

# Chapter 7 Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

## P261 - Problem set 7.1

1.  $2 \times 2$ :  $a_{11} \neq b_{11}, b_{12} \neq c_{12}, 2 \times 3$ :  $d_{11} \neq e_{11}$

2.  $a_{31} = 10, a_{13} = 81, a_{26} = 96, a_{33} = 0$

3. E1:  $3 \times 3, 3 \times 4$ ,

E2:  $3 \times 7$ ,

E3:  $2 \times 2, 2 \times 2, 2 \times 2, 2 \times 3, 2 \times 3$ ,

E5:  $3 \times 2$

4. 1A: 4, 0, 1

3A:  $a_{11}, a_{22}$

3B: 4, -1

5.  $B = \frac{1}{5}A$ ,

$B = \frac{1}{10}A$

6.  $B = \frac{1}{1.609}A$

7. No. No( $1 \times 1$  as exception?). Yes. Maybe not in math (how about  $1 \times 1$ ?) but OK in python. No.

8.  $2A + 4B = 4B + 2A = \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix}$

$B$

$0.4B - 4.2A = \begin{bmatrix} 0 & -6.4 & -16 \\ -23.2 & -19.8 & -19.4 \\ -5 & 1.6 & 11.8 \end{bmatrix}$

9.  $3A = \begin{bmatrix} 0 & 6 & 12 \\ 18 & 15 & 15 \\ 3 & 0 & -9 \end{bmatrix}$

$0.5B = \begin{bmatrix} 0 & 2.5 & 1 \\ 2.5 & 1.5 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

$3A + 0.5B = \begin{bmatrix} 0 & 8.5 & 13 \\ 20.5 & 16.5 & 17 \\ 2 & 2 & -10 \end{bmatrix}$

$3A + 0.5B + C$  is not defined.

$$10. (4 \bullet 3)A = 4(3A) = \begin{bmatrix} 0 & 24 & 48 \\ 72 & 60 & 60 \\ 12 & 0 & -36 \end{bmatrix}$$

$$14B - 3B = 11B = \begin{bmatrix} 0 & 55 & 22 \\ 55 & 33 & 44 \\ -22 & 44 & -22 \end{bmatrix}$$

$$11. 8C + 10D = 2(5D + 4C) = \begin{bmatrix} 0 & 26 \\ 34 & 32 \\ 28 & -10 \end{bmatrix}$$

$$0.6C - 0.6D = 0.6(C - D) = \begin{bmatrix} 5.4 & 0.6 \\ -4.2 & 2.4 \\ -0.6 & 0.6 \end{bmatrix}$$

$$12. (C + D) + E = (D + E) + C = \begin{bmatrix} 1 & 5 \\ 6 & 8 \\ 6 & -2 \end{bmatrix}$$

$$0(C - E) + 4D = 4D = \begin{bmatrix} -16 & 4 \\ 20 & 0 \\ 8 & -4 \end{bmatrix}$$

$A - 0C$ :  $3 \times 3$  can not minus  $3 \times 2$ , not defined

$$13. (2 \bullet 7)C = 2(7C) = \begin{bmatrix} 70 & 28 \\ -28 & 56 \\ 14 & 0 \end{bmatrix}$$

$$-D + 0E = -D = \begin{bmatrix} 4 & -1 \\ -5 & 0 \\ -2 & 1 \end{bmatrix}$$

$E - D + C + u$ : Since  $EDC$  are  $3 \times 2$  but  $u$  is  $3 \times 1$ , not defined.

$$14. (5u + 5v) - \frac{1}{2}w = \begin{bmatrix} 5 \\ 30 \\ -10 \end{bmatrix}$$

$$-20(u + v) + 2w = -4[(5u + 5v) - \frac{1}{2}w] = \begin{bmatrix} -20 \\ -120 \\ 40 \end{bmatrix}$$

$E - (u + v)$ :  $3 \times 2$  can not minus  $3 \times 1$ , not defined

$$10(u + v) + w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$15. (u + v) - w = u + (v - w) = \begin{bmatrix} 5.5 \\ 33 \\ -11 \end{bmatrix}$$

$C + 0w$ :  $3 \times 2$  can not minus  $3 \times 1$ , not defined

$0E + u - v$ :  $3 \times 2$  can not minus  $3 \times 1$ , not defined

$$16. 15v - 3w - 0u = -3w + 15v = \begin{bmatrix} 0 \\ 135 \\ 0 \end{bmatrix}$$

$D - u + 3C$ :  $3 \times 2$  can not minus  $3 \times 1$ , not defined

$$8.5w - 11.1u + 0.4v = \begin{bmatrix} 25.45 \\ 256.2 \\ 119.1 \end{bmatrix}$$

$$17. u + v + w = \begin{bmatrix} -4.5 \\ -27 \\ 9 \end{bmatrix}$$

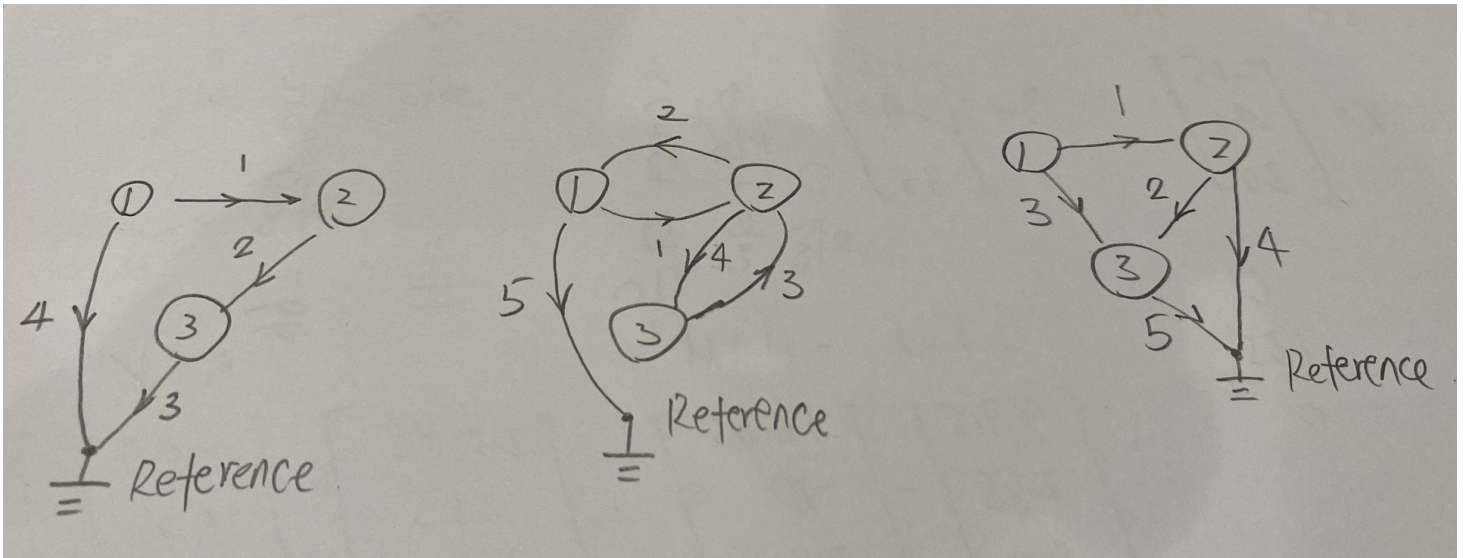
$$18. p = 0 - u - v - w = \begin{bmatrix} 4.5 \\ 27 \\ -9 \end{bmatrix}$$

19. Expand metrics with entries  $a_{ij}$ , then follow the basic arithmetic rule.

$$20. b^{-1}: \begin{bmatrix} -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$b^{-2}: \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

c:



## P270 - Problem set 7.2

Example 13. In the final stable situation (limit),  
 $I + C + R = 100$

$$0.7C + 0.1I = C$$

$$0.2C + 0.9I + 0.2R = I$$

$$0.1C + 0.8R = R$$

So we can get  $C=200/9$ ,  $I=200/3$ ,  $R=100/9$ .

Will revisit it after Sec. 8.2

1. Per definition, the number of the entries in the columns of the second matrix have to be same as the number of the entries in the rows of the first matrix. In short, if  $m \times n$  matrix multiple  $p \times q$ , then  $n=p$ . Or you won't be able to perform the dot product.

2. All entries or components are 0

3. No. All rows are proportional.

4. Min is 1 which is 0, and max is  $n(n-1) + 1$

Take 3x3 as example, 
$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

5. Min is 1 which is 0, and max is  $\frac{n(n+1)}{2}$

Take 3x3 as example, 
$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

6.  $U_1 + U_2, U_1 U_2, U_1^2$  are upper triangular matrices.  $L_1 + L_2$  is lower triangular.

7. 
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8.  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  for any  $m \geq 1, m \in N$ .  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$  for any  $m \geq 2, m \in N$ .

9. Expand metrics with entries  $a_{ij}$ , then follow the basic arithmetic rule.

10. Expand metrics with entries  $a_{ij}$ , then follow the basic arithmetic rule.

11. 
$$AB = AB^T = \begin{bmatrix} 10 & -14 & -6 \\ -5 & 7 & -12 \\ -5 & -1 & -4 \end{bmatrix}$$
  

$$BA = B^T A = \begin{bmatrix} 10 & -5 & -15 \\ -14 & 7 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

12. 
$$AA^T = \begin{bmatrix} 29 & 8 & 6 \\ 8 & 41 & 12 \\ 6 & 12 & 9 \end{bmatrix}, A^2 = \begin{bmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{bmatrix}, BB^T = B^2 = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

13. 
$$CC^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 13 & -6 \\ 0 & -6 & 4 \end{bmatrix}, BC = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}, CB \text{ not defined}, C^T B = \begin{bmatrix} -9 & 3 & 4 \\ -5 & -1 & 0 \end{bmatrix}$$

$$14. 3A - 2B = \begin{bmatrix} 10 & 0 & 9 \\ 0 & 1 & 18 \\ 3 & 6 & 10 \end{bmatrix}, (3A - 2B)^T = 3A^T - 2B^T = \begin{bmatrix} 10 & 0 & 3 \\ 0 & 1 & 6 \\ 9 & 18 & 10 \end{bmatrix},$$

$$(3A - 2B)^T a^T = \begin{bmatrix} 10 \\ -2 \\ -27 \end{bmatrix}$$

$$15. Aa \text{ not defined, } Aa^T = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}, (Ab)^T = b^T A^T = \begin{bmatrix} 7 & -11 & 3 \end{bmatrix}$$

$$16. BC = \text{Problem 13.2} = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}, BC^T \text{ not defined, } Bb = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix}, b^T B =$$

$$\begin{bmatrix} 0 & -8 & 2 \end{bmatrix}$$

$$17. ABC = \begin{bmatrix} -30 & -18 \\ 45 & 9 \\ 5 & -7 \end{bmatrix}, ABa \text{ not defined, } ABb = \begin{bmatrix} 22 \\ 4 \\ -12 \end{bmatrix}, Ca^T = \text{not defined.}$$

$$18. ab = 1, ba = \begin{bmatrix} 3 & -6 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix}, aA = \begin{bmatrix} 8 & -4 & -9 \end{bmatrix}, Bb = \text{problem 16.3} = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix}$$

$$19. 1.5a + 3.0b \text{ not defined. } 1.5a^T + 3.0b = \begin{bmatrix} 4.5 \\ -2 \\ -1 \end{bmatrix}, (A - B)b = Ab - Bb = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$$

$$20. b^T Ab = 7, aBa^T = 17, aCC^T = \begin{bmatrix} -3 & -24 & 12 \end{bmatrix}, C^T ba = \begin{bmatrix} 5 & -10 & 0 \\ 5 & -10 & 0 \end{bmatrix}$$

21. Expand metrics with entries  $a_{ij}$ , then follow the basic arithmetic rule.

$$22. A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}, AB = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

$$23. AB = A \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix}$$

$$24. AB = BA, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$2a_{11} + 3a_{12} = 2a_{11} + 3a_{21} \Rightarrow a_{12} = a_{21}$$

$$3a_{11} + 4a_{12} = 2a_{12} + 3a_{22} \Rightarrow 3a_{11} + 2a_{12} = 3a_{22}$$

$$2a_{21} + 3a_{22} = 3a_{11} + 4a_{21},$$

$$3a_{21} + 4a_{22} = 3a_{12} + 4a_{22}$$

$$\text{Let } A = \begin{bmatrix} x & y \\ y & \frac{3x+2y}{3} \end{bmatrix},$$

$$\text{Check: } AB = BA = \begin{bmatrix} 2x + 3y & 3x + 4y \\ 3x + 4y & 4x + 5\frac{2}{3}y \end{bmatrix}$$

25. a) Obvious.

$$b) C = [c_{ij}], C^T = [c_{ji}]$$

$D = C + C^T = [d_{ij}] = [c_{ij} + c_{ji}] = [c_{ji} + c_{ij}] = [d_{ji}]$ , so D is symmetric  
 $E = C - C^T = [e_{ij}] = [c_{ij} - c_{ji}] = -[c_{ji} - c_{ij}] = -[e_{ji}]$ , so E is skew-symmetric.

Let  $S = \frac{1}{2}D, T = \frac{1}{2}E$

$$S + T = \frac{1}{2}(D + E) = \frac{1}{2}(C + C^T + C - C^T) = C$$

$$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}, A^T = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 6 & 2 \end{bmatrix},$$

$$S = \frac{1}{2}(A + A^T) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix}, T = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B^T = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$S = \frac{1}{2}(B + B^T) = B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, T = \frac{1}{2}(B - B^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) symmetric:  $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}], \dots, M = [m_{ij}] = [m_{ji}]$

$$aA + bB + \dots + mM = a[a_{ij}] + b[b_{ij}] + \dots + m[m_{ij}] = a[a_{ji}] + b[b_{ji}] + \dots + m[m_{ji}].$$

Skew-symmetric:  $A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}], \dots, M = [m_{ij}] = -[m_{ji}]$

$$aA + bB + \dots + mM = a[a_{ij}] + b[b_{ij}] + \dots + m[m_{ij}] = -(a[a_{ji}] + b[b_{ji}] + \dots + m[m_{ji}])$$

.

d)  $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}]$

$$AB = [a_p b_q], \text{ if AB is symmetric, then } AB = [a_p b_q] = [a_q b_p] = [b_p a_q] = BA$$

vice verse.

e)  $A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}]$

$$AB = [a_p b_q], \text{ if AB is skew-symmetric, then } AB = [a_p b_q] = -[a_q b_p] = -[b_p a_q] = -BA$$

vice verse.

$$26. \text{ First day, status} = \begin{bmatrix} N \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ stochastic matrix} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

$$\text{Second day} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

$$\text{Two days after today} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$$

$$\text{Three days after today} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.722 \\ 0.278 \end{bmatrix}$$

The limit of N is  $\frac{5}{7}$

27. Reserve for future

$$28. \text{ Present} = \begin{bmatrix} \text{Subs.} \\ \text{Not} \end{bmatrix} = \begin{bmatrix} 1200 \\ 98800 \end{bmatrix}, \text{ stochastic matrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix}$$

$$\text{After 1 season} = \begin{bmatrix} \text{Subs.} \\ \text{Not} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1200 \\ 98800 \end{bmatrix} = \begin{bmatrix} 1278 \\ 98722 \end{bmatrix}, \text{ increase}$$

$$\text{After 2 seasons} = \begin{bmatrix} \text{Subs.} \\ \text{Not} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1278 \\ 98722 \end{bmatrix} = \begin{bmatrix} 1344 \\ 98656 \end{bmatrix}, \text{ increase}$$

$$\text{After 3 seasons} = \begin{bmatrix} \text{Subs.} \\ \text{Not} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1344 \\ 98656 \end{bmatrix} = \begin{bmatrix} 1407 \\ 98593 \end{bmatrix}, \text{ increase}$$

$$29. p = \begin{bmatrix} 35 \\ 62 \\ 30 \end{bmatrix}$$

$$v = Ap = \begin{bmatrix} 24,920 \\ 25,940 \end{bmatrix}$$

$$30. y = Ax$$

$$y_1 = x_1 \cos \theta - x_2 \sin \theta, y_2 = x_1 \sin \theta + x_2 \cos \theta$$

$$|y|^2 = (x_1 \cos \theta - x_2 \sin \theta)^2 + (x_1 \sin \theta + x_2 \cos \theta)^2 = x_1^2 + x_2^2 = |x|^2$$

$$\cos \alpha = \frac{x \cdot y}{|x||y|} = \frac{x_1^2 \cos \theta + x_2^2 \cos \theta}{x_1^2 + x_2^2} = \cos \theta$$

so x and y have the same length, and from x to y is counterclockwise rotate of  $\theta$

$$b) AA = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$c) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} =$$

$$\begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$$

$$d) [x_1, x_2, x_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = [3x_1, x_2, \frac{1}{2}x_3]$$

$$[x_1, x_2, x_3] \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} = [cx_1, cx_2, cx_3], \text{ Scalar matrix will amplify or squeeze the picture by } c.$$

$$e) [x_1, x_2, x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} = [x_1, x_2 \cos \theta + x_3 \sin \theta, -x_2 \sin \theta + x_3 \cos \theta]$$

$x_1$  remain the same. counterclockwise rotation of the Cartesian coordinate system  $x_2x_3$  in the plane about the origin by angle of  $\theta$

$$[x_1, x_2, x_3] \begin{bmatrix} \cos \varphi & 0 & -\sin \varphi \\ 0 & 1 & 0 \\ \sin \varphi & 0 & \cos \varphi \end{bmatrix} = [x_1 \cos \varphi + x_3 \sin \varphi, x_2, -x_1 \sin \varphi + x_3 \cos \varphi]$$

$x_2$  remain the same. counterclockwise rotation of the Cartesian coordinate system  $x_1x_3$  in the plane about the origin by angle of  $\varphi$

$$[x_1, x_2, x_3] \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = [x_1 \cos \psi + x_2 \sin \psi, -x_1 \sin \psi + x_2 \cos \psi, x_3]$$

$x_3$  remain the same. counterclockwise rotation of the Cartesian coordinate system  $x_1x_2$  in the plane about the origin by angle of  $\psi$

## P280 - Problem set 7.3

$$1. \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$$2. \begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{16}{5} \end{bmatrix}$$

$$3. \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$$4. \begin{bmatrix} 1 & -4 & 1 & -2 \\ 0 & 17 & -4 & 12 \\ 0 & -34 & 8 & -13 \end{bmatrix} = \text{No solution}$$

$$5. \begin{bmatrix} 1 & 33 & -225 \\ 0 & 139 & -973 \\ 0 & -376 & 2632 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -7 \\ 0 & 1 & -7 \end{bmatrix}$$



$$6. \begin{bmatrix} 1 & -2 & 2 & 9 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -5 & -20 \end{bmatrix} = \begin{bmatrix} 2t+1 \\ t \\ 4 \end{bmatrix}$$

$$7. \begin{bmatrix} 1 & 5 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3t \\ t \\ 2t \end{bmatrix}$$

$$8. \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 3 & 8 \\ 0 & 4 & 3 & 4 \end{bmatrix} = \text{No solution}$$

$$9. \begin{bmatrix} 3 & 4 & -5 & 13 \\ 0 & 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 3t-1 \\ 4-t \\ t \end{bmatrix}$$

$$10. \begin{bmatrix} 5 & -7 & 3 & 17 \\ 5 & -7 & 3 & -50/3 \end{bmatrix} = \text{No solution}$$

$$11. \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so we can get } \begin{bmatrix} 1 \\ 2m-n \\ n \\ m \end{bmatrix}$$

$$12. \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & -1 & 2 & -5/3 & -5 \end{bmatrix}, \text{ so we can get } \begin{bmatrix} n-2m \\ n \\ m \\ 3 \end{bmatrix}$$

$$13. \begin{bmatrix} 0 & 10 & 4 & -2 & -4 \\ -3 & -17 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 10 & 4 & -2 & -4 \\ 0 & -14 & 4 & 2 & 20 \\ 0 & 21 & -4 & 5 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -12 & 9 & 30 \\ 0 & 0 & -41 & 32 & 110 \\ 0 & 0 & -31 & 23 & 76 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

$$14. \begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 2 & 3 & 1 & -11 & 1 \\ 5 & -2 & 5 & -4 & 5 \\ 3 & 4 & -7 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 3 & -10 & 11 & -10 \\ 0 & 7 & -16 & 11 & -16 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

so we can get  $\begin{bmatrix} 0 \\ 3t \\ 1 + 2t \\ t \end{bmatrix}$

15. Expand metrics with entries  $a_{ij}$ , then follow the basic arithmetic rule.

16. Reserve for future

$$17. \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 16 \\ 0 & 4 & 1 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 32 \\ 0 & 0 & 6 & 48 \end{bmatrix}$$

so we can get  $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$

18. I am highly unsure about this one. Need a physical book

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

so we can get  $\begin{bmatrix} 9/5 \\ 12/5 \\ -3/5 \end{bmatrix}$

$$19. \begin{bmatrix} \frac{E_0}{R_2} + \frac{E_0}{R_1} \\ -\frac{E_0}{R_1} \\ \frac{E_0}{R_2} \end{bmatrix}$$

$$20. I_3 = I_x, I_1 = I_2$$

$$I_1 R_1 = I_x R_x, I_3 R_3 = I_2 R_2, \text{ so we can get}$$

$$R_x = R_3 R_1 / R_2$$

$$21. \begin{bmatrix} 1 & 1 & 0 & 0 & 1600 \\ 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 1 & 0 & 2200 \\ 0 & 0 & 1 & 1 & 1600 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 0 & 1 & 1 & 1600 \end{bmatrix}$$

Rank(3) < N (4), So the solutin is not unique.

$$22. 40 - 2P_1 - P_2 = 4P_1 - P_2 + 4, 6P_1 = 36, P_1 = 6$$

$$5P_1 - 2P_2 + 16 = 3P_2 - 4, P_1 = P_2 - 4, P_2 = 2$$

$$23. \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4 & -3 & 0 \end{bmatrix}$$

so we can get  $\begin{bmatrix} t \\ 5t \\ 3t \\ 4t \end{bmatrix}$

The smallest positive integers are 1, 5, 3, 4

$$24. a) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \\ 10 & 11 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

$$B = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ -5a_{11} + a_{21} & -5a_{12} + a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11} & a_{12} \\ -5a_{11} + a_{31} & -5a_{12} + a_{32} \\ a_{21} & a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

So  $B \neq C$

b) Obvious.

Row switch: reference E1

Row multiplication: reference E3

Row addition and subtraction: reference E2.

Expand metrics with entries  $a_{ij}$ , then follow the basic arithmetic rule.