W8 - Complex Trigonometric Functions

8A: Exponential Function

- Definition
  - $e^z = e^x(\cos y + i \sin y)$.
    - **(A)** $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.
    - **(B)** e^z is analytic for all z. (Proved in Example 2 of Sec. 13.4.)
    - **(C)** The derivative of e^z is e^z, that is,
  - series $1 + x + x^2/2! + x^3/3! + \cdots$
- Properties
  - $e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = e^{z_1+z_2}$
    - $e^{z+2\pi i} = e^z$ for all z
  - $e^x \neq 0$
  - fundamental region
    - $-\pi < y \leqq \pi$
- Euler formular
  - $e^z = e^x e^{iy}$.
    - $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$.
    - $|e^z| = e^x$.
      - $\arg e^z = y \pm 2n\pi \quad (n = 0, 1, 2, \cdots)$,
  - $e^{iy} = \cos y + i \sin y$.
    - $e^{2\pi i} = 1$
    - $e^{\pi i/2} = i$, $\quad e^{-\pi i/2} = -i$,
    - $e^{\pi i} = -1$, $\quad e^{-\pi i} = -1$.
  - $z = re^{i\theta}$.
    - $z = r(\cos\theta + i \sin\theta)$

8B: Trigonometric Function

- $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.
- $(\cos z)' = -\sin z$, $\qquad (\sin z)' = \cos z$, $\qquad (\tan z)' = \sec^2 z$,
- $\cos z = \cos x \cosh y - i \sin x \sinh y$
  $\sin z = \sin x \cosh y + i \cos x \sinh y$
  - $|\cos z|^2 = \cos^2 x + \sinh^2 y$
    $|\sin z|^2 = \sin^2 x + \sinh^2 y$
  - $\cosh y = \frac{1}{2}(e^y + e^{-y})$, $\qquad \sinh y = \frac{1}{2}(e^y - e^{-y})$;
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
  $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

Hyperbolic Function

- $\cosh z = \frac{1}{2}(e^z + e^{-z})$, $\qquad \sinh z = \frac{1}{2}(e^z - e^{-z})$.
  - $\tanh z = \frac{\sinh z}{\cosh z}$, $\quad \coth z = \frac{\cosh z}{\sinh z}$,
    $\operatorname{sech} z = \frac{1}{\cosh z}$, $\quad \operatorname{csch} z = \frac{1}{\sinh z}$.
  - $\cosh iz = \cos z$, $\qquad \sinh iz = i \sin z$.
  - $\cos iz = \cosh z$, $\qquad \sin iz = i \sinh z$.
- $(\cosh z)' = \sinh z$, $\qquad (\sinh z)' = \cosh z$,

8C: Logarithm and Principal Value

- $e^w = z$. $\quad e^w = e^{u+iv} = re^{i\theta}$. $\quad e^u = r$, $\quad v = \theta$.
- $\ln z = \ln r + i\theta \qquad (r = |z| > 0, \quad \theta = \arg z)$.
- the complex natural logarithm ln z (z != 0) is **infinitely** many-valued. **principal value**
  - z != 0, **principal value**
  - $\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z$
    - $-\pi < \operatorname{Arg} z \leqq \pi$.
  - $\ln z = \operatorname{Ln} z \pm 2n\pi i$
- Branch-cut
- $e^{\ln z} = z$
  - $\ln(e^z) = z \pm 2n\pi i$,
- Examples:

$\ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \cdots$	$\operatorname{Ln} 1 = 0$
$\ln 4 = 1.386294 \pm 2n\pi i$	$\operatorname{Ln} 4 = 1.386294$
$\ln(-1) = \pm\pi i, \pm 3\pi i, \pm 5\pi i, \cdots$	$\operatorname{Ln}(-1) = \pi i$
$\ln(-4) = 1.386294 \pm (2n + 1)\pi i$	$\operatorname{Ln}(-4) = 1.386294 + \pi i$
$\ln i = \pi i/2, -3\pi/2, 5\pi i/2, \cdots$	$\operatorname{Ln} i = \pi i/2$
$\ln 4i = 1.386294 + \pi i/2 \pm 2n\pi i$	$\operatorname{Ln} 4i = 1.386294 + \pi i/2$
$\ln(-4i) = 1.386294 - \pi i/2 \pm 2n\pi i$	$\operatorname{Ln}(-4i) = 1.386294 - \pi i/2$
$\ln(3 - 4i) = \ln 5 + i \arg(3 - 4i)$	$\operatorname{Ln}(3 - 4i) = 1.609438 - 0.927295i$
$= 1.609438 - 0.927295i \pm 2n\pi i$	(Fig. 337)

- $\ln(z_1 z_2) = \ln z_1 + \ln z_2$,
  - $\ln(z_1/z_2) = \ln z_1 - \ln z_2$
  - $\operatorname{Ln}(z_1 z_2)$
    May differ 2pi i
- $(\ln z)' = \frac{1}{z}$

General Power

- $z^c = e^{c \ln z}$
  - $(c \text{ complex}, z \neq 0)$.
  - $z^c = \sqrt[n]{z} = e^{(1/n) \ln z}$ $\quad (z \neq 0)$,
- $a^z = e^{z \ln a}$.