W11

Two-Factor, K_ij = 1

Concept

- Each of pair: Treatment
- Row mean: x_i. hat
- Column mean, grand mean

$\bar{X}_{i\cdot}$ = the average of measurements obtained when factor A is held at level i $= \dfrac{\sum_{j=1}^{J} X_{ij}}{J}$

$\bar{X}_{\cdot j}$ = the average of measurements obtained when factor B is held at level j $= \dfrac{\sum_{i=1}^{I} X_{ij}}{I}$

$\bar{X}_{\cdot\cdot}$ = the grand mean $= \dfrac{\sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij}}{IJ}$

Fixed Effects Model

$$X_{ij} = \mu_{ij} + \epsilon_{ij}$$

Assume the existence of I parameters $\alpha_1, \alpha_2, \ldots, \alpha_I$ and J parameters $\beta_1, \beta_2, \ldots, \beta_J$, such that

$$X_{ij} = \alpha_i + \beta_j + \epsilon_{ij} \quad (i = 1, \ldots, I, \quad j = 1, \ldots, J) \qquad (11.1)$$

so that

$$\mu_{ij} = \alpha_i + \beta_j \qquad (11.2)$$

Additive Model

$$\mu_{ij} - \mu_{i'j} = (\alpha_i + \beta_j) - (\alpha_{i'} + \beta_j) = \alpha_i - \alpha_{i'}$$

$$X_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} \qquad (11.3)$$

where $\sum_{i=1}^{I} \alpha_i = 0$, $\sum_{j=1}^{J} \beta_j = 0$, and the ϵ_{ij}'s are assumed independent, normally distributed, with mean 0 and common variance σ^2.

H_{0A}: $\alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$ versus H_{aA}: at least one $\alpha_i \neq 0$

H_{0B}: $\beta_1 = \beta_2 = \cdots = \beta_J = 0$ versus H_{aB}: at least one $\beta_j \neq 0$

Test Procedure

$$SST = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{\cdot\cdot})^2 \qquad df = IJ - 1$$

$$SSA = \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 = J\sum_{i=1}^{I}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 \qquad df = I - 1$$

$$SSB = \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})^2 = I\sum_{j=1}^{J}(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})^2 \qquad df = J - 1 \qquad (11.5)$$

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})^2 \qquad df = (I-1)(J-1)$$

The fundamental identity is

$$SST = SSA + SSB + SSE \qquad (11.6)$$

Hypotheses	Test Statistic Value	Rejection Region
H_{0A} versus H_{aA}	$f_A = \dfrac{MSA}{MSE}$	$f_A \geq F_{\alpha, I-1, (I-1)(J-1)}$
H_{0B} versus H_{aB}	$f_B = \dfrac{MSB}{MSE}$	$f_B \geq F_{\alpha, J-1, (I-1)(J-1)}$

Table 11.1 ANOVA Table for Example 11.3

Source of Variation	df	Sum of Squares	Mean Square	f
Factor A (brand)	$I - 1 = 2$	SSA = .1282	MSA = .0641	$f_A = 4.43$
Factor B (wash treatment)	$J - 1 = 3$	SSB = .4797	MSB = .1599	$f_B = 11.05$
Error	$(I-1)(J-1) = 6$	SSE = .0868	MSE = .01447	
Total	$IJ - 1 = 11$	SST = .6947		

Expected Mean Squares

For additive model

$$E(MSE) = \sigma^2$$

$$E(MSA) = \sigma^2 + \frac{J}{I-1}\sum_{i=1}^{I}\alpha_i^2$$

$$E(MSB) = \sigma^2 + \frac{I}{J-1}\sum_{j=1}^{J}\beta_j^2$$

Multiply Comparison

After rejecting either H_{0A} or H_{0B}, Tukey's procedure can be used to identify significant differences between the levels of the factor under investigation.

1. For comparing levels of factor A, obtain $Q_{\alpha, I, (I-1)(J-1)}$.
   For comparing levels of factor B, obtain $Q_{\alpha, J, (I-1)(J-1)}$.
2. Compute

$$w = Q \cdot (\text{estimated standard deviation of the sample means being compared})$$

$$= \begin{cases} Q_{\alpha, I, (I-1)(J-1)} \cdot \sqrt{MSE/J} & \text{for factor } A \text{ comparisons} \\ Q_{\alpha, J, (I-1)(J-1)} \cdot \sqrt{MSE/I} & \text{for factor } B \text{ comparisons} \end{cases}$$

   (because, e.g., the standard deviation of $\bar{X}_{i\cdot}$ is $\sigma/\sqrt{J}$).
3. Arrange the sample means in increasing order, underscore those pairs differing by less than w, and identify pairs not underscored by the same line as corresponding to significantly different levels of the given factor.

Randomized Block Design

P427 Example 11.5, Great example

Models with Random and mixed effects

Three-Factor

Fixed Effects Models

The fixed effects model for three-factor ANOVA with $L_{ijk} = L$ is

$$X_{ijkl} = \mu_{ijk} + \epsilon_{ijkl} \quad i = 1, \ldots, I, \quad j = 1, \ldots, J, \quad k = 1, \ldots, K, \quad l = 1, \ldots, L \qquad (11.12)$$

where the ϵ_{ijkl}'s are normally distributed with mean 0 and variance σ^2, and

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC} + \gamma_{ijk} \qquad (11.13)$$

Relevant sums of squares are

$$SST = \sum_i\sum_j\sum_k\sum_l (X_{ijkl} - \bar{X}_{\cdots\cdot})^2 \qquad df = IJKL - 1$$

$$SSA = \sum_i\sum_j\sum_k\sum_l \hat{\alpha}_i^2 = JKL\sum_i(\bar{X}_{i\cdots} - \bar{X}_{\cdots\cdot})^2 \qquad df = I - 1$$

$$SSAB = \sum_i\sum_j\sum_k\sum_l (\hat{\gamma}_{ij}^{AB})^2 \qquad df = (I-1)(J-1)$$

$$= KL\sum_i\sum_j(\bar{X}_{ij\cdot\cdot} - \bar{X}_{i\cdots} - \bar{X}_{\cdot j\cdot\cdot} + \bar{X}_{\cdots\cdot})^2$$

$$SSABC = \sum_i\sum_j\sum_k\sum_l \hat{\gamma}_{ijk}^2 = L\sum_i\sum_j\sum_k \hat{\gamma}_{ijk}^2 \qquad df = (I-1)(J-1)(K-1)$$

$$SSE = \sum_i\sum_j\sum_k\sum_l (X_{ijkl} - \bar{X}_{ijk\cdot})^2 \qquad df = IJK(L-1)$$

with the remaining main effect and two-factor interaction sums of squares obtained by symmetry. SST is the sum of the other eight SSs.

$$E(MSE) = \sigma^2$$

$$E(MSA) = \sigma^2 + \frac{JKL}{I-1}\sum_i \alpha_i^2$$

$$E(MSAB) = \sigma^2 + \frac{KL}{(I-1)(J-1)}\sum_i\sum_j(\gamma_{ij}^{AB})^2$$

$$E(MSABC) = \sigma^2 + \frac{L}{(I-1)(J-1)(K-1)}\sum_i\sum_j\sum_k(\gamma_{ijk})^2$$

Test Procedures

Null Hypothesis	Test Statistic Value	Rejection Region
H_{0A}: all α_i's $= 0$	$f_A = \dfrac{MSA}{MSE}$	$f_A \geq F_{\alpha, I-1, IJK(L-1)}$
H_{0AB}: all γ_{ij}^{AB}'s $= 0$	$f_{AB} = \dfrac{MSAB}{MSE}$	$f_{AB} \geq F_{\alpha, (I-1)(J-1), IJK(L-1)}$
H_{0ABC}: all γ_{ijk}'s $= 0$	$f_{ABC} = \dfrac{MSABC}{MSE}$	$f_{ABC} \geq F_{\alpha, (I-1)(J-1)(K-1), IJK(L-1)}$

Latin Square Experiment

2^p Factorial Experiements

Table 11.10 Signs for Computing Effect Contrasts

Experimental Condition	Cell Total	A	B	C	AB	AC	BC	ABC
		Factorial Effect						
(1)	$x_{111\cdot}$	−	−	−	+	+	+	−
a	$x_{211\cdot}$	+	−	−	−	−	+	+
b	$x_{121\cdot}$	−	+	−	−	+	−	+
ab	$x_{221\cdot}$	+	+	−	+	−	−	−
c	$x_{112\cdot}$	−	−	+	+	−	−	+
ac	$x_{212\cdot}$	+	−	+	−	+	−	−
bc	$x_{122\cdot}$	−	+	+	−	−	+	−
abc	$x_{222\cdot}$	+	+	+	+	+	+	+

$$AC \text{ contrast} = +x_{111\cdot} - x_{211\cdot} + x_{121\cdot} - x_{221\cdot} - x_{112\cdot} + x_{212\cdot} - x_{122\cdot} + x_{222\cdot}$$

$$SS(\text{effect}) = \frac{(\text{effect contrast})^2}{8n}$$

Table 11.11 Yates's Method of Computation

Treatment Condition	$x_{ijk\cdot}$	1	2	Effect Contrast	$SS = (\text{contrast})^2/16$
$(1) = x_{111\cdot}$	884	2233	4771	9735	
$a = x_{211\cdot}$	1349	2538	4964	2009	252,255.06
$b = x_{121\cdot}$	1037	2294	929	681	28,985.06
$ab = x_{221\cdot}$	1501	2670	1080	−233	3,393.06
$c = x_{112\cdot}$	819	465	305	193	2,328.06
$ac = x_{212\cdot}$	1475	464	376	151	1,425.06
$bc = x_{122\cdot}$	1123	656	−1	71	315.06
$abc = x_{222\cdot}$	1547	424	−232	−231	3,335.06
					292,036.42

Yates's method

Confounding

Frictional Replication

Anova: Two-Factor With Replication

Excel

Two-Factor, K_ij > 1

Fixed Effects Parameters and Hypotheses

$$\mu = \frac{1}{IJ}\sum_i\sum_j \mu_{ij} \qquad \mu_{i\cdot} = \frac{1}{J}\sum_j \mu_{ij} \qquad \mu_{\cdot j} = \frac{1}{I}\sum_i \mu_{ij}$$

$\alpha_i = \mu_{i\cdot} - \mu$ = the effect of factor A at level i

$\beta_j = \mu_{\cdot j} - \mu$ = the effect of factor B at level j (11.8)

$\gamma_{ij} = \mu_{ij} - (\mu + \alpha_i + \beta_j)$ = interaction between factor A at level i and factor B at level j

from which

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} \qquad (11.9)$$

H_{0AB}: $\gamma_{ij} = 0$ for all i, j versus H_{aAB}: at least one $\gamma_{ij} \neq 0$

H_{0A}: $\alpha_1 = \cdots = \alpha_I = 0$ versus H_{aA}: at least one $\alpha_i \neq 0$

H_{0B}: $\beta_1 = \cdots = \beta_J = 0$ versus H_{aB}: at least one $\beta_j \neq 0$

$$SST = \sum_i\sum_j\sum_k(X_{ijk} - \bar{X}_{\cdots})^2 \qquad df = IJK - 1$$

$$SSE = \sum_i\sum_j\sum_k(X_{ijk} - \bar{X}_{ij\cdot})^2 \qquad df = IJ(K-1)$$

$$SSA = \sum_i\sum_j\sum_k(\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdots})^2 \qquad df = I - 1$$

$$SSB = \sum_i\sum_j\sum_k(\bar{X}_{\cdot j\cdot} - \bar{X}_{\cdots})^2 \qquad df = J - 1$$

$$SSAB = \sum_i\sum_j\sum_k(\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdots})^2 \qquad df = (I-1)(J-1)$$

The fundamental identity is

$$SST = SSA + SSB + SSAB + SSE$$

SSAB is referred to as **interaction sum of squares.**

MS = SS/df

Test Procedure

$$E(MSE) = \sigma^2$$

$$E(MSA) = \sigma^2 + \frac{JK}{I-1}\sum_{i=1}^{I}\alpha_i^2 \qquad E(MSB) = \sigma^2 + \frac{IK}{J-1}\sum_{j=1}^{J}\beta_j^2$$

$$E(MSAB) = \sigma^2 + \frac{K}{(I-1)(J-1)}\sum_{i=1}^{I}\sum_{j=1}^{J}\gamma_{ij}^2$$

Hypotheses	Test Statistic Value	Rejection Region
H_{0A} versus H_{aA}	$f_A = \dfrac{MSA}{MSE}$	$f_A \geq F_{\alpha, I-1, IJ(K-1)}$
H_{0B} versus H_{aB}	$f_B = \dfrac{MSB}{MSE}$	$f_B \geq F_{\alpha, J-1, IJ(K-1)}$
H_{0AB} versus H_{aAB}	$f_{AB} = \dfrac{MSAB}{MSE}$	$f_{AB} \geq F_{\alpha, (I-1)(J-1), IJ(K-1)}$

Multiple Comparison

When the no-interaction hypothesis H_{0AB} is not rejected and at least one of the two main effect null hypotheses is rejected, Tukey's method can be used to identify significant differences in levels. For identifying differences among the α_i's when H_{0A} is rejected,

1. Obtain $Q_{\alpha, I, IJ(K-1)}$, where the second subscript I identifies the number of levels being compared and the third subscript refers to the number of degrees of freedom for error.
2. Compute $w = Q\sqrt{MSE/(JK)}$, where JK is the number of observations averaged to obtain each of the $\bar{x}_{i\cdot\cdot}$'s compared in Step 3.
3. Order the $\bar{x}_{i\cdot\cdot}$'s from smallest to largest and, as before, underscore all pairs that differ by less than w. Pairs not underscored correspond to significantly different levels of factor A.

Model with Mixed and Random effects

The **mixed effects model** when factor A is fixed and factor B is random is

$$X_{ijk} = \mu + \alpha_i + B_j + G_{ij} + \epsilon_{ijk}$$

$$i = 1, \ldots, I, \quad j = 1, \ldots, J, \quad k = 1, \ldots, K$$

H_{0A}: $\alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$ versus H_{aA}: at least one $\alpha_i \neq 0$

H_{0B}: $\sigma_B^2 = 0$ versus H_{aB}: $\sigma_B^2 > 0$

H_{0G}: $\sigma_G^2 = 0$ versus H_{aG}: $\sigma_G^2 > 0$

$$E(MSE) = \sigma^2$$

$$E(MSA) = \sigma^2 + K\sigma_G^2 + \frac{JK}{I-1}\sum \alpha_i^2$$

$$E(MSB) = \sigma^2 + K\sigma_G^2 + IK\sigma_B^2$$

$$E(MSAB) = \sigma^2 + K\sigma_G^2$$

For testing H_{0A} versus H_{aA} (A fixed, B random), the test statistic value is $f_A = MSA/MSAB$, and the rejection region is $f_A \geq F_{\alpha, I-1, (I-1)(J-1)}$. The test of H_{0B} versus H_{aB} utilizes $f_B = MSB/MSAB$, with rejection region $f_B \geq F_{\alpha, J-1, (I-1)(J-1)}$.