CS220 Discrete Math - Homework #6

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Question 1

In order to prove or disprove that $p_1p_2\cdots p_n + 1$ is prime for every n where $p_1p_2\cdots p_n$ are the n smallest prime numbers, we can test random numbers of the first few prime numbers.

$$
\begin{aligned}
n = 1 &: 2 + 1 = 3 \text{ prime} \\
2 &: 2 \times 3 + 1 = 7 \text{ prime} \\
3 &: 2 \times 3 \times 5 + 1 = 31 \text{ prime} \\
4 &: 2 \times 3 \times 5 \times 7 + 1 = 211 \text{ prime} \\
5 &: 2 \times 3 \times 5 \times 7 \times 11 + 1 = 2311 \text{ prime} \\
6 &: 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509 \text{ NOT prime}
\end{aligned}
$$

Because the statement fails for the 6 smallest prime numbers, we can say that $p_1p_2\cdots p_n + 1$ is not prime for every n.

Question 2

First, we need to find $\gcd(34, 89)$ using the Euclidean algorithm.

$$
\begin{aligned}
89 &= 2 \times 34 + 21 \\
34 &= 1 \times 21 + 13 \\
21 &= 1 \times 13 + 8 \\
13 &= 1 \times 8 + 5 \\
8 &= 1 \times 5 + 3 \\
5 &= 1 \times 3 + 2 \\
3 &= 1 \times 2 + \boxed{\mathbf{1}}
\end{aligned}
$$

Then we use Bézout's theorem to find the linear combination. We will work backwards from the previous operations.

$$
\begin{aligned}
1 &= 3 - (5 - 3) \\
&= 2 \times 3 - 5 \\
&= 2 \times (8 - 5) - 5 \\
&= 2 \times 8 - 3 \times 5 \\
&= 2 \times 8 - 3 \times (13 - 8) \\
&= 5 \times 8 - 3 \times 13 \\
&= 5 \times (21 - 13) - 3 \times 13 \\
&= 5 \times 21 - 8 \times 13 \\
&= 5 \times 21 - 8(34 - 21) \\
&= 13 \times 21 - 8 \times 34 \\
&= 13 \times (89 - 2 \times 34) - 8 \times 34 \\
&= 13 \times 89 - 26 \times 34 - 8 \times 34 \\
&= 13 \times 89 - \boxed{\mathbf{34}} \times 34
\end{aligned}
$$

The inverse of 34 modulo 89 is **-34** or **55**.

Question 3

Using a that we found previously, we can solve the given linear congruence.

$$
\begin{aligned}
34x &\equiv 77 \pmod{89} \\
1870x &= 55 \times 77 \pmod{89} \\
x &\equiv 4235 \equiv 47 \times 89 + 52 \equiv \boxed{\mathbf{52}} \pmod{89}
\end{aligned}
$$

Question 4

The pairs of positive integers that are less than 11 (that don't include 1 or 10) such that each pair are inverses of each other modulo 11 are shown below:

$$
\begin{aligned}
2 \times 6 = 12 = 1 \times 11 + 1 \equiv 1 \pmod{11} \\
3 \times 4 = 12 = 1 \times 11 + 1 \equiv 1 \pmod{11} \\
5 \times 9 = 45 = 4 \times 11 + 1 \equiv 1 \pmod{11} \\
7 \times 8 = 56 = 5 \times 11 + 1 \equiv 1 \pmod{11}
\end{aligned}
$$

Question 5

Fermat's Little Theorem states that, if a number p is prime and another number a is not divisible by p, then

$$a^{(p-1)} = 1 \pmod p$$

Therefore, we can solve $23^{1002} \bmod 41$ by:

$$\begin{aligned}
23^{1002} \pmod{41} &= (23^{40})^{23} \times 23^2 \pmod{41} \\
&= 1^{23} \times 23^2 \pmod{41} \\
&= 23^2 \pmod{41} \\
&= 529 \pmod{41} \\
529 &= 12 \times 41 + 37 \\
529 \pmod{41} &= \boxed{\mathbf{37}}
\end{aligned}$$