

Chapter 10 Vector Integral Calculus.

Integral Theorems

Selected Problem set 10.1

10.1 3.5.9.19

2-11 LINE INTEGRAL. WORK

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If \mathbf{F} is a force, this gives the work done by the force in the displacement along C . Show the details.

2. $\mathbf{F} = [y^2, -x^2]$, $C: y = 4x^2$ from $(0, 0)$ to $(1, 4)$
3. \mathbf{F} as in Prob. 2, C from $(0, 0)$ straight to $(1, 4)$. Compare.
4. $\mathbf{F} = [xy, x^2y^2]$, C from $(2, 0)$ straight to $(0, 2)$
5. \mathbf{F} as in Prob. 4, C the quarter-circle from $(2, 0)$ to $(0, 2)$ with center $(0, 0)$

$$\begin{aligned}
 3. \quad C: \mathbf{r}(t) &= [t, 4t] = t\mathbf{i} + 4t\mathbf{j} \\
 \mathbf{F}(\mathbf{r}(t)) &= [(4t)^2, -t^2] = [16t^2, -t^2] \\
 \mathbf{r}'(t) &= [1, 4] \\
 \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^1 [16t^2, -t^2] \cdot [1, 4] dt \\
 &= \int_0^1 (16t^2 - 4t^2) dt \\
 &= \int_0^1 12t^2 dt \\
 &= 4t^3 \Big|_0^1 = 4 - 0 = 4
 \end{aligned}$$

$$\begin{aligned}
 5. \quad C \text{ by } \mathbf{r}(t) &= [2\cos t, 2\sin t], \\
 \text{when } 0 &\leq t \leq \frac{\pi}{2} \\
 \mathbf{F}(\mathbf{r}(t)) &= [4\sin t \cos t, 16\sin^2 \cos^2 t] \\
 \mathbf{r}'(t) &= [-2\sin t, 2\cos t] \\
 \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} (-8\sin t \cos t + 32\sin^2 \cos^2 t) dt \\
 &= 8 \int_0^{\frac{\pi}{2}} (4\sin^2 \cos^2 t - \sin^2 \cos t) dt \\
 &= 8 \int_0^{\frac{\pi}{2}} \cos t (4\cos^2 t - 1) \sin^2 t dt \\
 &= 8 \int_0^{\frac{\pi}{2}} \cos t [-\sin^2 t (4\sin^2 t - 3)] dt \\
 u &= \sin t, \quad \frac{du}{dt} = \cos t, \quad dt = \frac{du}{\cos t} \\
 &= -8 \int_0^1 u^2 (4u^2 - 3) du \\
 &= -32 \int_0^1 u^4 du + 24 \int_0^1 u^2 du \\
 &= -\frac{32}{5} u^5 \Big|_0^1 + 24 \cdot \frac{1}{3} u^3 \Big|_0^1 \\
 &= -\frac{32}{5} + 8 = \frac{8}{5} = 1.6
 \end{aligned}$$

9. $\mathbf{F} = [x + y, y + z, z + x]$, $C: \mathbf{r} = [2t, 5t, t]$ from $t = 0$ to 1. Also from $t = -1$ to 1.

$$C: \mathbf{r} = [2t, 5t, t] \quad 0 \leq t \leq 1$$

$$\mathbf{r}' = [2, 5, 1]$$

$$\mathbf{F}(\mathbf{r}(t)) = [7t, 6t, 3t]$$

$$\int_C \mathbf{F}(\mathbf{r}) d\mathbf{r} = \int_0^1 [7t, 6t, 3t] [2, 5, 1] \cdot dt$$

$$= \int_0^1 47t \cdot dt$$

$$= \frac{47}{2} t^2 \Big|_0^1 = \frac{47}{2} = 23.5$$

$$-1 \leq t \leq 1$$

$$\int_{-1}^1 47t \cdot dt = \frac{47}{2} t^2 \Big|_{-1}^1 = 0$$

19. $f = xyz$, $C: \mathbf{r} = [4t, 3t^2, 12t]$, $-2 \leq t \leq 2$.
Sketch C .

$$C: \mathbf{r} = [4t, 3t^2, 12t] \quad -2 \leq t \leq 2$$

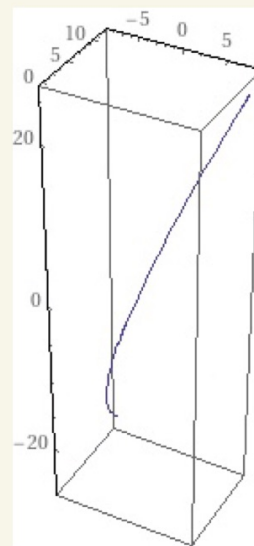
$$\mathbf{r}' = [4, 6t, 12]$$

$$\mathbf{F}(\mathbf{r}(t)) = 144t^4$$

$$\int_C f(\mathbf{r}) dt = \int_{-2}^2 144t^4 \cdot dt$$

$$= \frac{144}{5} t^5 \Big|_{-2}^2$$

$$= \frac{144}{5} \cdot 64 = 1843.2$$



10.2. 3. 5. 13. 15.

3-9 PATH INDEPENDENT INTEGRALS

Show that the form under the integral sign is exact in the plane (Probs. 3-4) or in space (Probs. 5-9) and evaluate the integral. Show the details of your work.

$$3. \int_{(\pi/2, \pi)}^{(\pi, 0)} \left(\frac{1}{2} \cos \frac{1}{2} x \cos 2y \, dx - 2 \sin \frac{1}{2} x \sin 2y \, dy \right)$$

$$4. \int_{(4, 0)}^{(6, 1)} e^{4y} (2x \, dx + 4x^2 \, dy)$$

$$5. \int_{(0, 0, \pi)}^{(2, 1/2, \pi/2)} e^{xy} (y \sin z \, dx + x \sin z \, dy + \cos z \, dz)$$

3. exactness:

$$(F_2)_x = -2 \sin 2y \cdot \frac{1}{2} \cdot \cos \frac{1}{2} x$$

$$= -\sin 2y \cdot \cos \frac{1}{2} x$$

$$(F_1)_y = \frac{1}{2} \cdot \cos \frac{1}{2} x \cdot (-\sin 2y) \cdot 2$$

$$= -\sin 2y \cos \frac{1}{2} x$$

Evaluate:

$$f_x = F_1 = \frac{1}{2} \cos \frac{1}{2} x \cdot \cos 2y$$

$$f_y = F_2 = -2 \sin \frac{1}{2} x \sin 2y$$

$$f = \cos 2y \cdot \sin \frac{1}{2} x + g(y)$$

$$f_y = \sin \frac{1}{2} x \cdot (-\sin 2y) \cdot 2 + g_y$$

$$f = \cos 2y \cdot \sin \frac{1}{2} x$$

$$\begin{aligned} f(\pi, 0) - f\left(\frac{\pi}{2}, \pi\right) &= 1 \cdot 1 - 1 \cdot \frac{\sqrt{2}}{2} \\ &= 1 - \frac{\sqrt{2}}{2} \end{aligned}$$

5. exactness:

$$(F_3)_y = x \cdot e^{xy} \cdot \cos z$$

$$(F_2)_z = e^{xy} \cdot x \cdot \cos z$$

$$(F_1)_z = e^{xy} \cdot y \cdot \cos z$$

$$(F_3)_x = \cos z \cdot y \cdot e^{xy}$$

$$(F_1)_y = \sin z (x \cdot e^{xy} \cdot y + e^{xy})$$

$$(F_2)_x = \sin z (y \cdot e^{xy} \cdot x + e^{xy})$$

Evaluate:

$$f_x = F_1 = e^{xy} \cdot y \cdot \sin z$$

$$f_y = F_2 = e^{xy} \cdot x \cdot \sin z$$

$$f_z = F_3 = e^{xy} \cdot \cos z$$

$$f = \sin z \cdot e^{xy} + g(y, z)$$

$$\begin{aligned} f_y &= x \cdot \sin z \cdot e^{xy} + g_y \\ &= x \cdot \sin z \cdot e^{xy} + h(z) \end{aligned}$$

$$f_z = e^{xy} \cdot \cos z + h'$$

$$h' = 0, \quad h = 0, \quad g = 0$$

$$f = \sin z \cdot e^{xy}$$

$$\begin{aligned} f\left(2, \frac{1}{2}, \frac{\pi}{2}\right) - f(0, 0, \pi) \\ = 1 \cdot e - 0 = e \end{aligned}$$

13-19 PATH INDEPENDENCE?

Check, and if independent, integrate from $(0, 0, 0)$ to (a, b, c) .

13. $2e^{x^2}(x \cos 2y \, dx - \sin 2y \, dy)$

check if independent

$$f_x = F_1 = 2e^{x^2} \cdot x \cdot \cos 2y$$

$$f_y = F_2 = -2e^{x^2} \cdot \sin 2y$$

$$f = \cos 2y \cdot e^{x^2} + g$$

$$f_y = e^{x^2} \cdot (-\sin 2y) \cdot 2 + g'$$

$$g' = 0 \quad g = 0, \text{ say}$$

$$f = \cos 2y \cdot e^{x^2}$$

Independent.

$$f(a, b, c) - f(0, 0, 0)$$

$$= \cos(2b) \cdot e^{a^2} - 1 \cdot e^0$$

$$= \cos(2b) \cdot e^{a^2} - 1$$

answer is wrong

15. $x^2y \, dx - 4xy^2 \, dy + 8z^2x \, dz$

check if independent

$$f_x = F_1 = x^2y$$

$$f_y = F_2 = -4xy^2$$

$$f_z = F_3 = 8z^2x$$

$$f = \frac{1}{3} \cdot y \cdot x^3 + g(y, z)$$

$$f_y = \frac{1}{3} \cdot x^3 + g_y$$

$$g_y = -4xy^2 - \frac{1}{3}x^3 \Rightarrow \text{dependent}$$

$$g(y, z) = -\frac{4}{3}xy^3 - \frac{1}{3}x^3y = 0$$

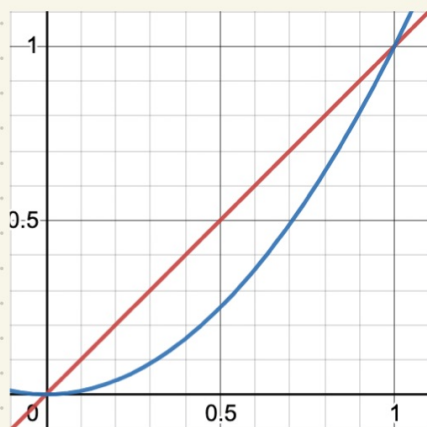
$$4xy^3 + x^3y = 0$$

$$4y^2 + x^2 = 0$$

10.3 5 9 15

5. $\int_0^1 \int_{x^2}^x (1 - 2xy) dy dx$

$$\begin{aligned}
 &= \int_0^1 \left[(y - xy^2) \Big|_{x^2}^x \right] dx \\
 &= \int_0^1 [x - x^3 - (x^2 - x^5)] dx \\
 &= \int_0^1 (x^5 - x^3 - x^2 + x) dx \\
 &= \frac{x^6}{6} - \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} \Big|_0^1 \\
 &= \frac{1}{6} - \frac{1}{4} - \frac{1}{3} + \frac{1}{2} \\
 &= \frac{2-3-4+6}{12} = \frac{1}{12}
 \end{aligned}$$



Red: $y = x$

Blue: $y = x^2$

$f(x, y) = 1 - 2xy$ not sure
how to show this...

9. The region beneath $z = 4x^2 + 9y^2$ and above the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 2)$, $(0, 2)$ in the xy -plane.

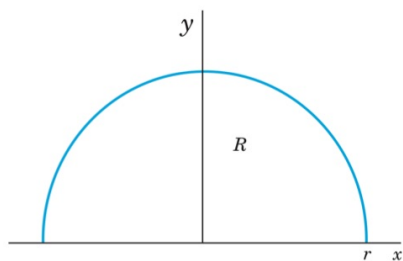
$$\begin{aligned}
 &\int_0^3 \int_0^2 (4x^2 + 9y^2) dy dx \\
 &= \int_0^3 \left[(4x^2 y + 3y^3) \Big|_0^2 \right] dx \\
 &= \int_0^3 (8x^2 + 24 - 0) dx \\
 &= \frac{8}{3} x^3 + 24x \Big|_0^3 \\
 &= 72 + 24 \times 3 = 144
 \end{aligned}$$

12-16

CENTER OF GRAVITY

Find the center of gravity (\bar{x}, \bar{y}) of a mass of density $f(x, y) = 1$ in the given region R .

15.



$$M = \iint_R f(x, y) dx dy = \int_0^\pi \int_0^r r dr d\theta = \int_0^\pi \frac{r^2}{2} d\theta = \frac{r}{2} \pi r^2$$

$$\bar{x} = \frac{1}{M} \iint_R x f(x, y) dx dy = 0, \text{ for reasons of symmetry.}$$

$$\begin{aligned} \bar{y} &= \frac{1}{M} \iint_R y f(x, y) dx dy = \frac{2}{\pi r^2} \int_0^\pi \int_0^r r \sin \theta r dr d\theta \\ &= \frac{2}{\pi r^2} \int_0^\pi \left(\sin \theta \cdot \frac{r^3}{3} \Big|_0^r \right) d\theta \\ &= \frac{2}{\pi r^2} \int_0^\pi \frac{r^3}{3} \sin \theta d\theta \\ &= \frac{2}{\pi r^2} \cdot \frac{r^3}{3} \cdot (-\cos \theta \Big|_0^\pi) \\ &= \frac{4r}{3\pi} \end{aligned}$$