

Chapter 9 Vector Differential Calculus. Grad, Div, Curl

P360 - Problem set 9.1

Problem 1-5

PS 9.1

1-5 COMPONENTS AND LENGTH

Find the components of the vector \mathbf{v} with initial point P and terminal point Q . Find $|\mathbf{v}|$. Sketch $|\mathbf{v}|$. Find the unit vector \mathbf{u} in the direction of \mathbf{v} .

1. $P: (1, 1, 0)$, $Q: (6, 2, 0)$
2. $P: (1, 1, 1)$, $Q: (2, 2, 0)$
3. $P: (-3.0, 4.0, -0.5)$, $Q: (5.5, 0, 1.2)$
4. $P: (1, 4, 2)$, $Q: (-1, -4, -2)$
5. $P: (0, 0, 0)$, $Q: (2, 1, -2)$

$$1. \vec{V} = [5, 1, 0]$$

$$|\mathbf{v}| = \sqrt{26}$$

$$\vec{U} = \left[\frac{5\sqrt{26}}{26}, \frac{\sqrt{26}}{26}, 0 \right]$$

$$2. \vec{V} = [1, 1, -1]$$

$$|\mathbf{v}| = \sqrt{3}$$

$$\vec{U} = \left[\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right]$$

$$3. \vec{V} = [8.5, -4, 1.7]$$

$$|\mathbf{v}| = \sqrt{72.25 + 16 + 2.89} \\ = \sqrt{91.14}$$

$$\vec{U} = \left[\frac{8.5}{\sqrt{91.14}}, \frac{-4}{\sqrt{91.14}}, \frac{1.7}{\sqrt{91.14}} \right]$$

$$4) \vec{V} = [-2, -8, -4]$$

$$|\mathbf{v}| = \sqrt{4 + 64 + 16} \\ = \sqrt{84}$$

$$\vec{U} = \left[-\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}} \right]$$

$$5) \vec{V} = [2, 1, -2]$$

$$|\mathbf{v}| = \sqrt{4 + 1 + 4} = 3$$

$$\vec{U} = \left[\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right]$$

6–10 Find the terminal point Q of the vector \mathbf{v} with components as given and initial point P . Find $|\mathbf{v}|$.

6. $4, 0, 0$; $P: (0, 2, 13)$
7. $\frac{1}{2}, 3, -\frac{1}{4}$; $P: (\frac{7}{2}, -3, \frac{3}{4})$
8. $13.1, 0.8, -2.0$; $P: (0, 0, 0)$
9. $6, 1, -4$; $P: (-6, -1, -4)$
10. $0, -3, 3$; $P: (0, 3, -3)$

Latex

Mistake

11–18 ADDITION, SCALAR MULTIPLICATION

Let $\mathbf{a} = [3, 2, 0] = 3\mathbf{i} + 2\mathbf{j}$; $\mathbf{b} = [-4, 6, 0] = 4\mathbf{i} + 6\mathbf{j}$,
 $\mathbf{c} = [5, -1, 8] = 5\mathbf{i} - \mathbf{j} + 8\mathbf{k}$, $\mathbf{d} = [0, 0, 4] = 4\mathbf{k}$.

Find:

6. $Q=[4, 2, 13]$, $|\mathbf{v}|=4$
7. $Q=[4, 0, 1/2]$, $|\mathbf{v}|=\sqrt{149}/4$

8. $Q=[13.1, 0.8, -2.0]$, $|v|=\sqrt{171.61 + 0.64 + 4}=\sqrt{176.25}$
9. $Q=[0, 0, 0]$, $|v|=\sqrt{53}$
10. $Q=[0, 0, 0]$, $|v|=3\sqrt{2}$
11. $2a=[6, 4, 0]$, $1/2a=[3/2, 1, 0]$, $-a = [-3, -2, 0]$
12. $(a+b)+c=a+(b+c)=[4, 7, 8]$ ***b is not consistant.***
13. $b+c=c+b = [1, 5, 8]$
14. $3c-6d=3(c-2d)=[15, -3, 0]$
15. $7(c-b)=7c-7b=7*[9, -7, 8]=[63, -49, 56]$
16. $\frac{9}{2}a - 3c=9(\frac{1}{2}a - \frac{1}{3}c)=[-3/2, 12, -24]$
17. $(7-3)a=7a-3a=4a=[12, 8, 0]$
18. $4a+3b = [0, 26, 0]$, $-4a-3b=-(4a+3b)=[0, -26, 0]$
19. 12-associative, 13-commutative, 14-16 scalar multiplication is distributive.
20. $a + b = [a_1, a_2, \dots, a_n] + [b_1, b_2, \dots, b_n] = [a_1 + b_1, a_2 + b_2, \dots, a_n + b_n] = [b_1 + a_1, b_2 + a_2, \dots, b_n + a_n] = b + a$

The rest can be approved in a simliar way.

21-25 FORCES, RESULTANT

Find the resultant in terms of components and its magnitude.

21. $\mathbf{p} = [2, 3, 0]$, $\mathbf{q} = [0, 6, 1]$, $\mathbf{u} = [2, 0, -4]$
22. $\mathbf{p} = [1, -2, 3]$, $\mathbf{q} = [3, 21, -16]$,
 $\mathbf{u} = [-4, -19, 13]$
23. $\mathbf{u} = [8, -1, 0]$, $\mathbf{v} = [\frac{1}{2}, 0, \frac{4}{3}]$, $\mathbf{w} = [-\frac{17}{2}, 1, \frac{11}{3}]$
24. $\mathbf{p} = [-1, 2, -3]$, $\mathbf{q} = [1, 1, 1]$, $\mathbf{u} = [1, -2, 2]$
25. $\mathbf{u} = [3, 1, -6]$, $\mathbf{v} = [0, 2, 5]$, $\mathbf{w} = [3, -1, -13]$

$$21) \vec{r} = \vec{p} + \vec{q} + \vec{u} \\ = [4, 9, -3]$$

$$|\vec{r}| = \sqrt{16+81+9} = \sqrt{106}$$

$$22) \vec{r} = \vec{p} + \vec{q} + \vec{u} \\ = [0, 0, 0]$$

$$|\vec{r}| = 0$$

$$23) \vec{r} = \vec{u} + \vec{v} + \vec{w} \\ = [0, 0, 5]$$

$$|\vec{r}| = 5$$

$$24) \vec{r} = \vec{p} + \vec{q} + \vec{u} \\ = [1, 1, 0]$$

$$|\vec{r}| = \sqrt{2}$$

$$25) \vec{r} = \vec{u} + \vec{v} + \vec{w}$$

$$= [6, 2, -14]$$

$$|\vec{r}| = \sqrt{36+4+196} = \sqrt{236}$$

26-37 FORCES, VELOCITIES

26. **Equilibrium.** Find \mathbf{v} such that $\mathbf{p}, \mathbf{q}, \mathbf{u}$ in Prob. 21 and \mathbf{v} are in equilibrium.
27. Find \mathbf{p} such that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in Prob. 23 and \mathbf{p} are in equilibrium.
28. **Unit vector.** Find the unit vector in the direction of the resultant in Prob. 24.

$$26) \vec{v} = 0 - \vec{r} = [-4, -9, 3]$$

$$27) \vec{p} = 0 - \vec{r} = [0, 0, -5]$$

$$28) \vec{u} = \frac{\vec{r}}{|\vec{r}|} = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right]$$

29. **Restricted resultant.** Find all \mathbf{v} such that the resultant of $\mathbf{v}, \mathbf{p}, \mathbf{q}, \mathbf{u}$ with $\mathbf{p}, \mathbf{q}, \mathbf{u}$ as in Prob. 21 is parallel to the xy -plane.
30. Find \mathbf{v} such that the resultant of $\mathbf{p}, \mathbf{q}, \mathbf{u}, \mathbf{v}$ with $\mathbf{p}, \mathbf{q}, \mathbf{u}$ as in Prob. 24 has no components in x - and y -directions.
31. For what k is the resultant of $[2, 0, -7], [1, 2, -3]$, and $[0, 3, k]$ parallel to the xy -plane?
32. If $|\mathbf{p}| = 6$ and $|\mathbf{q}| = 4$, what can you say about the magnitude and direction of the resultant? Can you think of an application to robotics?
33. Same question as in Prob. 32 if $|\mathbf{p}| = 9$, $|\mathbf{q}| = 6$, $|\mathbf{u}| = 3$.

29 Vector start from origin.
parallel to x - y means no component in z -direction.

Let result

$$\vec{z} = \vec{v} + \vec{p} + \vec{q} + \vec{u} = [x, y, 0]$$

So x and y are arbitrary

$$\vec{v} = \vec{z} - [4, 9, -3]$$

$$= [x-4, y-9, 3]$$

$$\text{Let } V_1 = x-4, V_2 = y-9.$$

$$\text{So } \vec{v} = [V_1, V_2, 3]$$

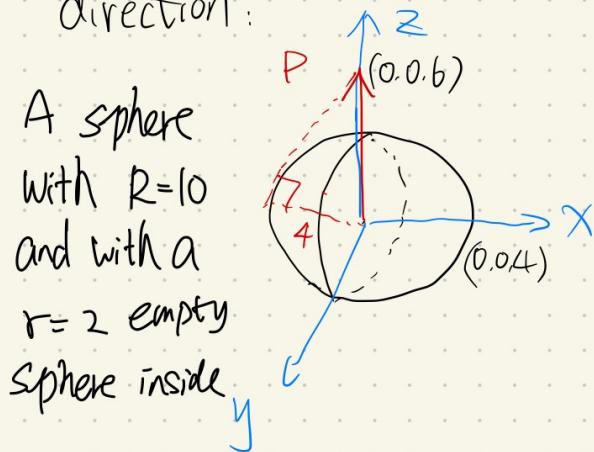
where V_1 and V_2 are arbitrary

$$30. \vec{v} = [0, 0, c] - [1, 1, 0]$$

$$= [-1, -1, c] \text{ where } c \text{ is a constant.}$$

$$31 \quad -7 - 3 + k = 0 \\ k = 10$$

32. magnitude, min=2, max=10
direction:



Robotic Vision, the area
where Robot can see/detect
or scan?

33. magnitude, min=0, max=18
direction

enhanced ice cream
without blind spot?

A solid sphere with
 $R=18$, no blind point.

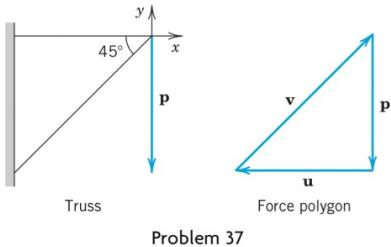
$$|\mathbf{p} + \mathbf{q} + \mathbf{u}| \leq 18$$

34. **Relative velocity.** If airplanes A and B are moving southwest with speed $|v_A| = 550$ mph, and northwest with speed $|v_B| = 450$ mph, respectively, what is the relative velocity $\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$ of B with respect to A ?

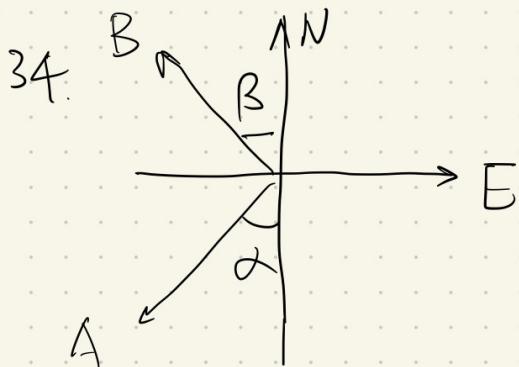
35. Same question as in Prob. 34 for two ships moving northeast with speed $|\mathbf{v}_A| = 22$ knots and west with speed $|\mathbf{v}_B| = 19$ knots.

36. **Reflection.** If a ray of light is reflected once in each of two mutually perpendicular mirrors, what can you say about the reflected ray?

37. **Force polygon.** Truss. Find the forces in the system of two rods (truss) in the figure, where $|\mathbf{p}| = 1000$ nt. Hint. Forces in equilibrium form a polygon, the *force polygon*.

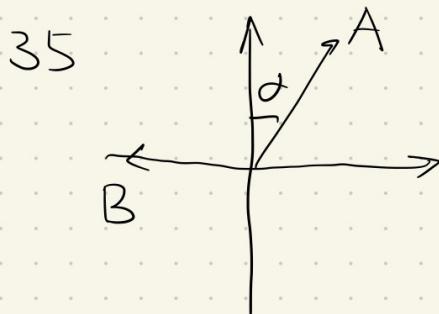


Problem 37



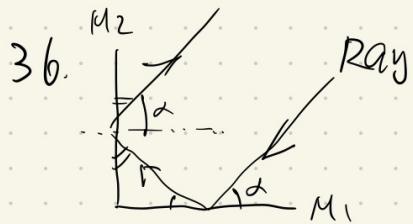
$$V = V_B - V_A \text{ on northeast.}$$

$$\begin{aligned} & [550 \sin \alpha - 450 \sin \beta, \\ & 550 \cos \alpha - 450 \cos \beta] \end{aligned}$$



$$V = V_B - V_A \text{ on northeast}$$

$$[22 \sin \alpha - 19, 22 \cos \alpha]$$



Reverse the direction.

$$37. P: [0, -1000]$$

$$U: [-1000, 0]$$

$$V: [1000, 1000]$$

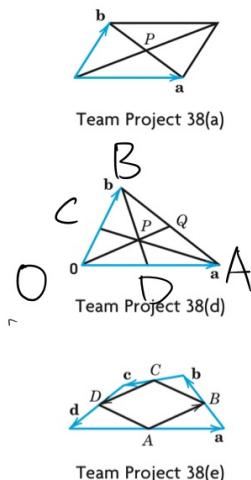
34, 35: it seems assume

α, β are 45°

maybe this is by default

38. TEAM PROJECT. Geometric Applications. To increase your skill in dealing with vectors, use vectors to prove the following (see the figures).

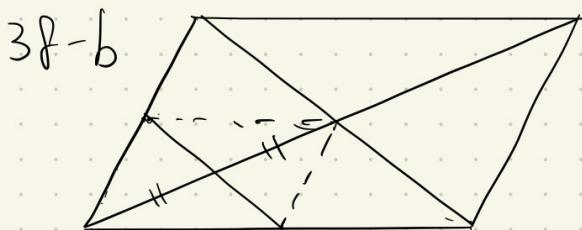
- The diagonals of a parallelogram bisect each other.
- The line through the midpoints of adjacent sides of a parallelogram bisects one of the diagonals in the ratio 1 : 3.
- Obtain (b) from (a).
- The three medians of a triangle (the segments from a vertex to the midpoint of the opposite side) meet at a single point, which divides the medians in the ratio 2 : 1.
- The quadrilateral whose vertices are the midpoints of the sides of an arbitrary quadrilateral is a parallelogram.
- The four space diagonals of a parallelepiped meet and bisect each other.
- The sum of the vectors drawn from the center of a regular polygon to its vertices is the zero vector.



$$\begin{aligned}\overrightarrow{OC} &= \frac{1}{2}\vec{b} \\ \overrightarrow{OB} &= \frac{1}{2}\vec{a} \\ \overrightarrow{DC} &= \frac{1}{2}\overrightarrow{AB} \quad \text{Vector Way?} \\ \triangle CDP &\sim \triangle ABP \\ \overrightarrow{DP} &= \frac{1}{2}\overrightarrow{PB}\end{aligned}$$

$$38-a \quad \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} = \frac{1}{2}(\vec{a} + \vec{b})$$

$$\frac{1}{2}\vec{a} - \frac{1}{2}\vec{b} = \frac{1}{2}(\vec{a} - \vec{b})$$



$$\frac{1}{4}\vec{a} - \frac{1}{4}\vec{b} = \frac{1}{4}(\vec{a} - \vec{b})$$

$$\frac{1}{4} : \frac{3}{4} = 1 : 3$$

38-c: Obvious.

$$38-d \quad \overrightarrow{AB} = \vec{a} - \vec{b}$$

$$\overrightarrow{AQ} = \frac{1}{2}(\vec{a} - \vec{b})$$

$$\overrightarrow{QO} = \vec{a} + \frac{1}{2}(\vec{a} - \vec{b})$$

$$\begin{aligned}38-e \quad \vec{a} - \vec{d} &= \vec{c} - \vec{b} \\ \frac{1}{2}(\vec{a} - \vec{d}) &= \frac{1}{2}(\vec{c} - \vec{b})\end{aligned}$$

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{BC} \\ \text{same} \Rightarrow \overrightarrow{AB} &= \overrightarrow{DC}\end{aligned}$$

38-f

38-g

PS 9-2

1-10 INNER PRODUCT

Let $\mathbf{a} = [1, -3, 5]$, $\mathbf{b} = [4, 0, 8]$, $\mathbf{c} = [-2, 9, 1]$.
Find:

1. $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{b} \cdot \mathbf{a}$, $\mathbf{b} \cdot \mathbf{c}$
2. $(-3\mathbf{a} + 5\mathbf{c}) \cdot \mathbf{b}$, $15(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b}$
3. $|\mathbf{a}|$, $|\mathbf{2b}|$, $|-c|$
4. $|\mathbf{a} + \mathbf{b}|$, $|\mathbf{a}| + |\mathbf{b}|$
5. $|\mathbf{b} + \mathbf{c}|$, $|\mathbf{b}| + |\mathbf{c}|$
6. $|\mathbf{a} + \mathbf{c}|^2 + |\mathbf{a} - \mathbf{c}|^2 - 2(|\mathbf{a}|^2 + |\mathbf{c}|^2)$
7. $|\mathbf{a} \cdot \mathbf{c}|$, $|\mathbf{a}||\mathbf{c}|$
8. $5\mathbf{a} \cdot 13\mathbf{b}$, $65\mathbf{a} \cdot \mathbf{b}$
9. $15\mathbf{a} \cdot \mathbf{b} + 15\mathbf{a} \cdot \mathbf{c}$, $15\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
10. $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c})$, $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c}$

$$5. |\mathbf{b} + \mathbf{c}| = \sqrt{2^2 + 9^2 + 9^2} = \sqrt{166}$$

$$|\mathbf{b}| + |\mathbf{c}| = 4\sqrt{5} + \sqrt{86}$$

$$6. (1+36+36) + (9+144+16) \\ - 2(35+86) = 0$$

$$7. |\mathbf{a} \cdot \mathbf{c}| = |-2 - 27 + 5| = 24 \\ |\mathbf{a}| |\mathbf{c}| = \sqrt{35} \cdot \sqrt{86}$$

$$8. 5\mathbf{a} \cdot 13\mathbf{b} = 65\mathbf{a} \cdot \mathbf{b} = 2860$$

$$9. 15\mathbf{a} \cdot \mathbf{b} + 15\mathbf{a} \cdot \mathbf{c} = 15\mathbf{a}(\mathbf{b} + \mathbf{c}) \\ = 15 \cdot \mathbf{a} \cdot (2, 9, 9) = 300$$

$$10. \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (16, -9, 7) \\ = 68$$

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = (-3, -3, -3) \cdot \mathbf{c} \\ = 6 - 27 - 3 = -24$$

$$1. \mathbf{a} \cdot \mathbf{b} = 4 + 0 + 40 = 44$$

$$\mathbf{b} \cdot \mathbf{a} = 44$$

$$\mathbf{b} \cdot \mathbf{c} = -8 + 0 + 8 = 0$$

$$2. (-3\mathbf{a} + 5\mathbf{c}) \cdot \mathbf{b} = (-3, 5, 4) \cdot \mathbf{b} \\ = -52 + 0 - 80 = -132$$

$$15(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} = 15(-3, -2, 4) \cdot \mathbf{b}$$

$$= 15 \cdot (-12 + 0 + 32) = 300$$

$$3. |\mathbf{a}| = \sqrt{1+9+25} = \sqrt{35}$$

$$|\mathbf{2b}| = 2 \cdot \sqrt{16+64} = 8\sqrt{5}$$

$$|-c| = \sqrt{4+81+1} = \sqrt{86}$$

$$4. |\mathbf{a} + \mathbf{b}| = \sqrt{25+9+164} = \sqrt{203}$$

$$|\mathbf{a}| + |\mathbf{b}| = \sqrt{35} + \sqrt{16+64} = \sqrt{35} + 4\sqrt{5}$$

11-16 GENERAL PROBLEMS

11. What laws do Probs. 1 and 4-7 illustrate?
12. What does $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ imply if $\mathbf{u} = \mathbf{0}$? If $\mathbf{u} \neq \mathbf{0}$?
13. Prove the Cauchy-Schwarz inequality.
14. Verify the Cauchy-Schwarz and triangle inequalities for the above \mathbf{a} and \mathbf{b} .
15. Prove the parallelogram equality. Explain its name.
16. **Triangle inequality.** Prove Eq. (7). Hint. Use Eq. (3) for $|\mathbf{a} + \mathbf{b}|$ and Eq. (6) to prove the square of Eq. (7), then take roots.

11-1 $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
 $\mathbf{a} \cdot \mathbf{b} = 0$, orthogonality

4-5) $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ (Triangle inequality).

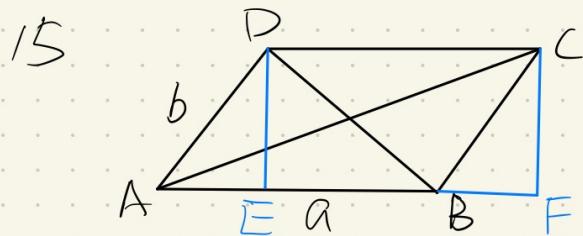
6) $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$ (Parallelogram equality).

7) $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|$ (Cauchy-Schwarz inequality).

12. if $\mathbf{u} = \mathbf{0}$, We know nothing about $\mathbf{u} \cdot \mathbf{w}$.
if $\mathbf{u} \neq \mathbf{0}$, $P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$. So \mathbf{v}, \mathbf{w} belongs
to a set B that the projections of
 \mathbf{u} on the vector multiple the magnitude of
the vector is a constant.

13. projection \leq itself. ($\cos \theta \leq 1$)

14. 4-5, and 7



$$1) AE = BF, DE = CF,$$

$$2) AC^2 + BD^2 = (AE + EF)^2 + CF^2 \\ + (a - AE)^2 + DE^2 \\ = 2(a^2 + b^2)$$

$$16. |a+b|^2 = (a+b)(a+b) \\ = a^2 + b^2 + 2ab$$

$$\leq |a|^2 + |b|^2 + 2|a||b| \\ = (|a| + |b|)^2$$

17-20 WORK

Mistake

Find the work done by a force \mathbf{p} acting on a body if the body is displaced along the straight segment \overline{AB} from A to B. Sketch \overline{AB} and \mathbf{p} . Show the details.

17. $\mathbf{p} = [2, 5, 0]$, A: (1, 3, 3), B: (3, 5, 5)

18. $\mathbf{p} = [-1, -2, 4]$, A: (0, 0, 0), B: (6, 7, 5)

19. $\mathbf{p} = [0, 4, 3]$, A: (4, 5, -1), B: (1, 3, 0)

20. $\mathbf{p} = [6, -3, -3]$, A: (1, 5, 2), B: (3, 4, 1)

21. **Resultant.** Is the work done by the resultant of two forces in a displacement the sum of the work done by each of the forces separately? Give proof or counterexample.

17. $W = \mathbf{P} \cdot \mathbf{d} = \mathbf{P} \cdot (2, 2, 2) = 14$

18. $W = \mathbf{P} \cdot (6, 7, 5) = 0$

19. $W = \mathbf{P} \cdot (-3, -2, 1) = -5$

Negative, because of direction.

20. $W = \mathbf{P} \cdot (2, -1, -1) = 18$

21. Yes.

$$W = \mathbf{P}_1 \cdot \mathbf{d} + \mathbf{P}_2 \cdot \mathbf{d}$$

$$= (\mathbf{P}_1 + \mathbf{P}_2) \cdot \mathbf{d}$$

22. $\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{3+2}{\sqrt{2} \sqrt{14}}$

$$\alpha = \arccos\left(\frac{\sqrt{5}}{14}\right)$$

23. $\cos \alpha = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}| |\mathbf{c}|} = \frac{3+2}{\sqrt{14} \sqrt{5}}$

$$\alpha = \arccos\left(\frac{\sqrt{70}}{14}\right)$$

24. $W \leq \frac{(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{c})}{|\mathbf{a} + \mathbf{c}| \cdot |\mathbf{b} + \mathbf{c}|}$

$$= \frac{(4, 3, 1) \cdot (4, 2, 3)}{\sqrt{16+9+1} \cdot \sqrt{16+4+9}}$$

$$= \frac{16+6+3}{\sqrt{26} \cdot \sqrt{29}} = \frac{25}{\sqrt{26} \sqrt{29}}$$

$$\alpha = \arccos\left(\frac{25}{\sqrt{26} \sqrt{29}}\right)$$

22-30 ANGLE BETWEEN VECTORS

Let $\mathbf{a} = [1, 1, 0]$, $\mathbf{b} = [3, 2, 1]$, and $\mathbf{c} = [1, 0, 2]$. Find the angle between:

22. \mathbf{a}, \mathbf{b}

23. \mathbf{b}, \mathbf{c}

24. $\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c}$

25. What will happen to the angle in Prob. 24 if we replace c by nc with larger and larger n ?

26. **Cosine law.** Deduce the law of cosines by using vectors \mathbf{a} , \mathbf{b} , and $\mathbf{a} - \mathbf{b}$.

27. **Addition law.** $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. Obtain this by using $\mathbf{a} = [\cos \alpha, \sin \alpha]$, $\mathbf{b} = [\cos \beta, \sin \beta]$ where $0 \leq \alpha \leq \beta \leq 2\pi$.

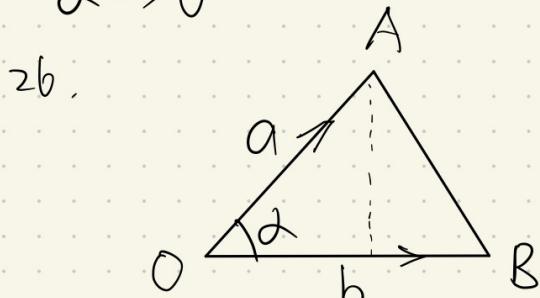
28. **Triangle.** Find the angles of the triangle with vertices $A: (0, 0, 2)$, $B: (3, 0, 2)$, and $C: (1, 1, 1)$. Sketch the triangle.

29. **Parallelogram.** Find the angles if the vertices are $(0, 0), (6, 0), (8, 3)$, and $(2, 3)$.

30. **Distance.** Find the distance of the point $A: (1, 0, 2)$ from the plane $P: 3x + y + z = 9$. Make a sketch.

$$25. \lim_{n \rightarrow \infty} \frac{(a+nc)(b+nc)}{|(a+nc)| \cdot |(b+nc)|} = \frac{c \cdot c}{|c|^2} = 1$$

$$\angle \rightarrow 0$$



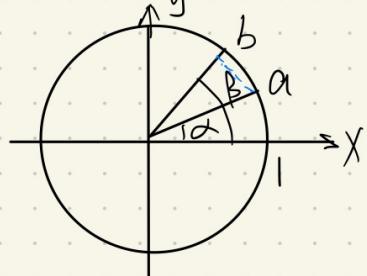
$$(\overrightarrow{BA})^2 = (a-b)^2 =$$

$$(a \sin \alpha)^2 + (b - a \cos \alpha)^2$$

$$= a^2 \sin^2 \alpha + b^2 + a^2 \cos^2 \alpha - 2ab \cos \alpha$$

$$= a^2 + b^2 - 2ab \cos \alpha$$

$$27. a \cdot b = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$



the projection of a on b is

$$P \approx a \cdot b = \cos(\beta - \alpha) = \cos(\alpha - \beta)$$

$$28. \overrightarrow{AB} = (3, 0, 0) = a$$

$$\overrightarrow{BC} = (-2, 1, -1) = b$$

$$\overrightarrow{CA} = (-1, -1, 1) = c$$

$$\cos \alpha = \frac{-ab}{|a||b|} = \frac{-6}{3\sqrt{6}} = \frac{\sqrt{6}}{3}$$

$$\cos \beta = \frac{-bc}{|b||c|} = \frac{2-1-1}{\sqrt{5}\sqrt{5}} = 0$$

$$\cos \gamma = \frac{-ca}{|c||a|} = \frac{+3}{3\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$29. \tan \alpha = \frac{3}{2} \quad \alpha = \arctan \frac{3}{2}$$

$$\begin{array}{c} \text{Diagram of a parallelogram with sides 2 and 3, and diagonal 6.} \\ \cos \alpha = \frac{(2 \cdot 3)}{\sqrt{4+9}} = \frac{2}{\sqrt{13}} \end{array}$$

$$30. \text{ moving to } (x_1, y_1, z_1)$$

$$A = (0, 0, 0)$$

$$P = 3x - 3 + y + z - 2 = 0$$

$$3x + y + z = 14$$

per example 6,

$$P = \frac{C}{|AT|} = \frac{14}{\sqrt{11}}$$

31-35 ORTHOGONALITY is particularly important, mainly because of orthogonal coordinates, such as **Cartesian coordinates**, whose **natural basis** [Eq. (9), Sec. 9.1], consists of three orthogonal unit vectors.

31. For what values of a_1 are $[a_1, 4, 3]$ and $[3, -2, 12]$ orthogonal?
32. **Planes.** For what c are $3x + z = 5$ and $8x - y + cz = 9$ orthogonal?
33. **Unit vectors.** Find all unit vectors $\mathbf{a} = [a_1, a_2]$ in the plane orthogonal to $[4, 3]$.
34. **Corner reflector.** Find the angle between a light ray and its reflection in three orthogonal plane mirrors, known as *corner reflector*.
35. **Parallelogram.** When will the diagonals be orthogonal? Give a proof.

$$31. 3a_1 - 8 + 36 = 0$$

$$a_1 = -\frac{28}{3}$$

$$32. (3, 0, 1) \cdot (8, -1, c) = 0$$

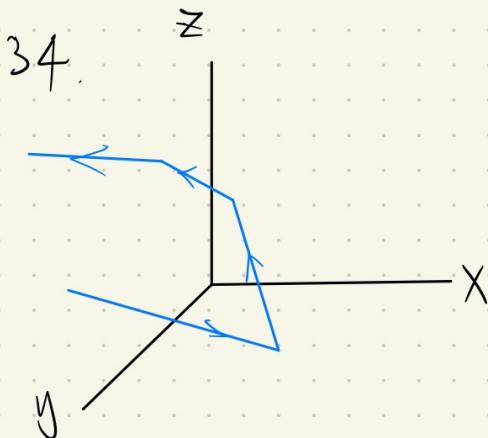
$$24 + c = 0 \quad c = -24$$

$$33. 4a_1 + 3a_2 = 0$$

$$a_2 = -\frac{4}{3}a_1$$

$$a_1^2 + \frac{16}{9}a_1^2 = 1$$

$$a_1 = \frac{3}{5}, \quad \left[\frac{3}{5}, -\frac{4}{5} \right]$$



$[a, b, c]$ Reflect on XY plane,

$\Rightarrow [a, b, -c]$ Reflect on XZ plane

$\Rightarrow [a, -b, -c]$ R on ZY plane

$[-a, -b, -c]$

$$35. (a+b)(a-b) = 0$$

$$a^2 - b^2 = 0 \quad a^2 = b^2$$

diamond shape. answer is wrong.

36-40 COMPONENT IN THE DIRECTION OF A VECTOR

Find the component of \mathbf{a} in the direction of \mathbf{b} . Make a sketch.

36. $\mathbf{a} = [1, 1, 1]$, $\mathbf{b} = [2, 1, 3]$

37. $\mathbf{a} = [3, 4, 0]$, $\mathbf{b} = [4, -3, 2]$

38. $\mathbf{a} = [8, 2, 0]$, $\mathbf{b} = [-4, -1, 0]$

39. When will the component (the projection) of \mathbf{a} in the direction of \mathbf{b} be equal to the component (the projection) of \mathbf{b} in the direction of \mathbf{a} ? First guess.

40. What happens to the component of \mathbf{a} in the direction of \mathbf{b} if you change the length of \mathbf{b} ?

$$36. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{2+1+3}{\sqrt{14}} = \frac{3\sqrt{14}}{7}$$

$$37. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{12-12+0}{\sqrt{5}} = 0 \quad \text{perpendicular}$$

$$38. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{-32-2}{\sqrt{17}} = -2\sqrt{17}$$

On the opp direction,

$$39. |\mathbf{a}| = |\mathbf{b}| \quad \underline{\text{or } \mathbf{a} \cdot \mathbf{b} = 0}$$

$$40. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a} \cdot k\mathbf{b}}{|k||\mathbf{b}|}$$

nothing change.

PS 9.3

1-10 GENERAL PROBLEMS

- Give the details of the proofs of Eqs. (4) and (5).
- What does $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ with $\mathbf{a} \neq \mathbf{0}$ imply?
- Give the details of the proofs of Eqs. (6) and (11).

(5)

$$(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$

$$(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c})$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} = [a_2(b_3 + c_2) - a_3(b_2 + c_2)] i - [a_1(b_3 + c_2) - a_3(b_1 + c_1)] j + [a_1(b_2 + c_3) - a_2(b_1 + c_1)] k$$

$$\mathbf{a} \times \mathbf{b}$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$$

$$(\mathbf{a} \times \mathbf{b})$$

$$\begin{vmatrix} i & j & k \\ (a_1, a_2, a_3) \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= ((a_2 b_3 - a_3 b_2) i$$

$$- (a_1 b_3 - a_3 b_1) j$$

$$+ (a_1 b_2 - a_2 b_1) k$$

$$= l [(a_2 b_3 - a_3 b_2) i$$

$$- (a_1 b_3 - a_3 b_1) j$$

$$+ (a_1 b_2 - a_2 b_1) k$$

Extend the rest, same.

$$\mathbf{a} \times \mathbf{c}$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{Reference above.}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

(β) , Same

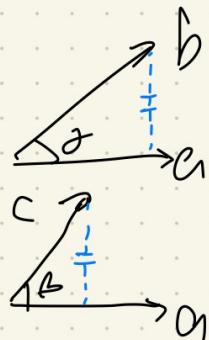
2. What does $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ with $\mathbf{a} \neq \mathbf{0}$ imply?
 3. Give the details of the proofs of Eqs. (6) and (11).

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \dots$$

$$\textcircled{2} |V| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|(|\mathbf{b}| \sin \alpha)$$

$$= |\mathbf{a} \times \mathbf{c}| = |\mathbf{a}|(|\mathbf{c}| \sin \beta)$$

$$|\mathbf{b}| \sin \alpha = |\mathbf{c}| \sin \beta.$$



and the direction of the result is same.

$$\textcircled{1} \quad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ x & y & z \end{vmatrix} =$$

$$(0-y)i - (z-x)j + yk$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \dots$$

$$\textcircled{2} \quad \mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = zi - xk$$

(6)

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$$

$$\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

exchange row, determinant $\times -1$

P295 . Sec 7.7 Theorem 1.

(11)

$$(\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2 c_3 - b_3 c_2) i - (b_1 c_3 - b_3 c_1) j + (b_1 c_2 - b_2 c_1) k$$

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (a_1 b_2 c_3 - a_1 b_3 c_2) \\ &\quad - (a_2 b_1 c_3 - a_2 b_3 c_1) \\ &\quad + (a_3 b_1 c_2 - a_3 b_2 c_1) \end{aligned}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \Rightarrow \mathbf{a} \rightarrow \mathbf{b}, \mathbf{b} \rightarrow \mathbf{c}, \mathbf{c} \rightarrow \mathbf{a}$$

$$= C_1 A_2 B_3 - C_1 A_3 B_2$$

$$- (C_2 A_1 B_3 - C_2 A_3 B_1)$$

$$+ (C_3 A_1 B_2 - C_3 A_2 B_1)$$

Geo perspective, same volume.

4. Lagrange's identity for $|\mathbf{a} \times \mathbf{b}|$. Verify it for $\mathbf{a} = [3, 4, 2]$ and $\mathbf{b} = [1, 0, 2]$. Prove it, using $\sin^2 \gamma = 1 - \cos^2 \gamma$. The identity is

$$(12) \quad |\mathbf{a} \times \mathbf{b}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2}.$$

5. What happens in Example 3 of the text if you replace \mathbf{p} by $-\mathbf{p}$?

in change the direction -

$$4. \{3, 4, 2\} \text{ cross } \{1, 0, 2\} = [8, -4, -4]$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{64 + 16 + 16} = \sqrt{96}$$

$$\begin{aligned} & \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2} \\ &= \sqrt{(9 + 16)(1 + 4) - (3 + 4)^2} \\ &= \sqrt{29 \times 5 - 49} = \sqrt{96} \end{aligned}$$

$$\text{Prove: } |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \alpha$$

$$= \sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}} \sin \alpha$$

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}}}$$

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}} \sqrt{1 - \cos^2 \alpha} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}} \cdot \sqrt{1 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})}} \\ &= \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2} \end{aligned}$$

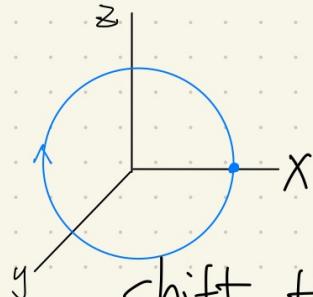
6. What happens in Example 5 if you choose a P at distance $2d$ from the axis of rotation?

ω remains the same

$|r| \sin \alpha$ become doubled.

so v get doubled.

7. Rotation. A wheel is rotating about the y -axis with angular speed $\omega = 20 \text{ sec}^{-1}$. The rotation appears clockwise if one looks from the origin in the positive y -direction. Find the velocity and speed at the point $[8, 6, 0]$. Make a sketch.



shift the center of wheel to $[0, 0, 0]$.

so $[8, 6, 0]$ equals to

$$[8, 0, 0]$$

$$s = 8 \times 20 = 160 \quad v = [0, 0, -160]$$

$$v = [0, 20, 0] \times [8, 0, 0]$$

$$= [0, 0, -160]$$

Not quite understand

- 8. Rotation.** What are the velocity and speed in Prob. 7 at the point $(4, 2, -2)$ if the wheel rotates about the line $y = x, z = 0$ with $\omega = 10 \text{ sec}^{-1}$?

$$⑧ y^2 + x^2 = 10^2$$

$$x^2 + x^2 = 100$$

$$x = \sqrt{50}$$

$$[\sqrt{50}, \sqrt{50}, 0] \times [4, 2, -2]$$

$$\begin{aligned} &= \begin{vmatrix} i & j & k \\ \sqrt{50} & \sqrt{50} & 0 \\ 4 & 2 & -2 \end{vmatrix} \\ &= -2\sqrt{50}i + 2\sqrt{50}j - 2\sqrt{50}k \end{aligned}$$

- 9. Scalar triple product.** What does $(\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = 0$ imply with respect to these vectors?

$$⑨ V=0$$

- 1. a, b, c on the same plane
- 2. b, c on the same line
- 3. one of $a, b, c = 0$.

$$⑩:$$

11-23

**VECTOR AND SCALAR
TRIPLE PRODUCTS**

With respect to right-handed Cartesian coordinates, let $\mathbf{a} = [2, 1, 0]$, $\mathbf{b} = [-3, 2, 0]$, $\mathbf{c} = [1, 4, -2]$, and $\mathbf{d} = [5, -1, 3]$. Showing details, find:

11. $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{a}$, $\mathbf{a} \cdot \mathbf{b}$

12. $3\mathbf{c} \times 5\mathbf{d}$, $15\mathbf{d} \times \mathbf{c}$, $15\mathbf{d} \cdot \mathbf{c}$, $15\mathbf{c} \cdot \mathbf{d}$

$$\begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \end{pmatrix} \times$$

$$(1) \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix}$$

$$= 0i - 0j + 7k$$

$$= 7k$$

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = -7k$$

$$\mathbf{a} \cdot \mathbf{b} = -6 + 2 + 0 = -4$$

$$(2) \quad 3\mathbf{c} \times 5\mathbf{d} = 15 \cdot \mathbf{c} \times \mathbf{d}$$

$$= 15 \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix}$$

$$= 15(10i - 13j - 21k)$$

$$15\mathbf{d} \times \mathbf{c} = -15\mathbf{c} \times \mathbf{d}$$

$$= 15(-10i + 3j + 24k)$$

$$15\mathbf{d} \cdot \mathbf{c} = 15 \cdot (5 - 4 - 6) \\ = -75 = 15\mathbf{c} \cdot \mathbf{d}$$

25-35 APPLICATIONS

25. **Moment m of a force \mathbf{p} .** Find the moment vector \mathbf{m} and m of $\mathbf{p} = [2, 3, 0]$ about $Q: (2, 1, 0)$ acting on a line through $A: (0, 3, 0)$. Make a sketch.

$$m = \mathbf{r} \times \mathbf{p}$$

$$= \overrightarrow{QA} \times \overrightarrow{P}$$

$$= (-2, 2, 0) \times (2, 3, 0)$$

$$= \begin{vmatrix} i & j & k \\ -2 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= -10k$$

26. **Moment.** Solve Prob. 25 if $\mathbf{p} = [1, 0, 3]$, $Q: (2, 0, 3)$, and $A: (4, 3, 5)$.

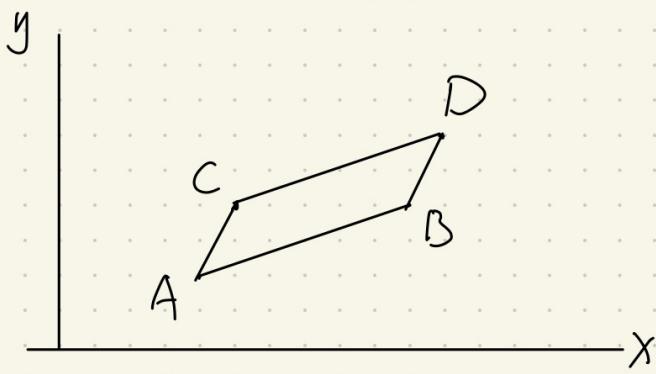
$$m = \mathbf{r} \times \mathbf{p}$$

$$= \overrightarrow{QA} \times \overrightarrow{P}$$

$$= [2, 3, 2] \times [1, 0, 3]$$

$$= \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 9i - 4j - 3k$$

27. **Parallelogram.** Find the area if the vertices are $(4, 2, 0)$, $(10, 4, 0)$, $(5, 4, 0)$, and $(11, 6, 0)$. Make a sketch.



$$\text{Area} = |\vec{AB} \times \vec{AC}|$$

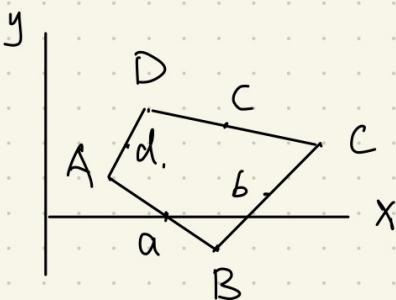
$$= \begin{vmatrix} i & j & k \\ 6 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= |10k| = 10$$

Answer is wrong.

Area is a scalar.

28. **A remarkable parallelogram.** Find the area of the quadrangle Q whose vertices are the midpoints of the sides of the quadrangle P with vertices $A: (2, 1, 0)$, $B: (5, -1, 0)$, $C: (8, 2, 0)$, and $D: (4, 3, 0)$. Verify that Q is a parallelogram.



Set a , b , c , d as the middle point, so we get

$$a: (3.5, 0, 0)$$

$$b: (6.5, 0.5, 0)$$

$$c: (6, 2.5, 0)$$

$$d: (3, 2, 0)$$

$$\vec{ab} = (3, 0.5, 0) = \vec{dc}$$

$$\vec{ad} = (-0.5, 2, 0) = \vec{bc}$$

$\Rightarrow Q$ is a parallelogram.

29. **Triangle.** Find the area if the vertices are $(0, 0, 1)$, $(2, 0, 5)$, and $(2, 3, 4)$.

$$\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} \begin{vmatrix} i & j & k \\ 2 & 0 & 4 \\ 2 & 3 & 3 \end{vmatrix}$$

$$= \frac{1}{2} |-12i + 2j + 6k|$$

$$= \frac{1}{2} \sqrt{12^2 + 2^2 + 6^2}$$

$$= \sqrt{46}$$

- 30. Plane.** Find the plane through the points $A: (1, 2, \frac{1}{4})$, $B: (4, 2, -2)$, and $C: (0, 8, 4)$.

For $ax + by + cz + d = 0$

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & \frac{1}{4} & 1 \\ 4 & 2 & -2 & 1 \\ 0 & 8 & 4 & 1 \end{vmatrix} = 0$$

$$\begin{aligned} &x(-4+8+2+16-\frac{1}{2}-8) \\ &-y(-2+16+0-0-4-1) \\ &+z(2+32+0-0-8-8) \\ &-1 \cdot (8+8+0-0+16-32) = 0 \end{aligned}$$

$$\begin{aligned} 13.5x - 9y + 18z &= 0 \\ 27x - 18y + 36z &= 0 \\ 3x - 2y + 4z &= 0 \end{aligned}$$

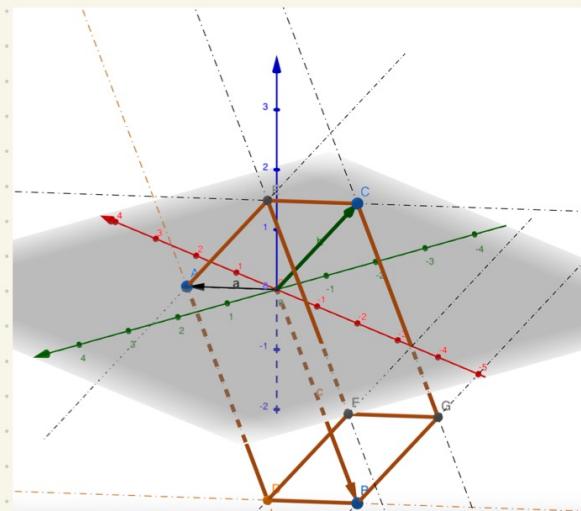
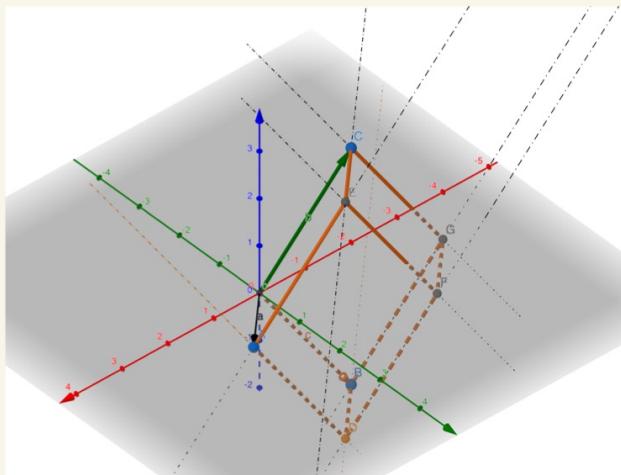
- 31. Plane.** Find the plane through $(1, 3, 4)$, $(1, -2, 6)$, and $(4, 0, 7)$.

Reference 30

$$3x - 2y - 5z + 23 = 0$$

Answer is wrong.

- 32. Parallelepiped.** Find the volume if the edge vectors are $\mathbf{i} + \mathbf{j}$, $-2\mathbf{i} + 2\mathbf{k}$, and $-2\mathbf{i} - 3\mathbf{k}$. Make a sketch.



$$(a, b, c) = a \cdot (b \times c)$$

$$= (\mathbf{i} + \mathbf{j}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ -2 & 0 & -3 \end{vmatrix}$$

$$= (\mathbf{i} + \mathbf{j}) \cdot [-\mathbf{j}(6+4)] = -10$$

$$V = |(a, b, c)| = 10$$

- 33. Tetrahedron.** Find the volume if the vertices are $(1, 1, 1)$, $(5, -7, 3)$, $(7, 4, 8)$, and $(10, 7, 4)$.

Per Wolfram or Wiki:

$$\begin{aligned}
 V &= \frac{1}{3!} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \frac{1}{3!} |(a, b, c)| \\
 &= \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & -7 & 3 & 1 \\ 7 & 4 & 8 & 1 \\ 10 & 7 & 4 & 1 \end{vmatrix} = \frac{1}{6} |(4, -8, 2) \cdot [(6, 3, 7) \times (9, 6, 3)]| \\
 &= \frac{1}{6} |(4, -8, 2) \cdot (33, 45, 9)| \\
 &= \frac{1}{6} \cdot 474 = 79
 \end{aligned}$$

Why?

This is intuitive.

- 34. Tetrahedron.** Find the volume if the vertices are $(1, 3, 6)$, $(3, 7, 12)$, $(8, 8, 9)$, and $(2, 2, 8)$.

$$V = \frac{1}{6} \cdot 90 = 15$$

35.

P.S. 9.4

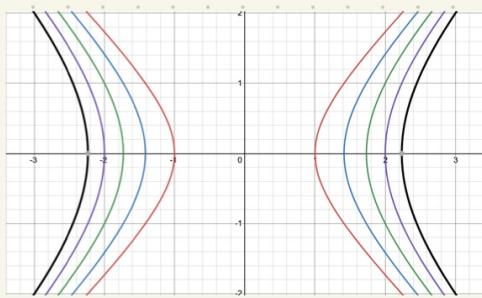
1-8 SCALAR FIELDS IN THE PLANE

Let the temperature T in a body be independent of z so that it is given by a scalar function $T = T(x, t)$. Identify the isotherms $T(x, y) = \text{const}$. Sketch some of them.

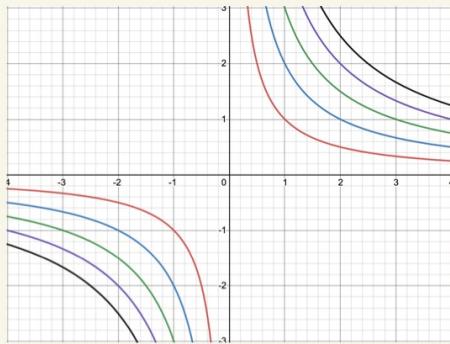
1. $T = x^2 - y^2$
2. $T = xy$
3. $T = 3x - 4y$
4. $T = \arctan(y/x) = C$
5. $T = y/(x^2 + y^2)$
6. $T = x/(x^2 + y^2)$
7. $T = 9x^2 + 4y^2$

Red = 1, black = 5.

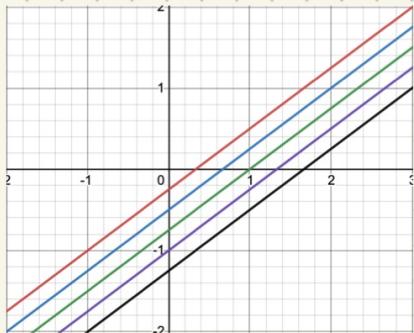
1. $T = x^2 - y^2 = C$



2. $T = xy = C$



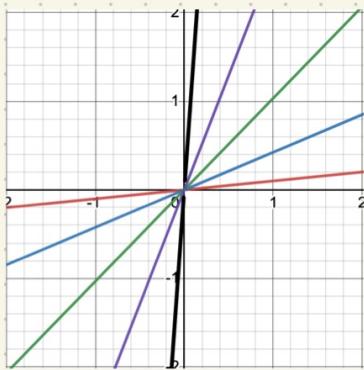
3. $T = 3x - 4y = C$



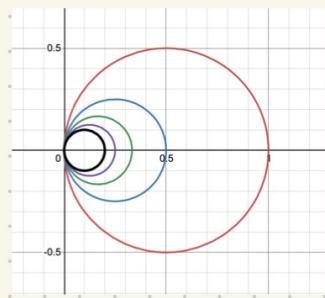
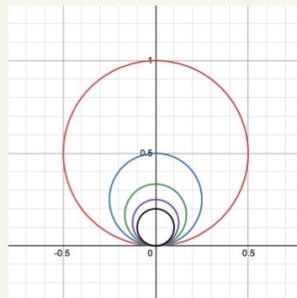
4. $T = \arctan(y/x) = C$

$C \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

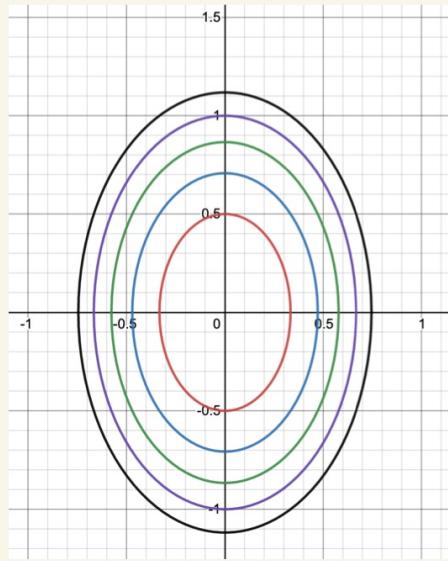
R: 0.1, B: 0.4 G: 0.8 P: 1.2, B: 1.5



5. $T = y/(x^2 + y^2) = C$

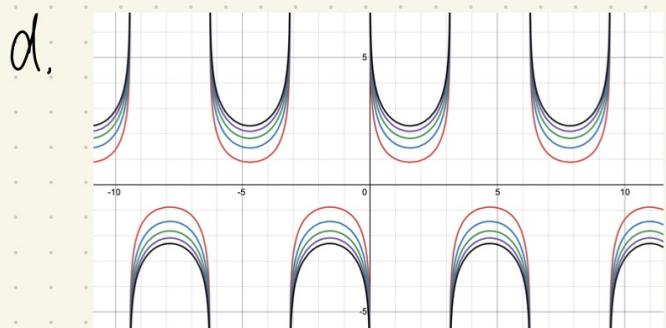
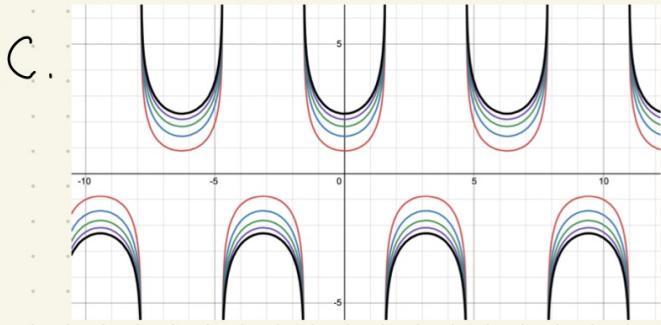
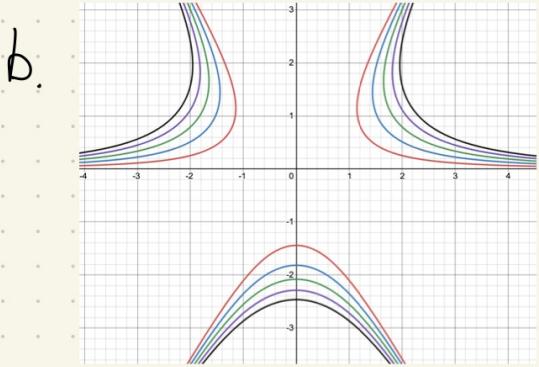
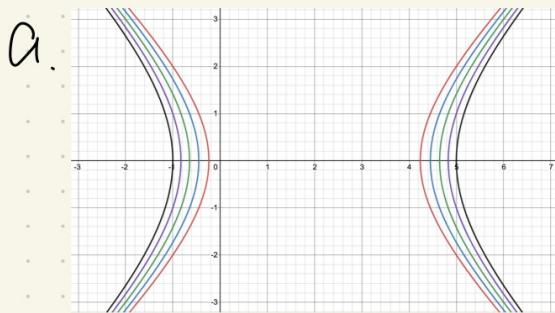


6. $T = x/(x^2 + y^2) = C$

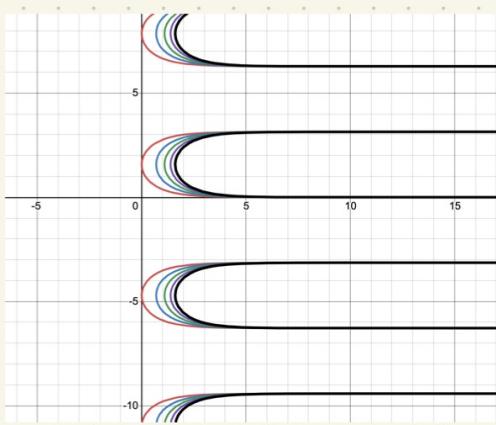


8. CAS PROJECT. Scalar Fields in the Plane. Sketch or graph isotherms of the following fields and describe what they look like.

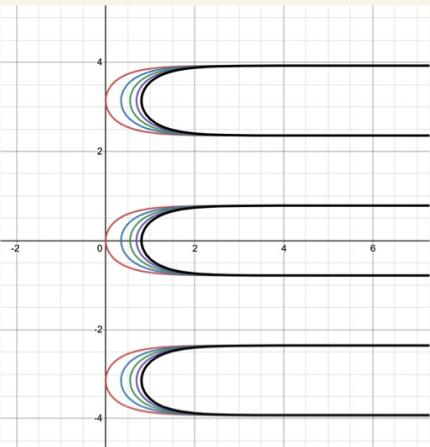
- (a) $x^2 - 4x - y^2$
- (b) $x^2y - y^3/3$
- (c) $\cos x \sinh y$
- (d) $\sin x \sinh y$
- (e) $e^x \sin y$
- (f) $e^{2x} \cos 2y$
- (g) $x^4 - 6x^2y^2 + y^4$
- (h) $x^2 - 2x - y^2$



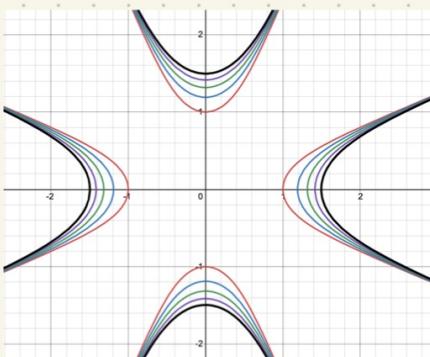
e.



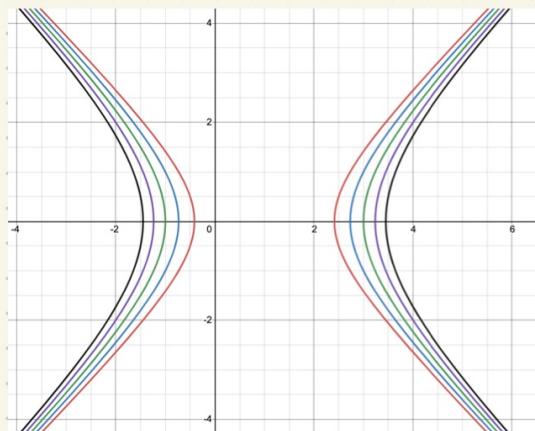
f.



g.



h.



9–14 SCALAR FIELDS IN SPACE

What kind of surfaces are the **level surfaces** $f(x, y, z) = \text{const}$?

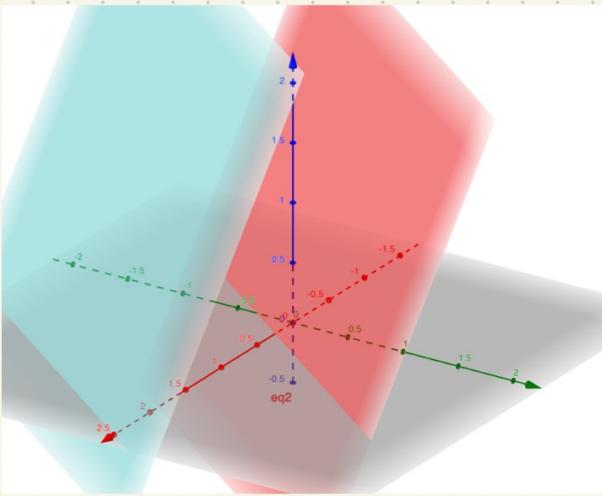
$$9. f = 4x - 3y + 2z$$

$$10. f = 9(x^2 + y^2) + z^2$$

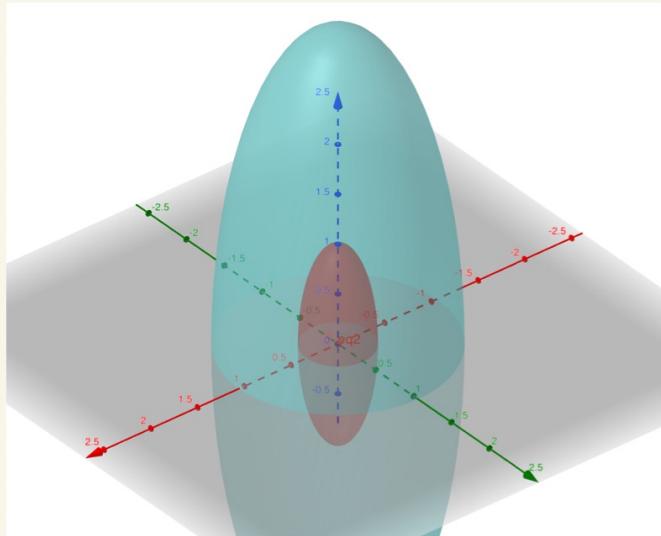
$$11. f = 5x^2 + 2y^2$$

$$12. f = z - \sqrt{x^2 + y^2}$$

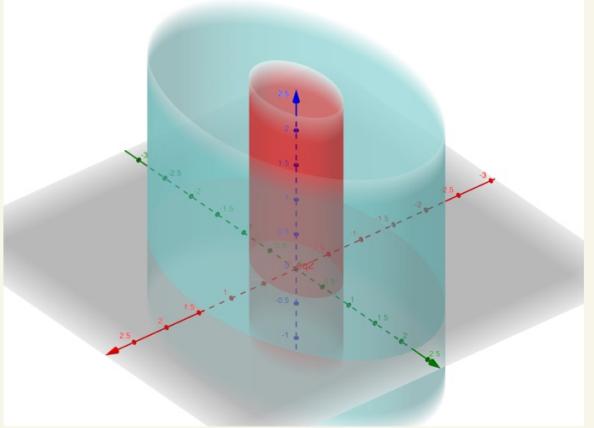
9. Teal: $f=10$ Red: $f=1$



10.



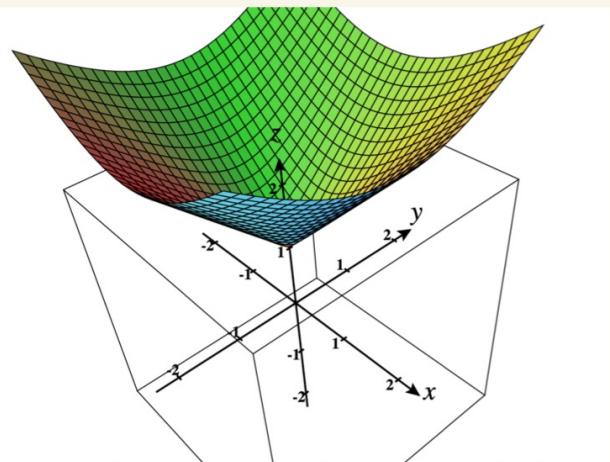
11.



12. Quite a lot of tools

Can not plot this.

$$f = 1$$



9–14 SCALAR FIELDS IN SPACE

What kind of surfaces are the **level surfaces** $f(x, y, z) = \text{const}$?

$$9. f = 4x - 3y + 2z$$

$$10. f = 9(x^2 + y^2) + z^2$$

$$11. f = 5x^2 + 2y^2$$

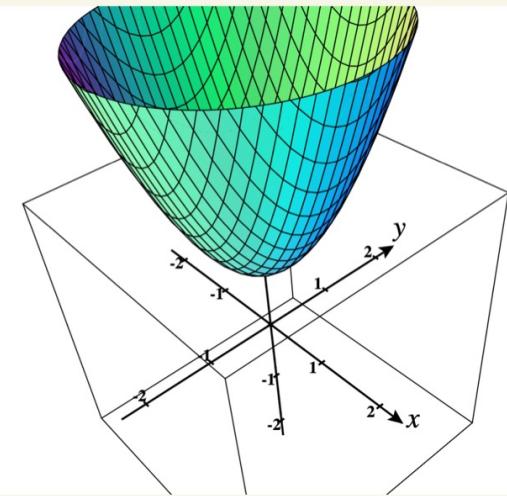
$$12. f = z - \sqrt{x^2 + y^2}$$

$$13. f = z - (x^2 + y^2)$$

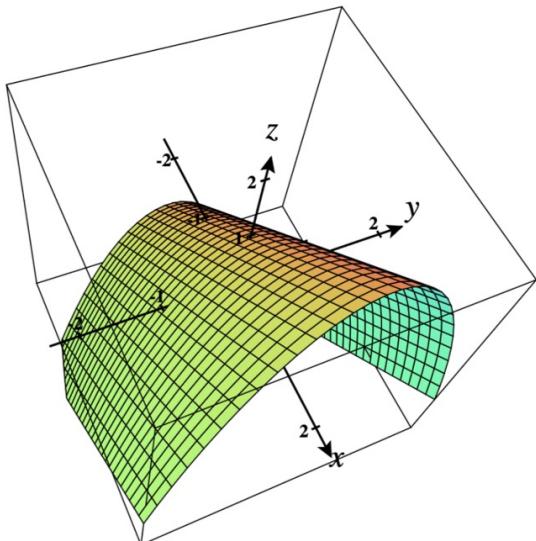
$$14. f = x - y^2$$

<https://c3d.libretexts.org/CalcPlot3D/index.html>

13.



14.



15–20 VECTOR FIELDS

Sketch figures similar to Fig. 198. Try to interpret the field of \mathbf{v} as a velocity field.

15. $\mathbf{v} = \mathbf{i} + \mathbf{j}$

17. $\mathbf{v} = x\mathbf{j}$

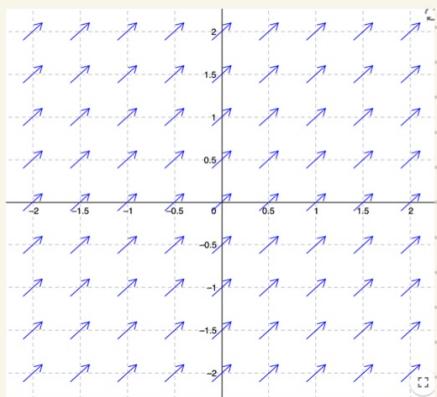
19. $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$

16. $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$

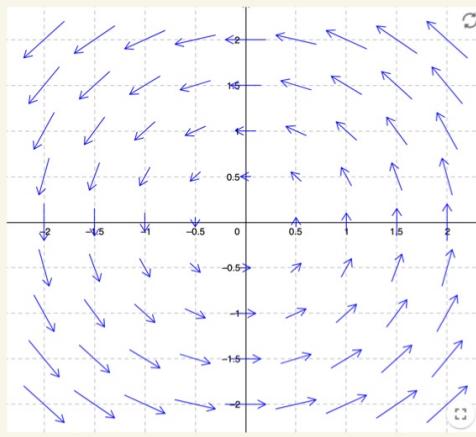
18. $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$

20. $\mathbf{v} = y\mathbf{i} - x\mathbf{j}$

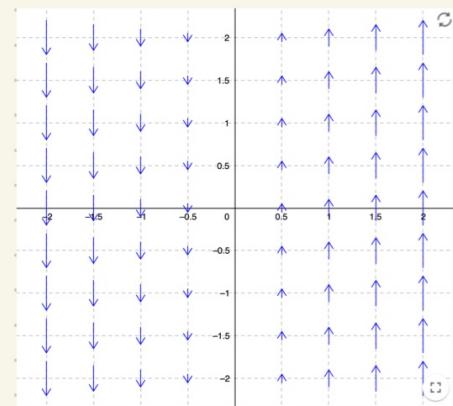
15.



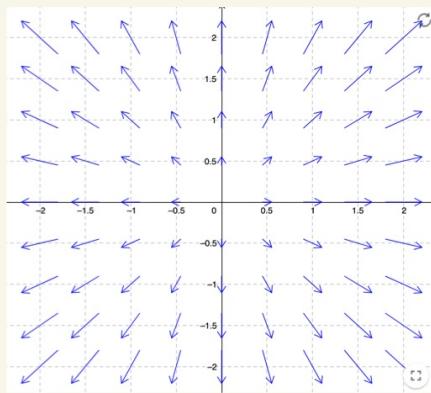
16.



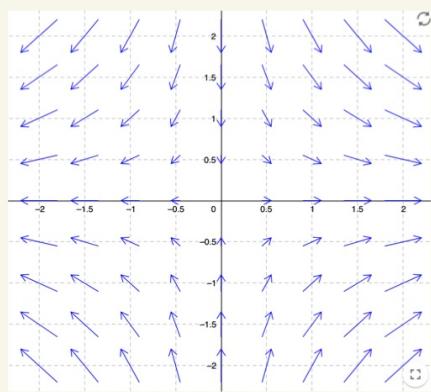
17.



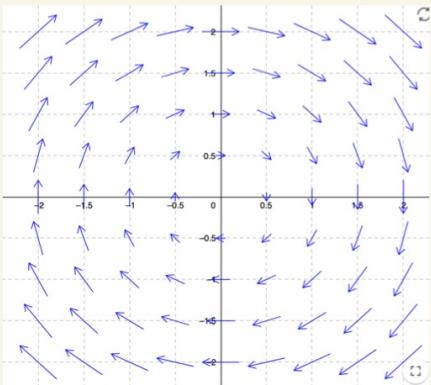
18.



19.



20.



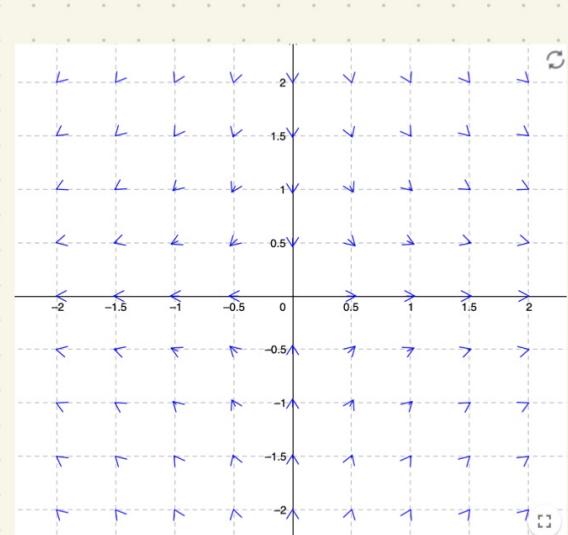
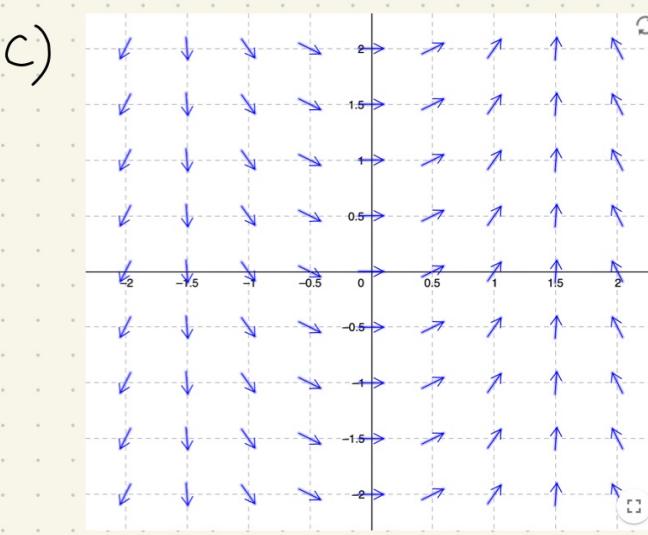
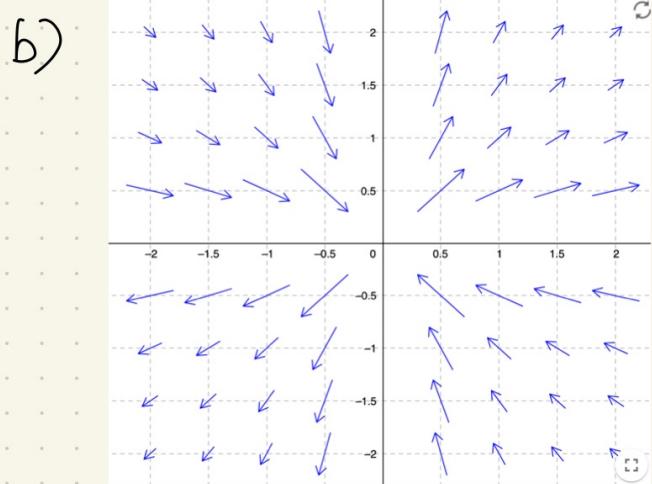
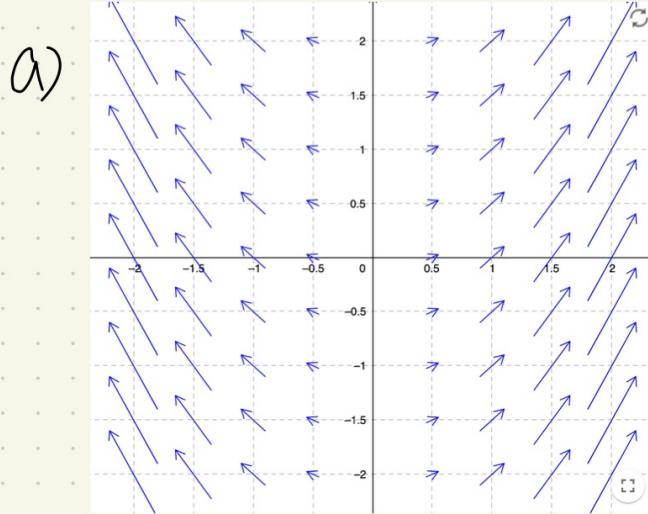
21. CAS PROJECT. Vector Fields. Plot by arrows:

(a) $\mathbf{v} = [x, x^2]$

(b) $\mathbf{v} = [1/y, 1/x]$

(c) $\mathbf{v} = [\cos x, \sin x]$

(d) $\mathbf{v} = e^{-(x^2+y^2)} [x, -y]$



22–25 DIFFERENTIATION

22. Find the first and second derivatives of $\mathbf{r} = [3 \cos 2t, 3 \sin 2t, 4t]$.
23. Prove (11)–(13). Give two typical examples for each formula.
24. Find the first partial derivatives of $\mathbf{v}_1 = [e^x \cos y, e^x \sin y]$ and $\mathbf{v}_2 = [\cos x \cosh y, -\sin x \sinh y]$.
25. **WRITING PROJECT. Differentiation of Vector Functions.** Summarize the essential ideas and facts and give examples of your own.

$$22. \mathbf{r} = [-6 \sin 2t, 6 \cos 2t, 4t]$$

$$\mathbf{r}' = [-12 \cos 2t, -12 \sin 2t, 0]$$

23.

$$24. \frac{\partial \mathbf{v}_1}{\partial x} = [e^x \cos y, e^x \sin y]$$

$$\frac{\partial \mathbf{v}_1}{\partial y} = [-e^x \sin y, e^x \cos y]$$

$$\frac{\partial \mathbf{v}_2}{\partial x} = [-\sin x \cosh y, -\cos x \sinh y]$$

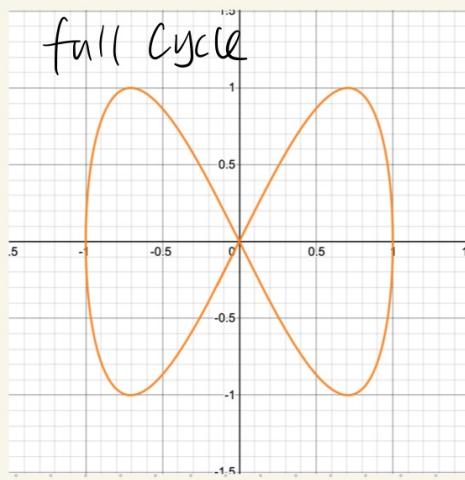
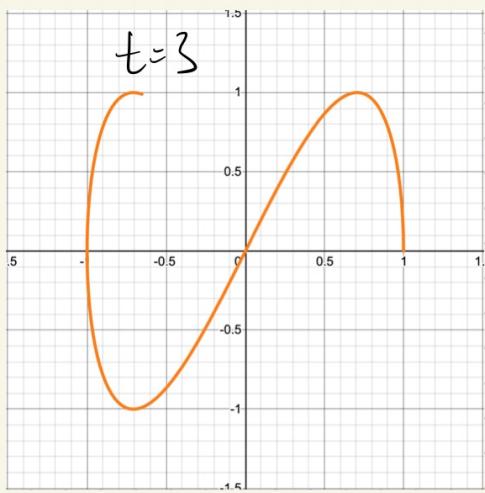
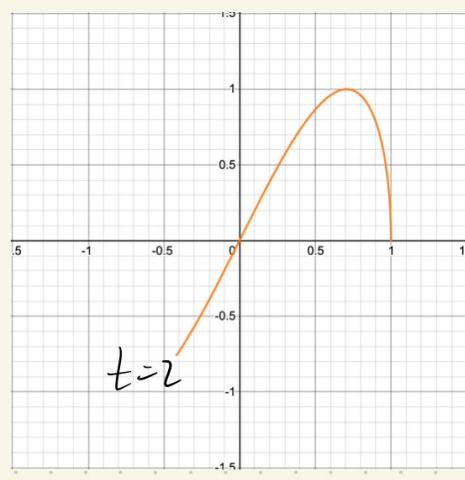
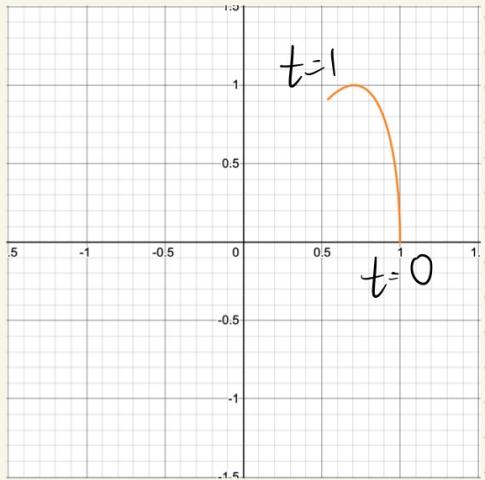
$$\frac{\partial \mathbf{v}_2}{\partial y} = [\cos x \sinh y, -\sin x \cosh y]$$

25.

P.S. 9.5

9. $[\cos t, \sin 2t, 0]$

$$0 \leq t \leq 1$$



11-20 FIND A PARAMETRIC REPRESENTATION

11. Circle in the plane $z = 1$ with center $(3, 2)$ and passing through the origin.
12. Circle in the yz -plane with center $(4, 0)$ and passing through $(0, 3)$. Sketch it.
13. Straight line through $(2, 1, 3)$ in the direction of $\mathbf{i} + 2\mathbf{j}$.

$$13. \quad \mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$$

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j}$$

$t=0$, on point $(2, 1, 3)$

$$\mathbf{r}(t) = [2, 1, 3] + t[1, 2, 0]$$

$$= [t+2, 2t+1, 3]$$

24-28 TANGENT $U(t)$

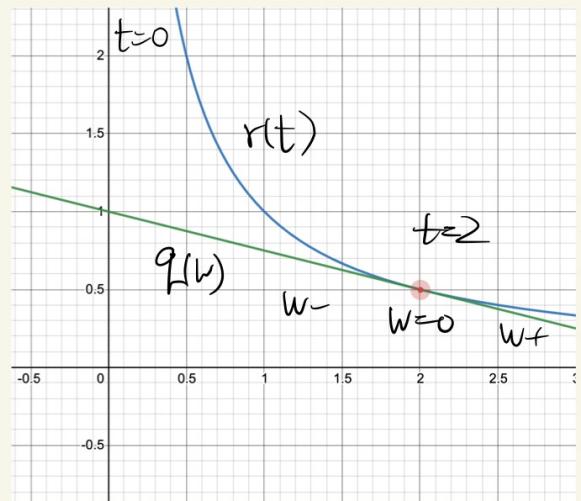
Given a curve $C: \mathbf{r}(t)$, find a tangent vector $\mathbf{r}'(t)$, a unit tangent vector $\mathbf{u}'(t)$, and the tangent of C at P . Sketch curve and tangent.

24. $\mathbf{r}(t) = [t, \frac{1}{2}t^2, 1]$, $P: (2, 2, 1)$
25. $\mathbf{r}(t) = [10 \cos t, 1, 10 \sin t]$, $P: (6, 1, 8)$
26. $\mathbf{r}(t) = [\cos t, \sin t, 9t]$, $P: (1, 0, 18\pi)$
27. $\mathbf{r}(t) = [t, 1/t, 0]$, $P: (2, \frac{1}{2}, 0)$

$$27. \quad \mathbf{r}'(t) = [1, -\frac{1}{t^2}, 0]$$

$$\begin{aligned} U(t) &= \frac{\mathbf{r}'}{|\mathbf{r}'|} \cdot \mathbf{r}' \\ &= \frac{1}{\sqrt{1 + \frac{1}{t^4}}} \cdot \mathbf{r}' \\ &= \frac{t^2}{t^4 + 1} \cdot \mathbf{r}' \\ &= \left[\frac{t^2}{t^4 + 1}, -\frac{1}{t^4 + 1}, 0 \right] \end{aligned}$$

$$\begin{aligned} q(w) &= \mathbf{r} + w\mathbf{r}' \\ &= \left[t, \frac{1}{t}, 0 \right] + \left[w, -\frac{w}{t^2}, 0 \right] \\ &= \left[2, \frac{1}{2}, 0 \right] + \left[w, -\frac{w}{4}, 0 \right] \\ &= \left[2+w, \frac{1}{2}-\frac{w}{4}, 0 \right] \end{aligned}$$



29-32 LENGTH

Find the length and sketch the curve.

31. Circle $\mathbf{r}(t) = [a \cos t, a \sin t]$ from $(a, 0)$ to $(0, a)$.

from $t=0$ to $t=\frac{\pi}{2}$

$$l = \int_0^{\frac{\pi}{2}} \sqrt{\mathbf{r}' \cdot \mathbf{r}'} dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(-a \sin t, a \cos t) \cdot (-a \sin t, a \cos t)} dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt$$

$$= \int_0^{\frac{\pi}{2}} |a| dt = |a| t \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} |a|$$

35-46

CURVES IN MECHANICS

Forces acting on moving objects (cars, airplanes, ships, etc.) require the engineer to know corresponding **tangential** and **normal accelerations**. In Probs. 35–38 find them, along with the **velocity** and **speed**. Sketch the path.

35. Parabola $\mathbf{r}(t) = [t, t^2, 0]$. Find \mathbf{v} and \mathbf{a} .

$$\mathbf{v} = \mathbf{r}'(t) = [1, 2t, 0]$$

$$\mathbf{a} = \mathbf{v}' = [0, 2, 0]$$

$$\text{Speed} = \sqrt{1^2 + 4t^2}$$

$$\text{Acceleration} = 2$$

47-55

CURVATURE AND TORSION

47. Circle. Show that a circle of radius a has curvature $1/a$.

$$\mathbf{r}(t) = [a \cos t, a \sin t], \quad a > 0$$

$$\mathbf{r}(s) = [a \cos(s/r), a \sin(s/r)]$$

$$\begin{aligned} K(s) &= |\mathbf{r}''(s)| = \left| \left[-\frac{\sin(s/a)}{a}, \frac{\cos(s/a)}{a} \right]' \right| \\ &= \left| \left[-\frac{1}{a} \left[-\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right) \right] \right]' \right| \\ &= \left| \frac{1}{a} \left[-\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right) \right] \right| \\ &= \frac{1}{a} \cdot 1 = \frac{1}{a} \end{aligned}$$

9.7

1-6 CALCULATION OF GRADIENTS

Find grad f . Graph some level curves $f = \text{const}$. Indicate ∇f by arrows at some points of these curves.

$$1. f = (x + 1)(2y - 1)$$

$$\nabla f = [2y - 1, 2(x+1)]$$

$$5. f = x^4 + y^4$$

$$\nabla f = [4x^3, 4y^3]$$

7-10 USEFUL FORMULAS FOR GRADIENT AND LAPLACIAN

Prove and illustrate by an example.

$$7. \nabla(f^n) = nf^{n-1}\nabla f$$

$$\text{let } g = f^n$$

$$\nabla g = g' = nf^{n-1} \cdot \nabla f$$

$$\nabla(f^2) = 2f \cdot \nabla f$$

11-15 USE OF GRADIENTS. ELECTRIC FORCE

The force in an electrostatic field given by $f(x, y, z)$ has the direction of the gradient. Find ∇f and its value at P .

$$11. f = xy, P: (-4, 5)$$

$$12. f = x/(x^2 + y^2), P: (1, 1)$$

$$13. f = \ln(x^2 + y^2), P: (8, 6)$$

$$13. \nabla f = \left[\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right]$$

$$\nabla f(P) = \left[\frac{16}{100}, \frac{12}{100} \right]$$

9.8. 1.5. 15

1-6 CALCULATION OF THE DIVERGENCE

Find $\operatorname{div} \mathbf{v}$ and its value at P .

1. $\mathbf{v} = [x^2, 4y^2, 9z^2], P: (-1, 0, \frac{1}{2})$ *Mistake*

$$\operatorname{div} \mathbf{v} = 2x + 8y + 18z$$

on point $P: (-1, 0, \frac{1}{2})$

$$\operatorname{div} \mathbf{v} = 2(-1) + 0 + 18 \cdot \frac{1}{2} = 7$$

15-20 LAPLACIAN

Calculate $\nabla^2 f$ by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

15. $f = \cos^2 x + \sin^2 y$

$$\begin{aligned}\operatorname{grad} f &= [2 \cos x \cdot (-\sin x), \\ &\quad 2 \sin y \cdot \cos y] \\ &= [-\sin 2x, \sin 2y]\end{aligned}$$

$$\begin{aligned}\nabla^2 f &= \operatorname{div}(\operatorname{grad} f) \\ &= -2 \cos 2x + 2 \cos 2y\end{aligned}$$

?

5. $\mathbf{v} = x^2y^2z^2[x, y, z], P: (3, -1, 4)$

$$\mathbf{v} = [x^3y^2z^2, x^2y^3z^2, x^2y^2z^3]$$

$$\begin{aligned}\operatorname{div} \mathbf{v} &= 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2 \\ &= 9x^2y^2z^2\end{aligned}$$

on point $(3, -1, 4)$

$$\operatorname{div} \mathbf{v} = 9 \cdot 9 \cdot 1 \cdot 16 = 1296$$

9.9 5.7. 11. 15

4-8 CALCULATION OF CURL

Find curl \mathbf{v} for \mathbf{v} given with respect to right-handed Cartesian coordinates. Show the details of your work.

4. $\mathbf{v} = [2y^2, 5x, 0]$

5. $\mathbf{v} = xyz[x, y, z]$

6. $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}[x, y, z]$

7. $\mathbf{v} = [0, 0, e^{-x} \sin y]$

8. $\mathbf{v} = [e^{-z^2}, e^{-x^2}, e^{-y^2}]$

5. $\mathbf{v} = [x^2yz, xy^2z, xy^2z^2]$

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz, xy^2z, xy^2z^2 \end{vmatrix}$$

$$= (xz^2 - xy^2)\mathbf{i} - (yz^2 - x^2y)\mathbf{j} + (y^2z - x^2z)\mathbf{k}$$

7. $\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & e^{-x} \sin y \end{vmatrix}$

$$= [e^{-x} \cos y]\mathbf{i} - [\sin y (-e^{-x})]\mathbf{j}$$

$$= e^{-x} \cos y \mathbf{i} + \sin y e^{-x} \mathbf{j} + 0 \mathbf{k}$$

$$= e^{-x} [\cos y, \sin y, 0]$$

9-13 FLUID FLOW

Let \mathbf{v} be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles). Hint. See the answers to Probs. 9 and 11 for a determination of a path.

9. $\mathbf{v} = [0, 3z^2, 0]$

10. $\mathbf{v} = [\sec x, \csc x, 0]$

11. $\mathbf{v} = [y, -2x, 0]$

$$11. \text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -2x & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} - 3\mathbf{k}$$

Not irrotational.

$\text{div } \mathbf{v} = 0$, incompressible.

Find the streamlines

15-20 DIV AND CURL

With respect to right-handed coordinates, let $\mathbf{u} = [y, z, x]$, $\mathbf{v} = [yz, zx, xy]$, $f = xyz$, and $g = x + y + z$. Find the given expressions. Check your result by a formula in Proj. 14 if applicable.

15. $\text{curl } (\mathbf{u} + \mathbf{v})$, $\text{curl } \mathbf{v}$

$$\text{curl } (\mathbf{u} + \mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+yz, z+zx, x+xy \end{vmatrix}$$

$$= (x-1-x)\mathbf{i} - (1+y-y)\mathbf{j} - \mathbf{k}$$

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz, zx, xy \end{vmatrix}$$

$$= (x-x)\mathbf{i} - (y-y)\mathbf{j} + (z-z)\mathbf{k}$$

$$= 0$$

Textbook is wrong