Chapter 13 - Complex Numbers and Functions. Complex Differentiation

Selected Problem set 13.1

13.1   1. 5. 11. 17

1. Powers of *i*. Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.

$z_1z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, \quad x_1y_2 + x_2y_1).$

$i^2 = (0,1)(0,1) = (0-1,\ 0+0) = (-1,0) = -1 + 0i = -1$

$i^3 = i^2 \cdot i = -1 \cdot i = (-1,0)(0,1) = (0-0,\ -1+0) = (0,-1) = -i$

$i^4 = (i^2)(i)^2 = -1 \cdot -1 = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i \cdot (-i) = -i^2 = 1 \Rightarrow 1/i = -i$.

$1/i^2 = 1/-1 = -1$.

$1/i^3 = 1/i^2 \cdot \frac{1}{i} = (-i) \cdot (-1) = i$.

5. Pure imaginary number. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.

$\bar{z} = x - iy = -z = -x - iy$.

$x = -x \Rightarrow x = 0 \Rightarrow z = iy$

z is a pure imaginary number

8–15 **COMPLEX ARITHMETIC**

Let $z_1 = -2 + 11i$, $z_2 = 2 - i$. Showing the details of your work, find, in the form $x + iy$:

8. $z_1 z_2$, $\overline{(z_1 z_2)}$ **9.** $\text{Re}\,(z_1^2)$, $(\text{Re}\, z_1)^2$

10. $\text{Re}\,(1/z_2^2)$, $1/\text{Re}\,(z_2^2)$

11. $(z_1 - z_2)^2/16$, $(z_1/4 - z_2/4)^2$

11. $(z_1 - z_2)^2/16 = (-4+12i)^2/16 = (16-144-96i)/16$

$= (-128 - 96i)/16 = -8 - 6i$

$(z_1/4 - z_2/4)^2 = [(-\frac{1}{2} + \frac{11}{4}i) - (\frac{1}{2} - \frac{1}{4}i)]^2$

$= (-1+3i)^2 = 1 - 9 - 6i = -8 - 6i$

16–20 Let $z = x + iy$. Showing details, find, in terms of x and y:

16. $\text{Im}\,(1/z)$, $\text{Im}\,(1/z^2)$ **17.** $\text{Re}\, z^4 - (\text{Re}\, z^2)^2$

17. $z^2 = (x^2 - y^2) + 2xy\, i$

$z^4 = [(x^2-y^2)^2 - 4x^2y^2] + 4xy(x^2-y^2)\, i$

$\text{Re}\, z^4 = (x^2-y^2)^2 - 4x^2y^2$

$(\text{Re}\, z^2)^2 = (x^2-y^2)^2$

$\text{Re}\, z^4 - (\text{Re}\, z^2)^2 = -4x^2y^2$

Selected Problem set 13.2

13.2 1. 3. 7. 11. 21. 29

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$ **2.** $-4 + 4i$

3. $2i, \quad -2i$ **4.** -5

5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$ **6.** $\dfrac{\sqrt{3} - 10i}{-\frac{1}{2}\sqrt{3} + 5i}$

7. $1 + \frac{1}{2}\pi i$ **8.** $\dfrac{-4 + 19i}{2 + 5i}$

1. $r = \sqrt{1+1} = \sqrt{2}$

$\theta = \arctan \frac{1}{1} = \frac{\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

3. $2i$: $r = 2$.

$\theta = \frac{\pi}{2}$

$2i = 2\cos\frac{\pi}{2} + 2i\sin\frac{\pi}{2}$

$-2i$: $r = 2$

$\theta = -\frac{\pi}{2}$

$-2i = 2\cos\left(-\frac{\pi}{2}\right) + 2i\sin\left(-\frac{\pi}{2}\right)$

7. $r = \sqrt{1 + \left(\frac{\pi}{2}\right)^2} = \sqrt{1 + \frac{\pi^2}{4}}$

$\theta = \arctan \frac{\frac{1}{2}\pi}{1} = \arctan\frac{\pi}{2}$

$1 + \frac{\pi}{2}i = \sqrt{1 + \frac{\pi^2}{4}}\left(\cos\left(\arctan\frac{\pi}{2}\right) + i\sin\left(\arctan\frac{\pi}{2}\right)\right)$

9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $-1 + i$ **10.** $-5, \quad -5 - i, \quad -5 + i$

11. $3 \pm 4i$ **12.** $-\pi - \pi i$

13. $(1 + i)^{20}$ **14.** $-1 + 0.1i, \quad -1 - 0.1i$

11. $r = \sqrt{3^2 + (\pm 4)^2} = 5$

$\operatorname{Arg} z = \arctan \frac{\pm 4}{3}$

21–27 ROOTS

Find and graph all roots in the complex plane.

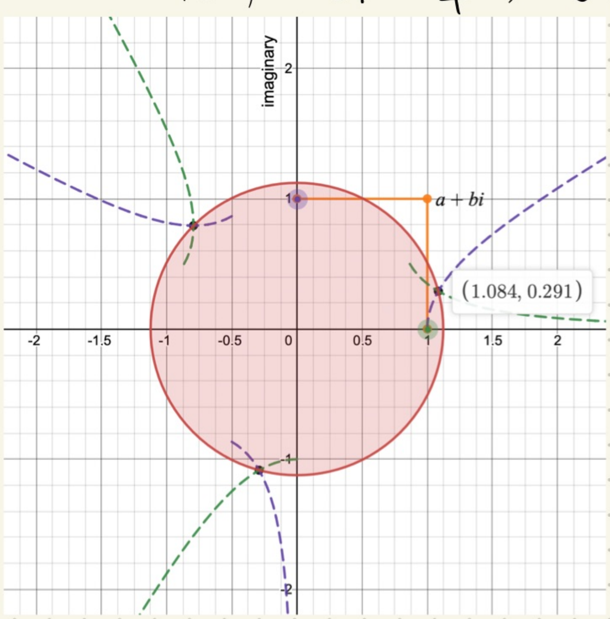
21. $\sqrt[3]{1 + i}$ **22.** $\sqrt[3]{3 + 4i}$

21. $r = \sqrt{1+1} = \sqrt{2}$.

$\operatorname{Arg} z = \arctan \frac{1}{1} = \frac{\pi}{4}$.

$\sqrt[3]{1+i} = \sqrt[6]{2}\left(\cos \frac{\frac{\pi}{4} + 2k\pi}{3} + i \sin \frac{\frac{\pi}{4} + 2k\pi}{3}\right)$.

$\theta = \frac{\pi}{12}, \quad \theta_1 = \frac{3}{4}\pi, \quad \theta_2 = \frac{17}{12}\pi$



28–31 EQUATIONS

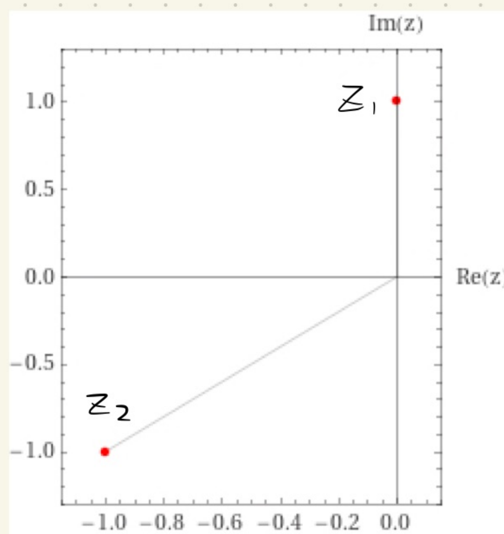
Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$

29. $z^2 + z + 1 - i = 0$

29. $z = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (1-i)}}{2}$

$z_1 = i \qquad z_2 = -1 - i$



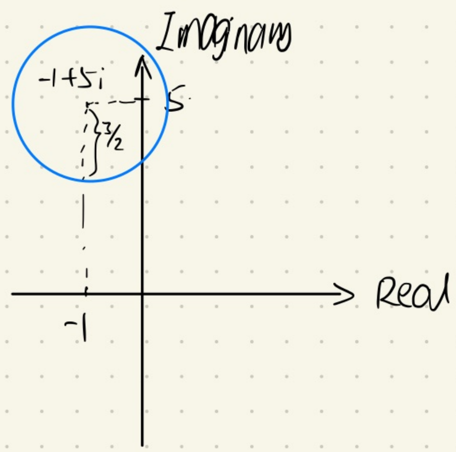
Selected Problem set 13.3

13.3 1, 11, 15, 23

1–8 REGIONS OF PRACTICAL INTEREST

Determine and sketch or graph the sets in the complex plane given by

1. $|z + 1 - 5i| \leqq \frac{3}{2}$



COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10–12 Function Values. Find Re f, and Im f and their values at the given point z.

10. $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$

11. $f(z) = 1/(1 - z)$ at $1 - i$

11. $z = x + iy$

$$f(z) = \frac{1}{1-z} = \frac{1}{(1-x)-iy}$$

$$= \frac{(1-x)+iy}{[(1-x)-iy][(1-x)+iy]}$$

$$= \frac{(1-x)+iy}{(1-x)^2+y^2}$$

$$= \frac{1-x}{(1-x)^2+y^2} + i\frac{y}{(1-x)^2+y^2}$$

$$\mathrm{Re}f = \frac{1-x}{(1-x)^2+y^2}. \quad \mathrm{Im}f = \frac{y}{(1-x)^2+y^2}$$

$$f(1-i) = \frac{1}{1-1+i} = \frac{1}{i}$$

$$= \frac{-i^2}{i} = -i$$

$$\mathrm{Re}\,f(1-i) = 0 \quad \mathrm{Im}\,f(1-i) = -1$$

14–17 Continuity. Find out, and give reason, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

14. $(\mathrm{Re}\, z^2)/|z|$

15. $|z|^2\, \mathrm{Im}\,(1/z)$

16. $(\mathrm{Im}\, z^2)/|z|^2$

17. $(\mathrm{Re}\, z)/(1 - |z|)$

15. $z = x + iy$.

$$|z|^2 = x^2 + y^2$$

$$\mathrm{Im}\left(\frac{1}{z}\right) = \mathrm{Im}\left(\frac{1}{x+iy}\right)$$

$$= \mathrm{Im}\left[\frac{x-iy}{(x+iy)(x-iy)}\right]$$

$$= \frac{-y}{x^2+y^2}$$

$$|z|^2 \cdot \mathrm{Im}\left(\frac{1}{z}\right) = -y$$

$$\lim_{z\to 0}(-y) = 0 = f(0)$$

Continuous.

18–23 **Differentiation.** Find the value of the derivative of

18. $(z - i)/(z + i)$ at i **19.** $(z - 4i)^8$ at $= 3 + 4i$

20. $(1.5z + 2i)/(3iz - 4)$ at any z. Explain the result.

21. $i(1 - z)^n$ at 0

22. $(iz^3 + 3z^2)^3$ at $2i$ **23.** $z^3/(z + i)^3$ at i

23.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

A

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\frac{(z_0+\Delta z)^3}{(z_0+\Delta z+i)^3} - \frac{z_0^3}{(z_0+i)^3}}{\Delta z}$$

$$= \frac{(z_0+\Delta z)^3 (z_0+i)^3 - z_0^3 (z_0+\Delta z+i)^3}{(z_0+\Delta z+i)^3 \cdot (z_0+i)^3 \cdot \Delta z}$$

$$= \frac{(z_0^2 + z_0 \cdot i + \Delta z \cdot z_0 + \Delta z \cdot i)^3 - (z_0^2 + \Delta z z_0 + z_0 i)^3}{(z_0+\Delta z+i)^3 (z_0+i)^3 \cdot \Delta z}$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

Lots of book keeping via original definition.

B

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$f'(z_0) = \frac{3z^2(z+i)^3 - z^3 \cdot 3(z+i)^2}{(z+i)^6} = \frac{z^2(3z+3i-3z)}{(z+i)^4}$$

$$= \frac{3z^2 i}{(z+i)^4} \qquad f'(i) = \frac{3 \cdot (-1) \cdot i}{(2i)^4} = -\frac{3}{16} i$$