数学建模

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浙江大学 ZheJiang University

数学与社会

稳定匹配

Nobel Prize 2012

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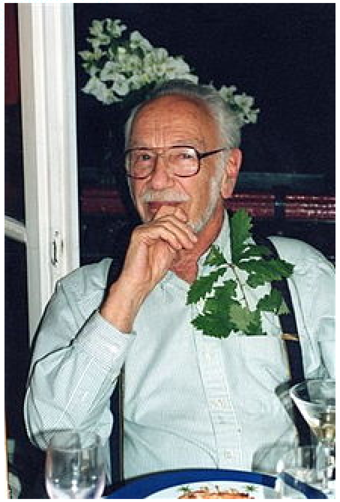

David Gale
（1921 - 2008）
美国数学家、经济学家

Prize motivation: for the theory of stable allocations and the practice of market design

Lloyd Stowell Shapley
（1923 - ）
美国数学家、经济学家

Alvin Elliot Roth
（1951 - ）
美国经济学家

稳定婚姻问题

- 现有 n 名男士 $m_1,m_2,\cdots,m_n$和 n名女士 $w_1,w_2,\cdots,w_n$。每位男士有一偏好顺序可对所有女士按其满意度进行排序，每位女士有一偏好顺序可对所有男士按其满意度进行排序
  - $w \succ_m w'$ 表示在男士 m 的偏好顺序中，w 优于 w'
- n个配对 $(m_{i_1},w_{j_1}),(m_{i_2},w_{j_2}),\cdots,(m_{i_n},w_{j_n})$ 组成的集合称为一组婚姻（**marriage**），其中 $i_1,i_2,\cdots,i_n$ 和 $j_1,j_2,\cdots,j_n$ 是 $1,2,\cdots,n$ 的两个排列
- 婚姻 $\mathcal{M}$ 称为不稳定（**unstable**）的，若存在不稳定组合 $\langle m_i,w_l\rangle$，$(m_i,w_j),(m_k,w_l)\in\mathcal{M}$，但 $w_l \succ_{m_i} w_j$ ，$m_i \succ_{w_l} m_k$
- 若一组婚姻不存在不稳定组合，则称为稳定（**stable**）的

稳定婚姻问题

数学建模

$$m_1 : w_2\ w_1\ w_3 \mid w_1 : m_1\ m_3\ m_2$$

$$m_2 : w_1\ w_2\ w_3 \mid w_2 : m_3\ m_1\ m_2$$

$$m_3 : w_1\ w_2\ w_3 \mid w_3 : m_1\ m_2\ m_3$$

稳定婚姻　不稳定婚姻

稳定婚姻	不稳定婚姻	
$(m_1,\ w_1)$	$(m_1,\ w_1)$	$m_1 \succ_{w_2} m_2$
$(m_2,\ w_3)$	$(m_2,\ w_2)$	$w_2 \succ_{m_1} w_1$
$(m_3,\ w_2)$	$(m_3,\ w_3)$	


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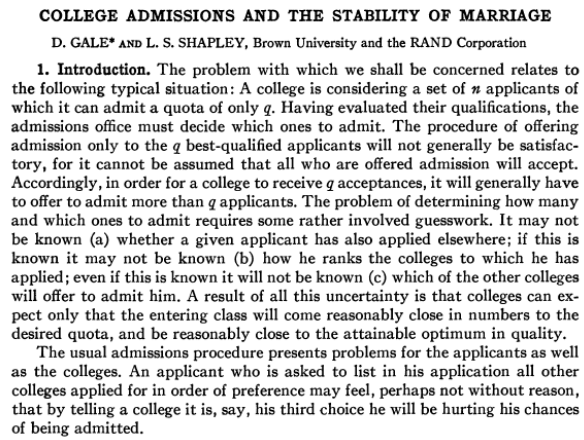
COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept. Accordingly, in order for a college to receive q acceptances, it will generally have to offer to admit more than q applicants. The problem of determining how many and which ones to admit requires some rather involved guesswork. It may not be known (a) whether a given applicant has also applied elsewhere; if this is known it may not be known (b) how he ranks the colleges to which he has applied; even if this is known it will not be known (c) which of the other colleges will offer to admit him. A result of all this uncertainty is that colleges can expect only that the entering class will come reasonably close in numbers to the desired quota, and be reasonably close to the attainable optimum in quality.

The usual admissions procedure presents problems for the applicants as well as the colleges. An applicant who is asked to list in his application all other colleges applied for in order of preference may feel, perhaps not without reason, that by telling a college it is, say, his third choice he will be hurting his chances of being admitted.

Gale, D., Shapley, L. S., College Admissions and the Stability of Marriage, *The American Mathematical Monthly*, 69, 9-15, 1962

MAA MATHEMATICAL ASSOCIATION OF AMERICA

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算法

- “男士选择，女士决定”（**deferred acceptance algorithm**）
  - 每位男士都选择他最钟爱的女士
  - 如果有女士被两位或者以上的男士选择，则这几位男士中除了她最喜欢的之外，对其他男士都表示拒绝
  - 被拒绝的那些男士转而考虑他（们）的除被拒绝之外的最满意女士。如果存在冲突（包括和之前选择某女士的男士发生冲突），则再由相应的女士决定拒绝哪些男士
  - 以上过程持续进行，直至不再出现冲突为止

$$m_1 : w_2 \ w_1 \ w_3 \,|\, w_1 : m_1 \ m_3 \ m_2$$

$$m_2 : \cancel{w_1} \ \cancel{w_2} \ w_3 \,|\, w_2 : m_3 \ m_1 \ m_2$$

$$m_3 : w_1 \ w_2 \ w_3 \,|\, w_3 : m_1 \ m_2 \ m_3$$

$$m_1 : w_2$$

$$m_2 : \cancel{w_1} \ \cancel{w_2} \ w_3$$

$$m_3 : w_1$$

MathSite

An Interactive Source for

Seeing, Hearing, Doing Mathematics

算法

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- 算法终止时给出一组婚姻。既不会出现冲突，也不会出现有女士未被男士选择情形
  - m和m'同时选择w，m'被拒绝；w'未被任意男士选择
  - 由于算法终止，m'已被所有女士，包括 w'拒绝
  - w'曾被优于m'的男士选择，算法终止时不会未被任意男士选择 矛盾

由于男士女士人数相等，两者必同时发生

- 算法给出的婚姻$\mathcal{M}$是稳定的
  - $\langle m, w' \rangle$ 是不稳定配对，$(m,w),(m',w') \in \mathcal{M}, w' \succ_m w, m \succ_{w'} m'$
  - m曾选择w'，但被w'拒绝
  - 存在男士m''，$m'' \succ_{w'} m$，由于$(m',w') \in \mathcal{M}$，故$m' \succ_{w'} m''$

$\Rightarrow m' \succ_{w'} m$ 矛盾

- 算法时间复杂性为$O(n^2)$
  - 任一男士不会多次选择同一女士

稳定婚姻数量

$$
\begin{array}{llll}
m_1: w_1\ \ w_2 \cdots & w_1: \cdots m_1 & (m_1, w_1) & (m_1, w_2) \\
m_2: w_2\ \ w_1 \cdots & w_2: \cdots m_2 & (m_2, w_2) & (m_2, w_1) \\
m_3: w_3\ \ w_4 \cdots & w_3: \cdots m_3 & (m_3, w_3) & (m_3, w_3) \\
m_4: w_4\ \ w_3 \cdots & w_4: \cdots m_4 & (m_4, w_4) & (m_4, w_4) \\
\vdots & \vdots & \vdots & \vdots \\
m_{2k-1}: w_{2k-1} w_{2k} \cdots & w_{2k-1}: \cdots m_{2k-1} & (m_{2k-1}, w_{2k-1}) & (m_{2k-1}, w_{2k-1}) \\
m_{2k}: w_{2k} w_{2k-1} \cdots & w_{2k}: \cdots m_{2k} & (m_{2k}, w_{2k}) & (m_{2k}, w_{2k})
\end{array}
$$

所有稳定婚姻数量至少为 2^k

最优性

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- 称一组稳定婚姻是男方最优（**man-optimal**）的，如果在该组婚姻中，每位男士都认为其配偶不比任何一组稳定婚姻中他的配偶来的差
  - 男方最优的稳定婚姻若存在，必是唯一的
- “男士选择，女士决定”算法给出的婚姻是男方最优的

  归纳法
  - 由于每位男士按照偏好从优到劣的顺序选择，只需证明任一被拒绝的配对不会出现在任何一组稳定婚姻中

最优性

- 男士 m 被女士 w 拒绝是算法运行过程中首次出现的男士被女士拒绝的情况
  - 存在男士 m', $m' \succ_w m$，且 w 位于 m 和 m' 偏好顺序的首位
  - 若 $(m,w),(m',w') \in \mathcal{M}$，则 $\langle m',w \rangle$ 为不稳定组合，$\mathcal{M}$ 不稳定
- 男士 m 被女士 w 拒绝。在此之前所有被拒绝的配对不会出现在任一组稳定婚姻中
  - 存在男士 m' 选择 w，$m' \succ_w m$
  - 若 $(m,w),(m',w') \in \mathcal{M}$
    - $w' \succ_{m'} w$，则 m' 在之前必被 w' 拒绝。由归纳假设，(m',w') 不会出现在稳定婚姻中，$\mathcal{M}$ 不稳定
    - $w \succ_{m'} w'$，则 $\langle m',w \rangle$ 为不稳定组合，$\mathcal{M}$ 不稳定

最劣性

- 男方最优的稳定婚姻必是女方最劣的，即在该组婚姻中，每位女士都认为其配偶不比任何一组稳定婚姻中她的配偶来的好
  - $\mathcal{M}$为男方最优稳定婚姻, $(m,w)\in\mathcal{M}$，若 $m \succ_w m'$, (m',w) 不会出现在任一组稳定婚姻中
  - 设$\mathcal{M}'$为另一组稳定婚姻, $(m',w)\in\mathcal{M}'$,$(m,w'')\in\mathcal{M}'$
    - 若 $w'' \succ_m w$，$\mathcal{M}$不为男方最优稳定婚姻
    - 若 $w \succ_m w''$，则$\langle m,w\rangle$为不稳定组合, $\mathcal{M}'$不稳定
- “女士选择，男士决定”算法给出的婚姻是女方最优、男方最劣的

稳定室友问题

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- 稳定室友问题（**Stable roommate problem**）
  - 某培训班有 $2n$ 名男性学员，两人合住一间标准间。每名学员有一偏好顺序可对其它 $2n-1$ 名学员进行排序。是否存在一组稳定的房间安排方案

$c_1 : c_2\ c_3\ c_4$

$c_2 : c_3\ c_1\ c_4$

$c_3 : c_1\ c_2\ c_4$

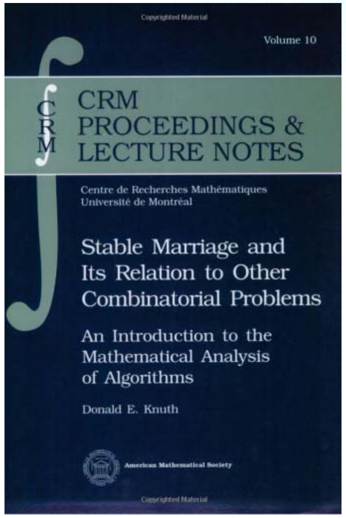
$c_4 :$ 任意

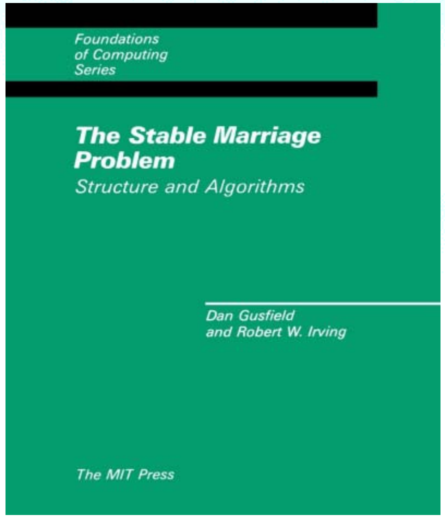
$$\begin{matrix}(c_1, c_2) \\ (c_3, c_4)\end{matrix} \longrightarrow \begin{matrix}(c_3, c_2) \\ (c_1, c_4)\end{matrix} \longrightarrow \begin{matrix}(c_2, c_4) \\ (c_1, c_3)\end{matrix}$$

稳定的安排方案未必存在

稳定室友问题

- 存在一时间复杂性为 $O(n^2)$ 的算法判断稳定的安排方案是否存在，并在存在时给出一组稳定的安排方案





男士和女士数目不同，存在不可接受组合，或出现对多人的满意度相同（**tie**）的情形

Gusfield, D. M., Irving, R. W., *The Stable Marriage Problem: Structure and Algorithms*, The MIT Press, 2003

Knuth, D. R., *Stable Marriage and Its Relation to Other Combinatorial Problems: An Introduction to the Mathematical Analysis of Algorithms*, The American Mathematical Society, 1996.

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三维稳定婚姻问题

- 现有分别由 n 名男士、n 名女士、n 件礼物组成的集合 M,W,D。任一集合中的每个元素对由另两个集合中元素两两组合而成的 n^2 个配对有给定的偏好顺序
- 一组三维婚姻为三元组的集合

$$\mathcal{M}=\{(m_i,w_i,d_i)\,|\,i=1,\cdots,n,\ \bigcup_{i=1}^{n}m_i=M,\ \bigcup_{i=1}^{n}w_i=W,\ \bigcup_{i=1}^{n}d_i=D\}$$

- 三维婚姻稳定，若对任一三元组 $(m,w,d)\notin\mathcal{M}$，若 $(m,w_1,d_1),(m_2,w,d_2),(m_3,w_3,d)\in\mathcal{M}$，则必有

$$(w_1,d_1)\succ_m(w,d),\ (m_2,d_2)\succ_w(m,d),\ (m_3,w_3)\succ_d(m,w)$$

三维稳定婚姻问题

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- 三维稳定婚姻未必存在。判断是否存在一组三维稳定婚姻是$\mathcal{NP}$-完全的

$m_1:(w_1d_2)\,(w_1d_1)\,(w_2d_2)\,(w_2d_1)$

$m_2:(w_2d_2)\,(w_1d_1)\,(w_2d_1)\,(w_1d_2)$

$w_1:(m_2d_1)\,(m_1d_2)\,(m_1d_1)\,(m_2d_2)$

$w_2:(m_2d_1)\,(m_1d_1)\,(m_2d_2)\,(m_1d_2)$

$d_1:(m_1w_2)\,(m_1w_1)\,(m_2w_1)\,(m_2w_2)$

$d_2:(m_1w_1)\,(m_2w_2)\,(m_1w_2)\,(m_2w_1)$

所有可能的婚姻	不稳定组合
$(m_1,w_1,d_1)\ (m_2,w_2,d_2)$	$\langle m_1,w_1,d_2\rangle$
$(m_1,w_1,d_2)\ (m_2,w_2,d_1)$	$\langle m_2,w_1,d_1\rangle$
$(m_1,w_2,d_1)\ (m_2,w_1,d_2)$	$\langle m_1,w_1,d_2\rangle$
$(m_1,w_2,d_2)\ (m_2,w_1,d_1)$	$\langle m_2,w_2,d_2\rangle$

NRMP

- 根据美国医生培养制度，医学院毕业生取得学位后需作为医院的住院医生（**Resident,** 旧称**Intern**）经过为期三年的实习期
- 二十世纪初期，医院和毕业生之间的双向选择呈无序状态
  - 医院为争夺毕业生，竞相提前开展招聘计划
  - 医院在给予毕业生职位时仅留极短时间供其考虑，以避免被毕业生拒绝后无法找到其它人选
- 通过**NRMP**计划实现双向选择的医院和毕业生曾达到**95%**。**Roth**研究后发现该计划使用的算法本质上与**Gale-Shapley**算法等价，能给出稳定分配方案是其成功的主要原

NRMP
National Resident Matching Program

The National Resident Matching Program (NRMP) is a private, not-for-profit corporation established in 1952 to provide a uniform date of appointment to positions in graduate medical education (GME) in the United States.

稳定分配问题

- “医院—毕业生”分配是多对一分配，每所医院存在招收毕业生数量的上界　　稳定的定义？
- 毕业生中的配偶对医院存在联合的偏好顺序，稳定分配未必存在

H_1	H_2	H_3	H_4
s_4	s_4	s_2	s_2
s_2	s_3	s_3	s_4
s_1	s_2	s_1	s_1
s_3	s_1	s_4	s_3

(H_1, s_1)　(H_2, s_2)

(H_3, s_3)　(H_4, s_4)

$\langle H_2, s_4 \rangle$

$\{s_1, s_2\}$	$\{s_3, s_4\}$
H_1H_2	H_4H_2
H_4H_1	H_4H_3
H_4H_3	H_4H_1
H_4H_2	H_3H_1
H_1H_4	H_3H_2
H_1H_3	H_3H_4
H_3H_4	H_2H_4
H_3H_1	H_2H_1
H_3H_2	H_2H_3
H_2H_3	H_1H_2
H_2H_4	H_1H_4
H_2H_1	H_1H_3

欺骗

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- 对稳定婚姻问题，若使用男方最优算法，女方可以通过提供虚假偏好获得更好的一组稳定婚姻。

$$m_1 : \cancel{w_2}\ w_1\ w_3 \mid w_1 : m_1\ m_2\ m_3$$
$$m_2 : \cancel{w_1}\ \cancel{w_2}\ w_3 \mid w_2 : m_3\ m_1\ m_2$$
$$m_3 : \cancel{w_1}\ w_2\ w_3 \mid w_3 : m_1\ m_2\ m_3$$

稳定婚姻

(m_1, w_2)

(m_2, w_3)

(m_3, w_1)

w_1 的配偶在其偏好顺序中居第二位

$$m_1 : \cancel{w_2}\ w_1$$
$$m_2 : \cancel{w_1}\ \cancel{w_2}\ w_3$$
$$m_3 : \cancel{w_1}\ w_2$$

(m_1, w_1)

(m_2, w_3)

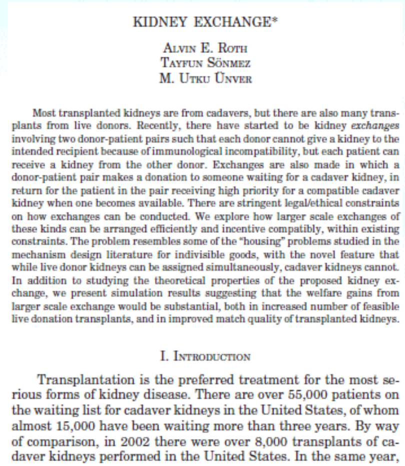
(m_3, w_2)

w_1 的配偶在其偏好顺序中居第一位

欺骗

数学建模

- 是否存在一种算法，能使参与者真实表达意愿，即参与者不会因为虚假表达意愿而获益
- 对任一稳定婚姻问题的算法，都存在部分参与者可通过提供虚假偏好顺序而获得更好的一组稳定婚姻
- 对给出男（女）方最优稳定婚姻的算法，男（女）方不可能通过提供虚假偏好顺序获得更好的一组稳定婚姻



KIDNEY EXCHANGE*

ALVIN E. ROTH
TAYFUN SÖNMEZ
M. UTKU ÜNVER

Most transplanted kidneys are from cadavers, but there are also many transplants from live donors. Recently, there have started to be kidney *exchanges* involving two donor-patient pairs such that each donor cannot give a kidney to the intended recipient because of immunological incompatibility, but each patient can receive a kidney from the other donor. Exchanges are also made in which a donor-patient pair makes a donation to someone waiting for a cadaver kidney, in return for the patient in the pair receiving high priority for a compatible cadaver kidney when one becomes available. There are stringent legal/ethical constraints on how exchanges can be conducted. We explore how larger scale exchanges of these kinds can be arranged efficiently and incentive compatibly, within existing constraints. The problem resembles some of the "housing" problems studied in the mechanism design literature for indivisible goods, with the novel feature that while live donor kidneys can be assigned simultaneously, cadaver kidneys cannot. In addition to studying the theoretical properties of the proposed kidney exchange, we present simulation results suggesting that the welfare gains from larger scale exchange would be substantial, both in increased number of feasible live donation transplants, and in improved match quality of transplanted kidneys.

I. INTRODUCTION

Transplantation is the preferred treatment for the most serious forms of kidney disease. There are over 55,000 patients on the waiting list for cadaver kidneys in the United States, of whom almost 15,000 have been waiting more than three years. By way of comparison, in 2002 there were over 8,000 transplants of cadaver kidneys performed in the United States. In the same year,

Roth, A. E., The Economics of Matching: Stability and Incentives. *Mathematics of Operations Research*, 7, 617-628, 1982.

Roth, A.E., Sonmez, T., Unver, M.U., Kidney exchange, *Quarterly Journal of Economics*, 119, 457–488, 2004.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