

# **Chapter 9 Vector Differential Calculus. Grad, Div, Curl**

## **P360 - Problem set 9.1**

Problem 1-5

# PS 9.1

## 1-5 COMPONENTS AND LENGTH

Find the components of the vector  $\mathbf{v}$  with initial point  $P$  and terminal point  $Q$ . Find  $|\mathbf{v}|$ . Sketch  $|\mathbf{v}|$ . Find the unit vector  $\mathbf{u}$  in the direction of  $\mathbf{v}$ .

1.  $P: (1, 1, 0)$ ,  $Q: (6, 2, 0)$
2.  $P: (1, 1, 1)$ ,  $Q: (2, 2, 0)$
3.  $P: (-3.0, 4.0, -0.5)$ ,  $Q: (5.5, 0, 1.2)$
4.  $P: (1, 4, 2)$ ,  $Q: (-1, -4, -2)$
5.  $P: (0, 0, 0)$ ,  $Q: (2, 1, -2)$

$$4) \vec{V} = [-2, -8, -4]$$

$$|\mathbf{V}| = \sqrt{4 + 64 + 16} \\ = \sqrt{84}$$

$$\vec{U} = \left[ -\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}} \right]$$

$$1. \vec{V} = [5, 1, 0]$$

$$|\mathbf{V}| = \sqrt{26}$$

$$\vec{U} = \left[ \frac{5\sqrt{26}}{26}, \frac{\sqrt{26}}{26}, 0 \right]$$

$$2. \vec{V} = [1, 1, -1]$$

$$|\mathbf{V}| = \sqrt{3}$$

$$\vec{U} = \left[ \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right]$$

$$3. \vec{V} = [8.5, -4, 1.7]$$

$$|\mathbf{V}| = \sqrt{72.25 + 16 + 2.89} \\ = \sqrt{91.14}$$

$$\vec{U} = \left[ \frac{8.5}{\sqrt{91.14}}, \frac{-4}{\sqrt{91.14}}, \frac{1.7}{\sqrt{91.14}} \right]$$

$$5) \vec{V} = [2, 1, -2]$$

$$|\mathbf{V}| = \sqrt{4 + 1 + 4} = 3$$

$$\vec{U} = \left[ \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right]$$

**6–10** Find the terminal point  $Q$  of the vector  $\mathbf{v}$  with components as given and initial point  $P$ . Find  $|\mathbf{v}|$ .

6.  $4, 0, 0$ ;  $P: (0, 2, 13)$
7.  $\frac{1}{2}, 3, -\frac{1}{4}$ ;  $P: (\frac{7}{2}, -3, \frac{3}{4})$
8.  $13.1, 0.8, -2.0$ ;  $P: (0, 0, 0)$
9.  $6, 1, -4$ ;  $P: (-6, -1, -4)$
10.  $0, -3, 3$ ;  $P: (0, 3, -3)$

Latex

Mistake

**11–18 ADDITION, SCALAR MULTIPLICATION**

Let  $\mathbf{a} = [3, 2, 0] = 3\mathbf{i} + 2\mathbf{j}$ ;  $\mathbf{b} = [-4, 6, 0] = 4\mathbf{i} + 6\mathbf{j}$ ,  
 $\mathbf{c} = [5, -1, 8] = 5\mathbf{i} - \mathbf{j} + 8\mathbf{k}$ ,  $\mathbf{d} = [0, 0, 4] = 4\mathbf{k}$ .

Find:

6.  $Q=[4, 2, 13]$ ,  $|\mathbf{v}|=4$
7.  $Q=[4, 0, 1/2]$ ,  $|\mathbf{v}|=\sqrt{149}/4$

$$8. Q=[13.1, 0.8, -2.0], |v|=\sqrt{171.61 + 0.64 + 4}=\sqrt{176.25}$$

$$9. Q=[0, 0, 0], |v|=\sqrt{53}$$

$$10. Q=[0, 0, 0], |v|=3\sqrt{2}$$

$$11. 2a=[6, 4, 0], 1/2a=[3/2, 1, 0], -a = [-3, -2, 0]$$

$$12. (a+b)+c=a+(b+c)=[4, 7, 8] \text{ } b \text{ is not consistant.}$$

$$13. b+c=c+b = [1, 5, 8]$$

$$14. 3c-6d=3(c-2d)=[15, -3, 0]$$

$$15. 7(c-b)=7c-7b=7*[9, -7, 8]=[63, -49, 56]$$

$$16. \frac{9}{2}a - 3c=9(\frac{1}{2}a - \frac{1}{3}c)=[-3/2, 12, -24]$$

$$17. (7-3)a=7a-3a=4a=[12, 8, 0]$$

$$18. 4a+3b = [0, 26, 0], -4a-3b=-(4a+3b)=[0, -26, 0]$$

19. 12-associative, 13-commutative, 14-16 scalar multiplication is distributive.

$$20. a + b = [a_1, a_2, \dots, a_n] + [b_1, b_2, \dots, b_n] = [a_1 + b_1, a_2 + b_2, \dots, a_n + b_n] = [b_1 + a_1, b_2 + a_2, \dots, b_n + a_n] = b + a$$

The rest can be approved in a simliar way.

**21-25 FORCES, RESULTANT**

Find the resultant in terms of components and its magnitude.

21.  $\mathbf{p} = [2, 3, 0]$ ,  $\mathbf{q} = [0, 6, 1]$ ,  $\mathbf{u} = [2, 0, -4]$
22.  $\mathbf{p} = [1, -2, 3]$ ,  $\mathbf{q} = [3, 21, -16]$ ,  
 $\mathbf{u} = [-4, -19, 13]$
23.  $\mathbf{u} = [8, -1, 0]$ ,  $\mathbf{v} = [\frac{1}{2}, 0, \frac{4}{3}]$ ,  $\mathbf{w} = [-\frac{17}{2}, 1, \frac{11}{3}]$
24.  $\mathbf{p} = [-1, 2, -3]$ ,  $\mathbf{q} = [1, 1, 1]$ ,  $\mathbf{u} = [1, -2, 2]$
25.  $\mathbf{u} = [3, 1, -6]$ ,  $\mathbf{v} = [0, 2, 5]$ ,  $\mathbf{w} = [3, -1, -13]$

$$21) \vec{r} = \vec{p} + \vec{q} + \vec{u} \\ = [4, 9, -3]$$

$$|\vec{r}| = \sqrt{16+81+9} = \sqrt{106}$$

$$22) \vec{r} = \vec{p} + \vec{q} + \vec{u} \\ = [0, 0, 0]$$

$$|\vec{r}| = 0$$

$$23) \vec{r} = \vec{u} + \vec{v} + \vec{w} \\ = [0, 0, 5]$$

$$|\vec{r}| = 5$$

$$24) \vec{r} = \vec{p} + \vec{q} + \vec{u} \\ = [1, 1, 0]$$

$$|\vec{r}| = \sqrt{2}$$

$$25) \vec{r} = \vec{u} + \vec{v} + \vec{w}$$

$$= [6, 2, -14]$$

$$|\vec{r}| = \sqrt{36+4+196} = \sqrt{236}$$

**26-37 FORCES, VELOCITIES**

26. **Equilibrium.** Find  $\mathbf{v}$  such that  $\mathbf{p}, \mathbf{q}, \mathbf{u}$  in Prob. 21 and  $\mathbf{v}$  are in equilibrium.
27. Find  $\mathbf{p}$  such that  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in Prob. 23 and  $\mathbf{p}$  are in equilibrium.
28. **Unit vector.** Find the unit vector in the direction of the resultant in Prob. 24.

$$26) \vec{v} = 0 - \vec{r} = [-4, -9, 3]$$

$$27) \vec{p} = 0 - \vec{r} = [0, 0, -5]$$

$$28) \vec{u} = \frac{\vec{r}}{|\vec{r}|} = \left[ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right]$$

29. **Restricted resultant.** Find all  $\mathbf{v}$  such that the resultant of  $\mathbf{v}, \mathbf{p}, \mathbf{q}, \mathbf{u}$  with  $\mathbf{p}, \mathbf{q}, \mathbf{u}$  as in Prob. 21 is parallel to the  $xy$ -plane.
30. Find  $\mathbf{v}$  such that the resultant of  $\mathbf{p}, \mathbf{q}, \mathbf{u}, \mathbf{v}$  with  $\mathbf{p}, \mathbf{q}, \mathbf{u}$  as in Prob. 24 has no components in  $x$ - and  $y$ -directions.
31. For what  $k$  is the resultant of  $[2, 0, -7], [1, 2, -3]$ , and  $[0, 3, k]$  parallel to the  $xy$ -plane?
32. If  $|\mathbf{p}| = 6$  and  $|\mathbf{q}| = 4$ , what can you say about the magnitude and direction of the resultant? Can you think of an application to robotics?
33. Same question as in Prob. 32 if  $|\mathbf{p}| = 9$ ,  $|\mathbf{q}| = 6$ ,  $|\mathbf{u}| = 3$ .

29 Vector start from origin.  
parallel to  $x$ - $y$  means no component in  $z$ -direction.

Let result

$$\vec{z} = \vec{v} + \vec{p} + \vec{q} + \vec{u} = [x, y, 0]$$

So  $x$  and  $y$  are arbitrary

$$\vec{v} = \vec{z} - [4, 9, -3]$$

$$= [x-4, y-9, 3]$$

$$\text{Let } V_1 = x-4, V_2 = y-9.$$

$$\text{So } \vec{v} = [V_1, V_2, 3]$$

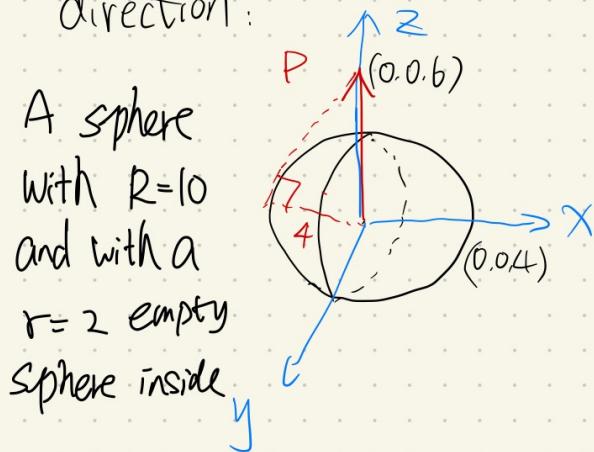
where  $V_1$  and  $V_2$  are arbitrary

$$30. \vec{v} = [0, 0, c] - [1, 1, 0]$$

$$= [-1, -1, c] \text{ where } c \text{ is a constant.}$$

$$31 \quad -7 - 3 + k = 0 \\ k = 10$$

32. magnitude, min=2, max=10  
direction:



Robotic Vision, the area  
where Robot can see/detect  
or scan?

33. magnitude, min=0, max=18  
direction

enhanced ice cream  
without blind spot?

A solid sphere with  
 $R=18$ , no blind point.

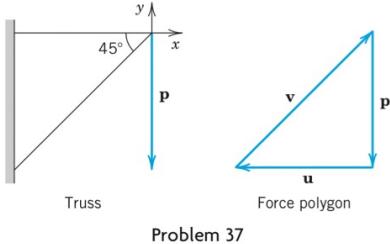
$$|\mathbf{p} + \mathbf{q} + \mathbf{u}| \leq 18$$

34. **Relative velocity.** If airplanes  $A$  and  $B$  are moving southwest with speed  $|v_A| = 550$  mph, and northwest with speed  $|v_B| = 450$  mph, respectively, what is the relative velocity  $\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$  of  $B$  with respect to  $A$ ?

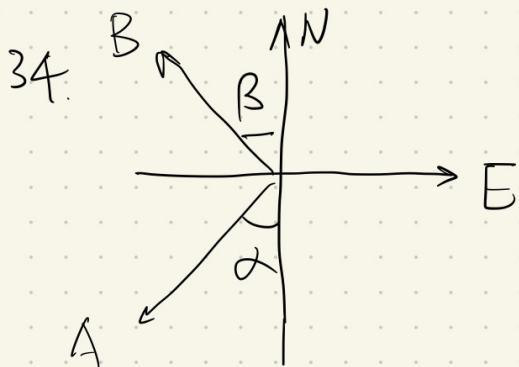
35. Same question as in Prob. 34 for two ships moving northeast with speed  $|\mathbf{v}_A| = 22$  knots and west with speed  $|\mathbf{v}_B| = 19$  knots.

36. **Reflection.** If a ray of light is reflected once in each of two mutually perpendicular mirrors, what can you say about the reflected ray?

37. **Force polygon.** Truss. Find the forces in the system of two rods (*truss*) in the figure, where  $|\mathbf{p}| = 1000$  nt. Hint. Forces in equilibrium form a polygon, the *force polygon*.

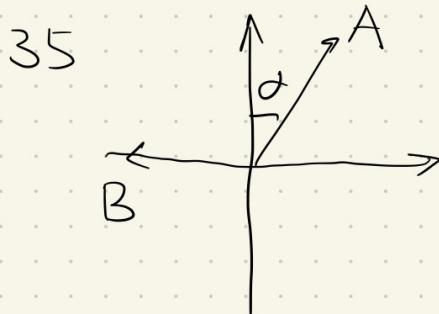


Problem 37



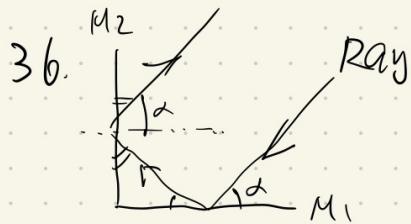
$$V = V_B - V_A \text{ on northeast.}$$

$$\begin{aligned} & [550 \sin \alpha - 450 \sin \beta, \\ & 550 \cos \alpha - 450 \cos \beta] \end{aligned}$$



$$V = V_B - V_A \text{ on northeast}$$

$$[22 \sin \alpha - 19, 22 \cos \alpha]$$



Reverse the direction.

$$37. P: [0, -1000]$$

$$U: [-1000, 0]$$

$$V: [1000, 1000]$$

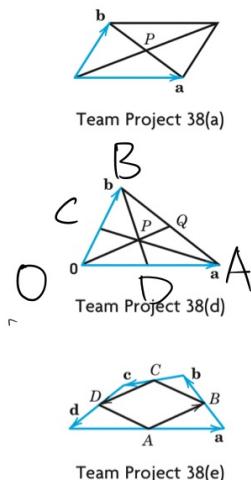
34, 35: it seems assume

$\alpha, \beta$  are  $45^\circ$

maybe this is by default

**38. TEAM PROJECT. Geometric Applications.** To increase your skill in dealing with vectors, use vectors to prove the following (see the figures).

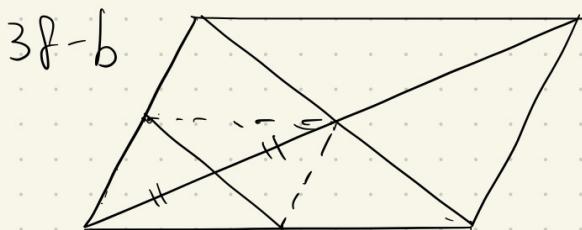
- The diagonals of a parallelogram bisect each other.
- The line through the midpoints of adjacent sides of a parallelogram bisects one of the diagonals in the ratio 1 : 3.
- Obtain (b) from (a).
- The three medians of a triangle (the segments from a vertex to the midpoint of the opposite side) meet at a single point, which divides the medians in the ratio 2 : 1.
- The quadrilateral whose vertices are the midpoints of the sides of an arbitrary quadrilateral is a parallelogram.
- The four space diagonals of a parallelepiped meet and bisect each other.
- The sum of the vectors drawn from the center of a regular polygon to its vertices is the zero vector.



$$\begin{aligned}\overrightarrow{OC} &= \frac{1}{2}\vec{b} \\ \overrightarrow{OB} &= \frac{1}{2}\vec{a} \\ \overrightarrow{DC} &= \frac{1}{2}\overrightarrow{AB} \quad \text{Vector Way?} \\ \triangle CDP &\sim \triangle ABP \\ \overrightarrow{DP} &= \frac{1}{2}\overrightarrow{PB}\end{aligned}$$

$$38-a \quad \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} = \frac{1}{2}(\vec{a} + \vec{b})$$

$$\frac{1}{2}\vec{a} - \frac{1}{2}\vec{b} = \frac{1}{2}(\vec{a} - \vec{b})$$



$$\frac{1}{4}\vec{a} - \frac{1}{4}\vec{b} = \frac{1}{4}(\vec{a} - \vec{b})$$

$$\frac{1}{4} : \frac{3}{4} = 1 : 3$$

38-c: Obvious.

$$38-d \quad \overrightarrow{AB} = \vec{a} - \vec{b}$$

$$\overrightarrow{AQ} = \frac{1}{2}(\vec{a} - \vec{b})$$

$$\overrightarrow{QO} = \vec{a} + \frac{1}{2}(\vec{a} - \vec{b})$$

$$\begin{aligned}38-e \quad \vec{a} - \vec{d} &= \vec{c} - \vec{b} \\ \frac{1}{2}(\vec{a} - \vec{d}) &= \frac{1}{2}(\vec{c} - \vec{b})\end{aligned}$$

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{BC} \\ \text{same} \Rightarrow \overrightarrow{AB} &= \overrightarrow{DC}\end{aligned}$$

38-f

38-g

# PS 9-2

## 1-10 INNER PRODUCT

Let  $\mathbf{a} = [1, -3, 5]$ ,  $\mathbf{b} = [4, 0, 8]$ ,  $\mathbf{c} = [-2, 9, 1]$ .  
Find:

1.  $\mathbf{a} \cdot \mathbf{b}$ ,  $\mathbf{b} \cdot \mathbf{a}$ ,  $\mathbf{b} \cdot \mathbf{c}$
2.  $(-3\mathbf{a} + 5\mathbf{c}) \cdot \mathbf{b}$ ,  $15(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b}$
3.  $|\mathbf{a}|$ ,  $|\mathbf{2b}|$ ,  $|-c|$
4.  $|\mathbf{a} + \mathbf{b}|$ ,  $|\mathbf{a}| + |\mathbf{b}|$
5.  $|\mathbf{b} + \mathbf{c}|$ ,  $|\mathbf{b}| + |\mathbf{c}|$
6.  $|\mathbf{a} + \mathbf{c}|^2 + |\mathbf{a} - \mathbf{c}|^2 - 2(|\mathbf{a}|^2 + |\mathbf{c}|^2)$
7.  $|\mathbf{a} \cdot \mathbf{c}|$ ,  $|\mathbf{a}||\mathbf{c}|$
8.  $5\mathbf{a} \cdot 13\mathbf{b}$ ,  $65\mathbf{a} \cdot \mathbf{b}$
9.  $15\mathbf{a} \cdot \mathbf{b} + 15\mathbf{a} \cdot \mathbf{c}$ ,  $15\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
10.  $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c})$ ,  $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c}$

$$5. |\mathbf{b} + \mathbf{c}| = \sqrt{2^2 + 9^2 + 9^2} = \sqrt{166}$$

$$|\mathbf{b}| + |\mathbf{c}| = 4\sqrt{5} + \sqrt{86}$$

$$6. (1+36+36) + (9+144+16) \\ - 2(35+86) = 0$$

$$7. |\mathbf{a} \cdot \mathbf{c}| = |-2 - 27 + 5| = 24 \\ |\mathbf{a}| |\mathbf{c}| = \sqrt{35} \cdot \sqrt{86}$$

$$8. 5\mathbf{a} \cdot 13\mathbf{b} = 65\mathbf{a} \cdot \mathbf{b} = 2860$$

$$9. 15\mathbf{a} \cdot \mathbf{b} + 15\mathbf{a} \cdot \mathbf{c} = 15\mathbf{a}(\mathbf{b} + \mathbf{c}) \\ = 15 \cdot \mathbf{a} \cdot (2, 9, 9) = 300$$

$$10. \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (16, -9, 7) \\ = 68$$

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = (-3, -3, -3) \cdot \mathbf{c} \\ = 6 - 27 - 3 = -24$$

$$1. \mathbf{a} \cdot \mathbf{b} = 4 + 0 + 40 = 44$$

$$\mathbf{b} \cdot \mathbf{a} = 44$$

$$\mathbf{b} \cdot \mathbf{c} = -8 + 0 + 8 = 0$$

$$2. (-3\mathbf{a} + 5\mathbf{c}) \cdot \mathbf{b} = (-3, 5, 4) \cdot \mathbf{b} \\ = -52 + 0 - 80 = -132$$

$$15(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} = 15(-3, -2, 4) \cdot \mathbf{b}$$

$$= 15 \cdot (-12 + 0 + 32) = 300$$

$$3. |\mathbf{a}| = \sqrt{1+9+25} = \sqrt{35}$$

$$|\mathbf{2b}| = 2 \cdot \sqrt{16+64} = 8\sqrt{5}$$

$$|-c| = \sqrt{4+81+1} = \sqrt{86}$$

$$4. |\mathbf{a} + \mathbf{b}| = \sqrt{25+9+164} = \sqrt{203}$$

$$|\mathbf{a}| + |\mathbf{b}| = \sqrt{35} + \sqrt{16+64} = \sqrt{35} + 4\sqrt{5}$$

**11-16 GENERAL PROBLEMS**

11. What laws do Probs. 1 and 4-7 illustrate?
12. What does  $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$  imply if  $\mathbf{u} = \mathbf{0}$ ? If  $\mathbf{u} \neq \mathbf{0}$ ?
13. Prove the Cauchy-Schwarz inequality.
14. Verify the Cauchy-Schwarz and triangle inequalities for the above  $\mathbf{a}$  and  $\mathbf{b}$ .
15. Prove the parallelogram equality. Explain its name.
16. **Triangle inequality.** Prove Eq. (7). Hint. Use Eq. (3) for  $|\mathbf{a} + \mathbf{b}|$  and Eq. (6) to prove the square of Eq. (7), then take roots.

11-1  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$   
 $\mathbf{a} \cdot \mathbf{b} = 0$ , orthogonality

4-5)  $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$  (Triangle inequality).

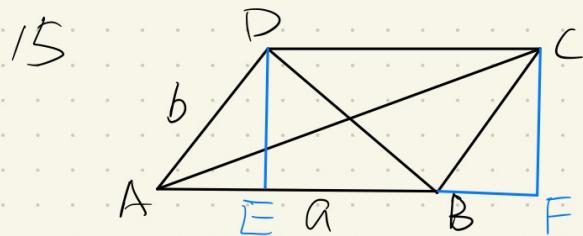
6)  $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$  (Parallelogram equality).

7)  $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|$  (Cauchy-Schwarz inequality).

12. if  $\mathbf{u} = \mathbf{0}$ , We know nothing about  $\mathbf{u} \cdot \mathbf{w}$ .  
if  $\mathbf{u} \neq \mathbf{0}$ ,  $P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$ . So  $\mathbf{v}, \mathbf{w}$  belongs  
to a set  $B$  that the projections of  
 $\mathbf{u}$  on the vector multiple the magnitude of  
the vector is a constant.

13. projection  $\leq$  itself. ( $\cos \theta \leq 1$ )

14. 4-5, and 7



$$1) \quad AE = BF, \quad DE = CF,$$

$$2) \quad AC^2 + BD^2 = (AE + BF)^2 + CF^2 \\ + (AF - AE)^2 + DE^2 \\ = 2(a^2 + b^2)$$

$$16. \quad |a+b|^2 = (a+b)(a+b) \\ = a^2 + b^2 + 2ab$$

$$\leq |a|^2 + |b|^2 + 2|a||b| \\ = (|a| + |b|)^2$$

17-20 WORK

**Mistake**

Find the work done by a force  $\mathbf{p}$  acting on a body if the body is displaced along the straight segment  $\overline{AB}$  from A to B. Sketch  $\overline{AB}$  and  $\mathbf{p}$ . Show the details.

17.  $\mathbf{p} = [2, 5, 0]$ , A: (1, 3, 3), B: (3, 5, 5)

18.  $\mathbf{p} = [-1, -2, 4]$ , A: (0, 0, 0), B: (6, 7, 5)

19.  $\mathbf{p} = [0, 4, 3]$ , A: (4, 5, -1), B: (1, 3, 0)

20.  $\mathbf{p} = [6, -3, -3]$ , A: (1, 5, 2), B: (3, 4, 1)

21. **Resultant.** Is the work done by the resultant of two forces in a displacement the sum of the work done by each of the forces separately? Give proof or counterexample.

17.  $W = \mathbf{P} \cdot \mathbf{d} = \mathbf{P} \cdot (2, 2, 2) = 14$

18.  $W = \mathbf{P} \cdot (6, 7, 5) = 0$

19.  $W = \mathbf{P} \cdot (-3, -2, 1) = -5$

Negative, because of direction.

20.  $W = \mathbf{P} \cdot (2, -1, -1) = 18$

21. Yes.

$$W = \mathbf{P}_1 \cdot \mathbf{d} + \mathbf{P}_2 \cdot \mathbf{d}$$

$$= (\mathbf{P}_1 + \mathbf{P}_2) \cdot \mathbf{d}$$

22.  $\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{3+2}{\sqrt{2} \sqrt{14}}$

$$\alpha = \arccos\left(\frac{\sqrt{5}}{14}\right)$$

23.  $\cos \alpha = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}| |\mathbf{c}|} = \frac{3+2}{\sqrt{14} \sqrt{5}}$

$$\alpha = \arccos\left(\frac{\sqrt{70}}{14}\right)$$

24.  $W \leq \frac{(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{c})}{|\mathbf{a} + \mathbf{c}| \cdot |\mathbf{b} + \mathbf{c}|}$

$$= \frac{(4, 3, 1) \cdot (4, 2, 3)}{\sqrt{16+9+1} \cdot \sqrt{16+4+9}}$$

$$= \frac{16+6+3}{\sqrt{26} \cdot \sqrt{29}} = \frac{25}{\sqrt{26} \sqrt{29}}$$

$$\alpha = \arccos\left(\frac{25}{\sqrt{26} \sqrt{29}}\right)$$

**22-30 ANGLE BETWEEN VECTORS**

Let  $\mathbf{a} = [1, 1, 0]$ ,  $\mathbf{b} = [3, 2, 1]$ , and  $\mathbf{c} = [1, 0, 2]$ . Find the angle between:

22.  $\mathbf{a}, \mathbf{b}$

23.  $\mathbf{b}, \mathbf{c}$

24.  $\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c}$

25. What will happen to the angle in Prob. 24 if we replace  $c$  by  $nc$  with larger and larger  $n$ ?

26. **Cosine law.** Deduce the law of cosines by using vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{a} - \mathbf{b}$ .

27. **Addition law.**  $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ . Obtain this by using  $\mathbf{a} = [\cos \alpha, \sin \alpha]$ ,  $\mathbf{b} = [\cos \beta, \sin \beta]$  where  $0 \leq \alpha \leq \beta \leq 2\pi$ .

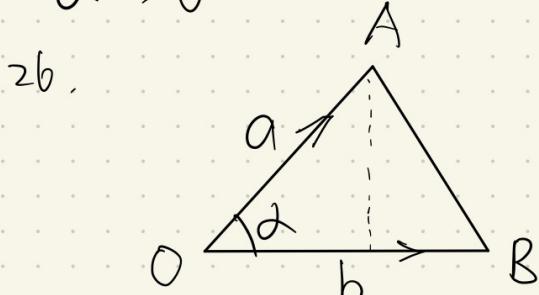
28. **Triangle.** Find the angles of the triangle with vertices  $A: (0, 0, 2)$ ,  $B: (3, 0, 2)$ , and  $C: (1, 1, 1)$ . Sketch the triangle.

29. **Parallelogram.** Find the angles if the vertices are  $(0, 0), (6, 0), (8, 3)$ , and  $(2, 3)$ .

30. **Distance.** Find the distance of the point  $A: (1, 0, 2)$  from the plane  $P: 3x + y + z = 9$ . Make a sketch.

$$25. \lim_{n \rightarrow \infty} \frac{(a+nc)(b+nc)}{|(a+nc)| \cdot |(b+nc)|} = \frac{c \cdot c}{|c|^2} = 1$$

$$\angle \rightarrow 0$$



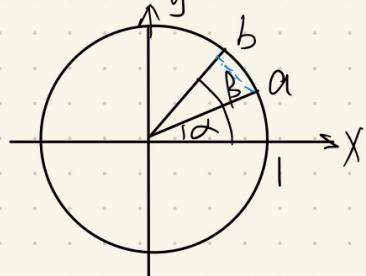
$$(\overrightarrow{BA})^2 = (a-b)^2 =$$

$$(a \sin \alpha)^2 + (b - a \cos \alpha)^2$$

$$= a^2 \sin^2 \alpha + b^2 + a^2 \cos^2 \alpha - 2ab \cos \alpha$$

$$= a^2 + b^2 - 2ab \cos \alpha$$

$$27. \mathbf{a} \cdot \mathbf{b} = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$



the projection of  $a$  on  $b$  is

$$P \approx a \cdot b = \cos(\beta - \alpha) = \cos(\alpha - \beta)$$

$$28. \overrightarrow{AB} = (3, 0, 0) = a$$

$$\overrightarrow{BC} = (-2, 1, -1) = b$$

$$\overrightarrow{CA} = (-1, -1, 1) = c$$

$$\cos \alpha = \frac{-ab}{|a||b|} = \frac{-6}{3\sqrt{6}} = \frac{\sqrt{6}}{3}$$

$$\cos \beta = \frac{-bc}{|b||c|} = \frac{2-1-1}{\sqrt{5}\sqrt{5}} = 0$$

$$\cos \gamma = \frac{-ca}{|c||a|} = \frac{+3}{3\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$29. \tan \alpha = \frac{3}{2} \quad \alpha = \arctan \frac{3}{2}$$

$$\cos \alpha = \frac{(2 \cdot 3)(6 \cdot 0)}{\sqrt{4+9} \cdot 6} = \frac{2}{\sqrt{13}}$$

30. mapping to  $(x_1, y_1, z_1)$

$$A = (0, 0, 0)$$

$$P = 3x - 3 + y + z - 2 = 0$$

$$3x + y + z = 14$$

per example 6,

$$P = \frac{C}{|AT|} = \frac{14}{\sqrt{11}}$$

**31-35 ORTHOGONALITY** is particularly important, mainly because of orthogonal coordinates, such as **Cartesian coordinates**, whose **natural basis** [Eq. (9), Sec. 9.1], consists of three orthogonal unit vectors.

31. For what values of  $a_1$  are  $[a_1, 4, 3]$  and  $[3, -2, 12]$  orthogonal?
32. **Planes.** For what  $c$  are  $3x + z = 5$  and  $8x - y + cz = 9$  orthogonal?
33. **Unit vectors.** Find all unit vectors  $\mathbf{a} = [a_1, a_2]$  in the plane orthogonal to  $[4, 3]$ .
34. **Corner reflector.** Find the angle between a light ray and its reflection in three orthogonal plane mirrors, known as *corner reflector*.
35. **Parallelogram.** When will the diagonals be orthogonal? Give a proof.

$$31. 3a_1 - 8 + 36 = 0$$

$$a_1 = -\frac{28}{3}$$

$$32. (3, 0, 1) \cdot (8, -1, c) = 0$$

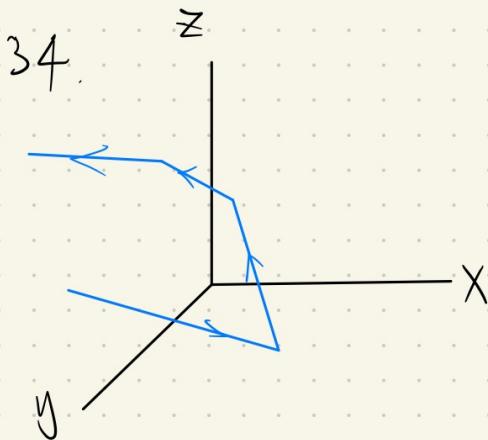
$$24 + c = 0 \quad c = -24$$

$$33. 4a_1 + 3a_2 = 0$$

$$a_2 = -\frac{4}{3}a_1$$

$$a_1^2 + \frac{16}{9}a_1^2 = 1$$

$$a_1 = \frac{3}{5}, \quad \left[ \frac{3}{5}, -\frac{4}{5} \right]$$



$[a, b, c]$  Reflect on XY plane,

$\Rightarrow [a, b, -c]$  Reflect on XZ plane

$\Rightarrow [a, -b, -c]$  R on ZY plane

$[-a, -b, -c]$

$$35. (a+b)(a-b) = 0$$

$$a^2 - b^2 = 0 \quad a^2 = b^2$$

diamond shape. answer is wrong.

**36-40 COMPONENT IN THE DIRECTION OF A VECTOR**

Find the component of  $\mathbf{a}$  in the direction of  $\mathbf{b}$ . Make a sketch.

36.  $\mathbf{a} = [1, 1, 1]$ ,  $\mathbf{b} = [2, 1, 3]$

37.  $\mathbf{a} = [3, 4, 0]$ ,  $\mathbf{b} = [4, -3, 2]$

38.  $\mathbf{a} = [8, 2, 0]$ ,  $\mathbf{b} = [-4, -1, 0]$

39. When will the component (the projection) of  $\mathbf{a}$  in the direction of  $\mathbf{b}$  be equal to the component (the projection) of  $\mathbf{b}$  in the direction of  $\mathbf{a}$ ? First guess.

40. What happens to the component of  $\mathbf{a}$  in the direction of  $\mathbf{b}$  if you change the length of  $\mathbf{b}$ ?

$$36. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{2+1+3}{\sqrt{14}} = \frac{3\sqrt{14}}{7}$$

$$37. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{12-12+0}{\sqrt{5}} = 0 \quad \text{perpendicular}$$

$$38. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{-32-2}{\sqrt{17}} = -2\sqrt{17}$$

On the opp direction,

$$39. |\mathbf{a}| = |\mathbf{b}| \quad \underline{\text{or } \mathbf{a} \cdot \mathbf{b} = 0}$$

$$40. P = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a} \cdot k\mathbf{b}}{|k||\mathbf{b}|}$$

nothing change.

# PS 9.3

## 1-10 GENERAL PROBLEMS

- Give the details of the proofs of Eqs. (4) and (5).
- What does  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$  with  $\mathbf{a} \neq \mathbf{0}$  imply?
- Give the details of the proofs of Eqs. (6) and (11).

(5)

$$(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$

$$(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c})$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} = [a_2(b_3 + c_2) - a_3(b_2 + c_2)] i - [a_1(b_3 + c_2) - a_3(b_1 + c_1)] j + [a_1(b_2 + c_3) - a_2(b_1 + c_1)] k$$

$$\mathbf{a} \times \mathbf{b}$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$$

$$(\mathbf{a} \times \mathbf{b})$$

$$\begin{vmatrix} i & j & k \\ (a_1, a_2, a_3) \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= ((a_2 b_3 - a_3 b_2) i$$

$$- (a_1 b_3 - a_3 b_1) j$$

$$+ (a_1 b_2 - a_2 b_1) k$$

$$= l [(a_2 b_3 - a_3 b_2) i$$

$$- (a_1 b_3 - a_3 b_1) j$$

$$+ (a_1 b_2 - a_2 b_1) k$$

Extend the rest, same.

$$\mathbf{a} \times \mathbf{c}$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{Reference above.}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

$(\beta)$ , Same

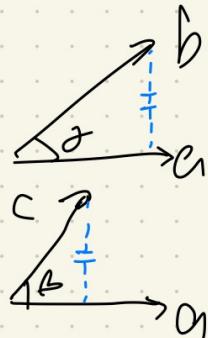
2. What does  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$  with  $\mathbf{a} \neq \mathbf{0}$  imply?  
 3. Give the details of the proofs of Eqs. (6) and (11).

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \dots$$

$$(2) |V| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|(|\mathbf{b}| \sin \alpha)$$

$$= |\mathbf{a} \times \mathbf{c}| = |\mathbf{a}|(|\mathbf{c}| \sin \beta)$$

$$|\mathbf{b}| \sin \alpha = |\mathbf{c}| \sin \beta.$$



and the direction of the result is same.

$$(2) \quad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ x & y & z \end{vmatrix} =$$

$$(0-y)i - (z-x)j + yk$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \dots$$

$$(2) \quad \mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = zi - xk$$

(6)

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$$

$$\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \text{ and } \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

exchange row, determinant  $\times -1$

P295 . Sec 7.7 Theorem 1.

(11)

$$(\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2 c_3 - b_3 c_2) i - (b_1 c_3 - b_3 c_1) j + (b_1 c_2 - b_2 c_1) k$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (a_1 b_2 c_3 - a_1 b_3 c_2) - (a_2 b_1 c_3 - a_2 b_3 c_1) + (a_3 b_1 c_2 - a_3 b_2 c_1)$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \Rightarrow \mathbf{a} \rightarrow \mathbf{b}, \mathbf{b} \rightarrow \mathbf{c}, \mathbf{c} \rightarrow \mathbf{a}$$

$$= C_1 A_2 B_3 - C_1 A_3 B_2$$

$$- (C_2 A_1 B_3 - C_2 A_3 B_1)$$

$$+ (C_3 A_1 B_2 - C_3 A_2 B_1)$$

Geo perspective, same volume.

4. Lagrange's identity for  $|\mathbf{a} \times \mathbf{b}|$ . Verify it for  $\mathbf{a} = [3, 4, 2]$  and  $\mathbf{b} = [1, 0, 2]$ . Prove it, using  $\sin^2 \gamma = 1 - \cos^2 \gamma$ . The identity is

$$(12) \quad |\mathbf{a} \times \mathbf{b}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2}.$$

5. What happens in Example 3 of the text if you replace  $\mathbf{p}$  by  $-\mathbf{p}$ ?

in change the direction -

$$4. \{3, 4, 2\} \text{ cross } \{1, 0, 2\} = [8, -4, -4]$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{64 + 16 + 16} = \sqrt{96}$$

$$\begin{aligned} & \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2} \\ &= \sqrt{(9 + 16)(1 + 4) - (3 + 4)^2} \\ &= \sqrt{29 \times 5 - 49} = \sqrt{96} \end{aligned}$$

$$\text{Prove: } |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \alpha$$

$$= \sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}} \sin \alpha$$

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}}}$$

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}} \sqrt{1 - \cos^2 \alpha} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}} \cdot \sqrt{1 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})}} \end{aligned}$$

$$= \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2}$$

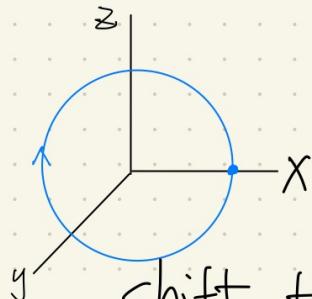
6. What happens in Example 5 if you choose a  $P$  at distance  $2d$  from the axis of rotation?

$\omega$  remains the same

$|r| \sin \alpha$  become doubled.

so  $v$  get doubled.

7. Rotation. A wheel is rotating about the  $y$ -axis with angular speed  $\omega = 20 \text{ sec}^{-1}$ . The rotation appears clockwise if one looks from the origin in the positive  $y$ -direction. Find the velocity and speed at the point  $[8, 6, 0]$ . Make a sketch.



shift the center of wheel to  $[0, 0, 0]$ .

so  $[8, 6, 0]$  equals to

$$[8, 0, 0]$$

$$s = 8 \times 20 = 160 \quad v = [0, 0, -160]$$

$$v = [0, 20, 0] \times [8, 0, 0]$$

$$= [0, 0, -160]$$

Not quite understand

8. **Rotation.** What are the velocity and speed in Prob. 7 at the point  $(4, 2, -2)$  if the wheel rotates about the line  $y = x, z = 0$  with  $\omega = 10 \text{ sec}^{-1}$ ?

$$⑧ y^2 + x^2 = 10^2$$

$$x^2 + x^2 = 100$$

$$x = \sqrt{50}$$

$$[\sqrt{50}, \sqrt{50}, 0] \times [4, 2, -2]$$

$$\begin{aligned} &= \begin{vmatrix} i & j & k \\ \sqrt{50} & \sqrt{50} & 0 \\ 4 & 2 & -2 \end{vmatrix} \\ &= -2\sqrt{50}i + 2\sqrt{50}j - 2\sqrt{50}k \end{aligned}$$

9. **Scalar triple product.** What does  $(\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = 0$  imply with respect to these vectors?

$$⑨ V = 0$$

- 1.  $a, b, c$  on the same plane
- 2.  $b, c$  on the same line
- 3. one of  $a, b, c = 0$ .

$$⑩:$$

11-23

**VECTOR AND SCALAR  
TRIPLE PRODUCTS**

With respect to right-handed Cartesian coordinates, let  $\mathbf{a} = [2, 1, 0]$ ,  $\mathbf{b} = [-3, 2, 0]$ ,  $\mathbf{c} = [1, 4, -2]$ , and  $\mathbf{d} = [5, -1, 3]$ . Showing details, find:

11.  $\mathbf{a} \times \mathbf{b}$ ,  $\mathbf{b} \times \mathbf{a}$ ,  $\mathbf{a} \cdot \mathbf{b}$

12.  $3\mathbf{c} \times 5\mathbf{d}$ ,  $15\mathbf{d} \times \mathbf{c}$ ,  $15\mathbf{d} \cdot \mathbf{c}$ ,  $15\mathbf{c} \cdot \mathbf{d}$

$$\begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \end{pmatrix} \times$$

$$(1) \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix}$$

$$= 0i - 0j + 7k$$

$$= 7k$$

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = -7k$$

$$\mathbf{a} \cdot \mathbf{b} = -6 + 2 + 0 = -4$$

$$(2) \quad 3\mathbf{c} \times 5\mathbf{d} = 15 \cdot \mathbf{c} \times \mathbf{d}$$

$$= 15 \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix}$$

$$= 15(10i - 13j - 21k)$$

$$15\mathbf{d} \times \mathbf{c} = -15\mathbf{c} \times \mathbf{d}$$

$$= 15(-10i + 3j + 24k)$$

$$15\mathbf{d} \cdot \mathbf{c} = 15 \cdot (5 - 4 - 6) \\ = -75 = 15\mathbf{c} \cdot \mathbf{d}$$

25-35 APPLICATIONS

25. **Moment  $m$  of a force  $\mathbf{p}$ .** Find the moment vector  $\mathbf{m}$  and  $m$  of  $\mathbf{p} = [2, 3, 0]$  about  $Q: (2, 1, 0)$  acting on a line through  $A: (0, 3, 0)$ . Make a sketch.

$$m = \mathbf{r} \times \mathbf{p}$$

$$= \overrightarrow{QA} \times \overrightarrow{P}$$

$$= (-2, 2, 0) \times (2, 3, 0)$$

$$= \begin{vmatrix} i & j & k \\ -2 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= -10k$$

26. **Moment.** Solve Prob. 25 if  $\mathbf{p} = [1, 0, 3]$ ,  $Q: (2, 0, 3)$ , and  $A: (4, 3, 5)$ .

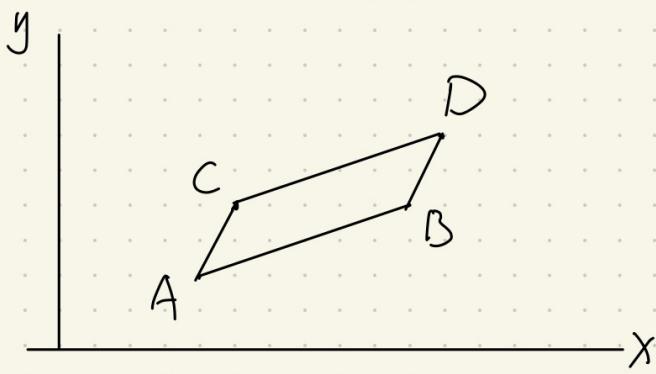
$$m = \mathbf{r} \times \mathbf{p}$$

$$= \overrightarrow{QA} \times \overrightarrow{P}$$

$$= [2, 3, 2] \times [1, 0, 3]$$

$$= \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 9i - 4j - 3k$$

27. **Parallelogram.** Find the area if the vertices are  $(4, 2, 0)$ ,  $(10, 4, 0)$ ,  $(5, 4, 0)$ , and  $(11, 6, 0)$ . Make a sketch.



$$\text{Area} = |\vec{AB} \times \vec{AC}|$$

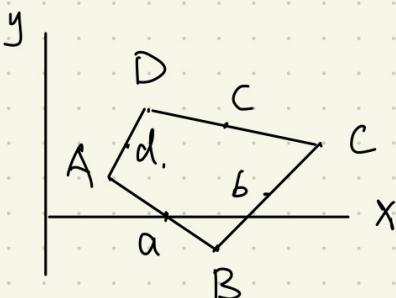
$$= \begin{vmatrix} i & j & k \\ 6 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= |10k| = 10$$

Answer is wrong.

Area is a scalar.

28. **A remarkable parallelogram.** Find the area of the quadrangle  $Q$  whose vertices are the midpoints of the sides of the quadrangle  $P$  with vertices  $A: (2, 1, 0)$ ,  $B: (5, -1, 0)$ ,  $C: (8, 2, 0)$ , and  $D: (4, 3, 0)$ . Verify that  $Q$  is a parallelogram.



Set  $a, b, c, d$  as the middle point, so we get

$$a: (3.5, 0, 0)$$

$$b: (6.5, 0.5, 0)$$

$$c: (6, 2.5, 0)$$

$$d: (3, 2, 0)$$

$$\vec{ab} = (3, 0.5, 0) = \vec{dc}$$

$$\vec{ad} = (-0.5, 2, 0) = \vec{bc}$$

$\Rightarrow Q$  is a parallelogram.

29. **Triangle.** Find the area if the vertices are  $(0, 0, 1)$ ,  $(2, 0, 5)$ , and  $(2, 3, 4)$ .

$$\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} \begin{vmatrix} i & j & k \\ 2 & 0 & 4 \\ 2 & 3 & 3 \end{vmatrix}$$

$$= \frac{1}{2} |-12i + 2j + 6k|$$

$$= \frac{1}{2} \sqrt{12^2 + 2^2 + 6^2}$$

$$= \sqrt{46}$$

- 30. Plane.** Find the plane through the points  $A: (1, 2, \frac{1}{4})$ ,  $B: (4, 2, -2)$ , and  $C: (0, 8, 4)$ .

For  $ax + by + cz + d = 0$

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & \frac{1}{4} & 1 \\ 4 & 2 & -2 & 1 \\ 0 & 8 & 4 & 1 \end{vmatrix} = 0$$

$$\begin{aligned} &x(-4+8+2+16-\frac{1}{2}-8) \\ &-y(-2+16+0-0-4-1) \\ &+z(2+32+0-0-8-8) \\ &-1 \cdot (8+8+0-0+16-32) = 0 \end{aligned}$$

$$\begin{aligned} 13.5x - 9y + 18z &= 0 \\ 27x - 18y + 36z &= 0 \\ 3x - 2y + 4z &= 0 \end{aligned}$$

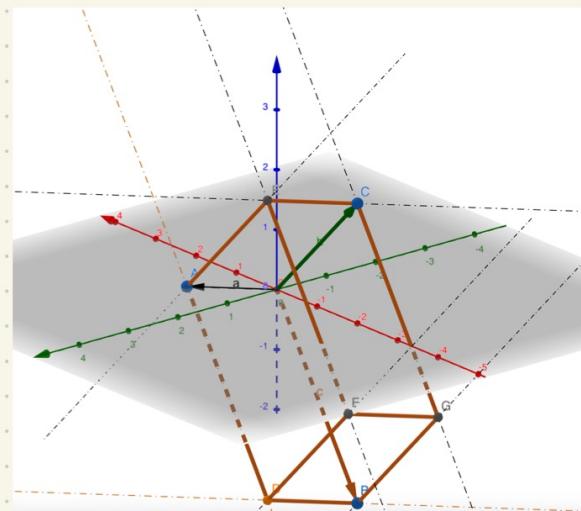
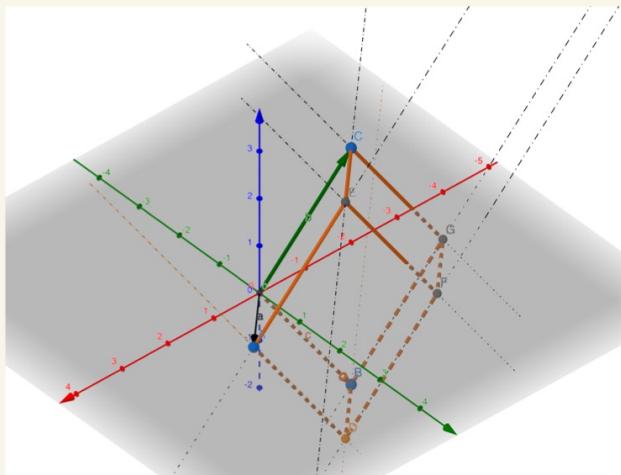
- 31. Plane.** Find the plane through  $(1, 3, 4)$ ,  $(1, -2, 6)$ , and  $(4, 0, 7)$ .

Reference 30

$$3x - 2y - 5z + 23 = 0$$

Answer is wrong.

- 32. Parallelepiped.** Find the volume if the edge vectors are  $\mathbf{i} + \mathbf{j}$ ,  $-2\mathbf{i} + 2\mathbf{k}$ , and  $-2\mathbf{i} - 3\mathbf{k}$ . Make a sketch.



$$(a, b, c) = a \cdot (b \times c)$$

$$= (\mathbf{i} + \mathbf{j}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ -2 & 0 & -3 \end{vmatrix}$$

$$= (\mathbf{i} + \mathbf{j}) \cdot [-\mathbf{j}(6+4)] = -10$$

$$V = |(a, b, c)| = 10$$

- 33. Tetrahedron.** Find the volume if the vertices are  $(1, 1, 1)$ ,  $(5, -7, 3)$ ,  $(7, 4, 8)$ , and  $(10, 7, 4)$ .

Per Wolfram or Wiki:

$$V = \frac{1}{3!} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \frac{1}{6} \left| (A - D) \left[ (B - D) \times (C - D) \right] \right|$$

$$= \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & -7 & 3 & 1 \\ 7 & 4 & 8 & 1 \\ 10 & 7 & 4 & 1 \end{vmatrix} = \frac{1}{6} \left| (4, -8, 2) \cdot [(6, 3, 7) \times (9, 6, 3)] \right|$$

$$= \frac{1}{6} \left| (4, -8, 2) \cdot (33, 45, 9) \right| = \frac{1}{6} \cdot 474 = 79$$

$$= \frac{1}{6} \cdot 474 = 79$$

This is intuitive.

Why?

- 34. Tetrahedron.** Find the volume if the vertices are  $(1, 3, 6)$ ,  $(3, 7, 12)$ ,  $(8, 8, 9)$ , and  $(2, 2, 8)$ .

$$V = \frac{1}{6} \cdot 90 = 15$$

35.

# P.S. 9.4

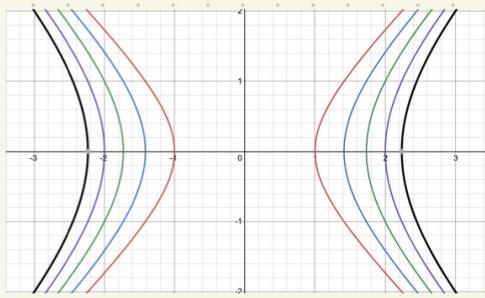
## 1-8 SCALAR FIELDS IN THE PLANE

Let the temperature  $T$  in a body be independent of  $z$  so that it is given by a scalar function  $T = T(x, t)$ . Identify the isotherms  $T(x, y) = \text{const}$ . Sketch some of them.

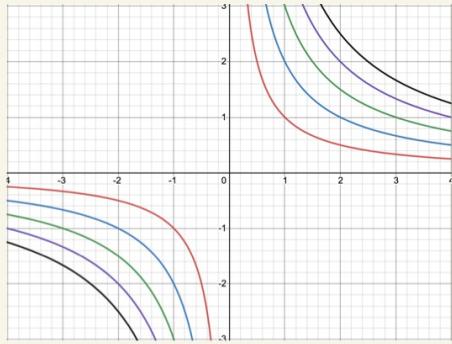
1.  $T = x^2 - y^2$
2.  $T = xy$
3.  $T = 3x - 4y$
4.  $T = \arctan(y/x) = C$
5.  $T = y/(x^2 + y^2)$
6.  $T = x/(x^2 + y^2)$
7.  $T = 9x^2 + 4y^2$

Red = 1, black = 5.

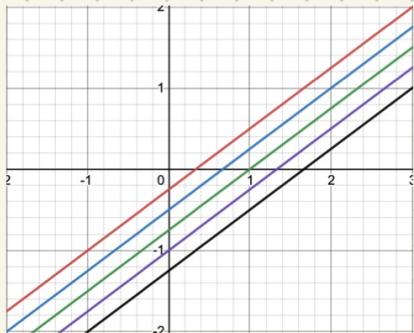
1.  $T = x^2 - y^2 = C$



2.  $T = xy = C$



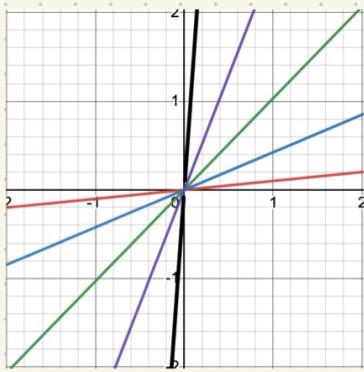
3.  $T = 3x - 4y = C$



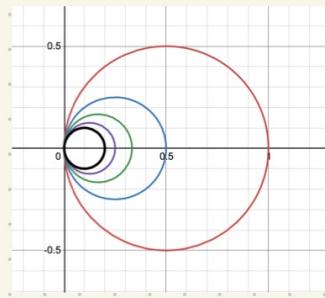
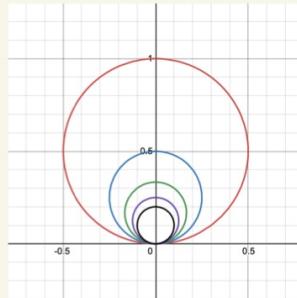
4.  $T = \arctan(y/x) = C$

$C \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

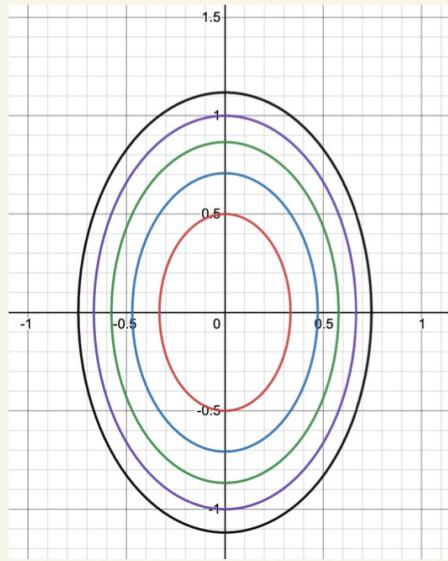
R: 0.1, B: 0.4 G: 0.8 P: 1.2, B: 1.5



5.  $T = y/(x^2 + y^2) = C$

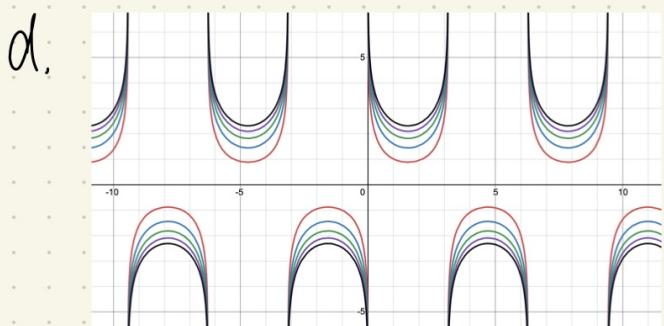
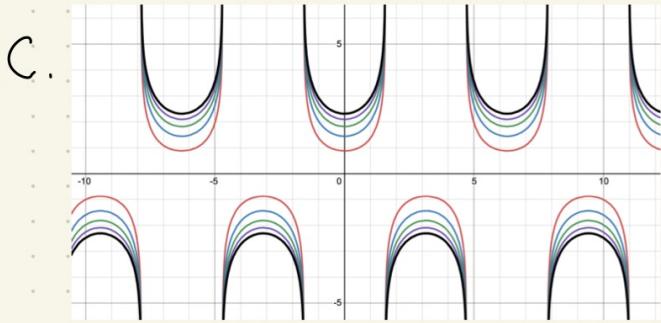
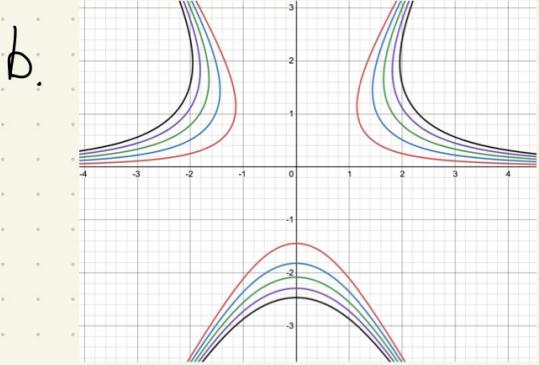
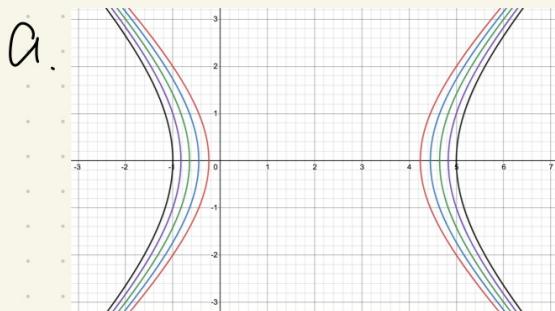


6.  $T = x/(x^2 + y^2) = C$

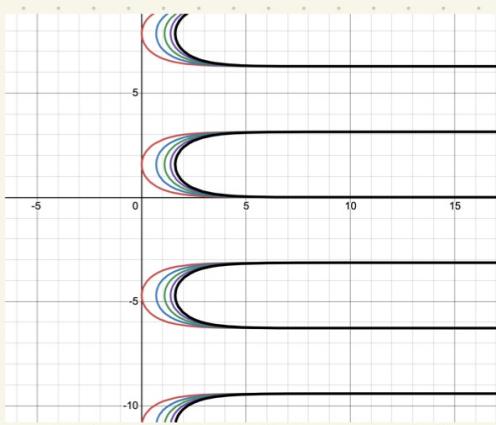


**8. CAS PROJECT. Scalar Fields in the Plane.** Sketch or graph isotherms of the following fields and describe what they look like.

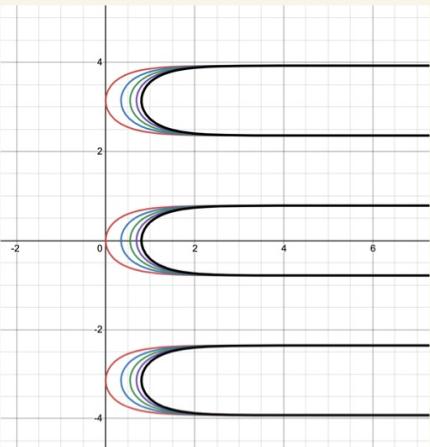
- (a)  $x^2 - 4x - y^2$
- (b)  $x^2y - y^3/3$
- (c)  $\cos x \sinh y$
- (d)  $\sin x \sinh y$
- (e)  $e^x \sin y$
- (f)  $e^{2x} \cos 2y$
- (g)  $x^4 - 6x^2y^2 + y^4$
- (h)  $x^2 - 2x - y^2$



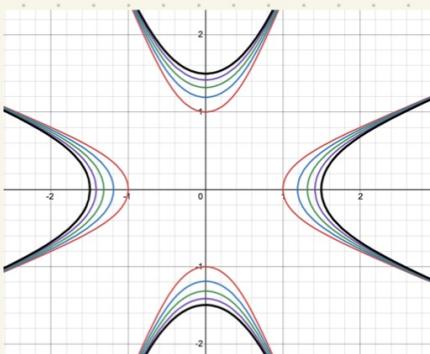
e.



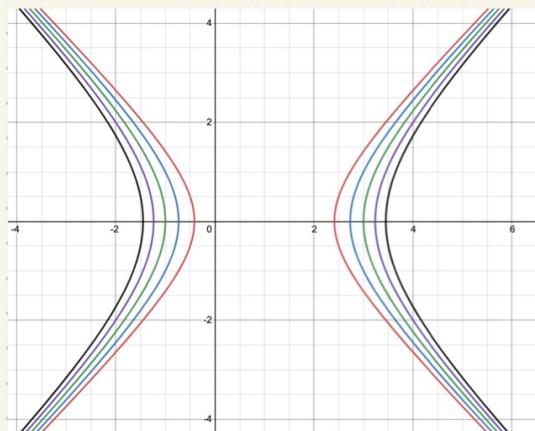
f.



g.



h.



**9-14 SCALAR FIELDS IN SPACE**

What kind of surfaces are the **level surfaces**  $f(x, y, z) = \text{const}$ ?

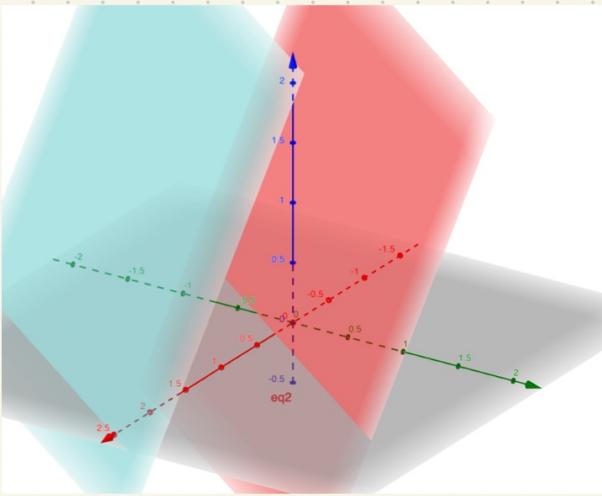
$$9. f = 4x - 3y + 2z$$

$$10. f = 9(x^2 + y^2) + z^2$$

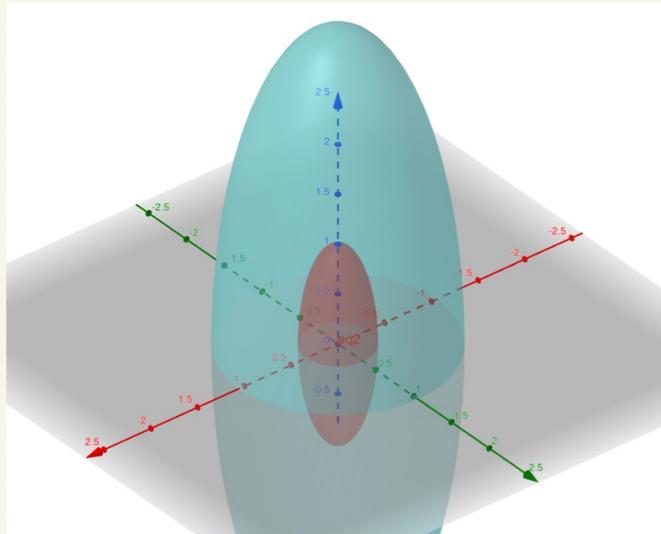
$$11. f = 5x^2 + 2y^2$$

$$12. f = z - \sqrt{x^2 + y^2}$$

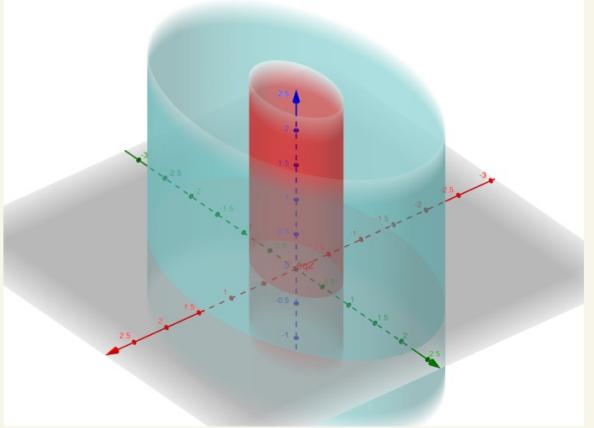
9. Teal:  $f=10$  Red:  $f=1$



10.



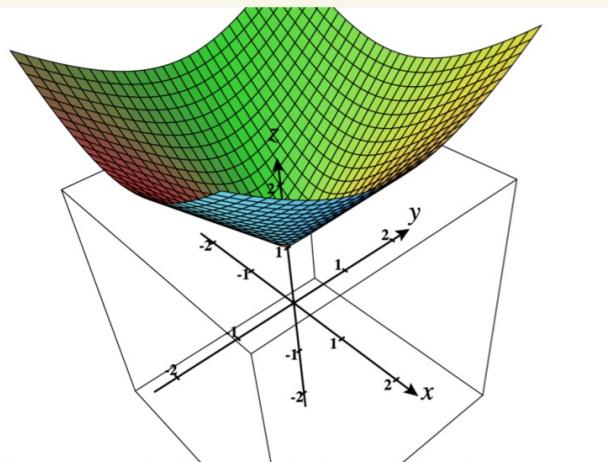
11.



12. Quite a lot of tools

Can not plot this.

$$f = 1$$



**9–14 SCALAR FIELDS IN SPACE**

What kind of surfaces are the **level surfaces**  $f(x, y, z) = \text{const}$ ?

$$9. f = 4x - 3y + 2z$$

$$10. f = 9(x^2 + y^2) + z^2$$

$$11. f = 5x^2 + 2y^2$$

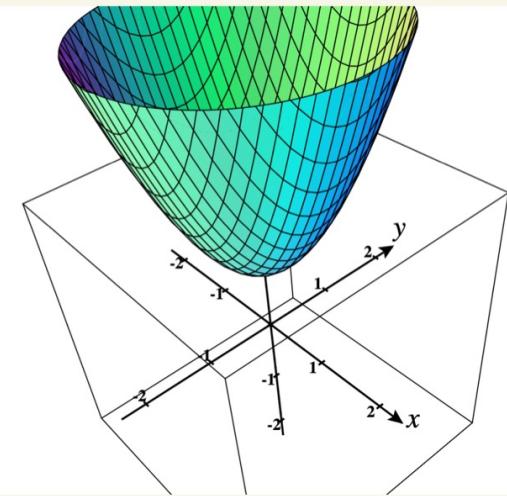
$$12. f = z - \sqrt{x^2 + y^2}$$

$$13. f = z - (x^2 + y^2)$$

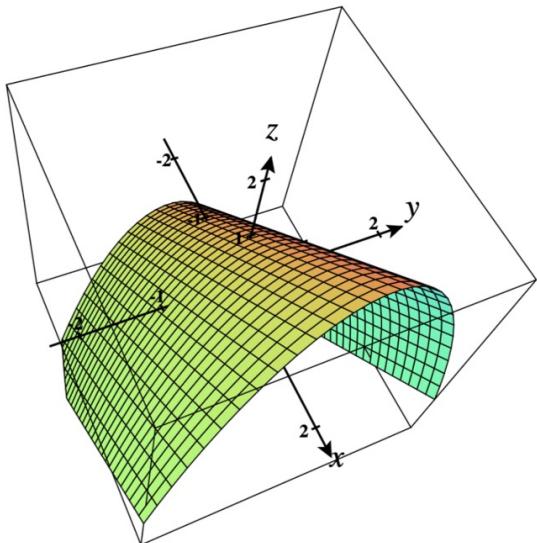
$$14. f = x - y^2$$

<https://c3d.libretexts.org/CalcPlot3D/index.html>

13.



14.



**15–20 VECTOR FIELDS**

Sketch figures similar to Fig. 198. Try to interpret the field of  $\mathbf{v}$  as a velocity field.

15.  $\mathbf{v} = \mathbf{i} + \mathbf{j}$

16.  $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$

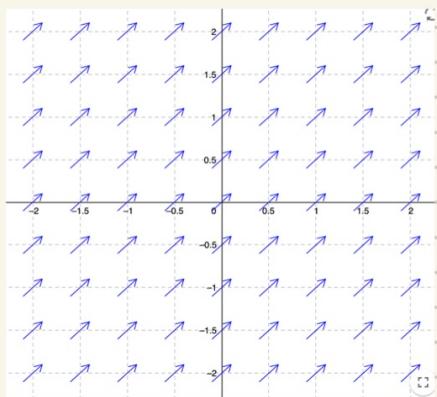
17.  $\mathbf{v} = x\mathbf{j}$

18.  $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$

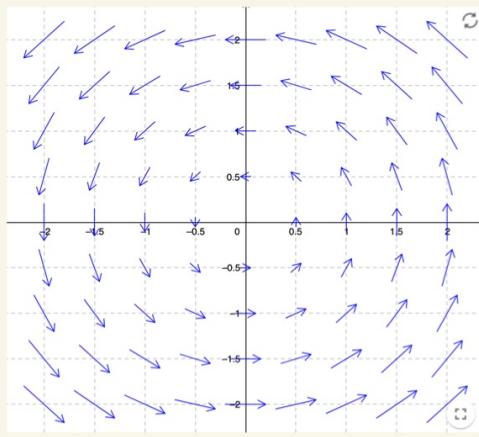
19.  $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$

20.  $\mathbf{v} = y\mathbf{i} - x\mathbf{j}$

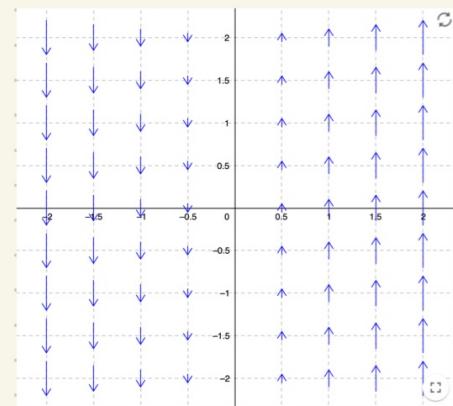
15.



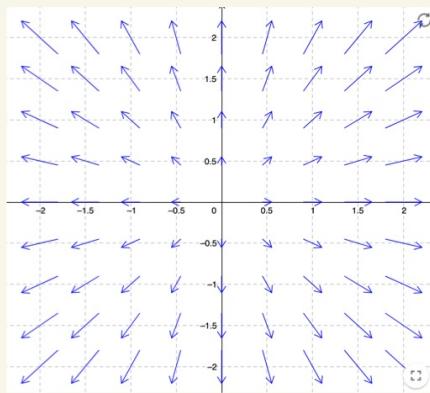
16.



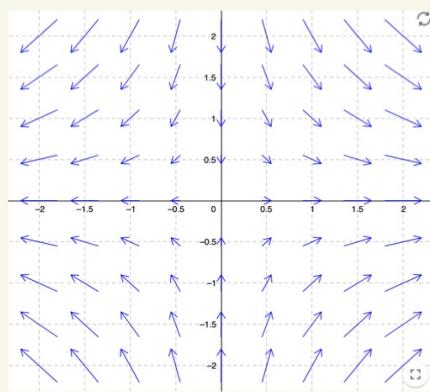
17.



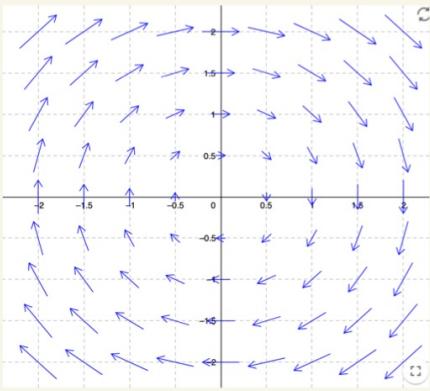
18.



19.



20.



Geogebra.org

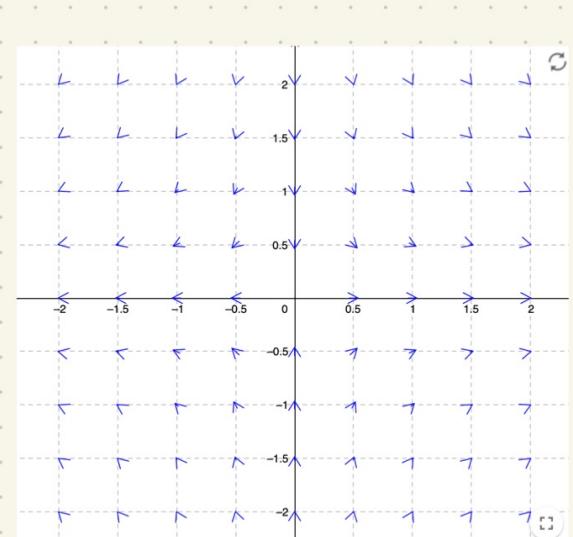
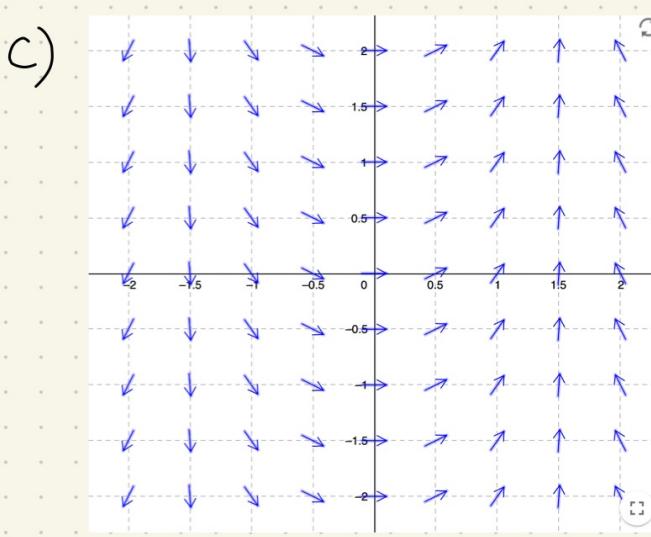
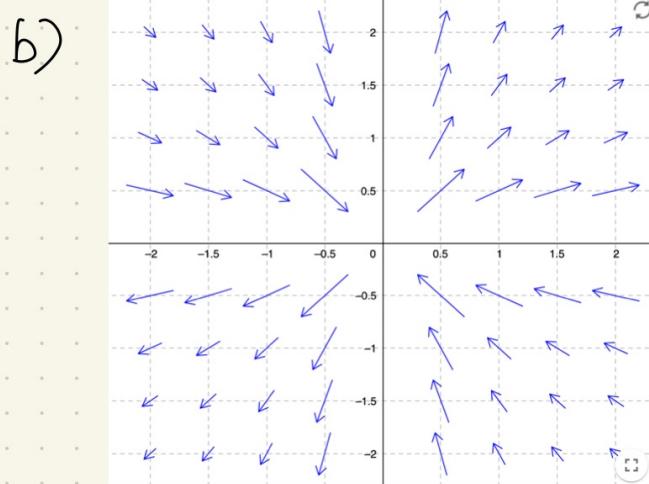
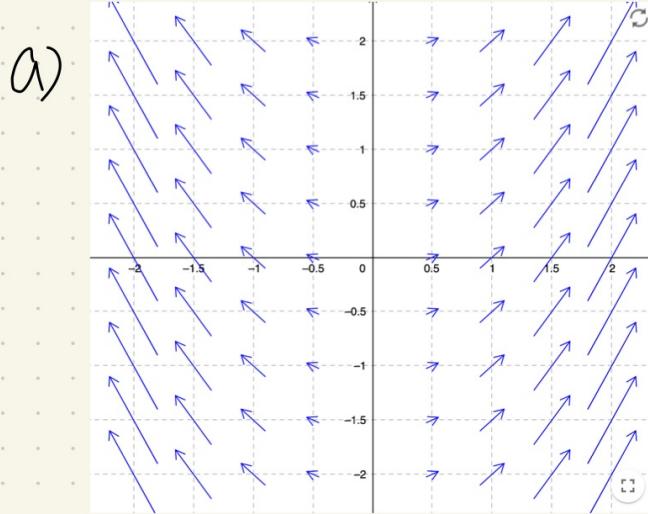
**21. CAS PROJECT. Vector Fields.** Plot by arrows:

(a)  $\mathbf{v} = [x, x^2]$

(b)  $\mathbf{v} = [1/y, 1/x]$

(c)  $\mathbf{v} = [\cos x, \sin x]$

(d)  $\mathbf{v} = e^{-(x^2+y^2)} [x, -y]$



**22–25 DIFFERENTIATION**

22. Find the first and second derivatives of  $\mathbf{r} = [3 \cos 2t, 3 \sin 2t, 4t]$ .
23. Prove (11)–(13). Give two typical examples for each formula.
24. Find the first partial derivatives of  $\mathbf{v}_1 = [e^x \cos y, e^x \sin y]$  and  $\mathbf{v}_2 = [\cos x \cosh y, -\sin x \sinh y]$ .
25. **WRITING PROJECT. Differentiation of Vector Functions.** Summarize the essential ideas and facts and give examples of your own.

$$22. \mathbf{r} = [-6 \sin 2t, 6 \cos 2t, 4t]$$

$$\mathbf{r}' = [-12 \cos 2t, -12 \sin 2t, 0]$$

23.

$$24. \frac{\partial \mathbf{v}_1}{\partial x} = [e^x \cos y, e^x \sin y]$$

$$\frac{\partial \mathbf{v}_1}{\partial y} = [-e^x \sin y, e^x \cos y]$$

$$\frac{\partial \mathbf{v}_2}{\partial x} = [-\sin x \cosh y, -\cos x \sinh y]$$

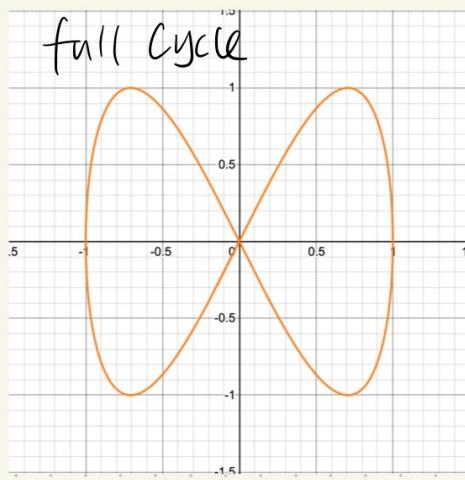
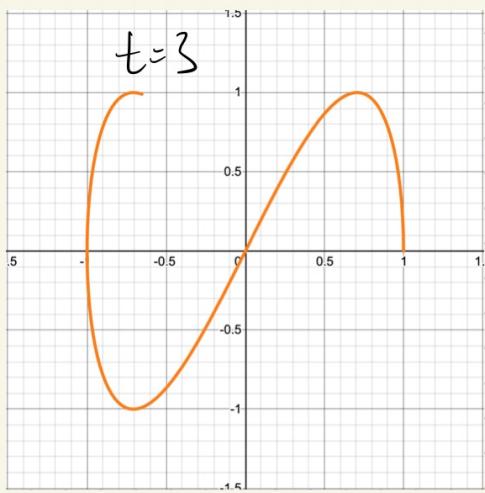
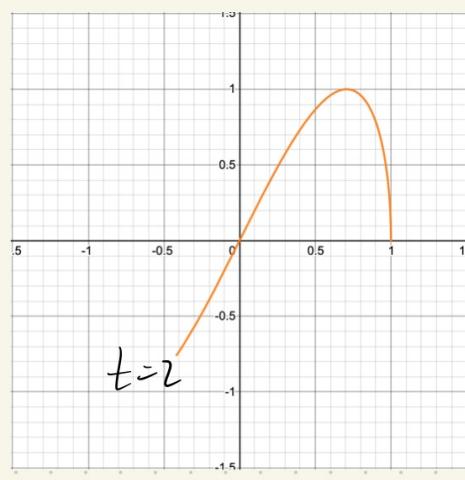
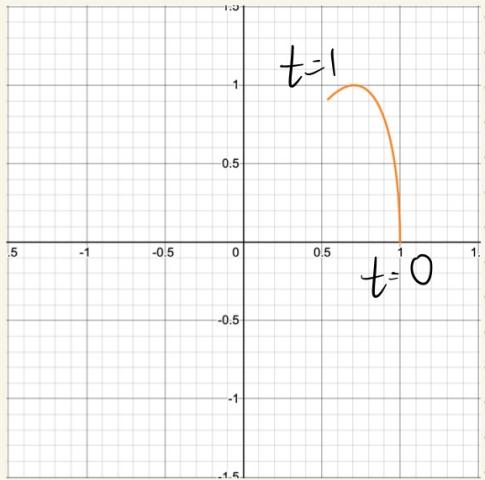
$$\frac{\partial \mathbf{v}_2}{\partial y} = [\cos x \sinh y, -\sin x \cosh y]$$

25.

# P.S. 9.5

9.  $[\cos t, \sin 2t, 0]$

$$0 \leq t \leq 1$$



**11-20 FIND A PARAMETRIC REPRESENTATION**

11. Circle in the plane  $z = 1$  with center  $(3, 2)$  and passing through the origin.
12. Circle in the  $yz$ -plane with center  $(4, 0)$  and passing through  $(0, 3)$ . Sketch it.
13. Straight line through  $(2, 1, 3)$  in the direction of  $\mathbf{i} + 2\mathbf{j}$ .

$$13. \quad \mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$$

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j}$$

$t=0$ , on point  $(2, 1, 3)$

$$\mathbf{r}(t) = [2, 1, 3] + t[1, 2, 0]$$

$$= [t+2, 2t+1, 3]$$

**24-28 TANGENT  $U(t)$**

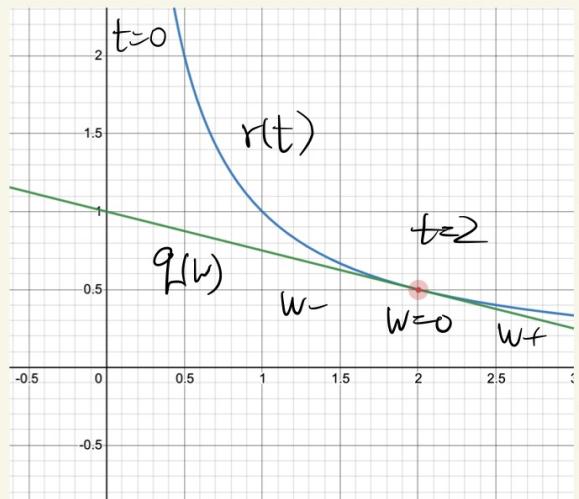
Given a curve  $C$ :  $\mathbf{r}(t)$ , find a tangent vector  $\mathbf{r}'(t)$ , a unit tangent vector  $\mathbf{u}'(t)$ , and the tangent of  $C$  at  $P$ . Sketch curve and tangent.

24.  $\mathbf{r}(t) = [t, \frac{1}{2}t^2, 1]$ ,  $P: (2, 2, 1)$
25.  $\mathbf{r}(t) = [10 \cos t, 1, 10 \sin t]$ ,  $P: (6, 1, 8)$
26.  $\mathbf{r}(t) = [\cos t, \sin t, 9t]$ ,  $P: (1, 0, 18\pi)$
27.  $\mathbf{r}(t) = [t, 1/t, 0]$ ,  $P: (2, \frac{1}{2}, 0)$

$$27. \quad \mathbf{r}'(t) = [1, -\frac{1}{t^2}, 0]$$

$$\begin{aligned} U(t) &= \frac{\mathbf{r}'}{|\mathbf{r}'|} \cdot \mathbf{r}' \\ &= \frac{1}{\sqrt{1 + \frac{1}{t^4}}} \cdot \mathbf{r}' \\ &= \frac{t^2}{t^4 + 1} \cdot \mathbf{r}' \\ &= \left[ \frac{t^2}{t^4 + 1}, -\frac{1}{t^4 + 1}, 0 \right] \end{aligned}$$

$$\begin{aligned} q(w) &= \mathbf{r} + w\mathbf{r}' \\ &= \left[ t, \frac{1}{t}, 0 \right] + \left[ w, -\frac{w}{t^2}, 0 \right] \\ &= \left[ 2, \frac{1}{2}, 0 \right] + \left[ w, -\frac{w}{4}, 0 \right] \\ &= \left[ 2+w, \frac{1}{2}-\frac{w}{4}, 0 \right] \end{aligned}$$



## 29-32 LENGTH

Find the length and sketch the curve.

31. Circle  $\mathbf{r}(t) = [a \cos t, a \sin t]$  from  $(a, 0)$  to  $(0, a)$ .

from  $t=0$  to  $t=\frac{\pi}{2}$

$$l = \int_0^{\frac{\pi}{2}} \sqrt{\mathbf{r}' \cdot \mathbf{r}'} dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(-a \sin t, a \cos t) \cdot (-a \sin t, a \cos t)} dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt$$

$$= \int_0^{\frac{\pi}{2}} |a| dt = |a| t \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} |a|$$

35-46

## CURVES IN MECHANICS

Forces acting on moving objects (cars, airplanes, ships, etc.) require the engineer to know corresponding **tangential** and **normal accelerations**. In Probs. 35–38 find them, along with the **velocity** and **speed**. Sketch the path.

35. Parabola  $\mathbf{r}(t) = [t, t^2, 0]$ . Find  $\mathbf{v}$  and  $\mathbf{a}$ .

$$\mathbf{v} = \mathbf{r}'(t) = [1, 2t, 0]$$

$$\mathbf{a} = \mathbf{v}' = [0, 2, 0]$$

$$\text{Speed} = \sqrt{1^2 + 4t^2}$$

$$\text{Acceleration} = 2$$

47-55

## CURVATURE AND TORSION

47. Circle. Show that a circle of radius  $a$  has curvature  $1/a$ .

$$\mathbf{r}(t) = [a \cos t, a \sin t], \quad a > 0$$

$$\mathbf{r}(s) = [a \cos(s/r), a \sin(s/r)]$$

$$\begin{aligned} K(s) &= |\mathbf{r}''(s)| = \left| \left[ -\frac{\sin(s/a)}{a}, \frac{\cos(s/a)}{a} \right]' \right| \\ &= \left| \left[ -\frac{1}{a} \left[ -\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right) \right] \right]' \right| \\ &= \left| \frac{1}{a} \left[ -\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right) \right] \right| \\ &= \frac{1}{a} \cdot 1 = \frac{1}{a} \end{aligned}$$

# 9.7

## 1-6 CALCULATION OF GRADIENTS

Find grad  $f$ . Graph some level curves  $f = \text{const}$ . Indicate  $\nabla f$  by arrows at some points of these curves.

$$1. f = (x + 1)(2y - 1)$$

$$\nabla f = [2y - 1, 2(x+1)]$$

$$5. f = x^4 + y^4$$

$$\nabla f = [4x^3, 4y^3]$$

## 7-10 USEFUL FORMULAS FOR GRADIENT AND LAPLACIAN

Prove and illustrate by an example.

$$7. \nabla(f^n) = nf^{n-1}\nabla f$$

$$\text{let } g = f^n$$

$$\nabla g = g' = nf^{n-1} \cdot \nabla f$$

$$\nabla(f^2) = 2f \cdot \nabla f$$

## 11-15 USE OF GRADIENTS. ELECTRIC FORCE

The force in an electrostatic field given by  $f(x, y, z)$  has the direction of the gradient. Find  $\nabla f$  and its value at  $P$ .

$$11. f = xy, P: (-4, 5)$$

$$12. f = x/(x^2 + y^2), P: (1, 1)$$

$$13. f = \ln(x^2 + y^2), P: (8, 6)$$

$$13. \nabla f = \left[ \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right]$$

$$\nabla f(P) = \left[ \frac{16}{100}, \frac{12}{100} \right]$$

9.8. 1.5. 15

**1-6 CALCULATION OF THE DIVERGENCE**

Find  $\operatorname{div} \mathbf{v}$  and its value at  $P$ .

1.  $\mathbf{v} = [x^2, 4y^2, 9z^2], P: (-1, 0, \frac{1}{2})$  *Mistake*

$$\operatorname{div} \mathbf{v} = 2x + 8y + 18z$$

on point  $P: (-1, 0, \frac{1}{2})$

$$\operatorname{div} \mathbf{v} = 2(-1) + 0 + 18 \cdot \frac{1}{2} = 7$$

**15-20 LAPLACIAN**

Calculate  $\nabla^2 f$  by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

15.  $f = \cos^2 x + \sin^2 y$

$$\begin{aligned}\operatorname{grad} f &= [2 \cos x \cdot (-\sin x), \\ &\quad 2 \sin y \cdot \cos y] \\ &= [-\sin 2x, \sin 2y]\end{aligned}$$

$$\begin{aligned}\nabla^2 f &= \operatorname{div}(\operatorname{grad} f) \\ &= -2 \cos 2x + 2 \cos 2y\end{aligned}$$

?

5.  $\mathbf{v} = x^2y^2z^2[x, y, z], P: (3, -1, 4)$

$$\mathbf{v} = [x^3y^2z^2, x^2y^3z^2, x^2y^2z^3]$$

$$\begin{aligned}\operatorname{div} \mathbf{v} &= 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2 \\ &= 9x^2y^2z^2\end{aligned}$$

on point  $(3, -1, 4)$

$$\operatorname{div} \mathbf{v} = 9 \cdot 9 \cdot 1 \cdot 16 = 1296$$

9.9 5.7. 11. 15

#### 4-8 CALCULATION OF CURL

Find curl  $\mathbf{v}$  for  $\mathbf{v}$  given with respect to right-handed Cartesian coordinates. Show the details of your work.

4.  $\mathbf{v} = [2y^2, 5x, 0]$

5.  $\mathbf{v} = xyz[x, y, z]$

6.  $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}[x, y, z]$

7.  $\mathbf{v} = [0, 0, e^{-x} \sin y]$

8.  $\mathbf{v} = [e^{-z^2}, e^{-x^2}, e^{-y^2}]$

5.  $\mathbf{v} = [x^2yz, xy^2z, xy^2z^2]$

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz, xy^2z, xy^2z^2 \end{vmatrix}$$

$$= (xz^2 - xy^2)\mathbf{i} - (yz^2 - x^2y)\mathbf{j} + (y^2z - x^2z)\mathbf{k}$$

7.  $\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & e^{-x} \sin y \end{vmatrix}$

$$= [e^{-x} \cos y]\mathbf{i} - [\sin y (-e^{-x})]\mathbf{j}$$

$$= e^{-x} \cos y \mathbf{i} + \sin y e^{-x} \mathbf{j} + 0 \mathbf{k}$$

$$= e^{-x} [\cos y, \sin y, 0]$$

#### 9-13 FLUID FLOW

Let  $\mathbf{v}$  be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles). Hint. See the answers to Probs. 9 and 11 for a determination of a path.

9.  $\mathbf{v} = [0, 3z^2, 0]$

10.  $\mathbf{v} = [\sec x, \csc x, 0]$

11.  $\mathbf{v} = [y, -2x, 0]$

$$11. \text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -2x & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} - 3\mathbf{k}$$

Not irrotational.

$\text{div } \mathbf{v} = 0$ , incompressible.

Find the streamlines

#### 15-20 DIV AND CURL

With respect to right-handed coordinates, let  $\mathbf{u} = [y, z, x]$ ,  $\mathbf{v} = [yz, zx, xy]$ ,  $f = xyz$ , and  $g = x + y + z$ . Find the given expressions. Check your result by a formula in Proj. 14 if applicable.

15.  $\text{curl}(\mathbf{u} + \mathbf{v})$ ,  $\text{curl } \mathbf{v}$

$$\text{curl}(\mathbf{u} + \mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+yz, z+zx, x+xy \end{vmatrix}$$

$$= (x-1-x)\mathbf{i} - (1+y-y)\mathbf{j} - \mathbf{k}$$

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz, zx, xy \end{vmatrix}$$

$$= (x-x)\mathbf{i} - (y-y)\mathbf{j} + (z-z)\mathbf{k}$$

$$= 0$$

Textbook is wrong