Chapter 13 - Complex Numbers and Functions. Complex Differentiation

Selected Problem set 13.1

13.1 1. 5. 11. 17

1. Powers of *i*. Show that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \cdots$ and $1/i = -i, 1/i^2 = -1, 1/i^3 = i, \cdots$.

$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, \quad x_1 y_2 + x_2 y_1).$

$i^2 = (0, 1)(0, 1) = (0-1, 0+0) = (-1, 0) = -1 + 0i = -1$

$i^3 = i^2 \cdot i = -1 \cdot i = (-1, 0)(0, 1) = (0-0, -1+0) = (0, -1) = -i$

$i^4 = (i^2)(i^2) = -1 \cdot -1 = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i \cdot (-i) = -i^2 = 1 \Rightarrow 1/i = -i.$

$1/i^2 = 1/-1 = -1.$

$1/i^3 = 1/i^2 \cdot \frac{1}{i} = (-i) \cdot (-1) = i.$

5. Pure imaginary number. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.

$\bar{z} = x - iy = -z = -x - iy.$

$x = -x \Rightarrow x = 0 \Rightarrow z = iy$

z is a pure imaginary number

8–15 **COMPLEX ARITHMETIC**

Let $z_1 = -2 + 11i$, $z_2 = 2 - i$. Showing the details of your work, find, in the form $x + iy$:

8. $z_1 z_2$, $\overline{(z_1 z_2)}$ **9.** $\text{Re}\,(z_1^2)$, $(\text{Re}\,z_1)^2$

10. $\text{Re}\,(1/z_2^2)$, $1/\text{Re}\,(z_2^2)$

11. $(z_1 - z_2)^2/16$, $(z_1/4 - z_2/4)^2$

11. $(z_1 - z_2)^2/16 = (-4+12i)^2/16 = (16-144-96i)/16$

$= (-128 - 96i)/16 = -8 - 6i$

$(z_1/4 - z_2/4)^2 = \left[\left(-\frac{1}{2} + \frac{11}{4}i\right) - \left(\frac{1}{2} - \frac{1}{4}i\right)\right]^2$

$= (-1+3i)^2 = 1 - 9 - 6i = -8 - 6i$

16–20 Let $z = x + iy$. Showing details, find, in terms of x and y:

16. $\text{Im}\,(1/z)$, $\text{Im}\,(1/z^2)$ **17.** $\text{Re}\,z^4 - (\text{Re}\,z^2)^2$

17. $z^2 = (x^2 - y^2) + 2xy\,i$

$z^4 = [(x^2-y^2)^2 - 4x^2y^2] + 4xy(x^2-y^2)\,i$

$\text{Re}\,z^4 = (x^2-y^2)^2 - 4x^2y^2$

$(\text{Re}\,z^2)^2 = (x^2-y^2)^2$

$\text{Re}\,z^4 - (\text{Re}\,z^2)^2 = -4x^2y^2$

Selected Problem set 13.2

13.2 1. 3. 7. 11. 21. 29

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$

2. $-4 + 4i$

3. $2i, \quad -2i$

4. -5

5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$

6. $\dfrac{\sqrt{3} - 10i}{-\frac{1}{2}\sqrt{3} + 5i}$

7. $1 + \frac{1}{2}\pi i$

8. $\dfrac{-4 + 19i}{2 + 5i}$

1. $r = \sqrt{1+1} = \sqrt{2}$

$\theta = \arctan \frac{1}{1} = \frac{\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

3. $2i$: $r = 2$.

$\theta = \frac{\pi}{2}$

$2i = 2\cos\frac{\pi}{2} + 2i\sin\frac{\pi}{2}$

$-2i$: $r = 2$

$\theta = -\frac{\pi}{2}$

$-2i = 2\cos\left(-\frac{\pi}{2}\right) + 2i\sin\left(-\frac{\pi}{2}\right)$

7. $r = \sqrt{1+\left(\frac{\pi}{2}\right)^2} = \sqrt{1+\frac{\pi^2}{4}}$

$\theta = \arctan \frac{\frac{1}{2}\pi}{1} = \arctan\frac{\pi}{2}$

$1+\frac{\pi}{2}i = \sqrt{1+\frac{\pi^2}{4}}\left(\cos\left(\arctan\frac{\pi}{2}\right) + i\sin\left(\arctan\frac{\pi}{2}\right)\right)$

9–14 **PRINCIPAL ARGUMENT**

Determine the principal value of the argument and graph it as in Fig. 325.

9. $-1 + i$ **10.** $-5, \quad -5 - i, \quad -5 + i$

11. $3 \pm 4i$ **12.** $-\pi - \pi i$

13. $(1 + i)^{20}$ **14.** $-1 + 0.1i, \quad -1 - 0.1i$

11. $r = \sqrt{3^2 + (\pm 4)^2} = 5$

$\operatorname{Arg} z = \arctan \frac{\pm 4}{3}$

21–27 **ROOTS**

Find and graph all roots in the complex plane.

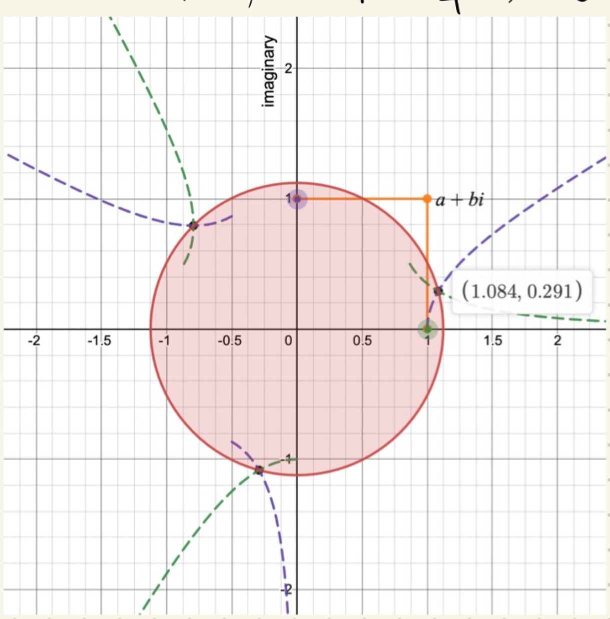
21. $\sqrt[3]{1 + i}$ **22.** $\sqrt[3]{3 + 4i}$

21. $r = \sqrt{1+1} = \sqrt{2}$.

$\operatorname{Arg} z = \arctan \frac{1}{1} = \frac{\pi}{4}$.

$\sqrt[3]{1+i} = \sqrt[6]{2} \left(\cos \frac{\frac{\pi}{4} + 2k\pi}{3} + i \sin \frac{\frac{\pi}{4} + 2k\pi}{3} \right)$.

$\theta = \frac{\pi}{12}, \quad \theta_1 = \frac{3}{4}\pi, \quad \theta_2 = \frac{17}{12}\pi$



28–31 **EQUATIONS**

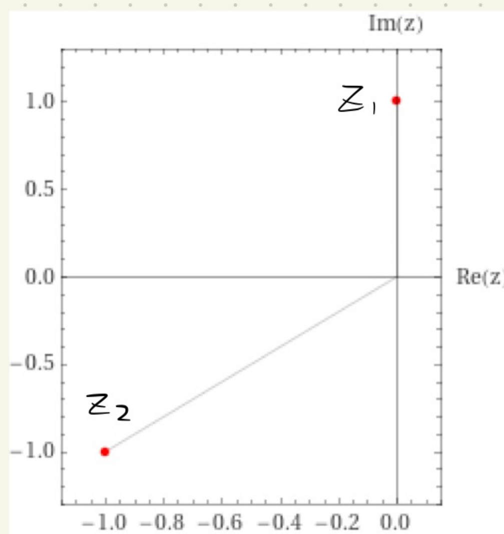
Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$

29. $z^2 + z + 1 - i = 0$

29. $z = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (1-i)}}{2}$

$z_1 = i \qquad z_2 = -1 - i$



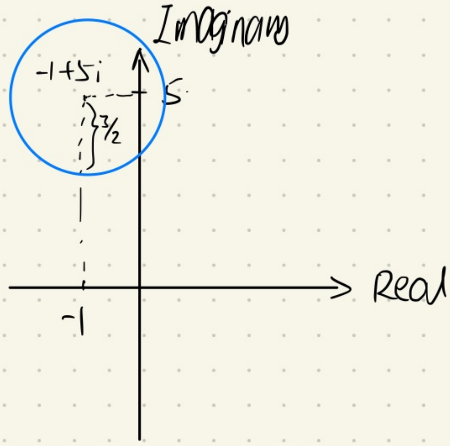
Selected Problem set 13.3

13.3 1, 11, 15, 23

1–8 REGIONS OF PRACTICAL INTEREST

Determine and sketch or graph the sets in the complex plane given by

1. $|z + 1 - 5i| \leqq \frac{3}{2}$



COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10–12 Function Values. Find Re *f*, and Im *f* and their values at the given point *z*.

10. $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$

11. $f(z) = 1/(1 - z)$ at $1 - i$

11. $z = x + iy$

$$f(z) = \frac{1}{1-z} = \frac{1}{(1-x) - iy}$$

$$= \frac{(1-x) + iy}{[(1-x) - iy][(1-x) + iy]}$$

$$= \frac{(1-x) + iy}{(1-x)^2 + y^2}$$

$$= \frac{1-x}{(1-x)^2 + y^2} + i\frac{y}{(1-x)^2 + y^2}$$

$$\mathrm{Re} f = \frac{1-x}{(1-x)^2 + y^2}. \quad \mathrm{Im} f = \frac{y}{(1-x)^2 + y^2}$$

$$f(1-i) = \frac{1}{1 - 1 + i} = \frac{1}{i}$$

$$= \frac{-i}{i^2} = -i$$

$$\mathrm{Re}\, f(1-i) = 0 \quad \mathrm{Im}\, f(1-i) = -1$$

14–17 Continuity. Find out, and give reason, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

14. $(\mathrm{Re}\, z^2)/|z|$

15. $|z|^2\, \mathrm{Im}\,(1/z)$

16. $(\mathrm{Im}\, z^2)/|z|^2$

17. $(\mathrm{Re}\, z)/(1 - |z|)$

15. $z = x + iy.$

$$|z|^2 = x^2 + y^2$$

$$\mathrm{Im}\left(\frac{1}{z}\right) = \mathrm{Im}\left(\frac{1}{x+iy}\right)$$

$$= \mathrm{Im}\left[\frac{x - iy}{(x+iy)(x-iy)}\right]$$

$$= \frac{-y}{x^2 + y^2}$$

$$|z|^2 \cdot \mathrm{Im}\left(\frac{1}{z}\right) = -y$$

$$\lim_{z \to 0} (-y) = 0 = f(0)$$

Continuous.

18–23 **Differentiation.** Find the value of the derivative of

18. $(z-i)/(z+i)$ at i

19. $(z-4i)^8$ at $= 3+4i$

20. $(1.5z+2i)/(3iz-4)$ at any z. Explain the result.

21. $i(1-z)^n$ at 0

22. $(iz^3+3z^2)^3$ at $2i$

23. $z^3/(z+i)^3$ at i

23.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

A

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\frac{(z_0+\Delta z)^3}{(z_0+\Delta z+i)^3} - \frac{z_0^3}{(z_0+i)^3}}{\Delta z}$$

$$= \frac{(z_0+\Delta z)^3 (z_0+i)^3 - z_0^3 (z_0+\Delta z+i)^3}{(z_0+\Delta z+i)^3 \cdot (z_0+i)^3 \cdot \Delta z}$$

$$= \frac{(z_0^2 + z_0 \cdot i + \Delta z \cdot z_0 + \Delta z \cdot i)^3 - (z_0^2 + \Delta z\, z_0 + z_0 i)^3}{(z_0+\Delta z+i)^3 (z_0+i)^3 \cdot \Delta z}$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

Lots of book keeping via original definition.

B

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$f'(z_0) = \frac{3z^2(z+i)^3 - z^3 \cdot 3(z+i)^2}{(z+i)^6} = \frac{z^2(3z+3i-3z)}{(z+i)^4}$$

$$= \frac{3z^2 i}{(z+i)^4} \qquad f'(i) = \frac{3\cdot(-1)\cdot i}{(2i)^4} = -\frac{3}{16} i$$

Selected Problem set 13.4

13.4 1. 5, 15, 21. 23

1. Cauchy–Riemann equations in polar form. Derive (7) from (1).

2–11 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? Use (1) or (7).

2. $f(z) = iz\bar{z}$

3. $f(z) = e^{-2x}(\cos 2y - i \sin 2y)$

4. $f(z) = e^{x}(\cos y - i \sin y)$

5. $f(z) = \text{Re}(z^2) - i\,\text{Im}(z^2)$

1. $$u_x = v_y, \qquad u_y = -v_x$$

We mention that, if we use the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r,\theta) + iv(r,\theta)$, then the **Cauchy–Riemann equations** are (Prob. 1)

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta \qquad (r > 0). \tag{7}$$

$f(z) = u(r,\theta) + iv(r,\theta)$

$u(r,\theta) = r\cos\theta$

$v(r,\theta) = r\sin\theta$

$u_r = \cos\theta$. $\Rightarrow u_r = \frac{1}{r}v_\theta$

$v_\theta = r\cos\theta$

$v_r = \sin\theta$ $\Rightarrow v_r = -\frac{1}{r}u_\theta$

$u_\theta = -r\sin\theta$

5. $z = x + iy$

$z^2 = (x^2 - y^2) + 2ixy$

$u = z^2 = x^2 - y^2$

$v = -z^2 = -2xy$

$u_x = 2x$
$v_y = -2x$ $\Rightarrow$ NO.

12–19 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

15. $u = x/(x^2 + y^2)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3} + \frac{6xy^2 - 2x^3}{(x^2+y^2)^3} = 0$$

harmonic.

$$u_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$u_y = x \cdot \frac{-2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_y = u_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$v_x = -u_y = \frac{2xy}{(x^2+y^2)^2}$$

$$v = \int\left[\frac{1}{(y^2+x^2)} - \frac{2x^2}{(y^2+x^2)^2}\right] dy$$

$$= -\frac{y}{y^2+x^2} + h(x)$$

$$v_x = \frac{2xy}{(x^2+y^2)^2} + \frac{dh}{dx}$$

$\frac{dh}{dx} = 0$ let $h(x) = c$

here it is real.

$$f(z) = u + iv = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2} + c\right)$$

In answer's format, C is Imaginary.

$$f(z) = \frac{1}{z} + C = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} + C$$

21–24 Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{\pi x} \cos av$ → y not v.

22. $u = \cos ax \cosh 2y$

23. $u = ax^3 + bxy$

21. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\pi^2 \cdot e^{\pi x} \cdot \cos ay - a^2 e^{\pi x} \cdot \cos ay = 0$

$a = \pm \pi$ ⇒ answer is wrong.

$U_x = \pi \cdot e^{\pi x} \cos ay$

$U_y = -e^{\pi x} \cdot a \cdot \sin ay$

$V_y = U_x = \pi \cdot e^{\pi x} \cos ay$

$V_x = -U_y = e^{\pi x} \cdot a \cdot \sin ay$

$V = \pm e^{\pi x} \cdot \sin ay + h(x)$

$V_x = \pm \pi e^{\pi x} \cdot \sin ay + \frac{dh}{dx}$

$\frac{dh}{dx} = 0$ let $h(x) = C$

$V = \pm e^{\pi x} \cdot \sin ay + C$ (Real)

23. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial(3ax^2 + by)}{\partial x} + \frac{\partial(bx)}{\partial y} = 0$

$6ax + 0 = 0$.

$a = 0$

$u = bxy$

$u_x = by$

$u_y = bx$

$V_y = U_x = by$

$V_x = -U_y = -bx$

$V = \frac{b}{2} y^2 + h(x)$

$V_x = 0 + \frac{dh}{dx} = -bx$

$h(x) = -\frac{b}{2} x^2 + C$

$V = \frac{b}{2}(y^2 - x^2) + C$

Selected Problem set 13.5

13.5 1, 3, 13, 17

1. e^z is entire. Prove this.

2–7 Function Values. Find e^z in the form $u + iv$ and $|e^z|$ if z equals

2. $3 + 4i$ **3.** $2\pi i(1 + i)$

1. e^z is a function differenciable:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{e^{(z_0 + \Delta z)} - e^{z_0}}{\Delta z}$$

$$= \frac{e^{z_0}(e^{\Delta z} - 1)}{\Delta z}$$

$$\ln f'(z_0) = \ln e^{z_0} + \ln(e^{\Delta z} - 1) - \ln \Delta z$$

$$= z_0 + \ln(e^{\Delta z} - 1) - \ln \Delta z$$

$$\lim_{\Delta z \to 0} = z_0 + \ln 0 - \ln 0$$

$$= z_0$$

$$\lim_{\Delta z \to 0} f'(z_0) = e^{z_0}$$

it works for a z in C

$\Rightarrow$ Entire

3. $2\pi i(1+i) = -2\pi + 2\pi i$

$x = -2\pi \quad y = 2\pi$

$$e^z = e^{-2\pi}(\cos 2\pi + i \sin 2\pi)$$

$$= e^{-2\pi} + i0$$

$$|e^z| = e^x = e^{-2\pi}$$

answer is a little bit weird. first = should be $\Rightarrow$

8–13 **Polar Form.** Write in exponential form (6):

8. $\sqrt[n]{z}$ **9.** $4 + 3i$

10. $\sqrt{i}, \quad \sqrt{-i}$ **11.** -6.3

12. $1/(1-z)$ **13.** $1 + i$

14–17 **Real and Imaginary Parts.** Find Re and Im of

14. $e^{-\pi z}$ **15.** $\exp(z^2)$

17. $\exp(z^3)$

13. $\sqrt{1^2+1^2} = \sqrt{2}$.

$$1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$$

$$e^x = \sqrt{2}$$

$$\cos y = \frac{\sqrt{2}}{2}, \quad \sin y = \frac{\sqrt{2}}{2}$$

$$y = \frac{\pi}{4}$$

$$e^z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$z = re^{i\theta}.$$

$$z = \sqrt{2} \cdot e^{i\frac{\pi}{4}}$$

17. $z = x + iy$

$$z^2 = (x^2 - y^2) + 2xyi$$

$$z^3 = (x^3 - xy^2 - 2xy^2)$$
$$+ [y(x^2 - y^2) + 2x^2y]\, i$$
$$= (x^3 - 3xy^2) + (3x^2y - y^3)i$$

$$\exp(z^3) = e^{x^3 - 3xy^2} \cdot e^{(3x^2y - y^3)i}$$

$$\Rightarrow r = e^{x^3 - 3xy^2} \quad \theta = 3x^2y - y^3$$

$$\exp(z^3) = e^{x^3 - 3xy^2} \cdot (\cos\theta + i\sin\theta)$$

$$\mathrm{Re}(\exp z^3) = e^{x^3 - 3x^2y} \cdot \cos(3x^2y - y^3)$$

$$\mathrm{Im}(\exp z^3) = ie^{x^3 - 3xy^2} \cdot \sin(3x^2y - y^3)$$