Chapter 10 Vector Integral Calculus. Integral Theorems

Selected Problem set 10.1

10.1 3.5.9.19

2–11 LINE INTEGRAL. WORK

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

2. $\mathbf{F} = [y^2, -x^2]$, C: $y = 4x^2$ from $(0, 0)$ to $(1, 4)$
3. $\mathbf{F}$ as in Prob. 2, C from $(0, 0)$ straight to $(1, 4)$. Compare.
4. $\mathbf{F} = [xy, x^2y^2]$, C from $(2, 0)$ straight to $(0, 2)$
5. $\mathbf{F}$ as in Prob. 4, C the quarter-circle from $(2, 0)$ to $(0, 2)$ with center $(0, 0)$

3. C: $r(t) = [t, 4t] = t\mathbf{i} + 4t\mathbf{j}$

$F(r(t)) = [(4t)^2, -t^2] = [16t^2, -t^2]$

$r'(t) = [1, 4]$

$$\int_C F(r) \cdot dr = \int_0^1 [16t^2, -t^2] \cdot [1, 4]\, dt$$

$$= \int_0^1 (16t^2 - 4t^2)\, dt$$

$$= \int_0^1 12t^2\, dt$$

$$= 4t^3 \Big|_0^1 = 4 - 0 = 4$$

5. C by $r(t) = [2\cos t, 2\sin t]$, when $0 \le t \le \frac{\pi}{2}$.

$F(r(t)) = [4\sin t \cos t, 16\sin^2 t \cos^2 t]$

$r'(t) = [-2\sin t, 2\cos t]$

$$\int_C F(r) \cdot dr = \int_0^{\frac{\pi}{2}} (-8\sin^2 t \cos t + 32\sin^2 t \cos^3 t)\, dt$$

$$= 8\int_0^{\frac{\pi}{2}} (4\sin^2 t \cos^3 t - \sin^2 t \cos t)\, dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t\, (4\cos^2 t - 1) \sin^2 t\, dt$$

$$= 8\int_0^{\frac{\pi}{2}} \cos t \left[-\sin^2 t (4\sin^2 t - 3)\right] dt$$

$u = \sin t$, $\frac{du}{dt} = \cos t$, $dt = \frac{du}{\cos t}$

$$= -8\int_0^1 u^2 (4u^2 - 3)\, du$$

$$= -32\int_0^1 u^4\, du + 24\int_0^1 u^2\, du$$

$$= -\frac{32}{5} u^5 \Big|_0^1 + 24 \cdot \frac{1}{3} u^3 \Big|_0^1$$

$$= -\frac{32}{5} + 8 = \frac{8}{5} = 1.6$$

9. $\mathbf{F} = [x + y, y + z, z + x]$, $C: \mathbf{r} = [2t, 5t, t]$ from $t = 0$ to 1. Also from $t = -1$ to 1.

$C: r = [2t, 5t, t] \quad 0 \le t \le 1$

$r' = [2, 5, 1]$

$F(r(t)) = [7t, 6t, 3t]$

$\int_C F(r)dr = \int_0^1 [7t, 6t, 3t][2, 5, 1] \cdot dt$

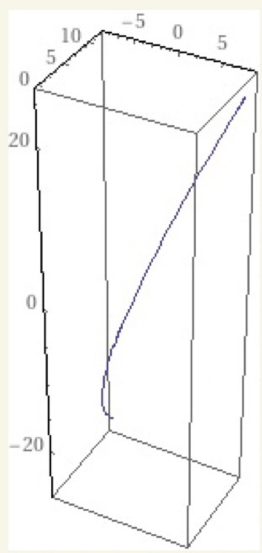
$= \int_0^1 47t \cdot dt$

$= \frac{47}{2} t^2 \Big|_0^1 = \frac{47}{2} = 23.5$

$-1 \le t \le 1$

$\int_{-1}^{1} 47t \cdot dt = \frac{47}{2} t^2 \Big|_{-1}^{1} = 0$

19. $f = xyz$, $C: \mathbf{r} = [4t, 3t^2, 12t]$, $-2 \leqq t \leqq 2$. Sketch C.



$C: r = [4t, 3t^2, 12t] \quad -2 \le t \le 2$

$r' = [4, 6t, 12]$

$F(r(t)) = 144t^4$

$\int_C f(r)dt = \int_{-2}^{2} 144t^4 \cdot dt$

$= \frac{144}{5} \cdot t^5 \Big|_{-2}^{2}$

$= \frac{144}{5} \cdot 64 = 1843.2$

Selected Problem set 10.2

10.2. 3. 5. 13. 15.

3–9 PATH INDEPENDENT INTEGRALS

Show that the form under the integral sign is exact in the plane (Probs. 3–4) or in space (Probs. 5–9) and evaluate the integral. Show the details of your work.

3. $\int_{(\pi/2,\,\pi)}^{(\pi,\,0)} (\frac{1}{2}\cos\frac{1}{2}x\cos 2y\,dx - 2\sin\frac{1}{2}x\sin 2y\,dy)$

4. $\int_{(4,\,0)}^{(6,\,1)} e^{4y}(2x\,dx + 4x^2\,dy)$

5. $\int_{(0,\,0,\,\pi)}^{(2,\,1/2,\,\pi/2)} e^{xy}(y\sin z\,dx + x\sin z\,dy + \cos z\,dz)$

3. exactness:

$(F_2)_x = -2\sin 2y \cdot \frac{1}{2}\cdot\cos\frac{1}{2}x$

$= -\sin 2y\cdot\cos\frac{1}{2}x$

$(F_1)_y = \frac{1}{2}\cdot\cos\frac{1}{2}x\cdot(-\sin 2y)\cdot 2$

$= -\sin 2y\cos\frac{1}{2}x$.

Evaluate:

$f_x = F_1 = \frac{1}{2}\cos\frac{1}{2}x\cdot\cos 2y$

$f_y = F_2 = -2\sin\frac{1}{2}x\sin 2y$

$f = \cos 2y\cdot\sin\frac{1}{2}x + g(y)$

$f_y = \sin\frac{1}{2}x\cdot(-\sin 2y)\cdot 2 + g_y$

$f = \cos 2y\cdot\sin\frac{1}{2}x$

$f(\pi,0) - f(\frac{\pi}{2},\pi) = 1\cdot 1 - 1\cdot\frac{\sqrt{2}}{2}$

$= 1 - \frac{\sqrt{2}}{2}$

5. exactness:

$(F_3)_y = x\cdot e^{xy}\cdot\cos z$

$(F_2)_z = e^{xy}\cdot x\cdot\cos z$

$(F_1)_z = e^{xy}\cdot y\cdot\cos z$

$(F_3)_x = \cos z\cdot y\cdot e^{xy}$

$(F_1)_y = \sin z\,(x\cdot e^{xy}\cdot y + e^{xy})$

$(F_2)_x = \sin z\,(y\cdot e^{xy}\cdot x + e^{xy})$

Evaluate:

$f_x = F_1 = e^{xy}\cdot y\cdot\sin z$

$f_y = F_2 = e^{xy}\cdot x\cdot\sin z$

$f_z = F_3 = e^{xy}\cdot\cos z$.

$f = \sin z\cdot e^{xy} + g(y,z)$

$f_y = x\cdot\sin z\cdot e^{xy} + g_y$

$= x\cdot\sin z\cdot e^{xy} + h(z)$

$f_z = e^{xy}\cdot\cos z + h'$

$h' = 0,\ h = 0,\ g = 0$

$f = \sin z\cdot e^{xy}$

$f(2,\frac{1}{2},\frac{\pi}{2}) - f(0,0,\pi)$

$= 1\cdot e - 0 = e$

13–19 PATH INDEPENDENCE?

Check, and if independent, integrate from (0, 0, 0) to (a, b, c).

13. $2e^{x^2}(x \cos 2y\, dx - \sin 2y\, dy)$

Check if independent

$f_x = F_1 = 2e^{x^2} \cdot x \cdot \cos 2y$

$f_y = F_2 = -2e^{x^2} \cdot \sin 2y$

$f = \cos 2y \cdot e^{x^2} + g$

$f_y = e^{x^2} \cdot (-\sin 2y) \cdot 2 + g'$

$g' = 0 \quad g = 0$, say.

$f = \cos 2y \cdot e^{x^2}$

Independent.

$f(a,b,c) - f(0,0,0)$

$= \cos(2b) \cdot e^{a^2} - 1 \cdot e^0$

$= \cos(2b) \cdot e^{a^2} - 1$

answer is wrong

15. $x^2 y\, dx - 4xy^2\, dy + 8z^2 x\, dz$

Check if independent

$f_x = F_1 = x^2 y$

$f_y = F_2 = -4xy^2$

$f_z = F_3 = 8z^2 x$

$f = \frac{1}{3} \cdot y \cdot x^3 + g(y,z)$

$f_y = \frac{1}{3} \cdot x^3 + g_y$

$g_y = -4xy^2 - \frac{1}{3}x^3 \Rightarrow$ dependent

$g(y,z) = -\frac{4}{3}xy^3 - \frac{1}{3}x^3 y = 0$

$4xy^3 + x^3 y = 0$

$4y^2 + x^2 = 0$

Selected Problem set 10.3

10.3 5 9 15

5. $\displaystyle\int_0^1\int_{x^2}^{x}(1-2xy)\,dy\,dx$

$= \int_0^1 \left[(y-xy^2)\Big|_{x^2}^{x}\right]dx$

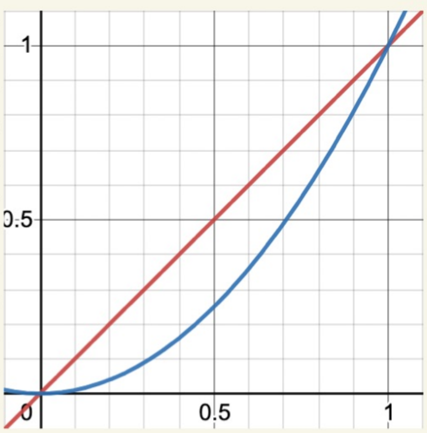
$= \int_0^1 [x-x^3-(x^2-x^5)]\,dx$

$= \int_0^1 (x^5-x^3-x^2+x)\,dx$

$= \frac{x^6}{6}-\frac{x^4}{4}-\frac{x^3}{3}+\frac{x^2}{2}\Big|_0^1$

$= \frac{1}{6}-\frac{1}{4}-\frac{1}{3}+\frac{1}{2}$

$= \frac{2-3-4+6}{12} = \frac{1}{12}$



Red: $y=x$

Blue: $y=x^2$

$f(x,y)=1-2xy$ not sure how to show this...

9. The region beneath $z=4x^2+9y^2$ and above the rectangle with vertices (0, 0), (3, 0), (3, 2), (0, 2) in the xy-plane.

$\int_0^3\int_0^2 (4x^2+9y^2)\,dy\,dx$

$= \int_0^3 \left[(4x^2y+3y^3)\Big|_0^2\right]dx$

$= \int_0^3 (8x^2+24-0)\,dx$

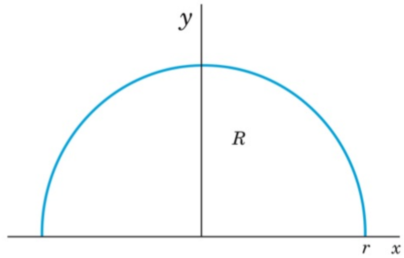
$= \frac{8}{3}x^3+24x\Big|_0^3$

$= 72+24\times3 = 144$

12–16 CENTER OF GRAVITY

Find the center of gravity $(\bar{x}, \bar{y})$ of a mass of density $f(x, y) = 1$ in the given region R.

15.



$$M = \iint_R f(x,y)\,dx\,dy = \int_0^{\pi}\int_0^{r} r\,dr\,d\theta = \int_0^{\pi} \frac{r^2}{2}\,d\theta = \frac{1}{2}\pi r^2$$

$$\bar{x} = \frac{1}{M}\iint_R x f(x,y)\,dx\,dy = 0,$$ for reasons of symmetry.

$$\bar{y} = \frac{1}{M}\iint_R y f(x,y)\,dx\,dy = \frac{2}{\pi r^2}\int_0^{\pi}\int_0^{r} r\sin\theta\, r\,dr\,d\theta$$

$$= \frac{2}{\pi r^2}\int_0^{\pi}\left(\sin\theta \cdot \frac{r^3}{3}\Big|_0^{r}\right)d\theta$$

$$= \frac{2}{\pi r^2}\int_0^{\pi}\frac{r^3}{3}\cdot\sin\theta\,d\theta$$

$$= \frac{2}{\pi r^2}\cdot\frac{r^3}{3}\left(-\cos\theta\Big|_0^{\pi}\right)$$

$$= \frac{4r}{3\pi}$$

Selected Problem set 10.4

10.4 3. 9. 17

1–10 LINE INTEGRALS: EVALUATION BY GREEN'S THEOREM

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R by Green's theorem, where

1. $\mathbf{F} = [y, -x]$, C the circle $x^2 + y^2 = 1/4$
2. $\mathbf{F} = [6y^2, 2x - 2y^4]$, R the square with vertices $\pm(2, 2)$, $\pm(2, -2)$
3. $\mathbf{F} = [x^2e^y, y^2e^x]$, R the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 3)$, $(0, 3)$

3. $$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \int_0^3 \int_0^2 (y^2e^x - x^2e^y)\, dx\,dy$$

$$= \int_0^3 \left(y^2e^x - \frac{x^3}{3}e^y\right)\Big|_{x=0}^{x=2} dy$$

$$= \int_0^3 \left[\left(y^2e^2 - \frac{8}{3}e^y\right) - (y^2 - 0)\right] dy$$

$$= \int_0^3 \left(y^2e^2 - \frac{8}{3}e^y - y^2\right) dy$$

$$= \frac{y^3}{3}(e^2-1) - \frac{8}{3}e^y \Big|_{y=0}^{y=3}$$

$$= \left[9(e^2-1) - \frac{8}{3}e^3\right] - \left(0 - \frac{8}{3}\cdot 1\right)$$

$$= 9(e^2-1) - \frac{8}{3}(e^3-1)$$

$$= -\frac{8}{3}e^3 + 9e^2 - \frac{19}{3}$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C F_1dx + F_2dy$$

$$= \oint_C x^2e^y \cdot dx + \oint_C y^2e^x dy$$

$$\oint_C x^2e^y \cdot dx = \int_0^2 x^2 \cdot e^0 dx - \int_0^2 x^2 \cdot e^3 dx$$

$$= \frac{x^3}{3}\Big|_0^2 - e^3 \cdot \frac{x^3}{3}\Big|_0^2$$

$$= (1-e^3) \cdot \frac{8}{3}$$

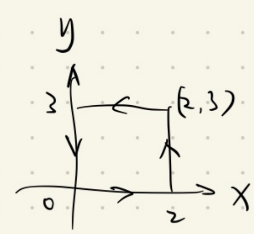
$$\oint_C y^2e^x dy = \int_0^3 y^2e^2 dy - \int_0^3 y^2 \cdot e^0 dy$$

$$= \frac{3^3}{3}(e^2-1)$$

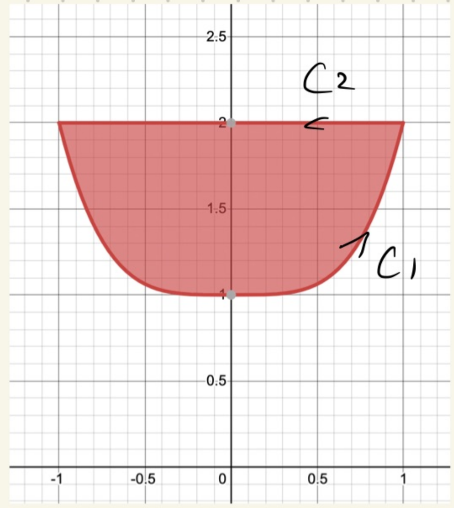
$$= 9(e^2-1)$$

$$\oint_C F_1dx + F_2dx$$

$$= (1-e^3) \cdot \frac{8}{3} + 9(e^2-1)$$



9. $\mathbf{F} = [e^{y/x}, e^{y} \ln x + 2x]$, R: $1 + x^{4} \leqq y \leqq 2$



$1+x^4 \le y \le 2 \qquad 1 \le y \le 2$

$1+x^4 \le 2 \qquad x^4 \le 1 \qquad -1 \le x \le 1$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{e^y}{x} + 2 - \frac{1}{x} \cdot e^{y/x}\right) dx\,dy$$

$$= \int_{-1}^{1} \int_{1+x^4}^{2} \left(\frac{e^y}{x} + 2 - \frac{e^{\frac{y}{x}}}{x}\right) \cdot dy\,dx$$

$$= \int_{-1}^{1} \left(\frac{e^y}{x} + 2y - e^{\frac{y}{x}}\right)\Big|_{y=1+x^4}^{y=2} dx$$

$$= \int_{-1}^{1} \left[\frac{e^2}{x} + 4 - e^{\frac{2}{x}} - \frac{e^{1+x^4}}{x} - 2(1+x^4) + e^{\frac{1+x^4}{x}}\right] dx = ?$$

$x \to 0$, ?

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C F_1 dx + F_2 dy$$

$$= \int_{-1}^{1} \int_{1+x^4}^{2} e^{\frac{y}{x}} \cdot dx + (e^y \ln x + 2x)\, dy + \int_{1}^{-1} e^{\frac{2}{x}} \cdot dx \qquad ?$$

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if $F = [e^y/x,\ e^y \ln x + 2x]$

$$\Rightarrow \int_{-1}^{1} \int_{1+x^4}^{2} \left(\frac{e^y}{x} + 2 - \frac{e^y}{x}\right) dy\,dx$$

$$= \int_{-1}^{1} \int_{1+x^4}^{2} 2\, dy\,dx$$

$$= \int_{1}^{1} \left(2y\Big|_{y=1+x^4}^{y=2}\right) dx$$

$$= \int_{-1}^{1} [4 - 2(1+x^4)]\, dx$$

$$= \int_{-1}^{1} (2 - 2x^4]\, dx$$

$$= 2x - \frac{2}{5}x^5 \Big|_{-1}^{1} = \frac{16}{5}$$

13–17 **INTEGRAL OF THE NORMAL DERIVATIVE**

Using (9), find the value of $\int_C \frac{\partial w}{\partial n}\, ds$ taken counterclockwise over the boundary C of the region R.

17. $w = x^3 - y^3, \quad 0 \leqq y \leqq x^2, \quad |x| \leqq 2$

(9) $$\iint_R \nabla^2 w \, dx\, dy = \oint_C \frac{\partial w}{\partial n}\, ds.$$

$$\oint_C \frac{\partial w}{\partial n} ds = \iint_R \nabla^2 w \, dx dy$$

$$= \int_{-2}^{2}\int_0^{x^2} (6x - 6y)\, dy dx$$

$$= \int_{-2}^{2} \left(6xy - 3y^2 \Big|_{y=0}^{y=x^2} \right) dx$$

$$= \int_{-2}^{2} \left[(6x \cdot x^2 - 3x^4) - (0-0) \right] dx$$

$$= 3 \int_{-2}^{2} (2x^3 - x^4)\, dx$$

$$= 3 \cdot \left(\frac{2}{4}x^4 - \frac{1}{5}x^5 \Big|_{-2}^{2} \right)$$

$$= 3 \left[\left(\frac{1}{2} \cdot 2^4 - \frac{1}{5} \cdot 2^5 \right) - \left(\frac{1}{2} \cdot 2^4 + \frac{1}{5} \cdot 2^5 \right) \right]$$

$$= 3 \cdot \left(-\frac{1}{5}\right) \cdot 2^6 = -\frac{192}{5}$$

Selected Problem set 10.5

10.5 5, 15

1–8 **PARAMETRIC SURFACE REPRESENTATION**

Familiarize yourself with parametric representations of important surfaces by deriving a representation (1), by finding the **parameter curves** (curves u = const and v = const) of the surface and a normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ of the surface. Show the details of your work.

5. Paraboloid of revolution $\mathbf{r}(u, v) = [u \cos v, \quad u \sin v, u^2]$

$x^2 + y^2 = z$

u = constant => z = constant => circle.

v = constant => parabola (half)

$r(u,v) = [u\cos v, u\sin v, u^2]$

$r_u = [\cos v, \sin v, 2u]$

$r_v = [-u\sin v, u\cos v, 0]$

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= -2u^2\cos v \cdot i - 2u^2\sin v\, j + u\, k$$

14–19 DERIVE A PARAMETRIC REPRESENTATION

Find a normal vector. The answer gives *one* representation; there are many. Sketch the surface and parameter curves.

14. Plane $4x + 3y + 2z = 12$

15. Cylinder of revolution $(x - 2)^2 + (y + 1)^2 = 25$

15. Centered Cylinder.

The circular cylinder $x^2 + y^2 = a^2$, $-1 \leqq z \leqq 1$, has radius a, height 2, and the z-axis as axis. A parametric representation is

$$\mathbf{r}(u, v) = [a \cos u, a \sin u, v] = a \cos u\,\mathbf{i} + a \sin u\,\mathbf{j} + v\mathbf{k} \qquad \text{(Fig. 242).}$$

So $r(u,v)$ of problem 15 is:

$r(u,v) = [5\cos u + 2,\ 5\sin u - 1,\ v]$

v is constant: Circle, center at $(2,-1)$, $r=5$

u is constant: line. Parallel to z-axis.

$r_u = [-5\sin u,\ 5\cos u,\ 0]$

$r_v = [0,\ 0,\ 1]$

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ -5\sin u & 5\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 5\cos u\, i + 5\sin u\, j$$