Chapter 8 Linear Algebra: Matrix Eigenvalue Problems

P329 - Problem set 8.1

P.S. 8.1

1–16 EIGENVALUES, EIGENVECTORS

Find the eigenvalues. Find the corresponding eigenvectors. Use the given λ or factor in Probs. 11 and 15.

1. $\begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}$ **2.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$(A-\lambda I)\cdot X=0$

1. $\det\begin{vmatrix} 3-\lambda & 0 \\ 0 & -0.6-\lambda \end{vmatrix}=0$

$(3-\lambda)(0.6+\lambda)=0$

$-\lambda^2+2.4\lambda+1.8=0$

$\lambda^2-2.4\lambda-1.8=0$

$(\lambda-3)(\lambda+0.6)=0$

1) $\lambda=3$ 2) $\lambda=-0.6$

$\begin{bmatrix} 0 & 0 \\ 0 & -3.6 \end{bmatrix}$ $\begin{bmatrix} 3.6 & 0 \\ 0 & 0 \end{bmatrix}$

$X=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $X=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2. $\det\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix}=0$

$\lambda^2=0$ $\lambda=0$

X could be any 2x1 vector.

P326 theorem 2, 0 allowed.

3. $\det\begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix}=0$

$-(5-\lambda)(6+\lambda)+18=0$

$\lambda^2+\lambda-30+18=0$

$\lambda^2+\lambda-12=0$

$(\lambda+4)(\lambda-3)=0$

1) $\lambda=-4$

$\begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix}=\begin{bmatrix} 9 & -2 \\ 0 & 0 \end{bmatrix}$

$X=\begin{bmatrix} 2 \\ 9 \end{bmatrix}$

2) $\lambda=3$ $\begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix}$

$=\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ 6. $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

4. $\det: \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$(\lambda-1)(\lambda-4)-4=0$

$\lambda^2-5\lambda=0$

$\lambda=0 \quad \lambda=5$

$\lambda=5: \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda=0 \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

5. $\begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0$

$\lambda^2+9=0$

$\lambda=\pm 3i$

$\lambda=3i \quad \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda=-3i: \begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} 1 \\ -i \end{bmatrix}$

6. $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$(\lambda-3)(\lambda-1)=0$

$\lambda=3 \quad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda=1 \quad \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$

$x=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 8. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

7 $\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$

$\lambda^2 = 0$

$\lambda = 0 \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

8. $\begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0$

$(a-\lambda)^2 + b^2 = 0$

$a-\lambda = \pm bi$

$\lambda = a \pm bi$

$\lambda = a + bi$

$\begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = a - bi \quad \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} -1 \\ i \end{bmatrix}$

9. $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ 10. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

9. $\begin{vmatrix} 0.8-\lambda & -0.6 \\ 0.6 & 0.8-\lambda \end{vmatrix} = 0$

$a = 0.8 \quad b = -0.6$

$\lambda = 0.8 - 0.6i \quad x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = 0.8 + 0.6i \quad x = \begin{bmatrix} -1 \\ i \end{bmatrix}$

10. $\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$

$a = \cos\theta \quad b = -\sin\theta$

$\lambda = \cos\theta - i\sin\theta \quad x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = \cos\theta + i\sin\theta \quad x = \begin{bmatrix} -1 \\ i \end{bmatrix}$

$$\textbf{11.}\ \begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}, \quad \lambda = 3$$

$$11.\ \begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\textbf{12.}\ \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$12.\ (3-\lambda)(4-\lambda)(1-\lambda) = 0$$

$\lambda = 3$

$$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \quad X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 4$

$$\begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \quad X = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad X = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$$

$$\textbf{13.}\ \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$$

$$13.\ \begin{vmatrix} 13-\lambda & 5 & 2 \\ 2 & 7-\lambda & -8 \\ 5 & 4 & 7-\lambda \end{vmatrix} = 0$$

$$(13-\lambda)\cdot(7-\lambda)^2 - 200 + 16 -$$
$$10(7-\lambda) - 32(\lambda - 13)$$
$$-10(7-\lambda) = 0$$
$$-\lambda^3 + 27\lambda^2 - 243\lambda + 729 = 0$$
$$(\lambda - 9)^3 = 0$$

$\lambda = 9$

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 9 & 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

14. $\begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$

14. $\lambda_1 = \frac{1}{2} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 3 \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$, $(\lambda + 1)^2$

15.

$\lambda_1 = 3 \quad \begin{bmatrix} -3 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -5 \begin{bmatrix} -3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = -1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

16. $\begin{bmatrix} -3 & 0 & 4 & 2 \\ 0 & 1 & -2 & 4 \\ 2 & 4 & -1 & -2 \\ 0 & 2 & -2 & 3 \end{bmatrix}$

16. $\lambda_1 = -1, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \begin{bmatrix} -11 \\ 1 \\ 5 \\ 1 \end{bmatrix}$

$\lambda_3 = 3, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

17. Counterclockwise rotation through the angle $\pi/2$ about the origin in R^2.

18. Reflection about the x_1-axis in R^2.

17. P271. Problem 30.

$A = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\lambda_1 = i, \ X_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}; \quad \lambda_2 = -i, \ X_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

18. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

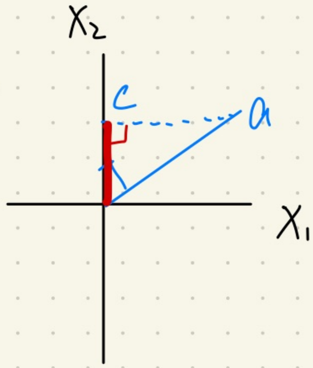
$a_{11}x_1 + a_{12}x_2 = x_1$
$a_{21}x_1 + a_{22}x_2 = -x_2$
$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Leftarrow$ $\lambda_1 = 1, \ X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\lambda_2 = -1, \ X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

19. Orthogonal projection (perpendicular projection) of R^2 onto the x_2-axis.

20. Orthogonal projection of R^3 onto the plane $x_2 = x_1$.

19.



Reference P365

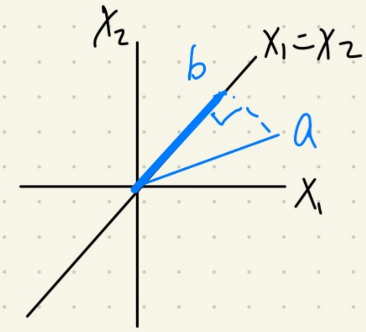
✓ if it means $\vec{c}$, then

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = 0,\ X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \lambda_2 = 1.\ X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

20



$$p = \frac{a \cdot b}{|b|} = \frac{X_1 \cdot X + X_2 \cdot X}{\sqrt{2} \cdot X}$$

$$= \sqrt{2} X.$$

$$(X_1 + X_2) \cdot X = 2X^2$$

$$X = \frac{X_1 + X_2}{2}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{X_1 + X_2}{2} \\ \frac{X_1 + X_2}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda_1 = 1 \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 0 \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

21–25 GENERAL PROBLEMS

21. Nonzero defect. Find further 2×2 and 3×3 matrices with positive defect. See Example 3.

22. Multiple eigenvalues. Find further 2×2 and 3×3 matrices with multiple eigenvalues. See Example 2.

23. Complex eigenvalues. Show that the eigenvalues of a real matrix are real or complex conjugate in pairs.

24. Inverse matrix. Show that $\mathbf{A}^{-1}$ exists if and only if the eigenvalues $\lambda_1, \cdots, \lambda_n$ are all nonzero, and then $\mathbf{A}^{-1}$ has the eigenvalues $1/\lambda_1, \cdots, 1/\lambda_n$.

25. Transpose. Illustrate Theorem 3 with examples of your own.

21. $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$(3-\lambda)^2 = 0 \quad \lambda = 3 \quad M_\lambda = 2$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad m_\lambda = 1 \quad \text{defect} = 1$

$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \lambda = 3, \quad M_\lambda = 3$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad m_\lambda = 2 \quad \text{defect} = 1$

Jordan block

22. $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \lambda_1 = 6 \quad \lambda_{2,3} = \pm\sqrt{2}$

23. Reference 22.

24. if A has $\lambda = 0$, $(A - 0I)X = 0$

$\det(A - 0I) = 0$

$\det A = 0 \Leftrightarrow A^{-1}$ not exist

$AX = \lambda X \Rightarrow A^{-1}AX = A^{-1}\lambda X$

$X = A^{-1} \cdot \lambda X = \lambda A^{-1} X$

$A^{-1}X = \frac{1}{\lambda} \cdot X$

$\Rightarrow \frac{1}{\lambda}$ is the eigenvalue.

25.

Eigenvalues of the Transpose

The transpose $\mathbf{A}^\top$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

Reference 22.

$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$

P333 - Problem set 8.2

PS 8.2

1–6 ELASTIC DEFORMATIONS

Given **A** in a deformation **y** = **Ax**, find the principal directions and corresponding factors of extension or contraction. Show the details.

1. $\begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$ **2.** $\begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$

1. $\begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0$

$9+\lambda^2-6\lambda-2.25=0$

$4\lambda^2-24\lambda+27=0$

$(2\lambda-3)(2\lambda-9)=0$

$\lambda_1 = \frac{3}{2}$

$\begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $135°$

$\lambda_2 = \frac{9}{2}$

$\begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $45°$

2. $\begin{vmatrix} 2-\lambda & 0.4 \\ 0.4 & 2-\lambda \end{vmatrix} = 0$

$4+\lambda^2-4\lambda-0.16=0$

$\lambda^2-4\lambda+3.84=0$

$(5\lambda-8)(5\lambda-12)=0$

$\lambda_1 = \frac{8}{5} = 1.6$

$\begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $135°$

$\lambda_2 = \frac{12}{5} = 2.4$

$\begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $45°$

3. $\begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$

3. $\begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0$

$14 + \lambda^2 - 9\lambda - 6 = 0$

$\lambda^2 - 9\lambda + 8 = 0$

$(\lambda - 8)(\lambda - 1) = 0$

$\lambda_1 = 8$

$\begin{bmatrix} -1 & \sqrt{6} \\ \sqrt{6} & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\sqrt{6} \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix} \Rightarrow \arctan\left(\frac{1}{\sqrt{6}}\right)$
$= 22.21^\circ$

$\lambda_2 = 1$

$\begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{\sqrt{6}}{6} \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} -1 \\ \sqrt{6} \end{bmatrix} \Rightarrow \arctan(-\sqrt{6})$
$= -67.79^\circ$

4. $\begin{vmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix} = 0$

$65 + \lambda^2 - 18\lambda - 4 = 0$

$\lambda^2 - 18\lambda + 61 = 0$

$\lambda = \frac{18 \pm \sqrt{18^2 - 4 \cdot 61}}{2}$

$= 9 \pm \sqrt{81 - 61} = 9 \pm 2\sqrt{5}$

$\lambda_1 = 9 + 2\sqrt{5}$

$\begin{bmatrix} -4-2\sqrt{5} & 2 \\ 2 & 4-2\sqrt{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 2+\sqrt{5} & -1 \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ 2+\sqrt{5} \end{bmatrix} \Rightarrow \arctan(2+\sqrt{5})$
$= 76.72^\circ$

$\lambda_2 = 9 - 2\sqrt{5}$

$\begin{bmatrix} 2\sqrt{5}-4 & 2 \\ 2 & 2\sqrt{5}+4 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{5}-2 & 1 \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 2-\sqrt{5} \end{bmatrix} \Rightarrow \arctan(2-\sqrt{5})$
$= -13.28^\circ$

5. $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$

5. $\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0$

$1+\lambda^2-2\lambda-\frac{1}{4}=0$

$4\lambda^2-8\lambda+3=0$

$(2\lambda-3)(2\lambda-1)=0$

$\lambda_1=\frac{3}{2}$

$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 45°

$\lambda_2=\frac{1}{2}$

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ -45°

6. $\begin{vmatrix} 1.25-\lambda & 0.75 \\ 0.75 & 1.25-\lambda \end{vmatrix} = 0$

$\frac{25}{16}+\lambda^2-\frac{5}{2}\lambda-\frac{9}{16}=0$

$2\lambda^2-5\lambda+2=0$

$(2\lambda-1)(\lambda-2)=0$

$\lambda_1=\frac{1}{2}$

$\begin{bmatrix} 0.75 & 0.75 \\ 0.75 & 0.75 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ -45°

$\lambda_2=2$

$\begin{bmatrix} -0.75 & 0.75 \\ 0.75 & -0.75 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 45°

7–9 MARKOV PROCESSES

Find the limit state of the Markov process modeled by the given matrix. Show the details.

7. $\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

8. $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$ **9.** $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix}$

7. $\begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{5}{8} \\ 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 5 \\ 8 \end{bmatrix}. \Rightarrow \begin{bmatrix} \frac{5}{13} \\ \frac{8}{13} \end{bmatrix}$

8. $\begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 3 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ So limit is $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

9. $\begin{bmatrix} -0.4 & 0.1 & 0.2 \\ 0.4 & -0.9 & 0.4 \\ 0 & 0.8 & -0.6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & -0.8 & 0.6 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 11 \\ 12 \\ 16 \end{bmatrix}$

So limit is $\begin{bmatrix} 11/39 \\ 12/39 \\ 16/39 \end{bmatrix}$

10–12 AGE-SPECIFIC POPULATION

Find the growth rate in the Leslie model (see Example 3) with the matrix as given. Show the details.

10. $\begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 3.45 & 0.60 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$

10. $\begin{vmatrix} -\lambda & 9 & 5 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 + 0.8 + 3.6\lambda = 0$

$5\lambda^3 - 18\lambda - 4 = 0$

$(\lambda-2)(5\lambda^2+10\lambda+2) = 0$

$\lambda_1 = 2$

$\lambda_{2,3} = \dfrac{-10 \pm \sqrt{60}}{10}$

$= -1 \pm \dfrac{\sqrt{15}}{5} < 0$

growth rate is 2.

11. $\begin{vmatrix} -\lambda & 3.45 & 0.6 \\ 0.9 & -\lambda & 0 \\ 0 & 0.45 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 + 0.54 \times 0.45 + 0.9 \times 3.45\lambda = 0$

$-\lambda^3 + 0.243 + 3.105\lambda = 0$

$(\lambda - 1.8)(\lambda^2 + 1.8\lambda + 0.135) = 0$

$\lambda_1 = 1.8$

$\lambda_{2,3} = \dfrac{-90 \pm \sqrt{90^2 - 50 \times 27}}{100} < 0$

growth rate is 1.8

12. $\begin{vmatrix} -\lambda & 3 & 2 & 2 \\ 0.5 & -\lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{vmatrix} = 0$

$= -\lambda \cdot (-\lambda^3) - 3 \cdot 0.5\lambda^2 + 2 \cdot (-0.25\lambda)$
$- 2 \cdot 0.025 = 0$

$\lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05 = 0$

$20\lambda^4 - 30\lambda^2 - 10\lambda - 1 = 0$

not sure how to move on
by pen & paper!
Maybe computer or numeric way

13–15 LEONTIEF MODELS[1]

13. Leontief input–output model. Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the 3×3 **consumption matrix**

$$\mathbf{A} = [a_{jk}] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

where a_{jk} is the fraction of the output of industry k consumed (purchased) by industry j. Let p_j be the price charged by industry j for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to $\mathbf{Ap} = \mathbf{p}$, where $\mathbf{p} = [p_1 \quad p_2 \quad p_3]^\mathsf{T}$, and find a solution $\mathbf{p}$ with nonnegative p_1, p_2, p_3.

14. Show that a consumption matrix as considered in Prob. 13 must have column sums 1 and always has the eigenvalue 1.

all the input and output consumed by the industries. it is a closed system

Σ input = Σ output

13. $$\begin{bmatrix} -0.9 & 0.5 & 0 \\ 0.8 & -1 & 0.4 \\ 0.1 & 0.5 & -0.4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & -4 \\ 0 & 50 & -36 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 0.4 \\ 0.72 \\ 1 \end{bmatrix}$$

15) $X - AX = [0.1 \quad 0.3 \quad 0.1]^T$

$(I - A)X = [0.1 \quad 0.3 \quad 0.1]^T$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.5 & 0 & 0.1 \\ 0.1 & 0.4 & 0.4 \end{bmatrix} \right) X = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$X = (I - A)^{-1} y$

$$= \begin{bmatrix} 0.9 & -0.4 & -0.2 \\ -0.5 & 1 & -0.1 \\ -0.1 & -0.4 & 0.6 \end{bmatrix}^{-1} y$$

$$= \begin{bmatrix} \frac{7}{4} & 1 & \frac{3}{4} \\ \frac{31}{32} & \frac{13}{8} & \frac{19}{32} \\ \frac{15}{16} & \frac{5}{4} & \frac{35}{16} \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 11/20 \\ 103/160 \\ 11/16 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.6438 \\ 0.6875 \end{bmatrix}$$

14. I don't get it.

the output doesn't necessary ALL consumed by those industries. (15) support this.

assume all output consumed within industries, then it means Σ column = 1

P338 - Problem set 8.3 1-15

PS. 8.3

1–10 SPECTRUM

Are the following matrices symmetric, skew-symmetric, or orthogonal? Find the spectrum of each, thereby illustrating Theorems 1 and 5. Show your work in detail.

1. $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$ **2.** $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

1. $a_{12} \neq a_{21}$

$a_{11} \neq -a_{11}$

$$\left[\begin{array}{cc|cc} \frac{4}{5} & \frac{3}{5} & 1 & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 5/4 & 0 \\ 1 & -\frac{4}{3} & 0 & -\frac{5}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 5/4 & 0 \\ 0 & 1 & 3/5 & 4/5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 4/5 & -3/5 \\ 0 & 1 & 3/5 & 4/5 \end{bmatrix}$$

$A^{-1} = A^T$. orthogonal.

$$\begin{vmatrix} 0.8-\lambda & 0.6 \\ -0.6 & 0.8-\lambda \end{vmatrix} = 0$$

$$0.64 + \lambda^2 - 1.6\lambda + 0.36 = 0$$

$$\lambda^2 - 1.6\lambda + 1 = 0$$

$$\lambda = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2}$$

$$= 0.8 \pm 0.6i$$

$$X_{1,2} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}$$

T3 $\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T \mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$

$$R_1 \begin{cases} 0.8 \cdot (-0.6) + 0.6 \cdot 0.8 = 0 \\ 0.8^2 + 0.6^2 = 1 \end{cases}$$

$$C \begin{cases} 0.8 \cdot 0.6 + (-0.6) \cdot 0.8 = 0 \\ 0.8^2 + (-0.6)^2 = 0.6^2 + 0.8^2 = 1 \end{cases}$$

T4 $\begin{vmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{vmatrix}$

$$= -0.64 + 0.36 = 1$$

T5 $\sqrt{0.8^2 + 0.6^2} = 1$

2. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

2. if $b=0$. symmetric

if $a=0$, Skew-symmetric

if $a^2+b^2=1$, orthonormal

$\lambda = a \pm bi \quad X = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$

T1 $b=0$, λ is Real.

$a=0$ λ is imaginary or 0.

3. $\begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3. Reference 2.

4. Reference 2

5. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ 6. $\begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix}$ Symmetric

5. $\lambda_1 = 6 \quad X_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

$\lambda_2 = 6 \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_3 = 1 \quad X_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

6. Symmetric

$\lambda_1 = a-k \quad v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = a-k \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = a+2k \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

if $a=k=0$. Skew-sym

7. $\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

Skew-symmetric

7. anti-Symmestic.

$\lambda_1 = 25i, \quad v_1 = \begin{bmatrix} -45+75i \\ -27-125i \\ 136 \end{bmatrix}$

$\lambda_2 = -25i \quad v_2 = \begin{bmatrix} -45-75i \\ -27+125i \\ 136 \end{bmatrix}$

$\lambda_3 = 0 \quad v_3 = \begin{bmatrix} 20 \\ 12 \\ 9 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

$\lambda_1 = 1 \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = \cos\theta - i\sin\theta$

$V_2 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

$\lambda_3 = \cos\theta + i\sin\theta$

$V_3 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

I already proved that I can use the tools properly to resolve problem. No mean to continue with more cases.

Skip 9-10

11. Reference mindmap and notes.

12 a) $A^T = A^{-1} \quad B^T = B^{-1}$

$(AB)^T = B^T \cdot A^T$

$= B^{-1} \cdot A^{-1}$

$= (AB)^{-1}$

$A^T A = A A^T = I.$

$(A^T A)^T = A^T A = A A^T = I.$

$(A^T)^{-1} = (A^{-1} \cdot A \cdot A^T)^{-1}$

$= (A \cdot A^T)^{-1} A$

$= I \cdot A = A$

Rotation Matrix - Wiki

(b) Rotation. Show that (6) is an orthogonal transformation. Verify that it satisfies Theorem 3. Find the inverse transformation.

$$\text{(6)} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A^T$

if $j = k$

$a_j \cdot a_k = \cos^2\theta + (-\sin\theta)^2$

$= \sin^2\theta + \cos^2\theta = 1$

if $j \neq k$

$a_j \cdot a_k = \cos\theta\sin\theta - \sin\theta\cos\theta$

$= 0$

(c) Powers. Write a program for computing powers $\mathbf{A}^m$ $(m = 1, 2, \cdots)$ of a 2×2 matrix $\mathbf{A}$ and their spectra. Apply it to the matrix in Prob. 1 (call it $\mathbf{A}$). To what rotation does $\mathbf{A}$ correspond? Do the eigenvalues of $\mathbf{A}^m$ have a limit as $m \to \infty$?

(d) Compute the eigenvalues of $(0.9\mathbf{A})^m$, where $\mathbf{A}$ is the matrix in Prob. 1. Plot them as points. What is their limit? Along what kind of curve do these points approach the limit?

(e) Find $\mathbf{A}$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$ is a counterclockwise rotation through 30° in the plane.

c). Per P.S. 8.2-18.

18. Scalar multiples, powers. $k\mathbf{A}$ has the eigenvalues $k\lambda_1, \cdots, k\lambda_n$. $\mathbf{A}^m(m = 1, 2, \cdots)$ has the eigenvalues $\lambda_1^m, \cdots, \lambda_n^m$. The eigenvectors are those of $\mathbf{A}$.

if $|\lambda|<1$, $\lim_{m\to\infty} \lambda^m = 0$

d) $(0.9A)^m = 0.9^m \cdot A^m$

$\lambda = 0.9^m \cdot \lambda_1^m$

if $|0.9\lambda_1| < 1$, $\lim \to 0$.

$0.9\lambda_1 = 1$ $\quad \to 1$

e) $\begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}$

13–20 GENERAL PROPERTIES

13. Verification. Verify the statements in Example 1.

14. Verify the statements in Examples 3 and 4.

15. Sum. Are the eigenvalues of $\mathbf{A} + \mathbf{B}$ sums of the eigenvalues of $\mathbf{A}$ and of $\mathbf{B}$?

are symmetric, skew-symmetric, and orthogonal, respectively, as you should verify. Every skew-symmetric matrix has all main diagonal entries zero. (Can you prove this?) ■

13). $a_{jk} = -a_{kj}$

$j=k$, $a_{jj} = -a_{jj} \Rightarrow 0$.

14). NO. if symmetric ⇒ eigenvalues GR but not ⇐.

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$j=k$.

$a_j \cdot a_k = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$

$j \neq k$

$a_j \cdot a_k = \frac{-4}{9} + \frac{2}{9} + \frac{2}{9}$

$= \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$

$\det(A) = -\frac{8}{27} + \frac{1}{27} - \frac{8}{27} - \frac{4}{27} - \frac{4}{27} - \frac{4}{27} = -1$

$$(6) \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$j=k$, $a_j \cdot a_k = \cos^2\theta + \sin^2\theta = 1$

$j \neq k$. $a_j a_k = \cos\theta\sin\theta - \sin\theta\cos\theta$

$= 0$

$\det(A) = \cos^2\theta + \sin^2\theta = 1$

15. PS. 8.2-16:

$\text{Trace}(A) = \sum \lambda_A$

$\text{Trace}(B) = \sum \lambda_B$

$A+B \Rightarrow \text{Trace}(A+B)$

$= \sum (\lambda_A + \lambda_B)$

16-20: Not finish yet

catch up

3. $\begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$

$2 \pm 8i$. answer is wrong.
TYPO.

9. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\lambda_1 = 1$, $V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = i$, $V_2 = \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$

$V_3 = -i$, $V_3 = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$

orthogonal.

P345 - Problem set 8.4

Ps 8.4

1–5 SIMILAR MATRICES HAVE EQUAL EIGENVALUES

Verify this for **A** and $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$. If **y** is an eigenvector of ~~**P**~~ $\hat{A}$, show that $\mathbf{x} = \mathbf{Py}$ are eigenvectors of **A**. Show the details of your work.

1. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$, $\quad \mathbf{P} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

Mistake: Verify A & D have equal eigenvalue $A \neq P^{-1}AP$.

1. $(3-\lambda)(-3-\lambda)-16=0.$

$(3-\lambda)(3+\lambda)+16=0$

$9-\lambda^2+16=0$

$\lambda=\pm 5$

$\lambda_1=5 \quad \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$x_1=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2=-5 \quad \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$x_2=\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$P: \begin{bmatrix} -4 & 2 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 2 & 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{3}{2} & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$

$\hat{A} = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} P$

$= \begin{bmatrix} 5.5 & -1 \\ 12.5 & 0 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} -22-3 & 11+1 \\ -50 & 25 \end{bmatrix}$

$= \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$

$(-25-\lambda)(25-\lambda)+600=0$

$625-\lambda^2+600=0$

$\lambda=\pm 5 \Rightarrow$ same as A

$\lambda_1=5: \begin{bmatrix} -30 & 12 \\ -50 & 20 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 0 & 0 \end{bmatrix}$

$y_1=\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$Py_1 = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = x_1$

$\lambda_2=-5: \begin{bmatrix} -20 & 12 \\ -50 & 30 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 0 & 0 \end{bmatrix}$

$y_2=\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$Py_2 = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = 2x_2$

2. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix}$

$A: \lambda_1 = -1 \quad x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -7 & 5 \\ -10 & 7 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix} P$

$= \begin{bmatrix} -29 & 20 \\ -42 & 29 \end{bmatrix} = \hat{A}$

$\lambda_1 = -1 \quad y_1 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$Py_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1$

$\lambda_2 = 1 \quad y_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$Py_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -x_2$

3. $\mathbf{A} = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{bmatrix}$

$A: \lambda_1 = 4 \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 6 \quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} \frac{8}{25} & -\frac{76}{25} \\ \frac{206}{25} & -\frac{82}{25} \end{bmatrix} P$

$= \begin{bmatrix} -\frac{376}{125} & -\frac{68}{125} \\ \frac{682}{125} & \frac{874}{125} \end{bmatrix} = \hat{A}$

$\lambda_1 = 4 \quad y_1 = \begin{bmatrix} -\frac{17}{31} \\ 1 \end{bmatrix}$

$Py_1 = \begin{bmatrix} \frac{25}{31} \\ \frac{25}{31} \end{bmatrix} = \frac{25}{31} \cdot x_1$

$\lambda_2 = 6 \quad y_2 = \begin{bmatrix} -\frac{2}{11} \\ 1 \end{bmatrix}$

$Py_2 = \begin{bmatrix} \frac{10}{11} \\ \frac{5}{11} \end{bmatrix} = \frac{5}{11} \cdot x_2$

WoW.

4. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$,
$\lambda_1 = 3$

A: $\lambda_1 = -1$ $\quad X_1 = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$

$\lambda_2 = 2$ $\quad X_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\lambda_3 = 3$ $\quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}$

$\hat{A} = P^{-1}AP = \begin{bmatrix} -3 & 0 & 7 \\ 0 & 3 & 2 \\ 2 & 0 & -4 \end{bmatrix} P$

$= \begin{bmatrix} 15 & 0 & 26 \\ 6 & 3 & 10 \\ -8 & 0 & -14 \end{bmatrix}$

$\lambda_1 = -1$ $\quad y_1 = \begin{bmatrix} -26 \\ -1 \\ 16 \end{bmatrix}$ $\quad Py_1 = X_1$

$\lambda_2 = 2$ $\quad y_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ $\quad Py_2 = -1 \cdot X_2$

$\lambda_3 = 3$ $\quad y_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad Py_3 = X_3$

5: ignore. Same approach as problem 1~4.

6-a Compare the co-effrent,
trace $= \sum \lambda$

1): $0 = 5 - 5$.
3) $10 = 4 + 6$
5) by pass.

6-b Theroms-3: $\text{trace}(A) = \text{trace}(\hat{A})$

$\text{trace}(AB) = \text{trace}(A^{-1}ABA)$
$= \text{trace}(BA)$

(c) Find a relationship between $\hat{\mathbf{A}}$ in (4) and $\hat{\mathbf{A}} = \mathbf{PAP}^{-1}$.

4). $\hat{A}_1 = P^{-1}AP$. $\quad \hat{A}_2 = PAP^{-1}$

$\hat{A}_1$ similar to A similar to $\hat{A}_2$

d) Yes. change the sequence of column vector (eignvector)

?

7. No basis. Find further 2×2 and 3×3 matrices without eigenbasis.

Not sure. if $\lambda \in \mathbb{C}$ then always have eigenvalue.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

with $\lambda = 0$, any eignvector works.

THEOREM 2 **Symmetric Matrices**

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

8. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\lambda_1 = -1 \quad V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3 \quad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$: orthonormal

9–16 **DIAGONALIZATION OF MATRICES**

Find an eigenbasis (a basis of eigenvectors) and diagonalize. Show the details.

9. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

9. $(1-\lambda)(4-\lambda) - 4 = 0$

$4 + \lambda^2 - 5\lambda - 4 = 0$

$\lambda^2 - 5\lambda = 0$

$\lambda_1 = 5 \quad \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 0 \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$X = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

$D = X^{-1}AX = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

11 $\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -6 & -2 \\ -6 & 3 \end{bmatrix}$

12 $\begin{bmatrix} -4.3 & 7.7 \\ 1.3 & 9.3 \end{bmatrix} = \begin{bmatrix} -11 & \frac{7}{13} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \frac{-13}{150} & \frac{7}{158} \\ \frac{13}{150} & \frac{143}{150} \end{bmatrix}$

13. $\begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{2}{3} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -4 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} -5 & -6 & 6 \\ -9 & -8 & 12 \\ -12 & -12 & 16 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{bmatrix}$

$= \begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

16 $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

17–23 PRINCIPAL AXES. CONIC SECTIONS

What kind of conic section (or pair of straight lines) is given by the quadratic form? Transform it to principal axes. Express $\mathbf{x}^\mathsf{T} = [x_1 \quad x_2]$ in terms of the new coordinate vector $\mathbf{y}^\mathsf{T} = [y_1 \quad y_2]$, as in Example 6.

17. $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

17. We have $Q = X^T A X$

$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$(7-\lambda)^2 - 9 = 0$

$\lambda_1 = 4 \quad \lambda_2 = 10$

$Q = 4y_1^2 + 10y_2^2 = 200$

$\frac{y_1^2}{50} + \frac{y_2^2}{20} = 1$

for A, $\lambda_1 = 4 \quad V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

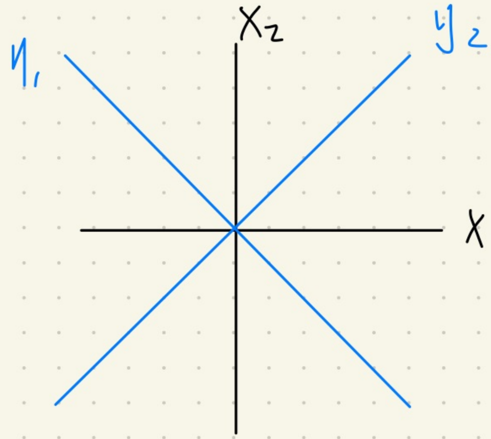
$U_1 = \frac{V_1}{|V_1|} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$\lambda_2 = 10 \quad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$U_2 = \frac{V_2}{|V_2|} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$x = Xy = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{cases} X_1 = -\frac{\sqrt{2}}{2} y_1 + \frac{\sqrt{2}}{2} y_2 \\ X_2 = \frac{\sqrt{2}}{2} y_1 + \frac{\sqrt{2}}{2} y_2 \end{cases}$$



So it rotate 45° and also enchang (X_1, X_2) sequence ?

18-23: Same as 17.
By pass them at the moment

$$Q = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k \qquad (a_{kj} = a_{jk})$$

24.

25.

5. $\mathbf{A} = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A: \lambda_{A1} = 0, \quad V_{A1} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_{A2} = 4, \quad V_{A2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_{A3} = 10, \quad V_{A3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\hat{A} = P^{-1}AP = \begin{bmatrix} 3 & 4 & -9 \\ -5 & 0 & 15 \\ -5 & 0 & 15 \end{bmatrix} P$

$= \begin{bmatrix} 4 & 3 & -9 \\ 0 & -5 & 15 \\ 0 & -5 & 15 \end{bmatrix}$

$\lambda_{\hat{A}1} = 0 \quad V_{\hat{A}1} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

$\lambda_{\hat{A}2} = 4 \quad V_{\hat{A}2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_{\hat{A}3} = 10 \quad V_{\hat{A}3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow \lambda_{\hat{A}}$ equals to λ_A

Similar.

$PV_{A1} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = V_{\hat{A}1}$

$PV_{A2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = V_{\hat{A}2}$

$PV_{A3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = V_{\hat{A}3}$

$\Rightarrow Px = y.$

P351 - Problem set 8.5

PS. 8.5

1–6 EIGENVALUES AND VECTORS

Is the given matrix Hermitian? Skew-Hermitian? Unitary? Find its eigenvalues and eigenvectors.

1. $\begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix}$ **2.** $\begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$

5. $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

1. $\lambda_1 = 7$ $V_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = 5$ $V_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 6 & -i \\ i & 6 \end{bmatrix}$

$\bar{A}^T = \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix} = A$

Hermitian.

$7 > 5 > 1$, not unitary.

$\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

$\bar{A}^T = -\bar{A} \Rightarrow$ Skew-Hermitian

$\lambda_1 = i$ $V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = i$ $V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_3 = -i$ $V_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = -1$

$\bar{A}^T = A^{-1}$, unitary.

9–12 COMPLEX FORMS

Is the matrix **A** Hermitian or skew-Hermitian? Find $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x}$. Show the details.

9. $\mathbf{A} = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -4i \\ 2+2i \end{bmatrix}$

11. $\mathbf{A} = \begin{bmatrix} i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 4 & 3+2i \\ 3-2i & -4 \end{bmatrix}$

$\bar{A}^T = A$ Hermitian

$\bar{X}^T = [4i \quad 2-2i]$

$\bar{X}^T A X = [2+6i \quad -16+20i] X$

$= -48$

$\bar{A} = \begin{bmatrix} -i & 1 & 2-i \\ -1 & 0 & -3i \\ -2-i & -3i & -i \end{bmatrix}$

$\bar{A}^T = -A \Rightarrow$ skew-Hermitian

$\bar{X}^T = [1 \quad -i \quad i]$

$\bar{X}^T \cdot A \cdot X = [-1 \quad -2 \quad 4+i] X$

$= -6i$