C8: Matrix Eigenvalue Problems

- 2A: The Eigenvalue Problem
  - How to find Eigenvalue and Eigenvectors
    - egenvalue could be 0, but we are looking for non 0 eigenvector
    - (A- λI)x=0, consistant, λ is the eigenvalue
    - Basis of eigen space
- 2B: Applications of Eigenvalue Problems
  - Stretching of an Elastic Membrane
- 2C: Symmetric, Skew-Symmetric, and Orthogonal Matrices
  - Symmetric
    - eingenvalue: Real
  - Skew-Symmetric
    - eingenvalue: Pure imaginary or 0
  - Orthogonal
    - Real or complex conjugate in pair and have abs value of 1
  - Attribute
    - $\mathbf{R} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$
    - $\mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).$
    - $\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T \mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$
      Orthonormal
  - $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$
    - b=0, symmetric
    - a=0, skew-symmetric
    - a^2+b^2=1, orthogonal
- 2D: Eigenbases
  - $$\begin{aligned} \mathbf{y} = \mathbf{Ax} &= \mathbf{A}(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) \\ &= c_1\mathbf{Ax}_1 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n \end{aligned}$$
  - We can decompose the complicated action of A into a sum of simple actions (on the eigenvectors of A)
  - Symmetric matrix has orthonormal basis of eigenvectors for R^n (P340, T2)
    - Wiki: A (real-valued) symmetric matrix is necessarily a normal matrix.
  - Similarity
    - $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$
      Any nonsingular P
    - Same eigenvalues
      - eigenvector: $\mathbf{P}^{-1}\mathbf{x}$
  - Diagonalization
    - $\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$ $\Rightarrow A = XDX^{-1}$
      - $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *is* $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$
        Sec 7.8, Page 304
- 2E: Complex Forms
  - Conjugate
    - $a_{jk} = \alpha + i\beta$
      with its complex conjugate
      $\bar{a}_{jk} = \alpha - i\beta.$
  - Hermitian
    - if $\bar{A}^T = -A$, that is, $\bar{a}_{kj} = -a_{jk}$
    - Symmetric. evalue: Real
  - Skew-hermitian
    - if $\bar{A}^T = A$, that is, $\bar{a}_{kj} = a_{jk}$
    - Skew-symmetric, evalue Imaginary or 0
  - Unitary
    - if $\bar{A}^T = A^{-1}$
    - Orthogonal.

C8.-

The order M_λ of an eigenvalue λ as a root of the characteristic polynomial is called the **algebraic multiplicity** of λ. The number m_λ of linearly independent eigenvectors corresponding to λ is called the **geometric multiplicity** of λ. Thus m_λ is the dimension of the eigenspace corresponding to this λ.

Since the characteristic polynomial has degree n, the sum of all the algebraic multiplicities must equal n. In Example 2 for $\lambda = -3$ we have $m_\lambda = M_\lambda = 2$. In general, $m_\lambda \leqq M_\lambda$, as can be shown. The difference $\Delta_\lambda = M_\lambda - m_\lambda$ is called the **defect** of λ. Thus $\Delta_{-3} = 0$ in Example 2, but positive defects Δ_λ can easily occur:

Theroms. 8.1

THEOREM 1

Eigenvalues

The eigenvalues of a square matrix $\mathbf{A}$ *are the roots of the characteristic equation* (4) *of* $\mathbf{A}$.

Hence an $n \times n$ *matrix has at least one eigenvalue and at most* n *numerically different eigenvalues.*

THEOREM 2

Eigenvectors, Eigenspace

If $\mathbf{w}$ *and* $\mathbf{x}$ *are eigenvectors of a matrix* $\mathbf{A}$ *corresponding to* ***the same*** *eigenvalue* λ, *so are* $\mathbf{w} + \mathbf{x}$ (provided $\mathbf{x} \neq -\mathbf{w}$) *and* $k\mathbf{x}$ *for any* $k \neq 0$.

Hence the eigenvectors corresponding to one and the same eigenvalue λ *of* $\mathbf{A}$, *together with* $\mathbf{0}$, *form a vector space* (cf. Sec. 7.4), *called the* **eigenspace** *of* $\mathbf{A}$ *corresponding to that* λ.

THEOREM 3

Eigenvalues of the Transpose

The transpose $\mathbf{A}^\mathsf{T}$ *of a square matrix* $\mathbf{A}$ *has the same eigenvalues as* $\mathbf{A}$.

Theorem 8.3

THEOREM 1 **Eigenvalues of Symmetric and Skew-Symmetric Matrices**

(a) *The eigenvalues of a symmetric matrix are real.*

(b) *The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.*

THEOREM 2 **Invariance of Inner Product**

An orthogonal transformation preserves the value of the **inner product** *of vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *in* R^n, *defined by*

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{a}^\mathsf{T}\mathbf{b} = [a_1 \quad \cdots \quad a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}. \tag{7}$$

That is, for any $\mathbf{a}$ *and* $\mathbf{b}$ *in* R^n, *orthogonal* $n \times n$ *matrix* $\mathbf{A}$, *and* $\mathbf{u} = \mathbf{Aa}, \mathbf{v} = \mathbf{Ab}$ *we have* $\mathbf{u} \bullet \mathbf{v} = \mathbf{a} \bullet \mathbf{b}$.

Hence the transformation also preserves the **length** *or* **norm** *of any vector* $\mathbf{a}$ *in* R^n *given by*

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \bullet \mathbf{a}} = \sqrt{\mathbf{a}^\mathsf{T}\mathbf{a}}. \tag{8}$$

THEOREM 3 **Orthonormality of Column and Row Vectors**

A real square matrix is orthogonal if and only if its column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ *(and also its row vectors) form an* **orthonormal system**, *that is,*

$$\mathbf{a}_j \bullet \mathbf{a}_k = \mathbf{a}_j^\mathsf{T}\mathbf{a}_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k. \end{cases} \tag{10}$$

THEOREM 4 **Determinant of an Orthogonal Matrix**

The determinant of an orthogonal matrix has the value $+1$ *or* -1.

THEOREM 5 **Eigenvalues of an Orthogonal Matrix**

The eigenvalues of an orthogonal matrix $\mathbf{A}$ *are real or complex conjugates in pairs and have absolute value* 1.

Theorem 8.4

THEOREM 1

Basis of Eigenvectors

If an $n \times n$ matrix $\mathbf{A}$ *has n* ***distinct*** *eigenvalues, then* $\mathbf{A}$ *has a basis of eigenvectors* $\mathbf{x}_1, \cdots, \mathbf{x}_n$ *for* R^n.

THEOREM 2

Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

THEOREM 3

Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ *is similar to* $\mathbf{A}$, *then* $\hat{\mathbf{A}}$ *has the same eigenvalues as* $\mathbf{A}$.

Furthermore, if $\mathbf{x}$ *is an eigenvector of* $\mathbf{A}$, *then* $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ *is an eigenvector of* $\hat{\mathbf{A}}$ *corresponding to the same eigenvalue.*

THEOREM 4

Diagonalization of a Matrix

If an $n \times n$ matrix $\mathbf{A}$ *has a basis of eigenvectors, then*

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} \quad (5)$$

$\Rightarrow A = XDX^{-1}$

is diagonal, with the eigenvalues of $\mathbf{A}$ *as the entries on the main diagonal. Here* $\mathbf{X}$ *is the matrix with these eigenvectors as column vectors. Also,*

$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X} \qquad (m = 2, 3, \cdots). \quad (5^*)$$

THEOREM 5

Principal Axes Theorem

The substitution (9) *transforms a quadratic form*

$$Q = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_jx_k \qquad (a_{kj} = a_{jk})$$

to the principal axes form or **canonical form** (10), *where* $\lambda_1, \cdots, \lambda_n$ *are the (not necessarily distinct) eigenvalues of the (symmetric!) matrix* $\mathbf{A}$, *and* $\mathbf{X}$ *is an orthogonal matrix with corresponding eigenvectors* $\mathbf{x}_1, \cdots, \mathbf{x}_n$, *respectively, as column vectors.*

Theorem 8.5

THEOREM 1

Eigenvalues

(a) *The eigenvalues of a Hermitian matrix (and thus of a symmetric matrix) are real.*

(b) *The eigenvalues of a skew-Hermitian matrix (and thus of a skew-symmetric matrix) are pure imaginary or zero.*

(c) *The eigenvalues of a unitary matrix (and thus of an orthogonal matrix) have absolute value* 1.

THEOREM 2

Invariance of Inner Product

A **unitary transformation**, *that is,* $\mathbf{y} = \mathbf{A}\mathbf{x}$ *with a unitary matrix* $\mathbf{A}$, *preserves the value of the inner product* (4), *hence also the norm* (5).

THEOREM 3

Unitary Systems of Column and Row Vectors

A complex square matrix is unitary if and only if its column vectors (and also its row vectors) form a unitary system.

THEOREM 4

Determinant of a Unitary Matrix

Let $\mathbf{A}$ *be a unitary matrix. Then its determinant has absolute value one, that is,* $|\det \mathbf{A}| = 1$.

THEOREM 5

Basis of Eigenvectors

A Hermitian, skew-Hermitian, or unitary matrix has a basis of eigenvectors for C^n *that is a unitary system.*

SUMMARY OF CHAPTER 8

Linear Algebra: Matrix Eigenvalue Problems

The practical importance of matrix eigenvalue problems can hardly be overrated. The problems are defined by the vector equation

$$\mathbf{Ax} = \lambda\mathbf{x}. \tag{1}$$

$\mathbf{A}$ is a given square matrix. All matrices in this chapter are ***square***. λ is a scalar. To *solve* the problem (1) means to determine values of λ, called **eigenvalues** (or **characteristic values**) of $\mathbf{A}$, such that (1) has a nontrivial solution $\mathbf{x}$ (that is, $\mathbf{x} \neq \mathbf{0}$), called an **eigenvector** of $\mathbf{A}$ corresponding to that λ. An $n \times n$ matrix has at least one and at most n numerically different eigenvalues. These are the solutions of the **characteristic equation** (Sec. 8.1)

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0. \tag{2}$$

$D(\lambda)$ is called the **characteristic determinant** of $\mathbf{A}$. By expanding it we get the **characteristic polynomial** of $\mathbf{A}$, which is of degree n in λ. Some typical applications are shown in Sec. 8.2.

Section 8.3 is devoted to eigenvalue problems for **symmetric** ($\mathbf{A}^\mathsf{T} = \mathbf{A}$), **skew-symmetric** ($\mathbf{A}^\mathsf{T} = -\mathbf{A}$), and **orthogonal matrices** ($\mathbf{A}^\mathsf{T} = \mathbf{A}^{-1}$). Section 8.4 concerns the diagonalization of matrices and the transformation of quadratic forms to principal axes and its relation to eigenvalues.

Section 8.5 extends Sec. 8.3 to the complex analogs of those real matrices, called **Hermitian** ($\overline{\mathbf{A}}^\mathsf{T} = \mathbf{A}$), **skew-Hermitian** ($\overline{\mathbf{A}}^\mathsf{T} = -\mathbf{A}$), and **unitary matrices** ($\overline{\mathbf{A}}^\mathsf{T} = \mathbf{A}^{-1}$). All the eigenvalues of a Hermitian matrix (and a symmetric one) are real. For a skew-Hermitian (and a skew-symmetric) matrix they are pure imaginary or zero. For a unitary (and an orthogonal) matrix they have absolute value 1.