Chapter 8 Linear Algebra: Matrix Eigenvalue Problems

P329 - Problem set 8.1

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P333 - Problem set 8.2

PS 8.2

1–6 ELASTIC DEFORMATIONS

Given **A** in a deformation $\mathbf{y} = \mathbf{Ax}$, find the principal directions and corresponding factors of extension or contraction. Show the details.

1. $\begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$ **2.** $\begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$

1. $\begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0$

$9+\lambda^2-6\lambda-2.25=0$

$4\lambda^2-24\lambda+27=0$

$(2\lambda-3)(2\lambda-9)=0$

$\lambda_1 = \frac{3}{2}$

$\begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{9}{2}$

$\begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $\begin{vmatrix} 2-\lambda & 0.4 \\ 0.4 & 2-\lambda \end{vmatrix} = 0$

$4+\lambda^2-4\lambda-0.16=0$

$\lambda^2-4\lambda+3.84=0$

$(5\lambda-8)(5\lambda-12)=0$

$\lambda_1 = \frac{8}{5} = 1.6$

$\begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{12}{5} = 2.4$

$\begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$ **4.** $\begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$

3. $\begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0$

$14 + \lambda^2 - 9\lambda - 6 = 0$

$\lambda^2 - 9\lambda + 8 = 0$

$(\lambda - 8)(\lambda - 1) = 0$

$\lambda_1 = 8$

$\begin{bmatrix} -1 & \sqrt{6} \\ \sqrt{6} & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\sqrt{6} \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$

$\lambda_2 = 1$

$\begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{\sqrt{6}}{6} \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} -1 \\ \sqrt{6} \end{bmatrix}$

4. $\begin{vmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix} = 0$

$65 + \lambda^2 - 18\lambda - 4 = 0$

$\lambda^2 - 18\lambda + 61 = 0$

$\lambda = \frac{18 \pm \sqrt{18^2 - 4 \cdot 61}}{2}$

$= 9 \pm \sqrt{81 - 61} = 9 \pm 2\sqrt{5}$

$\lambda_1 = 9 + 2\sqrt{5}$

$\begin{bmatrix} -4-2\sqrt{5} & 2 \\ 2 & 4-2\sqrt{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 2+\sqrt{5} & -1 \\ 0 & 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ 2+\sqrt{5} \end{bmatrix}$

$\lambda_2 = 9 - 2\sqrt{5}$

$\begin{bmatrix} 2\sqrt{5}-4 & 2 \\ 2 & 2\sqrt{5}+4 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{5}-2 & 1 \\ 0 & 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 2-\sqrt{5} \end{bmatrix}$

5. $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$

5. $\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0$

$1+\lambda^2-2\lambda-\frac{1}{4}=0$

$4\lambda^2-8\lambda+3=0$

$(2\lambda-3)(2\lambda-1)=0$

$\lambda_1=\frac{3}{2}$

$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2=\frac{1}{2}$

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6. $\begin{vmatrix} 1.25-\lambda & 0.75 \\ 0.75 & 1.25-\lambda \end{vmatrix} = 0$

$\frac{25}{16}+\lambda^2-\frac{5}{2}\lambda-\frac{9}{16}=0$

$2\lambda^2-5\lambda+2=0$

$(2\lambda-1)(\lambda-2)=0$

$\lambda_1=\frac{1}{2}$

$\begin{bmatrix} 0.75 & 0.75 \\ 0.75 & 0.75 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$X_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2=2$

$\begin{bmatrix} -0.75 & 0.75 \\ 0.75 & -0.75 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$X_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7–9 MARKOV PROCESSES

Find the limit state of the Markov process modeled by the given matrix. Show the details.

7. $\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

8. $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$ **9.** $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix}$

7. $\begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{5}{8} \\ 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$. $\Rightarrow \begin{bmatrix} \frac{5}{13} \\ \frac{8}{13} \end{bmatrix}$

8. $\begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 3 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ So limit is $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

9. $\begin{bmatrix} -0.4 & 0.1 & 0.2 \\ 0.4 & -0.9 & 0.4 \\ 0 & 0.8 & -0.6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & -0.8 & 0.6 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & -1 & -2 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} 11 \\ 12 \\ 16 \end{bmatrix}$

So limit is $\begin{bmatrix} 11/39 \\ 12/39 \\ 16/39 \end{bmatrix}$

10–12 AGE-SPECIFIC POPULATION

Find the growth rate in the Leslie model (see Example 3) with the matrix as given. Show the details.

10. $\begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 3.45 & 0.60 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$

10. $\begin{vmatrix} -\lambda & 9 & 5 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 + 0.8 + 3.6\lambda = 0$

$5\lambda^3 - 18\lambda - 4 = 0$

$(\lambda-2)(5\lambda^2+10\lambda+2) = 0$

$\lambda_1 = 2$

$\lambda_{2,3} = \frac{-10 \pm \sqrt{60}}{10}$

$= -1 \pm \frac{\sqrt{15}}{5} < 0$

growth rate is 2.

11. $\begin{vmatrix} -\lambda & 3.45 & 0.6 \\ 0.9 & -\lambda & 0 \\ 0 & 0.45 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 + 0.54 \times 0.45 + 0.9 \times 3.45\lambda = 0$

$-\lambda^3 + 0.243 + 3.105\lambda = 0$

$(\lambda-1.8)(\lambda^2+1.8\lambda+0.135) = 0$

$\lambda_1 = 1.8$

$\lambda_{2,3} = \frac{-90 \pm \sqrt{90^2 - 50\times 27}}{100} < 0$

growth rate is 1.8

12. $\begin{vmatrix} -\lambda & 3 & 2 & 2 \\ 0.5 & -\lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{vmatrix} = 0$

$= -\lambda \cdot (-\lambda^3) - 3 \cdot 0.5\lambda^2 + 2 \cdot (-0.25\lambda)$
$- 2 \cdot 0.025 = 0$

$\lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05 = 0$

$20\lambda^4 - 30\lambda^2 - 10\lambda - 1 = 0$

not sure how to move on
by pen & paper!
Maybe computer or numeric way

13–15 LEONTIEF MODELS[1]

13. Leontief input–output model. Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the 3×3 **consumption matrix**

$$\mathbf{A} = [a_{jk}] = \begin{array}{c} \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

where a_{jk} is the fraction of the output of industry k consumed (purchased) by industry j. Let p_j be the price charged by industry j for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to $\mathbf{Ap} = \mathbf{p}$, where $\mathbf{p} = [p_1 \;\; p_2 \;\; p_3]^\mathsf{T}$, and find a solution $\mathbf{p}$ with nonnegative p_1, p_2, p_3.

14. Show that a consumption matrix as considered in Prob. 13 must have column sums 1 and always has the eigenvalue 1.

all the input and output consumed by the industries. it is a closed system

Σ input = Σ output

13. $\begin{bmatrix} -0.9 & 0.5 & 0 \\ 0.8 & -1 & 0.4 \\ 0.1 & 0.5 & -0.4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 5 & -4 \\ 0 & 50 & -36 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow X = \begin{bmatrix} 0.4 \\ 0.72 \\ 1 \end{bmatrix}$

15) $X - AX = [0.1 \;\; 0.3 \;\; 0.1]^T$

$(I - A)X = [0.1 \;\; 0.3 \;\; 0.1]^T$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.5 & 0 & 0.1 \\ 0.1 & 0.4 & 0.4 \end{bmatrix} \right) X = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0.9 & -0.4 & -0.2 & 0.1 \\ -0.5 & 1 & -0.1 & 0.3 \\ -0.1 & -0.4 & 0.6 & 0.1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -6 & -1 \\ 0 & 3 & -3.1 & -0.2 \\ 0 & 3.2 & -5.2 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -6 & -1 \\ 0 & 3 & -3.1 & -0.2 \\ 0 & 0 & 1 & 0.6875 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 0.55 \\ 0.64375 \\ 0.6875 \end{bmatrix}$$

14. I don't get it.
the output doesn't necessary ALL consumed by those industries.(15) support this.
assume all output consumed within industries, then it means Σcolumn = 1

P338 - Problem set 8.3

PS. 8.3

1–10 SPECTRUM

Are the following matrices symmetric, skew-symmetric, or orthogonal? Find the spectrum of each, thereby illustrating Theorems 1 and 5. Show your work in detail.

1. $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$ **2.** $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

1. $a_{12} \neq a_{21}$

$a_{11} \neq -a_{11}$

$$\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 1 & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 5/4 & 0 \\ 1 & -\frac{4}{3} & 0 & -\frac{5}{3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 5/4 & 0 \\ 0 & 1 & 3/5 & 4/5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 4/5 & -3/5 \\ 0 & 1 & 3/5 & 4/5 \end{bmatrix}$$

$A^{-1} = A^T$. orthogonal.

$$\begin{vmatrix} 0.8-\lambda & 0.6 \\ -0.6 & 0.8-\lambda \end{vmatrix} = 0$$

$$0.64 + \lambda^2 - 1.6\lambda + 0.36 = 0$$

$$\lambda^2 - 1.6\lambda + 1 = 0$$

$$\lambda = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2}$$

$$= 0.8 \pm 0.6i$$

$$X_{1,2} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}$$

T3: $\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T \mathbf{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$

$$R_1 \begin{cases} 0.8 \cdot (-0.6) + 0.6 \cdot 0.8 = 0 \\ 0.8^2 + 0.6^2 = 1 \end{cases}$$

$$C \begin{cases} 0.8 \cdot 0.6 + (-0.6) \cdot 0.8 = 0 \\ 0.8^2 + (-0.6)^2 = 0.6^2 + 0.8^2 = 1 \end{cases}$$

T4: $\begin{vmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{vmatrix}$

$= -0.64 + 0.36 = 1$

T5: $\sqrt{0.8^2 + 0.6^2} = 1$

2. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

2. if $b=0$. symmetric

if $a=0$. skew-symmetric

if $a^2+b^2=1$. orthonormal

$\lambda = a \pm bi \quad X = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$

T1 $b=0$, λ is Real,

$a=0$ λ is imaginary or 0.

3. $\begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$ **4.** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3. Reference 2.

4. Reference 2

5. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ **6.** $\begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix}$ Symmetric

5. $\lambda_1 = 6 \quad X_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

$\lambda_2 = 6 \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_3 = 1 \quad X_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

6. Symmetric

$\lambda_1 = a-k \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = a-k \quad V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = a+2k \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

if $a=k=0$. skewsym