Simple Linear Regression: Inference

EC 320: Introduction to Econometrics

Winter 2022

Prologue

Housekeeping

- Lab today & Ex05 due today
- Midterm 1 solution posted
- Extra OH 7pm-8pm on Zoom

Last Time

We discussed the **classical assumptions of OLS:**

> 1. **Linearity:** The population relationship is linear in parameters with an additive error term.
> 2. **Sample Variation:** There is variation in X.
> 3. **Random Sampling:** We have a random sample from the population of interest.
> 4. **Exogeneity:** The X variable is exogenous (*i.e.*, $\mathbb{E}(u|X) = 0$).
> 5. **Homoskedasticity:** The error term has the same variance for each value of the independent variable (*i.e.*, $\mathrm{Var}(u|X) = \sigma^2$).
> 6. **Normality:** The population error term is normally distributed with mean zero and variance σ^2 (*i.e.*, $u \sim N(0, \sigma^2)$)

We restricted our attention to the first 5 assumptions.

Last Time

1. We used the first 4 assumptions to show that OLS is unbiased: $\mathbb{E}\left[\hat{\beta}\right] = \beta$

2. We used the first 5 assumptions to derive a formula for the **variance** of the OLS estimator: $\mathbf{Var}(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$.

Classical Assumptions

Today

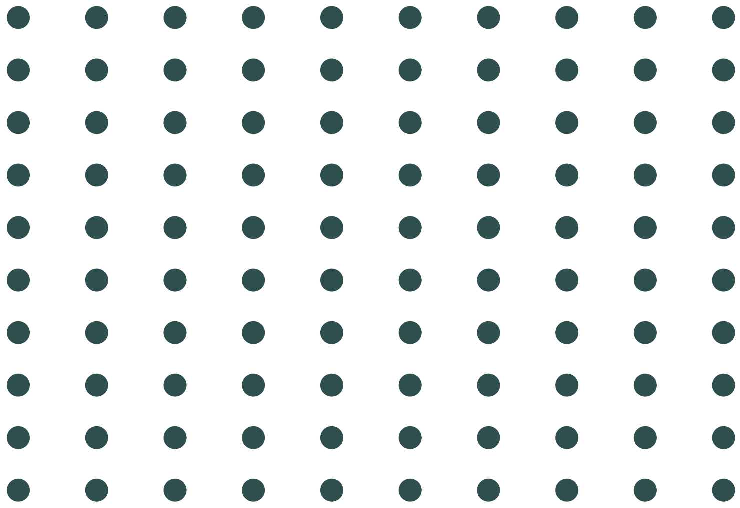
We will use the sampling distribution of $\hat{\beta}_1$ to conduct hypothesis tests.

- Can use all 6 classical assumptions to show that OLS is normally distributed:

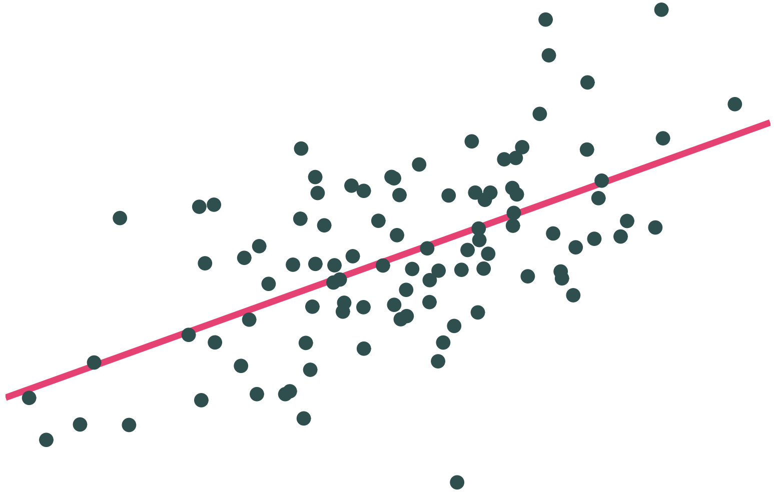
$$\hat{\beta} \sim N\left(\beta_1, \frac{\sigma^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right)$$

- We'll "prove" this using R.

Simulation



Population

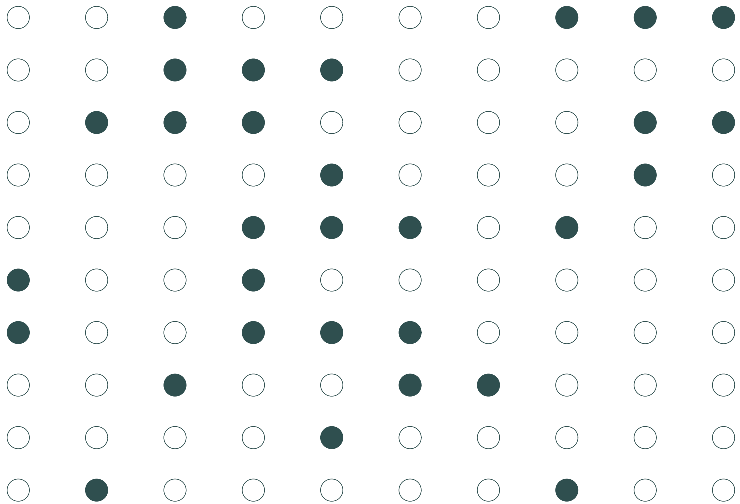


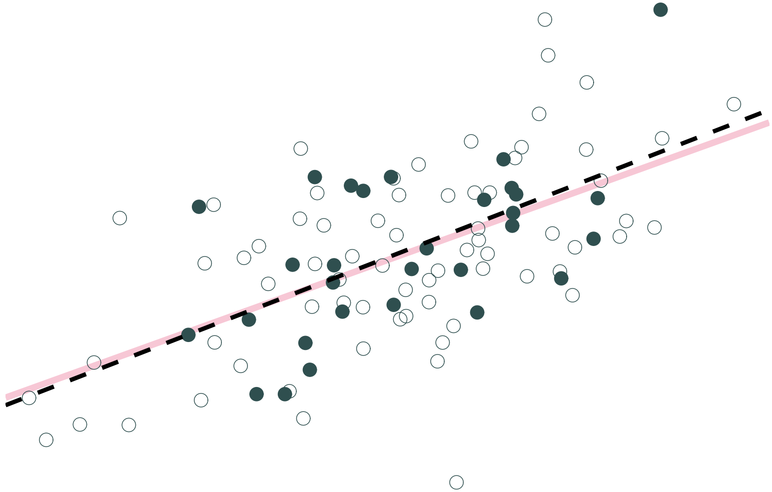
Population relationship

$$Y_i = 2.53 + 0.57X_i + u_i$$

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

Simulation





Sample 1: 30 random individuals

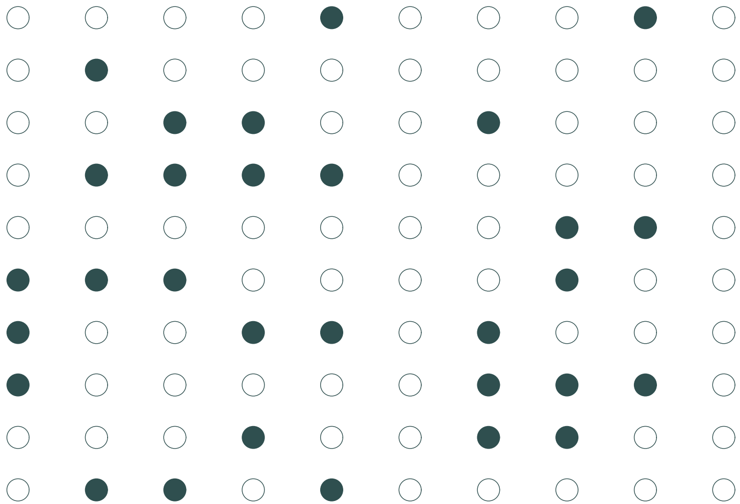
Population relationship

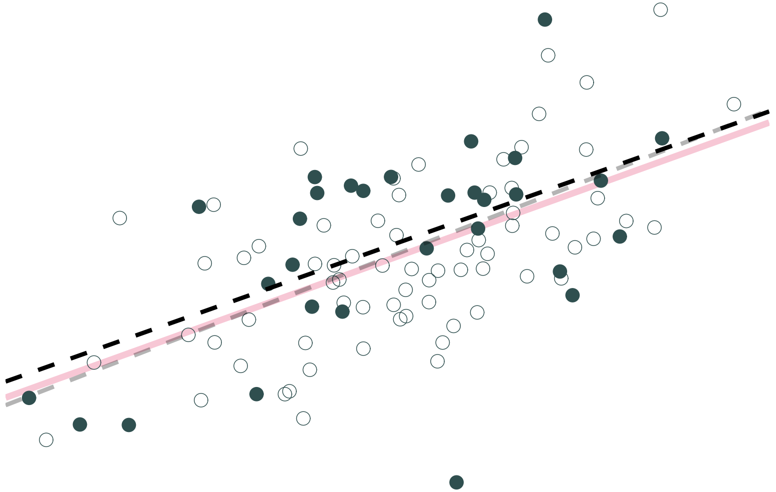
$$Y_i = 2.53 + 0.57X_i + u_i$$

Sample relationship

$$\hat{Y}_i = 2.36 + 0.61X_i$$

Simulation





Sample 2: 30 random individuals

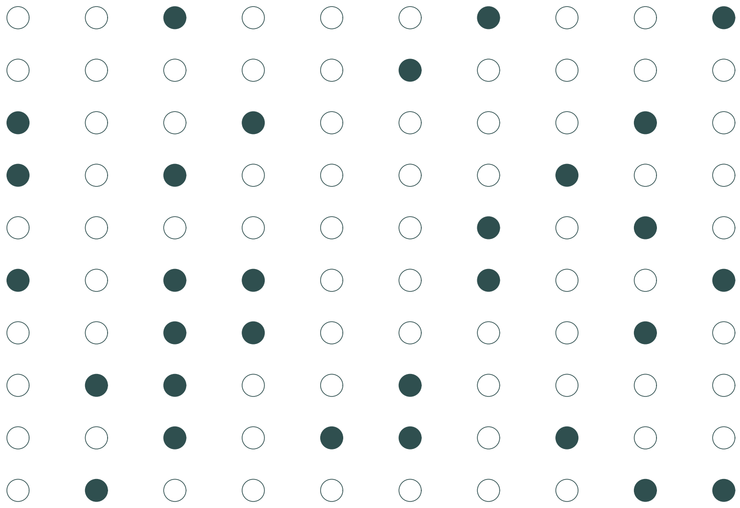
Population relationship

$$Y_i = 2.53 + 0.57 Y_i + u_i$$

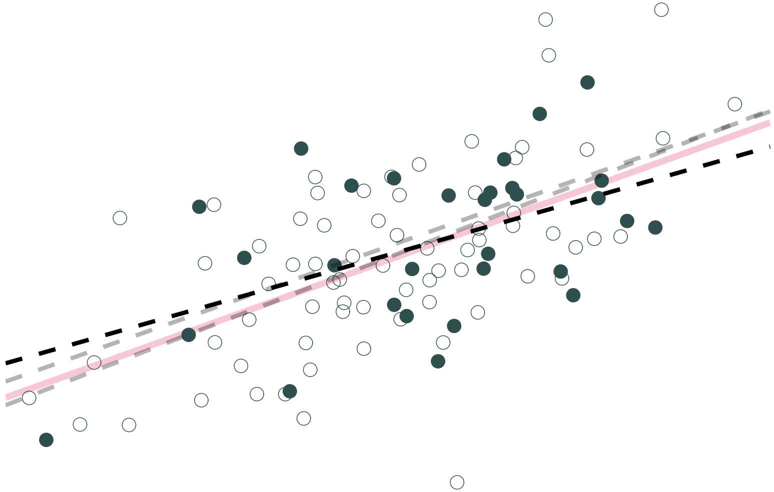
Sample relationship

$$\hat{Y}_i = 2.79 + 0.56 X_i$$

Simulation



Sample 3: 30 random individuals



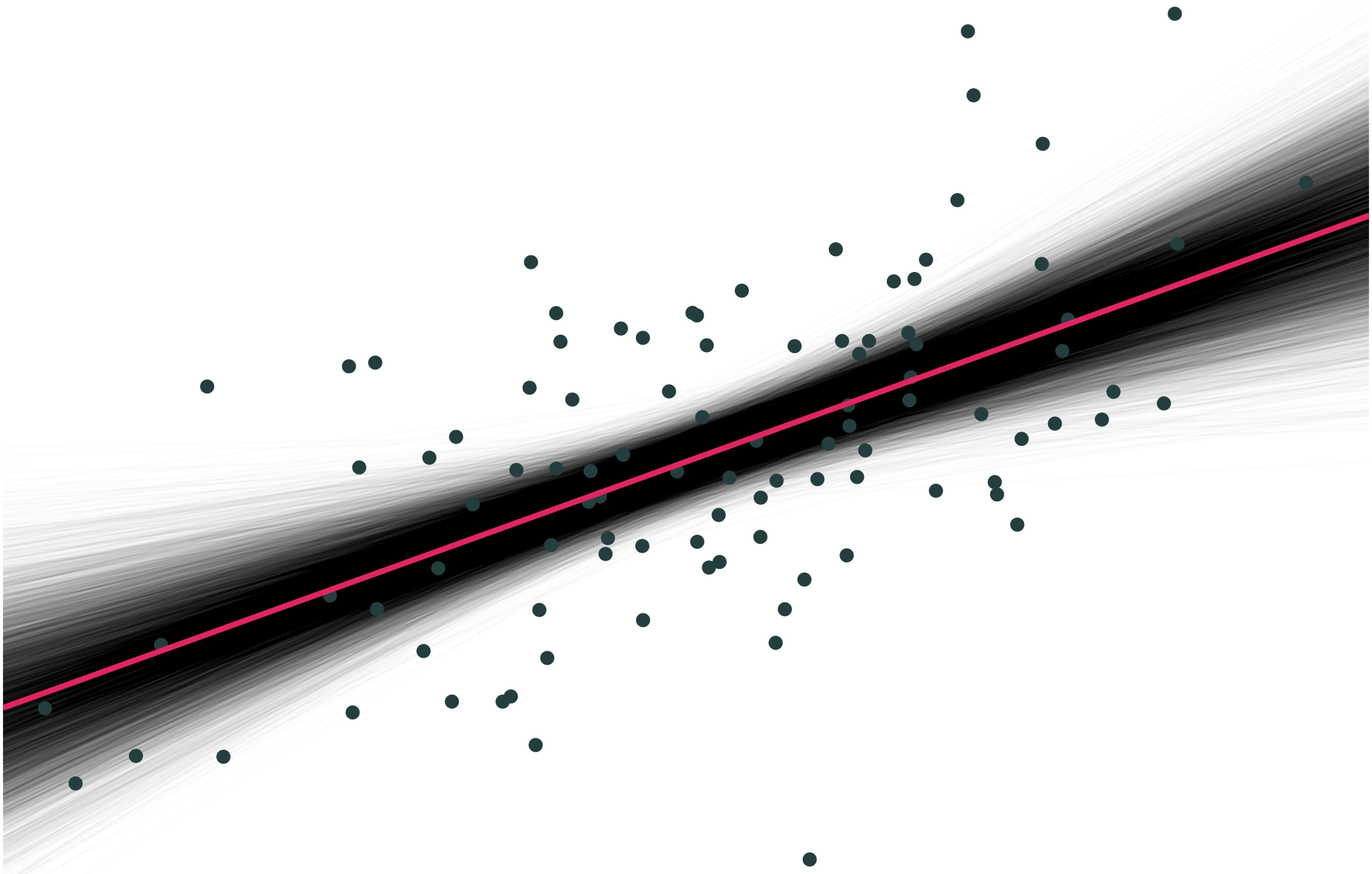
Population relationship

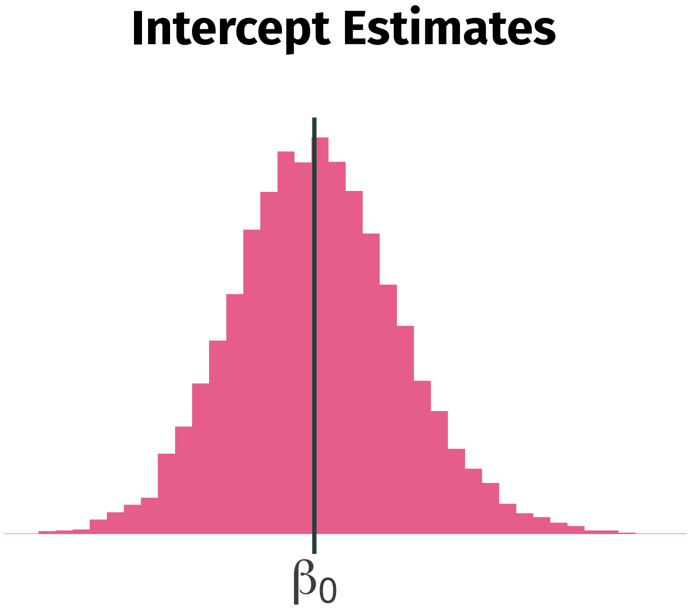
$$Y_i = 2.53 + 0.57X_i + u_i$$

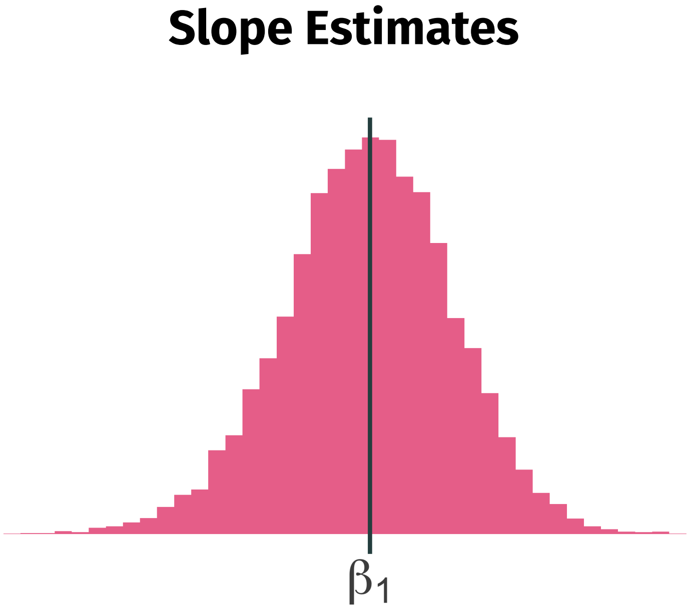
Sample relationship

$$\hat{Y}_i = 3.21 + 0.45X_i$$

Repeat **10,000 times** (Monte Carlo simulation).




Intercept Estimates
β_0


Slope Estimates
β_1

Simulation

Can you spot the classical assumptions?

```r
# Set population and sample sizes
n_p <- 100
n_s <- 30
# Generate population data
pop_df <- tibble(
  x = rnorm(n_p, mean = 5, sd = 1.5),
  e = rnorm(n_p, mean = 0, sd = 1),
  y = 2.53 + 0.57 * x + e
)
# Define simulation procedure
sim_ols <- function(x, size = n_s) {
  lm(y ~ x, data = pop_df %>% sample_n(size = size)) %>%
    tidy() %>%
    mutate(iteration = x)
}
# Run simulation
sim_df <- map_df(1:10000, ~sim_ols(.x, size = n_s))
```

Inference

Motivation

What does statistical evidence say about existing theories?

We want to test hypotheses posed by politicians, economists, scientists, people with foil hats, *etc.*

- Does building a giant wall **reduce crime**?
- Does shutting down a government **adversely affect the economy**?
- Does legal cannabis **reduce drunk driving** or **reduce opioid use**?
- Do air quality standards **improve health** or **reduce jobs**?

While uncertainty exists, we can still conduct *reliable* statistical tests (rejecting or failing to reject a hypothesis).

Inference

We know OLS has some nice properties, and we know how to estimate an intercept and slope coefficient using OLS.

Our current workflow:

- Get data (points with X and Y values).
- Regress Y on X.
- Plot the fitted values (*i.e.*, $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$) and report the estimates.

But how do we actually **learn** something from this exercise?

- Based upon our value of $\hat{\beta}_1$, can we rule out previously hypothesized values?
- How confident should we be in the precision of our estimates?

We need to be able to deal with uncertainty. Enter: **Inference.**

Inference

We use the standard error of $\hat{\beta}_1$, along with $\hat{\beta}_1$ itself, to learn about the parameter β_1.

After deriving the distribution of $\hat{\beta}_1$,† we have two (related) options for formal statistical inference (learning) about our unknown parameter β_1:

- **Hypothesis tests:** Determine whether there is statistically significant evidence to reject a hypothesized value or range of values.
- **Confidence intervals:** Use the estimate and its standard error to create an interval that, when repeated, will generally†† contain the true parameter.

† *Hint:* It's normal with mean β_1 and variance $\frac{\sigma^2}{\sum_{i=1}^n (X_i - \bar{X})^2}$.

†† *E.g.*, similarly constructed 95% confidence intervals will contain the true parameter 95% of the time.

OLS Variance

Hypothesis tests and confidence intervals require information about the variance of the OLS estimator:

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Problem

- The variance formula has a population parameter: σ^2 (a.k.a. error variance).

- We can't observe population parameters.

- **Solution:** Estimate σ^2.

Estimating Error Variance

Learning from our (prediction) errors

We can estimate the variance of u_i (a.k.a. σ^2) using the sum of squared residuals:

$$s_u^2 = \frac{\sum_i \hat{u}_i^2}{n - k}$$

where k gives the number of regression parameters.

- In a simple linear regression, $k = 2$.
- s_u^2 is an unbiased estimator of σ^2.

OLS Variance, Take 2

With $s_u^2 = \dfrac{\sum_i \hat{u}_i^2}{n-k}$, we can calculate

$$\text{Var}(\hat{\beta}_1) = \frac{s_u^2}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Taking the square root, we get the **standard error** of the OLS estimator:

$$\hat{\text{SE}}\left(\hat{\beta}_1\right) = \sqrt{\frac{s_u^2}{\sum_{i=1}^n (X_i - \bar{X})^2}}.$$

- Standard error = standard deviation of an estimator.

Standard Errors

R's `lm()` function estimates standard errors out of the box:

```
tidy(lm(y ~ x, pop_df))
```

```
#> # A tibble: 2 × 5
#>   term        estimate std.error statistic  p.value
#>   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
#> 1 (Intercept)    2.53     0.422       6.00 3.38e- 8
#> 2 x              0.567    0.0793      7.15 1.59e-10
```

I won't ask you to estimate standard errors by hand!

Hypothesis Tests

Hypothesis Tests

Null hypothesis (H_0): $\beta_1 = 0$

Alternative hypothesis (H_a): $\beta_1 \neq 0$

There are four possible outcomes of our test:

1. We **fail to reject** the null hypothesis and the null is true.
2. We **reject** the null hypothesis and the null is false.
3. We **reject** the null hypothesis, but the null is actually true (**Type I error**).
4. We **fail to reject** the null hypothesis, but the null is actually false (**Type II error**).

Hypothesis Tests

Goal: Make a statement about β_1 using information on $\hat{\beta}_1$.

$\hat{\beta}_1$ is random: it could be anything, even if $\beta_1 = \mathbf{0}$ is true.

- But if $\beta_1 = \mathbf{0}$ is true, then $\hat{\beta}_1$ is unlikely to take values far from zero.
- As the standard error shrinks, we are even less likely to observe "extreme" values of $\hat{\beta}_1$ (assuming $\beta_1 = \mathbf{0}$).

Our test should take extreme values of $\hat{\beta}_1$ as evidence against the null hypothesis, but it should also weight them by what we know about the variance of $\hat{\beta}_1$.

Hypothesis Tests

Null hypothesis

H_0: $\beta_1 = 0$

Alternative hypothesis

H_a: $\beta_1 \neq 0$

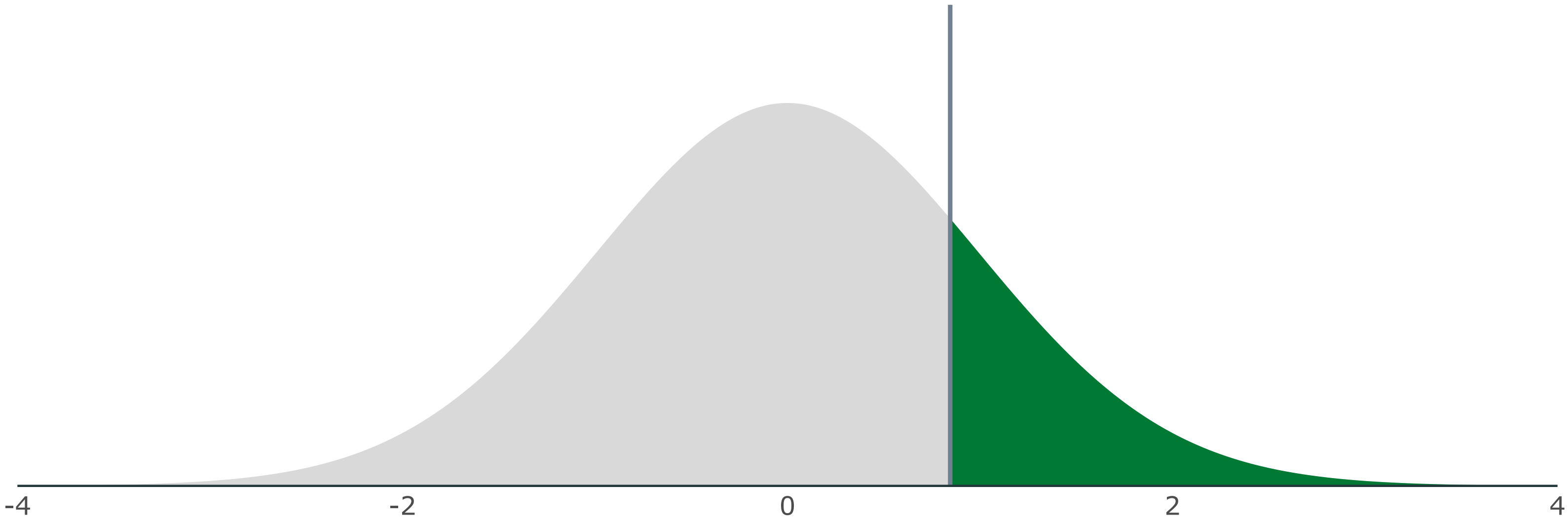
To conduct the test, we calculate a t-statistic:

$$t = \frac{\hat{\beta}_1 - \beta_1^0}{\hat{\text{SE}}\left(\hat{\beta}_1\right)}$$

- Distributed according to a t-distribution with $n - 2$ *degrees of freedom*.
- β_1^0 is the value of β_1 in our null hypothesis (*e.g.*, $\beta_1^0 = 0$).

Hypothesis Tests

Next, we use the t-**statistic** to calculate a p-**value**.



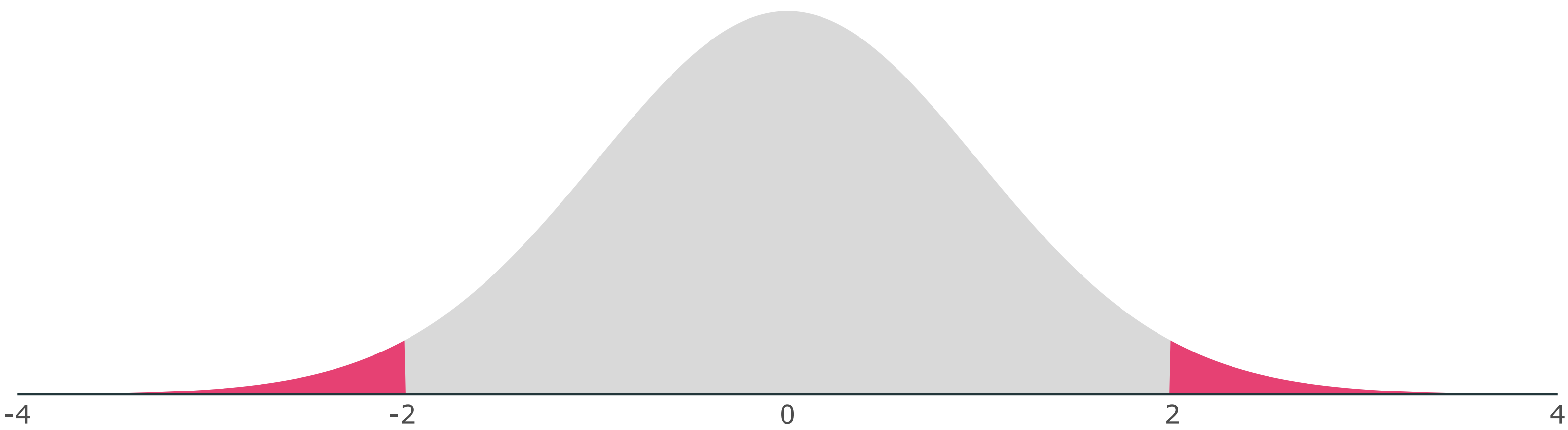
Describes the probability of seeing a t-statistic as extreme as the one we observe *if the null hypothesis is actually true.*

But...we still need some benchmark to compare our p-value against.

Hypothesis Tests

We worry mostly about false positives, so we conduct hypothesis tests based on the probability of making a Type I error.

How? We select a **significance level** α that specifies our tolerance for false positives. This is the probability of Type I error we choose to live with.



Hypothesis Tests

We then compare α to the p-value of our test.

- If the p-value is less than α, then we **reject the null hypothesis** at the $\alpha \cdot 100$ percent level.
- If the p-value is greater than α, then we **fail to reject the null hypothesis**.
- **Note:** *Fail to reject* $\neq$ *accept.*

Hypothesis Tests

Example: Are campus police associated with campus crime?

```
lm(crime ~ police, data = campus) %>% tidy()
```

```
#> # A tibble: 2 × 5
#>   term        estimate std.error statistic  p.value
#>   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
#> 1 (Intercept)    18.4       2.38      7.75 1.06e-11
#> 2 police          1.76      1.30      1.35 1.81e- 1
```

H_0: $\beta_{\text{Police}} = 0$ *v.s.* H_a: $\beta_{\text{Police}} \neq 0$

Significance level: $\alpha = 0.05$ (*i.e.*, 5 percent test)

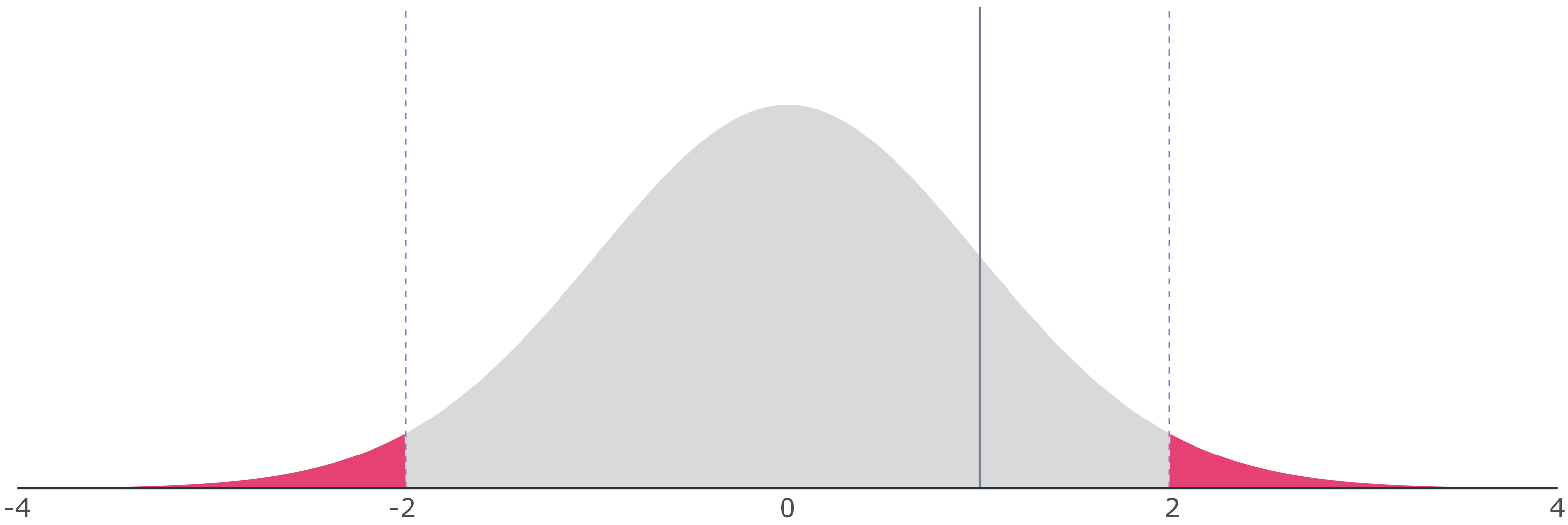
Test Condition: Reject H_0 if $p < \alpha$

$p = 0.18$. **Do we reject the null hypothesis?**

Hypothesis Tests

p-values are difficult to calculate by hand.

Alternative: Compare t**-statistic** to **critical values** from the t-distribution.



Hypothesis Tests

Notation: $t_{1-\alpha/2,n-2}$ or t_{crit}.

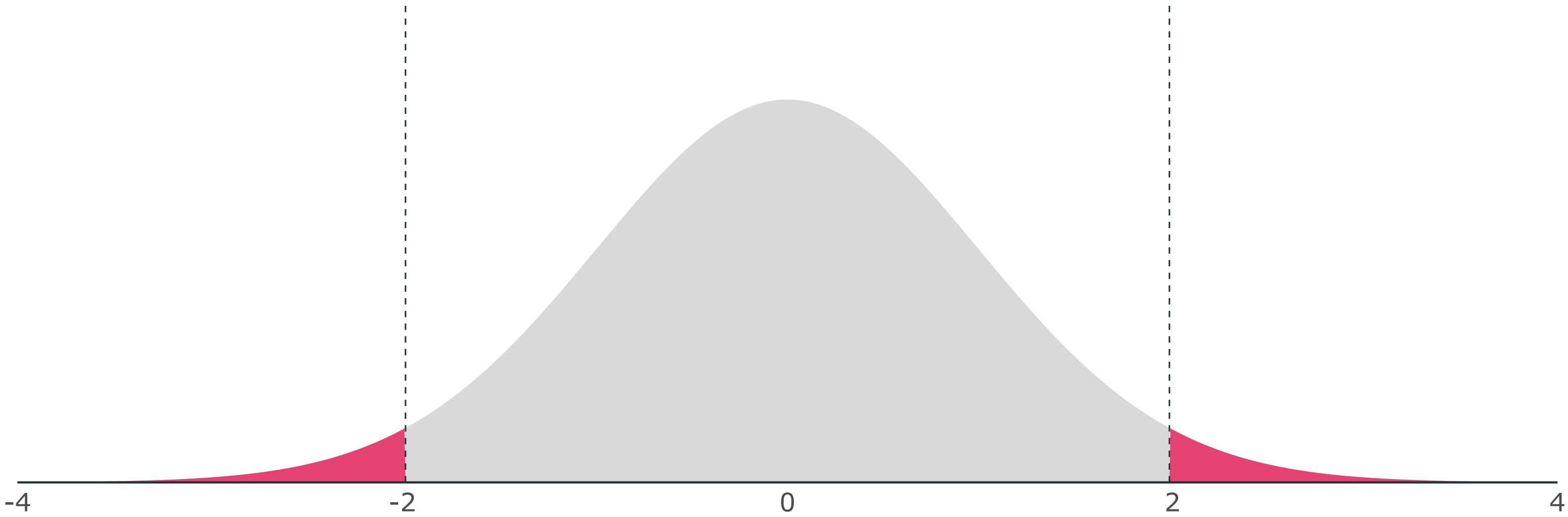
- Find in a t table using the significance level α and $n-2$ degrees of freedom.

Compare the the critical value to your t-statistic:

- If $|t| > |t_{1-\alpha/2,n-2}|$, then **reject the null**.
- If $|t| < |t_{1-\alpha/2,n-2}|$, then **fail to reject the null**.

Two-Sided Tests

Based on a critical value of $t_{1-\alpha/2,n-2} = t_{0.975,100} = 1.98$, we can identify a **rejection region** on the t-distribution.



If our t statistic is in the rejection region, then we reject the null hypothesis at the 5 percent level.

Two-Sided Tests

R defaults to testing hypotheses against the null hypothesis of zero.

```
lm(y ~ x, data = pop_df) %>% tidy()
```

```
#> # A tibble: 2 × 5
#>   term        estimate std.error statistic  p.value
#>   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
#> 1 (Intercept)    2.53     0.422       6.00 3.38e- 8
#> 2 x              0.567    0.0793      7.15 1.59e-10
```

H_0: $\beta_1 = 0$ *vs.* H_a: $\beta_1 \neq 0$

Significance level: $\alpha = 0.05$ (*i.e.*, 5 percent test)

$t_\text{stat} = 7.15$ and $t_{0.975,\,28} = 2.05$, which implies that $p < 0.05$.

Therefore, we **reject H_0** at the 5% level.

Two-Sided Tests

Example: Are campus police associated with campus crime?

```
lm(crime ~ police, data = campus) %>% tidy()
```

```
#> # A tibble: 2 × 5
#>   term        estimate std.error statistic  p.value
#>   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
#> 1 (Intercept)    18.4       2.38      7.75 1.06e-11
#> 2 police          1.76      1.30      1.35 1.81e- 1
```

H$_0$: $\beta_{\text{Police}} = 0$ *v.s.* H$_\text{a}$: $\beta_{\text{Police}} \neq 0$

Significance level: $\alpha = 0.1$ (*i.e.*, 10 percent test)

Test Condition: Reject H$_0$ if $|t| > t_{\text{crit}}$

$t = 1.35$ and $t_{\text{crit}} = 1.66$. **Do we reject the null hypothesis?**

One-Sided Tests

Sometimes we are confident that a parameter is non-negative or non-positive.

A **one-sided** test assumes that values on one side of the null hypothesis are impossible.

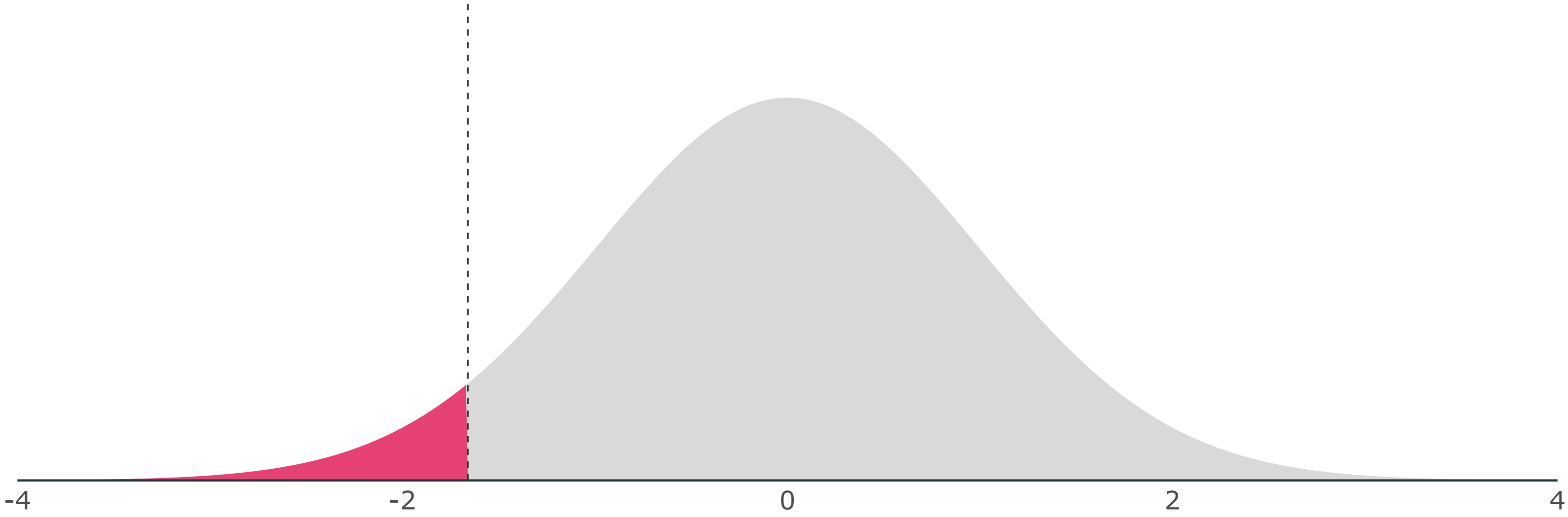
- **Option 1:** H_0: $\beta_1 = 0$ *vs.* H_a: $\beta_1 > 0$
- **Option 2:** H_0: $\beta_1 = 0$ *vs.* H_a: $\beta_1 < 0$

If this assumption is reasonable, then our rejection region changes.

- Same α.

One-Sided Tests

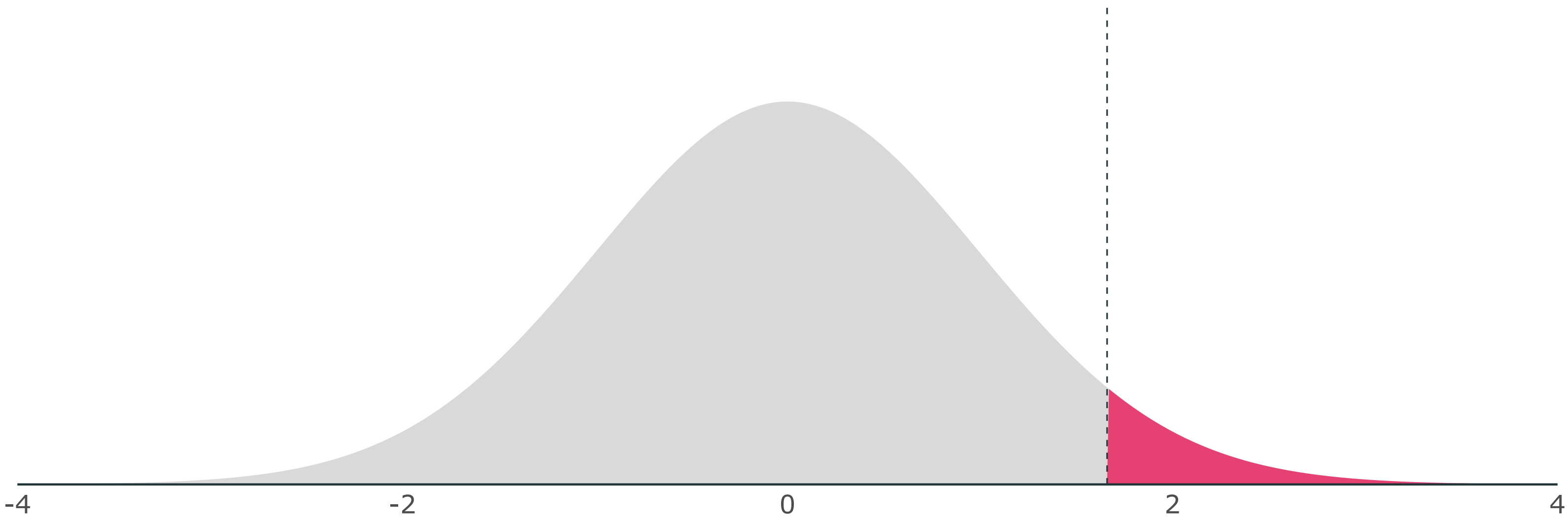
Left-tailed: Based on a critical value of $t_{1-\alpha,n-2} = t_{0.95,100} = 1.66$, we can identify a **rejection region** on the t-distribution.



If our t statistic is in the rejection region, then we reject the null hypothesis at the 5 percent level.

One-Sided Tests

Right-tailed: Based on a critical value of $t_{1-\alpha,n-2} = t_{0.95,100} = 1.66$, we can identify a **rejection region** on the t-distribution.



If our t statistic is in the rejection region, then we reject the null hypothesis at the 5 percent level.

One-Sided Tests

Example: Do campus police deter campus crime?

```
lm(crime ~ police, data = campus) %>% tidy()
```

```
#> # A tibble: 2 × 5
#>   term        estimate std.error statistic  p.value
#>   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
#> 1 (Intercept)    18.4       2.38      7.75 1.06e-11
#> 2 police          1.76      1.30      1.35 1.81e- 1
```

H$_0$: $\beta_{\text{Police}} = 0$ *v.s.* H$_a$: $\beta_{\text{Police}} < 0$

Significance level: $\alpha = 0.1$ (*i.e.*, 10 percent test)

Test Condition: Reject H$_0$ if $t < -t_{\text{crit}}$

$t = 1.35$ and $t_{\text{crit}} = 1.29$. **Do we reject the null hypothesis?**

Confidence Intervals

Confidence Intervals

Until now, we have considered **point estimates** of population parameters.

- Sometimes a range of values is more interesting/honest.

We can construct $(1-\alpha)\cdot 100$-percent level confidence intervals for β_1

$$\hat{\beta}_1 \pm t_{1-\alpha/2,n-2}\ \hat{\mathrm{SE}}\left(\hat{\beta}_1\right)$$

$t_{1-\alpha/2,n-2}$ denotes the $1-\alpha/2$ quantile of a t distribution with $n-2$ degrees of freedom.

Confidence Intervals

Q: Where does the confidence interval formula come from?

A: The confidence interval formula comes from the rejection condition of a two-sided test.

> Reject H_0 if $|t| > t_{\text{crit}}$

The test condition implies

> Fail to reject H_0 if $|t| \leq t_{\text{crit}}$

which is equivalent to

> Fail to reject H_0 if $-t_{\text{crit}} \leq t \leq t_{\text{crit}}$.

Confidence Intervals

Replacing t with its formula gives

> Fail to reject H_0 if $-t_{\text{crit}} \leq \frac{\hat{\beta}_1 - \beta_1^0}{\hat{\text{SE}}\left(\hat{\beta}_1\right)} \leq t_{\text{crit}}$.

Standard errors are always positive, so the inequalities do not flip when we multiply by $\hat{\text{SE}}\left(\hat{\beta}_1\right)$:

> Fail to reject H_0 if $-t_{\text{crit}}\, \hat{\text{SE}}\left(\hat{\beta}_1\right) \leq \hat{\beta}_1 - \beta_1^0 \leq t_{\text{crit}}\, \hat{\text{SE}}\left(\hat{\beta}_1\right)$.

Subtracting $\hat{\beta}_1$ yields

> Fail to reject H_0 if
> $-\hat{\beta}_1 - t_{\text{crit}}\, \hat{\text{SE}}\left(\hat{\beta}_1\right) \leq -\beta_1^0 \leq -\hat{\beta}_1 + t_{\text{crit}}\, \hat{\text{SE}}\left(\hat{\beta}_1\right)$.

Confidence Intervals

Multiplying by -1 and rearranging gives

> Fail to reject H_0 if
> $\hat{\beta}_1 - t_{\text{crit}}\,\hat{\text{SE}}\left(\hat{\beta}_1\right) \leq \beta_1^0 \leq \hat{\beta}_1 + t_{\text{crit}}\,\hat{\text{SE}}\left(\hat{\beta}_1\right).$

Replacing β_1^0 with β_1 and dropping the test condition yields the interval

$$\hat{\beta}_1 - t_{\text{crit}}\,\hat{\text{SE}}\left(\hat{\beta}_1\right) \leq \beta_1 \leq \hat{\beta}_1 + t_{\text{crit}}\,\hat{\text{SE}}\left(\hat{\beta}_1\right)$$

which is equivalent to

$$\hat{\beta}_1 \pm t_{\text{crit}}\;\hat{\text{SE}}\left(\hat{\beta}_1\right).$$

Confidence Intervals

Insight: A confidence interval is related to a two-sided hypothesis test.

- If a 95 percent confidence interval contains zero, then we fail to reject the null hypothesis at the 5 percent level.
- If a 95 percent confidence interval does not contain zero, then we reject the null hypothesis at the 5 percent level.
- **Generally:** A $(1-\alpha)\cdot 100$ percent confidence interval embeds a two-sided test at the $\alpha \cdot 100$ level.

Confidence Intervals

Example

```
lm(y ~ x, data = pop_df) %>% tidy()
```

```
#> # A tibble: 2 × 5
#>   term        estimate std.error statistic  p.value
#>   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
#> 1 (Intercept)    2.53     0.422       6.00 3.38e- 8
#> 2 x              0.567    0.0793      7.15 1.59e-10
```

```
# find degrees of freedom
dof <- summary(lm(y ~ x, data = pop_df))$df[2]
# return critical value
qt(0.975, dof)
```

```
#> [1] 1.984467
```

95% confidence interval for β_1 is $0.567 \pm 1.98 \times 0.0793 = [0.410,\ 0.724]$

Confidence Intervals

We have a confidence interval for β_1, *i.e.*, $[0.410,\ 0.724]$.

What does it mean?

Informally: The confidence interval gives us a region (interval) in which we can place some trust (confidence) for containing the parameter.

More formally: If we repeatedly sample from our population and construct confidence intervals for each of these samples, then $(1 - \alpha) \cdot 100$ percent of our intervals (*e.g.*, 95%) will contain the population parameter *somewhere in the interval*.

Now back to our simulation...

Confidence Intervals

We drew 10,000 samples (each of size $n = 30$) from our population and estimated our regression model for each sample:

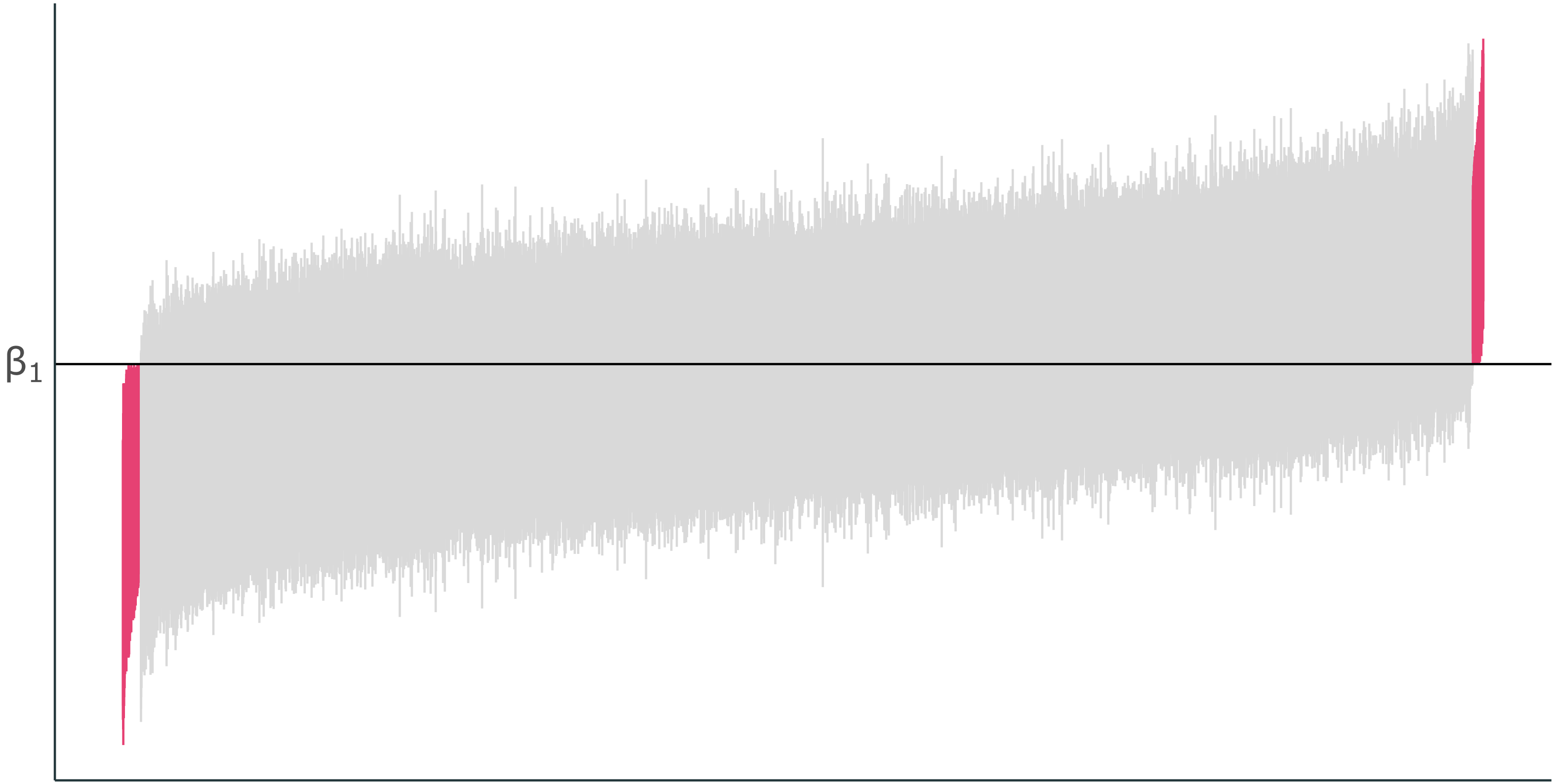
$$Y_i = \hat{\beta}_1 + \hat{\beta}_1 X_i + \hat{u}_i$$

(repeated 10,000 times)

Now, let's estimate 95% confidence intervals for each of these intervals…

Confidence Intervals

From our previous simulation: 97.9% of 95% confidence intervals contain the true parameter value of β_1.



Confidence Intervals

Example: Association of police with crime

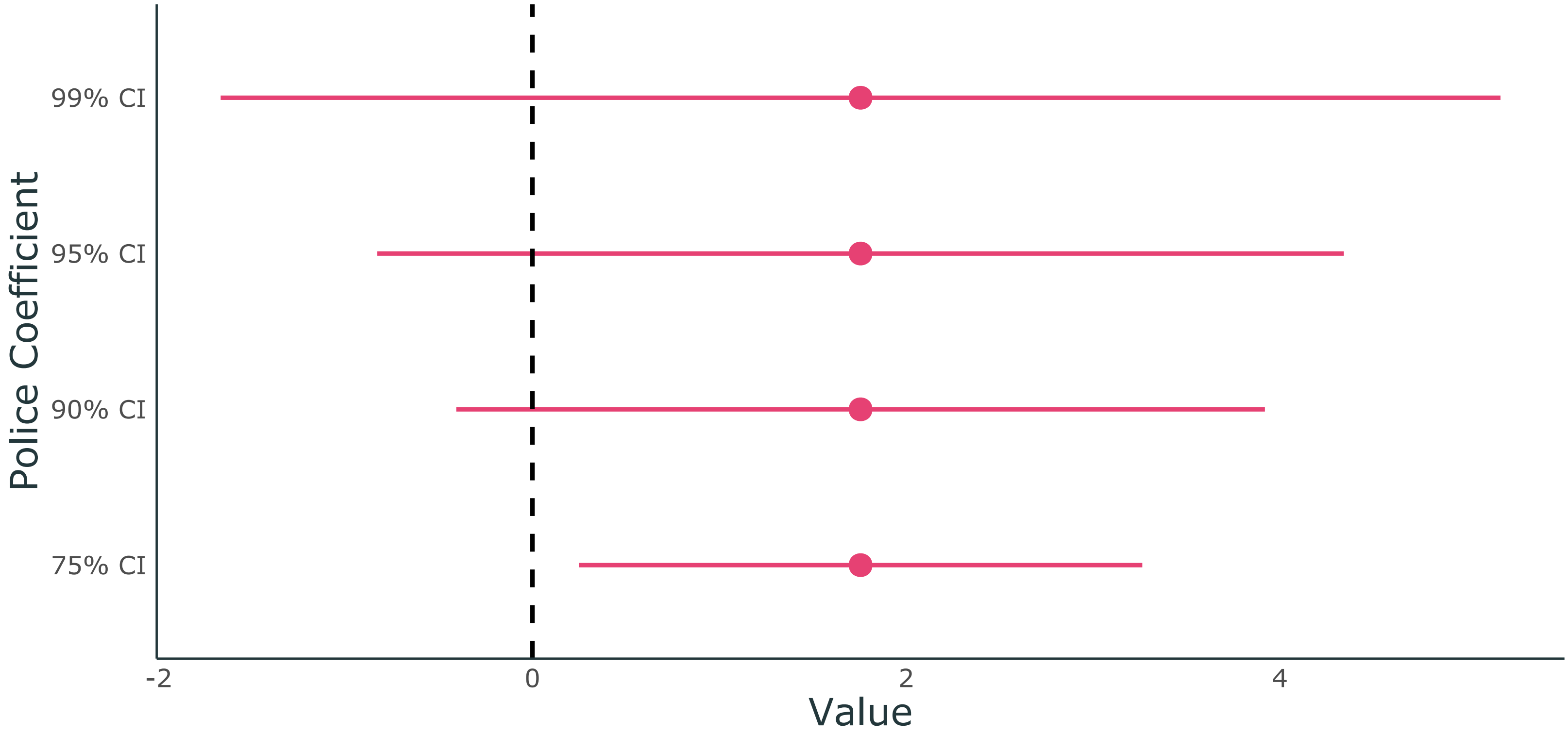
You can instruct `tidy` to return a 95 percent confidence interval for the association of campus police with campus crime:

```
lm(crime ~ police, data = campus) %>% tidy(conf.int = TRUE, conf.level = 0.95)
```

```
#> # A tibble: 2 × 7
#>   term        estimate std.error statistic  p.value conf.low conf.high
#>   <chr>          <dbl>     <dbl>     <dbl>    <dbl>    <dbl>     <dbl>
#> 1 (Intercept)    18.4       2.38      7.75 1.06e-11   13.7       23.1
#> 2 police          1.76      1.30      1.35 1.81e- 1   -0.830      4.34
```

Confidence Intervals

Example: Association of police with crime



Four confidence intervals for the same coefficient.