Chapter 13 - Complex Numbers and Functions. Complex Differentiation

Selected Problem set 13.1

13.1   1. 5. 11. 17

1. **Powers of *i*.** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\quad x_1 y_2 + x_2 y_1).$$

$i^2 = (0,1)(0,1) = (0-1,\ 0+0) = (-1,0) = -1 + 0i = -1$

$i^3 = i^2 \cdot i = -1 \cdot i = (-1,0)(0,1) = (0-0,\ -1+0) = (0,-1) = -i$

$i^4 = (i^2)(i^2) = -1 \cdot -1 = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i \cdot (-i) = -i^2 = 1 \Rightarrow 1/i = -i$.

$1/i^2 = 1/-1 = -1$.

$1/i^3 = 1/i^2 \cdot \frac{1}{i} = (-i)\cdot(-1) = i$.

5. **Pure imaginary number.** Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.

$\bar{z} = x - iy = -z = -x - iy$.

$x = -x \Rightarrow x = 0 \Rightarrow z = iy$

z is a pure imaginary number

8–15 **COMPLEX ARITHMETIC**

Let $z_1 = -2 + 11i$, $z_2 = 2 - i$. Showing the details of your work, find, in the form $x + iy$:

8. $z_1 z_2$, $\overline{(z_1 z_2)}$ **9.** $\text{Re}\,(z_1^2)$, $(\text{Re}\,z_1)^2$

10. $\text{Re}\,(1/z_2^2)$, $1/\text{Re}\,(z_2^2)$

11. $(z_1 - z_2)^2/16$, $(z_1/4 - z_2/4)^2$

$$11.\ (z_1 - z_2)^2/16 = (-4+12i)^2/16 = (16-144-96i)/16$$

$$= (-128-96i)/16 = -8-6i$$

$$(z_1/4 - z_2/4)^2 = \left[\left(-\tfrac{1}{2}+\tfrac{11}{4}i\right) - \left(\tfrac{1}{2}-\tfrac{1}{4}i\right)\right]^2$$

$$= (-1+3i)^2 = 1-9-6i = -8-6i$$

16–20 Let $z = x + iy$. Showing details, find, in terms of x and y:

16. $\text{Im}\,(1/z)$, $\text{Im}\,(1/z^2)$ **17.** $\text{Re}\,z^4 - (\text{Re}\,z^2)^2$

$$17.\ z^2 = (x^2-y^2) + 2xy\,i$$

$$z^4 = [(x^2-y^2)^2 - 4x^2y^2] + 4xy(x^2-y^2)\,i$$

$$\text{Re}\,z^4 = (x^2-y^2)^2 - 4x^2y^2$$

$$(\text{Re}\,z^2)^2 = (x^2-y^2)^2$$

$$\text{Re}\,z^4 - (\text{Re}\,z^2)^2 = -4x^2y^2$$

Selected Problem set 13.2

13.2 1. 3. 7. 11. 21. 29

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$
2. $-4 + 4i$
3. $2i, \quad -2i$
4. -5
5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$
6. $\dfrac{\sqrt{3} - 10i}{-\frac{1}{2}\sqrt{3} + 5i}$
7. $1 + \frac{1}{2}\pi i$
8. $\dfrac{-4 + 19i}{2 + 5i}$

1. $r = \sqrt{1+1} = \sqrt{2}$

$\theta = \arctan \frac{1}{1} = \frac{\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

3. $2i$: $r = 2$.

$\theta = \frac{\pi}{2}$

$2i = 2\cos\frac{\pi}{2} + 2i\sin\frac{\pi}{2}$

$-2i$: $r = 2$

$\theta = -\frac{\pi}{2}$

$-2i = 2\cos(-\frac{\pi}{2}) + 2i\sin(-\frac{\pi}{2})$

7. $r = \sqrt{1 + (\frac{\pi}{2})^2} = \sqrt{1 + \frac{\pi^2}{4}}$

$\theta = \arctan \dfrac{\frac{1}{2}\pi}{1} = \arctan\frac{\pi}{2}$

$1 + \frac{\pi}{2}i = \sqrt{1+\frac{\pi^2}{4}}\left(\cos(\arctan\frac{\pi}{2}) + i\sin(\arctan\frac{\pi}{2})\right)$

9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $-1 + i$ **10.** $-5, \quad -5 - i, \quad -5 + i$

11. $3 \pm 4i$ **12.** $-\pi - \pi i$

13. $(1 + i)^{20}$ **14.** $-1 + 0.1i, \quad -1 - 0.1i$

11. $r = \sqrt{3^2 + (\pm 4)^2} = 5$

$\operatorname{Arg} z = \arctan \frac{\pm 4}{3}$

21–27 ROOTS

Find and graph all roots in the complex plane.

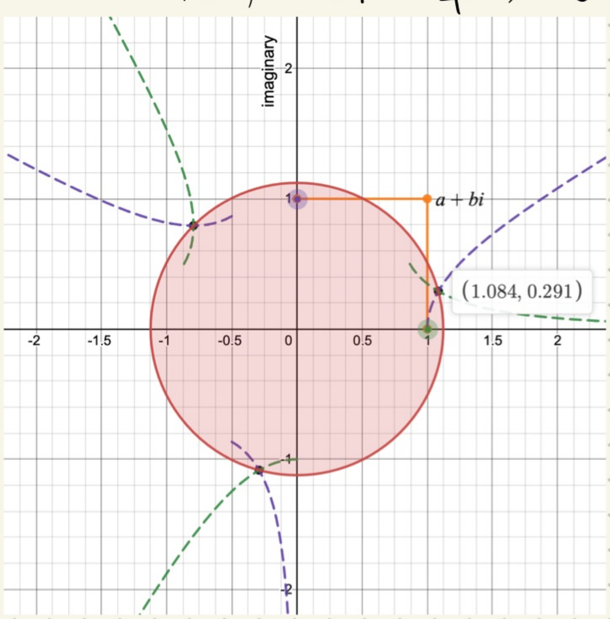
21. $\sqrt[3]{1 + i}$ **22.** $\sqrt[3]{3 + 4i}$

21. $r = \sqrt{1+1} = \sqrt{2}$.

$\operatorname{Arg} z = \arctan \frac{1}{1} = \frac{\pi}{4}$.

$\sqrt[3]{1+i} = \sqrt[6]{2}\left(\cos \frac{\frac{\pi}{4}+2k\pi}{3} + i \sin \frac{\frac{\pi}{4}+2k\pi}{3}\right)$.

$\theta = \frac{\pi}{12}, \quad \theta_1 = \frac{3}{4}\pi, \quad \theta_2 = \frac{17}{12}\pi$



28–31 EQUATIONS

Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$

29. $z^2 + z + 1 - i = 0$

29. $z = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (1-i)}}{2}$

$z_1 = i \quad z_2 = -1 - i$