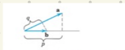
C9: Vector Differential Calculus, Vector Calculus

- Vector and Scalar Functions and Their Fields
  - Functions and Field
    - Vector function
    - Scalar function
      - f(P) - independent to coordinate system
      - Direction depends on coordinate system
    - Vector Field
      - Field of tangent vectors of a curve
      - Field of normal vectors of a surface
      - Velocity Field
      - Field of force, Gravitational field
      - Components depends on coordinate system, but vector field depdending only on P(point in space)
  - Vector Calculus
    - Convergence
    - Continuity
    - Differentiable
    - $\mathbf{v}'(t) = [v_1'(t), \quad v_2'(t), \quad v_3'(t)].$
    - $$\begin{gathered}(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \\ (\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' \\ (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}').\end{gathered}$$
    - The derivative of a vector function v(t) of constant length is either the zero vector or is perpendicular to v(t).
  - Demo
    - Isotherms
    - Derivativie
  - Partial Derivatives of a Vector Function
    - $$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$
- Curves. Arc Lenth. Curvature. Torsion
  - Definition
    - Differential geometry
    - Parametric representations
      - xyz all play an equal role
      - Induces an orientation on C
        - Positive sense
        - Negative sense
      - Application
        - Circle
        - Ellipse
        - Straight Line
        - Plane curve
        - Twisted curve
          - Circular Helix
        - Simple Curve
          - Arc
  - Tangent to a Curve
    - Tangent vector
      - Unit tangent vector
  - Length of a Curve
  - Arc Length s of a Curve
  - Curves in Mechanics
  - Curvature and Torsion
- Calculus Review
  - Chain Rules
  - Mean Value Theorems
- Gradient of a Scalar Field
  - Directional Derivative
  - Gradient is a vector. Maximum increase
  - Gradient as Surface Normal Vector
  - Vector Fields that are gradients of scalar fields (Potentials)
- Divergence of a Vector Field
- Curl of a Vector Field

C9: Vector Differential Calculus, Vector Algebra

- Vectors in 2-Space and 3-Space
  - Examples
    - Scalar
    - Vector
      - Initial and terminal
      - Length/Norm
        - Unit vector
      - Direction
      - Equality of Vectors
  - Components of a Vector
    - Cartesian Coordinate System
  - Vector Addition
    - Parallelogram Law
      - Resultant of two forces
      - Relative of velocity
    - Commutativity: a+b=b+a
    - Associativity: (u+v)+w=u+(v+w)
    - Difference of vectors: a-b = a+(-b)
  - Scalar Multiplication
  - Unit Vectors
    - Standard basis for R^3
- Inner Product. Dot
  - Orthogonality
    - Angle between vectors
      - $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma \quad \text{if} \quad \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$
      - $\mathbf{a} \cdot \mathbf{b} = 0 \quad \text{if} \quad \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}.$
    - CS inequality
      - $|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}||\mathbf{b}|$
    - Triangle Inequality
      - $|\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}|$
    - Parallelogram equality
      - $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$
  - Applications
    - Work done by a force
      - $W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d},$
    - Force in a rope
    - Component of a Force
      - $p = |\mathbf{a}| \cos \gamma.$
      - $p = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \quad (\mathbf{b} \neq \mathbf{0}).$
      - $p = \mathbf{a} \cdot \mathbf{b} \quad (|\mathbf{b}| = 1).$
      - 

        Projections p of **a** on **b** and q of **b** on **a**
    - Orthogonal Straight Lines
      - Normal vector
    - Normal Vector to a plane
      - Unit normal vector
- Vector Product. Cross
  - Geometry: Area
    - Right Handed Triple
      - Right-handed cartesian coordinate system
    - Vector product
      - $|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma,$
    - Anti-commutativity
      - $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$
    - In general, not associative
      - $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
  - Applications
    - Moment of a Force
      - $\mathbf{m} = \mathbf{r} \times \mathbf{p}$
    - Velocity of a Rotating Body
      - Angular speed
      - $\omega d = |\mathbf{w}||\mathbf{r}| \sin \gamma = |\mathbf{w} \times \mathbf{r}|.$
      - $\mathbf{v} = \mathbf{w} \times \mathbf{r}.$
  - Scalar Triple Product
    - Geometric interpretation: Volume of the parallelepiped
      - $(\mathbf{a} \;\; \mathbf{b} \;\; \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$
      - dot and cross are interchangable
        - $(\mathbf{a} \;\; \mathbf{b} \;\; \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$
    - Linear independence: if and only if (a, b, c) not zero
    - LaGrange Identity
      - $|\mathbf{a} \times \mathbf{b}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2}.$
    - Tetrahedron

C9

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Basic Properties of Vector Addition. Familiar laws for real numbers give immediately

(4)

(a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (*Commutativity*)

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (*Associativity*)

(c) $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

(d) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$

Basic Properties of Scalar Multiplication. From the definitions we obtain directly

(6)

(a) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

(b) $(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$

(c) $c(k\mathbf{a}) = (ck)\mathbf{a}$ (written $ck\mathbf{a}$)

(d) $1\mathbf{a} = \mathbf{a}.$

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma \quad \text{if} \quad \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$$

$$\mathbf{a} \bullet \mathbf{b} = 0 \quad \text{if} \quad \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}.$$

$$\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

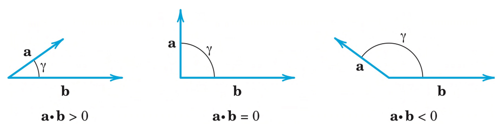


Fig. 178. Angle between vectors and value of inner product

THEOREM 1

Orthogonality Criterion

The inner product of two nonzero vectors is 0 *if and only if these vectors are perpendicular.*

Length and Angle. Equation (1) with $\mathbf{b} = \mathbf{a}$ gives $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$. Hence

(3) $$|\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}}.$$

From (3) and (1) we obtain for the angle γ between two nonzero vectors

(4) $$\cos \gamma = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a} \bullet \mathbf{b}}{\sqrt{\mathbf{a} \bullet \mathbf{a}}\sqrt{\mathbf{b} \bullet \mathbf{b}}}.$$

$$\text{(5)}\quad \begin{aligned} &\text{(a)} && (q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_1\mathbf{b} \cdot \mathbf{c} && \textit{(Linearity)} \\ &\text{(b)} && \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} && \textit{(Symmetry)} \\ &\text{(c)} && \left.\begin{array}{l} \mathbf{a} \cdot \mathbf{a} \geqq 0 \\ \mathbf{a} \cdot \mathbf{a} = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{array}\right\} && \textit{(Positive-definiteness)}. \end{aligned}$$

Hence *dot multiplication is commutative* as shown by (5b). Furthermore, it is *distributive with respect to vector addition.* This follows from (5a) with $q_1 = 1$ and $q_2 = 1$:

$$\text{(5a*)}\qquad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \qquad \textit{(Distributivity)}.$$

Furthermore, from (1) and $|\cos \gamma| \leqq 1$ we see that

$$\text{(6)}\qquad |\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}||\mathbf{b}| \qquad \textit{(Cauchy–Schwarz inequality)}.$$

Using this and (3), you may prove (see Prob. 16)

$$\text{(7)}\qquad |\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}| \qquad \textbf{(Triangle inequality)}.$$

Geometrically, (7) with $<$ says that one side of a triangle must be shorter than the other two sides together; this motivates the name of (7).

A simple direct calculation with inner products shows that

$$\text{(8)}\qquad |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \qquad \textit{(Parallelogram equality)}.$$

Work Done by a Force Expressed as an Inner Product

This is a major application. It concerns a body on which a *constant* force $\mathbf{p}$ acts. (For a *variable* force, see Sec. 10.1.) Let the body be given a displacement $\mathbf{d}$. Then the work done by $\mathbf{p}$ in the displacement is defined as

$$\text{(9)}\qquad W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d},$$

Example 3 is typical of applications that deal with the **component** or **projection** *of a vector* **a** *in the direction of a vector* **b** $(\neq \mathbf{0})$. If we denote by p the length of the orthogonal projection of **a** on a straight line l parallel to **b** as shown in Fig. 181, then

$$\text{(10)}\qquad p = |\mathbf{a}| \cos \gamma.$$

Here p is taken with the plus sign if $p\mathbf{b}$ has the direction of **b** and with the minus sign if $p\mathbf{b}$ has the direction opposite to **b**.

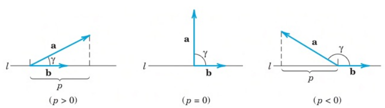


Fig. 181. Component of a vector **a** in the direction of a vector **b**

Multiplying (10) by $|\mathbf{b}|/|\mathbf{b}| = 1$, we have $\mathbf{a} \cdot \mathbf{b}$ in the numerator and thus

$$\text{(11)}\qquad p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \qquad (\mathbf{b} \neq \mathbf{0}).$$

If **b** is a unit vector, as it is often used for fixing a direction, then (11) simply gives

$$\text{(12)}\qquad p = \mathbf{a} \cdot \mathbf{b} \qquad (|\mathbf{b}| = 1).$$

Figure 182 shows the projection p of **a** in the direction of **b** (as in Fig. 181) and the projection $q = |\mathbf{b}| \cos \gamma$ of **b** in the direction of **a**.

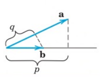


Fig. 182. Projections p of **a** on **b** and q of **b** on **a**

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma,$$

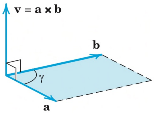


Fig. 185. Vector product

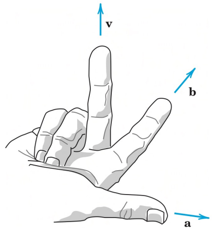


Fig. 186. Right-handed triple of vectors **a**, **b**, **v**

$$(2^{**}) \qquad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

General Properties of Vector Products

(a) *For every scalar l,*

$$(4) \qquad (l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) *Cross multiplication is distributive with respect to vector addition; that is,*

$$(5) \qquad \begin{aligned} &(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}), \\ &(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}). \end{aligned}$$

(c) *Cross multiplication is* ***not commutative*** *but* ***anticommutative;*** *that is,*

$$(6) \qquad \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \qquad \text{(Fig. 189).}$$

(d) *Cross multiplication is* ***not associative****; that is, in general,*

$$(7) \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

so that the parentheses cannot be omitted.

$$(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

THEOREM 2

Properties and Applications of Scalar Triple Products

(a) *In* (10) *the dot and cross can be interchanged:*

$$(11) \qquad (\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

(b) Geometric interpretation. *The absolute value* $|(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c})|$ *of* (10) *is the volume of the parallelepiped* (oblique box) *with* **a**, **b**, **c** *as edge vectors* (Fig. 193).

(c) Linear independence. *Three vectors in* R^3 *are linearly independent if and only if their scalar triple product is not zero.*

$$(c\mathbf{v})' = c\mathbf{v}' \qquad (c \text{ constant}),$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

$$(\mathbf{u} \bullet \mathbf{v})' = \mathbf{u}' \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{v}'$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

$$(\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}').$$

$$l = \int_a^b \sqrt{\mathbf{r}' \bullet \mathbf{r}'}\, dt \qquad \left(\mathbf{r}' = \frac{d\mathbf{r}}{dt}\right).$$

Constant

the *arc length function* or simply the **arc length** of C. Thus

(11) $$s(t) = \int_a^t \sqrt{\mathbf{r}' \bullet \mathbf{r}'}\, d\tilde{t} \qquad \left(\mathbf{r}' = \frac{d\mathbf{r}}{d\tilde{t}}\right).$$

(13) $$ds^2 = d\mathbf{r} \bullet d\mathbf{r} = dx^2 + dy^2 + dz^2.$$

ds is called the **linear element** of C.

$$\mathbf{u}(s) = \mathbf{r}'(s).$$

$$\kappa(s) = |\mathbf{u}'(s)| = |\mathbf{r}''(s)| \qquad ({}' = d/ds).$$

$$|\tau(s)| = |\mathbf{b}'(s)|.$$

$$\tau(s) = -\mathbf{p}(s) \bullet \mathbf{b}'(s).$$