UMass Boston CS220 Discrete Mathematics
Homework Questions

Professor: Ming Ouyang

Spring 2022

Homework 1 (due February 3, 2022)

1. If $p_1, p_2, \ldots, p_3$ are n propositions, explain why

$$\bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^{n} (\neg p_i \vee \neg p_j)$$

   is true if and only if at most one of $p_1, p_2, \ldots, p_3$ is true.

2. Construct the truth table of this compound proposition $(p \wedge q) \vee \neg r$.
3. Is the assertion "This statement is false" a proposition?
4. Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology.
5. Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.

Homework 2 (due February 10, 2022)

1. Let $P(x)$ be the predicate "x has a cellphone." Let $Q(x)$ be a predicate "x can solve quadratic equations." Let $R(x)$ be the predicate "x wants to be rich."
   (a) Let the domain be all students in CS 220. Translate the following statements into logical expressions using predicates, quantifiers, and logical connectives.
      i. Everyone in CS220 has a cellphone.
      ii. Somebody in CS220 can solve quadratic equations.
      iii. Somebody in CS220 does not want to be rich.
   (b) Let the domain be all people. Let $C(x)$ be the predicate "x is in CS220." Translate the statements above again.
2. Show that the argument with premises $(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, q, and $\neg s$ and conclusion r is valid by using rules of inference.
3. Let S be the conditional statement "If S is true, then unicorns live." Show that if S is a proposition, then "Unicorns live" is true. Show that it follows that S cannot be a proposition.
4. Use existential and universal quantifiers to express the statement "No one has more than two grandmothers" using the predicate $G(x, y)$, which represents "x is the grandmother of y," and the predicate $x \neq y$ for "x is not equal to y."
5. Prove or disprove that if A, B, and C are nonempty sets and $A \times B = A \times C$, then $B = C$.

Homework 3 (due February 17, 2022)

1. Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.
2. Let $f : R \to R$ and let $f(x) > 0$ for all $x \in R$.Show that $f(x)$ is strictly increasing if and only if the function $g(x) = \frac{1}{f(x)}$ is strictly decreasing.
3. A person deposits \$1,000 in an account that yields 9% interest compounded annually.
   (a) Set up a recurrence relation for the amount in the account at the end of n years.
   (b) Find an explicit formula for the amount in the account at the end of n years.
   (c) How much money will the account contain after 100 years?
4. Telescoping sum is the identity that $\sum_{i=1}^{n}(a_i - a_{i-1}) = a_n - a_0$. Use telescoping sum and the identity
$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$
   to compute
$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$
5. Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable.

Homework 4 (due February 24, 2022)

1. Use the definition of $f(n) = \mathrm{O}(g(n))$ to show that $2^n + 17 = \mathrm{O}(3^n)$.
2. Let k be a positive integer. Show that $1^k + 2^k + \cdots + n^k = \mathrm{O}(n^{k+1})$.
3. Arrange the functions $(1.5)^n$, n^{100}, $(\log n)^3$, $\sqrt{n}\log n$, 10^n, $(n!)^2$, and $n^{99} + n^{98}$ in a list so that each function is big-O of the next function.
4. The following C code comes form page 50 and Exercise 2-9 of the C Programming Language by Kernighan and Ritchie, the second edition.

```
int bitCount(unsigned x) {
    int count;

    for (count = 0; x != 0; x &= (x - 1))
        count++;
    return count;
}
```

   (a) Explain why it counts the number of 1 bits in the unsigned integer `x`.
   (b) How many iterations will the for-loop be executed?
5. To calculate the product of three integer matrices ABC, we can parenthesize the calculation as either $(AB)C$ or $A(BC)$. Which parenthesization uses fewer integer multiplications if A, B, and C have dimensions 3×9, 9×4, and 4×2, respectively?

Homework 5 (due March 3, 2022)

1. Write out the addition and multiplication tables for Z_7, where the sum $+_7$ and product $\cdot_7$ are *modulo* 7.

$+_7$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

$\cdot_7$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

2. Find the sum and product of $(20\mathrm{CBA})_{16}$ and $(\mathrm{A01})_{16}$. Express the answers in hexadecimal.
3. How many zeros are there at the end of 100!?
4. We call a positive integer perfect if it equals the sum of its positive divisors other than itself. Show that 6 and 28 are perfect.
5. Show that if a, b, and m are integers such that $m \geq 2$ and $a \equiv b \pmod{m}$, then $\gcd(a, m) = \gcd(b, m)$.

Homework 6 (due March 10, 2022)

1. Let $p_1, p_2, \ldots, p_n$ be the n smallest prime numbers. Prove or disprove that $p_1 p_2 \cdots p_n + 1$ is prime for every n.
2. Using Bézout's theorem, find an inverse of 34 modulo 89 — that is, solve $34a \equiv 1 \pmod{89}$.
3. Use the value of a from the previous question to solve $34x \equiv 77 \pmod{89}$.
4. Show that the positive integers less than 11, except 1 and 10, can be put in pairs such that each pair consists of integers that are inverses of each other modulo 11.
5. Use Fermat's little theorem to find $23^{1002} \bmod 41$.

Homework 7 (due March 24, 2022)

1. Prove that 6 divides $n^3 - n$ for all nonnegative integer n.
2. Let A and B be square matrices such that $AB = BA$. Prove that $AB^n = B^n A$ for all positive integer n.
3. Find the flaw with the following "proof" that $A^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0 = 1$ is true by the definition of a^0.
Inductive Step: Assume that $a_j = 1$ for all nonnegative integers j with $j \leq k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

4. Give a recursive definition of the set of bit strings that are palindromes.

5. Design a recursive algorithm to find a^{2^n}, where a is a real number and n is a positive integer. [*Hint*: Use the inequality $a^{2^{n+1}} = (a^{2^n})^2$.]

Homework 8 (due April 7, 2022)

1. Consider truth tables for a compound proposition of n variables. How many rows do the truth tables have? How many different truth tables exist? Use the product rule of counting to justify your answers.

2. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$?

3. Let $n_1, n_2, \ldots, n_t$ be positive integers. Show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some i, $i = 1, 2, \ldots, t$, the i-th box contains at least n_i objects.

4. How many permutations of the letters ABCDEFGH contain

   (a) the string ED?

   (b) the string CDE?

   (c) the strings BA and FGH?

   (d) the strings AB, DE, and GH?

   (e) the strings CAB and BED?

   (f) the strings BCA and ABF?

5. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

Homework 9 (due April 14, 2022)

1. Consider the expansion $\left(x + \frac{1}{x}\right)^{15}$. What are the coefficients of x^7 and x^8?

2. Use a combinatorial argument to show that, for a positive integer n,

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

3. In bridge, the 52 cards of a standard deck are dealt to four players. How many different ways are there to deal bridge hands to four players?

4. A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen? Hint: Represent the books that are chosen by bars and the books not chosen by stars. Count the number of sequences of five bars and seven stars so that no two bars are adjacent.
5. We distribute 5 balls into 7 distinguishable boxes. Each box can hold at most 1 ball. How many ways can we distribute the balls if
   (a) the balls are distinguishable
   (b) the balls are indistinguishable

Homework 10 (due April 21, 2022)

1. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?
2. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?
3. When we randomly select a permutation of $\{1, 2, 3\}$, what is the probability of the following events?
   (a) 1 precedes 3.
   (b) 3 precedes 1 and 3 precedes 2.
4. What is the probability of these events when we randomly permute the 26 lowercase letters?
   (a) The first 13 letters of the permutation are in alphabetical order.
   (b) a and z are next to each other in the permutation.
   (c) a and z are not next to each other in the permutation.
5. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

Homework 11 (due April 28, 2022)

1. E and F are events in a sample space, and $p(E) = \frac{2}{3}$, $p(F) = \frac{3}{4}$, and $p(F|E) = \frac{5}{8}$. Find $p(E|F)$.
2. Suppose that E, F_1, F_2, and F_3 are events from a sample space S and that F_1, F_2, and F_3 are pairwise disjoint and their union is S. Find $p(F_2|E)$ if $p(E|F_1) = \frac{2}{7}$, $p(E|F_2) = \frac{3}{8}$, $p(E|F_3) = \frac{1}{2}$, $p(F_1) = \frac{1}{6}$, $p(F_2) = \frac{1}{2}$, and $p(F_3) = \frac{1}{3}$.
3. What is expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \ldots, 50\}$ and the purchaser wins nothing otherwise?
4. What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?

5. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.6 is tossed n times deviates from the mean by more than $\sqrt{n}$.

Homework 12 (due May 5, 2022)

1. Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix representing R^2.

2. Find the smallest relation containing the relation $\{(1,2),(1,4),(3,3),(4,1)\}$ that is
   (a) reflexive and transitive.
   (b) symmetric and transitive.
   (c) reflexive, symmetric, and transitive.

3. Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

4. Determine whether the relations represented by these zero-one matrices are equivalence relations.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. Determine whether the relations represented by these zero-one matrices are partial orders.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$