CS 188: Artificial Intelligence

Constraint Satisfaction Problems

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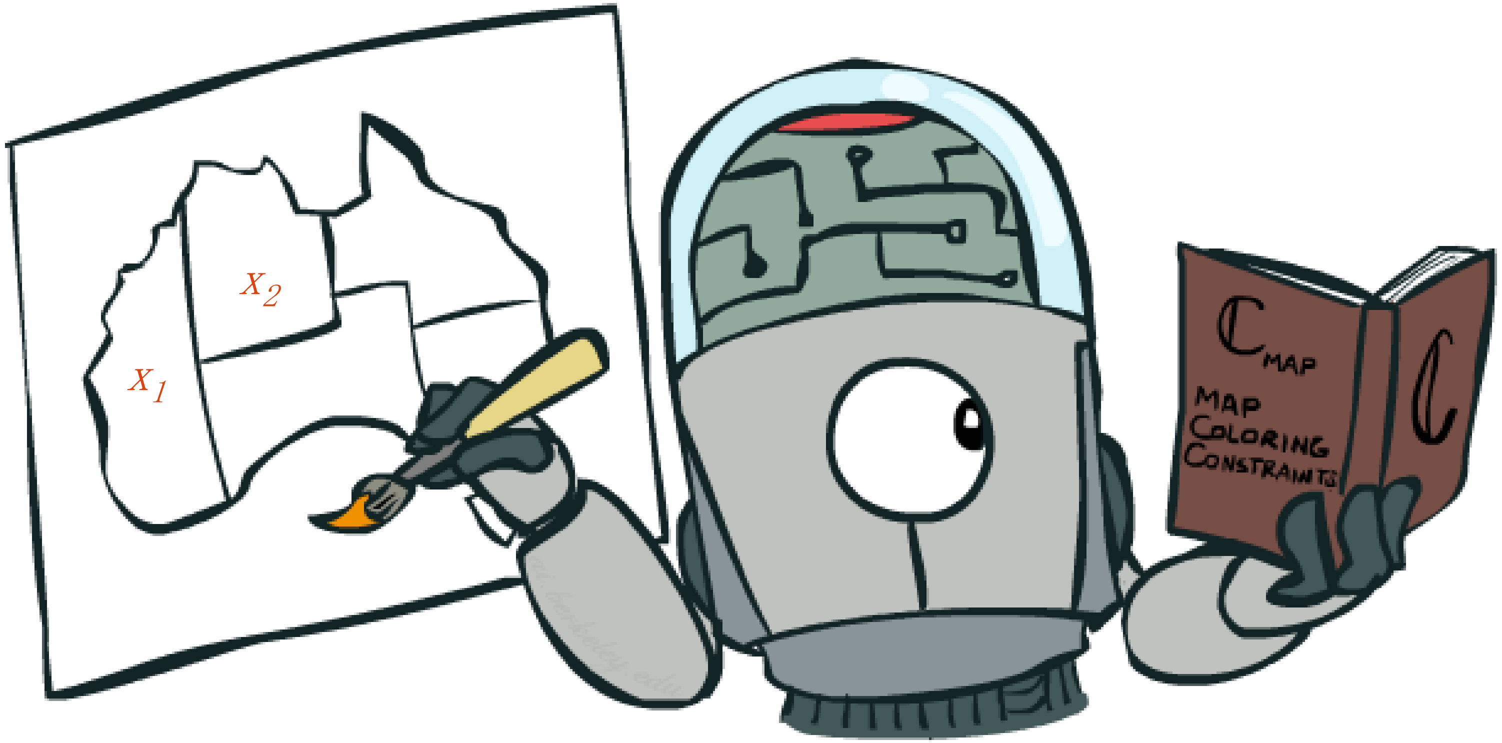
[These slides adapted from Dan Klein and Pieter Abbeel]

Constraint Satisfaction Problems

N variables

domain D

constraints



states

partial assignment

goal test

complete; satisfies constraints

successor function

assign an unassigned variable

What is Search For?

- Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space

- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - **Heuristics** give problem-specific guidance

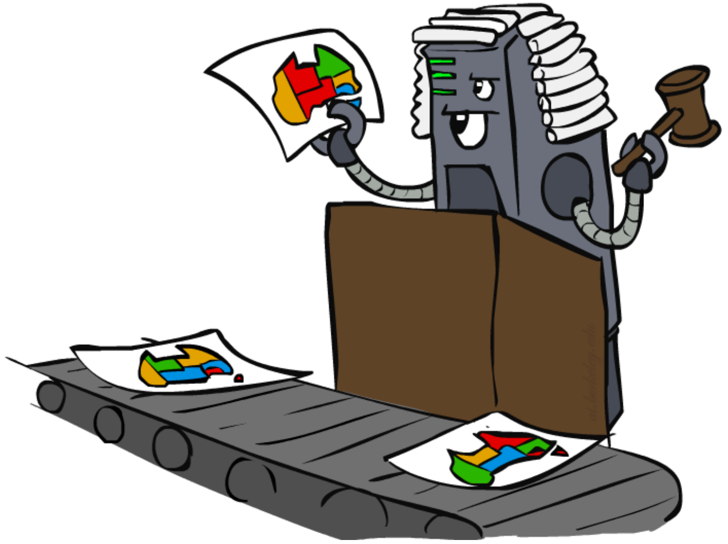
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems

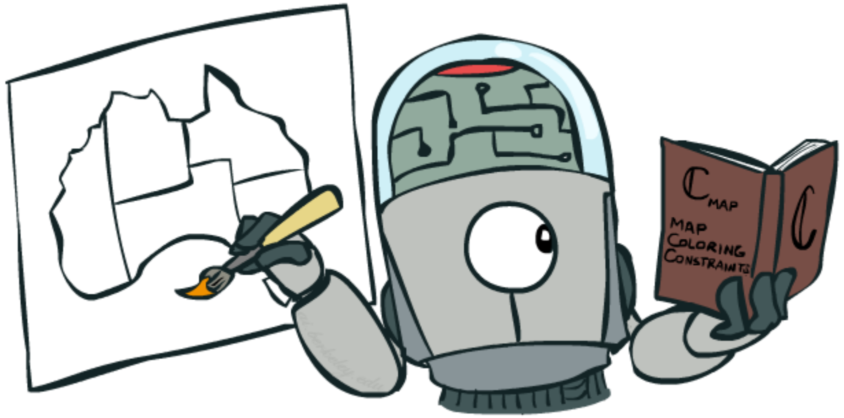
Constraint Satisfaction Problems

- Standard search problems:
  - State is a “black box”: arbitrary data structure
  - Goal test can be any function over states
  - Successor function can also be anything

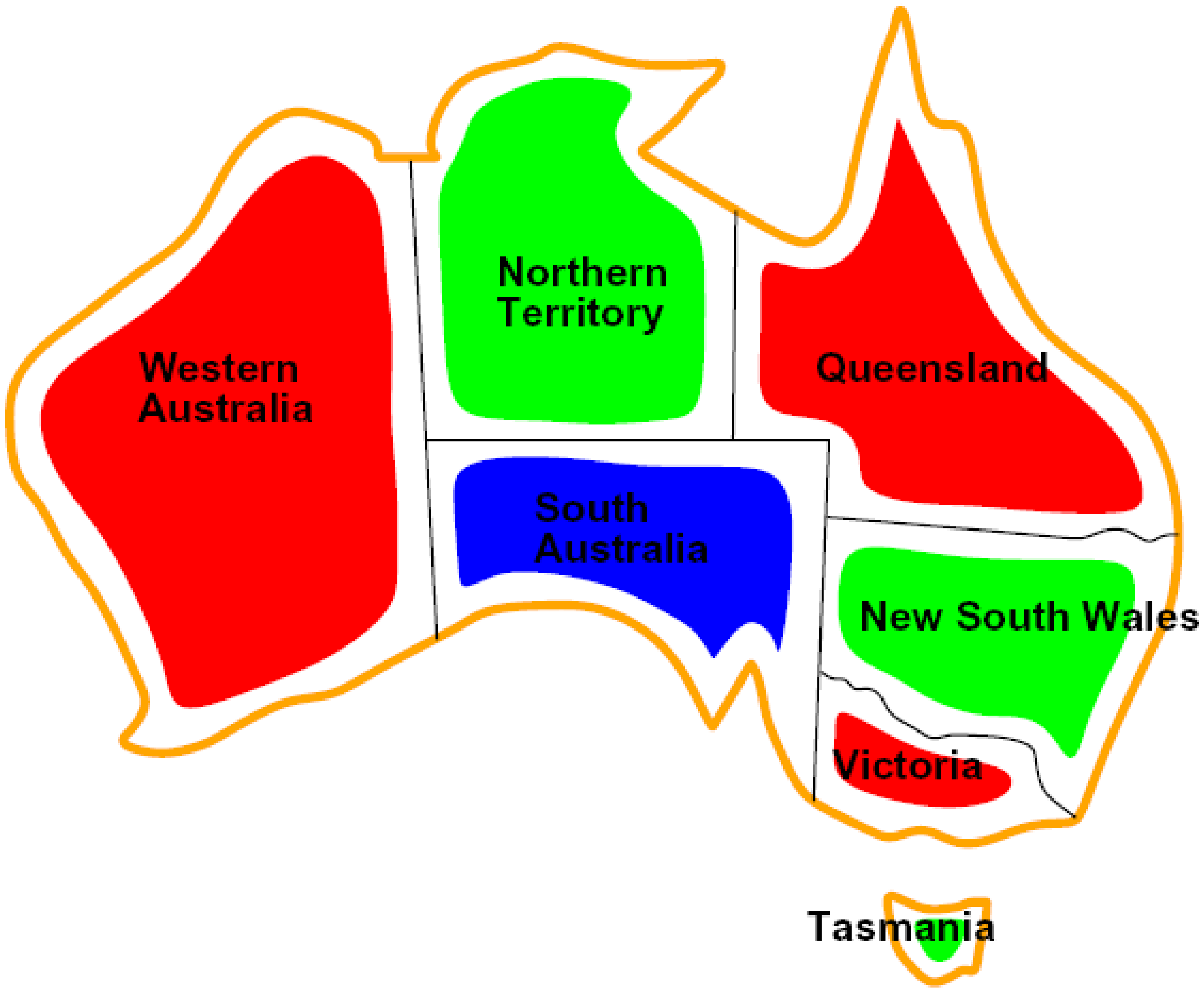
- Constraint satisfaction problems (CSPs):
  - A **special subset** of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Allows useful **general-purpose algorithms** with **more power** than standard search algorithms





CSP Examples



Example: Map Coloring

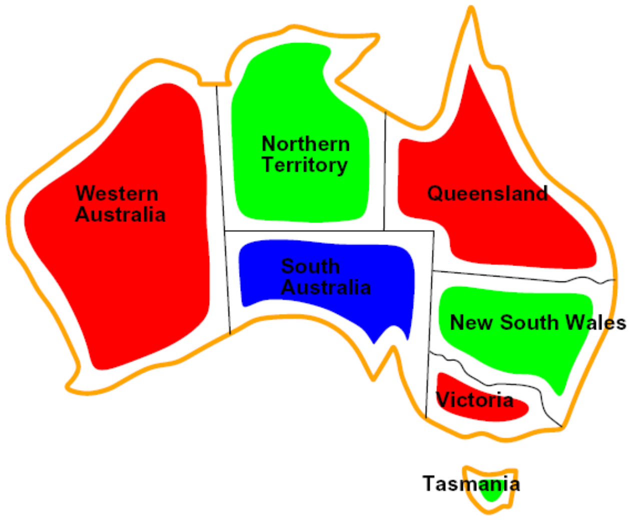
- Variables WA, NT, Q, NSW, V, SA, T
- Domains: $D = \{\text{red}, \text{green}, \text{blue}\}$
- Constraints: adjacent regions must have different colors

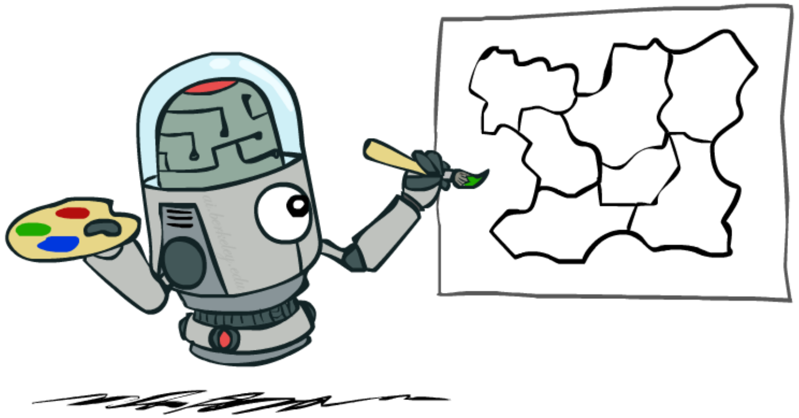
  Implicit: $WA \neq NT$

  Explicit: $(WA, NT) \in \{(\text{red}, \text{green}), (\text{red}, \text{blue}), \ldots\}$

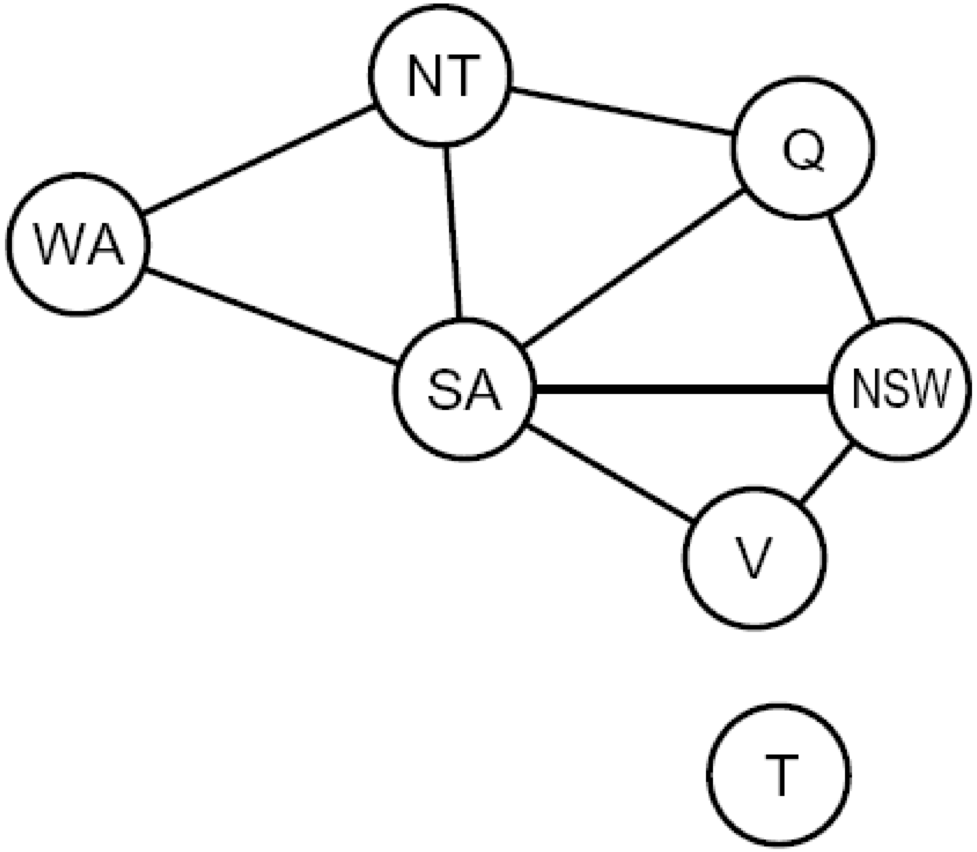
- Solutions are *assignments satisfying all constraints*, e.g.:

  {WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}





Constraint Graphs

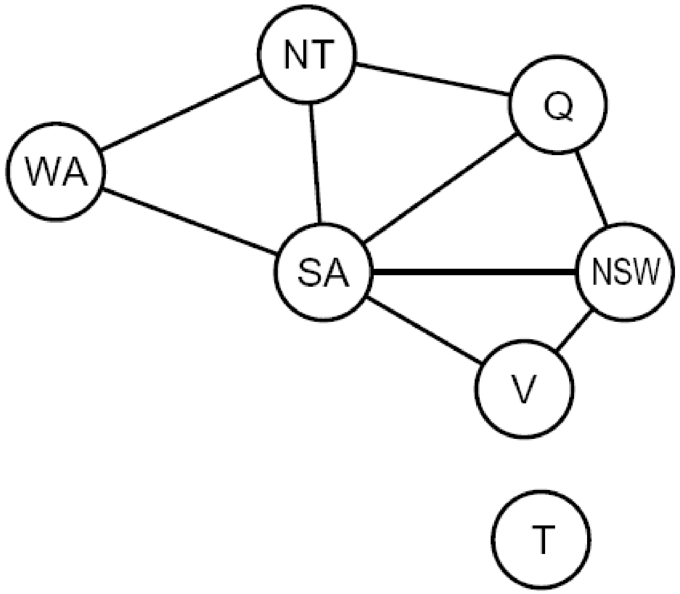


Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables

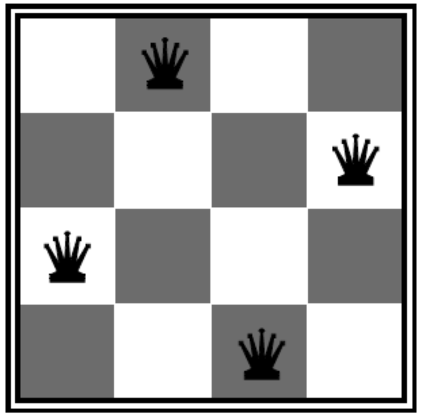
- Binary constraint graph: nodes are variables, arcs show constraints

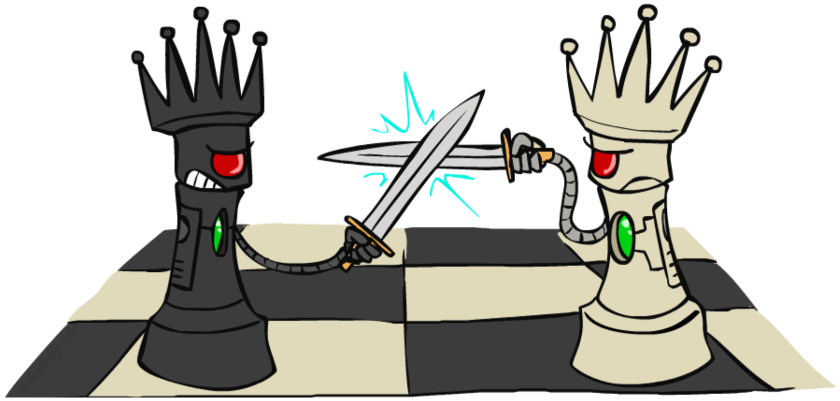
- General-purpose CSP algorithms use the **graph structure** to speed up search. E.g., Tasmania is an independent subproblem!



Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints





$$\forall i, j, k \ \ (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

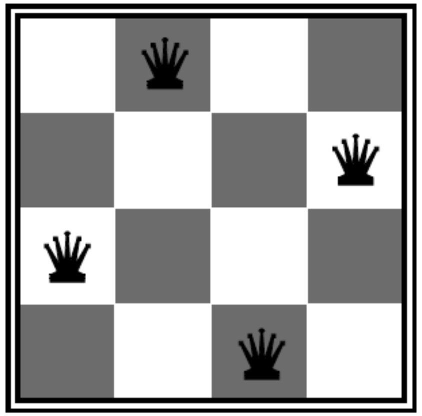
- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:

Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

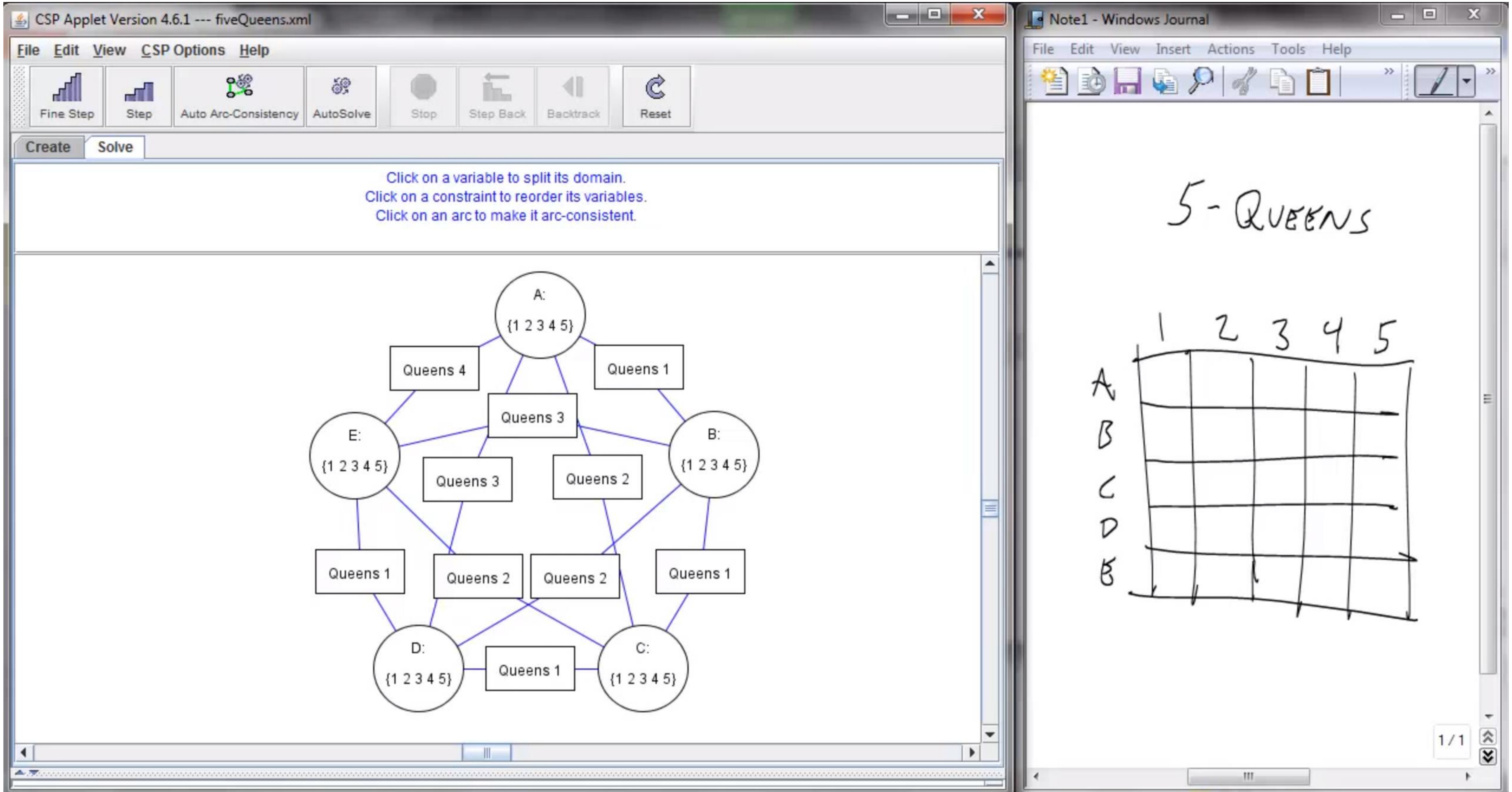
Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \ldots\}$

$\cdots$

Q_1
Q_2
Q_3
Q_4



Screenshot of Demo N-Queens



Example: Cryptarithmetic

- Variables:

  $F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$

- Domains:

  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- Constraints:

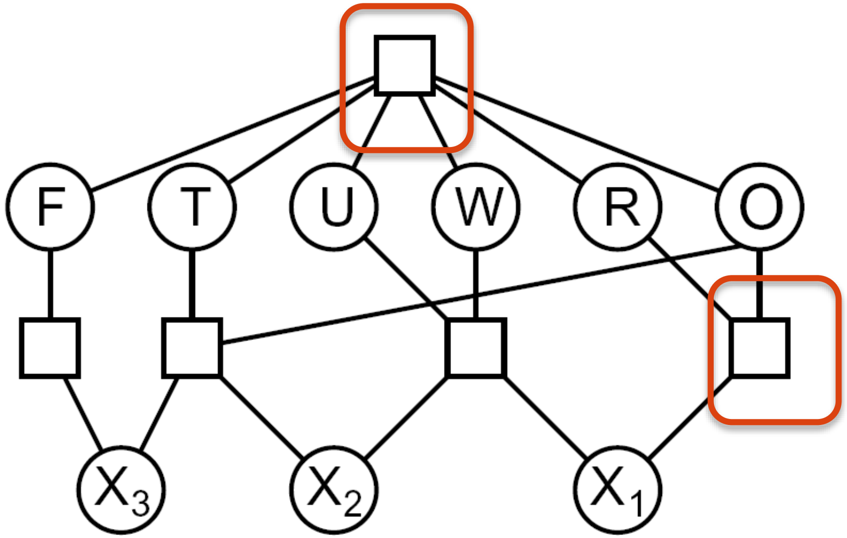
  $\text{alldiff}(F, T, U, W, R, O)$

  $O + O = R + 10 \cdot X_1$

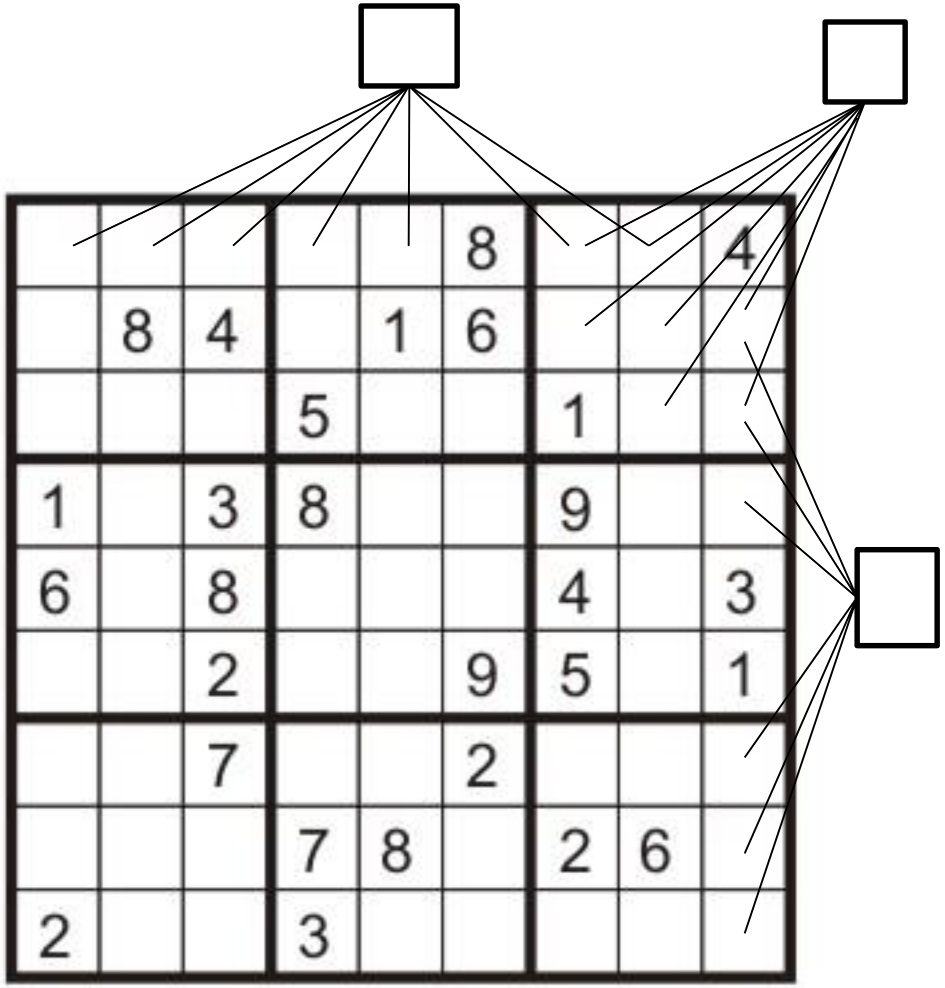
  $\cdots$

X_1

$$
\begin{array}{cccc}
 & T & W & O \\
+ & T & W & O \\
\hline
F & O & U & R
\end{array}
$$



Example: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

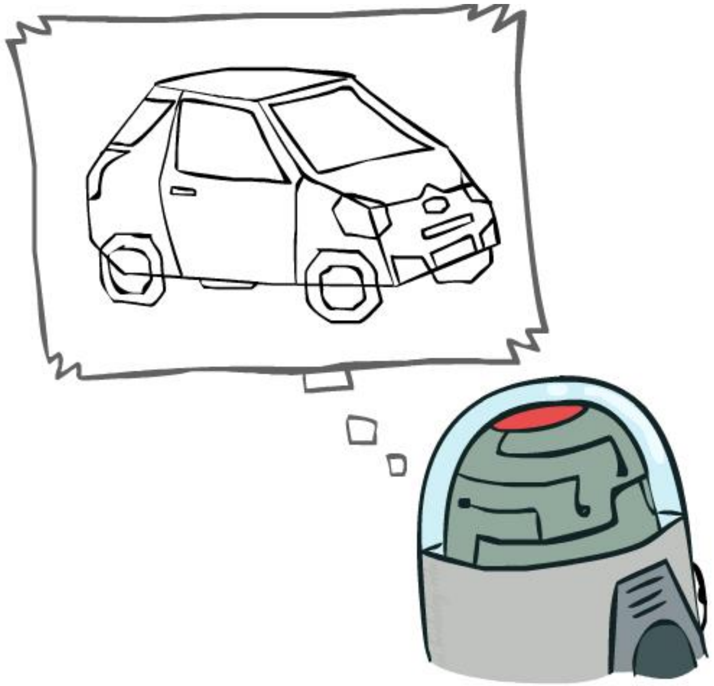
9-way alldiff for each row

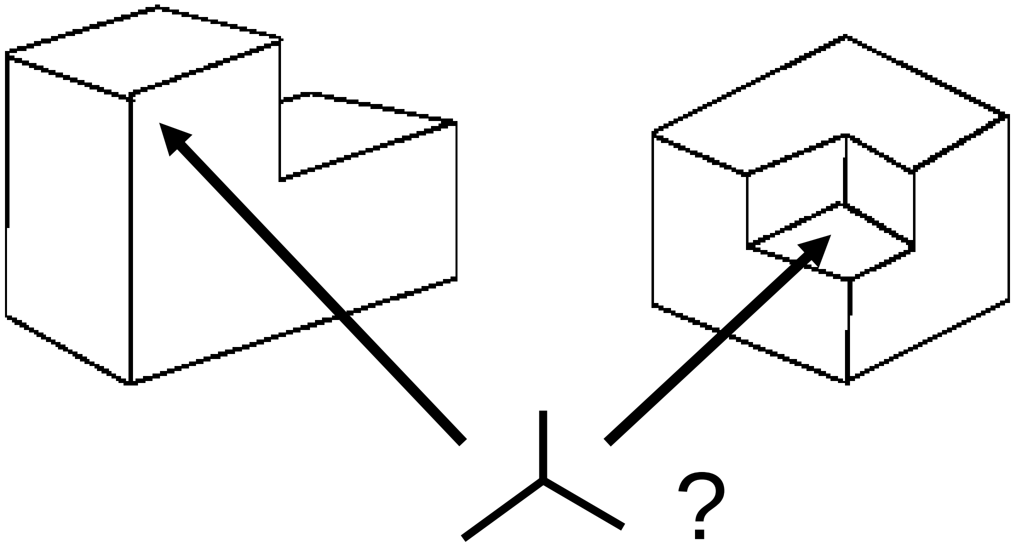
9-way alldiff for each region

(or can have a bunch of pairwise inequality constraints)

Example: The Waltz Algorithm

- The **Waltz algorithm** is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP





- Approach:
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
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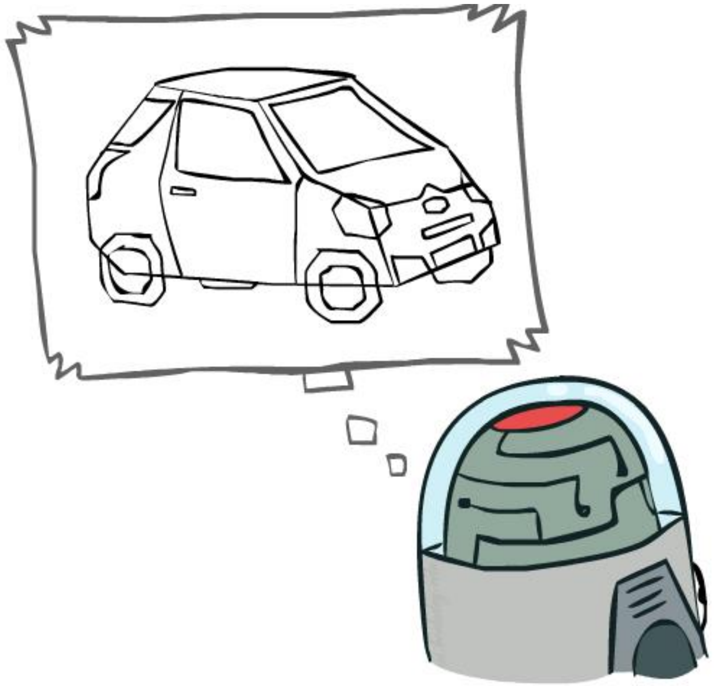
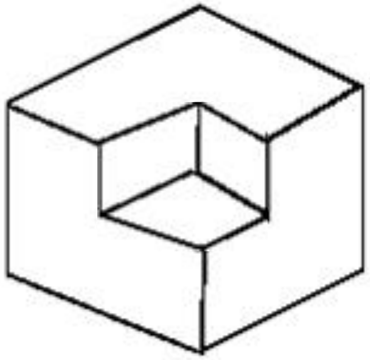
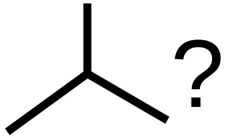




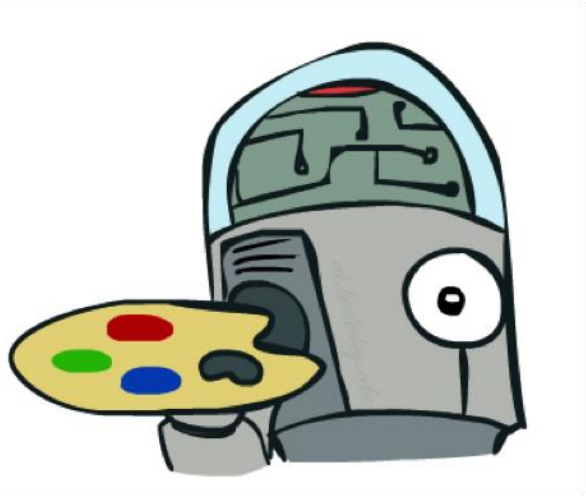


- Approach:
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Varieties of CSPs and Constraints

Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including **Boolean satisfiability** (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable



- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by LP methods (see cs170 for a bit of this theory)

Varieties of Constraints

- Varieties of Constraints
  - **Unary constraints** involve a single variable (equivalent to reducing domains), e.g.:

    $$SA \neq green$$

  - **Binary constraints** involve pairs of variables, e.g.:

    $$SA \neq WA$$

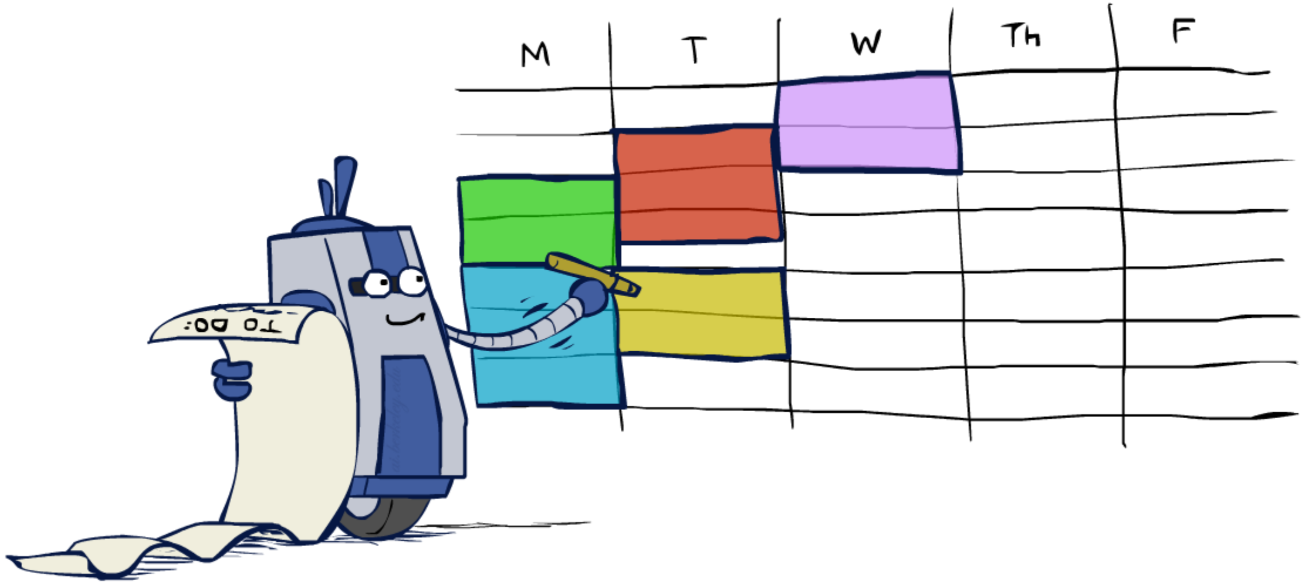
  - **Higher-order constraints** involve **3 or more variables:**

    e.g., cryptarithmetic column constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often **representable by a cost** for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes'

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!



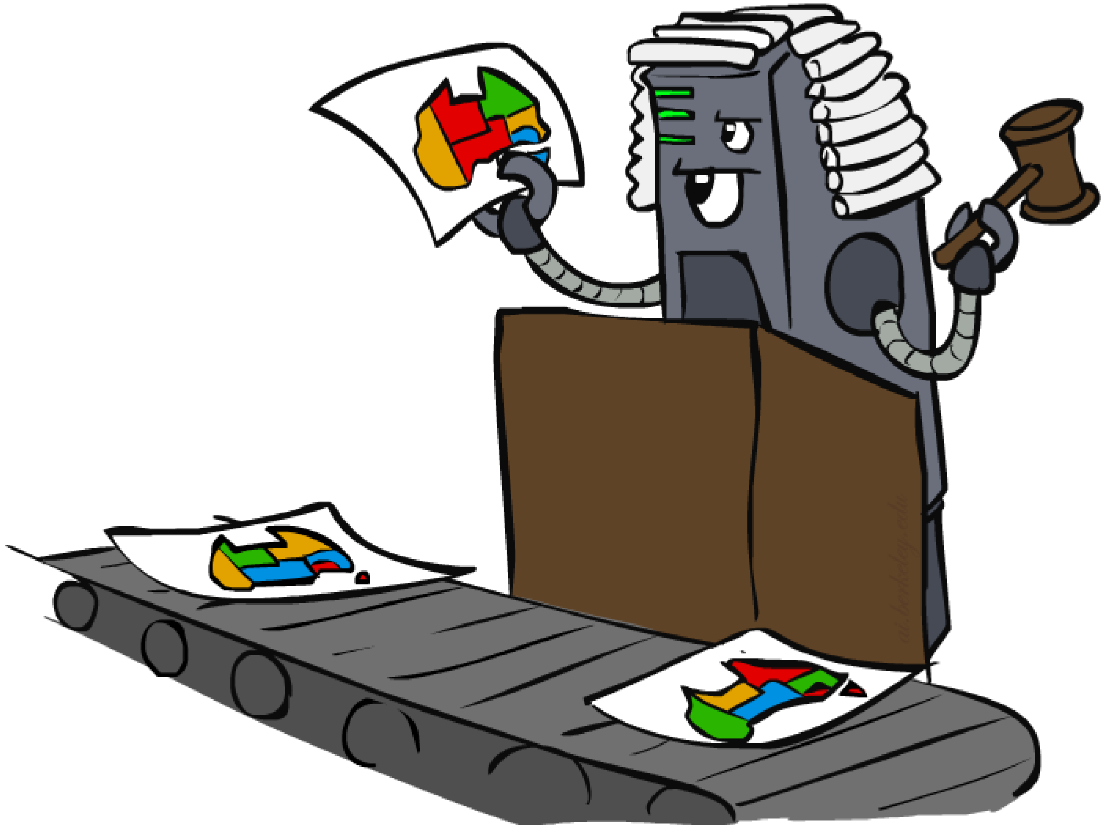
- Many real-world problems involve real-valued variables…

Solving CSPs

Standard Search Formulation

- Standard search formulation of CSPs

- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - **Successor function**: assign a value to an unassigned variable
  - **Goal test**: the current assignment is complete and satisfies all constraints
- We' ll start with the straightforward, naïve approach,

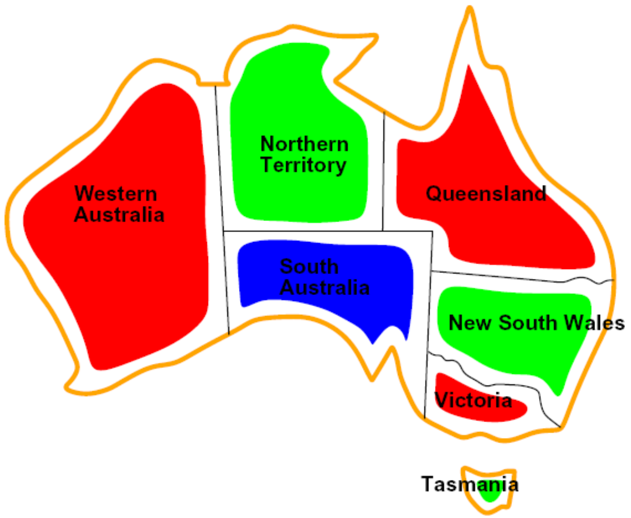


Search Methods

- What would BFS do?

$\{\}$

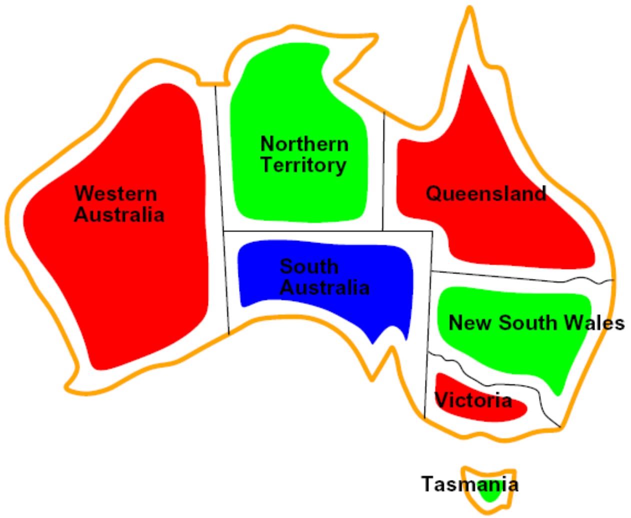
$\{WA=g\}$ $\{WA=r\}$ … $\{NT=g\}$ …



[Demo: coloring -- dfs]

Search Methods

- What would BFS do?

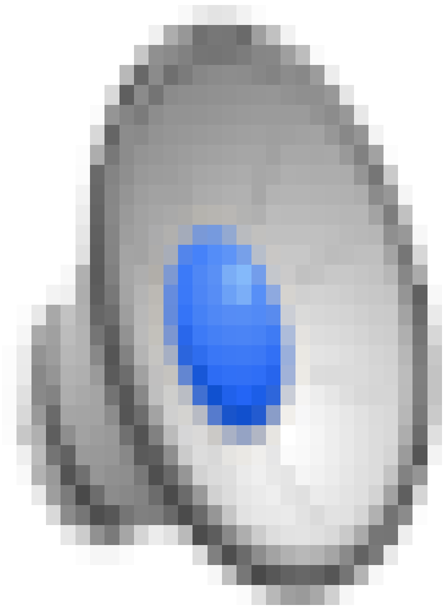


- What would DFS do?
  - let’s see!

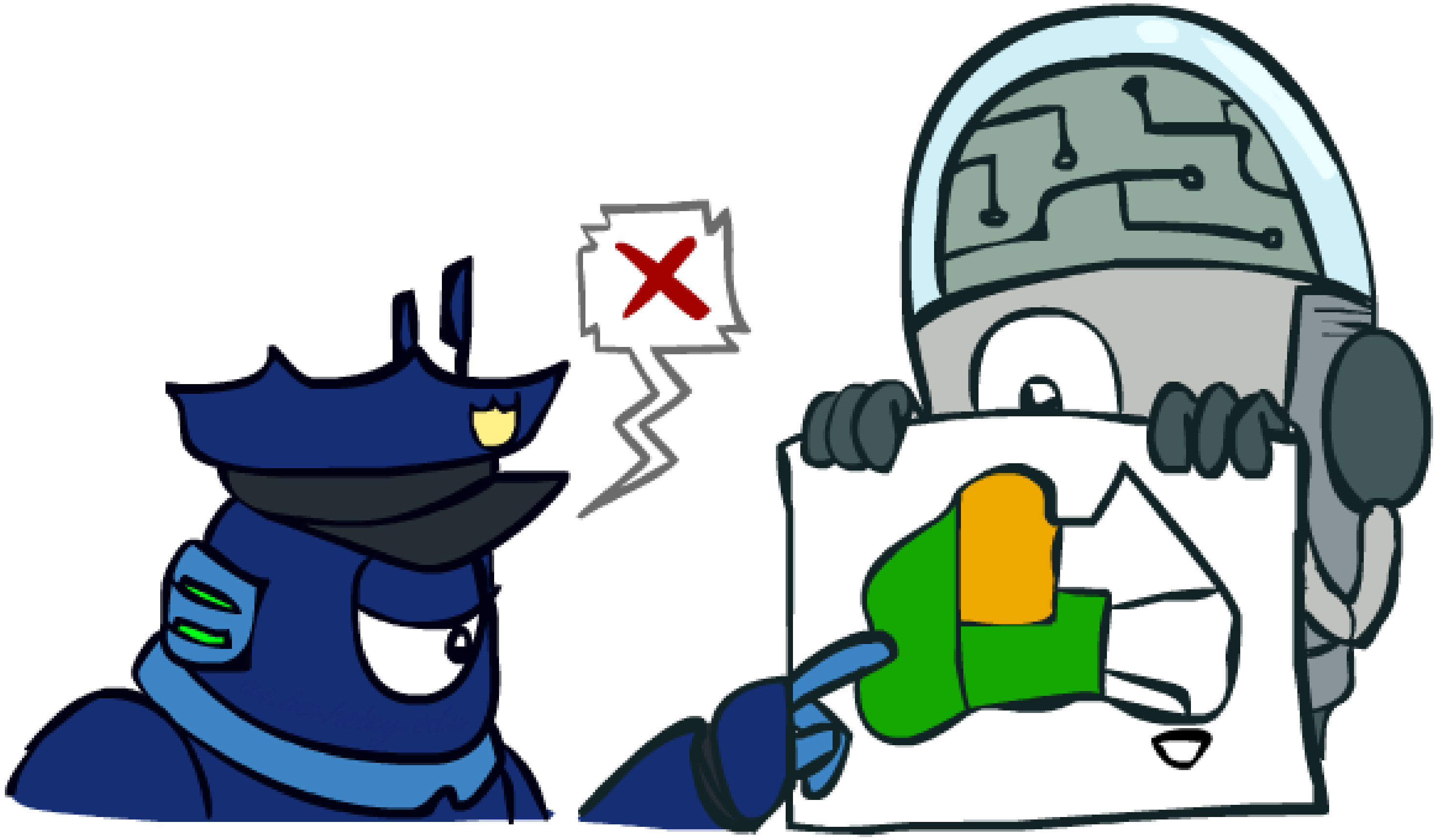
- What problems does naïve search have?

[Demo: coloring -- dfs]

Video of Demo Coloring -- DFS

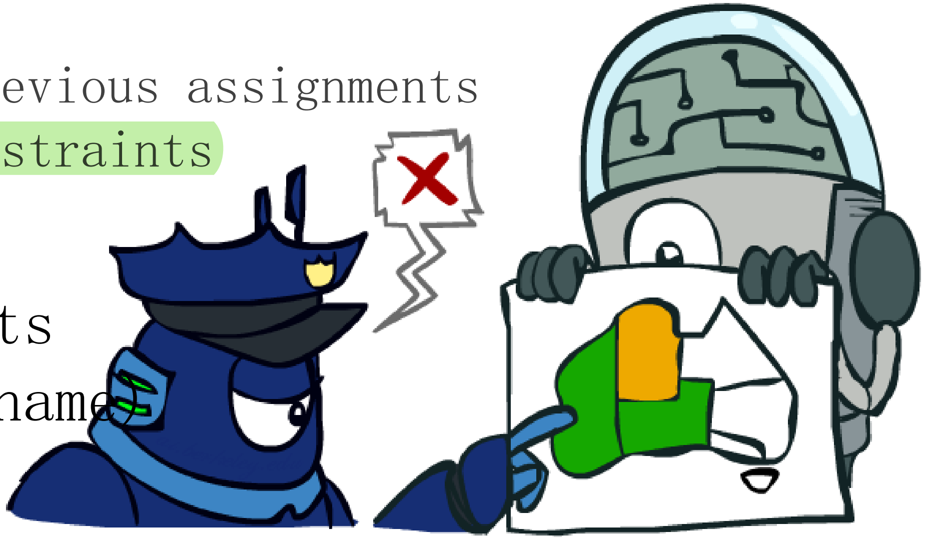


Backtracking Search

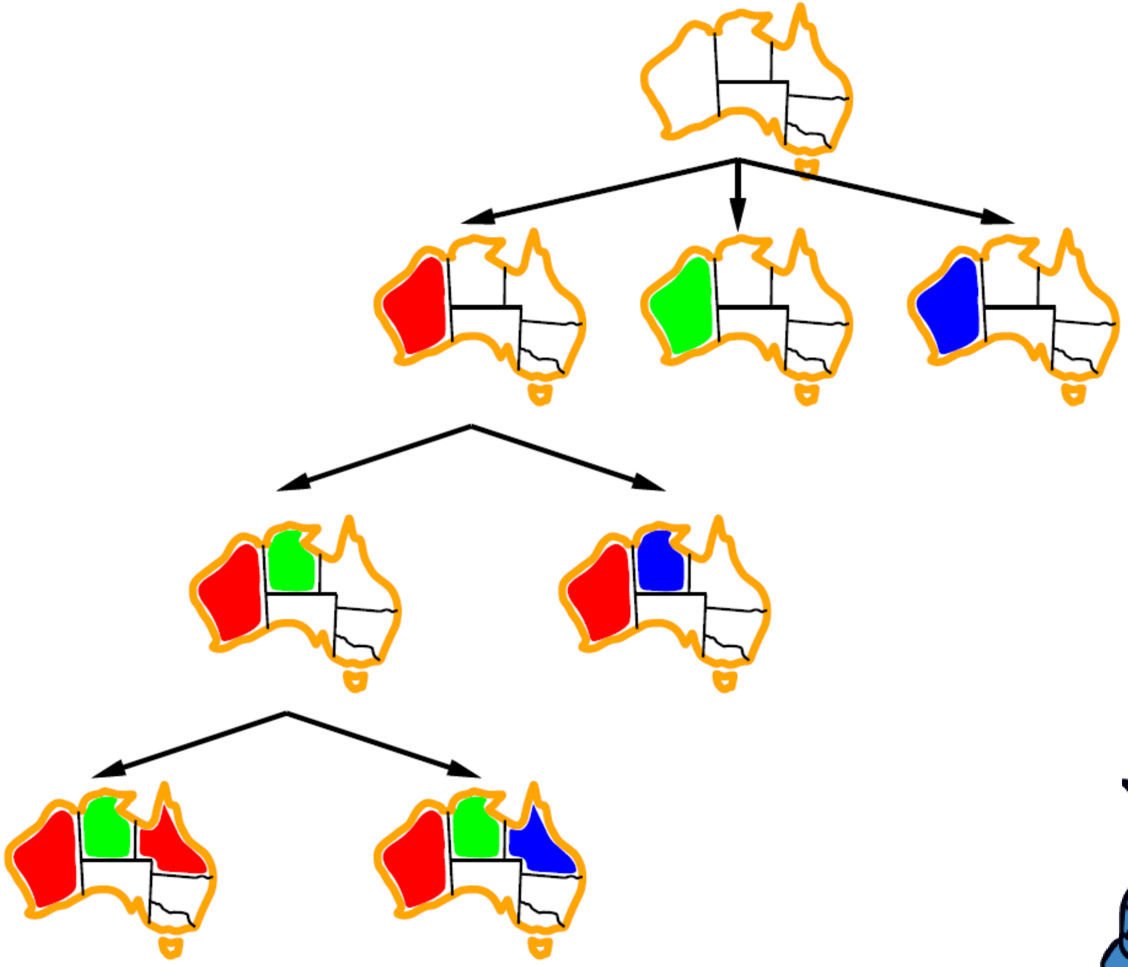


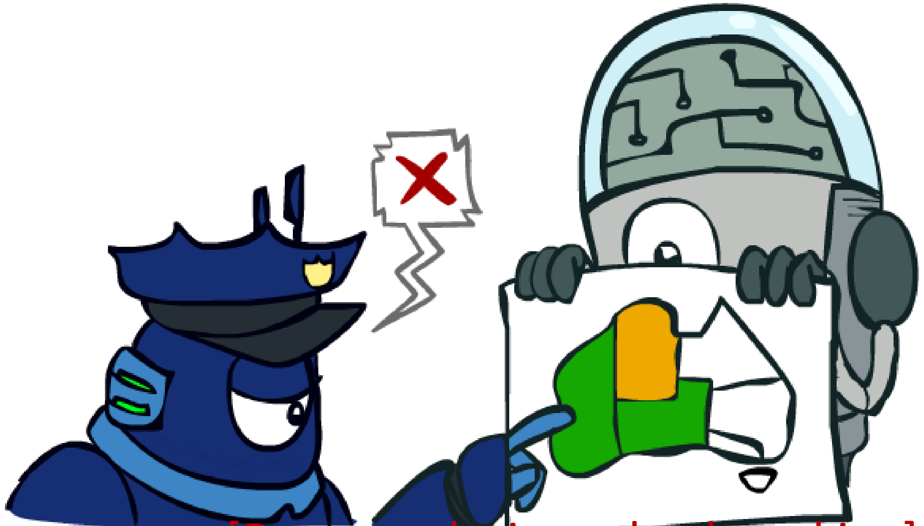
Backtracking Search

- **Backtracking search** is the basic uninformed algorithm for solving CSPs → 走一步看一步
- Idea 1: One variable at a time
  - Variable assignments are commutative, so fix ordering -> **better branching factor!**
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to **a single variable at each step**
- Idea 2: Check constraints as you go
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to check the constraints
  - "Incremental goal test"
- Depth-first search with these two improvements is called *backtracking search* (not the best name)
- Can solve n-queens for n ≈ 25



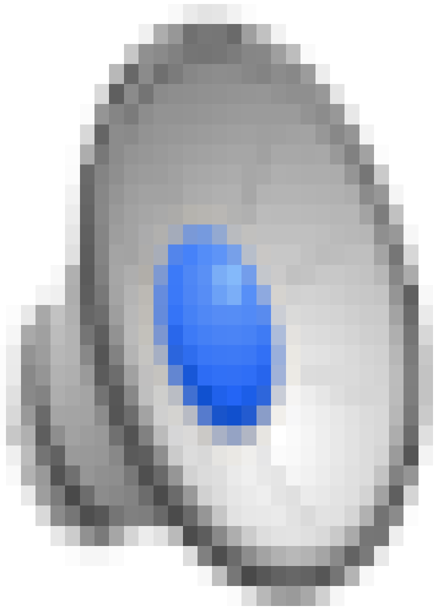
Backtracking Example





[Demo: coloring -- backtracking]

Video of Demo Coloring – Backtracking



Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

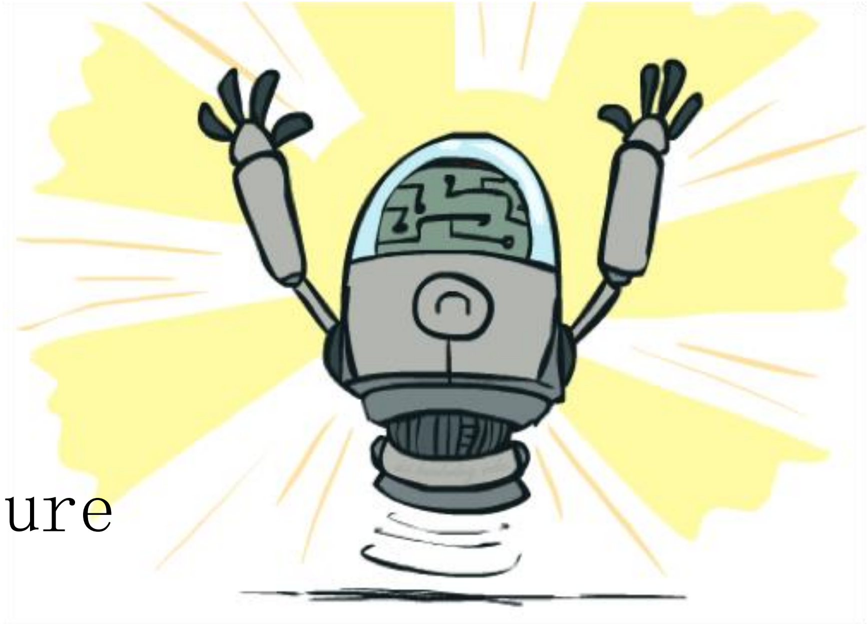
- Backtracking = DFS + variable-ordering + **fail-on-violation**
- What are the choice points?

Improving Backtracking

- General-purpose ideas give huge gains in speed

- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?

- Filtering: Can we detect inevitable failure early?

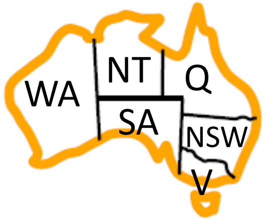


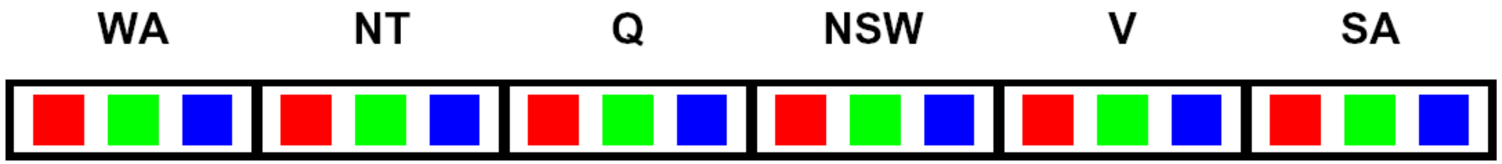
Filtering

Keep track of domains for **unassigned variables** and **cross off bad options**

Filtering: Forward Checking

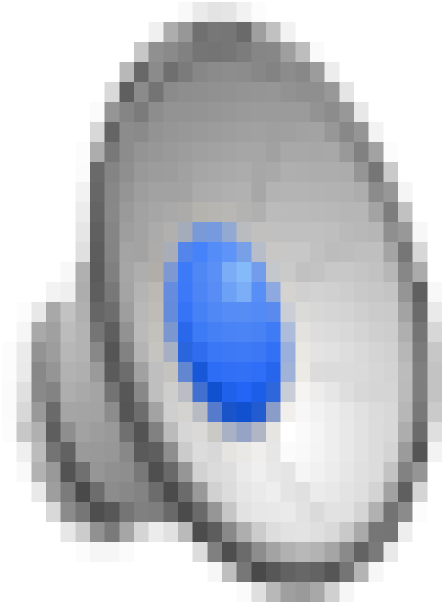
- Filtering: Keep track of domains for unassigned variables and cross of bad options
- Forward checking: **Cross off values** that violate a constraint when added to the existing assignment





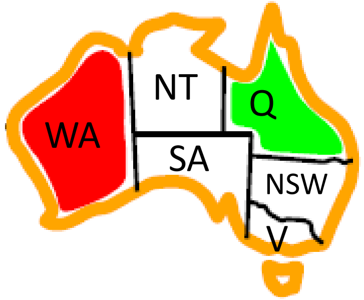
[Demo: coloring -- forward checking]

Video of Demo Coloring – Backtracking with Forward Checking



Filtering: Constraint Propagation

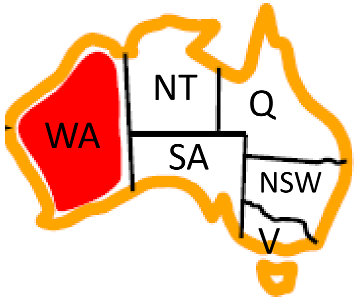
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

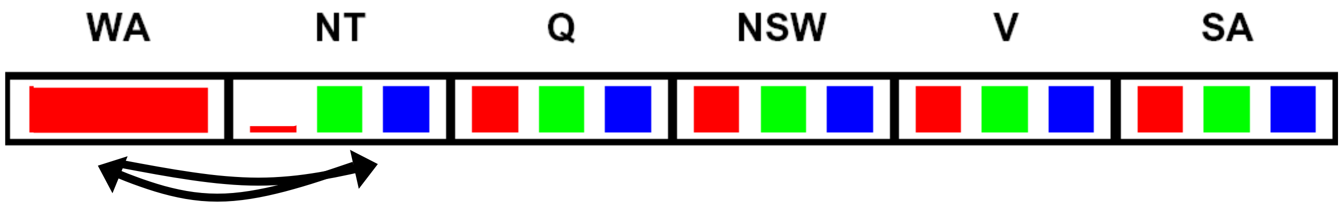


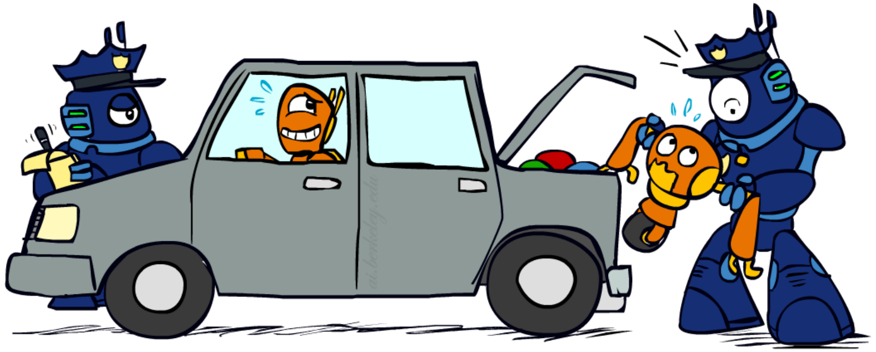
- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Consistency of A Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint







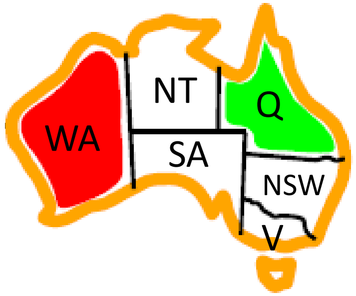
Delete from the tail!

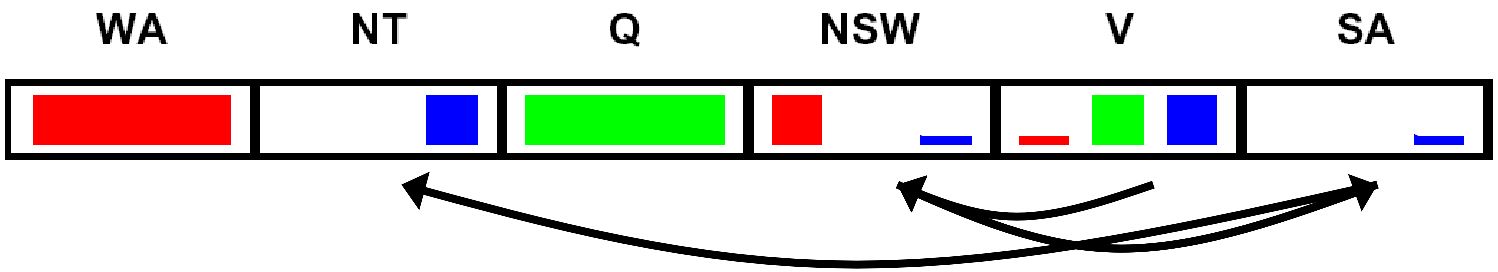
Forward checking?

Enforcing consistency of arcs pointing to each new assignment

Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:





- Important: If X loses a value, neighbors of X need to be rechecked!
- **Arc consistency detects failure earlier than forward checking**
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

Enforcing Arc Consistency in a CSP

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

Queue 's head has at most d values, then remove-inconsistent-value need to execute d*n times, then Neighbors[Xi] has at most d values, then we may add d*n*d times, for the fact that the while loop can run at most n times, thus we havt O(n^2d^2)

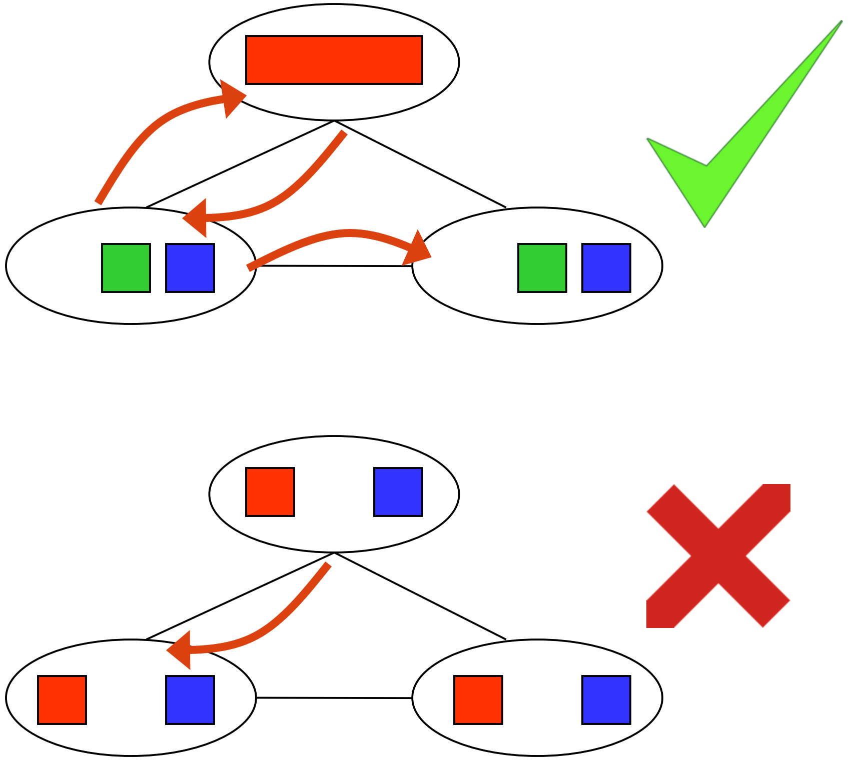
---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  removed ← *false*
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN[X_i]; *removed* ← *true*
  return *removed*

- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard - why?

Limitations of Arc Consistency

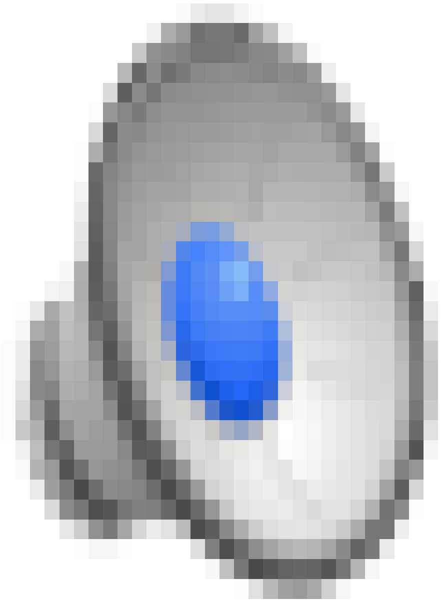
- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)



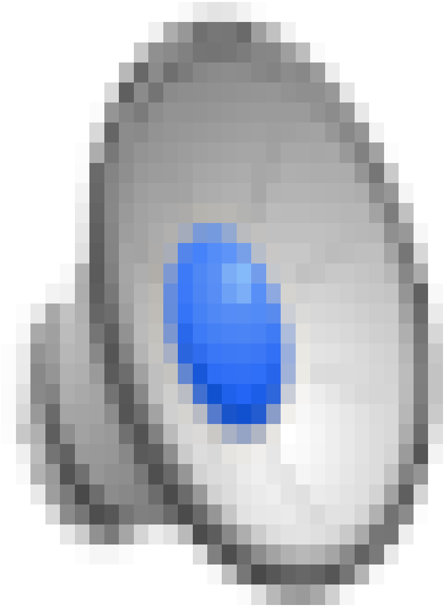
- Arc consistency still runs inside a backtracking search!

[Demo: coloring -- forward checking]
[Demo: coloring -- arc consistency]

Video of Demo Coloring – Backtracking with Forward Checking – Complex Graph

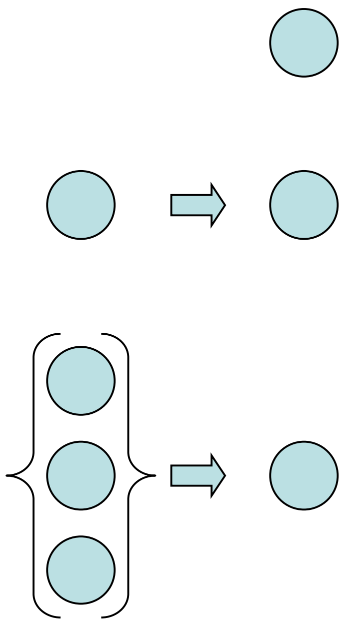


Video of Demo Coloring – Backtracking with Arc Consistency – Complex Graph

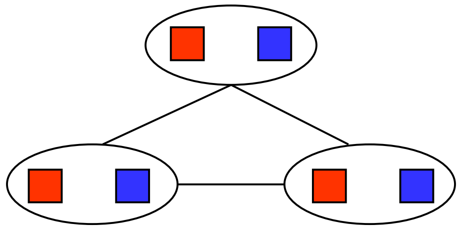


K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



Strong K-Consistency

- Strong k-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)