Chapter 16 - Laurent Series. Residue Integration

Selected Problem set 16.1

16.1 1. 9. 19

1–8 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. Show the details of your work.

1. $\dfrac{\cos z}{z^4}$ 2. $\dfrac{\exp(-1/z^2)}{z^2}$

1. $$z^{-4}\cos z = z^{-4}\cdot \sum_{n=0}^{\infty}(-1)^n \cdot \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{z^{2n-4}}{(2n!)}$$

$$= \frac{1}{z^4} - \frac{1}{2!z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots$$

principle part: $\dfrac{1}{z^4} - \dfrac{1}{2z^2}$ $0 < z < \infty$

9–16 LAURENT SERIES NEAR A SINGULARITY AT z_0

Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

9. $\dfrac{e^z}{(z-1)^2}$, $z_0 = 1$ 10. $\dfrac{z^2 - 3i}{(z-3)^2}$, $z_0 = 3$

9. $$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

$$e \cdot e^{z-1} = e\cdot\left(1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right)$$

$$\frac{e^z}{(z-1)^2} = \frac{e\cdot e^{z-1}}{(z-1)^2} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \frac{(z-1)^2}{4!} + \cdots$$

$0 < |z-1| < R$

19–25 TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

19. $\dfrac{1}{1-z^2}$, $z_0 = 0$ **20.** $\dfrac{1}{z}$, $z_0 = 1$

19. $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad |z| < 1$

$\dfrac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} = 1 + z^2 + z^4 + \cdots \qquad |z^2| < 1 \Rightarrow |z| < 1$

$\dfrac{1}{1-z^2} = \dfrac{-1}{z^2(1-z^{-2})} = -z^{-2} \sum_{n=0}^{\infty} z^{-2n} = -\sum_{n=0}^{\infty} z^{-2n-2} \qquad |z| > 1$

Selected Problem set 16.2

16.2 1, 5, 15

1–10 ZEROS

Determine the location and order of the zeros.

1. $\sin^4 \frac{1}{2}z$
2. $(z^4 - 81)^3$
3. $(z + 81i)^4$
4. $\tan^2 2z$
5. $z^{-2} \sin^2 \pi z$
6. $\cosh^4 z$

1. Let $X = 0 + 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$

$f(X) = 0$

$f'(z) = 2\cos(\frac{z}{2})\sin^3(\frac{z}{2})$, $f'(x) = 0$

$f''(z) = 3\cos^2(\frac{z}{2})\sin^2(\frac{z}{2}) - \sin^4(\frac{z}{2})$, $f''(x) = 0$

$f^{(3)}(z) = 3\cos^3(\frac{z}{2})\sin(\frac{z}{2}) - 5\cos(\frac{z}{2})\sin^3(\frac{z}{2})$ $f^{(3)}(x) = 0$

$f^{(4)}(z) = \frac{5}{2}\sin^4(\frac{z}{2}) - 12\cos^2(\frac{z}{2})\sin^2(\frac{z}{2}) + \frac{3}{2}\cos^4(\frac{z}{2})$ $f^{(4)}(x) \neq 0$

order: 4. Location: $0 + 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$

5. Let $x = n$, $n = \pm 1, \pm 2, \dots$

$f(x) = 0$

$$f'(z) = -\frac{2\sin(\pi z)[\sin(\pi z) - \pi z\cos(\pi z)]}{z^3}, \quad f'(x) = 0$$

$$f''(z) = \frac{2[(\pi^2 z^2 - 3)\sin^2(\pi z) + 4\pi z\cos(\pi z)\sin(\pi z) - \pi^2 z^2\cos^2(\pi z)]}{z^4}$$

$f''(1) = 2\pi^2 \neq 0$

order: 2 Location: $\pm 1, \pm 2, \dots$

13–22 SINGULARITIES

Determine the location of the singularities, including those at infinity. For poles also state the order. Give reasons.

13. $\frac{1}{(z+2i)^2} - \frac{z}{z-i} + \frac{z+1}{(z-i)^2}$

14. $e^{z-i} + \frac{2}{z-i} - \frac{8}{(z-i)^3}$

15. $z \exp(1/(z-1-i)^2)$ **16.** $\tan \pi z$

Simple pole at ∞ essential.

15. $f(z) = z \cdot e^{\frac{1}{(z-1-i)^2}}$

what is essential singularity point?

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$z-1-i=0$, $z=1+i$ is singularity point

$$f(z) = z \cdot \left[1 + \frac{1}{(z-1-i)^2} + \frac{1}{2\cdot(z-1-i)^4} + \frac{1}{3!(z-1-i)^6} + \cdots\right]$$

$$= [(z-1-i)+(1+i)]\,[\qquad]$$

$$= (z-1-i) + \frac{1}{(z-1-i)} + \frac{1}{2\cdot(z-1-i)^3} + \cdots$$

$$+ (1+i) + \frac{1+i}{(z-1-i)^2} + \frac{1+i}{2(z-1-i)^4} + \cdots$$

$$= z + \frac{1}{z-1-i} + \frac{1+i}{(z-1-i)^2} + \frac{1}{2(z-1-i)^3}$$

$$+ \frac{1+i}{2(z-1-i)^4} + \cdots$$

part (1) has finity many term $\Rightarrow$ Isolated essential singularity

Pole: $z=1+i$

part (2) infinity ?

Selected Problem set 16.3

16.3 5, 9, 21, 23

3–12 RESIDUES

Find all the singularities in the finite plane and the corresponding residues. Show the details.

3. $\dfrac{\sin 2z}{z^6}$

4. $\dfrac{\cos z}{z^4}$

5. $\dfrac{8}{1+z^2}$

6. $\tan z$

5. $z = \pm i$

$$\operatorname*{Res}_{z=i} \frac{8}{(z+i)(z-i)} = \frac{8}{z+i}\bigg|_{z=i} = \frac{4}{i} = -4i$$

$$\operatorname*{Res}_{z=-i} \frac{8}{(z+i)(z-i)} = \frac{8}{z-i}\bigg|_{z=-i} = \frac{4}{-i} = 4i$$

9. $\dfrac{1}{1-e^z}$

$e^z = 1$

$e^{iy} = \cos y + i \sin y$

let $\frac{z}{i} = y$

$e^z = \cos\left(\frac{z}{i}\right) + i\sin\left(\frac{z}{i}\right)$

$z = 2n\pi i \quad n = 0, \pm 1, \pm 2 \ldots$ **not only 0.**

$$\operatorname*{Res}_{z=0} \frac{1}{1-e^z} = \frac{1}{-e^z}\bigg|_{z=0} = -1$$

21. $\oint_C \frac{\cos \pi z}{z^5}\, dz, \quad C: |z| = \frac{1}{2}$

$z = 0.$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$

$$\frac{\cos \pi z}{z^5} = \frac{1}{z^5} - \frac{\pi^2}{2!\, z^3} + \frac{\pi^4}{4!\, z} - \frac{\pi^6 z}{6!}$$

$$b_1 = \frac{\pi^4}{4!} = \frac{\pi^4}{24}$$

$$\oint_C \frac{\cos \pi z}{z^5}\, dz = 2\pi i b = 2\pi \cdot i \cdot \frac{\pi^4}{24} = \frac{\pi^5}{12} i$$

23. $\oint_C \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)}\, dz, \quad C$ the unit circle

$z_1 = \frac{1}{2}, \qquad z_2 = \frac{1}{3}$

$$\oint_C \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)}\, dz = \boxed{\frac{1}{12} \oint \frac{30z^2 - 23z + 5}{(z-\frac{1}{2})^2 (z - \frac{1}{3})}}$$

$$= 2\pi i \cdot \frac{1}{12} \left[\operatorname*{Res}_{z_1 = \frac{1}{2}} \frac{30z^2 - 23z + 5}{(z-\frac{1}{2})^2(z-\frac{1}{3})} + \operatorname*{Res}_{z_2 = \frac{1}{3}} \frac{30z^2 - 23z + 5}{(z-\frac{1}{2})^2(z-\frac{1}{3})} \right]$$

$$= \frac{\pi i}{6} \left[\lim_{z \to \frac{1}{2}} \left(\frac{30z^2 - 23z + 5}{z - \frac{1}{3}} \right)' + \lim_{z \to \frac{1}{3}} \frac{30z^2 - 23z + 5}{(z - \frac{1}{2})^2} \right]$$

$$= \frac{\pi i}{6} \left[\frac{6(45z^2 - 30z + 4)}{(3z-1)^2} \Bigg|_{z = \frac{1}{2}} + \frac{30z^2 - 23z + 5}{(z - \frac{1}{2})^2} \Bigg|_{z = \frac{1}{3}} \right]$$

$$= \frac{\pi i}{6} \left[6 + 24 \right] = 5\pi i$$

Selected Problem set 16.4