

# **Chapter 8 Linear Algebra: Matrix Eigenvalue Problems**

**P329 - Problem set 8.1**

# P.S. 8.1

## 1-16 EIGENVALUES, EIGENVECTORS

Find the eigenvalues. Find the corresponding eigenvectors. Use the given  $\lambda$  or factor in Probs. 11 and 15.

1.  $\begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}$

2.  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3.  $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$

4.  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$(A - \lambda I) \cdot X = 0$$

1.  $\det \begin{vmatrix} 3-\lambda & 0 \\ 0 & -0.6-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(0.6+\lambda) = 0$$

$$-\lambda^2 + 2.4\lambda + 1.8 = 0$$

$$\lambda^2 - 2.4\lambda - 1.8 = 0$$

$$(\lambda-3)(\lambda+0.6) = 0$$

1)  $\lambda = 3$       2)  $\lambda = -0.6$

$$\begin{bmatrix} 0 & 0 \\ 0 & -3.6 \end{bmatrix} \quad \begin{bmatrix} 3.6 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2.  $\det \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$

$$\lambda^2 = 0 \quad \lambda = 0$$

$X$  could be any  $2 \times 1$  vector.

P326 theorem 2. 0 allowed.

3.  $\det \begin{vmatrix} 5\lambda & -2 \\ 9 & -6\lambda \end{vmatrix} = 0$

$$-(5-\lambda)(6+\lambda) + 18 = 0$$

$$\lambda^2 + \lambda - 30 + 18 = 0$$

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda+4)(\lambda-3) = 0$$

1)  $\lambda = -4$

$$\begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

2)  $\lambda = 3$        $\begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix}$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$3. \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$$

$$4. \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$5. \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$6. \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$5. \begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 9 = 0$$

$$4. \det: \begin{vmatrix} -\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-4) - 4 = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda=0 \quad \lambda=5$$

$$\lambda=5: \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda=0 \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\lambda = \pm 3i$$
$$\lambda = 3i: \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda = -3i: \begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$6. \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(\lambda-3)(\lambda-1)=0$$

$$\lambda=3 \quad \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda=1 \quad \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$7. \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$8. \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\lambda = a - bi \quad \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$7. \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = 0$$

$$X = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$9. \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

$$10. \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\lambda = 0 \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$9. \begin{vmatrix} 0.8 - \lambda & -0.6 \\ 0.6 & 0.8 - \lambda \end{vmatrix} = 0$$

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$a = 0.8 \quad b = -0.6$$

$$\lambda = 0.8 - 0.6i \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$8. \begin{vmatrix} a - \lambda & b \\ -b & a - \lambda \end{vmatrix} = 0$$

$$\lambda = 0.8 + 0.6i \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$(a - \lambda)^2 + b^2 = 0$$

$$10. \begin{vmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{vmatrix} = 0$$

$$a - \lambda = \pm bi$$

$$a = \cos \theta \quad b = -\sin \theta$$

$$\lambda = a \pm bi$$

$$\lambda = \cos \theta - i \sin \theta \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda = a + bi$$

$$\lambda = \cos \theta + i \sin \theta \quad X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

11.  $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}, \lambda = 3$

$$\lambda = 1$$

$$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$$

11.  $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$

13.  $\begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

13.  $\begin{vmatrix} 13-\lambda & 5 & 2 \\ 2 & 7-\lambda & -8 \\ 5 & 4 & 7-\lambda \end{vmatrix} = 0$

$$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

12.  $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

$$(13-\lambda)(7-\lambda)^2 - 200 + 16 - 10(7-\lambda) - 32(\lambda-13) - 10(7-\lambda) = 0$$

12.  $(3-\lambda)(4-\lambda)(1-\lambda) = 0$

$$-\lambda^3 + 27\lambda^2 - 243\lambda + 729 = 0$$

$$(\lambda-9)^3 = 0$$

$$\lambda = 3$$

$$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 9$$

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 9 & 18 \end{bmatrix}$$

$$\lambda = 4$$

$$\begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$14. \begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$15. \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}, (\lambda + 1)^2$$

$$\lambda_1 = 3, \begin{bmatrix} -3 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5, \begin{bmatrix} -3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -1, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \leftarrow$$

$$\lambda_2 = -1, \quad \lambda_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$16. \begin{bmatrix} -3 & 0 & 4 & 2 \\ 0 & 1 & -2 & 4 \\ 2 & 4 & -1 & -2 \\ 0 & 2 & -2 & 3 \end{bmatrix}$$

$$\lambda_1 = -1, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -5, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 3, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

17. Counterclockwise rotation through the angle  $\pi/2$  about the origin in  $R^2$ .

18. Reflection about the  $x_1$ -axis in  $R^2$ .

17. P271. Problem 30.

$$A = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

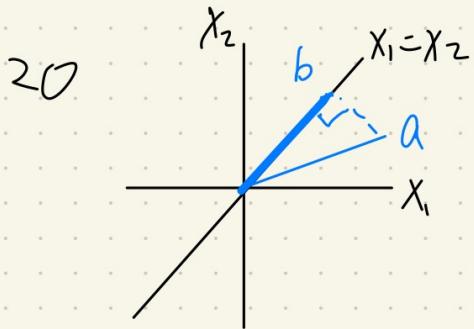
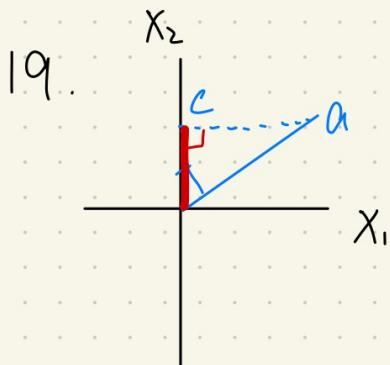
$$\lambda_1 = i, \quad \lambda_2 = -i; \quad \lambda_3 = -i, \quad \lambda_4 = i$$

$$18. \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 = x_1 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

19. Orthogonal projection (perpendicular projection) of  $R^2$  onto the  $x_2$ -axis.

20. Orthogonal projection of  $R^3$  onto the plane  $x_2 = x_1$ .



$$P = \frac{a \cdot b}{\|b\|} = \frac{x_1 \cdot x_1 + x_2 \cdot x_2}{\sqrt{2} \cdot x_1}$$

$$= \sqrt{2} x_1.$$

$$(x_1 + x_2) \cdot x = 2x^2$$

$$x = \frac{x_1 + x_2}{2}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2}{2} \\ \frac{x_1 + x_2}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = 0, \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 1, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_1 = 1 \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0 \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

**21–25 GENERAL PROBLEMS**

21. **Nonzero defect.** Find further  $2 \times 2$  and  $3 \times 3$  matrices with positive defect. See Example 3.
22. **Multiple eigenvalues.** Find further  $2 \times 2$  and  $3 \times 3$  matrices with multiple eigenvalues. See Example 2.
23. **Complex eigenvalues.** Show that the eigenvalues of a real matrix are real or complex conjugate in pairs.
24. **Inverse matrix.** Show that  $A^{-1}$  exists if and only if the eigenvalues  $\lambda_1, \dots, \lambda_n$  are all nonzero, and then  $A^{-1}$  has the eigenvalues  $1/\lambda_1, \dots, 1/\lambda_n$ .
25. **Transpose.** Illustrate Theorem 3 with examples of your own.

$$21. \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$(3-\lambda)^2 = 0 \quad \lambda = 3 \quad M_\lambda = 2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad m_x = 1 \quad \text{defect} = 1$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \lambda = 3, \quad M_\lambda = 3$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad M_\lambda = 2 \quad \text{defect} = 1$$

Jordan block

$$22. \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \lambda_1 = 6 \quad \lambda_{2,3} = \pm \sqrt{2}$$

23. Reference 22.

24. If  $A$  has  $\lambda=0$ ,  $(A-0I)x=0$

$$\det(A-0I) = 0$$

$\det A = 0 \Leftrightarrow A^{-1}$  not exist

$$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x$$

$$x = A^{-1} \cdot \lambda x = \lambda A^{-1}x$$

$$A^{-1}x = \frac{1}{\lambda} \cdot x$$

$\Rightarrow \frac{1}{\lambda}$  is the eigenvalue.

25.

**Eigenvalues of the Transpose**

The transpose  $A^T$  of a square matrix  $A$  has the same eigenvalues as  $A$ .

Reference 22.

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = -1$$

# PS 8.2

## 1-6 ELASTIC DEFORMATIONS

Given  $\mathbf{A}$  in a deformation  $\mathbf{y} = \mathbf{Ax}$ , find the principal directions and corresponding factors of extension or contraction. Show the details.

$$1. \begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$$

$$2. \begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$$

$$1. \begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0$$

$$9 + \lambda^2 - 6\lambda - 2.25 = 0$$

$$4\lambda^2 - 24\lambda + 27 = 0$$

$$(2\lambda - 3)(2\lambda - 9) = 0$$

$$\lambda_1 = \frac{3}{2}$$

$$\begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{9}{2}$$

$$\begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$2. \begin{vmatrix} 2-\lambda & 0.4 \\ 0.4 & 2-\lambda \end{vmatrix} = 0$$

$$4 + \lambda^2 - 4\lambda - 0.16 = 0$$

$$\lambda^2 - 4\lambda + 3.84 = 0$$

$$(5\lambda - 8)(5\lambda - 12) = 0$$

$$\lambda_1 = \frac{8}{5} = 1.6$$

$$\begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{12}{5} = 2.4$$

$$\begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$3. \begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$$

$$4. \begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$$

$$3. \begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0$$

$$14 + \lambda^2 - 9\lambda - 6 = 0$$

$$\lambda^2 - 9\lambda + 8 = 0$$

$$(\lambda - 8)(\lambda - 1) = 0$$

$$\lambda_1 = 8$$

$$\begin{bmatrix} -1 & \sqrt{6} \\ \sqrt{6} & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\sqrt{6} \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1$$

$$\begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{\sqrt{6}}{6} \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -1 \\ \sqrt{6} \end{bmatrix}$$

$$4. \begin{vmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix} = 0$$

$$65 + \lambda^2 - 18\lambda - 4 = 0$$

$$\lambda^2 - 18\lambda + 61 = 0$$

$$\lambda = \frac{18 \pm \sqrt{18^2 - 4 \cdot 61}}{2}$$

$$= 9 \pm \sqrt{81 - 61} = 9 \pm 2\sqrt{5}$$

$$\lambda_1 = 9 + 2\sqrt{5}$$

$$\begin{bmatrix} -4-2\sqrt{5} & 2 \\ 2 & 4-2\sqrt{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 2+\sqrt{5} & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 2+\sqrt{5} \end{bmatrix}$$

$$\lambda_2 = 9 - 2\sqrt{5}$$

$$\begin{bmatrix} 2\sqrt{5}-4 & 2 \\ 2 & 2\sqrt{5}+4 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{5}-2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 2-\sqrt{5} \end{bmatrix}$$

$$5. \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$6. \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$$

$$5. \begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$$1 + \lambda^2 - 2\lambda - \frac{1}{4} = 0$$

$$4\lambda^2 - 8\lambda + 3 = 0$$

$$(2\lambda - 3)(2\lambda - 1) = 0$$

$$\lambda_1 = \frac{3}{2}$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$6. \begin{vmatrix} 1.25-\lambda & 0.75 \\ 0.75 & 1.25-\lambda \end{vmatrix} = 0$$

$$\frac{25}{16} + \lambda^2 - \frac{5}{2}\lambda - \frac{9}{16} = 0$$

$$2\lambda^2 - 5\lambda + 2 = 0$$

$$(2\lambda - 1)(\lambda - 2) = 0$$

$$\lambda_1 = \frac{1}{2}$$

$$\begin{bmatrix} 0.75 & 0.75 \\ 0.75 & 0.75 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2$$

$$\begin{bmatrix} -0.75 & 0.75 \\ 0.75 & -0.75 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

## 7-9 MARKOV PROCESSES

Find the limit state of the Markov process modeled by the given matrix. Show the details.

7.  $\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

8.  $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$     9.  $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & -0.8 & 0.6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$7. \begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{5}{8} \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{5}{13} \\ \frac{8}{13} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 12 \\ 16 \end{bmatrix}$$

$$\text{So limit is } \begin{bmatrix} 1/39 \\ 2/39 \\ 16/39 \end{bmatrix}$$

$$8. \begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 3 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ So limit is } \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

9.  $\begin{bmatrix} -0.4 & 0.1 & 0.2 \\ 0.4 & -0.9 & 0.4 \\ 0 & 0.8 & -0.6 \end{bmatrix}$

**10-12 AGE-SPECIFIC POPULATION**

Find the growth rate in the Leslie model (see Example 3) with the matrix as given. Show the details.

$$10. \begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \quad 11. \begin{bmatrix} 0 & 3.45 & 0.6 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$$

$$12. \begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$11. \begin{vmatrix} -\lambda & 3.45 & 0.6 \\ 0.9 & -\lambda & 0 \\ 0 & 0.45 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 0.54 \times 0.45 + 0.9 \times 3.45 \lambda = 0$$

$$10. \begin{vmatrix} -\lambda & 9 & 5 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 0.8 + 3.6\lambda = 0$$

$$5\lambda^3 - 18\lambda - 4 = 0$$

$$(\lambda - 2)(5\lambda^2 + 10\lambda + 2) = 0$$

$$\lambda_1 = 2$$

$$\lambda_{2,3} = \frac{-10 \pm \sqrt{60}}{10}$$

$$= -1 \pm \frac{\sqrt{15}}{5} < 0$$

Growth rate is 2.

$$-\lambda^3 + 0.243 + 3.105\lambda = 0$$

$$(\lambda - 1.8)(\lambda^2 + 1.8\lambda + 0.135) = 0$$

$$\lambda_1 = 1.8$$

$$\lambda_{2,3} = \frac{-90 \pm \sqrt{90^2 - 5 \times 27}}{100} < 0$$

Growth rate is 1.8

$$12. \begin{vmatrix} -\lambda & 3 & 2 & 2 \\ 0.5 - \lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{vmatrix} = 0$$

$$= -\lambda \cdot (-\lambda^3) - 3 \cdot 0.5\lambda^2 + 2 \cdot (-0.25\lambda)$$

$$- 2 \cdot 0.025 = 0$$

$$\lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05 = 0$$

$$20\lambda^4 - 30\lambda^2 - 10\lambda - 1 = 0$$

not sure how to move on

by pen & paper.

Maybe computer or numeric way

13-15 LEONTIEF MODELS<sup>1</sup>

13. Leontief input-output model. Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the  $3 \times 3$  consumption matrix

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} 1 & 2 & 3 \\ 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$$

where  $a_{jk}$  is the fraction of the output of industry  $k$  consumed (purchased) by industry  $j$ . Let  $p_j$  be the price charged by industry  $j$  for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to  $\mathbf{Ap} = \mathbf{p}$ , where  $\mathbf{p} = [p_1 \ p_2 \ p_3]^T$ , and find a solution  $\mathbf{p}$  with nonnegative  $p_1, p_2, p_3$ .

14. Show that a consumption matrix as considered in Prob. 13 must have column sums 1 and always has the eigenvalue 1.

all the input and output consumed by the industries, it is a closed system  
 $\sum \text{input} = \sum \text{output}$

$$15) \quad X \mathbf{AX} = [0.1 \ 0.3 \ 0.1]^T$$

$$(\mathbf{I} - \mathbf{A}) \mathbf{X} = [0.1 \ 0.3 \ 0.1]^T$$

$$13. \quad \begin{bmatrix} -0.9 & 0.5 & 0 \\ 0.8 & -1 & 0.4 \\ 0.1 & 0.5 & -0.4 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.5 & 0 & 0.1 \\ 0.1 & 0.4 & 0.4 \end{bmatrix} \right) \mathbf{X} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & -4 \\ 0 & 50 & -36 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.9 & -0.4 & -0.2 & 0.1 \\ -0.5 & 1 & -0.1 & 0.3 \\ -0.1 & -0.4 & 0.6 & 0.1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -6 & -1 \\ 0 & 3 & -3.1 & -0.2 \\ 0 & 3.2 & -5.2 & -1 \end{bmatrix}$$

$$14. \quad \begin{bmatrix} 1 & 4 & -6 & -1 \\ 0 & 3 & -3.1 & -0.2 \\ 0 & 0 & 1 & 0.6875 \end{bmatrix}$$

$$\Rightarrow \mathbf{X} = \begin{bmatrix} 0.55 \\ 0.64375 \\ 0.6875 \end{bmatrix}$$

14. I don't get it.  
 -the output doesn't necessarily  
ALL consumed by those  
 industries. (15) support this.  
 assume all output consumed within  
 industries, then it means  $\sum \text{Column} = 1$

# PS. 8.3

## 1-10 SPECTRUM

Are the following matrices symmetric, skew-symmetric, or orthogonal? Find the spectrum of each, thereby illustrating Theorems 1 and 5. Show your work in detail.

$$1. \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$$

$$2. \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$1. a_{12} \neq a_{21}$$

$$a_{11} \neq -a_{11}$$

$$\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 1 & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{5}{4} & 0 \\ 1 & -\frac{4}{3} & 0 & -\frac{5}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 1 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$A^{-1} = A^T$ , orthogonal.

$$\begin{vmatrix} 0.8-\lambda & 0.6 \\ -0.6 & 0.8-\lambda \end{vmatrix} = 0$$

$$0.64 + \lambda^2 - 1.6\lambda + 0.36 = 0$$

$$\lambda^2 - 1.6\lambda + 1 = 0$$

$$\lambda = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2}$$

$$= 0.8 \pm 0.6i$$

$$\lambda_{1,2} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}$$

T3

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^\top \mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

$$R_1 \left\{ \begin{array}{l} 0.8 \cdot (-0.6) + 0.6 \cdot 0.8 = 0 \\ 0.8^2 + 0.6^2 = 1 \end{array} \right.$$

$$C \left\{ \begin{array}{l} 0.8 \cdot 0.6 + (-0.6) \cdot 0.8 = 0 \\ 0.8^2 + (-0.6)^2 = 0.6^2 + 0.8^2 = 1 \end{array} \right.$$

T4:  $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$

$$= -0.64 + 0.36 = 1$$

T5  $\sqrt{0.8^2 + 0.6^2} = 1$

$$2. \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

2. if  $b=0$ , symmetric

if  $a=0$ , skew-symmetric

if  $a^2+b^2=1$ , orthonormal

$$\lambda = a \pm bi \quad X = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

(T1)  $b=0$ ,  $\lambda$  is real

$a=0$   $\lambda$  is imaginary or 0.

$$3. \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$$

$$4. \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

3. Reference 2.

4. Reference 2

$$5. \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

$$6. \begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix}$$

Symmetric

$$5. \lambda_1 = 6 \quad X_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 6 \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 1 \quad X_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

6. Symmetric

$$\lambda_1 = a-k \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = a+k \quad V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = a+2k \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

if  $a=k=0$ , showsym

$$7. \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

skew-symmetric  
7. Anti-Symmetric

$$\lambda_1 = 25i, \quad V_1 = \begin{bmatrix} -45+75i \\ -27-125i \\ 136 \end{bmatrix}$$

$$\lambda_2 = -25i, \quad V_2 = \begin{bmatrix} -45-75i \\ -27+125i \\ 136 \end{bmatrix}$$

$$\lambda_3 = 0, \quad V_3 = \begin{bmatrix} 20 \\ 12 \\ 9 \end{bmatrix}$$

$$8. \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$\lambda_1 = 1 \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \cos \theta - i \sin \theta$$

$$v_2 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

$$\lambda_3 = \cos \theta + i \sin \theta$$

$$v_3 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

I already proved that I can use the tools property to resolve problem. No mean to continue with more cases.

Skip 9-10

11. Reference mindmap and notes.

$$(2) a) A^T = A^{-1} \quad B^T = B^{-1}$$

$$(AB)^T = B^T \cdot A^T$$

$$= B^{-1} \cdot A^{-1}$$

$$= (AB)^{-1}$$

$$A^T A = A A^T = I$$

$$(A^T A)^T = A^T A = A A^T = I$$

$$(A^T)^{-1} = (A^{-1} \cdot A \cdot A^T)^{-1}$$

$$= (A \cdot A^T)^{-1} A$$

$$= I \cdot A = A$$

Rotation Matrix - wiki

(b) **Rotation.** Show that (6) is an orthogonal transformation. Verify that it satisfies Theorem 3. Find the inverse transformation.

$$(6) \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = A^T$$

if  $j=k$

$$a_j \cdot a_k = \cos^2 \theta + (-\sin \theta)^2$$

$$= \sin^2 \theta + \cos^2 \theta = 1$$

if  $j \neq k$

$$a_j \cdot a_k = \cos \theta \sin \theta - \sin \theta \cos \theta$$

$$= 0$$

(c) **Powers.** Write a program for computing powers  $\mathbf{A}^m$  ( $m = 1, 2, \dots$ ) of a  $2 \times 2$  matrix  $\mathbf{A}$  and their spectra. Apply it to the matrix in Prob. 1 (call it  $\mathbf{A}$ ). To what rotation does  $\mathbf{A}$  correspond? Do the eigenvalues of  $\mathbf{A}^m$  have a limit as  $m \rightarrow \infty$ ?

(d) Compute the eigenvalues of  $(0.9\mathbf{A})^m$ , where  $\mathbf{A}$  is the matrix in Prob. 1. Plot them as points. What is their limit? Along what kind of curve do these points approach the limit?

(e) Find  $\mathbf{A}$  such that  $\mathbf{y} = \mathbf{Ax}$  is a counterclockwise rotation through  $30^\circ$  in the plane.

are symmetric, skew-symmetric, and orthogonal, respectively, as you should verify. Every skew-symmetric matrix has all main diagonal entries zero. (Can you prove this?)

c). Per P.S. 8.2-ff,

**18. Scalar multiples, powers.**  $k\mathbf{A}$  has the eigenvalues  $k\lambda_1, \dots, k\lambda_n$ .  $\mathbf{A}^m$  ( $m = 1, 2, \dots$ ) has the eigenvalues  $\lambda_1^m, \dots, \lambda_n^m$ . The eigenvectors are those of  $\mathbf{A}$ .

$$\text{if } |\lambda| < 1, \lim_{m \rightarrow \infty} \lambda^m = 0$$

$$d) (0.9\mathbf{A})^m = 0.9^m \cdot \mathbf{A}^m$$

$$\lambda = 0.9^m \cdot \lambda_1^m$$

$$\text{if } |0.9\lambda_1| < 1, \lim \rightarrow 0.$$

$$0.9\lambda_1 = 1 \rightarrow 1$$

$$e) \begin{bmatrix} \cos \frac{\pi}{6} & -\sin \frac{\pi}{6} \\ \sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{bmatrix}$$

### 13-20 GENERAL PROPERTIES

13. **Verification.** Verify the statements in Example 1.

14. Verify the statements in Examples 3 and 4.

15. **Sum.** Are the eigenvalues of  $\mathbf{A} + \mathbf{B}$  sums of the eigenvalues of  $\mathbf{A}$  and of  $\mathbf{B}$ ?

$$(3) a_{jk} = -a_{kj}$$

$$j=k, a_{jj} = -a_{jj} \Rightarrow 0.$$

(4) NO. if symmetric  $\Rightarrow$  eigenvalues GR but not  $\Leftarrow$ .

$$j \neq k:$$

$$a_j \cdot a_k = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

$$j \neq k$$

$$a_j \cdot a_k = \frac{-4}{9} + \frac{2}{9} + \frac{2}{9}$$

$$= \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

$$\det(\mathbf{A}) = -\frac{8}{27} + \frac{1}{27} - \frac{8}{27} - \frac{4}{27} - \frac{4}{27} - \frac{4}{27} = -1$$

$$(6) \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$j=k, a_j \cdot a_k = \cos^2 \theta + \sin^2 \theta = 1$$

$$j \neq k, a_j \cdot a_k = \cos \theta \sin \theta - \sin \theta \cos \theta = 0$$

$$\det(\mathbf{A}) = \cos^2 \theta + \sin^2 \theta = 1$$

15. PS. 8.2-16:

$$\text{Trace}(A) = \sum \lambda_A$$

$$\text{Trace}(B) = \sum \lambda_B$$

$$A+B \Rightarrow \text{Trace}(A+B)$$

$$= \sum (\lambda_A + \lambda_B)$$

16 - 20: Not finish yet