CS220 Test 1 Study Guide

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1 The First Question (50 points)

The first question comes directly from the textbook and lecture notes.

1. Is $p \to q$ equivalent to $\neg p \to \neg q$?
2. Is $\neg(p \leftrightarrow q)$ equivalent to $p \leftrightarrow \neg q$?
3. Describe *modus ponens*.
4. Let S be a set and $|S| = k$, where k is a nonnegative integer. What is $|\mathcal{P}(S)|$?
5. Let r be the common ratio of a geometric series, $0 < r < 1$. Find the sum $\sum_{i=0}^{n} ar^i$.
6. Demonstrate $3\log n + 5 \in O(\log n)$.
7. What is the worst-case and average-case runtime of binary search?
8. Find Bézout's coefficients of 252 and 198.
9. Find an inverse of 5 modulo 11.
10. Use mathematical induction to prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

2 The Second Question (30 points)

The second question is easier than homework questions.

1. Prove or disprove that $(p \to q) \to r$ and $p \to (q \to r)$ are equivalent.
2. Show that the set of irrational numbers is an uncountable set.
3. Prove that there is no smallest positive rational number.
4. Determine which relationship $(\subseteq, =, \supseteq)$ is true for the pair of sets.
   (a) $A \cup B$, $A \cup (B - A)$
   (b) $A \cup (B \cap C)$, $(A \cup B) \cap C$
   (c) $(A - B) \cup (A - C)$. $A - (B \cap C)$
   (d) $(A - C) - (B - C)$, $A - B$
5. Prove or disprove that if A, B, and C are sets then $A - (B \cap C) = (A - B) \cap (A - C)$.
6. What is the asymptotic relationship between $n^{\frac{3}{4}}(\log n)^{\frac{4}{3}}$ and $n^{\frac{4}{3}}(\log n)^{\frac{3}{4}}$?
7. Show that an integer is divisible by 9 if and only if the sum of its decimal digits is divisible by 9.
8. Prove that there are no solutions in positive integers to the equation $x^4 + y^4 = 100$.

9. Use the Euclidean algorithm to find $\gcd(203, 101)$ and $\gcd(34, 21)$.
10. Use mathematical induction to show that $\sum_{i=0}^{n}(i+1) = \frac{(n+1)(n+2)}{2}$ whenever n is a nonnegative integer.

3 The Third Question (20 points)

The third question is similar to homework questions in difficulty.

1. Prove that $(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p$ is a tautology using propositional equivalence and the laws of logic.
2. Recall that two sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. Prove that the open interval $(0, 1)$ has the same cardinality as the open interval $(0, 2)$.
3. Arrange the functions in a list so that each function is big-O of the next function:

   $n \cdot 2^n$, $\log(n^n)$, $(n^{100})^n$, $\log(n!)$, 3^n, $2^{n \log n}$, $n^{\frac{3}{2}}$
4. Show that the function $f(n) = (n+2)\log(n^2+1) + \log(n^3+1)$ is O($n \log n$).
5. Take a positive integer and we can write it in the octal number system. Prove or disprove: The positive integer is divisible 7 if and only if the sum of its octal digits is divisible by 7.
6. Use mathematical induction to prove that $2 \mid (n^2 + n)$ for all $n \geq 0$. [*Note*: this means $n^2 + n$ is an even number.]
7. Let m be a positive integer, and let a, b, and c be integers. Show that if $a \equiv b \pmod{m}$, then $a - c \equiv b - c \pmod{m}$.
8. Use the telescoping sum technique to derive $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
9. Show that $3^n < n!$ whenever n is an integer with $n \geq 7$.
10. Suppose that the only currency were 3-dollar bills and 10-dollar bills. Show that every amount greater than 17 dollars could be made from a combination of these bills.

4 Additional Practice Questions

1. Is the assertion "This statement is false" a proposition?
2. Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (p \vee \neg p)$ is true when p, q, and r have the same truth value and it is false otherwise.
3. The n-th statement in a list of 100 statements is "Exactly n of the statements in this list are false."
   (a) What conclusions can you draw from these statements?
   (b) Answer part (a) if the n-th statement is "At least n of the statements in this list are false."
   (c) Answer part (b) assuming that the list contains 99 statements.

4. Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [*Hint*: Work backward, assuming that you did end up with nine zeros.]

5. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. If A, B, C. and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?

6. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

   (a) 3, 6, 11, 18, 27, 18, 38, 51, 66, 83, 102, ...

   (b) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...

   (c) 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, ...

7. Let a_n be the n-th term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, ..., be constructed by including the integer k exactly k times. Show that

$$a_n = \lfloor \sqrt{2n} + \frac{1}{2} \rfloor.$$

8. The ternary search algorithm locates an element in a list of increasing integers by successively splitting the list into three sublists of equal (or as close to equal as possible) size, and restricting the search to the appropriate piece. Specify the steps of this algorithm.

9. Show that if there were a coin worth 12 cents, the greedy algorithm using quarters, 12-cent coins, dimes, nickels, and pennies would not always produce change using the fewest coins possible.

10. Give the big-O of these functions.

    (a) $n \log(n^2 + 1) + n^2 \log n$

    (b) $(n \log n + 1)^2 + (\log n + 1)(n^2 + 1)$

    (c) $n^{2^n} + n^{n^2}$

11. Show that if a, bm k, and m are integers such that $k \geq 1$, $m \geq 2$, and $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$.

12. Show that if a, bm c, and m are integers such that $m \geq 2$, $c > 0$, and $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{m}$.

13. Show that if a and b are both positive integers, then $(2^a - 1) \bmod (2^b - 1) = 2^{a \bmod b} - 1$.

14. Solve the system of congruence $x \equiv 3 \pmod{6}$ and $x \equiv 4 \pmod{7}$.

15. Prove that for every positive integer n,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

16. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

17. Show that the set S defined by $1 \in S$ and $s + t \in S$ whenever $s \in S$ and $t \in S$ is the set of positive integers.
18. Use structural induction to show that $n(T) \geq 2h(T) + 1$, where T is a full binary tree, $n(T)$ is the number of vertices of T, and $h(T)$ is the height of T.
19. Devise a recursive algorithm to find a^{2^n}, where a is a real number and n is a positive integer. [*Hint*: Use the equality $a^{2^{n+1}} = (a^{2^n})^2$.]

5 Sample Test 1

1. Solve $4x \equiv 5 \pmod 9$.
2. Show that if there were a coin worth 12 cents, the greedy algorithm using quarters, 12-cent coins, dimes, nickels, and pennies would not always produce change using the fewest coins possible.
3. Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when p, q, and r have the same truth value and it is false otherwise.

6 Real Test 1 Questions

The following questions were the questions I received for test 1.

1. Use mathematical induction to prove that $n^2 + n < n^3$ for all $n > 2$.
2. Given that a, b, and c are nonnegative integers, show that if $a|c$ and $b|c$, then $ab|c^2$. [*Hint*: $x|y$ means that x divides y or that y is divisible by x.]
3. Prove that the open interval $(0, 1)$ has the same cardinality as the open interval $(0, 2)$.