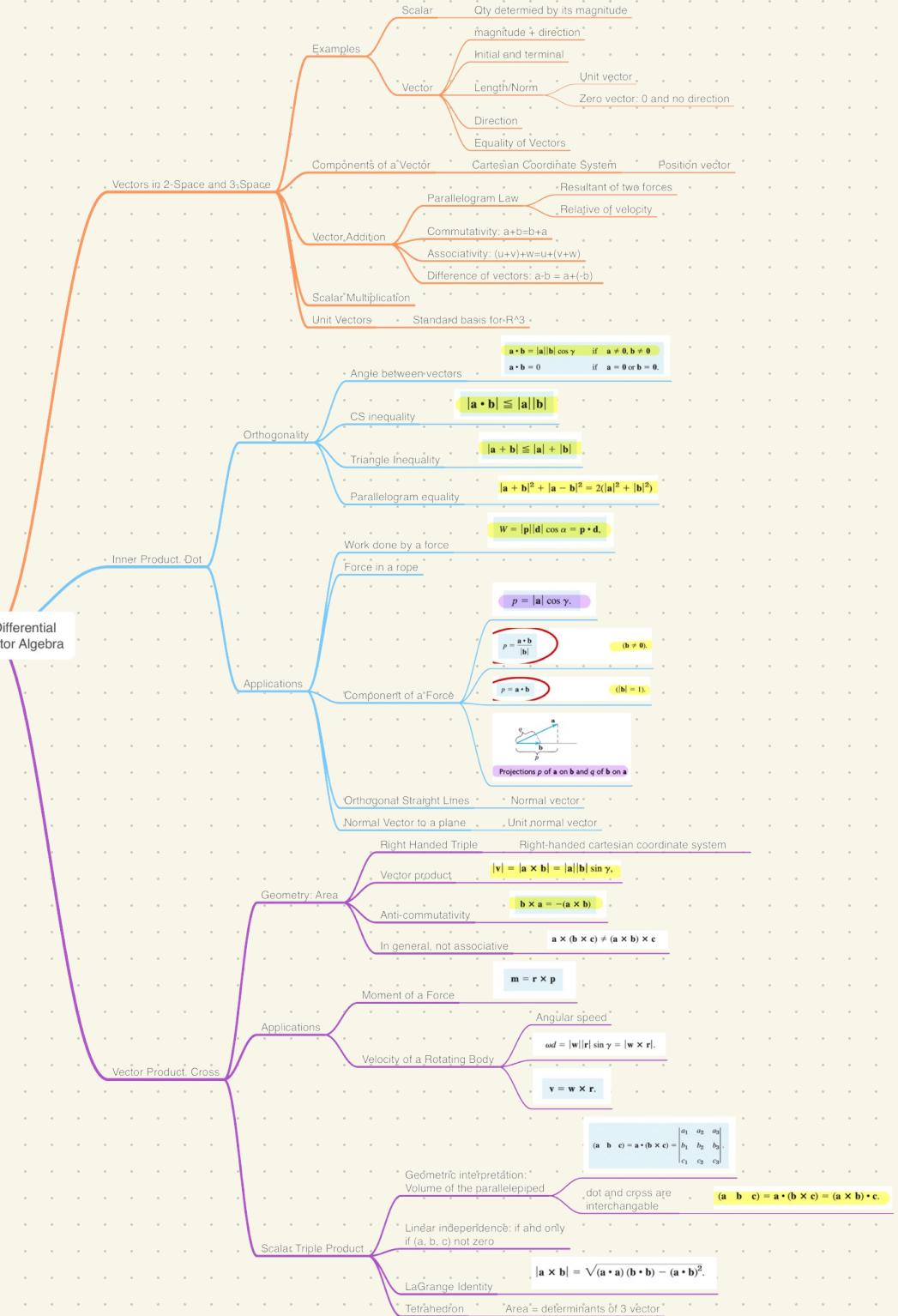
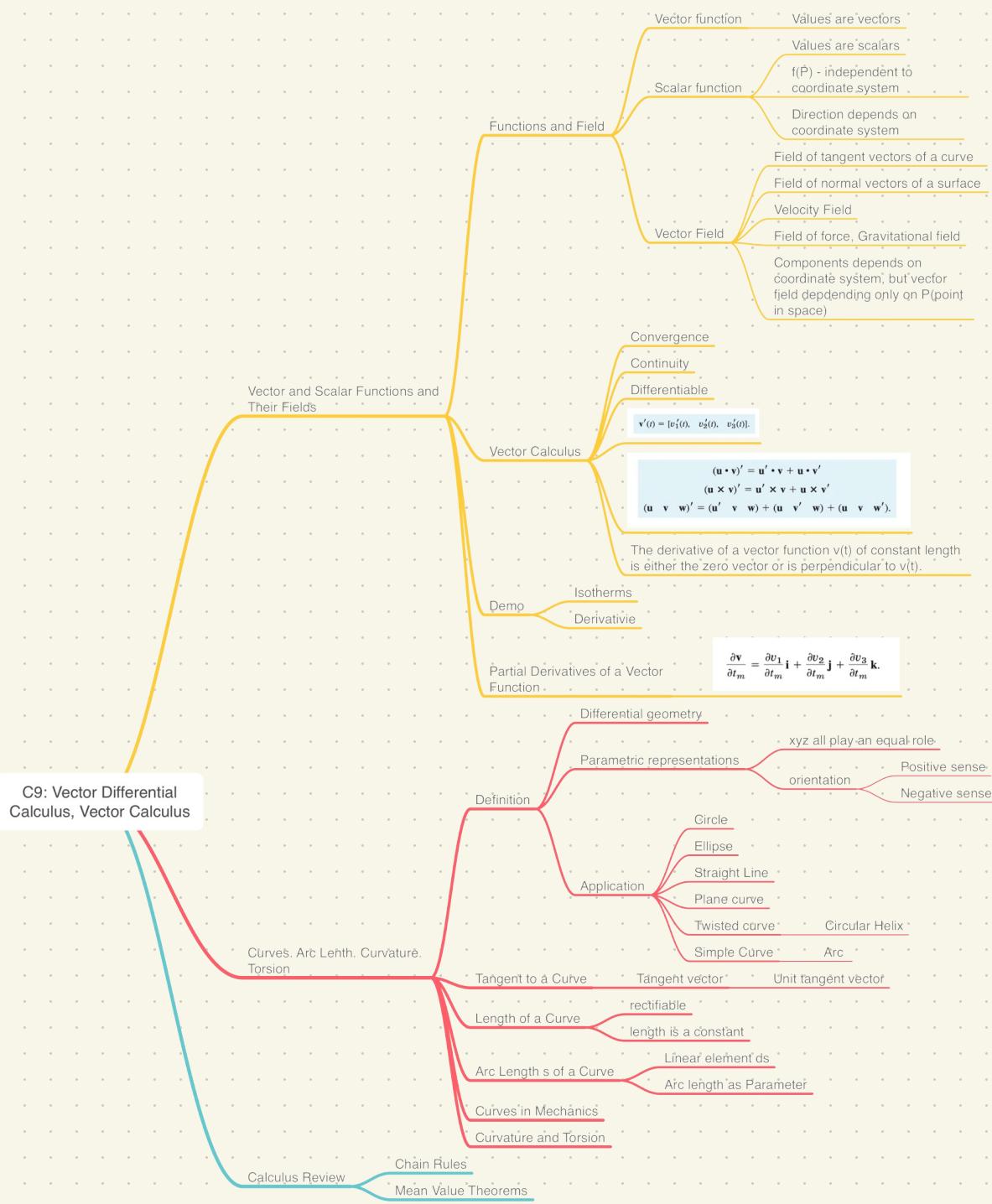


# W3

## C9: Vector Differential Calculus, Vector Algebra





C9

**Basic Properties of Scalar Multiplication.** From the definitions we obtain directly

- (a)  $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$   
 (b)  $(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$   
 (c)  $c(k\mathbf{a}) = (ck)\mathbf{a}$   
 (written  $ck\mathbf{a}$ )  
 (d)  $1\mathbf{a} = \mathbf{a}.$

**Basic Properties of Vector Addition.** Familiar laws for real numbers give immediately

- (4) (a)  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$  (Commutativity)  
 (b)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$  (Associativity)  
 (c)  $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$   
 (d)  $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \gamma \quad \text{if } \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$$

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \text{if } \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}.$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

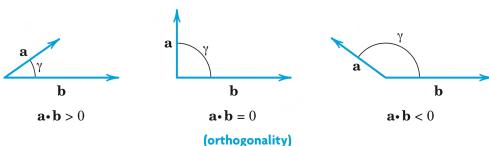


Fig. 178. Angle between vectors and value of inner product

### THEOREM 1

#### Orthogonality Criterion

The inner product of two nonzero vectors is 0 if and only if these vectors are perpendicular.

**Length and Angle.** Equation (1) with  $\mathbf{b} = \mathbf{a}$  gives  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ . Hence

$$(3) \quad |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

From (3) and (1) we obtain for the angle  $\gamma$  between two nonzero vectors

$$(4) \quad \cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}}}.$$

$$\begin{aligned}
 (a) \quad & (q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_2\mathbf{b} \cdot \mathbf{c} && (\text{Linearity}) \\
 (5) \quad (b) \quad & \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} && (\text{Symmetry}) \\
 (c) \quad & \mathbf{a} \cdot \mathbf{a} \geq 0 && \\
 & \mathbf{a} \cdot \mathbf{a} = 0 \quad \text{if and only if } \mathbf{a} = \mathbf{0} && \left. \right\} (\text{Positive-definiteness}).
 \end{aligned}$$

Hence dot multiplication is commutative as shown by (5b). Furthermore, it is distributive with respect to vector addition. This follows from (5a) with  $q_1 = 1$  and  $q_2 = 1$ :

$$(5a^*) \quad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \quad (\text{Distributivity}).$$

Furthermore, from (1) and  $|\cos \gamma| \leq 1$  we see that

$$(6) \quad |\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}| \quad (\text{Cauchy-Schwarz inequality}).$$

Using this and (3), you may prove (see Prob. 16)

$$(7) \quad |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \quad (\text{Triangle inequality}).$$

Geometrically, (7) with  $<$  says that one side of a triangle must be shorter than the other two sides together; this motivates the name of (7).

A simple direct calculation with inner products shows that

$$(8) \quad |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad (\text{Parallelogram equality}).$$

### Work Done by a Force Expressed as an Inner Product

This is a major application. It concerns a body on which a *constant* force  $\mathbf{p}$  acts. (For a *variable* force, see Sec. 10.1.) Let the body be given a displacement  $\mathbf{d}$ . Then the work done by  $\mathbf{p}$  in the displacement is defined as

$$(9) \quad W = |\mathbf{p}| |\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d},$$

Example 3 is typical of applications that deal with the **component or projection of a vector  $\mathbf{a}$  in the direction of a vector  $\mathbf{b}$**  ( $\neq \mathbf{0}$ ). If we denote by  $p$  the length of the orthogonal projection of  $\mathbf{a}$  on a straight line  $l$  parallel to  $\mathbf{b}$  as shown in Fig. 181, then

$$(10) \quad p = |\mathbf{a}| \cos \gamma.$$

Here  $p$  is taken with the plus sign if  $p\mathbf{b}$  has the direction of  $\mathbf{b}$  and with the minus sign if  $p\mathbf{b}$  has the direction opposite to  $\mathbf{b}$ .

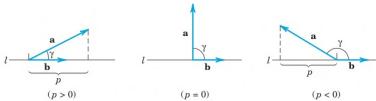


Fig. 181. Component of a vector  $\mathbf{a}$  in the direction of a vector  $\mathbf{b}$

Multiplying (10) by  $|\mathbf{b}|/|\mathbf{b}| = 1$ , we have  $\mathbf{a} \cdot \mathbf{b}$  in the numerator and thus

$$(11) \quad p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \quad (\mathbf{b} \neq \mathbf{0}).$$

If  $\mathbf{b}$  is a unit vector, as it is often used for fixing a direction, then (11) simply gives

$$(12) \quad p = \mathbf{a} \cdot \mathbf{b} \quad (|\mathbf{b}| = 1).$$

Figure 182 shows the projection  $p$  of  $\mathbf{a}$  in the direction of  $\mathbf{b}$  (as in Fig. 181) and the projection  $q$  of  $\mathbf{b}$  in the direction of  $\mathbf{a}$ .

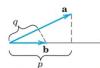


Fig. 182. Projections  $p$  of  $\mathbf{a}$  on  $\mathbf{b}$  and  $q$  of  $\mathbf{b}$  on  $\mathbf{a}$

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \gamma,$$

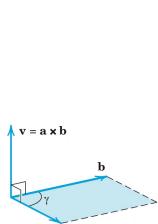


Fig. 185. Vector product

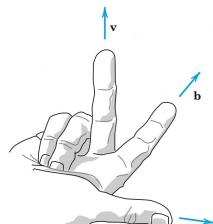


Fig. 186. Right-handed triple of vectors  $\mathbf{a}, \mathbf{b}, \mathbf{v}$

$$(2**)\quad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

### General Properties of Vector Products

(a) For every scalar  $l$ ,

$$(4) \quad (l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) Cross multiplication is distributive with respect to vector addition; that is,

$$(5) \quad (\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$

$$(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

(c) Cross multiplication is not commutative but anticommutative; that is,

$$(6) \quad \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \quad (\text{Fig. 189}).$$

(d) Cross multiplication is not associative; that is, in general,

$$(7) \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

so that the parentheses cannot be omitted.

$$(\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

### THEOREM 2

#### Properties and Applications of Scalar Triple Products

(a) In (10) the dot and cross can be interchanged:

$$(11) \quad (\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

(b) Geometric interpretation. The absolute value  $|(\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c})|$  of (10) is the volume of the parallelepiped (oblique box) with  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  as edge vectors (Fig. 193).

(c) Linear independence. Three vectors in  $R^3$  are linearly independent if and only if their scalar triple product is not zero.

$$(c\mathbf{v})' = c\mathbf{v}'$$

( $c$  constant),

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

$$(\mathbf{u} - \mathbf{v} - \mathbf{w})' = (\mathbf{u}' - \mathbf{v} - \mathbf{w}) + (\mathbf{u} - \mathbf{v}' - \mathbf{w}) + (\mathbf{u} - \mathbf{v} - \mathbf{w}').$$

$$l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'} dt$$

$$\left( \mathbf{r}' = \frac{d\mathbf{r}}{dt} \right).$$

Constant

the **arc length function** or simply the **arc length** of  $C$ . Thus

$$(11) \quad s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'} d\tilde{t} \quad \left( \mathbf{r}' = \frac{d\mathbf{r}}{d\tilde{t}} \right).$$

$$(13) \quad ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 + dz^2.$$

$ds$  is called the **linear element** of  $C$ .

$$\mathbf{u}(s) = \mathbf{r}'(s).$$

$$\kappa(s) = |\mathbf{u}'(s)| = |\mathbf{r}''(s)| \quad (' = d/ds).$$

$$|\tau(s)| = |\mathbf{b}'(s)|.$$

$$\tau(s) = -\mathbf{p}(s) \cdot \mathbf{b}'(s).$$