Chapter 16 - Laurent Series. Residue Integration

Selected Problem set 16.1

16.1 1. 9. 19

1–8 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. Show the details of your work.

1. $\dfrac{\cos z}{z^4}$ 2. $\dfrac{\exp(-1/z^2)}{z^2}$

1. $$z^{-4}\cos z = z^{-4}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{z^{2n-4}}{(2n!)}$$

$$= \frac{1}{z^4} - \frac{1}{2!z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \dots$$

principle part: $\dfrac{1}{z^4} - \dfrac{1}{2z^2}$ $\qquad 0 < z < \infty$

9–16 LAURENT SERIES NEAR A SINGULARITY AT z_0

Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

9. $\dfrac{e^z}{(z-1)^2}$, $z_0 = 1$ 10. $\dfrac{z^2 - 3i}{(z-3)^2}$, $z_0 = 3$

9. $$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \dots$$

$$e\cdot e^{z-1} = e\cdot\left(1 + (z-1) + \frac{(z-1)^2}{2!} + \dots\right)$$

$$\frac{e^z}{(z-1)^2} = \frac{e\cdot e^{z-1}}{(z-1)^2} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \frac{(z-1)^2}{4!} + \dots$$

$$0 < |z-1| < R$$

19–25 **TAYLOR AND LAURENT SERIES**

Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

19. $\frac{1}{1-z^2}$, $z_0 = 0$ **20.** $\frac{1}{z}$, $z_0 = 1$

19. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad |z| < 1$

$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} = 1 + z^2 + z^4 + \cdots \qquad |z^2| < 1 \Rightarrow |z| < 1$

$\frac{1}{1-z^2} = \frac{-1}{z^2(1-z^{-2})} = -z^{-2} \sum_{n=0}^{\infty} z^{-2n} = -\sum_{n=0}^{\infty} z^{-2n-2} \qquad |z| > 1$

Selected Problem set 16.2

16.2 1. 5. 15

1–10 ZEROS

Determine the location and order of the zeros.

1. $\sin^4 \frac{1}{2}z$
2. $(z^4 - 81)^3$
3. $(z + 81i)^4$
4. $\tan^2 2z$
5. $z^{-2} \sin^2 \pi z$
6. $\cosh^4 z$

1. Let $X = 0+2n\pi$, $n = 0, \pm 1, \pm 2, \dots$

$f(X) = 0$

$f'(z) = 2\cos(\frac{z}{2})\sin^3(\frac{z}{2})$, $f'(x) = 0$

$f''(z) = 3\cos^2(\frac{z}{2})\sin^2(\frac{z}{2}) - \sin^4(\frac{z}{2})$, $f''(x) = 0$

$f^{(3)}(z) = 3\cos^3(\frac{z}{2})\sin(\frac{z}{2}) - 5\cos(\frac{z}{2})\sin^3(\frac{z}{2})$ $f^{(3)}(x) = 0$

$f^{(4)}(z) = \frac{5}{2}\sin^4(\frac{z}{2}) - 12\cos^2(\frac{z}{2})\sin^2(\frac{z}{2}) + \frac{3}{2}\cos^4(\frac{z}{2})$ $f^{(4)}(x) \neq 0$

order: 4. Location: $0 + 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$

5. Let $x = n$, $n = \pm 1, \pm 2, \dots$

$f(x) = 0$

$f'(z) = -\dfrac{2\sin(\pi z)[\sin(\pi z) - \pi z\cos(\pi z)]}{z^3}$, $f'(x) = 0$

$f''(z) = \dfrac{2[(\pi^2 z^2 - 3)\sin^2(\pi z) + 4\pi z\cos(\pi z)\sin(\pi z) - \pi^2 z^2\cos^2(\pi z)]}{z^4}$

$f''(1) = 2\pi^2 \neq 0$

order: 2 Location: $\pm 1, \pm 2 \dots$

13–22 **SINGULARITIES**

Determine the location of the singularities, including those at infinity. For poles also state the order. Give reasons.

13. $\frac{1}{(z+2i)^2} - \frac{z}{z-i} + \frac{z+1}{(z-i)^2}$

14. $e^{z-i} + \frac{2}{z-i} - \frac{8}{(z-i)^3}$

15. $z \exp(1/(z-1-i)^2)$ **16.** $\tan \pi z$

Simple pole at ∞ essential.

15. $f(z) = z \cdot e^{\frac{1}{(z-1-i)^2}}$

what is essential singularity point?

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$z-1-i=0$, $z=1+i$ is singularity point

$$f(z) = z \cdot \left[1 + \frac{1}{(z-1-i)^2} + \frac{1}{2\cdot(z-1-i)^4} + \frac{1}{3!(z-1-i)^6} + \cdots\right]$$

$$= [(z-1-i)+(1+i)]\,[\quad]$$

$$= (z-1-i) + \frac{1}{(z-1-i)} + \frac{1}{2\cdot(z-1-i)^3} + \cdots$$

$$+ (1+i) + \frac{1+i}{(z-1-i)^2} + \frac{1+i}{2(z-1-i)^4} + \cdots$$

$$= z + \frac{1}{z-1-i} + \frac{1+i}{(z-1-i)^2} + \frac{1}{2(z-1-i)^3}$$

$$+ \frac{1+i}{2(z-1-i)^4} + \cdots$$

part (1) has finity many term $\Rightarrow$ Isolated essential singularity

Pole: $z=1+i$

part (2) infinity ?

Selected Problem set 16.3

Selected Problem set 16.4