Chapter 7 Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

P261 - Problem set 7.1

1. 2x2: $a_{11} \neq b_{11}, b_{12} \neq c_{12}$, 2x3: $d_{11} \neq e_{11}$
2. $a_{31} = 10,\ \ a_{13} = 81\ \ , a_{26} = 96,\ \ a_{33} = 0$
3. E1: 3x3, 3x4,
   E2: 3x7,
   E3: 2x2, 2x2, 2x2, 2x3, 2x3,
   E5:3x2
4. 1A: 4, 0, 1
   3A: a_{11}, a_{22}
   3B: 4, -1
5. $B = \frac{1}{5}A$,
   $B = \frac{1}{10}A$
6. $B = \frac{1}{1.609}A$
7. No. No(1x1 as exception?). Yes. Maybe not in math (how about 1x1?) but OK in python. No.
8. $2A + 4B = 4B + 2A = \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix}$
   B
   $0.4B - 4.2A = \begin{bmatrix} 0 & -6.4 & -16 \\ -23.2 & -19.8 & -19.4 \\ -5 & 1.6 & 11.8 \end{bmatrix}$
9. $3A = \begin{bmatrix} 0 & 6 & 12 \\ 18 & 15 & 15 \\ 3 & 0 & -9 \end{bmatrix}$
   $0.5B = \begin{bmatrix} 0 & 2.5 & 1 \\ 2.5 & 1.5 & 2 \\ -1 & 2 & -1 \end{bmatrix}$
   $3A + 0.5B = \begin{bmatrix} 0 & 8.5 & 13 \\ 20.5 & 16.5 & 17 \\ 2 & 2 & -10 \end{bmatrix}$
   3A+0.5B+C is not defined.

10. $(4 \bullet 3)A = 4(3A) = \begin{bmatrix} 0 & 24 & 48 \\ 72 & 60 & 60 \\ 12 & 0 & -36 \end{bmatrix}$

$14B - 3B = 11B = \begin{bmatrix} 0 & 55 & 22 \\ 55 & 33 & 44 \\ -22 & 44 & -22 \end{bmatrix}$

11. $8C + 10D = 2(5D + 4C) = \begin{bmatrix} 0 & 26 \\ 34 & 32 \\ 28 & -10 \end{bmatrix}$

$0.6C - 0.6D = 0.6(C - D) = \begin{bmatrix} 5.4 & 0.6 \\ -4.2 & 2.4 \\ -0.6 & 0.6 \end{bmatrix}$

12. $(C + D) + E = (D + E) + C = \begin{bmatrix} 1 & 5 \\ 6 & 8 \\ 6 & -2 \end{bmatrix}$

$0(C - E) + 4D = 4D = \begin{bmatrix} -16 & 4 \\ 20 & 0 \\ 8 & -4 \end{bmatrix}$

$A - 0C$: 3x3 can not minus 3x2, not defined

13. $(2 \bullet 7)C = 2(7C) = \begin{bmatrix} 70 & 28 \\ -28 & 56 \\ 14 & 0 \end{bmatrix}$

$-D + 0E = -D = \begin{bmatrix} 4 & -1 \\ -5 & 0 \\ -2 & 1 \end{bmatrix}$

$E - D + C + u$: Since EDC are 3x2 but u is 3x1, not defined.

14. $(5u + 5v) - \frac{1}{2}w = \begin{bmatrix} 5 \\ 30 \\ -10 \end{bmatrix}$

$-20(u + v) + 2w = -4[(5u + 5v) - \frac{1}{2}w] = \begin{bmatrix} -20 \\ -120 \\ 40 \end{bmatrix}$

$E - (u + v)$: 3x2 can not minus 3x1, not defined

$10(u + v) + w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

15. $(u + v) - w = u + (v - w) = \begin{bmatrix} 5.5 \\ 33 \\ -11 \end{bmatrix}$

$C + 0w$: 3x2 can not minus 3x1, not defined

$0E + u - v$: 3x2 can not minus 3x1, not defined

16. $15v - 3w - 0u = -3w + 15v = \begin{bmatrix} 0 \\ 135 \\ 0 \end{bmatrix}$

$D - u + 3C$: 3x2 can not minus 3x1, not defined

$8.5w - 11.1u + 0.4v = \begin{bmatrix} 25.45 \\ 256.2 \\ 119.1 \end{bmatrix}$

17. $u + v + w = \begin{bmatrix} -4.5 \\ -27 \\ 9 \end{bmatrix}$

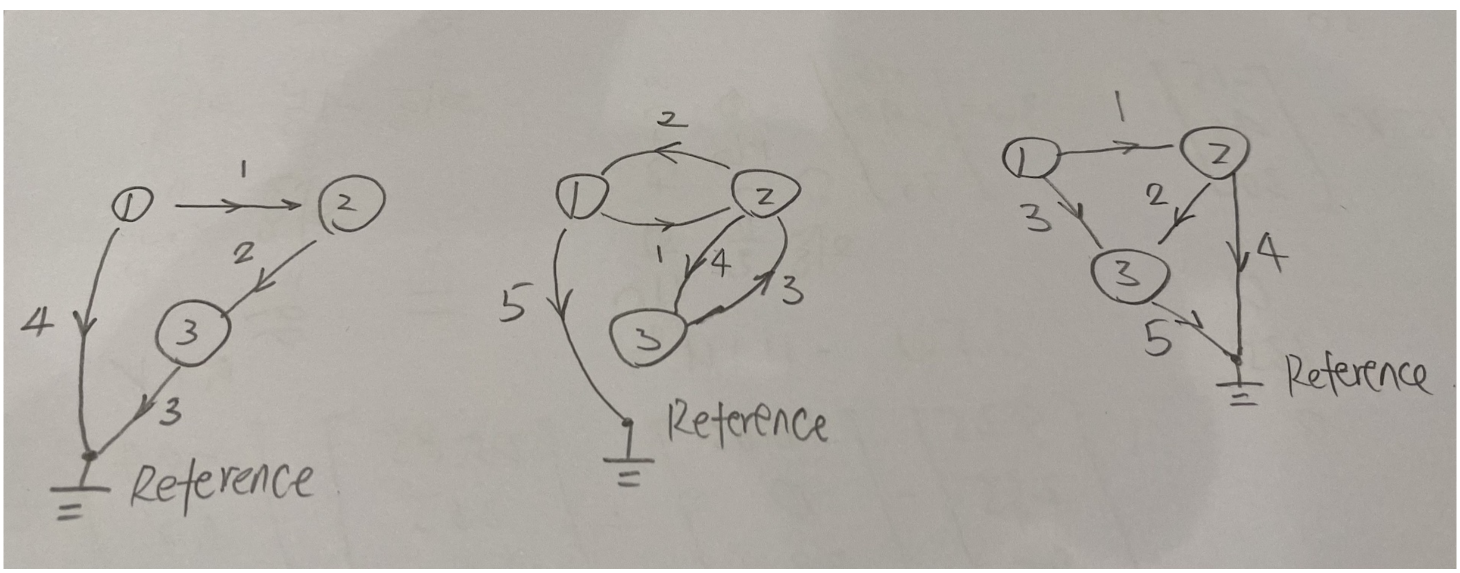
18. $p = 0 - u - v - w = \begin{bmatrix} 4.5 \\ 27 \\ -9 \end{bmatrix}$

19. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

20. b-1: $\begin{bmatrix} -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

b-2: $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

c:



P270 - Problem set 7.2

Example13. In the final stable situation(limit),

$I + C + R = 100$

$0.7C + 0.1I = C$
$0.2C + 0.9I + 0.2R = I$
$0.1C + 0.8R = R$

So we can get C=200/9, I=200/3, R=100/9.

Will revisit it after Sec. 8.2

1. Per definition, the number of the entries in the columns of the second matrix have to be same as the number of the entries in the rows of the first matrix. In short, if mxn matrix multiple pxq, then n=p. Or you won't be able to perform the dot product.
2. All entries or components are 0
3. No. All rows are proportional.
4. Min is 1 which is 0, and max is $n(n-1)+1$
   Take 3x3 as example, $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$
5. Min is 1 which is 0, and max is $\frac{n(n+1)}{2}$
   Take 3x3 as example, $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$
6. $U_1 + U_2, U_1U_2, U_1^2$ are upper triangular matrices.$L_1 + L_2$ is lower triangular.
7. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
8. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for any $m \geq 1, m \in N$. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ for any $m \geq 2, m \in N$.
9. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
10. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
11. $AB = AB^T = \begin{bmatrix} 10 & -14 & -6 \\ -5 & 7 & -12 \\ -5 & -1 & -4 \end{bmatrix}$

$$BA = B^TA = \begin{bmatrix} 10 & -5 & -15 \\ -14 & 7 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

12. $AA^T = \begin{bmatrix} 29 & 8 & 6 \\ 8 & 41 & 12 \\ 6 & 12 & 9 \end{bmatrix}, A^2 = \begin{bmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{bmatrix}, BB^T = B^2 = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

13. $CC^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 13 & -6 \\ 0 & -6 & 4 \end{bmatrix}$, $BC = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}$, CB not defined, $C^T B = \begin{bmatrix} -9 & 3 & 4 \\ -5 & -1 & 0 \end{bmatrix}$

14. $3A - 2B = \begin{bmatrix} 10 & 0 & 9 \\ 0 & 1 & 18 \\ 3 & 6 & 10 \end{bmatrix}$, $(3A - 2B)^T = 3A^T - 2B^T = \begin{bmatrix} 10 & 0 & 3 \\ 0 & 1 & 6 \\ 9 & 18 & 10 \end{bmatrix}$,

$(3A - 2B)^T a^T = \begin{bmatrix} 10 \\ -2 \\ -27 \end{bmatrix}$

15. Aa not defined, $Aa^T = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$, $(Ab)^T = b^T A^T = \begin{bmatrix} 7 & -11 & 3 \end{bmatrix}$

16. $BC = Problem 13.2 = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}$, BC^T not defined, $Bb = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix}$, $b^T B =$

$\begin{bmatrix} 0 & -8 & 2 \end{bmatrix}$

17. $ABC = \begin{bmatrix} -30 & -18 \\ 45 & 9 \\ 5 & -7 \end{bmatrix}$, ABa not defined, $ABb = \begin{bmatrix} 22 \\ 4 \\ -12 \end{bmatrix}$, Ca^T= not defined.

18. $ab = 1$, $ba = \begin{bmatrix} 3 & -6 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$, $aA = \begin{bmatrix} 8 & -4 & -9 \end{bmatrix}$, $Bb = problem 16.3 = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix}$

19. $1.5a + 3.0b$ not defined. $1.5a^T + 3.0b = \begin{bmatrix} 4.5 \\ -2 \\ -1 \end{bmatrix}$, $(A - B)b = Ab - Bb = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$

20. $b^T Ab$=7, aBa^T=17, $aCC^T = \begin{bmatrix} -3 & -24 & 12 \end{bmatrix}$, $C^T ba = \begin{bmatrix} 5 & -10 & 0 \\ 5 & -10 & 0 \end{bmatrix}$

21. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

22. $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$, $AB = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$

23. $AB = A \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix}$

24. $AB = BA$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$2a_{11} + 3a_{12} = 2a_{11} + 3a_{21}$=> $a_{12} = a_{21}$

$3a_{11} + 4a_{12} = 2a_{12} + 3a_{22}$=> $3a_{11} + 2a_{12} = 3a_{22}$

$2a_{21} + 3a_{22} = 3a_{11} + 4a_{21}$,

$3a_{21} + 4a_{22} = 3a_{12} + 4a_{22}$

Let $A = \begin{bmatrix} x & y \\ y & \frac{3x+2y}{3} \end{bmatrix}$,

Check: $AB = BA = \begin{bmatrix} 2x+3y & 3x+4y \\ 3x+4y & 4x+5\frac{2}{3}y \end{bmatrix}$

25. a) Obvious.

b) $C = [c_{ij}], C^T = [c_{ji}]$

$D = C + C^T = [d_{ij}] = [c_{ij} + c_{ji}] = [c_{ji} + c_{ij}] = [d_{ji}]$, so D is symmetric

$E = C - C^T = [e_{ij}] = [c_{ij} - c_{ji}] = -[c_{ji} - c_{ij}] = -[e_{ji}]$, so E is skew-symmetric.

Let $S = \frac{1}{2}D, T = \frac{1}{2}E$

$S + T = \frac{1}{2}(D + E) = \frac{1}{2}(C + C^T + C - C^T) = C$

$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}, A^T = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 6 & 2 \end{bmatrix},$

$S = \frac{1}{2}(A + A^T) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix}, T = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B^T = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$S = \frac{1}{2}(B + B^T) = B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, T = \frac{1}{2}(B - B^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c) symmetric: $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}], ..., M = [m_{ij}] = [m_{ji}]$

$aA + bB + ... + mM = a[a_{ij}] + b[b_{ij}] + ... + m[m_{ij}] = a[a_{ji}] + b[b_{ji}] + +... + m[m_{ji}]$.

Skew-symmetric: $A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}], ..., M = [m_{ij}] = -[m_{ji}]$

$aA + bB + ... + mM = a[a_{ij}] + b[b_{ij}] + ... + m[m_{ij}] = -(a[a_{ji}] + b[b_{ji}] + +... + m[m_{ji}])$

.

d) $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}]$

$AB = [a_p b_q]$, if AB is symmetric, then $AB = [a_p b_q] = [a_q b_p] = [b_p a_q] = BA$

vice verse.

e) $A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}]$

$AB = [a_p b_q]$, if AB is skew-symmetric, then $AB = [a_p b_q] = -[a_q b_p] = -[b_p a_q] = -BA$

vice verse.

26. First day, status = $\begin{bmatrix} N \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, stochastic matrix = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$

Second day = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$

Two days after today = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$

Three days after today = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.722 \\ 0.278 \end{bmatrix}$

The limit of N is $\frac{5}{7}$

27. Reserve for future

28. Present = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 1200 \\ 98800 \end{bmatrix}$, stochastic matrix = $\begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix}$

After 1 season = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1200 \\ 98800 \end{bmatrix} = \begin{bmatrix} 1278 \\ 98722 \end{bmatrix}$, increase

After 2 seasons = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1278 \\ 98722 \end{bmatrix} = \begin{bmatrix} 1344 \\ 98656 \end{bmatrix}$, increase

After 3 seasons = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1344 \\ 98656 \end{bmatrix} = \begin{bmatrix} 1407 \\ 98593 \end{bmatrix}$, increase

29. $p = \begin{bmatrix} 35 \\ 62 \\ 30 \end{bmatrix}$

$v = Ap = \begin{bmatrix} 24,920 \\ 25,940 \end{bmatrix}$

30.$y = Ax$

$y_1 = x_1 \cos\theta - x_2 \sin\theta, y_2 = x_1 \sin\theta + x_2 \cos\theta$

$|y|^2 = (x_1 \cos\theta - x_2 \sin\theta)^2 + (x_1 \sin\theta + x_2 \cos\theta)^2 = x_1^2 + x_2^2 = |x|^2$

$\cos\alpha = \frac{x*y}{|x||y|} = \frac{x_1^2 \cos\theta + x_2^2 \cos\theta}{x_1^2 + x_2^2} = \cos\theta$

so x and y have the same length, and from x to y is counterclockwise rotate of θ

b)$AA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} =$
$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

c)$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} =$
$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

d)$[x_1, x_2, x_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = [3x_1, x_2, \frac{1}{2}x_3]$

$[x_1, x_2, x_3]\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} = [cx_1, cx_2, cx_3]$, Scalar matrix will amplify or squeeze the picture by c.

e) $[x_1, x_2, x_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = [x_1, x_2\cos\theta + x_3\sin\theta, -x_2\sin\theta + x_3\cos\theta]$

$x_1$remain the same. counterclockwise rotation of the Cartesian coordinate system x_2x_3 in the plane about the origin by angle of θ

$$[x_1, x_2, x_3]\begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} = [x_1\cos\varphi + x_3\sin\varphi, x_2, -x_1\sin\varphi + x_3\cos\varphi]$$

$x_2$remain the same. counterclockwise rotation of the Cartesian coordinate system x_1x_3 in the plane about the origin by angle of φ

$$[x_1, x_2, x_3]\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = [x_1\cos\psi + x_2\sin\psi, -x_1\sin\psi + x_2\cos\psi, x_3]$$

$x_3$remain the same. counterclockwise rotation of the Cartesian coordinate system x_1x_2 in the plane about the origin by angle of ψ

P280 - Problem set 7.3

1. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{6}{5} \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -4 & 1 & -2 \\ 0 & 17 & -4 & 12 \\ 0 & -34 & 8 & -13 \end{bmatrix}$ = No solution

5. $\begin{bmatrix} 1 & 33 & -225 \\ 0 & 139 & -973 \\ 0 & -376 & 2632 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -7 \\ 0 & 1 & -7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -2 & 2 & 9 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -5 & -20 \end{bmatrix} = \begin{bmatrix} 2t+1 \\ t \\ 4 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 5 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3t \\ t \\ 2t \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 3 & 8 \\ 0 & 4 & 3 & 4 \end{bmatrix}$ = No solution

9. $\begin{bmatrix} 3 & 4 & -5 & 13 \\ 0 & 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 3t-1 \\ 4-t \\ t \end{bmatrix}$

10. $\begin{bmatrix} 5 & -7 & 3 & 17 \\ 5 & -7 & 3 & -50/3 \end{bmatrix}$ = No solution

11. $\begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so we can get $\begin{bmatrix} 1 \\ 2m-n \\ n \\ m \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & -1 & 2 & -5/3 & -5 \end{bmatrix}$, so we can get $\begin{bmatrix} n-2m \\ n \\ m \\ 3 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 10 & 4 & -2 & -4 \\ -3 & -17 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 10 & 4 & -2 & -4 \\ 0 & -14 & 4 & 2 & 20 \\ 0 & 21 & -4 & 5 & 22 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -12 & 9 & 30 \\ 0 & 0 & -41 & 32 & 110 \\ 0 & 0 & -31 & 23 & 76 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 2 & 3 & 1 & -11 & 1 \\ 5 & -2 & 5 & -4 & 5 \\ 3 & 4 & -7 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 3 & -10 & 11 & -10 \\ 0 & 7 & -16 & 11 & -16 \end{bmatrix} =$
$\begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$

so we can get $\begin{bmatrix} 0 \\ 3t \\ 1+2t \\ t \end{bmatrix}$

15. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
16. Reserve for future
17. $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 16 \\ 0 & 4 & 1 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 32 \\ 0 & 0 & 6 & 48 \end{bmatrix}$

    so we can get $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$
18. I am highly unsure about this one. Need a physical book
    $\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

    so we can get $\begin{bmatrix} 9/5 \\ 12/5 \\ -3/5 \end{bmatrix}$
19. $\begin{bmatrix} \frac{E_0}{R_2} + \frac{E_0}{R_1} \\ -\frac{E_0}{R_1} \\ \frac{E_0}{R_2} \end{bmatrix}$
20. $I_3 = I_x, I_1 = I_2$
    $I_1 R_1 = I_x R_x, I_3 R_3 = I_2 R_2$, so we can get
    $R_x = R_3 R_1 / R_2$
21. $\begin{bmatrix} 1 & 1 & 0 & 0 & 1600 \\ 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 1 & 0 & 2200 \\ 0 & 0 & 1 & 1 & 1600 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 0 & 1 & 1 & 1600 \end{bmatrix}$

Rank(3) < N (4), So the solutin is not unique.

22. $40 - 2P_1 - P2 = 4P_1 - P_2 + 4, 6P_1 = 36, P_1 = 6$
$5P_1 - 2P_2 + 16 = 3P_2 - 4, P_1 = P_2 - 4, P_2 = 2$

23. $\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4 & -3 & 0 \end{bmatrix}$

so we can get $\begin{bmatrix} t \\ 5t \\ 3t \\ 4t \end{bmatrix}$

The smallest positive integers are 1, 5, 3, 4

24. a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \\ 10 & 11 & 12 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

$$B = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ -5a_{11} + a_{21} & -5a_{12} + a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11} & a_{12} \\ -5a_{11} + a_{31} & -5a_{12} + a_{32} \\ a_{21} & a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

So $B \neq C$

b)Natually.

Row switch: reference E1

Row multiplication: reference E3 (replace by c)

Row addition and subtraction: reference E2.

Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

P287 - Problem Set 7.4

1. $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, Rank=1, $\{[2, -1, 3]\}$

$A^T = \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$,$\{[2,-1]^T\}$

2. $\begin{bmatrix} a & b \\ a & \frac{a^2}{b} \end{bmatrix}$,
   if $a = b = 0$, rank = 0, $\{0\}$, $\{0\}$
   if $b = \pm a$, rank = 1, $\{[1,-1]\}$, $\{[1,-1]^T\}$
   The rest, rank = 2, $\{[a,b],[b,a]\}$, $\{[a,b]^T,[b,a]^T\}$

3. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix}$, rank = 3, $\{[1,0,0],[0,1,0],[0,0,1]\}$,
   $\{[1,0,0]^T,[0,1,0]^T,[0,0,1]^T\}$,

4. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, rank = 3, $\{[1,0,0],[0,1,0],[0,0,1]\}$,
   $\{[1,0,0]^T,[0,1,0]^T,[0,0,1]^T\}$,

5. $\begin{bmatrix} 1 & 0 & -21 \\ 0 & 11 & -3 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -21 \\ 0 & 11 & -3 \\ 0 & 0 & 1 \end{bmatrix}$, rank = 3, $\{[1,0,0],[0,1,0],[0,0,1]\}$,
   $\{[1,0,0]^T,[0,1,0]^T,[0,0,1]^T\}$,

6. $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, rank = 2, $\{[1,1,4],[0,1,0]\}$,

$A^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
$\{[1,-1,4]^T,[0,1,0]^T,\}$,

7. $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, rank = 2, $\{[2,0,1,0],[0,1,0,2]\}$,

$$A^T = \begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \{[2,0,1]^T, [0,1,0]^T\},$$

8. $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 12 & 30 & 63 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 12 & 30 & 63 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 30 & 75 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rank = 4,

$\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$,

$\{[1,0,0,0]^T,[0,1,0,0]^T,[0,0,1,0]^T,[0,0,0,1]^T\}$

9. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 9 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 9 & 8 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, rank = 3, $\{[1,1,1,1],[0,9,8,9],[0,0,1,0]\}$,

$$A^T = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \{[9,0,1,0]^T,[0,0,1,0]^T,[1,1,1,1]^T\}$$

10. $\begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 5 & -2 & 1 & 0 \\ -2 & 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{bmatrix}$, rank = 3,

$\{[1,-4,-11,2],[0,1,2,0],[0,0,2,-1]\}$,

$$A^T = A, \{[1,-4,-11,2]^T,[0,1,2,0]^T,[0,0,2,-1]^T\}$$

11. New row 1 = row 2 - row 1 = [1, 1, ... , 1]

Add new row 1 to row k will get row k+1. so rank = 2, base is row 1 and row 2.

b) Same

c) All rows similar to row 1, just matter of factor 2^k. So rank = 1

12. $Rank(AB) = Rank[(AB)^T] = Rank(B^T A^T)$

13. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

14. Let A is a $m \times n$ matrix, and assume $m > n$

    $Rank(A) \leq n < m$. so A is linearly dependent on the row vectors

    verse vise, L.D on the column vectors
15. n = Rank of row = rank of column
16. Matrix A, B, AB.

    Let A as the base of the vector space V(A), then V(AB) is the subset of V(A).

    $Rank(A) = dim[V(A)] \geq dim[V(AB)] = Rank(AB)$

    If B is nonsingular, then Rank(A)=Rank(AB)

    Vise verse on B.
17. $\begin{bmatrix} 1 & 16 & -12 & -22 \\ 3 & 4 & 0 & 2 \\ 2 & -1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 16 & -12 & -22 \\ 0 & 11 & -9 & -17 \\ 0 & 33 & -37 & -51 \end{bmatrix} = \begin{bmatrix} 1 & 16 & -12 & -22 \\ 0 & 11 & -9 & -17 \\ 0 & 0 & 10 & 0 \end{bmatrix}$

    Linear independent.
18. $\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 30 & 20 & 15 & 12 \\ 20 & 15 & 12 & 10 \\ 105 & 84 & 70 & 60 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1 & 1 & 0.9 \\ 0 & 15 & 16 & 15 \\ 0 & 126 & 140 & 135 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1 & 1 & 0.9 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 14 & 21.6 \end{bmatrix}$

    Rank = 4, Linear independent.
19. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

    Rank = 3, Linear independent.
20. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

    Rank = 2, Linear Dependent.
21. $\begin{bmatrix} 2 & 0 & 0 & 7 \\ 2 & 0 & 0 & 8 \\ 2 & 0 & 0 & 9 \\ 2 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

    Rank = 3, Linear Dependent.
22. V1 * 30/4 = V3, rank=1. Linear Dependent.
23. $\begin{bmatrix} 9 & 8 & 7 & 6 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

    Rank = 2, Linear independent.
24. 4 rows 3 column, Linear Dependent.

25. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 6 & 0 & -1 & 3 \\ 2 & 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 6 & 7 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

    Rank = 3, Linear independent.

26. $V_4 = 2V_1$, discard V_4

    $\begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -2 & -4 \end{bmatrix} discardV_3

27. Yes, dimension=2, {[-2, 0, 1], [0, 2, 1]}
28. k=0, Yes, dimension=2, {[1, 0, 0], [0, 1, -3]}

    if k!=0, No. 2*V is not in the set.
29. No.
30. n = 2, dimension = 1, {0}.

    $n > 2.$ dimension = 2, {[0, 0 ... 1, 0], [0, 0 ... 0, 1]}
31. No. -1 * V not include in the set.
32. Yes, dimension=1, {[-5/4, 1, -23/4]}
33. Yes, dimension=1, {[1, 10/3, 3]}
34. No. 2 * V not in the set.
35. Yes, dimension=1, {[1, 1/2, 1/3, 1/4]}