Chapter 15 - Power Series, Taylor Series

Selected Problem set 15.1

15.1 1. 5, 9, 17, 19, 23, 25

bound.
Convergent
Limit

Convergent, hence bounded

1–10 SEQUENCES

Is the given sequence $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find its limit points. Show your work in detail.

1. $z_n = (1+i)^{2n}/2^n$ **2.** $z_n = (3+4i)^n/n!$

1. $\dfrac{z_{n+1}}{z_n} = \dfrac{(1+i)^{2(n+1)}}{2^{n+1}} \cdot \dfrac{2^n}{(1+i)^{2n}} = \dfrac{(1+i)^2}{2} = \dfrac{1+i^2+2i}{2} = i$

$z_1 = \dfrac{(1+i)^2}{2} = i,\ z_2 = -1,\ z_3 = -i,\ z_4 = 1,\ z_5 = z_1$

bounded.

Theorem 7. $\left|\dfrac{z_{n+1}}{z_n}\right| = 1 \geq 1$ diverges

5. $z_n = (-1)^n + 10i$

$z_1 = -1+10i \quad z_2 = 1+10i \quad z_3 = z_1 \quad z_4 = z_2 \ \ldots$

bounded.

for $t = 1, 2, \ldots$

$$\left|\frac{z_{2t+1}}{z_{2t}}\right| = \left|\frac{(1+10i)^2}{(-1+10i)(1+10i)}\right| = \left|\frac{1-100+20i}{-100-1}\right| = \left|+\frac{99}{101} - \frac{20}{101}i\right|$$

≥ 1

diverges.

9. $z_n = (3 + 3i)^{-n}$

$|z_1| = \left|\frac{1}{6}(1-i)\right| = \frac{\sqrt{2}}{6}$ $\quad |z_2| = \left|-\frac{i}{18}\right| = \frac{1}{18}$

$|z_3| = \left|-\frac{1}{108}(1+i)\right| = \frac{\sqrt{2}}{108}$

$|z_{n+1}| = \left|z_n \cdot \left(\frac{1}{3+3i}\right)\right| = |z_n| \cdot \frac{\sqrt{2}}{6} \le |z_1| = \frac{\sqrt{2}}{6}$ bounded.

$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(3+3i)^{-(n+1)}}{(3+3i)^{-n}}\right| = \left|(3+3i)^{-1}\right| = \frac{\sqrt{2}}{6} < 1$ converges

16–25 SERIES

Is the given series convergent or divergent? Give a reason. Show details.

16. $\sum_{n=0}^{\infty} \frac{(20 + 30i)^n}{n!}$ $\quad$ **17.** $\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$

17. $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{(-1)^n}\right| = \left|(-1) \cdot \frac{\ln n}{\ln(n+1)}\right| = \left|\frac{\ln n}{\ln(n+1)}\right|$

$n^{n+1} > (n+1)^n$ for $n \ge 3$. $\Rightarrow \frac{\ln n}{\ln(n+1)} > \frac{n}{n+1}$

$\Rightarrow \left|\frac{z_{n+1}}{z_n}\right| > \left|\frac{n}{n+1}\right|$ diverges

19. $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{(n+1)^2-i} \cdot \frac{n^2-i}{i^n}\right| = \left|i \cdot \frac{n^2-i}{(n+1)^2-i}\right|$$

$$= \left|\frac{(n^2-i)[(n+1)^2+i]}{[(n+1)^2-i][(n+1)^2+i]}\right|$$

$$= \left|\frac{(n^2-i)(n^2+2n+1+i)}{(n+1)^4+1}\right|$$

$$= \left|\frac{n^4+2n^3+n^2+\cancel{in^2}-\cancel{in^2}-2ni-i+1}{(n+1)^4+1}\right|$$

$$= \left|\frac{1+\frac{2}{n}+\frac{1}{n^2}-\frac{2i}{n^3}+\frac{1-i}{n^4}}{(1+\frac{1}{n})^4+\frac{1}{n}}\right|$$

$\lim \to 1 \quad \Rightarrow$ don't know

$$|z_n|^{\frac{1}{n}} = \left|\frac{i}{\sqrt[n]{n^2-i}}\right| = \left|\frac{i}{(n^2+1)^{\frac{1}{n}}}\right| < \left|\frac{i}{n^{\frac{2}{n}}}\right| = 1$$

converge absolutely

23. $\sum_{n=0}^{\infty} \frac{(-1)^n(1+i)^{2n}}{(2n)!}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}(1+i)^{2n+2}}{(2n+2)!} \cdot \frac{2n!}{(-1)^n(1+i)^{2n}}\right|$$

$$= \left|\frac{-1\cdot(1+i)^2}{(2n+2)(2n+1)}\right|$$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0 < 1 \quad \text{converge absolutely}$$

25. $\sum_{n=1}^{\infty} \frac{i^n}{n}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{n+1}\cdot\frac{n}{i^n}\right| = \left|\frac{n\,i}{n+1}\right| = \left|\frac{n}{n+1}\right| \quad \begin{matrix}\lim \Rightarrow 1\\ \text{not sure}\end{matrix}$$

$$|z_n^{\frac{1}{n}}| = \left|\frac{1}{\sqrt[n]{n}}\right| \Rightarrow \lim_{n\to\infty}|z_n|^{\frac{1}{n}} = 1 \quad \text{not sure}$$

Per example 3. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots$ conditionally converge.

so $i, -1, -i, 1.$

$\{z_{4t+1}+z_{4t+3}\}$ t is even $\Rightarrow$ converge.

$\{z_{4t+2}+z_{4t+4}\}$ t is odd $\Rightarrow$ converge

$\Rightarrow S_n$ converge ?

Selected Problem set 15.2