Chapter 16 - Laurent Series. Residue Integration

Selected Problem set 16.1

16.1   1. 9. 19

1–8 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. Show the details of your work.

1. $\frac{\cos z}{z^4}$ 2. $\frac{\exp(-1/z^2)}{z^2}$

1. $z^{-4}\cos z = z^{-4}\cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n-4}}{(2n!)}$

$$= \frac{1}{z^4} - \frac{1}{2!z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots$$

principle part: $\frac{1}{z^4} - \frac{1}{2z^2}$ $0<z<\infty$

9–16 LAURENT SERIES NEAR A SINGULARITY AT z_0

Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

9. $\frac{e^z}{(z-1)^2}$, $z_0 = 1$ 10. $\frac{z^2 - 3i}{(z-3)^2}$, $z_0 = 3$

9. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$

$e \cdot e^{z-1} = e \cdot \left(1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right)$

$$\frac{e^z}{(z-1)^2} = \frac{e \cdot e^{z-1}}{(z-1)^2} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \frac{(z-1)^2}{4!} + \cdots$$

$0 < |z-1| < R$

19–25 **TAYLOR AND LAURENT SERIES**

Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

19. $\dfrac{1}{1-z^2}$, $z_0 = 0$ **20.** $\dfrac{1}{z}$, $z_0 = 1$

19. $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots \qquad |z|<1$$

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} = 1+z^2+z^4+\cdots \qquad |z^2|<1 \Rightarrow |z|<1$$

$$\frac{1}{1-z^2} = \frac{-1}{z^2(1-z^{-2})} = -z^{-2}\sum_{n=0}^{\infty} z^{-2n} = -\sum_{n=0}^{\infty} z^{-2n-2} \qquad |z|>1$$

Selected Problem set 16.2

16.2 1. 5. 15

1–10 ZEROS

Determine the location and order of the zeros.

1. $\sin^4 \frac{1}{2}z$
2. $(z^4 - 81)^3$
3. $(z + 81i)^4$
4. $\tan^2 2z$
5. $z^{-2} \sin^2 \pi z$
6. $\cosh^4 z$

1. Let $X = 0 + 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$

$f(X) = 0$

$f'(z) = 2\cos(\frac{z}{2})\sin^3(\frac{z}{2})$, $f'(x) = 0$

$f''(z) = 3\cos^2(\frac{z}{2})\sin^2(\frac{z}{2}) - \sin^4(\frac{z}{2})$, $f''(x) = 0$

$f^{(3)}(z) = 3\cos^3(\frac{z}{2})\sin(\frac{z}{2}) - 5\cos(\frac{z}{2})\sin^3(\frac{z}{2})$ $f^{(3)}(x) = 0$

$f^{(4)}(z) = \frac{5}{2}\sin^4(\frac{z}{2}) - 12\cos^2(\frac{z}{2})\sin^2(\frac{z}{2}) + \frac{3}{2}\cos^4(\frac{z}{2})$ $f^{(4)}(x) \neq 0$

order: 4. Location: $0 + 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$

5. Let $x = n$, $n = \pm 1, \pm 2, \dots$

$f(x) = 0$

$f'(z) = -\dfrac{2\sin(\pi z)[\sin(\pi z) - \pi z\cos(\pi z)]}{z^3}$, $f'(x) = 0$

$f''(z) = \dfrac{2[(\pi^2 z^2 - 3)\sin^2(\pi z) + 4\pi z\cos(\pi z)\sin(\pi z) - \pi^2 z^2\cos^2(\pi z)]}{z^4}$

$f''(1) = 2\pi^2 \neq 0$

order: 2 Location: $\pm 1, \pm 2 \dots$

13–22 **SINGULARITIES**

Determine the location of the singularities, including those at infinity. For poles also state the order. Give reasons.

13. $\frac{1}{(z+2i)^2} - \frac{z}{z-i} + \frac{z+1}{(z-i)^2}$

14. $e^{z-i} + \frac{2}{z-i} - \frac{8}{(z-i)^3}$

15. $z \exp(1/(z-1-i)^2)$ **16.** $\tan \pi z$

Simple pole at ∞ essential.

15. $f(z) = z \cdot e^{\frac{1}{(z-1-i)^2}}$

what is essential singularity point?

P724. 16.3

$z-1-i=0$, $z=1+i$ is singularity point

$$f(z) = z \cdot \left[1 + \frac{1}{(z-1-i)^2} + \frac{1}{2\cdot(z-1-i)^4} + \frac{1}{3!(z-1-i)^6} + \cdots\right]$$

$$= [(z-1-i) + (1+i)]\,[\qquad]$$

$$= (z-1-i) + \frac{1}{(z-1-i)} + \frac{1}{2\cdot(z-1-i)^3} + \cdots$$

$$+ (1+i) + \frac{1+i}{(z-1-i)^2} + \frac{1+i}{2(z-1-i)^4} + \cdots$$

$$= z + \frac{1}{z-1-i} + \frac{1+i}{(z-1-i)^2} + \frac{1}{2(z-1-i)^3}$$

$$+ \frac{1+i}{2(z-1-i)^4} + \cdots$$

part (1) has finity many term $\Rightarrow$ Isolated essential singularity

Pole: $z = 1+i$

part (2) infinity ?

Selected Problem set 16.3

16.3 5, 9, 21, 23

3–12 RESIDUES

Find all the singularities in the finite plane and the corresponding residues. Show the details.

3. $\frac{\sin 2z}{z^6}$ **4.** $\frac{\cos z}{z^4}$

5. $\frac{8}{1+z^2}$ **6.** $\tan z$

5. $z = \pm i$

$$\operatorname*{Res}_{z=i} \frac{8}{(z+i)(z-i)} = \frac{8}{z+i}\Big|_{z=i} = \frac{4}{i} = -4i$$

$$\operatorname*{Res}_{z=-i} \frac{8}{(z+i)(z-i)} = \frac{8}{z-i}\Big|_{z=-i} = \frac{4}{-i} = 4i$$

9. $\frac{1}{1-e^z}$

$e^z = 1$

$e^{iy} = \cos y + i\sin y$

let $\frac{z}{i} = y$

$e^z = \cos\left(\frac{z}{i}\right) + i\sin\left(\frac{z}{i}\right)$

$z = 2n\pi i \quad n = 0, \pm 1, \pm 2 \ldots$ not only 0.

$$\operatorname*{Res}_{z=0} \frac{1}{1-e^z} = \frac{1}{-e^z}\Big|_{z=0} = -1$$

21. $\oint_C \frac{\cos \pi z}{z^5}\,dz, \quad C: |z| = \frac{1}{2}$

$z=0.$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$

$$\frac{\cos \pi z}{z^5} = \frac{1}{z^5} - \frac{\pi^2}{2!\,z^3} + \frac{\pi^4}{4!\,z} - \frac{\pi^6 z}{6!}$$

$$b_1 = \frac{\pi^4}{4!} = \frac{\pi^4}{24}$$

$$\oint_C \frac{\cos \pi z}{z^5}\,dz = 2\pi i b = 2\pi \cdot i \cdot \frac{\pi^4}{24} = \frac{\pi^5}{12} i$$

23. $\oint_C \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)}\,dz, \quad C$ the unit circle

$z_1 = \frac{1}{2}, \qquad z_2 = \frac{1}{3}$

$$\oint_C \frac{30z^2-23z+5}{(2z-1)^2(3z-1)}\,dz = \boxed{\frac{1}{12}\oint \frac{30z^2-23z+5}{(z-\frac{1}{2})^2(z-\frac{1}{3})}}$$

$$= 2\pi i \cdot \frac{1}{12}\left[\operatorname*{Res}_{z_1=\frac{1}{2}} \frac{30z^2-23z+5}{(z-\frac{1}{2})^2(z-\frac{1}{3})} + \operatorname*{Res}_{z_2=\frac{1}{3}} \frac{30z^2-23z+5}{(z-\frac{1}{2})^2(z-\frac{1}{3})}\right]$$

$$= \frac{\pi i}{6}\left[\lim_{z\to\frac{1}{2}} \left(\frac{30z^2-23z+5}{z-\frac{1}{3}}\right)' + \lim_{z\to\frac{1}{3}} \frac{30z^2-23z+5}{(z-\frac{1}{2})^2}\right]$$

$$= \frac{\pi i}{6}\left[\frac{6(45z^2-30z+4)}{(3z-1)^2}\bigg|_{z=\frac{1}{2}} + \frac{30z^2-23z+5}{(z-\frac{1}{2})^2}\bigg|_{z=\frac{1}{3}}\right]$$

$$= \frac{\pi i}{6}\left[6+24\right] = 5\pi i$$

Selected Problem set 16.4

16.4 1, 5, 11, 13.

1–9 INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals and show the details of your work.

1. $\displaystyle\int_0^{\pi} \frac{2\,d\theta}{k-\cos\theta}$ **2.** $\displaystyle\int_0^{\pi} \frac{d\theta}{\pi+3\cos\theta}$

1. let $\alpha = 2\theta$

$$\int_0^{\pi} \frac{2d\theta}{k-\cos\theta} = \int_0^{2\pi} \frac{2\cdot d\frac{\alpha}{2}}{k-\cos\frac{\alpha}{2}} = \int_0^{2\pi} \frac{d\alpha}{k-\cos\frac{\alpha}{2}} = J$$

set $e^{i\frac{\alpha}{2}} = z$

$$e^{i\frac{\alpha}{2}} = \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}$$

$$e^{-i\frac{\alpha}{2}} = \cos(-\tfrac{\alpha}{2}) + i\sin(-\tfrac{\alpha}{2})$$

$$\cos\frac{\alpha}{2} = \frac{1}{2}\left(e^{i\frac{\alpha}{2}} + e^{-i\frac{\alpha}{2}}\right) = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$e^{i\frac{\alpha}{2}} = z \Rightarrow \frac{i}{2}\cdot e^{i\frac{\alpha}{2}}\,d\alpha = dz \Rightarrow \frac{dz}{d\alpha} = \frac{i}{2}z$$

$$J = \oint_C \frac{2dz/(iz)}{k-\frac{1}{2}(z+\frac{1}{z})} = \frac{2}{i}\oint_C \frac{dz}{kz-\frac{1}{2}(z^2+1)}$$

$$= \frac{-4}{i}\oint_C \frac{dz}{z^2-2kz+1}$$

$$= 4i\oint_C \frac{dz}{z^2-2kz+1}$$

$$= 4i\oint_C \frac{dz}{[z-(k+\sqrt{k^2-1})][z-(k-\sqrt{k^2-1})]}$$

C is the unit circle

$k>1 \quad \left|k-\sqrt{k^2-1}\right| = \left|\frac{1}{k+\sqrt{k^2-1}}\right| < 1.$

$q'(z) = 2z-2k$

$\operatorname*{Res}_{z=k-\sqrt{k^2-1}} = -\frac{1}{2\sqrt{k^2-1}} \qquad J = 4i \cdot 2\pi i \cdot -\frac{1}{2\sqrt{k^2-1}}$

$= \frac{4\pi}{\sqrt{k^2-1}}$

$k<-1 \quad \left|k+\sqrt{k^2-1}\right| = \left|\frac{1}{k-\sqrt{k^2-1}}\right| < 1$

$q'(z) = 2z-2k$

$\operatorname*{Res}_{z=k+\sqrt{k^2-1}} = \frac{1}{2k\sqrt{k^2-1}} \qquad J = 4i \cdot 2\pi i \cdot \frac{1}{2\sqrt{k^2-1}}$

$= \frac{-4\pi}{\sqrt{k^2-1}}$

$-1 \le k \le 1.$ Can $z_0 \in C$?

$k \in C$? $k=1$: $\int_0^{\pi} \frac{2}{1-\cos\theta}\, d\theta$

$\Rightarrow F(x) = -\frac{2(\cos\theta+1)}{\sin\theta} + C$

$\Rightarrow -\frac{0}{0} + \frac{2}{0}$

$\Rightarrow$ divergent or undefined.

5. $\displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{5 - 4\cos\theta}\, d\theta$

let $e^{i\theta} = z$ $\qquad \cos\theta = \frac{1}{2}(z + \frac{1}{z}) \qquad dz/d\theta = iz$

$$\int_0^{2\pi} \frac{\cos^2\theta}{5 - 4\cos\theta}\, d\theta = \oint_c \frac{\frac{1}{4}(z+\frac{1}{z})^2}{5 - 2(z+\frac{1}{z})} \cdot \frac{dz}{iz}$$

$$= \frac{1}{4i} \oint_c \frac{(z^2+1)^2}{5z^3 - 2(z^2+1)\cdot z^2} \cdot dz$$

$$= -\frac{i}{4} \oint_c \frac{(z^2+1)^2}{2z^4 - 5z^3 + 2z^2} \cdot dz$$

$$= \frac{i}{4} \oint_c \frac{(z^2+1)^2}{z^2(2z^2 - 5z + 2)} \cdot dz$$

$$= \frac{i}{8} \oint \frac{(z^2+1)^2}{z^2(z-\frac{1}{2})(z-2)}\, dz$$

$z_1 = 0$: order $= 2$; $\quad z_2 = \frac{1}{2}$; order $= 1$; $\quad z_3 = 2$ out of unit circle

for a second-order pole ($m = 2$),

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \{[(z - z_0)^2 f(z)]'\}.$$

$$\operatorname*{Res}_{z_1=0} f(z) = \lim_{z \to 0} \left[\frac{(z^2+1)^2}{(z-\frac{1}{2})(z-2)}\right]' = \lim_{z \to 0} \frac{2(z^2+1)(4z^3 - 15z^2 + 4z + 5)}{(z-2)^2(2z-1)^2}$$

$$= \frac{2 \cdot 1 \cdot 5}{4 \cdot 1} = \frac{5}{2}$$

$$\operatorname*{Re}_{z_2=\frac{1}{2}} z = \frac{p(z_0)}{q_2'(z_0)} = \frac{\left[\left(\frac{1}{2}\right)^2+1\right]^2}{4\cdot\left(\frac{1}{2}\right)^3-\frac{15}{2}\left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}} = \frac{\frac{25}{16}}{-\frac{3}{8}} = -\frac{25}{6}$$

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{5}{2}+-\frac{25}{6}\right) = -\frac{10}{3}\pi i$$

$$\int_0^{2\pi}\frac{\cos^2\theta}{5-4\cos\theta}\,d\theta = \frac{i}{8}\oint_C f(z)\,dz = \frac{i}{8}\cdot\frac{-10}{3}\pi i = \frac{5}{12}\pi$$

10–22 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate the following integrals and show details of your work.

10. $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$

11. $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

12. $\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+5)^2}$

13. $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)}\,dx$

11. $(1+x^2)^2=0 \Rightarrow x_1 = i \quad x_2=-i$ (ignore. not in upper half-plane)

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \int_{-\infty}^{+\infty} \frac{dx}{(x+i)^2(x-i)^2}$$

$p(x)=1 \qquad q(x)=(x+i)^2(x-i)^2$

$$\frac{1}{(1+x^2)^2} = -\frac{1}{4(x-i)^2} - \frac{i}{4(x-i)} + \frac{3}{16} + \frac{i}{8}(x-i) + \cdots$$

order $=2$

$$\operatorname*{Res}_{x\to i} f(x) = \lim_{x\to i}\left[\frac{1}{(x+i)^2}\right]' = \lim_{x\to i}\left[-2(x+i)^{-3}\right] = -\frac{i}{4}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

$$= 2\pi i \cdot -\frac{i}{4} = \frac{\pi}{2}$$

10–22 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate the following integrals and show details of your work.

10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$ **11.** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

12. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+5)^2}$ **13.** $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)}\,dx$

13. $X_1 = i$ $X_2 = 2i$ ($-i$ and $-2i$ ignore)

$$\frac{X}{(X^2+1)(X^2+4)} = \frac{1}{6(X-i)} - \frac{7i}{36} + \frac{235i(X-i)^2}{1296} + \cdots$$

$$= -\frac{1}{6(X-2i)} - \frac{13i}{72} + \frac{151}{864}(X-2i) + \cdots$$

order $= 1$.

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

$p(X) = X$ $q(X) = (X^2+1)(X^2+4)$

$q'(X) = 4X^3 + 10X$

$$\operatorname*{Res}_{X_1=i} f(X) = \frac{X}{4X^3+10X} = \frac{1}{4X^2+10} = \frac{1}{6}$$

$$\operatorname*{Res}_{X_2=2i} f(X) = \frac{1}{4X^2+10} = -\frac{1}{6}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

$$= 2\pi i\left(\frac{1}{6} - \frac{1}{6}\right) = 0$$