# Chapter 15 - Power Series, Taylor Series

## Selected Problem set 15.1

15.1 1. 5, 9, 17. 19. 23. 25

bound.
Convergent
Limit

Convergent, hence bounded

**1–10 SEQUENCES**

Is the given sequence $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find its limit points. Show your work in detail.

**1.** $z_n = (1+i)^{2n}/2^n$ **2.** $z_n = (3+4i)^n/n!$

1. $\dfrac{z_{n+1}}{z_n} = \dfrac{(1+i)^{2(n+1)}}{2^{n+1}} \cdot \dfrac{2^n}{(1+i)^{2n}} = \dfrac{(1+i)^2}{2} = \dfrac{1+i^2+2i}{2} = i$

$z_1 = \dfrac{(1+i)^2}{2} = i$, $z_2 = -1$, $z_3 = -i$, $z_4 = 1$, $z_5 = z_1$

bounded.

Theorem 7. $\left|\dfrac{z_{n+1}}{z_n}\right| = 1 \geq 1$ diverges

**5.** $z_n = (-1)^n + 10i$

$z_1 = -1+10i$ $z_2 = 1+10i$ $z_3 = z_1$ $z_4 = z_2$ ...

bounded.

for $t = 1, 2, \ldots$

$$\left|\frac{z_{2t+1}}{z_{2t}}\right| = \left|\frac{(1+10i)^2}{(-1+10i)(1+10i)}\right| = \left|\frac{1-100+20i}{-100-1}\right| = \left|+\frac{99}{101} - \frac{20}{101}i\right| \geq 1$$

diverges.

**9.** $z_n = (3+3i)^{-n}$

$|z_1| = \left|\frac{1}{6}(1-i)\right| = \frac{\sqrt{2}}{6}$ $\quad |z_2| = \left|-\frac{i}{18}\right| = \frac{1}{18}$

$|z_3| = \left|-\frac{1}{108}(1+i)\right| = \frac{\sqrt{2}}{108}$

$|z_{n+1}| = \left|z_n \cdot \left(\frac{1}{3+3i}\right)\right| = |z_n| \cdot \frac{\sqrt{2}}{6} \le |z_1| = \frac{\sqrt{2}}{6}$ bounded.

$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(3+3i)^{-(n+1)}}{(3+3i)^{-n}}\right| = \left|(3+3i)^{-1}\right| = \frac{\sqrt{2}}{6} < 1$ converges

**16–25 SERIES**

Is the given series convergent or divergent? Give a reason. Show details.

**16.** $\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$ **17.** $\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$

17. $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{(-1)^n}\right| = \left|(-1) \cdot \frac{\ln n}{\ln(n+1)}\right| = \left|\frac{\ln n}{\ln(n+1)}\right|$

$n^{n+1} > (n+1)^n$ for $n \ge 3$. $\Rightarrow \frac{\ln n}{\ln(n+1)} > \frac{n}{n+1}$

$\Rightarrow \left|\frac{z_{n+1}}{z_n}\right| > \left|\frac{n}{n+1}\right|$ diverges

**19.** $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{(n+1)^2 - i} \cdot \frac{n^2 - i}{i^n}\right| = \left| i \cdot \frac{n^2-i}{(n+1)^2 - i}\right|$$

$$= \left|\frac{(n^2-i)[(n+1)^2+i]}{[(n+1)^2-i][(n+1)^2+i]}\right|$$

$$= \left|\frac{(n^2-i)(n^2+2n+1+i)}{(n+1)^4+1}\right|$$

$$= \left|\frac{n^4+2n^3+n^2+\cancel{in^2}-\cancel{in^2}-2ni-i+1}{(n+1)^4+1}\right|$$

$$= \left|\frac{1+\frac{2}{n}+\frac{1}{n^2}-\frac{2i}{n^3}+\frac{1-i}{n^4}}{(1+\frac{1}{n})^4+\frac{1}{n}}\right|$$

lim → 1 ⇒ don't know

$$|z_n|^{\frac{1}{n}} = \left|\frac{i}{\sqrt[n]{n^2-i}}\right| = \left|\frac{i}{(n^2+1)^{\frac{1}{n}}}\right| < \left|\frac{i}{n^{\frac{2}{n}}}\right| = 1$$

converge absolutely

**23.** $\sum_{n=0}^{\infty} \frac{(-1)^n(1+i)^{2n}}{(2n)!}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}\,(1+i)^{2n+2}}{(2n+2)!} \cdot \frac{2n!}{(-1)^n\,(1+i)^{2n}}\right|$$

$$= \left|\frac{-1\cdot(1+i)^2}{(2n+2)(2n+1)}\right|$$

$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0 < 1$ converge absolutely

**25.** $\sum_{n=1}^{\infty} \frac{i^n}{n}$

$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{n+1} \cdot \frac{n}{i^n}\right| = \left|\frac{n\,i}{n+1}\right| = \left|\frac{n}{n+1}\right|$ $\lim \Rightarrow 1$ not sure

$\left|z_n^{\frac{1}{n}}\right| = \left|\frac{1}{\sqrt[n]{n}}\right| \Rightarrow \lim_{n\to\infty}|z_n|^{\frac{1}{n}} = 1$ not sure

Per example 3. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ conditionally converge.

so $1, -1, -i, 1.$

$\{z_{4t+1} + z_{4t+3}\}$ $t$ is even $\Rightarrow$ converge. $x_i$

$\{z_{4t+2} + z_{4t+4}\}$ $t$ is odd $\Rightarrow$ converge $y$

$\Rightarrow$ $S_n$ jumping from $x_i$ to $y$, diverge?

# Selected Problem set 15.2

15.2  7. 9. 11. 13. 17

**6–18  RADIUS OF CONVERGENCE**

Find the center and the radius of convergence.

**6.** $\sum_{n=0}^{\infty} 4^n(z+1)^n$  **7.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z-\frac{1}{2}\pi\right)^{2n}$

$$\sqrt{\left|\frac{a_{n+1}(z-\frac{\pi}{2})^{2n+2}}{a_n(z-\frac{\pi}{2})^{2n}}\right|} = \sqrt{\left|\frac{a_{n+1}}{a_n}\right| \cdot \left|(z-\frac{\pi}{2})^2\right|}$$

$$= \sqrt{\left|\frac{(-1)^{n+1}}{(2n+2)!} \cdot \frac{2n!}{(-1)^n}\right|} \cdot \left|(z-\frac{\pi}{2})\right|$$

$$= \sqrt{\left|\frac{1}{(2n+2)(2n+1)}\right|} \cdot \left|(z-\frac{\pi}{2})\right|$$

$L = L^* \cdot (z-\frac{\pi}{2})$.  $\lim_{n\to\infty} L^* = 0$,  $\lim_{n\to+\infty} L = 0$

Converge for all z by the ratio test

**9.** $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n}$

$$\sqrt{\left|\frac{a_{n+1}(z-i)^{2n+2}}{a_n(z-i)^{2n}}\right|} = \sqrt{\frac{a_{n+1}}{a_n}} \cdot |z-i|$$

$$= \sqrt{\frac{(n+1)\cdot n}{3^{n+1}} \cdot \frac{3^n}{n(n-}} \cdot |z-i|$$

$$= \sqrt{\frac{n+1}{n-1} \cdot \frac{1}{3}} \quad |z-i|$$

$L = L^* \; |z-i|$.  $\lim_{n\to\infty} L^* = \frac{\sqrt{3}}{3}$.  $\lim_{n\to+\infty} L = \sqrt{3}$  Center $= i$

**11.** $\sum_{n=0}^{\infty} \left(\frac{2-i}{1+5i}\right) z^n$

$|z| \geq 1$: divergence.

$|z| < 1$ Converge

$$S_n = \left(\frac{2-i}{1+5i}\right) \cdot z^0 + \left(\frac{2-i}{1+5i}\right) \cdot z + \cdots \left(\frac{2-i}{1+5i}\right) z^n$$

$$z S_n = \left(\frac{2-i}{1+5i}\right) z^1 + \cdots \left(\frac{2-i}{1+5i}\right) \cdot z^{n+1}$$

$$(1-z) S_n = \frac{2-i}{1+5i} \cdot 1 - \left(\frac{2-i}{1+5i}\right) \cdot z^{n+1}$$

$n \to +\infty$ $z^{n+1} \to 0$

$$S_n = \frac{2-i}{(1-z)(1+5i)}$$

Center: 0

Radiu: 1

**13.** $\sum_{n=0}^{\infty} 16^n (z+i)^{4n}$

$$\sqrt[4]{\left|\frac{a_{n+1}(z+i)^{4n+4}}{a_n (z+i)^{4n}}\right|} = \sqrt[4]{\left|\frac{a_{n+1}}{a_n}\right|} \cdot |z+i|$$

$$= \sqrt[4]{16} \cdot |z+i| = 2|z+i|$$

$\lim_{n\to\infty} L^* = 2$ $\lim_{n\to\infty} L = \frac{1}{2}$ Center $-i$

**17.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n(n+1)} z^{2n+1}$

$$\sqrt{\left|\frac{a_{n+1} \cdot z^{2n+3}}{a_n \cdot z^{2n+1}}\right|} = \sqrt{\frac{a_{n+1}}{a_n}} \cdot |z|$$

$$= \sqrt{\frac{2^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{2^n}}\ |z|$$

$$= \sqrt{\frac{2n}{n+2}} \cdot |z|$$

$\lim_{n\to\infty} L^* = \sqrt{2}$ $\qquad$ $\lim_{n\to\infty} L = \frac{\sqrt{2}}{2}$ $\qquad$ center : 0

# Selected Problem set 15.3

15.3 5, 7, 9, 11, 15

**5–15 RADIUS OF CONVERGENCE BY DIFFERENTIATION OR INTEGRATION**

Find the radius of convergence in two ways: **(a)** directly by the Cauchy–Hadamard formula in Sec. 15.2, and **(b)** from a series of simpler terms by using Theorem 3 or Theorem 4.

**5.** $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}(z-2i)^n$ **6.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1}$

5. (a) $R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ **(Cauchy–Hadamard formula[1]).**

$$R = \lim_{n\to\infty}\left|\frac{n(n-1)}{2^n}\cdot\frac{2^{n+1}}{(n+1)\cdot n}\right| = \lim_{n\to\infty}\left|\frac{2(n-1)}{n+1}\right| = 2$$

(b) $$f(z) = \sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}(z-2i)^n = \sum_{n=2}^{\infty} n(n-1)\cdot\left(\frac{z-2i}{2}\right)^n$$

$$g(x) = \sum_{n=2}^{\infty} x^n \quad \Rightarrow R=1$$

$$g^{(2)}(x) = \sum_{n=2}^{\infty} n(n-1)\cdot x^{n-2} \quad \Rightarrow R=1.$$

$$\left|\frac{z-2i}{2}\right| < 1 \qquad |z-2i| < 2 \quad \Rightarrow R=2$$

$$7. \sum_{n=1}^{\infty} \frac{n}{3^n}(z+2i)^{2n}$$

a) per P683.

$$\left|\frac{\frac{n+1}{3^{n+1}}(z+2i)^{2(n+1)}}{\frac{n}{3^n}(z+2i)^{2n}}\right| = \left|\frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n}\right| \cdot \left|(z+2i)^2\right| = \left|\frac{n+1}{3n}\right| \cdot \left|(z+2i)^2\right|$$

The limit is $L = L^* \; |z+2i| = \frac{\sqrt{3}}{3}|z+2i|$

$R = \frac{1}{L^*} = \sqrt{3}$ or $\frac{\sqrt{3}}{3}|z+2i| < 1$, $|z+2i| < \sqrt{3}$.

b) $f(x) = 3 \cdot \left(\frac{x}{3}\right)^n = \frac{1}{3^{n-1}} \cdot x^n$

$f'(x) = \frac{n}{3^{n-1}} \cdot x^{n-1}$

$f'(x) \cdot \frac{x}{3} = \frac{n}{3^n} x^n$

$f(x)$: $R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{3^n}{3^{n-1}}\right| = 3$

$f'(x)$: $R = 3$ $\qquad f'(x) \cdot \frac{x}{3}$: $R = 3$.

$\Rightarrow R = \sqrt{3}$

**9.** $\sum_{n=1}^{\infty} \frac{(-2)^n}{n(n+1)(n+2)} z^{2n}$

a) $\left| \frac{(-2)^{n+1} \cdot z^{2(n+1)}}{(n+1)(n+2)(n+3)} \cdot \frac{n(n+1)(n+2)}{(-2)^n \cdot z^{2n}} \right| = \left| \frac{n}{n+3} \cdot (-2) \cdot z^2 \right|$

$= \left| \frac{2n}{n+3} \right| \cdot |z^2|$

The lim is $2|z^2| < 1$, $|z| < \frac{\sqrt{2}}{2}$, $R = \frac{\sqrt{2}}{2}$

b). $f(x) = x^{n+2}$ $|x| < 1$

$f^{(3)}(x) = n(n+1)(n+2)x^n$ $|x| < 1$

$|-2z^2| < 1$ $|z| < \frac{\sqrt{2}}{2}$ $R = \frac{\sqrt{2}}{2}$

**11.** $\sum_{n=1}^{\infty} \frac{3^n n(n+1)}{7^n} (z+2)^{2n}$

a) $\left| \frac{3^{n+1}(n+1)(n+2) \cdot (z+2)^{2n+2}}{7^{n+1}} \cdot \frac{7^n}{(z+2)^{2n} \cdot 3^n \cdot n(n+1)} \right|$

$= \left| \frac{3(n+2)}{7n} \right| \cdot \left| (z+2)^2 \right|$

the lim is $\frac{3}{7} \cdot \left| (z+2)^2 \right| < 1$ $\quad R: \sqrt{\frac{7}{3}}$

b). $f(x) = \left(\frac{3}{7}x\right)^{n+1} = \left(\frac{3}{7}\right)^{n+1} \cdot x^{n+1}$ $\quad \left| \frac{3}{7}x \right| < 1$

$f^{(2)}(x) = n(n+1) \left(\frac{3}{7}\right)^{n+1} \cdot x^{n-1} = \frac{3^{n+1} \cdot n(n+1)}{7^{n+1}} \cdot x^{n-1}$

$f^{(2)}(x) \cdot \frac{7}{3} \cdot x = \frac{3^n n(n+1)}{7^n} \cdot x^n$

$\left| (z+2)^2 \cdot \frac{3}{7} \right| < 1$ $\quad \left| (z+2)^2 \right| < \frac{7}{3}$ $\quad R: \sqrt{\frac{7}{3}}$

**15.** $\sum_{n=2}^{\infty} \frac{4^n n(n-1)}{3^n}(z-i)^n$

a) $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to+\infty} \left| \frac{4^n n(n-1)}{3^n} \cdot \frac{3^{n+1}}{4^{n+1}(n+1)\cdot n} \right|$

$= \lim_{n\to+\infty} \left| \frac{3(n-1)}{4(n+1)} \right| = \frac{3}{4}$

b) $f(x) = \left(\frac{4}{3}x\right)^n = \left(\frac{4}{3}\right)^n \cdot x^n$ $\quad\quad \left|\frac{4}{3}x\right| < 1$

$f^{(2)}(x) = \left(\frac{4}{3}\right)^n \cdot n(n-1) \cdot x^{n-2}$

$f^{(2)}(x) \cdot x^2 = \frac{4^n n(n-1)}{3^n} \cdot x^n$

$\left|\frac{4}{3}(z-i)\right| < 1 \quad\quad R = \frac{3}{4}$

# Selected Problem set 15.4

15.4 3, 5, 7, 19, 21, 23

**3–10 MACLAURIN SERIES**

Find the Maclaurin series and its radius of convergence.

**3.** $\sin 2z^2$ **4.** $\frac{z+2}{1-z^2}$

3. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin 2z^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(2z^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} \cdot z^{4n+2}}{(2n+1)!} = 2z^2 - \frac{8z^6}{3!} + \frac{32z^{10}}{5!} - \cdots$

**5.** $\frac{1}{2+z^4}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$\frac{1}{2+z^4} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{1}{2}z^4)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{1}{2}z^4\right)^n = -\left(-\frac{1}{2}\right)^{n+1} \cdot z^{4n}$

$= \frac{1}{2} - \frac{1}{4}z^4 + \frac{1}{8}z^8 - \cdots$

**7.** $\cos^2 \frac{1}{2}z$

$\cos^2 x = \frac{1}{2}[\cos(2x) + 1] = 1 - x^2 + \frac{x^4}{3} + \cdots$

$\cos^2 \frac{1}{2}z = 1 - \left(\frac{1}{2}z\right)^2 + \frac{\left(\frac{1}{2}z\right)^4}{3} + \cdots$

$= 1 - \frac{z^2}{4} + \frac{z^4}{48} - \cdots$

**18–25** **TAYLOR SERIES**

Find the Taylor series with center $z_0$ and its radius of convergence.

**18.** $1/z, \quad z_0 = i$ **19.** $1/(1-z), \quad z_0 = i$

**20.** $\cos^2 z, \quad z_0 = \pi/2$ **21.** $\sin z, \quad z_0 = \pi/2$

**22.** $\cosh(z - \pi i), \quad z_0 = \pi i$

**23.** $1/(z+i)^2, \quad z_0 = i$ **24.** $e^{z(z-2)}, \quad z_0 = 1$

19.

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots.$$

$$f(z) = f(z_0) + \frac{z - z_0}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

$$f(z) = \frac{1}{1-z} = \frac{1}{1-i} + \frac{z-z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} \cdot f''(z_0) + \cdots$$

$$= \frac{1}{1-i} + \frac{z-z_0}{1} \cdot \frac{1}{(1-i)^2} + \frac{(z-z_0)^2}{2!} \cdot \frac{2!}{(1-i)^3}$$

$$+ \frac{(z-z_0)^3}{3!} \frac{3!}{(1-i)^4} + \cdots$$

$$= \frac{1+i}{2} + \frac{i}{2}(z-i) + \left(-\frac{1-i}{4}\right)(z-i)^2 + -\frac{1}{4}(z-i)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}} \cdot (z-z_0)^n$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(1-i)^{n+2}}{(1-i)^{n+1}}\right| = |1-i| = \sqrt{2}$$

**21.** $\sin z, \quad z_0 = \pi/2$

$$f(z) = \sin z = f\left(\tfrac{\pi}{2}\right) + \frac{z-\frac{\pi}{2}}{1!} \cdot f'\left(\tfrac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^2}{2!} \cdot f''\left(\tfrac{\pi}{2}\right) + \cdots$$

$$= 1 + \frac{z-\frac{\pi}{2}}{1!} \cdot \cos\left(\tfrac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^2}{2!} \cdot \left(-\sin\tfrac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^3}{3!} \cdot \left(-\cos\tfrac{\pi}{2}\right) + \cdots$$

$$= 1 + 0 - \frac{1}{2!}\left(z-\tfrac{\pi}{2}\right)^2 + 0 + \frac{1}{4!}\left(z-\tfrac{\pi}{2}\right)^4 + 0 - \frac{\left(z-\frac{\pi}{2}\right)^6}{6!}$$

or:

$$\sin z = \sin\left(z - \tfrac{\pi}{2} + \tfrac{\pi}{2}\right) = \sin\left(z-\tfrac{\pi}{2}\right)\cos\tfrac{\pi}{2} + \cos\left(z-\tfrac{\pi}{2}\right)\sin\tfrac{\pi}{2}$$

$$= \cos\left(z-\tfrac{\pi}{2}\right)$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(z-\frac{\pi}{2}\right)^{2n}}{2n!}$$

$\cos z$: $R = \infty$. $\Rightarrow R = \infty$.

**23.** $1/(z+i)^2, \quad z_0 = i$

$$\frac{1}{(z+i)^2} = \frac{1}{[(z-i)+2i]^2} = \frac{1}{(2i)^2} \frac{1}{\left(\frac{z-i}{2i}+1\right)^2}$$

P696:

$$\frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$

$$= 1 - mz + \frac{m(m+1)}{2!}z^2 - \frac{m(m+1)(m+2)}{3!}z^3 + \cdots$$

$$= -\frac{1}{4} \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{z-i}{2i}\right)^n$$

$\left|\frac{z-i}{2i}\right| < 1 \qquad |z-i| < 2 \qquad R = 2$