W10 - Cauchy's Integral Formula

10A: Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig. 356),

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Multiply connected domains

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z - z_0}\, dz + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z - z_0}\, dz,$$

10B: Derivatives of Analytic Functions

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3}\, dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 1, 2, \cdots);$$

So we can get:

$$\oint_c \frac{f(z)dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Cauchy's Inequality

$$|f^{(n)}(z_0)| \leqq \frac{n!M}{r^n}.$$

Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.

Morera's[2] Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\, dz = 0 \qquad (3)$$

for every closed path in D, then $f(z)$ is analytic in D.