数学建模

浙江大学数学系　谈之奕

tanzy@zju.edu.cn

基本数学模型

传染病模型

传染病

浙江大学 ZheJiang University

数学建模

- 传染病（infectious deseases）是由各种病原体引起的，能在人与人、动物与动物、人与动物之间互相传播的一种疾病
- 传染病得以在某一人群中发生和传播，必须具备传染源、传播途径和易感人群三个基本环节
- 流行病学（epidemiology）是研究疾病在人群中发生、发展和分布规律的科学
- 地方病（endemic diseases）指局限在某些地方发生的疾病

传染病模型

浙江大学 ZheJiang University

数学建模

- 自1927年起，英国数学家、流行病学家**William Ogilvy Kermack**（**1898–1970**）和**Anderson Gray McKendrick**（**1876–1943**）在***Proceedings of the Royal Society of London A***上先后发表三篇论文，提出了揭示传染病传播规律的仓室（**compartment**）模型
  - **A Contribution to the Mathematical Theory of Epidemics, 115, 700-721, 1927**
  - **II. The Problem of Endemicity, 138, 55-83, 1932**
  - **III. Further Studies of the Problem of Endemicity, 141, 94-122, 1933**

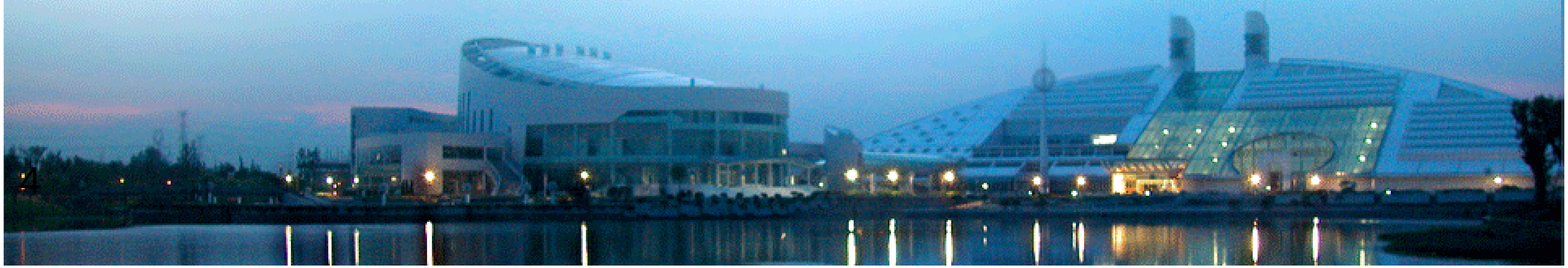
创刊于**1800**年，**1854**年改用现名。**Maxwell**电磁场理论，**DNA**双螺旋结构等重要论文均发表在该刊上

基本假设

浙江大学 ZheJiang University

数学建模

- 人群分类
  - 易感者（**Susceptible**）：易受疾病感染但尚未发病
  - 感染者（**Infective**）：已感染且具传染性
- 在疾病传播期内所考察地区总人数 N 保持不变
- t 时刻易感者和感染者人数所占比例分别为 $S(t)$ 和 $I(t)$，$S(t)+I(t)=1$
- 每个感染者单位时间内可使数量为 βN 的人受到感染，其中易感者数量为 βS，β 称为有效接触率

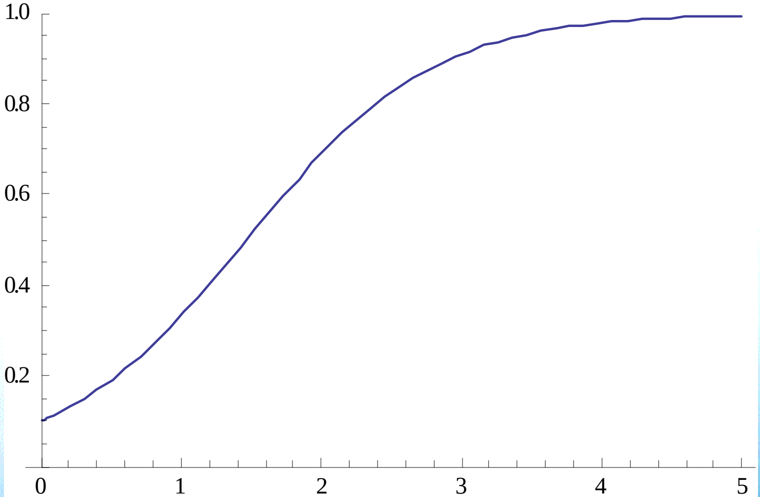


SI模型

$$N\frac{dI}{dt}=\beta NSI \Rightarrow \frac{dI}{dt}=\beta I(1-I) \Rightarrow \frac{1}{I(1-I)}dI=\beta dt$$

$$S(t)+I(t)=1 \qquad I(0)=I_0$$

$$I(t)=\frac{1}{1+\left(\frac{1}{I_0}-1\right)e^{-\beta t}}$$



模型分析

数学建模

- $I(t)$单调增加，$\lim_{t\to\infty} I(t)=1$，所有人最终都将被感染
- 当 $I(t)=\frac{1}{2}$，即 $t_m=\frac{1}{\beta}\ln\left(\frac{1}{I_0}-1\right)$ 时，$\frac{dI}{dt}$ 达到最大值，该时刻为感染者增加最快时刻，t_m与 β 成反比
- 传染病防治水平越高，β越小，传染病高峰达到时刻越晚

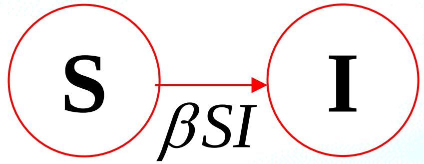
$$\boxed{\frac{dI}{dt}=\beta I(1-I)}$$

$$\frac{d^2I}{dt^2}=\beta(1-2I)\frac{dI}{dt}$$

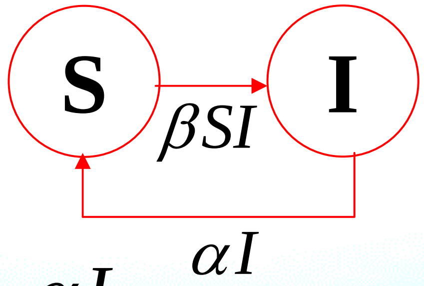
$$\frac{dI}{dt}=\frac{\beta e^{-\beta t}\left(\frac{1}{I_0}-1\right)}{\left(1+\left(\frac{1}{I_0}-1\right)e^{-\beta t}\right)^2}$$

治愈

- 感染者经治疗后可以痊愈，重新成为易感者



- 单位时间内被治愈的感染者占总数的比例为 α



$$N\frac{dI}{dt} = \beta NSI - \alpha NI \Rightarrow \frac{dI}{dt} = \beta I(1-I) - \alpha I$$

$$\sigma = \frac{\beta}{\alpha} \qquad \Rightarrow \frac{dI}{dt} = -\beta I\left(I - \left(1 - \frac{1}{\sigma}\right)\right)$$

浙江大学
ZheJiang University
数学建模

SIS模型

$$\begin{cases} \dfrac{dI}{dt} = \beta I(1-I) - \alpha I \\ I(0) = I_0 \end{cases}$$

$$\left(\frac{\beta}{(\beta-\alpha)-\beta I} + \frac{1}{I}\right) dI = (\beta-\alpha)dt$$

$$I(t) = \begin{cases} \left(\left(\dfrac{1}{I_0} - \dfrac{1}{1-\dfrac{1}{\sigma}}\right) e^{-(\beta-\alpha)t} + \dfrac{1}{1-\dfrac{1}{\sigma}}\right)^{-1}, & \beta \neq \alpha \\ \dfrac{I_0}{\beta t I_0 + 1}, & \beta = \alpha \end{cases}$$

$$\sigma = \frac{\beta}{\alpha}$$

$$\lim_{t\to\infty} I(t) = \begin{cases} 1-\dfrac{1}{\sigma} & \sigma > 1 \\ 0 & \sigma \le 1 \end{cases}$$

SIS模型

$$\frac{dI}{dt} = -\beta I\left(I - \left(1 - \frac{1}{\sigma}\right)\right)$$

- $\sigma > 1$时
  - $I_0 < 1 - \frac{1}{\sigma}$ 时, $I(t)$单调递增
  - $I_0 > 1 - \frac{1}{\sigma}$ 时, $I(t)$单调递减
- $\sigma \le 1$ 时, $\frac{dI}{dt} \le 0$, $I(t)$单调递减
- 为遏制传染病，必须减小σ值
  - 增大 α 值，提高医疗水平，加快治愈进程
  - 减少 β 值，提高防治水平，控制传播途径

$\sigma \le 1$

$\beta > \alpha$

移出

数学建模

- 某些传染病治愈后会产生免疫力，形成既非易感者，也非感染者的群体，称为移出者（**Removed**）
  - 移出的原因包括发病后隔离、死亡等
- 记 t 时刻移出者所占比例为 $R(t)$

$$N\frac{dR}{dt}=\alpha NI$$

$$\mathbf{S} \xrightarrow{\beta SI} \mathbf{I} \xrightarrow{\alpha I} \mathbf{R}$$

SIR模型

$$
\begin{cases}
\dfrac{dS}{dt} = -\beta SI \\
\dfrac{dI}{dt} = \beta SI - \alpha I \\
\dfrac{dR}{dt} = \alpha I
\end{cases}
$$

$$S(t) + I(t) + R(t) = 1$$

$$
\begin{cases}
\dfrac{dS}{dt} = -\beta SI \\
\dfrac{dI}{dt} = (\beta S - \alpha) I
\end{cases}
$$

$$S(0) = S_0, I(0) = I_0, R(0) = R_0$$

一阶非线性常微分方程组

定性理论

数学建模

- 微分方程解的分析方法
  - 求方程的解析解
  - 利用数值方法求方程的数值解
  - 对解的性状作定性分析
- **19世纪末，Poincaré和Lyapunov分别创立了常微分方程定性理论和稳定性理论**

Jules Henri Poincaré（1854－1912）
法国数学家、物理学家

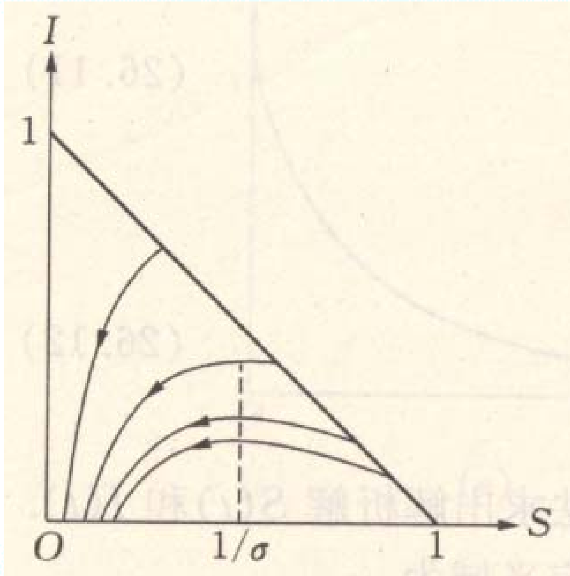
Aleksandr Mikhailovich Lyapunov（1857－1918）
苏联数学家、物理学家

自治系统

- 记 $\mathbf{x}=(x_1,x_2)^{\mathrm{T}}$，$\mathbf{F}(t,\mathbf{x})=(f_1(t,\mathbf{x}),f_2(t,\mathbf{x}))^{\mathrm{T}}$，一阶常微分方程组 $\dfrac{d\mathbf{x}}{dt}=\mathbf{F}(t,\mathbf{x})$ 称为自治（**autonomous**）的，若 $\mathbf{F}(t,\mathbf{x})$ 不显含 t
  - $\mathbf{x}$ 所属空间 $\mathbb{R}^2$ 称为相空间
  - 方程组的解在平面上对应的曲线称为相轨线，并按 t 增加的方向定义正向
  - 全体轨线所构成的图称为相图
  - 满足 $\mathbf{F}(\mathbf{x})=\mathbf{0}$ 的点称为奇点

SIR模型

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI \\ \dfrac{dI}{dt} = (\beta S - \alpha) I \end{cases} \Rightarrow \frac{dI}{dS} = \frac{1}{\sigma S} - 1$$



$$D = \{(S, I) \mid S \ge 0, I \ge 0, S + I \le 1\}$$

$$I(t) = S_0 + I_0 - S(t) + \frac{1}{\sigma} \ln \frac{S(t)}{S_0}$$

SIR模型

数学建模

- $\lim_{t\to\infty} I(t) = 0$，即不论初始条件为何，感染者终将消失
  - 由 $S(t) \ge 0, \dfrac{dS}{dt} \le 0, \lim_{t\to\infty} S(t)$ 存在，记为 S_∞
  - 由 $R(t) \le 1, \dfrac{dR}{dt} \ge 0, \lim_{t\to\infty} R(t)$ 存在
  - 由 $S(t)+I(t)+R(t) \equiv 1, \lim_{t\to\infty} I(t)$ 存在，记为 I_∞
  - 若 $I_\infty = \varepsilon > 0$，则对充分大的 t，$\dfrac{dR}{dt} > \alpha\dfrac{\varepsilon}{2}, \lim_{t\to\infty} R(t) = \infty$

    矛盾

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI \\ \dfrac{dI}{dt} = \beta SI - \alpha I \\ \dfrac{dR}{dt} = \alpha I \end{cases}$$

SIR模型

数学建模

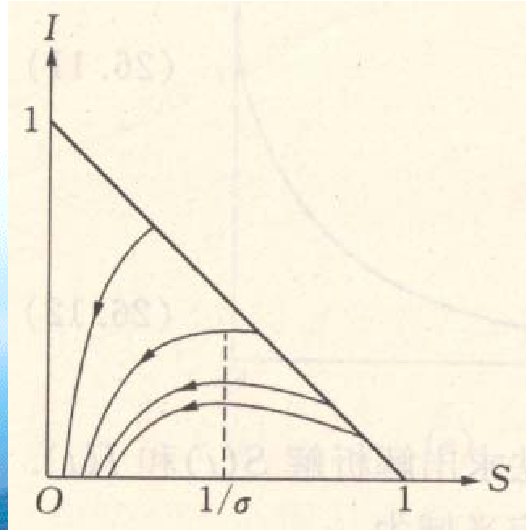
- 若 $S_0 > \frac{1}{\sigma}$ ，由 $\frac{dI}{dt} = \beta(S - \frac{1}{\sigma})I$
  - $\frac{1}{\sigma} < S(t) < S_0$ 时，$I(t)$ 单调增加，在 $S(t) = \frac{1}{\sigma}$ 时达到最大值

$$S_0 + I_0 - \frac{1}{\sigma}(1 + \ln \sigma S_0)$$

  - $S(t) < \frac{1}{\sigma}$ 时，$I(t)$ 单调减小至 0

$$\boxed{I(t) = S_0 + I_0 - S(t) + \frac{1}{\sigma}\ln\frac{S(t)}{S_0}}$$

$$\boxed{\begin{cases} \frac{dS}{dt} = -\beta SI \\ \frac{dI}{dt} = (\beta S - \alpha)I \end{cases}}$$



SIR模型

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI \\ \dfrac{dI}{dt} = (\beta S - \alpha) I \end{cases}$$

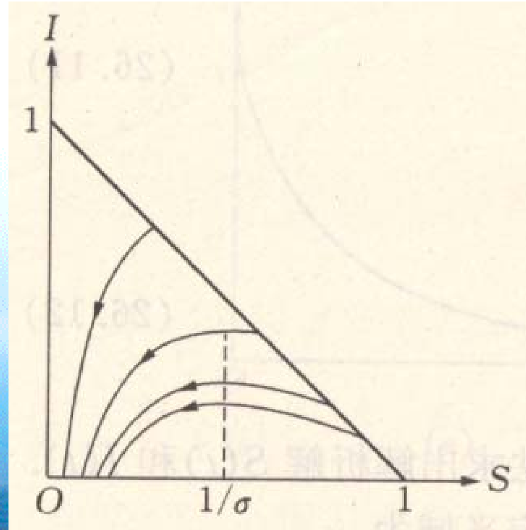
- 若 $s_0 \le \frac{1}{\sigma}$，$I(t)$ 单调减小至 0，$S(t)$ 单调减小至 S_∞
- S_∞ 为最终未被感染的比例，为方程

$$S_0 + I_0 - S_\infty + \frac{1}{\sigma}\ln\frac{S_\infty}{S_0} = 0$$

的根。若忽略 I_0，则 $\sigma \approx \dfrac{\ln S_0 - \ln S_\infty}{S_0 - S_\infty}$

该式可用于在某次传染期结束后估计 σ 的值

$$\boxed{I(t) = S_0 + I_0 - S(t) + \frac{1}{\sigma}\ln\frac{S(t)}{S_0}}$$



预防免疫

浙江大学 ZheJiang University

数学建模

- 当 $S_0 < \dfrac{1}{\sigma}$ 时，$I(t)$ 不会增加，可以认为传染病没有蔓延
- 忽略 I_0，则 $R_0 \geq 1 - \dfrac{1}{\sigma}$ 时，传染病不会蔓延，提高 R_0 可通过预防免疫来实现
  - 现有数据显示，天花的 σ 值较小，麻疹等传染病的 σ 值较大，目前全世界已消灭天花疾病

模型验证

$$
\begin{cases}
\dfrac{dS}{dt} = -\beta SI \\
\dfrac{dI}{dt} = \beta SI - \alpha I \\
\dfrac{dR}{dt} = \alpha I
\end{cases}
$$

- 孟买某岛（**1905.12.17-1906.7.21**）（**Kermack,McKendrick,1926**）
  - 该岛上**80%－90%**的感染者死亡，视为移出者

$$\frac{dS}{dR} = -\sigma S \Rightarrow S(t) = S_0 e^{-\sigma R(t)}$$

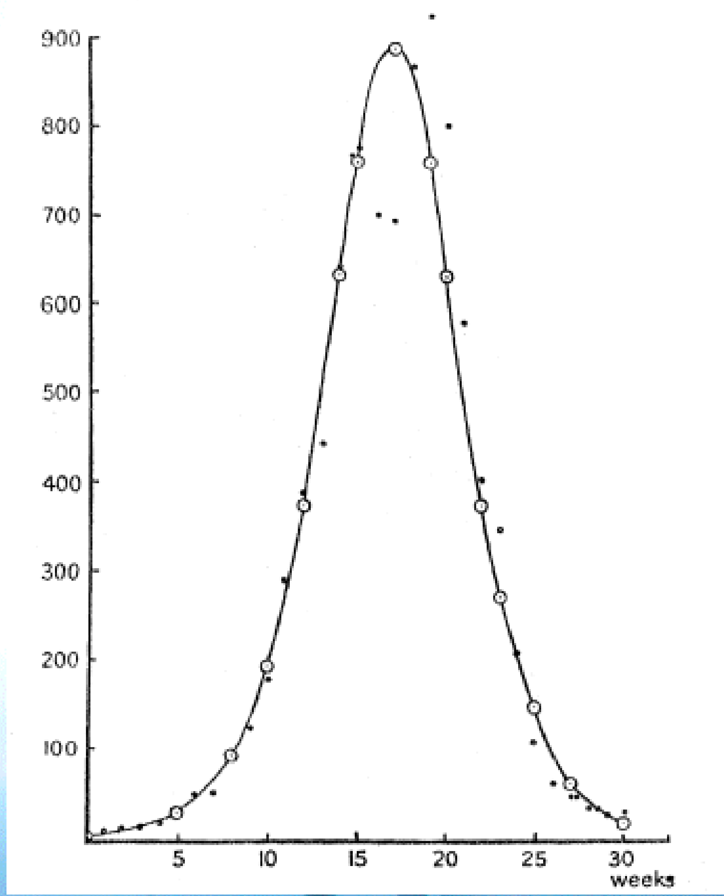
$$\frac{dR}{dt} = \alpha\left(1 - R(t) - S_0 e^{-\sigma R(t)}\right)$$

1903年，印度广大地区发生瘟疫，死亡**60**万人，**1904－1905**年，死亡**100**万人。**1906－1907**年，死亡**167.27**万人

模型验证

- 当 $\sigma R(t) \ll 1$时，将 $e^{-\sigma R(t)}$ 用**Taylor**展开，取前三项，求解常微分方程，选定参数，作图并与现实数据比较

$$\boxed{\frac{dR}{dt} = \alpha\left(1 - R - S_0 e^{-\sigma R}\right)}$$



模型推广

- 潜伏期、对已确诊感染者进行隔离等更多种类人群划分
- 因出生、死亡、迁徙等原因造成的总人口变动
- 具有时滞效应、考虑年龄结构的传染病模型
- 与传染病传播相似问题的建模与分析

基本数学模型

种间关系

种群

数学建模

- 种群（**population**）指同种生物在一定空间范围内同时生活着所有个体的集群
- 种间关系
  - 竞争（**competition**）：两物种个体为争夺共同资源而相互施加不利影响
  - 捕食（**predation**）： 一种生物以另一种生物为食
  - 共生（**symbiosis**）：两种不同生物个体之间任何形式的共同生活

个体（**individual**）
↓
种群（**population**）
↓
群落（**community**）
↓
生态系统（**ecosystem**）

Volterra和D’Ancona

浙江大学
ZheJiang University
数学建模

- 二十世纪二十年代，意大利生物学家**Umberto D’Ancona**观察到**Trieste**鱼市上鲨鱼和食用鱼所占比例的波动情况，这一数据基本反映了**Adriatic**海中两类鱼的比例
- 在第一次世界大战期间，捕鱼量大幅下降，数据显示对作为捕食者的鲨鱼更为有利

年份	1914	1915	1916	1917	1918	1919	1920	1921	1922	1923
鲨鱼比例(%)	11.9	21.4	22.1	21.2	36.4	27.3	16.0	15.9	14.8	10.7

Volterra和D’Ancona

数学建模

- **D’Ancona**将此问题求教于**Volterra**，后者建立了数学模型，对这一现象作出了解释
- 稍早时，美国数学家**Alfred James Lotka**（**1880-1949**）也给出了相同的模型，这一模型现被称作**Lotka-Volterra**模型

12 NATURE [JANUARY 1, 1927

tubes, which on closer examination proved to occur always in pairs and to project from the openings of the burrows formed by the shipworms. Plainly the tubes had been formed around the siphons of the Teredo. They were of varying length, depending presumably on the thickness of the deposits, the longest being some two-fifths of an inch. The general appearance of the wood is shown in Fig. 1.

Photo.] [A. J. Smith.

FIG. 1.—Portion of wood badly infected with *Teredo norvegica*. The white objects are the protruding calcareous siphonal tubes which appeared after the fæcal deposits had been washed off. In several cases the paired tubes can plainly be distinguished.

Normally the external openings of the tubes of Teredo are very difficult to distinguish, consisting of a pair of minute openings ringed with calcareous matter out of which project the siphons and within which these are immediately withdrawn on stimulation. The presence of fæcal deposits, which had accumulated to an abnormal degree owing to the lack of water currents to remove them, would tend to obstruct the passage of the siphons and so endanger the life of the animals within. The response of the animals to this abnormal and dangerous state of affairs was to lay down calcareous tubes around the siphons, which by this means were able to maintain free contact with the water.

Dr. W. T. Calman has directed my attention to the fact that the giant shipworm, *Kuphus arenarius*, which lives vertically embedded in the mud of mangrove swamps in the Pacific, normally has the siphons encased in this manner, a fact which was known to

Fluctuations in the Abundance of a Species considered Mathematically.

WITH regard to Prof. Volterra's interesting article, "Fluctuations in the Abundance of a Species considered Mathematically," in NATURE of October 16, page 558, I may be permitted to point to certain prior publications on the subject, of which Prof. Volterra seems to be unaware. The general theory as well as a number of special cases have been set forth in "Elements of Physical Biology" (published by Williams and Wilkins, Baltimore, 1925), in which work a considerable number of references to the journal literature are given. Among other things Prof. Volterra's diagram "Fig. 2" will be found on page 90 of the book cited; the expression for the period of isochronous small oscillations in the case of two species is also found on the same page. Prof. Volterra refers to certain applications of his analysis to problems of sea fisheries, to a passage in Darwin's "Origin of Species," to extinction of species, to pathogenic germs, and to parasitology. An application to sea fisheries is found in the book cited on page 95; to a passage in Herbert Spencer on page 61; to the extinction of species on pages 94, 95; to pathogenic germs on pages 77, 79, 147 *et seq.*; to parasitology on page 83.

The effect of introducing a third species into a system of two species is discussed on page 94; the effect on equilibrium of changing various factors is treated in Chap. xxii., "Displacement of Equilibrium," and, in particular, the effect on equilibrium between food and feeding species is analysed on page 289. The distinction between oscillatory and aperiodic systems, and its relation to certain quadratic forms, is referred to on pages 146, 148, and 159.

It would be gratifying if Prof. Volterra's publication should direct attention to a field and method of inquiry which apparently has hitherto passed almost unnoticed.

ALFRED J. LOTKA.

Metropolitan Life Insurance Company,
New York City, October 29.

In the above letter from Dr. Lotka, which is in accordance with our preceding correspondence, following upon the publication of my article in NATURE, he justly observes that he had obtained the differential equations in the case of two species, one of which feeds upon the other, that he had given, as well as myself, the same diagram of the integral, and also the period in the case of small fluctuations. In this I recognize his priority, and am sorry not to have known his work, and therefore not to have been able to mention it.

Nature（**No 2983, 12-13, 1927**）上**Lotka**和**Volterra**关于模型的通信

Vito Volterra（**1860－1940**）意大利数学家

捕食者—食饵系统

- 记 $x(t), y(t)$分别为t时刻鲨鱼（捕食者）和食用鱼（食饵）的种群数量
  - 由于海洋资源丰富，食用鱼独立生存时以常数增长率增长；鲨鱼的存在使食用鱼增长率减少，程度与鲨鱼数量呈正比
  - 鲨鱼缺乏食用鱼时死亡率为常数；食用鱼的存在使鲨鱼死亡率降低，程度与食饵数量呈正比

$$\frac{dx}{dt} = rx - axy \qquad \frac{dy}{dt} = -dy + bxy$$

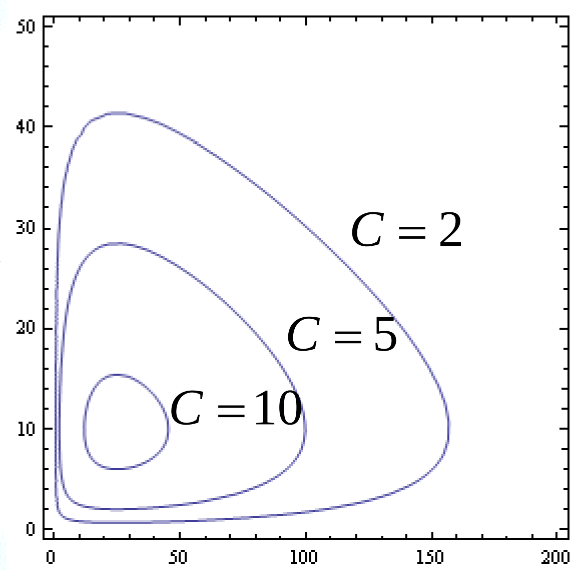
相轨线

$$\boxed{\frac{dx}{dt}=rx-axy \qquad \frac{dy}{dt}=-dy+bxy}$$

$$\frac{dx}{dy}=\frac{x(r-ay)}{y(-d+bx)} \Rightarrow \frac{-d+bx}{x}dx=\frac{r-ay}{y}dy$$

$$\Rightarrow -d\ln x+bx=r\ln y-ay+c$$

$$\Rightarrow (x^d e^{-bx})(y^r e^{-ay})=C$$



- 若 $x(0)=x_0, y(0)=y_0$，则

$$C=(x_0^d e^{-bx_0})(y_0^r e^{-ay_0})$$

$r=1, d=0.5$

$a=0.1, b=0.02$

相轨线

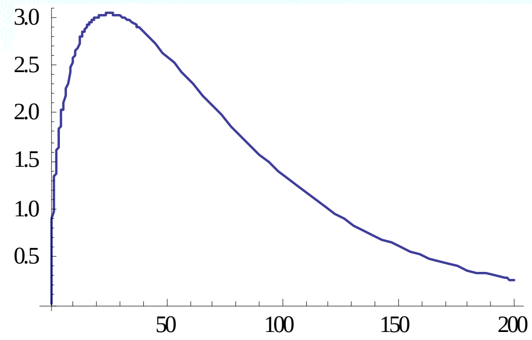
浙江大学
ZheJiang University
数学建模

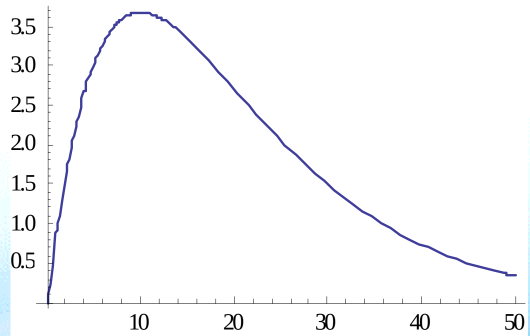
- 令 $f(x)=x^d e^{-bx}$，$f(x)$在$x_m=\frac{d}{b}$处取得极大值 $f_{\max}$； 令 $g(y)=y^r e^{-ay}$，$g(y)$在 $y_m=\frac{r}{a}$处取得极大值 $g_{\max}$，

$$0 \le C = f(x)g(y) \le f_{\max} g_{\max}$$

- 若$C=f_{\max}g_{\max}$，则 $x=x_m, y=y_m$，相轨线 $\mathscr{L}$ 退化为点 (x_m, y_m)

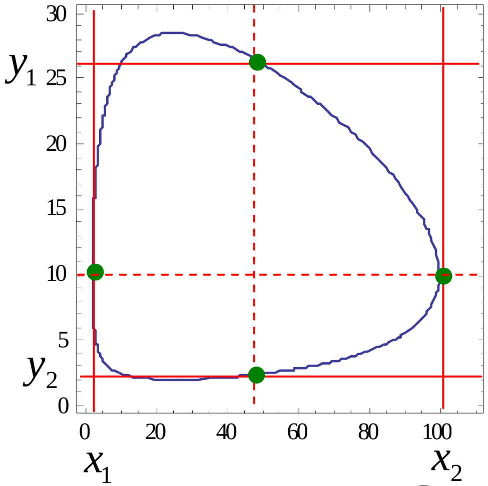
$$\boxed{(x^d e^{-bx})(y^r e^{-ay})=C}$$

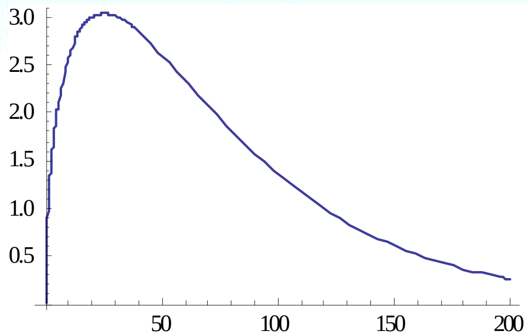


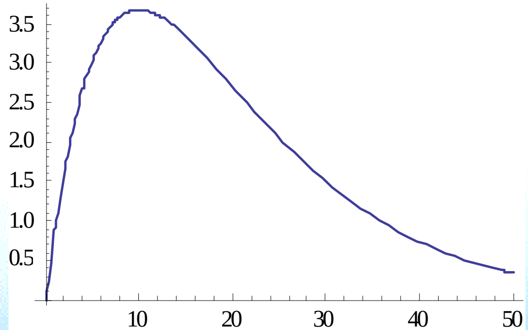


相轨线

- 任取 $0 < C < f_{\max} g_{\max}$，记 $p = \dfrac{C}{g_{\max}} < f_{\max}$，则存在 x_1, x_2，$x_1 < x_m < x_2$ 且 $f(x_1) = f(x_2) = p$，$\mathscr{L}$ 通过点 $(x_1, y_m), (x_2, y_m)$







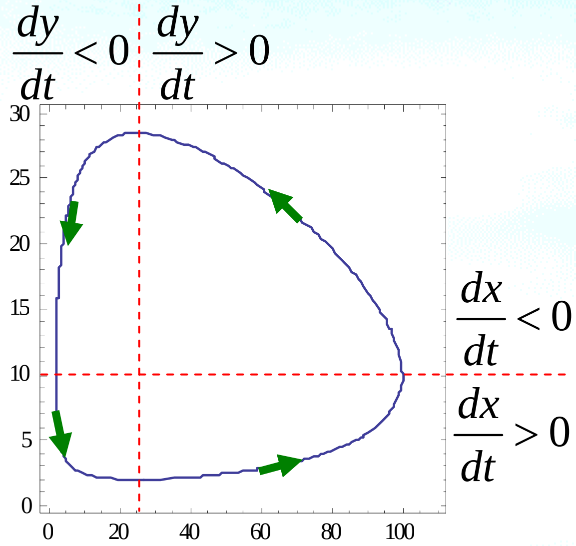
- 对任一 $x \in (x_1, x_2)$，$f(x) > p$。记 $q = \dfrac{C}{f(x)}$，则 $q = \dfrac{p g_{\max}}{f(x)} < g_{\max}$，存在 $y_1, y_2, y_1 < y_m < y_2$，且 $g(y_1) = g(y_2) = q$，$\mathscr{L}$ 通过点 $(x, y_1), (x, y_2)$

$$\boxed{(x^d e^{-bx})(y^r e^{-ay}) = C}$$

相轨线

数学建模

- 对任意 C，相轨线是一条封闭曲线。当 C 自 $f_{\max}g_{\max}$ 开始逐渐变小时，相轨线从点 (x_m, y_m) 开始不断向外扩展
- 直线 $x = x_m, y = y_m$ 将相轨线分为四段，各段的 $\frac{dx}{dt}$ 和 $\frac{dy}{dt}$ 符号不全相同，由此可确定轨线的方向

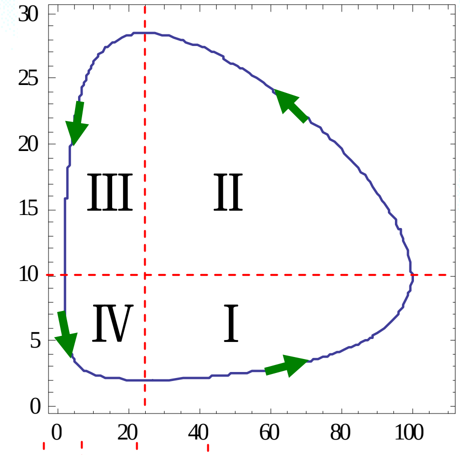


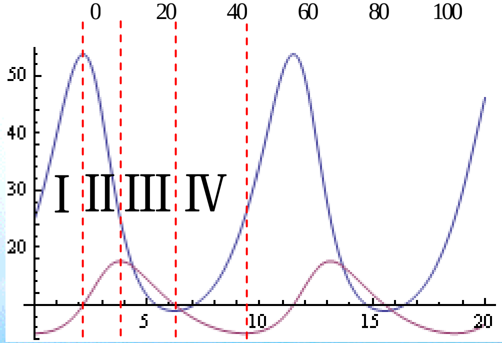
$$\frac{dx}{dt} = x(r - ay)$$
$$\frac{dy}{dt} = y(-d + bx)$$

相轨线

数学建模

- 由于相轨线是封闭曲线，函数 $x(t), y(t)$ 均为周期函数，记周期为 T
- 周期 T 可相应分为四段，各段内 $x(t)$ 和 $y(t)$ 的增减性不全相同
- 在一个周期内食饵先于捕食者到达最大值或最小值





平均值

数学建模

$$x(t)=\frac{1}{by(t)}\frac{dy}{dt}+\frac{d}{b}$$

$$\bar{x}=\frac{1}{T}\int_0^T x(t)dt=\frac{1}{T}\left(\frac{1}{b}\ln y(t)+\frac{d}{b}t\right)\Big|_0^T$$

$$=\frac{1}{T}\left(\frac{\ln y(T)-\ln y(0)}{b}+\frac{d}{b}T\right)\Big|_0^T=\frac{d}{b}$$

$$y(t)=-\frac{1}{ax(t)}\frac{dx}{dt}+\frac{r}{a}$$

$$\bar{y}=\frac{1}{T}\int_0^T y(t)dt=\frac{1}{T}\left(-\frac{1}{a}\ln x(t)+\frac{r}{a}t\right)\Big|_0^T=\frac{r}{a}$$

$$\frac{dx}{dt}=x(r-ay)$$
$$\frac{dy}{dt}=y(-d+bx)$$

$$x_m=\frac{d}{b}$$
$$y_m=\frac{r}{a}$$

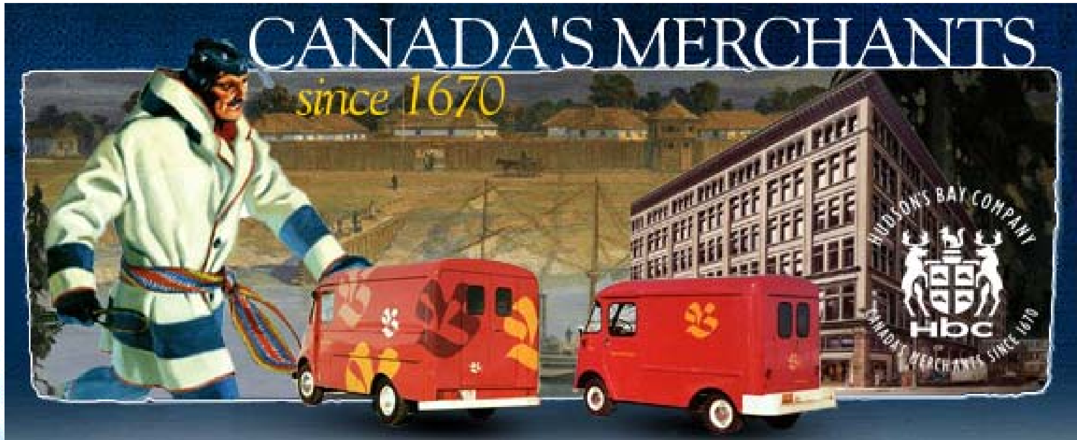
捕捞

- 设捕捞系数为 e，即由于人为捕捞，食用鱼增长率由 r 下降为 $r-e$，鲨鱼死亡率由 d 上升为 $d+e$，则 $\overline{x}(e)=\dfrac{d+e}{b}>\overline{x}(0),\overline{y}(e)=\dfrac{r-e}{a}<\overline{y}(0)$
- $\overline{x}(e)$ 为 e 的增函数，$\overline{y}(e)$ 为 e 的减函数。因此当捕捞系数减小时，鲨鱼所占比例增加
- 害虫和它的天敌益虫构成食饵－捕食者系统。如果一种杀虫剂杀死益虫和害虫的效力相当，长期使用将导致害虫增加

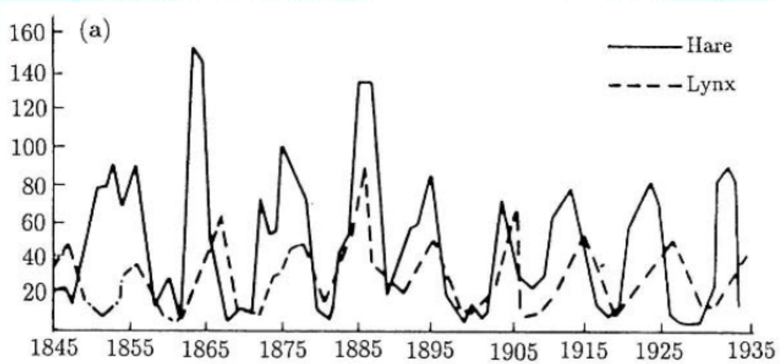
山猫与野兔

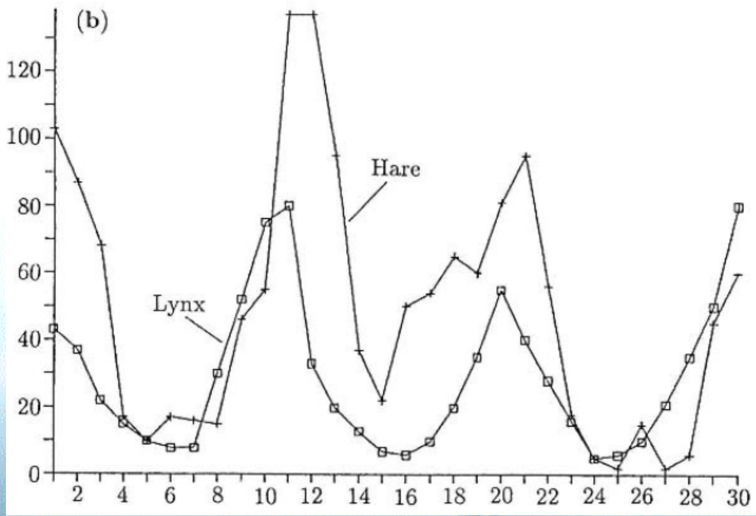
浙江大学
ZheJiang University
数学建模

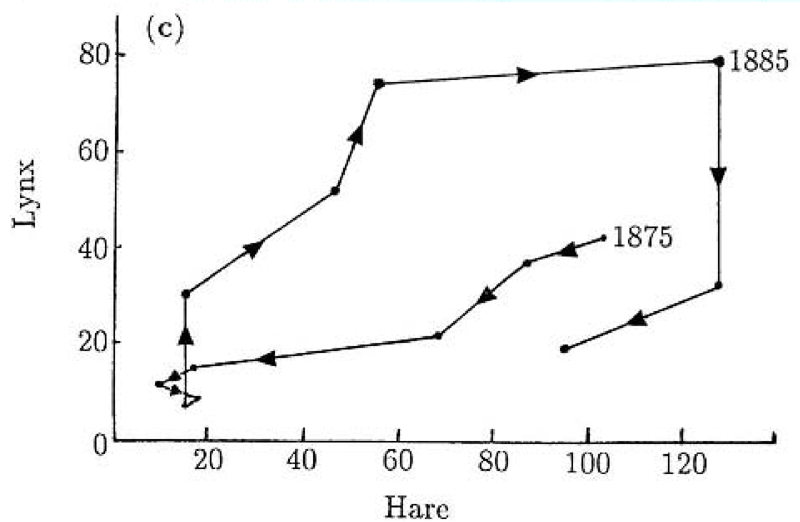
- 加拿大**Hudson Bay Company**长期从事皮毛贸易，存有**1845**－**1930**年在北美捕获的**Lynx canadensis**和**Lepus americanus (snowshoe hare)**两种动物的数量数据。这组数据常被用来分析捕食者－食饵系统



山猫与野兔







顺时针！野兔吃山猫？

（a）1845－1935年山猫，野兔－时间图

（b）1875－1904年山猫，野兔－时间图

（c）1875－1904年山猫－野兔图

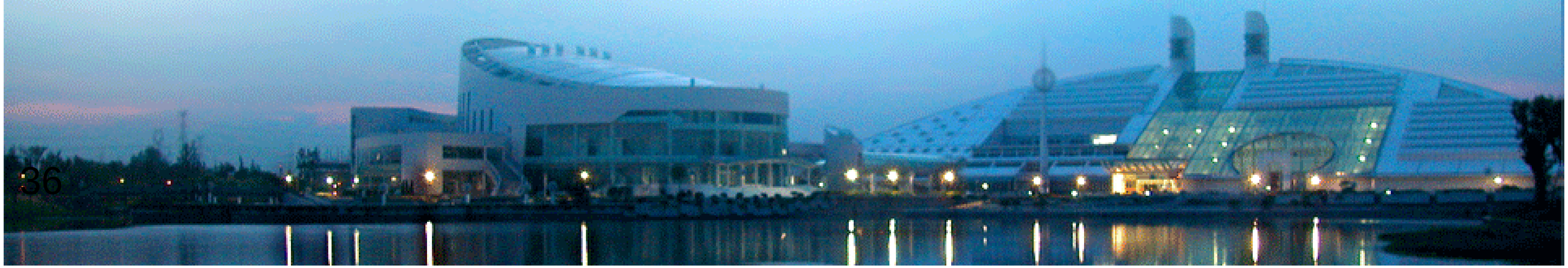
（单位均为千只）

浙江大学 ZheJiang University

数学建模

一般双种群模型

- $$\begin{cases} \dfrac{dx}{dt} = x(a_{10} + a_{11}x + a_{12}y) \\ \dfrac{dy}{dt} = y(a_{20} + a_{21}x + a_{22}y) \end{cases}$$
- $x(t), y(t)$ 分别表示种群X和Y在t时刻的数量
- a_{10}, a_{20}分别表示种群 X和 Y的生长率（出生率与死亡率之差）
  - $a_{10} > 0$ 表示 X 可依靠系统外食物为生
  - $a_{10} < 0$ 表示 X必须依赖 Y为食才能生存



一般双种群模型

$$\begin{cases} \dfrac{dx}{dt} = x(a_{10} + a_{11}x + a_{12}y) \\ \dfrac{dy}{dt} = y(a_{20} + a_{21}x + a_{22}y) \end{cases}$$

- a_{11}, a_{22} 分别表示种群 X 和 Y 的密度制约项
  - $a_{11} = 0$ 表示 X 是非密度制约的
  - $a_{11} < 0$ 表示 X 是密度制约的
- a_{12}, a_{21} 反映种群 X 和 Y 之间的关系
  - X 为食饵，Y 为捕食者 $a_{12} < 0, a_{21} > 0$
  - X 为寄主，Y 为寄生物 $a_{12} < 0, a_{21} > 0$
  - 相互竞争 $a_{12} < 0, a_{21} < 0$
  - 互惠共存 $a_{12} > 0, a_{21} > 0$


浙江大学
ZheJiang University
谢 谢