CS310: Advanced Data Structures and Algorithms

Fall 2022 Assignment 3

Due: Monday, October 31, 2022 on Gradescope

Goal

Practice hash tables and graphs.

Questions

1. **Hashing:** Quadratic probing is an alternative way to resolve collisions. It is similar to linear probing in the sense that if a collision occurs, another index will be probed. but if, say, the key is hashed to index i, instead of probing $(i + 1)\%m$, $(i + 2)\%m$, $(i + 3)\%m$, etc., the probing is done in quadratically larger jumps, so the probing chain in this case is $(i + 1)\%m$, $(i + 4)\%m$, $(i + 9)\%m$, etc., and in general the k^{th} probe step is $(i + k^2)\%m$.

   Given a hash table of size 11, the key data is the following identifiers for some inventory: A29, C42, E12, D31, F08 and G34, B10. The hash function is H(LN) = ((L-'A') + N) % 11, where L is the letter, N is the number and 'A' is the ASCII value for A. For example, H("C29") = (2 + 29)% 11 = 9 since 'C' - 'A' = 2. For your convenience– the multiples of 11 are 11, 22, 33, 44, 55 etc. (such that the two digits are the same).

   (a) Draw the final configuration of the table after all the elements are inserted in the following illustration using linear probing. Do not rehash.

   The hash of each of the identifiers are as shown:

   - H("A29") = (0 + 29) % 11 = 7
   - H("C42") = (2 + 42) % 11 = 0
   - H("E12") = (4 + 12) % 11 = 5
   - H("D31") = (3 + 31) % 11 = 1
   - H("F08") = (5 + 8) % 11 = 2
   - H("G34") = (6 + 34) % 11 = 7 (index 7 is already filled here so index 8 is filled)
   - H("B10") = (1 + 10) % 11 = 0 (index 0, 1, and 2 are filled here so index 3 is filled)

C42	D31	F08	B10		E12		A29	G34		

   (b) Do the same for quadratic probing. Again, do not rehash.

   We will follow a similar idea to the previous question.

   - H("A29") = (0 + 29) % 11 = 7
   - H("C42") = (2 + 42) % 11 = 0

- H("E12") = (4 + 12) % 11 = 5
- H("D31") = (3 + 31) % 11 = 1
- H("F08") = (5 + 8) % 11 = 2
- H("G34") = (6 + 34) % 11 = 7 (index 7 is already filled here so index at $(7+1)\%11$ is filled)
- H("B10") = (1 + 10) % 11 = 0 (index 0 and 1 ($i+1$) are filled so $i+4$ is filled)

C42	D31	F08		B10	E12		A29	G34		

2. **Graphs: K&T, Ch. 3 q.4:** inspired by the example of that great Cornellian, Vladimir Nabokov, some of your friends have become amateur lepidopterists (they study butterflies). Often when they return from a trip with specimens of butterflies, it is very difficult for them to tell how many distinct species they've caught – thanks to the fact that many species look very similar to one another.

   One day they return with n butterflies, and they believe that each belongs to one of two different species, which we'll call A and B for purposes of this discussion. They'd like to divide the n specimens into two groups–those that belong to A and those that belong to B – but it's very hard for them to directly label any one specimen. So they decide to adopt the following approach.

   For each pair of specimens i and j, they study them carefully side by side. If they're confident enough in their judgment, then they label the pair (i,j) either "same" (meaning they believe them both to come from the same species) or "different" (meaning they believe them to come from different species). They also have the option of rendering no judgment on a given pair, in which case we'll call the pair *ambiguous*.

   So now they have the collection of n specimens, as well as a collection of m judgments (either "same" or "different") for the pairs that were not declared to be ambiguous. They'd like to know if this data is consistent with the idea that each butterfly is from one of species A or B. So more concretely, we'll declare the m judgments to be consistent if it is possible to label each specimen either A or B in such a way that for each pair (i,j) labeled "same," it is the case that i and j have the same label; and for each pair (i,j) labeled "different," it is the case that i and j have different labels. They're in the middle of tediously working out whether their judgments are consistent, when one of them realizes that you probably have an algorithm that would answer this question right away.

   Give an algorithm with running time $O(m+n)$ that determines whether the m judgments are consistent. **Hint:** There is an underlying undirected graph problem here. Try to find out what are the vertices, what are the edges, and what algorithm(s) we showed in class can solve this problem.

   In order to solve this problem, we need to create an underlying graph where the vertices are each of the specimens and the edges indicate any judgments relating any two vertices. So given n specimens and m judgments, the algorithm would be as follows:

   Input: Graph = $G = (m, n)$

   1. Run BFS on G starting from an arbitrary vertex v.
   2. During each iteration of the BFS, we check the judgments of any connecting nodes to the current vertex and label the vertices accordingly.

3. Then we check each edge. If the judgment was "same" but the labels of the vertices attached to the judgment are different, then we have an inconsistency (so return `false`).
4. If the judgment was "different" but the labels of the vertices attached to the judgment are the same, then we also have an inconsistency (so return `false`).
5. If the BFS doesn't end prematurely by returning `false`, then return `true`.

3. **Graphs: K&T, Ch. 3 q.9:** There's a natural intuition that two nodes that are far apart in a communication network– separated by many hops– have a more "tenuous" connection than two nodes that are close together. There are a number of algorithmic results that are based to some extent on different ways of making this notion precise. Here's one that involves the susceptibility of paths to the deletion of nodes. Suppose that an n-node undirected graph $G = (V, E)$ contains two nodes s and t such that the distance between s and t is strictly greater than n/2.

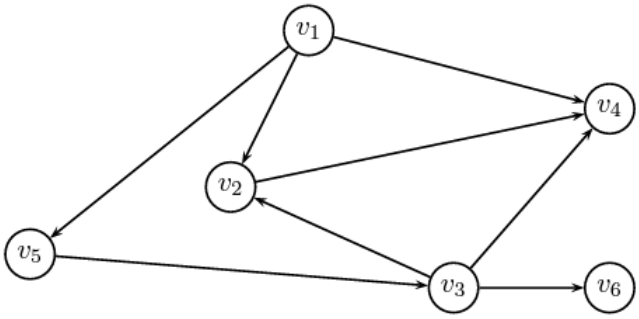
   (a) Show that there must exist some node v, not equal to either s or t, such that deleting v from G destroys all s-t paths. (In other words, the graph obtained from G by deleting v contains no path from s to t.) **Hint:** The proof is not very complicated. Just reason about the distance between s and t being greater than $n/2$, assume such a node v does not exist and prove by contradiction why it cannot be.

   - We know that the minimum distance between nodes s and t must be $\frac{n}{2} + 1$
   - We assume that there is no node v such that deleting it destroys all paths between s and t.
   - If this was the case, then we would have the case that each layer between s and t would have 2 nodes within them (deleting a node of a layer still leaves a path remaining). This means that the total number of nodes in the tree NOT including s or t would be n $(2 \cdot \frac{n}{2})$. This would be impossible as the given size of the graph is n and the size of the previously described graph would be of size $n + 2$.
   - This means that there must be a layer that only contains one node within it. We know that deleting a layer with only one node would destroy all paths from s to t.

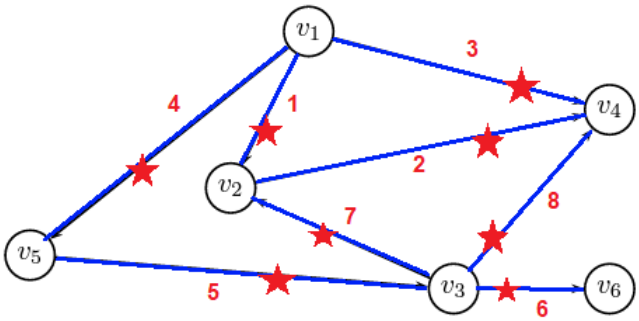
   (b) Give an algorithm with running time $O(m + n)$ to find such a node v. **Hint:** The algorithm involves some graph traversal. The algorithm centers around BFS.

   1. We can BFS from s and t.
   2. If we find a node v that the BFS finds from s and t, then we return `true`.
   3. If not, then we return `false`.

4. **Directed Graphs:** Trace Depth-First search (DFS) on the following graph, starting from v_1. Mark all vertices and edges in order of visitation. To make the search fully specified, if two or more options exist, break tie by alphabetical order. Mark by * the edges that participate in the DFS tree.



The order of visitation would be as follows: $v_1 \to v_2$, $v_2 \to v_4$, $v_1 \to v_4$, $v_1 \to v_5$, $v_5 \to v_3$, $v_3 \to v_6$, $v_3 \to v_2$, $v_3 \to v_4$



5. **DAG:** A *Hamiltonian path* in a graph is a graph that visits every vertex exactly once.

   (a) Show that a DAG (directed acyclic graph) has a hamiltonian path if and only if it has just one topological ordering (you may assume the graph is connected). Remember this is an if and only if proof, so show both sides.

   (b) Give an $O(m+n)$ algorithm to find whether a DAG has a hamiltonian path.

   Given a DAG $G = (V, E)$, the algorithm would consist of:

   1. Sort the graph topologically using DFS.
   2. We can use DFS again to search through every vertex in the graph to ensure that there is at least one edge between consecutive vertices.
   3. If there isn't, we can return `false`.
   4. If the DFS ends without returning `false`, then we must have a Hamiltonian path (and thus, we return `true`).

6. **DAG:** A student suggested to topologically sort a DAG $G = (V, E)$ by first running BFS from a vertex s with an in-degree of 0, and then listing the vertices in the order of their distance from s. Show a simple example where this algorithm will not work. You may assume G is indeed a DAG, so no wise-assery.