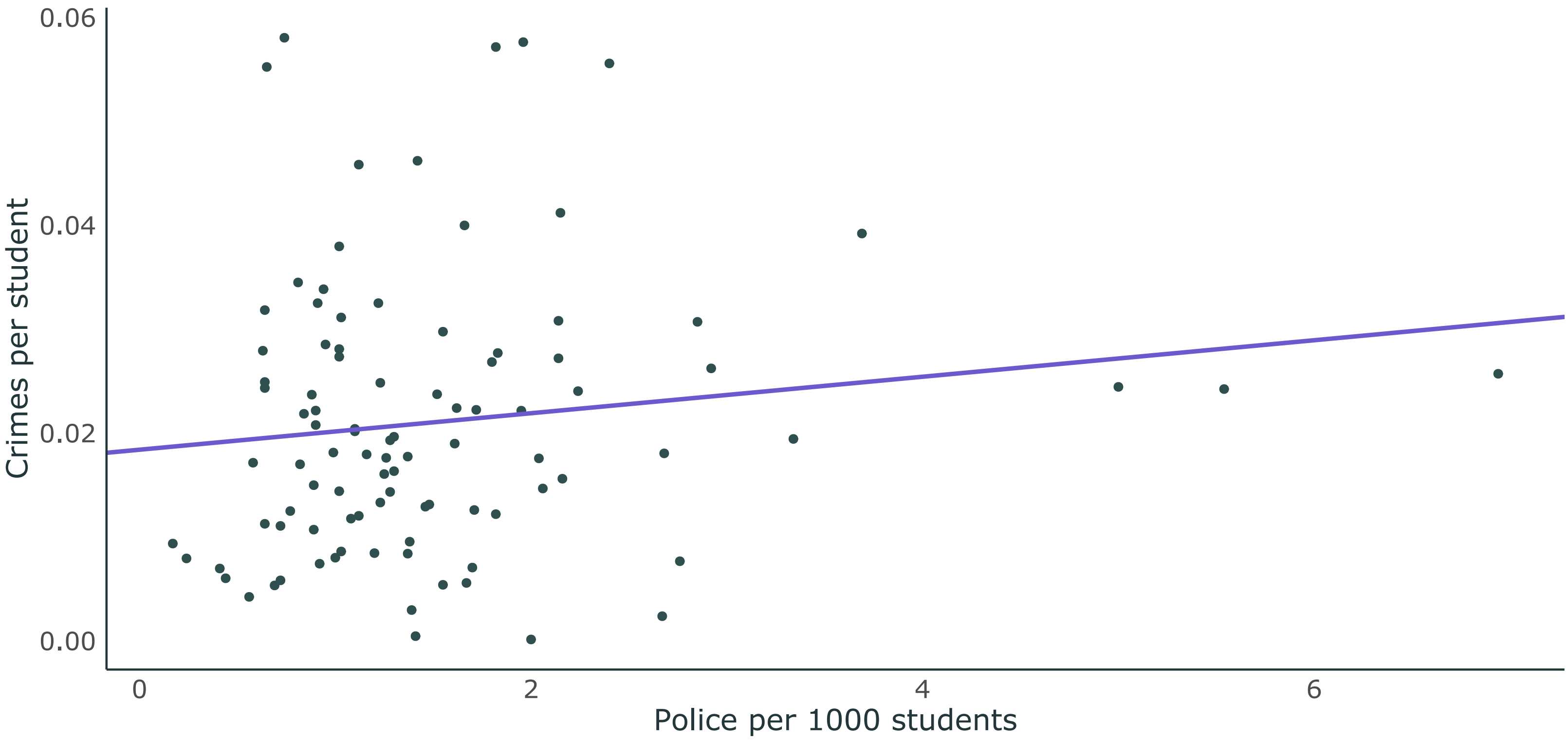
Last Time

We ran a regression of crimes per 1000 students on police per 1000 students. We found that $\hat{\beta}_1 = 18.41$ and $\hat{\beta}_2 = 1.76$.



Last Time

What if we had run a regression of crimes per student on police per 1000 students? What would happen to the slope?



$\hat{\beta}_2$ = 0.001756.

Demeaning

Practice problem

Suppose that, before running a regression of Y_i on X_i, you decided to *demean* each variable by subtracting off the mean from each observation. This gave you $\tilde{Y}_i = Y_i - \bar{Y}$ and $\tilde{X}_i = X_i - \bar{X}$.

Then you decide to estimate

$$\tilde{Y}_i = \beta_1 + \beta_2 \tilde{X}_i + u_i.$$

What will you get for your intercept estimate $\hat{\beta}_1$?