Chapter 10 Vector Integral Calculus. Integral Theorems

Selected Problem set 10.1

10.1 3. 5. 9. 19.

2–11 LINE INTEGRAL. WORK

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

2. $\mathbf{F} = [y^2, -x^2]$, C: $y = 4x^2$ from (0, 0) to (1, 4)

3. $\mathbf{F}$ as in Prob. 2, C from (0, 0) straight to (1, 4). Compare.

4. $\mathbf{F} = [xy, x^2y^2]$, C from (2, 0) straight to (0, 2)

5. $\mathbf{F}$ as in Prob. 4, C the quarter-circle from (2, 0) to (0, 2) with center (0, 0)

3. C: $r(t) = [t, 4t] = ti + 4tj$

$F(r(t)) = [(4t)^2, -t^2] = [16t^2, -t^2]$

$r'(t) = [1, 4]$

$$\int_C F(r)\cdot dr = \int_0^1 [16t^2, -t^2]\cdot[1, 4]\,dt$$

$$= \int_0^1 (16t^2 - 4t^2)\,dt$$

$$= \int_0^1 12t^2\,dt$$

$$= 4t^3\Big|_0^1 = 4 - 0 = 4$$

5. C by $r(t) = [2\cos t, 2\sin t]$, when $0 \le t \le \frac{\pi}{2}$.

$F(r(t)) = [4\sin t\cos t, 16\sin^2 t\cos^2 t]$

$r'(t) = [-2\sin t, 2\cos t]$

$$\int_C F(r)\cdot dr = \int_0^{\frac{\pi}{2}} (-8\sin^2 t\cos t + 32\sin^2 t\cos^3 t)\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} (4\sin^2 t\cos^3 t - \sin^2 t\cos t)\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t\,(4\cos^2 t - 1)\sin^2 t\,dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t\cdot[-\sin^2 t(4\sin^2 t - 3)]\,dt$$

$u = \sin t$, $\frac{du}{dt} = \cos t$, $dt = \frac{du}{\cos t}$

$$= -8\int_0^1 u^2(4u^2 - 3)\,du$$

$$= -32\int_0^1 u^4\,du + 24\int_0^1 u^2\,du$$

$$= -\frac{32}{5}u^5\Big|_0^1 + 24\cdot\frac{1}{3}u^3\Big|_0^1$$

$$= -\frac{32}{5} + 8 = \frac{8}{5} = 1.6$$

9. $\mathbf{F} = [x + y, y + z, z + x]$, $C: \mathbf{r} = [2t, 5t, t]$ from $t = 0$ to 1. Also from $t = -1$ to 1.

$C: r = [2t, 5t, t] \quad 0 \le t \le 1$

$r' = [2, 5, 1]$

$F(r(t)) = [7t, 6t, 3t]$

$$\int_C F(r)dr = \int_0^1 [7t, 6t, 3t]\,[2, 5, 1]\cdot dt$$

$$= \int_0^1 47t \cdot dt$$

$$= \frac{47}{2}t^2\Big|_0^1 = \frac{47}{2} = 23.5$$

$-1 \le t \le 1$

$$\int_{-1}^1 47t \cdot dt = \frac{47}{2}t^2\Big|_{-1}^1 = 0$$

19. $f = xyz$, $C: \mathbf{r} = [4t, 3t^2, 12t]$, $-2 \le t \le 2$. Sketch C.

$C: r = [4t, 3t^2, 12t] \quad -2 \le t \le 2$

$r' = [4, 6t, 12]$

$F(r(t)) = 144t^4$

$$\int_C f(r)dt = \int_{-2}^2 144t^4 \cdot dt$$

$$= \frac{144}{5} \cdot t^5\Big|_{-2}^2$$

$$= \frac{144}{5} \cdot 64 = 1843.2$$