

Chapter 8 Linear Algebra: Matrix Eigenvalue Problems

P329 - Problem set 8.1

P.S. 8.1

1-16 EIGENVALUES, EIGENVECTORS

Find the eigenvalues. Find the corresponding eigenvectors. Use the given λ or factor in Probs. 11 and 15.

1. $\begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$(A - \lambda I) \cdot X = 0$$

1. $\det \begin{vmatrix} 3-\lambda & 0 \\ 0 & -0.6-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(0.6+\lambda) = 0$$

$$-\lambda^2 + 2.4\lambda + 1.8 = 0$$

$$\lambda^2 - 2.4\lambda - 1.8 = 0$$

$$(\lambda-3)(\lambda+0.6) = 0$$

1) $\lambda = 3$ 2) $\lambda = -0.6$

$$\begin{bmatrix} 0 & 0 \\ 0 & -3.6 \end{bmatrix} \quad \begin{bmatrix} 3.6 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. $\det \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$

$$\lambda^2 = 0 \quad \lambda = 0$$

X could be any 2×1 vector.

P326 theorem 2. 0 allowed.

3. $\det \begin{vmatrix} 5\lambda & -2 \\ 9 & -6\lambda \end{vmatrix} = 0$

$$-(5-\lambda)(6+\lambda) + 18 = 0$$

$$\lambda^2 + \lambda - 30 + 18 = 0$$

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda+4)(\lambda-3) = 0$$

1) $\lambda = -4$

$$\begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

2) $\lambda = 3$ $\begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix}$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$3. \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$$

$$4. \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$5. \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$6. \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$5. \begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 9 = 0$$

$$4. \det \begin{vmatrix} -\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-4) - 4 = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda = 0 \quad \lambda = 5$$

$$\lambda = 5: \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = 0 \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\lambda = \pm 3i$$
$$\lambda = 3i: \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda = -3i: \begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$6. \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$$\lambda = 3 \quad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 1 \quad \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$7. \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$8. \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\lambda = a - bi \quad \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$7. \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = 0$$

$$X = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$9. \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

$$10. \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\lambda = 0 \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$9. \begin{vmatrix} 0.8 - \lambda & -0.6 \\ 0.6 & 0.8 - \lambda \end{vmatrix} = 0$$

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$a = 0.8 \quad b = -0.6$$

$$\lambda = 0.8 - 0.6i \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$8. \begin{vmatrix} a - \lambda & b \\ -b & a - \lambda \end{vmatrix} = 0$$

$$\lambda = 0.8 + 0.6i \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$(a - \lambda)^2 + b^2 = 0$$

$$10. \begin{vmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{vmatrix} = 0$$

$$a - \lambda = \pm bi$$

$$a = \cos \theta \quad b = -\sin \theta$$

$$\lambda = a \pm bi$$

$$\lambda = \cos \theta - i \sin \theta \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda = a + bi$$

$$\lambda = \cos \theta + i \sin \theta \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

11. $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}, \lambda = 3$

$$\lambda = 1$$

$$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$$

11. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

13. $\begin{vmatrix} 13-\lambda & 5 & 2 \\ 2 & 7-\lambda & -8 \\ 5 & 4 & 7-\lambda \end{vmatrix} = 0$

$$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

12. $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

$$(13-\lambda)(7-\lambda)^2 - 200 + 16 - 10(7-\lambda) - 32(\lambda-13) - 10(7-\lambda) = 0$$

12. $(3-\lambda)(4-\lambda)(1-\lambda) = 0$

$$-\lambda^3 + 27\lambda^2 - 243\lambda + 729 = 0$$

$$(\lambda-9)^3 = 0$$

$$\lambda = 3$$

$$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 9$$

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 9 & 18 \end{bmatrix}$$

$$\lambda = 4$$

$$\begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$14. \begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$15. \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}, (\lambda + 1)^2$$

$$\lambda_1 = 3, \begin{bmatrix} -3 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5, \begin{bmatrix} -3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -1, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \leftarrow$$

$$\lambda_2 = -1, \quad \lambda_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$16. \begin{bmatrix} -3 & 0 & 4 & 2 \\ 0 & 1 & -2 & 4 \\ 2 & 4 & -1 & -2 \\ 0 & 2 & -2 & 3 \end{bmatrix}$$

$$\lambda_1 = -1, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 3, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

17. Counterclockwise rotation through the angle $\pi/2$ about the origin in R^2 .

18. Reflection about the x_1 -axis in R^2 .

17. P271. Problem 30.

$$A = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

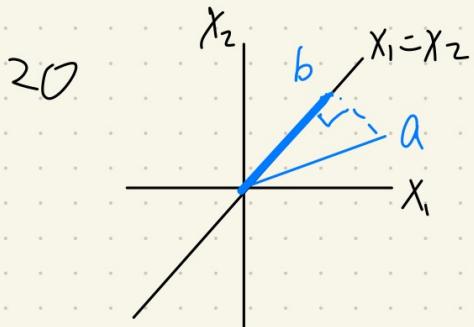
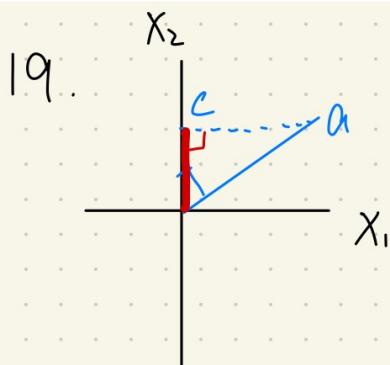
$$\lambda_1 = i, \quad \lambda_2 = -i; \quad \lambda_3 = -i, \quad \lambda_4 = i$$

$$18. \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 = x_1 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

19. Orthogonal projection (perpendicular projection) of R^2 onto the x_2 -axis.

20. Orthogonal projection of R^3 onto the plane $x_2 = x_1$.



$$P = \frac{a \cdot b}{\|b\|} = \frac{x_1 \cdot x_1 + x_2 \cdot x_2}{\sqrt{2} \cdot x_1}$$

$$= \sqrt{2} x_1.$$

$$(x_1 + x_2) \cdot x = 2x^2$$

$$x = \frac{x_1 + x_2}{2}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2}{2} \\ \frac{x_1 + x_2}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = 0, \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 1, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_1 = 1 \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0 \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

21–25 GENERAL PROBLEMS

21. **Nonzero defect.** Find further 2×2 and 3×3 matrices with positive defect. See Example 3.
22. **Multiple eigenvalues.** Find further 2×2 and 3×3 matrices with multiple eigenvalues. See Example 2.
23. **Complex eigenvalues.** Show that the eigenvalues of a real matrix are real or complex conjugate in pairs.
24. **Inverse matrix.** Show that A^{-1} exists if and only if the eigenvalues $\lambda_1, \dots, \lambda_n$ are all nonzero, and then A^{-1} has the eigenvalues $1/\lambda_1, \dots, 1/\lambda_n$.
25. **Transpose.** Illustrate Theorem 3 with examples of your own.

$$21. \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$(3-\lambda)^2 = 0 \quad \lambda = 3 \quad M_\lambda = 2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad m_x = 1 \quad \text{defect} = 1$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \lambda = 3, \quad M_\lambda = 3$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad M_\lambda = 2 \quad \text{defect} = 1$$

Jordan block

$$22. \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \lambda_1 = 6 \quad \lambda_{2,3} = \pm \sqrt{2}$$

23. Reference 22.

24. If A has $\lambda=0$, $(A-0I)x=0$

$$\det(A-0I) = 0$$

$\det A = 0 \Leftrightarrow A^{-1}$ not exist

$$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x$$

$$x = A^{-1} \cdot \lambda x = \lambda A^{-1}x$$

$$A^{-1}x = \frac{1}{\lambda} \cdot x$$

$\Rightarrow \frac{1}{\lambda}$ is the eigenvalue.

25.

Eigenvalues of the Transpose

The transpose A^T of a square matrix A has the same eigenvalues as A .

Reference 22.

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$$

PS 8.2

1-6 ELASTIC DEFORMATIONS

Given \mathbf{A} in a deformation $\mathbf{y} = \mathbf{Ax}$, find the principal directions and corresponding factors of extension or contraction. Show the details.

$$1. \begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$$

$$2. \begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$$

$$1. \begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0$$

$$9 + \lambda^2 - 6\lambda - 2.25 = 0$$

$$4\lambda^2 - 24\lambda + 27 = 0$$

$$(2\lambda - 3)(2\lambda - 9) = 0$$

$$\lambda_1 = \frac{3}{2}$$

$$\begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad 135^\circ$$

$$\lambda_2 = \frac{9}{2}$$

$$\begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad 45^\circ$$

$$2. \begin{vmatrix} 2-\lambda & 0.4 \\ 0.4 & 2-\lambda \end{vmatrix} = 0$$

$$4 + \lambda^2 - 4\lambda - 0.16 = 0$$

$$\lambda^2 - 4\lambda + 3.84 = 0$$

$$(5\lambda - 8)(5\lambda - 12) = 0$$

$$\lambda_1 = \frac{8}{5} = 1.6$$

$$\begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad 135^\circ$$

$$\lambda_2 = \frac{12}{5} = 2.4$$

$$\begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad 45^\circ$$

$$3. \begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$$

$$4. \begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$$

$$3. \begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0$$

$$14 + \lambda^2 - 9\lambda - 6 = 0$$

$$\lambda^2 - 9\lambda + 8 = 0$$

$$(\lambda - 8)(\lambda - 1) = 0$$

$$\lambda_1 = 8$$

$$\begin{bmatrix} -1 & \sqrt{6} \\ \sqrt{6} & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\sqrt{6} \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix} \Rightarrow \arctan\left(\frac{1}{\sqrt{6}}\right) = 22.21^\circ$$

$$\lambda_2 = 1$$

$$\begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{\sqrt{6}}{6} \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -1 \\ \sqrt{6} \end{bmatrix} \Rightarrow \arctan(\sqrt{6})$$

$$= -67.79^\circ$$

$$4. \begin{vmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix} = 0$$

$$65 + \lambda^2 - 18\lambda - 4 = 0$$

$$\lambda^2 - 18\lambda + 61 = 0$$

$$\lambda = \frac{18 \pm \sqrt{18^2 - 4 \cdot 61}}{2}$$

$$= 9 \pm \sqrt{81 - 61} = 9 \pm 2\sqrt{5}$$

$$\lambda_1 = 9 + 2\sqrt{5}$$

$$\begin{bmatrix} -4-2\sqrt{5} & 2 \\ 2 & 4-2\sqrt{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 2+\sqrt{5} & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 2+\sqrt{5} \end{bmatrix} \Rightarrow \arctan(2+\sqrt{5}) = 76.72^\circ$$

$$\lambda_2 = 9 - 2\sqrt{5}$$

$$\begin{bmatrix} 2\sqrt{5}-4 & 2 \\ 2 & 2\sqrt{5}+4 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{5}-2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 2-\sqrt{5} \end{bmatrix} \Rightarrow \arctan(2-\sqrt{5}) = -13.28^\circ$$

$$5. \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$6. \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$$

$$5. \begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$$1 + \lambda^2 - 2\lambda - \frac{1}{4} = 0$$

$$4\lambda^2 - 8\lambda + 3 = 0$$

$$(2\lambda - 3)(2\lambda - 1) = 0$$

$$\lambda_1 = \frac{3}{2}$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$6. \begin{vmatrix} 1.25 - \lambda & 0.75 \\ 0.75 & 1.25 - \lambda \end{vmatrix} = 0$$

$$\frac{25}{16} + \lambda^2 - \frac{5}{2}\lambda - \frac{9}{16} = 0$$

$$2\lambda^2 - 5\lambda + 2 = 0$$

$$(2\lambda - 1)(\lambda - 2) = 0$$

$$\lambda_1 = \frac{1}{2}$$

$$\begin{bmatrix} 0.75 & 0.75 \\ 0.75 & 0.75 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

-45°

$$\lambda_2 = 2$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

45°

$$\lambda_2 = \frac{1}{2}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

45°

$$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad -45^\circ$$

$$\begin{bmatrix} -0.75 & 0.75 \\ 0.75 & -0.75 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

7-9 MARKOV PROCESSES

Find the limit state of the Markov process modeled by the given matrix. Show the details.

7. $\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

8. $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$ 9. $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & -0.8 & 0.6 \\ 0 & 0 & 0 \end{bmatrix}$$

7. $\begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{5}{8} \\ 0 & 0 \end{bmatrix}$

$$X = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{5}{13} \\ \frac{8}{13} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 12 \\ 16 \end{bmatrix}$$

So limit is $\begin{bmatrix} 1/39 \\ 2/39 \\ 16/39 \end{bmatrix}$

8. $\begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 3 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ So limit is } \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

9. $\begin{bmatrix} -0.4 & 0.1 & 0.2 \\ 0.4 & -0.9 & 0.4 \\ 0 & 0.8 & -0.6 \end{bmatrix}$

10-12 AGE-SPECIFIC POPULATION

Find the growth rate in the Leslie model (see Example 3) with the matrix as given. Show the details.

$$10. \begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \quad 11. \begin{bmatrix} 0 & 3.45 & 0.6 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$$

$$12. \begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$11. \begin{vmatrix} -\lambda & 3.45 & 0.6 \\ 0.9 & -\lambda & 0 \\ 0 & 0.45 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 0.54 \times 0.45 + 0.9 \times 3.45 \lambda = 0$$

$$10. \begin{vmatrix} -\lambda & 9 & 5 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 0.8 + 3.6\lambda = 0$$

$$5\lambda^3 - 18\lambda - 4 = 0$$

$$(\lambda - 2)(5\lambda^2 + 10\lambda + 2) = 0$$

$$\lambda_1 = 2$$

$$\lambda_{2,3} = \frac{-10 \pm \sqrt{60}}{10}$$

$$= -1 \pm \frac{\sqrt{15}}{5} < 0$$

Growth rate is 2.

$$-\lambda^3 + 0.243 + 3.105\lambda = 0$$

$$(\lambda - 1.8)(\lambda^2 + 1.8\lambda + 0.135) = 0$$

$$\lambda_1 = 1.8$$

$$\lambda_{2,3} = \frac{-90 \pm \sqrt{90^2 - 5 \times 27}}{100} < 0$$

Growth rate is 1.8

$$12. \begin{vmatrix} -\lambda & 3 & 2 & 2 \\ 0.5 - \lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{vmatrix} = 0$$

$$= -\lambda \cdot (-\lambda^3) - 3 \cdot 0.5\lambda^2 + 2 \cdot (-0.25\lambda)$$

$$- 2 \cdot 0.025 = 0$$

$$\lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05 = 0$$

$$20\lambda^4 - 30\lambda^2 - 10\lambda - 1 = 0$$

not sure how to move on

by pen & paper.

Maybe computer or numeric way

13-15 LEONTIEF MODELS¹

13. Leontief input-output model. Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the 3×3 consumption matrix

$$A = [a_{jk}] = \begin{bmatrix} 1 & 2 & 3 \\ 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$$

where a_{jk} is the fraction of the output of industry k consumed (purchased) by industry j . Let p_j be the price charged by industry j for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to $Ap = p$, where $p = [p_1 \ p_2 \ p_3]^T$, and find a solution p with nonnegative p_1, p_2, p_3 .

14. Show that a consumption matrix as considered in Prob. 13 must have column sums 1 and always has the eigenvalue 1.

all the input and output consumed by the industries, it is a closed system
 $\sum \text{input} = \sum \text{output}$

$$15) XAX = [0.1 \ 0.3 \ 0.1]^T$$

$$(I - A)X = [0.1 \ 0.3 \ 0.1]^T$$

$$13. \begin{bmatrix} -0.9 & 0.5 & 0 \\ 0.8 & -1 & 0.4 \\ 0.1 & 0.5 & -0.4 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.5 & 0 & 0.1 \\ 0.1 & 0.4 & 0.4 \end{bmatrix} \right) X = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & -4 \\ 0 & 50 & -36 \\ 0 & 0 & 0 \end{bmatrix} \quad X = (I - A)^{-1} y$$

$$\Rightarrow X = \begin{bmatrix} 0.4 \\ 0.72 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & -0.4 & -0.2 \\ -0.5 & 1 & -0.1 \\ -0.1 & -0.4 & 0.6 \end{bmatrix}^{-1} y$$

14. I don't get it.

The output doesn't necessarily

ALL consumed by those

industries. (15) support this.

assume all output consumed within industries, then it means $\sum \text{column} = 1$

$$= \begin{bmatrix} \frac{7}{4} & 1 & \frac{3}{4} \\ \frac{31}{32} & \frac{13}{8} & \frac{19}{32} \\ \frac{15}{16} & \frac{5}{4} & \frac{35}{16} \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{180}{160} \\ \frac{103}{160} \\ \frac{11}{16} \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.6438 \\ 0.6875 \end{bmatrix}$$

PS. 8.3

1-10 SPECTRUM

Are the following matrices symmetric, skew-symmetric, or orthogonal? Find the spectrum of each, thereby illustrating Theorems 1 and 5. Show your work in detail.

$$1. \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$$

$$2. \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$1. a_{12} \neq a_{21}$$

$$a_{11} \neq -a_{11}$$

$$\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 1 & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{5}{4} & 0 \\ 1 & -\frac{4}{3} & 0 & -\frac{5}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 1 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$A^{-1} = A^T$, orthogonal.

$$\begin{vmatrix} 0.8-\lambda & 0.6 \\ -0.6 & 0.8-\lambda \end{vmatrix} = 0$$

$$0.64 + \lambda^2 - 1.6\lambda + 0.36 = 0$$

$$\lambda^2 - 1.6\lambda + 1 = 0$$

$$\lambda = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2}$$

$$= 0.8 \pm 0.6i$$

$$\lambda_{1,2} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}$$

T3

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^\top \mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

$$R_1 \left\{ \begin{array}{l} 0.8 \cdot (-0.6) + 0.6 \cdot 0.8 = 0 \\ 0.8^2 + 0.6^2 = 1 \end{array} \right.$$

$$C \left\{ \begin{array}{l} 0.8 \cdot 0.6 + (-0.6) \cdot 0.8 = 0 \\ 0.8^2 + (-0.6)^2 = 0.6^2 + 0.8^2 = 1 \end{array} \right.$$

T4: $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$

$$= -0.64 + 0.36 = 1$$

T5 $\sqrt{0.8^2 + 0.6^2} = 1$

$$2. \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

2. if $b=0$, symmetric

if $a=0$, skew-symmetric

if $a^2+b^2=1$, orthonormal

$$\lambda = a \pm bi \quad X = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

(T1) $b=0$, λ is real

$a=0$ λ is imaginary or 0.

$$3. \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$$

$$4. \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

3. Reference 2.

4. Reference 2

$$5. \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

$$6. \begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix}$$

Symmetric

$$5. \lambda_1 = 6 \quad X_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 6 \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 1 \quad X_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

6. Symmetric

$$\lambda_1 = a-k \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = a+k \quad V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = a+2k \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

if $a=k=0$, showsym

$$7. \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

skew-symmetric
7. Anti-Symmetric

$$\lambda_1 = 25i, \quad V_1 = \begin{bmatrix} -45+75i \\ -27-125i \\ 136 \end{bmatrix}$$

$$\lambda_2 = -25i, \quad V_2 = \begin{bmatrix} -45-75i \\ -27+125i \\ 136 \end{bmatrix}$$

$$\lambda_3 = 0, \quad V_3 = \begin{bmatrix} 20 \\ 12 \\ 9 \end{bmatrix}$$

$$8. \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$\lambda_1 = 1 \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \cos \theta - i \sin \theta$$

$$v_2 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

$$\lambda_3 = \cos \theta + i \sin \theta$$

$$v_3 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

I already proved that I can use the tools property to resolve problem. No mean to continue with more cases.

Skip 9-10

11. Reference mindmap and notes.

$$(2) a) A^T = A^{-1} \quad B^T = B^{-1}$$

$$(AB)^T = B^T \cdot A^T$$

$$= B^{-1} \cdot A^{-1}$$

$$= (AB)^{-1}$$

$$A^T A = A A^T = I$$

$$(A^T A)^T = A^T A = A A^T = I$$

$$(A^T)^{-1} = (A^{-1} \cdot A \cdot A^T)^{-1}$$

$$= (A \cdot A^T)^{-1} A$$

$$= I \cdot A = A$$

Rotation Matrix - wiki

(b) **Rotation.** Show that (6) is an orthogonal transformation. Verify that it satisfies Theorem 3. Find the inverse transformation.

$$(6) \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = A^T$$

if $j=k$

$$a_j \cdot a_k = \cos^2 \theta + (-\sin \theta)^2$$

$$= \sin^2 \theta + \cos^2 \theta = 1$$

if $j \neq k$

$$a_j \cdot a_k = \cos \theta \sin \theta - \sin \theta \cos \theta$$

$$= 0$$

(c) **Powers.** Write a program for computing powers \mathbf{A}^m ($m = 1, 2, \dots$) of a 2×2 matrix \mathbf{A} and their spectra. Apply it to the matrix in Prob. 1 (call it \mathbf{A}). To what rotation does \mathbf{A} correspond? Do the eigenvalues of \mathbf{A}^m have a limit as $m \rightarrow \infty$?

(d) Compute the eigenvalues of $(0.9\mathbf{A})^m$, where \mathbf{A} is the matrix in Prob. 1. Plot them as points. What is their limit? Along what kind of curve do these points approach the limit?

(e) Find \mathbf{A} such that $\mathbf{y} = \mathbf{Ax}$ is a counterclockwise rotation through 30° in the plane.

are symmetric, skew-symmetric, and orthogonal, respectively, as you should verify. Every skew-symmetric matrix has all main diagonal entries zero. (Can you prove this?)

c). Per P.S. 8.2-ff,

18. Scalar multiples, powers. $k\mathbf{A}$ has the eigenvalues $k\lambda_1, \dots, k\lambda_n$. \mathbf{A}^m ($m = 1, 2, \dots$) has the eigenvalues $\lambda_1^m, \dots, \lambda_n^m$. The eigenvectors are those of \mathbf{A} .

$$\text{if } |\lambda| < 1, \lim_{m \rightarrow \infty} \lambda^m = 0$$

$$d) (0.9\mathbf{A})^m = 0.9^m \cdot \mathbf{A}^m$$

$$\lambda = 0.9^m \cdot \lambda_1^m$$

$$\text{if } |0.9\lambda_1| < 1, \lim \rightarrow 0.$$

$$0.9\lambda_1 = 1 \rightarrow 1$$

$$e) \begin{bmatrix} \cos \frac{\pi}{6} & -\sin \frac{\pi}{6} \\ \sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{bmatrix}$$

13-20 GENERAL PROPERTIES

13. **Verification.** Verify the statements in Example 1.

14. Verify the statements in Examples 3 and 4.

15. **Sum.** Are the eigenvalues of $\mathbf{A} + \mathbf{B}$ sums of the eigenvalues of \mathbf{A} and of \mathbf{B} ?

$$(3) a_{jk} = -a_{kj}$$

$$j=k, a_{jj} = -a_{jj} \Rightarrow 0.$$

(4) NO. if symmetric \Rightarrow eigenvalues GR but not \Leftarrow

$$j \neq k:$$

$$a_j \cdot a_k = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

$$j \neq k$$

$$a_j \cdot a_k = \frac{-4}{9} + \frac{2}{9} + \frac{2}{9}$$

$$= \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

$$\det(\mathbf{A}) = -\frac{8}{27} + \frac{1}{27} - \frac{8}{27} - \frac{4}{27} - \frac{4}{27} - \frac{4}{27} = -1$$

$$(6) \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$j=k, a_j \cdot a_k = \cos^2 \theta + \sin^2 \theta = 1$$

$$j \neq k, a_j \cdot a_k = \cos \theta \sin \theta - \sin \theta \cos \theta = 0$$

$$\det(\mathbf{A}) = \cos^2 \theta + \sin^2 \theta = 1$$

15. PS. 8.2-16:

$$\text{Trace}(A) = \sum \lambda_A$$

$$\text{Trace}(B) = \sum \lambda_B$$

$$A+B \Rightarrow \text{Trace}(A+B)$$

$$= \sum (\lambda_A + \lambda_B)$$

16 - 20: Not finish yet

Catch up

$$3. \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$$

$2 \pm 8i$. Answer is wrong.

Type.

$$9. \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\lambda_1 = 1, \quad V_1 = \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}$$

$$\lambda_2 = i, \quad V_2 = \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -i, \quad V_3 = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$$

orthogonal.

PS 8.4

Mistake: Verify

$A \& D$ have
equal eigenvalues

$A \neq P^{-1}AP$.

1-5 SIMILAR MATRICES HAVE EQUAL EIGENVALUES

Verify this for A and $D = P^{-1}AP$. If y is an eigenvector of P^{-1} , show that $x = Py$ are eigenvectors of A . Show the details of your work.

$$1. A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, \quad P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$1. (3-\lambda)(-3-\lambda) - 16 = 0.$$

$$(3-\lambda)(3+\lambda) + 16 = 0$$

$$9 - \lambda^2 + 16 = 0$$

$$\lambda = \pm 5$$

$$\lambda_1 = 5 : \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5 : \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$P : \begin{bmatrix} -4 & 2 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{3}{2} & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

$$A = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} P$$

$$= \begin{bmatrix} 5.5 & -1 \\ 12.5 & 0 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -22-3 & 11+1 \\ -50 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$$

$$(-25-\lambda)(25-\lambda) + 600 = 0$$

$$625 - \lambda^2 + 600 = 0$$

$$\lambda = \pm 5 \Rightarrow \text{same as } A$$

$$\lambda_1 = 5 : \begin{bmatrix} -30 & 12 \\ -50 & 20 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 0 & 0 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$Py_1 = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = X_1$$

$$\lambda_2 = -5 : \begin{bmatrix} -20 & 12 \\ -50 & 30 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 0 & 0 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$Py_2 = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -27 \\ 4 \end{bmatrix} = 2X_2$$

$$2. \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix}$$

$$3. \mathbf{A} = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{bmatrix}$$

$$A: \lambda_1 = -1 \quad x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} -7 & 5 \\ -10 & 7 \end{bmatrix}$$

$$\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix} \mathbf{P}$$

$$= \begin{bmatrix} -29 & 20 \\ -42 & 29 \end{bmatrix} = \hat{\mathbf{A}}$$

$$\lambda_1 = -1 \quad y_1 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\mathbf{P} y_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1$$

$$\lambda_2 = 1 \quad y_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{P} y_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -x_2$$

$$A: \lambda_1 = 4 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 6 \quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{bmatrix}$$

$$\mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{bmatrix} \frac{8}{25} & -\frac{76}{25} \\ \frac{206}{25} & -\frac{82}{25} \end{bmatrix} \mathbf{P}$$

$$= \begin{bmatrix} \frac{376}{125} & -\frac{68}{125} \\ \frac{682}{125} & \frac{874}{125} \end{bmatrix} = \hat{\mathbf{A}}$$

$$\lambda_1 = 4 \quad y_1 = \begin{bmatrix} -\frac{17}{31} \\ 1 \end{bmatrix}$$

$$\mathbf{P} y_1 = \begin{bmatrix} \frac{25}{31} \\ \frac{25}{31} \end{bmatrix} = \frac{25}{31} \cdot x_1$$

$$\lambda_2 = 6 \quad y_2 = \begin{bmatrix} -\frac{2}{9} \\ 1 \end{bmatrix}$$

$$\mathbf{P} y_2 = \begin{bmatrix} \frac{10}{11} \\ \frac{5}{11} \end{bmatrix} = \frac{5}{11} \cdot x_2$$

WOW.

$$4. \mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}, \quad \lambda_1 = 3$$

6-a Compare the co-efficient,
trace = $\sum \lambda$

$$\Lambda: \lambda_1 = -1 \quad X_1 = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$$

1) $0 = 5 - 5$.

3) $10 = 4 + 6$

5) by pass.

$$\lambda_2 = 2 \quad X_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

6-b Theorems-3; trace(A) = trace(\hat{A})

$$\text{trace}(AB) = \text{trace}(A^{-1}ABA) \\ = \text{trace}(BA)$$

$$P^{-1} = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

(c) Find a relationship between \hat{A} in (4) and $\hat{A} = PAP^{-1}$.

$$\hat{A} = P^{-1}AP = \begin{bmatrix} -3 & 0 & 7 \\ 0 & 3 & 2 \\ 2 & 0 & -4 \end{bmatrix} P$$

$$4). \hat{A}_1 = P^{-1}AP. \quad \hat{A}_2 = PAP^{-1}$$

$$= \begin{bmatrix} 15 & 0 & 26 \\ 6 & 3 & 10 \\ -8 & 0 & -14 \end{bmatrix}$$

\hat{A}_1 similar to A similar to \hat{A}_2

$$\lambda_1 = -1 \quad y_1 = \begin{bmatrix} -26 \\ -1 \\ 16 \end{bmatrix} \quad Py_1 = X_1$$

d) Yes. change the sequence of
column vector (eigenvector)

$$\lambda_2 = 2 \quad y_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \quad Py_2 = -1 \cdot X_2 \quad ?$$

7. No basis. Find further 2×2 and 3×3 matrices without eigenbasis.

$$\lambda_3 = 3 \quad y_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Py_3 = X_3$$

NOT SURE. if $\lambda \in C$ then
 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ always have eigenvalue-
 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with $\lambda = 0$, any
 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ eigenvector works.

5: ignore. Same approach
as problem 1~4.

THEOREM 2

Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n .

$$8. \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\lambda_1 = -1 \quad V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3 \quad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}: \text{orthonormal}$$

9-16 DIAGONALIZATION OF MATRICES

Find an eigenbasis (a basis of eigenvectors) and diagonalize. Show the details.

$$9. \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$10. \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$9. (1-\lambda)(4-\lambda) - 4 = 0$$

$$4 + \lambda^2 - 5\lambda - 4 = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda_1 = 5 \quad \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 0 \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$D = X^{-1} A X = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

$$10. \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$11. \begin{bmatrix} -14 & 7 \\ -42 & 16 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -6 & 3 \end{bmatrix}$$

$$12. \begin{bmatrix} -4.3 & 7.7 \\ 1.3 & 9.3 \end{bmatrix} = \begin{bmatrix} -11 & \frac{7}{13} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-13}{150} & \frac{7}{150} \\ \frac{13}{150} & \frac{143}{150} \end{bmatrix}$$

$$13. \begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{2}{3} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -4 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

$$14. \begin{bmatrix} -5 & -6 & 6 \\ -9 & -8 & 12 \\ -12 & -12 & 16 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & 2 & -2 \\ -2 & -3 & 1 \end{bmatrix}$$

$$15. \begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$16 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

17-23 PRINCIPAL AXES. CONIC SECTIONS

What kind of conic section (or pair of straight lines) is given by the quadratic form? Transform it to principal axes. Express $\mathbf{x}^T = [x_1 \ x_2]$ in terms of the new coordinate vector $\mathbf{y}^T = [y_1 \ y_2]$, as in Example 6.

$$17. 7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$$

17. We have $Q = \mathbf{x}^T A \mathbf{x}$

$$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(7-\lambda)^2 - 9 = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 10$$

$$Q = 4y_1^2 + 10y_2^2 = 200$$

$$\frac{y_1^2}{50} + \frac{y_2^2}{20} = 1$$

$$\text{for } A, \quad \lambda_1 = 4 \quad V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

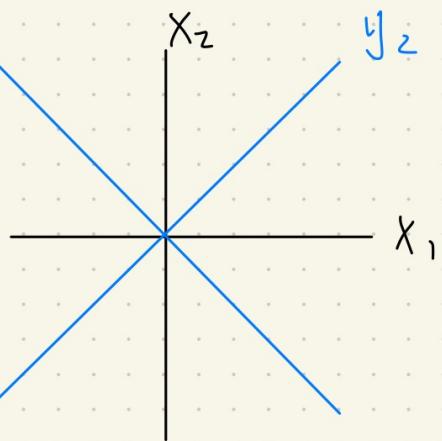
$$U_1 = \frac{V_1}{|V_1|} = \left[\begin{array}{c} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{array} \right]$$

$$\lambda_2 = 10 \quad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$U_2 = \frac{V_2}{|V_2|} = \left[\begin{array}{c} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{array} \right]$$

$$x = Xy = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{cases} x_1 = -\frac{\sqrt{2}}{2}y_1 + \frac{\sqrt{2}}{2}y_2 \\ x_2 = \frac{\sqrt{2}}{2}y_1 + \frac{\sqrt{2}}{2}y_2 \end{cases}$$



So it rotate 45° and
also exchange (x_1, x_2)
Sequence?

18-23: Same Q5 17.
By pass them at the moment

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{j=1}^n \sum_{k=1}^n a_{jk} x_j x_k \quad (a_{kj} = a_{jk})$$

24.

25.

$$5. A = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda_A \text{ equals to } \lambda_A$$

Similar.

$$A: \lambda_{A_1} = 0, \quad V_{A_1} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$PV_{A_1} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = V_{A_1}$$

$$\lambda_{A_2} = 4, \quad V_{A_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$PV_{A_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = V_{A_2}$$

$$\lambda_{A_3} = 10, \quad V_{A_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$PV_{A_3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = V_{A_3}$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P_x = y.$$

$$\hat{A} = P^{-1}AP = \begin{bmatrix} 3 & 4 & -9 \\ -5 & 0 & 15 \\ -5 & 0 & 15 \end{bmatrix} P$$

$$= \begin{bmatrix} 4 & 3 & -9 \\ 0 & -5 & 15 \\ 0 & -5 & 15 \end{bmatrix}$$

$$\lambda_{\hat{A}_1} = 0 \quad V_{\hat{A}_1} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

$$\lambda_{\hat{A}_2} = 4 \quad V_{\hat{A}_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_{\hat{A}_3} = 10 \quad V_{\hat{A}_3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

PS. 8.5

1-6 EIGENVALUES AND VECTORS

Is the given matrix Hermitian? Skew-Hermitian? Unitary?
Find its eigenvalues and eigenvectors.

$$1. \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix}$$

$$2. \begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$$

$$5. \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$$1. \lambda_1 = 7 \quad V_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = 5 \quad V_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 6 & -i \\ i & 6 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix} = A$$

Hermitian.

$7 > 5 > 1$, not unitary.

$$\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$\bar{A}^T = -\bar{A} \Rightarrow \text{skew-Hermitian}$$

$$\lambda_1 = i \quad V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -i \quad V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_3 = i \quad V_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = -1$$

$$\bar{A}^T = A^{-1}, \text{ unitary.}$$

9-12 COMPLEX FORMS

Is the matrix \mathbf{A} Hermitian or skew-Hermitian? Find $\bar{\mathbf{x}}^T \mathbf{A} \mathbf{x}$. Show the details.

$$9. \mathbf{A} = \begin{bmatrix} 4 & 3 - 2i \\ 3 + 2i & -4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -4i \\ 2 + 2i \end{bmatrix}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} 4 & 3+2i \\ 3-2i & -4 \end{bmatrix}$$

$$\bar{\mathbf{A}}^T = \mathbf{A} \quad \text{Hermitian}$$

$$\bar{\mathbf{x}}^T = [4i \ 2-2i]$$

$$\bar{\mathbf{x}}^T \mathbf{A} \mathbf{x} = [0+14i, 20i] \mathbf{x}$$

$$= 16$$

$$11. \mathbf{A} = \begin{bmatrix} i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} -i & 1 & 2-i \\ -1 & 0 & -3i \\ 2-i & -3i & -i \end{bmatrix}$$

$$\bar{\mathbf{A}}^T = -\mathbf{A} \rightarrow \text{skew-Hermitian}$$

$$\bar{\mathbf{x}}^T = [1 \ -i \ i]$$

$$\bar{\mathbf{x}}^T \cdot \mathbf{A} \cdot \mathbf{x} = [-1 \ -2 \ 4+i] \mathbf{x}$$

$$= -6i$$