Chapter 8 Linear Algebra: Matrix Eigenvalue Problems

P329 - Problem set 8.1

P.S. 8.1

1–16 EIGENVALUES, EIGENVECTORS

Find the eigenvalues. Find the corresponding eigenvectors. Use the given λ or factor in Probs. 11 and 15.

1. $\begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}$ **2.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$(A-\lambda I)\cdot x=0$

1. $\det\begin{vmatrix} 3-\lambda & 0 \\ 0 & -0.6-\lambda \end{vmatrix}=0$

$(3-\lambda)(0.6+\lambda)=0$

$-\lambda^2+2.4\lambda+1.8=0$

$\lambda^2-2.4\lambda-1.8=0$

$(\lambda-3)(\lambda+0.6)=0$

1) $\lambda=3$ 2) $\lambda=-0.6$

$\begin{bmatrix} 0 & 0 \\ 0 & -3.6 \end{bmatrix}$ $\begin{bmatrix} 3.6 & 0 \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $x=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2. $\det\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix}=0$

$\lambda^2=0$ $\lambda=0$

X could be any 2x1 vector.

P326 theorem 2. 0 allowed.

3. $\det\begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix}=0$

$-(5-\lambda)(6+\lambda)+18=0$

$\lambda^2+\lambda-30+18=0$

$\lambda^2+\lambda-12=0$

$(\lambda+4)(\lambda-3)=0$

1) $\lambda=-4$

$\begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix}=\begin{bmatrix} 9 & -2 \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} 2 \\ 9 \end{bmatrix}$

2) $\lambda=3$ $\begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix}$

$=\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ 6. $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

4. det: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$(\lambda-1)(\lambda-4) - 4 = 0$

$\lambda^2 - 5\lambda = 0$

$\lambda = 0 \quad \lambda = 5$

$\lambda = 5$: $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 0 \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

5. $\begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0$

$\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$\lambda = 3i \quad \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda = -3i$: $\begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

6. $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$(\lambda-3)(\lambda-1) = 0$

$\lambda = 3 \quad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1 \quad \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **8.** $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

7 $\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$

$\lambda^2 = 0$

$\lambda = 0$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

8. $\begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0$

$(a-\lambda)^2 + b^2 = 0$

$a - \lambda = \pm bi$

$\lambda = a \pm bi$

$\lambda = a + bi$

$\begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = a - bi$ $\begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} -1 \\ i \end{bmatrix}$

9. $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ **10.** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

9. $\begin{vmatrix} 0.8-\lambda & -0.6 \\ 0.6 & 0.8-\lambda \end{vmatrix} = 0$

$a = 0.8$ $b = -0.6$

$\lambda = 0.8 - 0.6i$ $X = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = 0.8 + 0.6i$ $X = \begin{bmatrix} -1 \\ i \end{bmatrix}$

10. $\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$

$a = \cos\theta$ $b = -\sin\theta$

$\lambda = \cos\theta - i\sin\theta$ $X = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = \cos\theta + i\sin\theta$ $X = \begin{bmatrix} -1 \\ i \end{bmatrix}$

11. $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$, $\lambda = 3$

11. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

12. $(3-\lambda)(4-\lambda)(1-\lambda)=0$

$\lambda=3$

$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$ $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda=4$

$\begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix}$ $X = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

$\lambda=1$

$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ $X = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$

13. $\begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$

13. $\begin{vmatrix} 13-\lambda & 5 & 2 \\ 2 & 7-\lambda & -8 \\ 5 & 4 & 7-\lambda \end{vmatrix} = 0$

$(13-\lambda)\cdot(7-\lambda)^2 - 200 + 16 -$

$10(7-\lambda) - 32(\lambda-13)$

$-10(7-\lambda) = 0$

$-\lambda^3 + 27\lambda^2 - 243\lambda + 729 = 0$

$(\lambda-9)^3 = 0$

$\lambda = 9$

$\begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 9 & 18 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$

16. $\begin{bmatrix} -3 & 0 & 4 & 2 \\ 0 & 1 & -2 & 4 \\ 2 & 4 & -1 & -2 \\ 0 & 2 & -2 & 3 \end{bmatrix}$

14. $\lambda_1 = \frac{1}{2} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 3 \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

16. $\lambda_1 = -1, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -5, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 1 \end{bmatrix}$

$\lambda_3 = 3, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}, \quad (\lambda + 1)^2$

15.

$\lambda_1 = 3 \quad \begin{bmatrix} -3 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -5 \begin{bmatrix} -3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = -1 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

17. Counterclockwise rotation through the angle $\pi/2$ about the origin in R^2.

18. Reflection about the x_1-axis in R^2.

17. P271. Problem 30.

$A = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\lambda_1 = i, \ x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}; \quad \lambda_2 = -i, \ x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

18. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

$a_{11}x_1 + a_{12}x_2 = x_1$

$a_{21}x_1 + a_{22}x_2 = -x_2$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

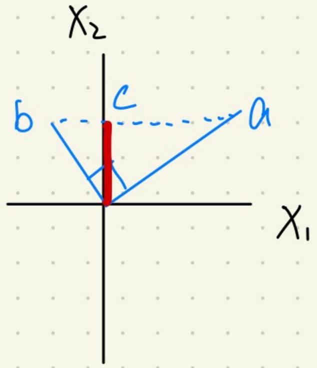
$\Leftarrow \quad \lambda_1 = 1, \ x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1, \ x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

19. Orthogonal projection (perpendicular projection) of R^2 onto the x_2-axis.

20. Orthogonal projection of R^3 onto the plane $x_2 = x_1$.

19.



Reference
P365

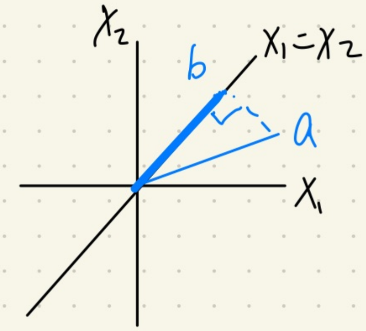
✓ if it means $\vec{c}$, the

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = 0,\ x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \lambda_2 = 1.\ x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

20



$$p = \frac{a \cdot b}{|b|} = \frac{X_1 \cdot X + X_2 \cdot X}{\sqrt{2} \cdot X}$$

$$= \sqrt{2}\, X.$$

$$(X_1 + X_2) \cdot X = 2X^2$$

$$X = \frac{X_1 + X_2}{2}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2}{2} \\ \frac{x_1 + x_2}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda_1 = 1 \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 0 \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

21–25 GENERAL PROBLEMS

21. Nonzero defect. Find further 2×2 and 3×3 matrices with positive defect. See Example 3.

22. Multiple eigenvalues. Find further 2×2 and 3×3 matrices with multiple eigenvalues. See Example 2.

23. Complex eigenvalues. Show that the eigenvalues of a real matrix are real or complex conjugate in pairs.

24. Inverse matrix. Show that $\mathbf{A}^{-1}$ exists if and only if the eigenvalues $\lambda_1, \cdots, \lambda_n$ are all nonzero, and then $\mathbf{A}^{-1}$ has the eigenvalues $1/\lambda_1, \cdots, 1/\lambda_n$.

25. Transpose. Illustrate Theorem 3 with examples of your own.

21. $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$(3-\lambda)^2 = 0 \quad \lambda = 3 \quad M_\lambda = 2$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad m_\lambda = 1 \quad \text{defect} = 1$

$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \lambda = 3, \quad M_\lambda = 3$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad m_\lambda = 2 \quad \text{defect} = 1$

Jordan block

22. $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \lambda_1 = 6 \quad \lambda_{2,3} = \pm\sqrt{2}$

23. Reference 22.

24. if A has $\lambda = 0$, $(A - 0I)X = 0$

$\det(A - 0I) = 0$

$\det A = 0 \Leftrightarrow A^{-1}$ not exist

$AX = \lambda X \Rightarrow A^{-1}AX = A^{-1}\lambda X$

$X = A^{-1} \cdot \lambda X = \lambda A^{-1} X$

$A^{-1}X = \frac{1}{\lambda} \cdot X$

$\Rightarrow \frac{1}{\lambda}$ is the eigenvalue.

25.

Eigenvalues of the Transpose

The transpose $\mathbf{A}^\mathsf{T}$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

Reference 22.

$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$

P333 - Problem set 8.2

PS 8.2

1–6 ELASTIC DEFORMATIONS

Given **A** in a deformation **y** = **Ax**, find the principal directions and corresponding factors of extension or contraction. Show the details.

1. $\begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$ **2.** $\begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$

1. $\begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0$

$9+\lambda^2-6\lambda-2.25=0$

$4\lambda^2-24\lambda+27=0$

$(2\lambda-3)(2\lambda-9)=0$

$\lambda_1=\frac{3}{2}$

$\begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2=\frac{9}{2}$

$\begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $\begin{vmatrix} 2-\lambda & 0.4 \\ 0.4 & 2-\lambda \end{vmatrix} = 0$

$4+\lambda^2-4\lambda-0.16=0$

$\lambda^2-4\lambda+3.84=0$

$(5\lambda-8)(5\lambda-12)=0$

$\lambda_1=\frac{8}{5}=1.6$

$\begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2=\frac{12}{5}=2.4$

$\begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$ **4.** $\begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$

3. $\begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0$

$14+\lambda^2-9\lambda-6=0$

$\lambda^2-9\lambda+8=0$

$(\lambda-8)(\lambda-1)=0$

$\lambda_1=8$

$\begin{bmatrix} -1 & \sqrt{6} \\ \sqrt{6} & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\sqrt{6} \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$

$\lambda_2=1$

$\begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{\sqrt{6}}{6} \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} -1 \\ \sqrt{6} \end{bmatrix}$

4. $\begin{vmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix} = 0$

$65+\lambda^2-18\lambda-4=0$

$\lambda^2-18\lambda+61=0$

$\lambda=\frac{18\pm\sqrt{18^2-4\cdot 61}}{2}$

$=9\pm\sqrt{81-61}=9\pm2\sqrt{5}$

$\lambda_1=9+2\sqrt{5}$

$\begin{bmatrix} -4-2\sqrt{5} & 2 \\ 2 & 4-2\sqrt{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 2+\sqrt{5} & -1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} 1 \\ 2+\sqrt{5} \end{bmatrix}$

$\lambda_2=9-2\sqrt{5}$

$\begin{bmatrix} 2\sqrt{5}-4 & 2 \\ 2 & 2\sqrt{5}+4 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{5}-2 & 1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ 2-\sqrt{5} \end{bmatrix}$

5. $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$

5. $\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0$

$1+\lambda^2-2\lambda-\frac{1}{4}=0$

$4\lambda^2-8\lambda+3=0$

$(2\lambda-3)(2\lambda-1)=0$

$\lambda_1=\frac{3}{2}$

$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2=\frac{1}{2}$

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6. $\begin{vmatrix} 1.25-\lambda & 0.75 \\ 0.75 & 1.25-\lambda \end{vmatrix} = 0$

$\frac{25}{16}+\lambda^2-\frac{5}{2}\lambda-\frac{9}{16}=0$

$2\lambda^2-5\lambda+2=0$

$(2\lambda-1)(\lambda-2)=0$

$\lambda_1=\frac{1}{2}$

$\begin{bmatrix} 0.75 & 0.75 \\ 0.75 & 0.75 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2=2$

$\begin{bmatrix} -0.75 & 0.75 \\ 0.75 & -0.75 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7–9 MARKOV PROCESSES

Find the limit state of the Markov process modeled by the given matrix. Show the details.

7. $\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

8. $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$ **9.** $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix}$

7. $\begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{5}{8} \\ 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{5}{13} \\ \frac{8}{13} \end{bmatrix}$

8. $\begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 3 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ So limit is $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

9. $\begin{bmatrix} -0.4 & 0.1 & 0.2 \\ 0.4 & -0.9 & 0.4 \\ 0 & 0.8 & -0.6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & -0.8 & 0.6 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 11 \\ 12 \\ 16 \end{bmatrix}$

So limit is $\begin{bmatrix} 11/39 \\ 12/39 \\ 16/39 \end{bmatrix}$

10–12 AGE-SPECIFIC POPULATION

Find the growth rate in the Leslie model (see Example 3) with the matrix as given. Show the details.

10. $\begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 3.45 & 0.60 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$

10. $\begin{vmatrix} -\lambda & 9 & 5 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 + 0.8 + 3.6\lambda = 0$

$5\lambda^3 - 18\lambda - 4 = 0$

$(\lambda - 2)(5\lambda^2 + 10\lambda + 2) = 0$

$\lambda_1 = 2$

$\lambda_{2,3} = \frac{-10 \pm \sqrt{60}}{10}$

$= -1 \pm \frac{\sqrt{15}}{5} < 0$

growth rate is 2.

11. $\begin{vmatrix} -\lambda & 3.45 & 0.6 \\ 0.9 & -\lambda & 0 \\ 0 & 0.45 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 + 0.54 \times 0.45 + 0.9 \times 3.45\lambda = 0$

$-\lambda^3 + 0.243 + 3.105\lambda = 0$

$(\lambda - 1.8)(\lambda^2 + 1.8\lambda + 0.135) = 0$

$\lambda_1 = 1.8$

$\lambda_{2,3} = \frac{-90 \pm \sqrt{90^2 - 50 \times 27}}{100} < 0$

growth rate is 1.8

12. $\begin{vmatrix} -\lambda & 3 & 2 & 2 \\ 0.5 & -\lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{vmatrix} = 0$

$= -\lambda \cdot (-\lambda^3) - 3 \cdot 0.5\lambda^2 + 2 \cdot (-0.25\lambda)$
$- 2 \cdot 0.025 = 0$

$\lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05 = 0$

$20\lambda^4 - 30\lambda^2 - 10\lambda - 1 = 0$

not sure how to move on
by pen & paper!
Maybe computer or numeric way

13–15 LEONTIEF MODELS[1]

13. Leontief input–output model. Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the 3×3 **consumption matrix**

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \mathbf{A} = [a_{jk}] = & \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

where a_{jk} is the fraction of the output of industry k consumed (purchased) by industry j. Let p_j be the price charged by industry j for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to $\mathbf{Ap} = \mathbf{p}$, where $\mathbf{p} = [p_1 \quad p_2 \quad p_3]^\mathsf{T}$, and find a solution $\mathbf{p}$ with nonnegative p_1, p_2, p_3.

14. Show that a consumption matrix as considered in Prob. 13 must have column sums 1 and always has the eigenvalue 1.

all the input and output consumed by the industries. it is a closed system

Σ input = Σ output

15) $X - AX = [0.1 \ \ 0.3 \ \ 0.1]^T$

$(I-A)X = [0.1 \ \ 0.3 \ \ 0.1]^T$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.5 & 0 & 0.1 \\ 0.1 & 0.4 & 0.4 \end{bmatrix}\right) X = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0.9 & -0.4 & -0.2 & 0.1 \\ -0.5 & 1 & -0.1 & 0.3 \\ -0.1 & -0.4 & 0.6 & 0.1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -6 & -1 \\ 0 & 3 & -3.1 & -0.2 \\ 0 & 3.2 & -5.2 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -6 & -1 \\ 0 & 3 & -3.1 & -0.2 \\ 0 & 0 & 1 & 0.6875 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 0.55 \\ 0.64375 \\ 0.6875 \end{bmatrix}$$

13. $\begin{bmatrix} -0.9 & 0.5 & 0 \\ 0.8 & -1 & 0.4 \\ 0.1 & 0.5 & -0.4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 5 & -4 \\ 0 & 50 & -36 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 0.4 \\ 0.72 \\ 1 \end{bmatrix}$$

14. I don't get it.

the output doesn't necessary <u>ALL</u> consumed by those industries. (15) support this.

assume all output consumed within industries, then it means Σ column = 1

P338 - Problem set 8.3

PS. 8.3

1–10 SPECTRUM

Are the following matrices symmetric, skew-symmetric, or orthogonal? Find the spectrum of each, thereby illustrating Theorems 1 and 5. Show your work in detail.

1. $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$ **2.** $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

1. $a_{12} \neq a_{21}$

$a_{11} \neq -a_{11}$

$$\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 1 & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 5/4 & 0 \\ 1 & -\frac{4}{3} & 0 & -\frac{5}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 5/4 & 0 \\ 0 & 1 & 3/5 & 4/5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 4/5 & -3/5 \\ 0 & 1 & 3/5 & 4/5 \end{bmatrix}$$

$A^{-1} = A^T$. orthogonal.

$$\begin{vmatrix} 0.8-\lambda & 0.6 \\ -0.6 & 0.8-\lambda \end{vmatrix} = 0$$

$$0.64 + \lambda^2 - 1.6\lambda + 0.36 = 0$$

$$\lambda^2 - 1.6\lambda + 1 = 0$$

$$\lambda = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2}$$

$$= 0.8 \pm 0.6i$$

$$X_{1,2} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}$$

T3 $\quad \mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T \mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$

$$R_1 \begin{cases} 0.8 \cdot (-0.6) + 0.6 \cdot 0.8 = 0 \\ 0.8^2 + 0.6^2 = 1 \end{cases}$$

$$C \begin{cases} 0.8 \cdot 0.6 + (-0.6) \cdot 0.8 = 0 \\ 0.8^2 + (-0.6)^2 = 0.6^2 + 0.8^2 = 1 \end{cases}$$

T4 $\begin{vmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{vmatrix}$

$= -0.64 + 0.36 = 1$

T5 $\sqrt{0.8^2 + 0.6^2} = 1$

2. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

2. if $b=0$. symmetric

if $a=0$, skew-symmetric

if $a^2+b^2=1$, orthonormal

$\lambda = a \pm bi \quad X = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$

T1 $b=0$, λ is Real,

$a=0$ λ is imaginary or 0.

3. $\begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$ **4.** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3. Reference 2.

4. Reference 2

5. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ **6.** $\begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix}$ Symmetric

5. $\lambda_1 = 6 \quad X_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

$\lambda_2 = 6 \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_3 = 1 \quad X_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

6. Symmetric

$\lambda_1 = a-k \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = a-k \quad V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = a+2k \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

if $a=k=0$. skewsym