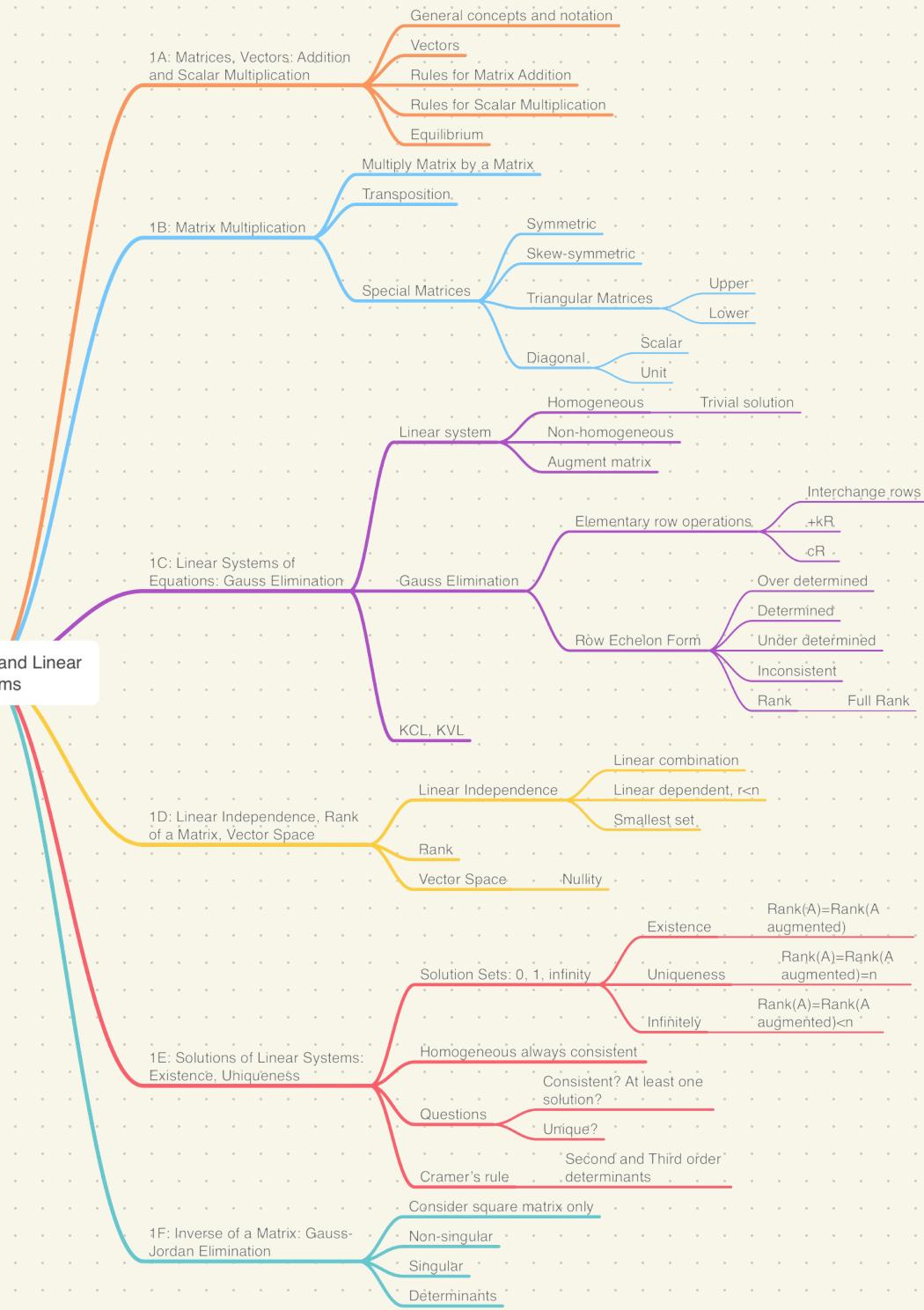


## C7: Matrices and Linear Systems



# Chapter 7

## - Caution

### Inverse of a Matrix by Determinants

The inverse of a nonsingular  $n \times n$  matrix  $\mathbf{A} = [a_{jk}]$  is given by

$$(4) \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} [\mathbf{C}_{jk}]^T = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

where  $C_{jk}$  is the cofactor of  $a_{jk}$  in  $\det \mathbf{A}$  (see Sec. 7.7). (CAUTION! Note well that in  $\mathbf{A}^{-1}$ , the cofactor  $C_{jk}$  occupies the same place as  $a_{kj}$  (not  $a_{jk}$ ) does in  $\mathbf{A}$ .)

In particular, the inverse of

$$(4*) \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{is} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

- (10)
- (a)  $(\mathbf{A}^T)^T = \mathbf{A}$
  - (b)  $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
  - (c)  $(c\mathbf{A})^T = c\mathbf{A}^T$
  - (d)  $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ .

**CAUTION!** Note that in (10d) the transposed matrices are *in reversed order*. We leave the proofs as an exercise in Probs. 9 and 10.

### Elementary Row Operations for Matrices:

*Interchange of two rows*

*Addition of a constant multiple of one row to another row*

*Multiplication of a row by a nonzero constant  $c$*

**CAUTION!** These operations are for rows, *not for columns*! They correspond to the following

**CAUTION!**  $\det(c\mathbf{A}) = c^n \det \mathbf{A}$  (not  $c \det \mathbf{A}$ ). Explain why.

$$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

### THEOREM 1

### Behavior of an $n$ th-Order Determinant under Elementary Row Operations

- (a) *Interchange of two rows multiplies the value of the determinant by  $-1$ .*
- (b) *Addition of a multiple of a row to another row does not alter the value of the determinant.*
- (c) *Multiplication of a row by a nonzero constant  $c$  multiplies the value of the determinant by  $c$ . (This holds also when  $c = 0$ , but no longer gives an elementary row operation.)*

# -Theorems & Rules

**Rules for Matrix Addition and Scalar Multiplication.** From the familiar laws for the addition of numbers we obtain similar laws for the addition of matrices of the same size  $m \times n$ , namely,

- (3)
- (a)  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - (b)  $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$  (written  $\mathbf{A} + \mathbf{B} + \mathbf{C}$ )
  - (c)  $\mathbf{A} + \mathbf{0} = \mathbf{A}$
  - (d)  $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}.$

Hence matrix addition is *commutative* and *associative* [by (3a) and (3b)].

Similarly, for scalar multiplication we obtain the rules

- (4)
- (a)  $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$
  - (b)  $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$
  - (c)  $c(k\mathbf{A}) = (ck)\mathbf{A}$  (written  $ck\mathbf{A}$ )
  - (d)  $1\mathbf{A} = \mathbf{A}.$

- (a)  $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$  written  $k\mathbf{AB}$  or  $\mathbf{AkB}$
- (b)  $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$  written  $\mathbf{ABC}$
- (c)  $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
- (d)  $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$

## THEOREM 3

### Rank in Terms of Column Vectors

The rank  $r$  of a matrix  $\mathbf{A}$  equals the maximum number of linearly independent column vectors of  $\mathbf{A}$ .

Hence  $\mathbf{A}$  and its transpose  $\mathbf{A}^T$  have the same rank.

### Rank in Terms of Determinants

Consider an  $m \times n$  matrix  $\mathbf{A} = [a_{jk}]$ :

- (1)  $\mathbf{A}$  has rank  $r \geq 1$  if and only if  $\mathbf{A}$  has an  $r \times r$  submatrix with a nonzero determinant.
- (2) The determinant of any square submatrix with more than  $r$  rows, contained in  $\mathbf{A}$  (if such a matrix exists!) has a value equal to zero.

Furthermore, if  $m = n$ , we have:

- (3) An  $n \times n$  square matrix  $\mathbf{A}$  has rank  $n$  if and only if

$$\det \mathbf{A} \neq 0.$$

### Cramer's Theorem (Solution of Linear Systems by Determinants)

- (a) If a linear system of  $n$  equations in the same number of unknowns  $x_1, \dots, x_n$

$$(6) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

has a nonzero coefficient determinant  $D = \det \mathbf{A}$ , the system has precisely one solution. This solution is given by the formulas

$$(7) \quad x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \cdots, \quad x_n = \frac{D_n}{D} \quad (\text{Cramer's rule})$$

where  $D_k$  is the determinant obtained from  $D$  by replacing in  $D$  the  $k$ th column by the column with the entries  $b_1, \dots, b_n$ .

- (b) Hence if the system (6) is homogeneous and  $D \neq 0$ , it has only the trivial solution  $x_1 = 0, x_2 = 0, \dots, x_n = 0$ . If  $D = 0$ , the homogeneous system also has nontrivial solutions.

### Existence of the Inverse

The inverse  $\mathbf{A}^{-1}$  of an  $n \times n$  matrix  $\mathbf{A}$  exists if and only if  $\text{rank } \mathbf{A} = n$ , thus (by Theorem 3, Sec. 7.7) if and only if  $\det \mathbf{A} \neq 0$ . Hence  $\mathbf{A}$  is nonsingular if  $\text{rank } \mathbf{A} = n$ , and is singular if  $\text{rank } \mathbf{A} < n$ .

### Determinant of a Product of Matrices

For any  $n \times n$  matrices  $\mathbf{A}$  and  $\mathbf{B}$ ,

$$(10) \quad \det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}.$$

This product is called the **inner product** or **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$ . Other notations for it are  $(\mathbf{a}, \mathbf{b})$  and  $\mathbf{a} \cdot \mathbf{b}$ . Thus

$$\mathbf{a}^\top \mathbf{b} = (\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = [a_1 \cdots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^n a_i b_i = a_1 b_1 + \cdots + a_n b_n.$$

$$(3) \quad |(\mathbf{a}, \mathbf{b})| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad (\text{Cauchy-Schwarz inequality}).$$

From this follows

$$(4) \quad \|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \quad (\text{Triangle inequality}).$$

A simple direct calculation gives

$$(5) \quad \|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2) \quad (\text{Parallelogram equality}).$$

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} = \sqrt{\mathbf{a}^\top \mathbf{a}} = \sqrt{a_1^2 + \cdots + a_n^2}.$$

# Linear Algebra: Matrices, Vectors, Determinants.

## Linear Systems

An  $m \times n$  **matrix**  $\mathbf{A} = [a_{jk}]$  is a rectangular array of numbers or functions (“entries,” “elements”) arranged in  $m$  horizontal **rows** and  $n$  vertical **columns**. If  $m = n$ , the matrix is called **square**. A  $1 \times n$  matrix is called a **row vector** and an  $m \times 1$  matrix a **column vector** (Sec. 7.1).

The **sum**  $\mathbf{A} + \mathbf{B}$  of matrices of the same **size** (i.e., both  $m \times n$ ) is obtained by adding corresponding entries. The **product** of  $\mathbf{A}$  by a scalar  $c$  is obtained by multiplying each  $a_{jk}$  by  $c$  (Sec. 7.1).

The **product**  $\mathbf{C} = \mathbf{AB}$  of an  $m \times n$  matrix  $\mathbf{A}$  by an  $r \times p$  matrix  $\mathbf{B} = [b_{jk}]$  is defined only when  $r = n$ , and is the  $m \times p$  matrix  $\mathbf{C} = [c_{jk}]$  with entries

$$(1) \quad c_{jk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \quad (\text{row } j \text{ of } \mathbf{A} \text{ times column } k \text{ of } \mathbf{B}).$$

This multiplication is motivated by the composition of **linear transformations** (Secs. 7.2, 7.9). It is associative, but is *not commutative*: if  $\mathbf{AB}$  is defined,  $\mathbf{BA}$  may not be defined, but even if  $\mathbf{BA}$  is defined,  $\mathbf{AB} \neq \mathbf{BA}$  in general. Also  $\mathbf{AB} = \mathbf{0}$  may not imply  $\mathbf{A} = \mathbf{0}$  or  $\mathbf{B} = \mathbf{0}$  or  $\mathbf{BA} = \mathbf{0}$  (Secs. 7.2, 7.8). Illustrations:

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ [1 \ 2] \begin{bmatrix} 3 \\ 4 \end{bmatrix} &= [11], \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} [1 \ 2] = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}. \end{aligned}$$

The **transpose**  $\mathbf{A}^T$  of a matrix  $\mathbf{A} = [a_{jk}]$  is  $\mathbf{A}^T = [a_{kj}]$ ; rows become columns and conversely (Sec. 7.2). Here,  $\mathbf{A}$  need not be square. If it is and  $\mathbf{A} = \mathbf{A}^T$ , then  $\mathbf{A}$  is called **symmetric**; if  $\mathbf{A} = -\mathbf{A}^T$ , it is called **skew-symmetric**. For a product,  $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$  (Sec. 7.2).

A main application of matrices concerns **linear systems of equations**

$$(2) \quad \mathbf{Ax} = \mathbf{b} \quad (\text{Sec. 7.3})$$

( $m$  equations in  $n$  unknowns  $x_1, \dots, x_n$ ;  $\mathbf{A}$  and  $\mathbf{b}$  given). The most important method of solution is the **Gauss elimination** (Sec. 7.3), which reduces the system to “triangular” form by *elementary row operations*, which leave the set of solutions unchanged. (Numeric aspects and variants, such as *Doolittle's* and *Cholesky's methods*, are discussed in Secs. 20.1 and 20.2.)

**Cramer's rule** (Secs. 7.6, 7.7) represents the unknowns in a system (2) of  $n$  equations in  $n$  unknowns as quotients of determinants; for numeric work it is impractical. **Determinants** (Sec. 7.7) have decreased in importance, but will retain their place in eigenvalue problems, elementary geometry, etc.

The **inverse**  $\mathbf{A}^{-1}$  of a square matrix satisfies  $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ . It exists if and only if  $\det \mathbf{A} \neq 0$ . It can be computed by the *Gauss–Jordan elimination* (Sec. 7.8).

The **rank**  $r$  of a matrix  $\mathbf{A}$  is the maximum number of linearly independent rows or columns of  $\mathbf{A}$  or, equivalently, the number of rows of the largest square submatrix of  $\mathbf{A}$  with nonzero determinant (Secs. 7.4, 7.7).

The system (2) has solutions if and only if  $\text{rank } \mathbf{A} = \text{rank } [\mathbf{A} \quad \mathbf{b}]$ , where  $[\mathbf{A} \quad \mathbf{b}]$  is the **augmented matrix** (Fundamental Theorem, Sec. 7.5).

The **homogeneous system**

$$(3) \quad \mathbf{Ax} = \mathbf{0}$$

has solutions  $\mathbf{x} \neq \mathbf{0}$  (“nontrivial solutions”) if and only if  $\text{rank } \mathbf{A} < n$ , in the case  $m = n$  equivalently if and only if  $\det \mathbf{A} = 0$  (Secs. 7.6, 7.7).

Vector spaces, inner product spaces, and linear transformations are discussed in Sec. 7.9. See also Sec. 7.4.