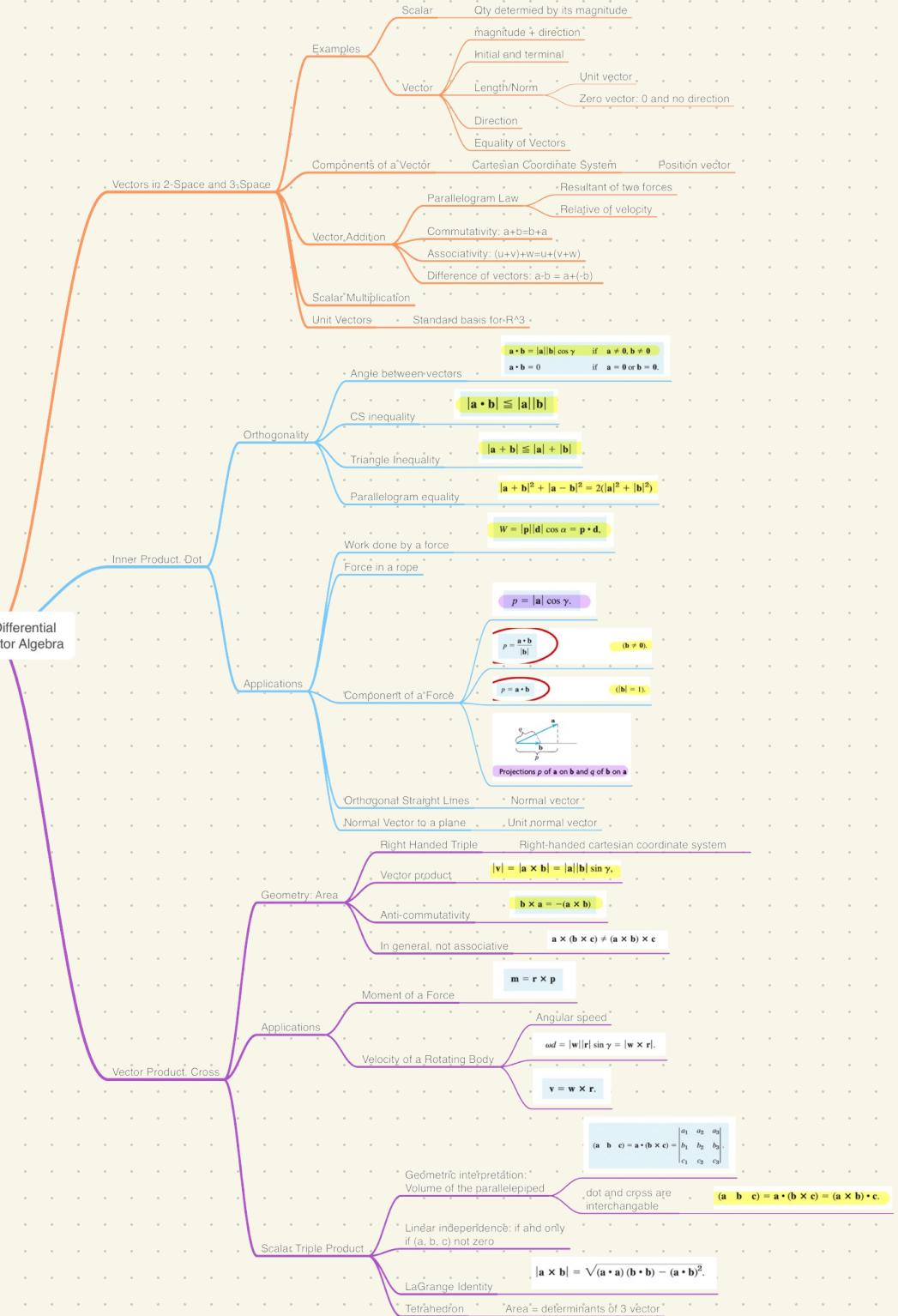
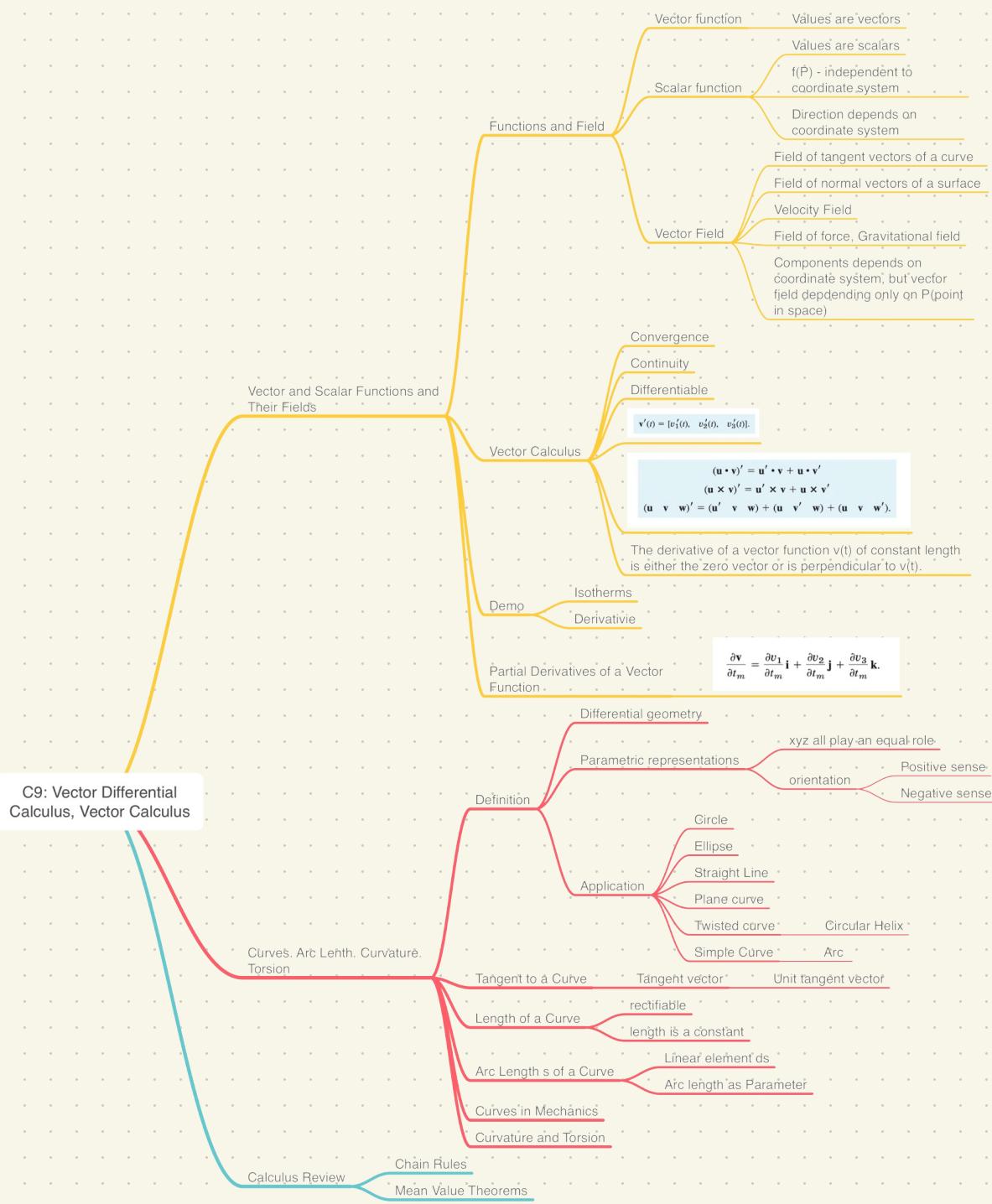


W3

C9: Vector Differential Calculus, Vector Algebra





C9

Basic Properties of Scalar Multiplication. From the definitions we obtain directly

- (a) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
 (b) $(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$
 (c) $c(k\mathbf{a}) = (ck)\mathbf{a}$
 (written $ck\mathbf{a}$)
 (d) $1\mathbf{a} = \mathbf{a}.$

Basic Properties of Vector Addition. Familiar laws for real numbers give immediately

- (4) (a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (Commutativity)
 (b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity)
 (c) $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$
 (d) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \gamma \quad \text{if } \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$$

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \text{if } \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}.$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

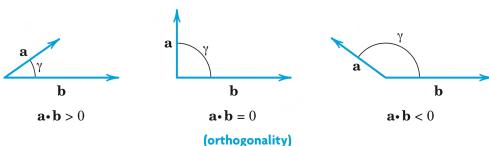


Fig. 178. Angle between vectors and value of inner product

THEOREM 1

Orthogonality Criterion

The inner product of two nonzero vectors is 0 if and only if these vectors are perpendicular.

Length and Angle. Equation (1) with $\mathbf{b} = \mathbf{a}$ gives $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$. Hence

$$(3) \quad |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

From (3) and (1) we obtain for the angle γ between two nonzero vectors

$$(4) \quad \cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}}}.$$

$$\begin{aligned}
 (a) \quad & (q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_2\mathbf{b} \cdot \mathbf{c} && (\text{Linearity}) \\
 (5) \quad (b) \quad & \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} && (\text{Symmetry}) \\
 (c) \quad & \mathbf{a} \cdot \mathbf{a} \geq 0 && \\
 & \mathbf{a} \cdot \mathbf{a} = 0 \quad \text{if and only if } \mathbf{a} = \mathbf{0} && \left. \right\} (\text{Positive-definiteness})
 \end{aligned}$$

Hence dot multiplication is commutative as shown by (5b). Furthermore, it is distributive with respect to vector addition. This follows from (5a) with $q_1 = 1$ and $q_2 = 1$:

$$(5a^*) \quad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \quad (\text{Distributivity}).$$

Furthermore, from (1) and $|\cos \gamma| \leq 1$ we see that

$$(6) \quad |\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}| \quad (\text{Cauchy-Schwarz inequality}).$$

Using this and (3), you may prove (see Prob. 16)

$$(7) \quad |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \quad (\text{Triangle inequality}).$$

Geometrically, (7) with $<$ says that one side of a triangle must be shorter than the other two sides together; this motivates the name of (7).

A simple direct calculation with inner products shows that

$$(8) \quad |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad (\text{Parallelogram equality}).$$

Work Done by a Force Expressed as an Inner Product

This is a major application. It concerns a body on which a *constant* force \mathbf{p} acts. (For a *variable* force, see Sec. 10.1.) Let the body be given a displacement \mathbf{d} . Then the work done by \mathbf{p} in the displacement is defined as

$$(9) \quad W = |\mathbf{p}| |\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d},$$

Example 3 is typical of applications that deal with the **component or projection of a vector \mathbf{a} in the direction of a vector \mathbf{b}** ($\neq \mathbf{0}$). If we denote by p the length of the orthogonal projection of \mathbf{a} on a straight line l parallel to \mathbf{b} as shown in Fig. 181, then

$$(10) \quad p = |\mathbf{a}| \cos \gamma.$$

Here p is taken with the plus sign if $p\mathbf{b}$ has the direction of \mathbf{b} and with the minus sign if $p\mathbf{b}$ has the direction opposite to \mathbf{b} .

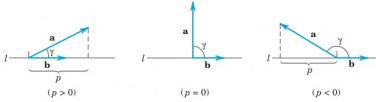


Fig. 181. Component of a vector \mathbf{a} in the direction of a vector \mathbf{b}

Multiplying (10) by $|\mathbf{b}|/|\mathbf{b}| = 1$, we have $\mathbf{a} \cdot \mathbf{b}$ in the numerator and thus

$$(11) \quad p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \quad (\mathbf{b} \neq \mathbf{0}).$$

If \mathbf{b} is a unit vector, as it is often used for fixing a direction, then (11) simply gives

$$(12) \quad p = \mathbf{a} \cdot \mathbf{b} \quad (|\mathbf{b}| = 1).$$

Figure 182 shows the projection p of \mathbf{a} in the direction of \mathbf{b} (as in Fig. 181) and the projection $q = |\mathbf{b}| \cos \gamma$ of \mathbf{b} in the direction of \mathbf{a} .

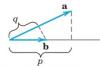


Fig. 182. Projections p of \mathbf{a} on \mathbf{b} and q of \mathbf{b} on \mathbf{a}

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \gamma,$$

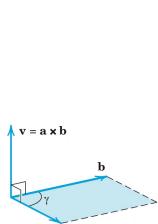


Fig. 185. Vector product

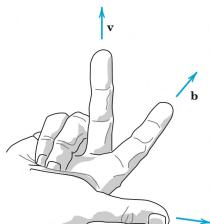


Fig. 186. Right-handed triple of vectors $\mathbf{a}, \mathbf{b}, \mathbf{v}$

$$(2**)\quad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

General Properties of Vector Products

(a) For every scalar l ,

$$(4) \quad (l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) Cross multiplication is distributive with respect to vector addition; that is,

$$(5) \quad (\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$

$$(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

(c) Cross multiplication is not commutative but anticommutative; that is,

$$(6) \quad \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \quad (\text{Fig. 189}).$$

(d) Cross multiplication is not associative; that is, in general,

$$(7) \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

so that the parentheses cannot be omitted.

$$(\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

THEOREM 2

Properties and Applications of Scalar Triple Products

(a) In (10) the dot and cross can be interchanged:

$$(11) \quad (\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

(b) Geometric interpretation. The absolute value $|(\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c})|$ of (10) is the volume of the parallelepiped (oblique box) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as edge vectors (Fig. 193).

(c) Linear independence. Three vectors in R^3 are linearly independent if and only if their scalar triple product is not zero.

$$(c\mathbf{v})' = c\mathbf{v}'$$

(c constant),

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

$$(\mathbf{u} - \mathbf{v} - \mathbf{w})' = (\mathbf{u}' - \mathbf{v} - \mathbf{w}) + (\mathbf{u} - \mathbf{v}' - \mathbf{w}) + (\mathbf{u} - \mathbf{v} - \mathbf{w}').$$

$$l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'} dt$$

$$\left(\mathbf{r}' = \frac{d\mathbf{r}}{dt} \right).$$

Constant

the **arc length function** or simply the **arc length** of C . Thus

$$(11) \quad s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'} d\tilde{t} \quad \left(\mathbf{r}' = \frac{d\mathbf{r}}{d\tilde{t}} \right).$$

$$(13) \quad ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 + dz^2.$$

ds is called the **linear element** of C .

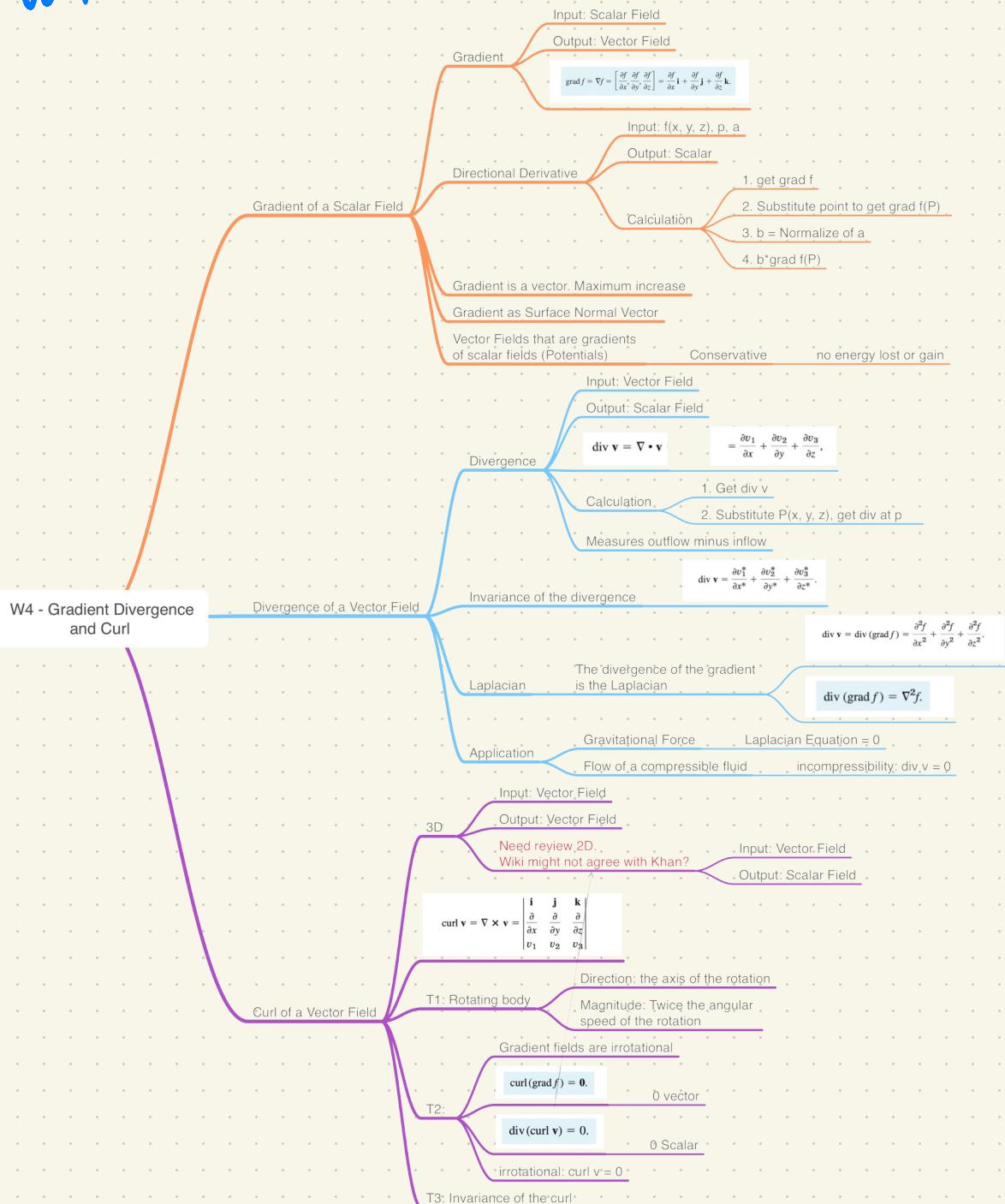
$$\mathbf{u}(s) = \mathbf{r}'(s).$$

$$\kappa(s) = |\mathbf{u}'(s)| = |\mathbf{r}''(s)| \quad (' = d/ds).$$

$$|\tau(s)| = |\mathbf{b}'(s)|.$$

$$\tau(s) = -\mathbf{p}(s) \cdot \mathbf{b}'(s).$$

w4



9.6

chain rule

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}.$$

9.7. Grad.

$$\text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

$$D_{\mathbf{b}} f = \frac{df}{ds} = \mathbf{b} \cdot \text{grad } f$$

($|\mathbf{b}| = 1$).

$$D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \text{grad } f.$$

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{p} = \frac{k}{r^3} \mathbf{r}$$

(Coulomb's law⁶).

All vectors of the form $\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ constitute the **real vector space R^3** with componentwise vector addition

$$(1) \quad [a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

and componentwise scalar multiplication (c a scalar, a real number)

$$(2) \quad c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3] \quad (\text{Sec. 9.1}).$$

For instance, the *resultant* of forces \mathbf{a} and \mathbf{b} is the sum $\mathbf{a} + \mathbf{b}$.

The **inner product** or **dot product** of two vectors is defined by

$$(3) \quad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \gamma = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (\text{Sec. 9.2})$$

where γ is the angle between \mathbf{a} and \mathbf{b} . This gives for the **norm** or **length** $|\mathbf{a}|$ of \mathbf{a}

$$(4) \quad |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

as well as a formula for γ . If $\mathbf{a} \cdot \mathbf{b} = 0$, we call \mathbf{a} and \mathbf{b} **orthogonal**. The dot product is suggested by the *work* $W = \mathbf{p} \cdot \mathbf{d}$ done by a force \mathbf{p} in a displacement \mathbf{d} .

The **vector product** or **cross product** $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is a vector of length

$$(5) \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \gamma \quad (\text{Sec. 9.3})$$

and perpendicular to both \mathbf{a} and \mathbf{b} such that $\mathbf{a}, \mathbf{b}, \mathbf{v}$ form a *right-handed triple*. In terms of components with respect to right-handed coordinates,

$$(6) \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (\text{Sec. 9.3}).$$

The vector product is suggested, for instance, by moments of forces or by rotations. **CAUTION!** This multiplication is *anticommutative*, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and is *not associative*.

An (oblique) box with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$ has volume equal to the absolute value of the **scalar triple product**

$$(7) \quad (\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Sections 9.4–9.9 extend differential calculus to vector functions

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$$

and to vector functions of more than one variable (see below). The derivative of $\mathbf{v}(t)$ is

$$(8) \quad \mathbf{v}' = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = [v'_1, v'_2, v'_3] = v'_1\mathbf{i} + v'_2\mathbf{j} + v'_3\mathbf{k}.$$

Differentiation rules are as in calculus. They imply (Sec. 9.4)

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \quad (\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'.$$

Curves C in space represented by the position vector $\mathbf{r}(t)$ have $\mathbf{r}'(t)$ as a **tangent vector** (the **velocity** in mechanics when t is time), $\mathbf{r}'(s)$ (s arc length, Sec. 9.5) as the **unit tangent vector**, and $|\mathbf{r}''(s)| = \kappa$ as the **curvature** (the **acceleration** in mechanics).

Vector functions $\mathbf{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$ represent vector fields in space. Partial derivatives with respect to the Cartesian coordinates x, y, z are obtained componentwise, for instance,

$$\frac{\partial \mathbf{v}}{\partial x} = \left[\frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial x}, \frac{\partial v_3}{\partial x} \right] = \frac{\partial v_1}{\partial x}\mathbf{i} + \frac{\partial v_2}{\partial x}\mathbf{j} + \frac{\partial v_3}{\partial x}\mathbf{k} \quad (\text{Sec. 9.6}).$$

The **gradient** of a scalar function f is

$$(9) \quad \text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \quad (\text{Sec. 9.7}).$$

The **directional derivative** of f in the direction of a vector \mathbf{a} is

$$(10) \quad D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \nabla f \quad (\text{Sec. 9.7}).$$

The **divergence** of a vector function \mathbf{v} is

$$(11) \quad \text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}. \quad (\text{Sec. 9.8}).$$

The **curl** of \mathbf{v} is

$$(12) \quad \text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \quad (\text{Sec. 9.9})$$

or minus the determinant if the coordinates are left-handed.

Some basic formulas for grad, div, curl are (Secs. 9.7–9.9)

$$(13) \quad \begin{aligned} \nabla(fg) &= f\nabla g + g\nabla f \\ \nabla(f/g) &= (1/g^2)(g\nabla f - f\nabla g) \end{aligned}$$

$$(14) \quad \begin{aligned} \text{div}(f\mathbf{v}) &= f \text{div } \mathbf{v} + \mathbf{v} \cdot \nabla f \\ \text{div}(f\nabla g) &= f\nabla^2 g + \nabla f \cdot \nabla g \end{aligned}$$

$$(15) \quad \begin{aligned} \nabla^2 f &= \text{div}(\nabla f) \\ \nabla^2(fg) &= g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g \end{aligned}$$

$$(16) \quad \begin{aligned} \text{curl}(f\mathbf{v}) &= \nabla f \times \mathbf{v} + f \text{curl } \mathbf{v} \\ \text{div}(\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \text{curl } \mathbf{u} - \mathbf{u} \cdot \text{curl } \mathbf{v} \end{aligned}$$

$$(17) \quad \begin{aligned} \text{curl}(\nabla f) &= \mathbf{0} \\ \text{div}(\text{curl } \mathbf{v}) &= 0. \end{aligned}$$

For grad, div, curl, and ∇^2 in **curvilinear coordinates** see App. A3.4.