Chapter 14 - Complex Integration

Selected Problem set 14.1

14.1 1. 5. 7. 11. 13. 23. 25

1–10 **FIND THE PATH** and sketch it.

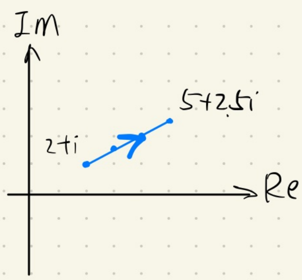
1. $z(t) = (1 + \frac{1}{2}i)t \quad (2 \leqq t \leqq 5)$

$z(2) = 2+i$

$z(3) = 3+1.5i$

$z(4) = 4+2i$

$z(5) = 5+2.5i$



add direction.

5. $z(t) = 3 - i + \sqrt{10}e^{-it} \quad (0 \leqq t \leqq 2\pi)$

$z(t) = 3 - i + \sqrt{10}\left[\cos(-\theta) + i\sin(-\theta)\right]$

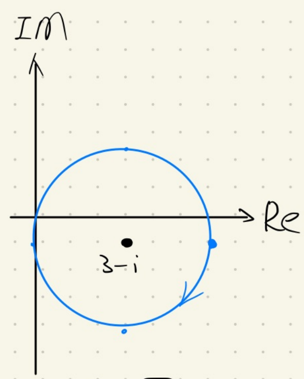
$z(0) = 3+\sqrt{10} - i$

$z(\frac{\pi}{2}) = 3 - (1+\sqrt{10})i$

$z(\pi) = (3-\sqrt{10}) - i$

$z(\frac{3}{2}\pi) = 3 + (\sqrt{10}-1)i$

$z(2\pi) = 3+\sqrt{10} - i$



$r = \sqrt{10}$

pay attension to special point.

x-Intersection. y-Intersection

0 ...

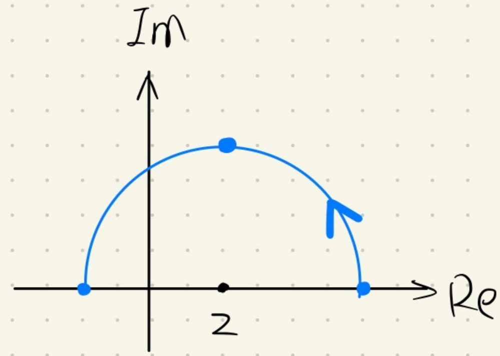
7. $z(t) = 2 + 4e^{\pi i t/2} \quad (0 \leqq t \leqq 2)$

$$e^{\frac{\pi t}{2} \cdot i} = \cos\left(\frac{\pi}{2}t\right) + i \sin\left(\frac{\pi}{2}t\right)$$

$$z(0) = 2 + 4(1 + 0i) = 6$$

$$z(1) = 2 + 4(0 + i) = 2 + 4i$$

$$z(2) = 2 + 4(-1 + 0i) = -2$$



11–20 **FIND A PARAMETRIC REPRESENTATION** and sketch the path.

11. Segment from $(-1, 1)$ to $(1, 3)$

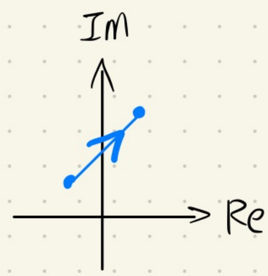
$$m = \frac{3-1}{1-(-1)} = 1$$

$$(-1+t,\ 1+t) \qquad 0 \le t \le 2$$

or $z(t) = (-1, 1) + i\,t(2, 2)$

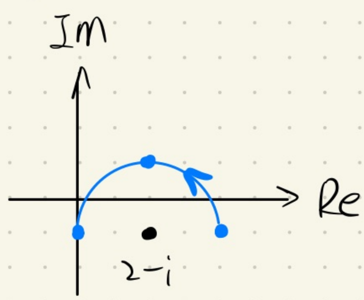
$= (-1+2t) + i(1+2t)$

$0 \le t \le 1$



13. Upper half of $|z - 2 + i| = 2$ from $(4, -1)$ to $(0, -1)$

$$z(t) = 2 - i + 2e^{it} \qquad 0 \le t \le \pi$$



or

$$z(t) = (2 + 2\cos t) + i(-1 + 2\sin t)$$

$$0 \le t \le \pi$$

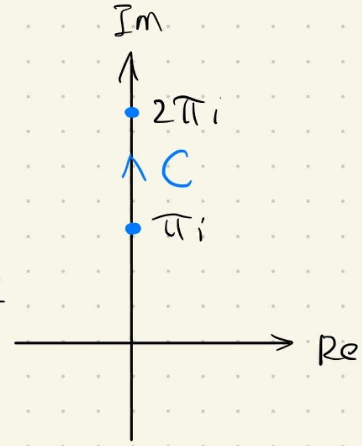
21–30 INTEGRATION

Integrate by the first method or state why it does not apply and use the second method. Show the details.

23. $\int_C e^z\,dz$, C the shortest path from πi to $2\pi i$

$C: z(t) = t\pi i \qquad 1 \le t \le 2$

$\dot{z}(t) = \pi i$



$$e^z \Big|_{\pi i}^{2\pi i} = e^{2\pi i} - e^{\pi i} = 1 + 0i - (-1 + 0i) = 2$$

$$\int_1^2 e^{t\pi i} \cdot \pi i\,dt$$

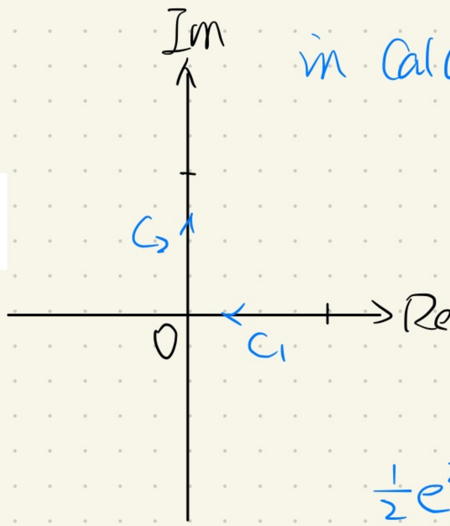
$$= \pi i \int_1^2 [\cos(t\pi) + i\sin(t\pi)]\,dt$$

$$= \pi i \left(0 + -\frac{2}{\pi} \cdot i\right)$$

$$= 2$$

Simply connected.
$\Rightarrow$ Apply theorem 1.
which is simpler in calculation.

25. $\int_C z \exp(z^2)\,dz$, C from 1 along the axes to i



$C_1: z_1(t) = 1 - t \qquad 0 \le t \le 1$

$\dot{z}_1(t) = -1$

$C_2 \quad z_2(t) = ti \qquad 0 \le t \le 1$

$\dot{z}_2(t) = i$

$\frac{1}{2} e^{z^2} \Big|_1^i = -\sinh 1$

$$\int_C z\exp(z^2) = \int_{C_1} z_1 \exp(z_1^2) + \int_{C_2} z_2 \exp(z_2^2)$$

$$= \int_0^1 (1-t) \cdot e^{(1-t)^2} \cdot (-1)\,dt + \int_0^1 ti \cdot e^{(ti)^2} \cdot i\,dt$$

$$= \int_0^1 (t-1) \cdot e^{(1-t)^2} - t \cdot e^{-t^2}\,dt = -\sinh 1$$

☆
1. prove function is analytic $\Rightarrow$ path independent
2. $(e^{z^2})' = 2z \cdot e^{z^2}$
use first evaluation
$\int_1^i z e^{z^2} dz = \frac{1}{2}\int_1^{-1} e^{z^2} dz^2 = -\sinh 1$

-1.1752 very complex. numerical approach

Selected Problem set 14.2

14.2 9, 11, 15, 21, 23, 25

9–19 CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

9. $f(z) = \exp(-z^2)$ **10.** $f(z) = \tan \frac{1}{4} z$

$z = x + iy$.

$f(z) = e^{-(x^2 - y^2 + 2xyi)} = e^{y^2 - x^2 - 2xyi}$

$= e^{y^2 - x^2} \cdot e^{-2xyi}$

$= e^{y^2-x^2} \cdot [\cos(-2xy) + i\sin(-2xy)]$

$= e^{y^2-x^2} [\cos(2xy) - i\sin(2xy)]$

$u = e^{y^2-x^2} \cos(2xy)$

$v = -e^{y^2-x^2} \sin(2xy)$

$u_x = -2x \cdot e^{y^2-x^2} \cdot \cos 2xy - e^{y^2-x^2} \cdot 2y \cdot \sin 2xy$

$u_y = 2y \cdot e^{y^2-x^2} \cdot \cos 2xy - e^{y^2-x^2} \cdot 2x \cdot \sin 2xy$

$v_x = 2x \cdot e^{y^2-x^2} \cdot \sin 2xy - e^{y^2-x^2} \cdot 2y \cdot \cos 2xy$

$v_y = -2y \cdot e^{y^2-x^2} \cdot \sin 2xy - e^{y^2-x^2} \cdot 2x \cdot \cos 2xy$

$u_x = v_y$. $u_y = -v_x$. $f(z)$ is analytic.

$\oint_C f(z)\,dz = 0$

or: $g(x) = e^z$

$h(z) = -z^2$

are both analytic

$\Rightarrow$ composition

$f(z) = g(h(z))$ analytic

11. $f(z) = 1/(2z - 1)$

9–19 CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

not analytic at $z = \frac{1}{2}$

$\oint_C \frac{dx}{x} = 2\pi i$ per page 648. for C as unit circle.

or $\int_C (z-z_0)^m dz = 2\pi i$

P649

Let $2z-1 = x$ $\quad 2dz = dx$

$\oint_C \frac{1}{2z-1} dz = \oint_C \frac{1}{x} \cdot \frac{1}{2} \cdot dx$

$= \frac{1}{2} \oint_C \frac{dx}{x} = \frac{1}{2} \cdot 2\pi i = \pi i$

15. $f(z) = \text{Im}\, z$

$z(t) = \cos t + i \sin t = e^{it} \quad 0 \le t \le 2\pi$

$f(z) = \sin t \qquad \dot{z}(t) = i e^{it}$

$\oint_C f(z) \cdot dz = \int_0^{2\pi} \sin t \cdot i e^{it} dt$

For $\int \sin t \cdot i e^{it} \cdot dt$ ①

let $u = it \qquad dt = -i du$

① $= -i \int e^u \sinh(u) \cdot du$

$= -i \int \frac{e^{-2u}(e^{2u}-1)}{4} \cdot 2 \cdot e^{2u} du$

Let $v = e^u \qquad du = \frac{e^{-2u}}{2} dv$

① $= -i \cdot \frac{1}{4} \int \frac{v-1}{v} dv = -i\left(\frac{v}{4} - \frac{\ln v}{4}\right) + C$

① $= -i\left(\frac{v}{4} - \frac{\ln v}{4}\right) + C$

$= -i\left(\frac{e^{2u}}{4} - \frac{u}{2}\right) + C$

$= -\frac{ie^{2it}}{4} - \frac{t}{2} + C$

$\int_0^{2\pi} \sin t \cdot i e^{it} dt$

$= -\frac{ie^{2it}}{4} - \frac{t}{2} \Big|_0^{2\pi}$

$= -\frac{t}{2} \Big|_0^{2\pi} = -\pi$

20–30 FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

20. $\oint_C \mathrm{Ln}\,(1-z)\,dz$, C the boundary of the parallelogram with vertices $\pm i$, $\pm(1+i)$.

21. $\oint_C \frac{dz}{z-3i}$, C the circle $|z| = \pi$ counterclockwise.

A: Very complex

B: Good.

C: apply the known result. → easiest

21 $f(z) = \frac{1}{z-3i}$ not analytic at $z = 3i$

$z(t) = e^{\pi}\cdot(\cos t + i\sin t) = e^{\pi}e^{it} \quad 0 \le t \le 2\pi$

$\dot{z}(t) = e^{\pi}\cdot i e^{it}$

$$f(z(t)) = \frac{1}{e^{\pi}\cos t + (e^{\pi}\sin t - 3)i}$$

$$\oint f(z)dz = \int_0^{2\pi} \frac{1}{e^{\pi}\cos t + (e^{\pi}\sin t - 3)i}\cdot e^{\pi}\cdot i\cdot e^{it}\,dt$$

$$= \int_0^{2\pi} \frac{[e^{\pi}\cos t - (e^{\pi}\sin t - 3)i]\cdot e^{\pi}\cdot i\cdot e^{it}}{e^{2\pi}\cos^2 t + (e^{\pi}\sin t - 3)^2}\,dt$$

$$= \ln\left|e^{it+\pi} - 3i\right|\Big|_0^{2\pi} = 2\pi i$$

C equation 3, example 6.

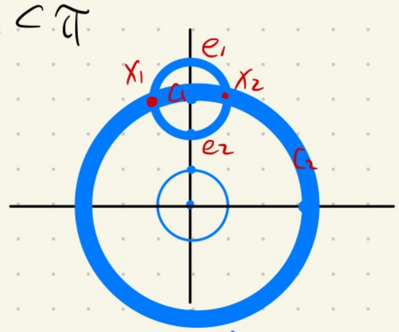
$\oint (z-z_0)^{-1}dz = 2\pi i$

B let $z - 3i = x$. $dz = dx$. $|3i| = 3 < \pi$

$2\pi i = \oint_D \frac{dx}{x}$ for D the unit circle

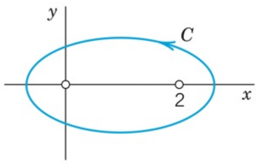
$\oint_E \frac{dz}{z-3i}$ for E: $\cos t + i(\sin t + 3)$

$\oint_C \frac{dz}{z-3i} = \oint_E \frac{dz}{z-3i} = \oint_D \frac{dx}{x} = 2\pi i$



$\oint_e = \oint_{e_1} + \oint_{e_2} = \oint_{c_1} + \oint_{c_2} = \oint_C$

23. $\oint_C \frac{2z-1}{z^2-z}\,dz$, C:



Use partial fractions.

$$\frac{2z-1}{z^2-z} = \frac{z-1+z}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$

$$\oint_C \frac{2z-1}{z^2-z}\,dz = \oint \frac{1}{z}\,dz + \oint \frac{1}{z-1}\,dz$$

$$= 2\pi i + 2\pi i = 4\pi i$$

$$\frac{a}{z} + \frac{b}{z-1} = \frac{az-a+bz}{z(z-1)}$$

$a+b=2$

$a=1 \quad b=1$

25. $\oint_C \frac{e^z}{z}\,dz$, C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

e^z analytic.

$\frac{1}{z}$ analytic except $z=0$.

$\frac{e^z}{z}$ analytic except $z=0$.

$$C = \{|z|=2\} - \{|z|=1\}$$

$$= (\{|z|=2\} - \{|z|=0\}) - (\{|z|=1\} - \{|z|=0\})$$

$$= 0 - 0 = 0$$

or: (6) $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$ (Fig. 353) P658

Selected Problem set 14.3

14.3 1, 3, 7, 13

1–4 CONTOUR INTEGRATION

Integrate $z^2/(z^2-1)$ by Cauchy's formula counterclockwise around the circle.

1. $|z+1| = 1$ **2.** $|z-1-i| = \pi/2$
3. $|z+i| = 1.4$ **4.** $|z+5-5i| = 7$

1. $g(z)$ not analytic at ± 1. -1 in the domain, 1 is not

$$\oint_C \frac{z^2}{z^2-1}\,dz = \oint_C \frac{1}{(z+1)}\cdot\frac{z^2}{(z-1)}\,dz$$

$$= 2\pi i \left(\frac{z^2}{z-1}\right)\Big|_{z=-1} = -\pi i$$

or

$$\frac{1}{2}\int_C \frac{z^2}{z-1}\,dz - \frac{1}{2}\int_C \frac{z^2}{z+1}\,dz$$

apply $\int_C \frac{f(x)}{x-x_0}\,dx = 2\pi i f(x_0)$

$\Rightarrow 0 - \pi i = -\pi i$

3. $g(z) = \frac{z^2}{z^2-1}$ not analytic at ± 1.

$|z+i| = 1.4$ does not cover ± 1 $(\sqrt{2} \doteq 1.414)$

So $g(z)$ is analytic in domain

So $\oint_C \frac{z^2}{z^2-1}\,dz = 0$

5–8 Integrate the given function around the unit circle.

5. $(\cos 3z)/(6z)$ **6.** $e^{2z}/(\pi z - i)$

7. $z^3/(2z - i)$ **8.** $(z^2 \sin z)/(4z - 1)$

7. $g(z)$ not analytic at $\frac{i}{2}$ which is in the domain.

$$\oint_C \frac{z^3}{2z-i} dz = \frac{1}{2} \oint_C \frac{z^3}{z-\frac{1}{2}i} dz$$

$$= \frac{1}{2} \cdot 2\pi i \left. z^3 \right|_{z=\frac{1}{2}i}$$

$$= \pi i \cdot \left(\frac{1}{2}i\right)^3 = \frac{\pi}{8}$$

11–19 **FURTHER CONTOUR INTEGRALS**

Integrate counterclockwise or as indicated. Show the details.

13. $\oint_C \frac{z+2}{z-2} dz, \quad C: |z-1| = 2$

$g(z)$ not analytic at 2. which is covered in the domain.

$$\oint_C \frac{z+2}{z-2} dz = 2\pi i \left. (z+2) \right|_{z=2}$$

$$= 2\pi i \cdot 4 = 8\pi i$$

Selected Problem set 14.4

14.4 3, 13

1–7 CONTOUR INTEGRATION. UNIT CIRCLE

Integrate counterclockwise around the unit circle.

1. $\oint_C \frac{\sin z}{z^4}\, dz$ **2.** $\oint_C \frac{z^6}{(2z-1)^6}\, dz$

3. $\oint_C \frac{e^z}{z^n}\, dz, \quad n = 1, 2, \cdots$ **4.** $\oint_C \frac{e^z \cos z}{(z-\pi/4)^3}\, dz$

8–19 INTEGRATION. DIFFERENT CONTOURS

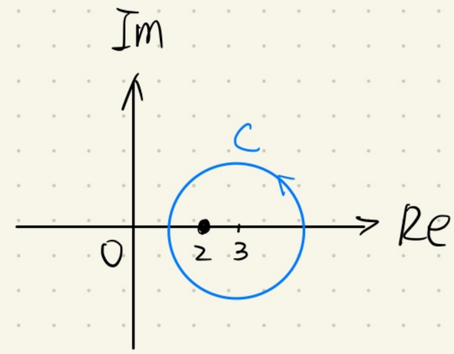
Integrate. Show the details. *Hint.* Begin by sketching the contour. Why?

13. $\oint_C \frac{\operatorname{Ln} z}{(z-2)^2}\, dz, \quad C: |z-3| = 2$ counterclockwise.

3.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{f^{(n)}(z_0)\cdot 2\pi i}{n!}$$



$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz$$

$$\oint_C \frac{e^z}{(z-0)^n} \cdot dz$$

$$= \frac{f^{(n-1)}(0) \cdot 2\pi i}{(n-1)!}$$

$$= \frac{1 \cdot 2\pi i}{(n-1)!}$$

$f(z) = \operatorname{Ln} z, \quad n = 1$

$$\oint_C \frac{\operatorname{Ln} z}{(z-2)^2}\, dz = (\operatorname{Ln}(2))' \cdot 2\pi i$$

$$= \frac{1}{2} \cdot 2\pi i$$

$$= \pi i$$