Chapter 10 Vector Integral Calculus. Integral Theorems

Selected Problem set 10.1

10.1 3. 5. 9. 19

2–11 LINE INTEGRAL. WORK

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

2. $\mathbf{F} = [y^2, -x^2]$, C: $y = 4x^2$ from (0, 0) to (1, 4)
3. $\mathbf{F}$ as in Prob. 2, C from (0, 0) straight to (1, 4). Compare.
4. $\mathbf{F} = [xy, x^2y^2]$, C from (2, 0) straight to (0, 2)
5. $\mathbf{F}$ as in Prob. 4, C the quarter-circle from (2, 0) to (0, 2) with center (0, 0)

3. C: $r(t) = [t, 4t] = t\,i + 4t\,j$

$F(r(t)) = [(4t)^2, -t^2] = [16t^2, -t^2]$

$r'(t) = [1, 4]$

$$\int_C F(r)\cdot dr = \int_0^1 [16t^2, -t^2]\cdot[1, 4]\,dt$$

$$= \int_0^1 (16t^2 - 4t^2)\,dt$$

$$= \int_0^1 12t^2\,dt$$

$$= 4t^3\Big|_0^1 = 4 - 0 = 4$$

5. C by $r(t) = [2\cos t, 2\sin t]$, when $0 \le t \le \frac{\pi}{2}$.

$F(r(t)) = [4\sin t\cos t, 16\sin^2 t\cos^2 t]$

$r'(t) = [-2\sin t, 2\cos t]$

$$\int_C F(r)\cdot dr = \int_0^{\frac{\pi}{2}} (-8\sin^2 t\cos t + 32\sin^2 t\cos^3 t)\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} (4\sin^2 t\cos^3 t - \sin^2 t\cos t)\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t\,(4\cos^2 t - 1)\sin^2 t\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t\,[-\sin^2 t\,(4\sin^2 t - 3)]\,dt$$

$u = \sin t$, $\frac{du}{dt} = \cos t$, $dt = \frac{du}{\cos t}$

$$= -8\int_0^1 u^2(4u^2 - 3)\,du$$

$$= -32\int_0^1 u^4\,du + 24\int_0^1 u^2\,du$$

$$= -\frac{32}{5}u^5\Big|_0^1 + 24\cdot\frac{1}{3}u^3\Big|_0^1$$

$$= -\frac{32}{5} + 8 = \frac{8}{5} = 1.6$$

9. $\mathbf{F} = [x + y, y + z, z + x]$, $C: \mathbf{r} = [2t, 5t, t]$ from $t = 0$ to 1. Also from $t = -1$ to 1.

$C: \ r = [2t, 5t, t] \quad 0 \le t \le 1$

$r' = [2, 5, 1]$

$F(r(t)) = [7t, 6t, 3t]$

$\int_C F(r)\,dr = \int_0^1 [7t, 6t, 3t][2, 5, 1] \cdot dt$

$= \int_0^1 47t \cdot dt$

$= \frac{47}{2} t^2 \Big|_0^1 = \frac{47}{2} = 23.5$

$-1 \le t \le 1$

$\int_{-1}^{1} 47t \cdot dt = \frac{47}{2} t^2 \Big|_{-1}^{1} = 0$

19. $f = xyz$, $C: \mathbf{r} = [4t, 3t^2, 12t]$, $-2 \leqq t \leqq 2$. Sketch C.

$C: \ r = [4t, 3t^2, 12t] \quad -2 \le t \le 2$

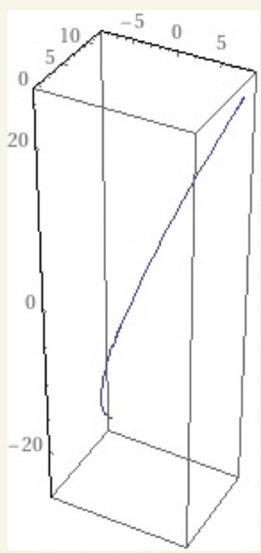
$r' = [4, 6t, 12]$

$F(r(t)) = 144t^4$

$\int_C f(r)\,dt = \int_{-2}^{2} 144t^4 \cdot dt$

$= \frac{144}{5} \cdot t^5 \Big|_{-2}^{2}$

$= \frac{144}{5} \cdot 64 = 1843.2$



Selected Problem set 10.2

10.2. 3. 5. 13. 15.

3–9 PATH INDEPENDENT INTEGRALS

Show that the form under the integral sign is exact in the plane (Probs. 3–4) or in space (Probs. 5–9) and evaluate the integral. Show the details of your work.

3. $\int_{(\pi/2,\, \pi)}^{(\pi,\, 0)} \left(\tfrac{1}{2}\cos\tfrac{1}{2}x \cos 2y\, dx - 2\sin\tfrac{1}{2}x \sin 2y\, dy\right)$

4. $\int_{(4,\, 0)}^{(6,\, 1)} e^{4y}(2x\, dx + 4x^2\, dy)$

5. $\int_{(0,\, 0,\, \pi)}^{(2,\, 1/2,\, \pi/2)} e^{xy}(y \sin z\, dx + x \sin z\, dy + \cos z\, dz)$

3. exactness:

$(F_2)_x = -2\sin 2y \cdot \frac{1}{2} \cdot \cos\frac{1}{2}x$

$= -\sin 2y \cdot \cos\frac{1}{2}x$

$(F_1)_y = \frac{1}{2} \cdot \cos\frac{1}{2}x \cdot (-\sin 2y) \cdot 2$

$= -\sin 2y \cos\frac{1}{2}x$.

Evaluate:

$f_x = F_1 = \frac{1}{2}\cos\frac{1}{2}x \cdot \cos 2y$

$f_y = F_2 = -2\sin\frac{1}{2}x \sin 2y$

$f = \cos 2y \cdot \sin\frac{1}{2}x + g(y)$

$f_y = \sin\frac{1}{2}x \cdot (-\sin 2y) \cdot 2 + g_y$

$f = \cos 2y \cdot \sin\frac{1}{2}x$

$f(\pi, 0) - f(\frac{\pi}{2}, \pi) = 1 \cdot 1 - 1 \cdot \frac{\sqrt{2}}{2}$

$= 1 - \frac{\sqrt{2}}{2}$

5. exactness:

$(F_3)_y = x \cdot e^{xy} \cdot \cos z$

$(F_2)_z = e^{xy} \cdot x \cdot \cos z$

$(F_1)_z = e^{xy} \cdot y \cdot \cos z$

$(F_3)_x = \cos z \cdot y \cdot e^{xy}$

$(F_1)_y = \sin z\,(x \cdot e^{xy} \cdot y + e^{xy})$

$(F_2)_x = \sin z\,(y \cdot e^{xy} \cdot x + e^{xy})$

Evaluate:

$f_x = F_1 = e^{xy} \cdot y \cdot \sin z$

$f_y = F_2 = e^{xy} \cdot x \cdot \sin z$

$f_z = F_3 = e^{xy} \cdot \cos z$.

$f = \sin z \cdot e^{xy} + g(y, z)$

$f_y = x \cdot \sin z \cdot e^{xy} + g_y$

$= x \cdot \sin z \cdot e^{xy} + h(z)$

$f_z = e^{xy} \cdot \cos z + h'$

$h' = 0,\ h = 0,\ g = 0$

$f = \sin z \cdot e^{xy}$

$f(2, \frac{1}{2}, \frac{\pi}{2}) - f(0, 0, \pi)$

$= 1 \cdot e - 0 = e$

13–19 PATH INDEPENDENCE?

Check, and if independent, integrate from (0, 0, 0) to (a, b, c).

13. $2e^{x^2}(x \cos 2y\, dx - \sin 2y\, dy)$

15. $x^2 y\, dx - 4xy^2\, dy + 8z^2 x\, dz$

Check if independent

$f_x = F_1 = 2e^{x^2} \cdot x \cdot \cos 2y$

$f_y = F_2 = -2e^{x^2} \cdot \sin 2y$

$f = \cos 2y \cdot e^{x^2} + g$

$f_y = e^{x^2} \cdot (-\sin 2y) \cdot 2 + g'$

$g' = 0 \quad g = 0$, say.

$f = \cos 2y \cdot e^{x^2}$

Independent.

$f(a,b,c) - f(0,0,0)$

$= \cos(2b) \cdot e^{a^2} - 1 \cdot e^0$

$= \cos(2b) \cdot e^{a^2} - 1$

answer is wrong

Check if independent

$f_x = F_1 = x^2 y$

$f_y = F_2 = -4xy^2$

$f_z = F_3 = 8z^2 x$

$f = \frac{1}{3} \cdot y \cdot x^3 + g(y,z)$

$f_y = \frac{1}{3} \cdot x^3 + g_y$

$g_y = -4xy^2 - \frac{1}{3}x^3 \Rightarrow$ dependent

$g(y,z) = -\frac{4}{3}xy^3 - \frac{1}{3}x^3 y = 0$

$4xy^3 + x^3 y = 0$

$4y^2 + x^2 = 0$