Chapter 10 Vector Integral Calculus. Integral Theorems

Selected Problem set 10.1

10.1 3. 5. 9. 19

2–11 LINE INTEGRAL. WORK

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

2. $\mathbf{F} = [y^2, -x^2]$, C: $y = 4x^2$ from (0, 0) to (1, 4)
3. $\mathbf{F}$ as in Prob. 2, C from (0, 0) straight to (1, 4). Compare.
4. $\mathbf{F} = [xy, x^2y^2]$, C from (2, 0) straight to (0, 2)
5. $\mathbf{F}$ as in Prob. 4, C the quarter-circle from (2, 0) to (0, 2) with center (0, 0)

3. C: $r(t) = [t, 4t] = t\,i + 4t\,j$

$F(r(t)) = [(4t)^2, -t^2] = [16t^2, -t^2]$

$r'(t) = [1, 4]$

$$\int_C F(r)\cdot dr = \int_0^1 [16t^2, -t^2]\cdot[1, 4]\,dt$$

$$= \int_0^1 (16t^2 - 4t^2)\,dt$$

$$= \int_0^1 12t^2\,dt$$

$$= 4t^3\Big|_0^1 = 4 - 0 = 4$$

5. C by $r(t) = [2\cos t, 2\sin t]$, when $0 \le t \le \frac{\pi}{2}$.

$F(r(t)) = [4\sin t\cos t, 16\sin^2 t\cos^2 t]$

$r'(t) = [-2\sin t, 2\cos t]$

$$\int_C F(r)\cdot dr = \int_0^{\frac{\pi}{2}} (-8\sin^2 t\cos t + 32\sin^2 t\cos^3 t)\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} (4\sin^2 t\cos^3 t - \sin^2 t\cos t)\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t\,(4\cos^2 t - 1)\sin^2 t\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t\,[-\sin^2 t(4\sin^2 t - 3)]\,dt$$

$u = \sin t$, $\frac{du}{dt} = \cos t$, $dt = \frac{du}{\cos t}$

$$= -8\int_0^1 u^2(4u^2 - 3)\,du$$

$$= -32\int_0^1 u^4\,du + 24\int_0^1 u^2\,du$$

$$= -\frac{32}{5}u^5\Big|_0^1 + 24\cdot\frac{1}{3}u^3\Big|_0^1$$

$$= -\frac{32}{5} + 8 = \frac{8}{5} = 1.6$$

9. $\mathbf{F} = [x + y, y + z, z + x]$, $C: \mathbf{r} = [2t, 5t, t]$ from $t = 0$ to 1. Also from $t = -1$ to 1.

$C: \ r = [2t, 5t, t] \quad 0 \le t \le 1$

$r' = [2, 5, 1]$

$F(r(t)) = [7t, 6t, 3t]$

$$\int_C F(r) dr = \int_0^1 [7t, 6t, 3t][2, 5, 1] \cdot dt$$

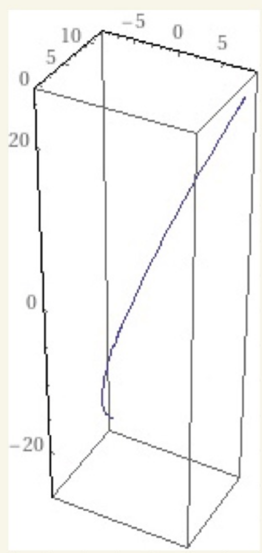
$$= \int_0^1 47t \cdot dt$$

$$= \frac{47}{2} t^2 \Big|_0^1 = \frac{47}{2} = 23.5$$

$-1 \le t \le 1$

$$\int_{-1}^{1} 47t \cdot dt = \frac{47}{2} t^2 \Big|_{-1}^{1} = 0$$

19. $f = xyz$, $C: \mathbf{r} = [4t, 3t^2, 12t]$, $-2 \leqq t \leqq 2$. Sketch C.



$C: \ r = [4t, 3t^2, 12t] \quad -2 \le t \le 2$

$r' = [4, 6t, 12]$

$F(r(t)) = 144t^4$

$$\int_C f(r) dt = \int_{-2}^{2} 144t^4 \cdot dt$$

$$= \frac{144}{5} \cdot t^5 \Big|_{-2}^{2}$$

$$= \frac{144}{5} \cdot 64 = 1843.2$$

Selected Problem set 10.2

10.2. 3. 5. 13. 15.

3–9 PATH INDEPENDENT INTEGRALS

Show that the form under the integral sign is exact in the plane (Probs. 3–4) or in space (Probs. 5–9) and evaluate the integral. Show the details of your work.

3. $\int_{(\pi/2,\,\pi)}^{(\pi,\,0)} (\frac{1}{2}\cos\frac{1}{2}x\cos 2y\,dx - 2\sin\frac{1}{2}x\sin 2y\,dy)$

4. $\int_{(4,\,0)}^{(6,\,1)} e^{4y}(2x\,dx + 4x^2\,dy)$

5. $\int_{(0,\,0,\,\pi)}^{(2,\,1/2,\,\pi/2)} e^{xy}(y\sin z\,dx + x\sin z\,dy + \cos z\,dz)$

3. exactness:

$(F_2)_x = -2\sin 2y \cdot \frac{1}{2} \cdot \cos\frac{1}{2}x$

$= -\sin 2y \cdot \cos\frac{1}{2}x$

$(F_1)_y = \frac{1}{2} \cdot \cos\frac{1}{2}x \cdot (-\sin 2y) \cdot 2$

$= -\sin 2y \cos\frac{1}{2}x$

Evaluate:

$f_x = F_1 = \frac{1}{2}\cos\frac{1}{2}x \cdot \cos 2y$

$f_y = F_2 = -2\sin\frac{1}{2}x\sin 2y$

$f = \cos 2y \cdot \sin\frac{1}{2}x + g(y)$

$f_y = \sin\frac{1}{2}x \cdot (-\sin 2y) \cdot 2 + g_y$

$f = \cos 2y \cdot \sin\frac{1}{2}x$

$f(\pi, 0) - f(\frac{\pi}{2}, \pi) = 1 \cdot 1 - 1 \cdot \frac{\sqrt{2}}{2}$

$= 1 - \frac{\sqrt{2}}{2}$

5. exactness:

$(F_3)_y = x \cdot e^{xy} \cdot \cos z$

$(F_2)_z = e^{xy} \cdot x \cdot \cos z$

$(F_1)_z = e^{xy} \cdot y \cdot \cos z$

$(F_3)_x = \cos z \cdot y \cdot e^{xy}$

$(F_1)_y = \sin z\,(x \cdot e^{xy} \cdot y + e^{xy})$

$(F_2)_x = \sin z\,(y \cdot e^{xy} \cdot x + e^{xy})$

Evaluate:

$f_x = F_1 = e^{xy} \cdot y \cdot \sin z$

$f_y = F_2 = e^{xy} \cdot x \cdot \sin z$

$f_z = F_3 = e^{xy} \cdot \cos z$

$f = \sin z \cdot e^{xy} + g(y, z)$

$f_y = x \cdot \sin z \cdot e^{xy} + g_y$

$= x \cdot \sin z \cdot e^{xy} + h(z)$

$f_z = e^{xy} \cdot \cos z + h'$

$h' = 0, \quad h = 0, \quad g = 0$

$f = \sin z \cdot e^{xy}$

$f(2, \frac{1}{2}, \frac{\pi}{2}) - f(0, 0, \pi)$

$= 1 \cdot e - 0 = e$

13–19 PATH INDEPENDENCE?

Check, and if independent, integrate from (0, 0, 0) to (a, b, c).

13. $2e^{x^2}(x \cos 2y\, dx - \sin 2y\, dy)$

15. $x^2y\, dx - 4xy^2\, dy + 8z^2x\, dz$

Check if independent

$f_x = F_1 = 2e^{x^2} \cdot x \cdot \cos 2y$

$f_y = F_2 = -2e^{x^2} \cdot \sin 2y$

$f = \cos 2y \cdot e^{x^2} + g$

$f_y = e^{x^2} \cdot (-\sin 2y) \cdot 2 + g'$

$g' = 0 \quad g = 0$, say.

$f = \cos 2y \cdot e^{x^2}$

Independent.

$f(a,b,c) - f(0,0,0)$

$= \cos(2b) \cdot e^{a^2} - 1 \cdot e^0$

$= \cos(2b) \cdot e^{a^2} - 1$

answer is wrong

Check if independent

$f_x = F_1 = x^2y$

$f_y = F_2 = -4xy^2$

$f_z = F_3 = 8z^2x$

$f = \frac{1}{3} \cdot y \cdot x^3 + g(y,z)$

$f_y = \frac{1}{3} \cdot x^3 + g_y$

$g_y = -4xy^2 - \frac{1}{3}x^3 \Rightarrow$ dependent

$g(y,z) = -\frac{4}{3}xy^3 - \frac{1}{3}x^3y = 0$

$4xy^3 + x^3y = 0$

$4y^2 + x^2 = 0$

Selected Problem set 10.3

10.3   5   9   15

5. $\int_0^1 \int_{x^2}^{x} (1 - 2xy)\, dy\, dx$

$= \int_0^1 \left[(y - xy^2) \Big|_{x^2}^{x} \right] dx$

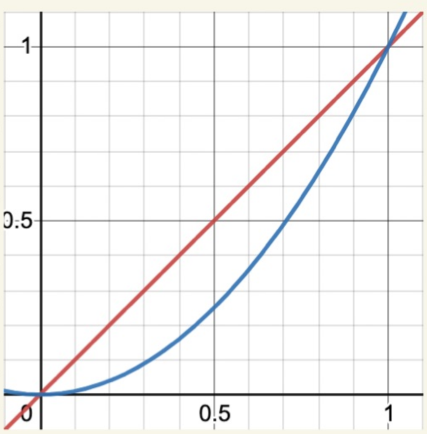
$= \int_0^1 [x - x^3 - (x^2 - x^5)]\, dx$

$= \int_0^1 (x^5 - x^3 - x^2 + x)\, dx$

$= \frac{x^6}{6} - \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} \Big|_0^1$

$= \frac{1}{6} - \frac{1}{4} - \frac{1}{3} + \frac{1}{2}$

$= \frac{2-3-4+6}{12} = \frac{1}{12}$



Red: $y = x$

Blue: $y = x^2$

$f(x,y) = 1 - 2xy$ not sure how to show this...

9. The region beneath $z = 4x^2 + 9y^2$ and above the rectangle with vertices (0, 0), (3, 0), (3, 2), (0, 2) in the xy-plane.

$\int_0^3 \int_0^2 (4x^2 + 9y^2)\, dy\, dx$

$= \int_0^3 \left[(4x^2 y + 3y^3) \Big|_0^2 \right] dx$

$= \int_0^3 (8x^2 + 24 - 0)\, dx$

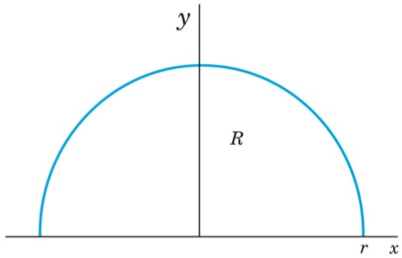
$= \frac{8}{3}x^3 + 24x \Big|_0^3$

$= 72 + 24 \times 3 = 144$

12–16 **CENTER OF GRAVITY**

Find the center of gravity $(\bar{x}, \bar{y})$ of a mass of density $f(x, y) = 1$ in the given region R.

15.



$$M = \iint_R f(x,y)\,dx\,dy = \int_0^{\pi}\int_0^{r} r\,dr\,d\theta = \int_0^{\pi} \frac{r^2}{2}\,d\theta = \frac{1}{2}\pi r^2$$

$$\bar{x} = \frac{1}{M}\iint_R x f(x,y)\,dx\,dy = 0,$$ for reasons of symmetry.

$$\bar{y} = \frac{1}{M}\iint_R y f(x,y)\,dx\,dy = \frac{2}{\pi r^2}\int_0^{\pi}\int_0^{r} r\sin\theta\, r\,dr\,d\theta$$

$$= \frac{2}{\pi r^2}\int_0^{\pi}\left(\sin\theta \cdot \frac{r^3}{3}\Big|_0^r\right)d\theta$$

$$= \frac{2}{\pi r^2}\int_0^{\pi} \frac{r^3}{3} \cdot \sin\theta\,d\theta$$

$$= \frac{2}{\pi r^2}\cdot\frac{r^3}{3}\left(-\cos\theta\Big|_0^{\pi}\right)$$

$$= \frac{4r}{3\pi}$$

Selected Problem set 10.4

10.4 3. 9. 17

1–10 LINE INTEGRALS: EVALUATION BY GREEN'S THEOREM

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem, where

1. $\mathbf{F} = [y, -x]$, C the circle $x^2 + y^2 = 1/4$
2. $\mathbf{F} = [6y^2, 2x - 2y^4]$, R the square with vertices $\pm(2, 2)$, $\pm(2, -2)$
3. $\mathbf{F} = [x^2e^y, y^2e^x]$, R the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 3)$, $(0, 3)$

3. $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

$= \int_0^3 \int_0^2 (y^2e^x - x^2e^y)\, dx\,dy$

$= \int_0^3 \left(y^2e^x - \frac{x^3}{3}e^y\right)\Big|_{x=0}^{x=2} dy$

$= \int_0^3 \left[\left(y^2e^2 - \frac{8}{3}e^y\right) - (y^2 - 0)\right] dy$

$= \int_0^3 \left(y^2e^2 - \frac{8}{3}e^y - y^2\right) dy$

$= \frac{y^3}{3}(e^2-1) - \frac{8}{3}e^y \Big|_{y=0}^{y=3}$

$= \left[9(e^2-1) - \frac{8}{3}e^3\right] - \left(0 - \frac{8}{3}\cdot 1\right)$

$= 9(e^2-1) - \frac{8}{3}(e^3-1)$

$= -\frac{8}{3}e^3 + 9e^2 - \frac{19}{3}$

$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C F_1dx + F_2dy$

$= \oint_C x^2e^y\, dx + \oint_C y^2e^x dy$

$\oint_C x^2e^y\, dx = \int_0^2 x^2 e^0 dx - \int_0^2 x^2 \cdot e^3 dx$

$= \frac{x^3}{3}\Big|_0^2 - e^3 \cdot \frac{x^3}{3}\Big|_0^2$

$= (1-e^3)\cdot\frac{8}{3}$

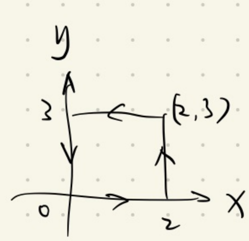
$\oint_C y^2e^x dy = \int_0^3 y^2e^2dy - \int_0^3 y^2\cdot e^0 dy$

$= \frac{3^3}{3}(e^2-1)$

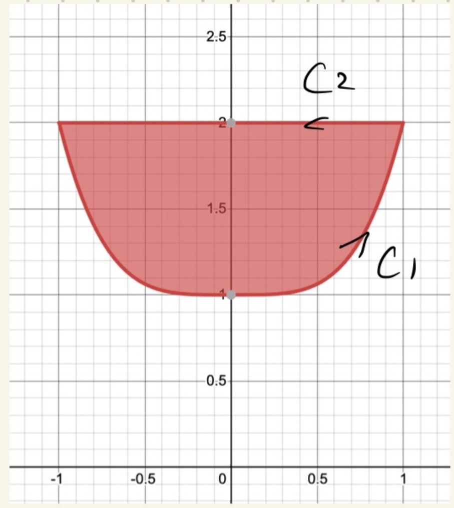
$= 9(e^2-1)$

$\oint_C F_1dx + F_2dx$

$= (1-e^3)\cdot\frac{8}{3} + 9(e^2-1)$



9. $\mathbf{F} = [e^{y/x}, e^y \ln x + 2x]$, R: $1 + x^4 \leqq y \leqq 2$



$1+x^4 \le y \le 2 \qquad 1 \le y \le 2$

$1+x^4 \le 2 \qquad x^4 \le 1 \qquad -1 \le x \le 1$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{e^y}{x} + 2 - \frac{1}{x} \cdot e^{y/x}\right) dx\,dy$$

$$= \int_{-1}^{1} \int_{1+x^4}^{2} \left(\frac{e^y}{x} + 2 - \frac{e^{\frac{y}{x}}}{x}\right) \cdot dy\,dx$$

$$= \int_{-1}^{1} \left(\frac{e^y}{x} + 2y - e^{\frac{y}{x}}\right) \Bigg|_{y=1+x^4}^{y=2} dx$$

$$= \int_{-1}^{1} \left[\frac{e^2}{x} + 4 - e^{\frac{2}{x}} - \frac{e^{1+x^4}}{x} - 2(1+x^4) + e^{\frac{1+x^4}{x}}\right] dx = ?$$

$x \to 0$, ?

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C F_1 dx + F_2 dy$$

$$= \int_{-1}^{1} \int_{1+x^4}^{2} e^{\frac{y}{x}} \cdot dx + (e^y \ln x + 2x)\,dy + \int_{1}^{-1} e^{\frac{2}{x}} \cdot dx \quad ?$$

if $F = [e^y / x,\ e^y \ln x + 2x]$

$$\Rightarrow \int_{-1}^{1} \int_{1+x^4}^{2} \left(\frac{e^y}{x} + 2 - \frac{e^y}{x}\right) dy\,dx$$

$$= \int_{-1}^{1} \int_{1+x^4}^{2} 2\,dy\,dx$$

$$= \int_{-1}^{1} \left(2y \Big|_{y=1+x^4}^{y=2}\right) dx$$

$$= \int_{-1}^{1} [4 - 2(1+x^4)]\,dx$$

$$= \int_{-1}^{1} (2 - 2x^4]\,dx$$

$$= 2x - \frac{2}{5}x^5 \Big|_{-1}^{1} = \frac{16}{5}$$

13–17 INTEGRAL OF THE NORMAL DERIVATIVE

Using (9), find the value of $\int_C \frac{\partial w}{\partial n}\, ds$ taken counterclockwise over the boundary C of the region R.

17. $w = x^3 - y^3, \quad 0 \leqq y \leqq x^2, \quad |x| \leqq 2$

(9) $$\iint_R \nabla^2 w \, dx\, dy = \oint_C \frac{\partial w}{\partial n}\, ds.$$

$$\oint_C \frac{\partial w}{\partial n} \cdot ds = \iint_R \nabla^2 w \, dx dy$$

$$= \int_{-2}^{2} \int_0^{x^2} (6x - 6y)\, dy dx$$

$$= \int_{-2}^{2} \left(6xy - 3y^2 \Big|_{y=0}^{y=x^2} \right) dx$$

$$= \int_{-2}^{2} \left[(6x \cdot x^2 - 3x^4) - (0-0) \right] dx$$

$$= 3 \int_{-2}^{2} (2x^3 - x^4)\, dx$$

$$= 3 \cdot \left(\frac{2}{4} x^4 - \frac{1}{5} x^5 \Big|_{-2}^{2} \right)$$

$$= 3 \left[\left(\frac{1}{2} \cdot 2^4 - \frac{1}{5} \cdot 2^5 \right) - \left(\frac{1}{2} \cdot 2^4 + \frac{1}{5} \cdot 2^5 \right) \right]$$

$$= 3 \cdot \left(-\frac{1}{5}\right) \cdot 2^6 = -\frac{192}{5}$$

Selectd Problem set 10.5

10.5 5, 15

1–8 PARAMETRIC SURFACE REPRESENTATION

Familiarize yourself with parametric representations of important surfaces by deriving a representation (1), by finding the **parameter curves** (curves u = const and v = const) of the surface and a normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ of the surface. Show the details of your work.

5. Paraboloid of revolution $\mathbf{r}(u, v) = [u \cos v, \quad u \sin v, \; u^2]$

$x^2 + y^2 = z$

u = constant ⇒ z = constant ⇒ circle.

v = constant ⇒ parabola (half)

$r(u,v) = [u\cos v, u\sin v, u^2]$

$r_u = [\cos v, \sin v, 2u]$

$r_v = [-u\sin v, u\cos v, 0]$

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= -2u^2\cos v \cdot i - 2u^2\sin v\, j + u\, k$$

14–19 **DERIVE A PARAMETRIC REPRESENTATION**

Find a normal vector. The answer gives *one* representation; there are many. Sketch the surface and parameter curves.

14. Plane $4x + 3y + 2z = 12$

15. Cylinder of revolution $(x - 2)^2 + (y + 1)^2 = 25$

15. Centered Cylinder.

The circular cylinder $x^2 + y^2 = a^2$, $-1 \leqq z \leqq 1$, has radius a, height 2, and the z-axis as axis. A parametric representation is

$$\mathbf{r}(u, v) = [a \cos u, a \sin u, v] = a \cos u\,\mathbf{i} + a \sin u\,\mathbf{j} + v\mathbf{k} \qquad \text{(Fig. 242).}$$

So $r(u,v)$ of problem 15 is:

$r(u,v) = [5\cos u + 2,\ 5\sin u - 1,\ v]$

v is constant: Circle, center at $(2,-1)$, $r=5$

u is constant: line. Parallel to z-axis.

$r_u = [-5\sin u,\ 5\cos u,\ 0]$

$r_v = [0,\ 0,\ 1]$

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ -5\sin u & 5\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 5\cos u\, i + 5\sin u\, j$$

Selected Problem set 10.6

10.6 3. 13

1–10 FLUX INTEGRALS (3) $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$

Evaluate the integral for the given data. Describe the kind of surface. Show the details of your work.

1. $\mathbf{F} = [-x^2, y^2, 0]$, $S: \mathbf{r} = [u, v, 3u - 2v]$, $0 \leqq u \leqq 1.5$, $-2 \leqq v \leqq 2$
2. $\mathbf{F} = [e^y, e^x, 1]$, $S: x + y + z = 1$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$
3. $\mathbf{F} = [0, x, 0]$, $S: x^2 + y^2 + z^2 = 1$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$

A sphere $x^2 + y^2 + z^2 = a^2$ can be represented in the form

(3) $\mathbf{r}(u, v) = a \cos v \cos u\,\mathbf{i} + a \cos v \sin u\,\mathbf{j} + a \sin v\,\mathbf{k}$

$u: [0, \frac{\pi}{2}]$ $\quad v: [0, \frac{\pi}{2}]$

$r_u' = [\cos v(-\sin u), \cos v \cos u, 0]$

$r_v' = [-\sin v \cdot \cos u, -\sin v \sin u, \cos v]$

$$N = \begin{vmatrix} i & j & k \\ -\cos v \sin u & \cos v \cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & \cos v \end{vmatrix}$$

$= [\cos^2 v \cos u, \ \cos^2 v \sin u, \ \cos v \sin v \sin^2 u + \sin v \cos v \cos^2 u]$

$= [\cos^2 v \cos u, \ \cos^2 v \sin u, \ \sin v \cos v]$

$F(r(u,v)) \cdot N(u,v) = \cos v \cos u \cdot \cos^2 v \sin u$

$= \cos^3 v \cos u \sin u$

$$\iint_S F \cdot n\, dA = \iint_R \cos^3 v \cos u \sin u\, du\, dv$$

$$\int_0^{\frac{\pi}{2}} \cos u \sin u\, du = \frac{\sin^2 u}{2} \Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} \cos^3 v\, dv = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos v(1 - \sin^2 v)\, dv$$

let $t = \sin v$

$= \frac{1}{2} \int_0^1 (1 - t^2)\, dt$

$= \frac{1}{2} \int_0^1 1\, dt - \frac{1}{2} \int_0^1 t^2\, dt$

$= (\frac{1}{2} - 0) - (\frac{1}{2} \cdot \frac{1}{3} - 0)$

$= \frac{1}{2} - \frac{1}{6}$

$= \frac{1}{3}$

12–16 **SURFACE INTEGRALS (6)** $\iint_S G(\mathbf{r})\, dA$

Evaluate these integrals for the following data. Indicate the kind of surface. Show the details.

12. $G = \cos x + \sin x$, S the portion of $x + y + z = 1$ in the first octant

13. $G = x + y + z$, $z = x + 2y$, $0 \leqq x \leqq \pi$, $0 \leqq y \leqq x$

$$\iint_S G(\mathbf{r})\, dA = \iint_R G(\mathbf{r}(u, v))|\mathbf{N}(u, v)|\, du\, dv.$$

let $x = u$, $y = v$, $z = u + 2v$

$\vec{r} = [x, y, z] = [u, v, u+2v]$

$\vec{r}_u = [1, 0, 1]$

$\vec{r}_v = [0, 1, 2]$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$= (0-1)i - (2-0)j + (1-0)k$

$= -i - 2j + k$

$|N(u,v)| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

$G(r(u,v)) \cdot \sqrt{6} = \sqrt{6}(2u + 3v)$

$$\iint_S G(r)\, dA$$

$$= \int_0^{\pi}\int_0^{u} \sqrt{6}\,(2u + 3v) \cdot dv\, du$$

$$= \int_0^{\pi} \sqrt{6}\left(2uv + \frac{3}{2}v^2\right)\Big|_{v=0}^{v=u} \cdot du$$

$$= \sqrt{6}\int_0^{\pi}\left(2u^2 + \frac{3}{2}u^2\right) du$$

$$= \sqrt{6} \cdot \frac{7}{2 \times 3} \cdot u^3 \Big|_0^{\pi}$$

$$= \frac{7}{6}\sqrt{6}\,\pi^3$$

Selected Problem set 10.7

10.7 1. 9. 17

1–8 APPLICATION: MASS DISTRIBUTION

Find the total mass of a mass distribution of density σ in a region T in space.

1. $\sigma = x^2 + y^2 + z^2$, T the box $|x| \leqq 4$, $|y| \leqq 1$, $0 \leqq z \leqq 2$

$F = x^2 + y^2 + z^2$

$-4 \leq x \leq 4,\quad -1 \leq y \leq 1,\quad 0 \leq z \leq 2$

$$\iiint_T \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dA.$$

$\operatorname{div} F = 2x + 2y + 2z$

$$\iiint_T \operatorname{div} F\, dV = \int_0^2 \int_{-1}^1 \int_{-4}^4 (2x+2y+2z)\, dx\, dy\, dz$$

$$= \int_0^2 \int_{-1}^1 \left[x^2 + (2y+2z)x \Big|_{x=-4}^{x=4} \right] dy\, dz$$

$$= \int_0^2 \int_{-1}^1 16(y+z)\, dy\, dz$$

$$= 8 \int_0^2 \left(y^2 + 2yz \Big|_{-1}^{1} \right) dz$$

$$= 16 \int_0^2 2z\, dz$$

$$= 16 \cdot z^2 \Big|_0^2$$

$$= 16 \cdot 4 = 64$$

Think before applying formula.

$$\iiint_T F \cdot dV = \int_0^2 \int_{-1}^1 \int_{-4}^4 (x^2+y^2+z^2)\, dx\, dy\, dz$$

$$= \int_0^2 \int_{-1}^1 \left(\frac{x^3}{3} + (y^2+z^2)x \Big|_{-4}^{4} \right) dy\, dz$$

$$= \int_0^2 \int_{-1}^1 \left[\frac{128}{3} + 8(y^2+z^2) \right] dy\, dz$$

$$= \int_0^2 \left(\frac{128}{3} y + \frac{8}{3} y^3 + 8z^2 y \Big|_{-1}^{1} \right) dz$$

$$= \int_0^2 \left(\frac{272}{3} + 16z^2 \right) dz$$

$$= \frac{272}{3} z + \frac{16z^3}{3} \Big|_0^2$$

$$= \frac{544}{3} + \frac{128}{3} = 224$$

9–18 **APPLICATION OF THE DIVERGENCE THEOREM**

Evaluate the surface integral $\iint_S \mathbf{F} \bullet \mathbf{n}\, dA$ by the divergence theorem. Show the details.

9. $\mathbf{F} = [x^2, 0, z^2]$, S the surface of the box $|x| \leqq 1$, $|y| \leqq 3$, $0 \leqq z \leqq 2$

$$\iiint_T \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dA.$$

$\operatorname{div} F = 2x + 0 + 2z$

$$\iiint_T \operatorname{div} F \cdot dV = \int_0^2 \int_{-3}^3 \int_{-1}^1 (2x+2z) \cdot dx\, dy\, dz.$$

$$= \int_0^2 \int_{-3}^3 \left(x^2 + 2zx \Big|_{x=-1}^{x=1} \right) dy\, dz$$

$$= \int_0^2 \int_{-3}^3 4z \cdot dy\, dz$$

$$= \int_0^2 \left(4zy \Big|_{y=-3}^{y=3} \right) dz$$

$$= \int_0^2 24z \cdot dz$$

$$= 12z^2 \Big|_0^2$$

$$= 48$$

17. $\mathbf{F} = [x^2, y^2, z^2]$, S the surface of the cone $x^2 + y^2 \leqq z^2$, $0 \leqq z \leqq h$

$\operatorname{div} F = 2x + 2y + 2z$

$$\iiint_T \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dA.$$

$y^2 \le z^2 - x^2 \le h^2 - x^2$

$$\iiint_T \operatorname{div} F\, dV = \int_0^h \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_0^h \int_{-z}^{z} \left[x^2 + (2y+2z)x \Big|_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} \right] dy\, dz$$

$$= \int_0^h \int_{-z}^{z} \left[4(y+z)\sqrt{z^2-y^2} \right] dy\, dz$$

$$f(y) = 4(y+z)\sqrt{z^2-y^2}$$

$$F^*(y) = \frac{6z^3 \arcsin\left(\frac{y}{z}\right) + \sqrt{z^2-y^2}\,(4y^2 + 6|z|y - 4z^2)}{3} + C$$

$$= \int_0^h 2\pi z^3\, dz$$

$$= \frac{2\pi}{4} \cdot z^4 \Big|_0^h = \frac{\pi}{2} h^4$$

Something wrong. try polar in next page

$$\iiint_T \operatorname{div} F \, dV = \iiint_T (2x+2y+2z)\,dx\,dy\,dz$$

$$= \int_0^h G(z)\cdot dz.$$

$$G(z) = \iint_S (2x+2y+2z)\,dx\,dy$$

$$x^2+y^2 \le z^2, \quad z \ge 0, \quad \Rightarrow \quad x = r\cos\theta, \quad y = r\sin\theta$$

$$0 \le \theta \le 2\pi \qquad 0 \le r \le z.$$

$$\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$G(z) = \int_0^{2\pi}\int_0^{z} (2r\cos\theta + 2r\sin\theta + 2z)\, r \cdot dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{z} \left[2(\cos\theta+\sin\theta) r^2 + 2zr\right] dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{2}{3}(\cos\theta+\sin\theta) r^3 + z r^2 \Big|_{r=0}^{r=z} \right] \cdot d\theta$$

$$= \int_0^{2\pi} \left[\frac{2}{3}(\cos\theta+\sin\theta) z^3 + z^3 \right] d\theta$$

$$= \frac{2}{3} z^3 \cdot \left(\int_0^{2\pi} \cos\theta\, d\theta + \int_0^{2\pi} \sin\theta\, d\theta \right) + z^3 \theta \Big|_0^{2\pi}$$

$$= \frac{2}{3} z^3 (0+0) + 2\pi z^3$$

$$\iiint_T \operatorname{div} F \cdot dV = \int_0^h 2\pi z^3\, dz = \frac{\pi}{2}\cdot h^4$$

Selected Problem set 10.9

10.9 1. 3. 5.

From 10.6

1–10 DIRECT INTEGRATION OF SURFACE INTEGRALS

Evaluate the surface integral $\iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dA$ directly for the given $\mathbf{F}$ and S.

1. $\mathbf{F} = [z^2, -x^2, 0]$, S the rectangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 4, 4)$, $(1, 4, 4)$

at every point (except perhaps for some edges or cusps, as for a cube or cone). For a given vector function $\mathbf{F}$ we can now define the **surface integral** over S by

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \bullet \mathbf{N}(u, v)\, du\, dv. \tag{3}$$

Here $\mathbf{N} = |\mathbf{N}|\mathbf{n}$ by (2), and $|\mathbf{N}| = |\mathbf{r}_u \times \mathbf{r}_v|$ is the area of the parallelogram with sides $\mathbf{r}_u$ and $\mathbf{r}_v$, by the definition of cross product. Hence

$$\mathbf{n}\, dA = \mathbf{n}|\mathbf{N}|\, du\, dv = \mathbf{N}\, du\, dv. \tag{3*}$$

And we see that $dA = |\mathbf{N}|du\, dv$ is the element of area of S.

$\text{curl } F = 2z\,j - 2x\,k$.

4 points plane equation:

$0x - 4y + 4z = 0 \Rightarrow y = z$.

$N = \text{grad}(y - z) = [0, 1, -1]$

$\iint_S (\text{curl}(F))\, n\, dV = \iint_S \text{curl } F \cdot N \cdot dxdy$

$= \iint_R [0, 2z, -2x] \cdot [0, 1, -1] \cdot dxdy$.

$= \int_0^1 \int_0^4 (2z + 2x)\, dy\, dx$

$= \int_0^1 \int_0^4 (2y + 2x)\, dy\, dx$

$= \int_0^1 \left(y^2 + 2xy \Big|_0^4 \right) dx$

$= \int_0^1 (16 + 8x)\, dx$

$= 16x + 4x^2 \Big|_0^1 = 20$

Since we are not specify the direction, So N could be both direction aka $\pm N$.

So result $= \pm 20$

3. $\mathbf{F} = [e^{-z}, e^{-z}\cos y, e^{-z}\sin y]$, S: $z = y^2/2$, $-1 \leqq x \leqq 1$, $0 \leqq y \leqq 1$

$$\operatorname{Curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-z} & e^{-z}\cdot\cos y & e^{-z}\sin y \end{vmatrix}$$

$$= (e^{-z}\cdot\cos y + e^{-z}\cdot\cos y)\, i$$

$$- (0 + e^{-z})\, j$$

$$+ (0 - 0)\, k$$

$$= 2\cdot e^{-z}\cdot\cos y\ i - e^{-z}\, j$$

P399: Gradient as Surface Normal vector.

$$f(x, y, z) = y^2/2 - z = 0$$

$$N = \operatorname{grad} f = [0, y, -1]$$

$$\iint_R \operatorname{Curl} F \cdot N \cdot dx\,dy$$

$$= \iint_R -y e^{-z}\, dx\,dy$$

$$= \int_{-1}^{1}\int_0^1 -y e^{-\frac{y^2}{2}}\, dy\,dx$$

$$= \int_{-1}^{1} \left(\frac{1}{\sqrt{e}} - 1\right) dx$$

$$= \frac{2}{\sqrt{e}} - 2$$

We do not specify the direction of the S, so N could be $\pm$

result $= \pm\left(\frac{2}{\sqrt{e}} - 2\right)$

5. $\mathbf{F} = [z^2, \frac{3}{2}x, 0]$, S: $0 \leqq x \leqq a$, $0 \leqq y \leqq a$, $z = 1$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & \frac{3}{2}x & 0 \end{vmatrix}$$

$$= (0-0)i - (0-2z)j + (\tfrac{3}{2}-0)k$$

$$= 2z \cdot j + \tfrac{3}{2}k.$$

assume $a > 0$.

$N = n = [0, 0, 1]$

$$\iint_S \operatorname{curl} F \cdot n \cdot dA$$

$$= \iint_S \frac{3}{2}\, dx\, dy$$

$$= \int_0^a \int_0^a \frac{3}{2}\, dx\, dy$$

$$= \int_0^a \left(\frac{3}{2}x \Big|_0^a \right) dy$$

$$= \int_0^a \frac{3}{2}a\, dy$$

$$= \frac{3}{2}a y \Big|_0^a$$

$$= \frac{3}{2}a^2$$

Since we have not define the direction of s. so the result is $\pm \frac{3}{2}a^2$