W4

W4 - Gradient Divergence and Curl

- Gradient of a Scalar Field
  - Gradient
    - Input: Scalar Field
    - Output: Vector Field
    - $\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$
  - Directional Derivative
    - Input: f(x, y, z), p, a
    - Output: Scalar
    - Calculation
      1. get grad f
      2. Substitute point to get grad f(P)
      3. b = Normalize of a
      4. b*grad f(P)
  - Gradient is a vector. Maximum increase
  - Gradient as Surface Normal Vector
  - Vector Fields that are gradients of scalar fields (Potentials)
    - Conservative
      - no energy lost or gain
- Divergence of a Vector Field
  - Divergence
    - Input: Vector Field
    - Output: Scalar Field
    - $\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v}$ $= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$
    - Calculation
      1. Get div v
      2. Substitute P(x, y, z), get div at p
    - Measures outflow minus inflow
  - Invariance of the divergence
    - $\operatorname{div} \mathbf{v} = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*}.$
  - Laplacian
    - The divergence of the gradient is the Laplacian
      - $\operatorname{div} \mathbf{v} = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$
      - $\operatorname{div}(\operatorname{grad} f) = \nabla^2 f.$
  - Application
    - Gravitational Force
      - Laplacian Equation = 0
    - Flow of a compressible fluid
      - incompressibility: div v = 0
- Curl of a Vector Field
  - 3D
    - Input: Vector Field
    - Output: Vector Field
    - Need review 2D. Wiki might not agree with Khan?
      - Input: Vector Field
      - Output: Scalar Field
  - $\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$
  - T1: Rotating body
    - Direction: the axis of the rotation
    - Magnitude: Twice the angular speed of the rotation
  - T2:
    - Gradient fields are irrotational
    - $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}.$
      - 0 vector
    - $\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0.$
      - 0 Scalar
    - irrotational: curl v = 0
  - T3: Invariance of the curl

9.6

chain rule

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

9.7. Grad.

$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

$$D_{\mathbf{b}} f = \frac{df}{ds} = \mathbf{b} \cdot \operatorname{grad} f \qquad (|\mathbf{b}| = 1).$$

$$D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|}\mathbf{a} \cdot \operatorname{grad} f.$$

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{p} = \frac{k}{r^3}\mathbf{r} \qquad \text{(Coulomb's law}^6\text{)}.$$

All vectors of the form $\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ constitute the **real vector space** R^3 with componentwise vector addition

$$[a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3] \tag{1}$$

and componentwise scalar multiplication (c a scalar, a real number)

$$c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3] \qquad \text{(Sec. 9.1).} \tag{2}$$

For instance, the *resultant* of forces $\mathbf{a}$ and $\mathbf{b}$ is the sum $\mathbf{a} + \mathbf{b}$.

The **inner product** or **dot product** of two vectors is defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(Sec. 9.2)} \tag{3}$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This gives for the **norm** or **length** $|\mathbf{a}|$ of $\mathbf{a}$

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{4}$$

as well as a formula for γ. If $\mathbf{a} \cdot \mathbf{b} = 0$, we call $\mathbf{a}$ and $\mathbf{b}$ **orthogonal**. The dot product is suggested by the *work* $W = \mathbf{p} \cdot \mathbf{d}$ done by a force $\mathbf{p}$ in a displacement $\mathbf{d}$.

The **vector product** or **cross product** $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is a vector of length

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma \qquad \text{(Sec. 9.3)} \tag{5}$$

and perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$ form a *right-handed* triple. In terms of components with respect to right-handed coordinates,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{(Sec. 9.3).} \tag{6}$$

The vector product is suggested, for instance, by moments of forces or by rotations. CAUTION! This multiplication is *anti*commutative, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and is *not* associative.

An (oblique) box with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ has volume equal to the absolute value of the **scalar triple product**

$$(7) \qquad (\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Sections 9.4–9.9 extend differential calculus to vector functions

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$$

and to vector functions of more than one variable (see below). The derivative of $\mathbf{v}(t)$ is

$$(8) \qquad \mathbf{v}' = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = [v_1', v_2', v_3'] = v_1'\mathbf{i} + v_2'\mathbf{j} + v_3'\mathbf{k}.$$

Differentiation rules are as in calculus. They imply (Sec. 9.4)

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \qquad (\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'.$$

Curves C in space represented by the position vector $\mathbf{r}(t)$ have $\mathbf{r}'(t)$ as a **tangent vector** (the **velocity** in mechanics when t is time), $\mathbf{r}'(s)$ (s arc length, Sec. 9.5) as the *unit tangent vector,* and $|\mathbf{r}''(s)| = \kappa$ as the *curvature* (the *acceleration* in mechanics).

Vector functions $\mathbf{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$ represent vector fields in space. Partial derivatives with respect to the Cartesian coordinates x, y, z are obtained componentwise, for instance,

$$\frac{\partial \mathbf{v}}{\partial x} = \left[\frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial x}, \frac{\partial v_3}{\partial x}\right] = \frac{\partial v_1}{\partial x}\mathbf{i} + \frac{\partial v_2}{\partial x}\mathbf{j} + \frac{\partial v_3}{\partial x}\mathbf{k} \qquad \text{(Sec. 9.6).}$$

The **gradient** of a scalar function f is

$$(9) \qquad \operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \qquad \text{(Sec. 9.7).}$$

The **directional derivative** of f in the direction of a vector $\mathbf{a}$ is

$$(10) \qquad D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \nabla f \qquad \text{(Sec. 9.7).}$$

The **divergence** of a vector function $\mathbf{v}$ is

$$(11) \qquad \operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}. \qquad \text{(Sec. 9.8).}$$

The **curl** of **v** is

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{12}$$ (Sec. 9.9)

or minus the determinant if the coordinates are left-handed.

Some basic formulas for grad, div, curl are (Secs. 9.7–9.9)

$$\begin{aligned} \nabla(fg) &= f\nabla g + g\nabla f \\ \nabla(f/g) &= (1/g^2)(g\nabla f - f\nabla g) \end{aligned} \tag{13}$$

$$\begin{aligned} \text{div}\,(f\mathbf{v}) &= f\,\text{div}\,\mathbf{v} + \mathbf{v} \cdot \nabla f \\ \text{div}\,(f\nabla g) &= f\nabla^2 g + \nabla f \cdot \nabla g \end{aligned} \tag{14}$$

$$\begin{aligned} \nabla^2 f &= \text{div}\,(\nabla f) \\ \nabla^2(fg) &= g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g \end{aligned} \tag{15}$$

$$\begin{aligned} \text{curl}\,(f\mathbf{v}) &= \nabla f \times \mathbf{v} + f\,\text{curl}\,\mathbf{v} \\ \text{div}\,(\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v} \end{aligned} \tag{16}$$

$$\begin{aligned} \text{curl}\,(\nabla f) &= \mathbf{0} \\ \text{div}\,(\text{curl}\,\mathbf{v}) &= 0. \end{aligned} \tag{17}$$

For grad, div, curl, and ∇^2 in **curvilinear coordinates** see App. A3.4.