W5 - Green's Integral Theorem

- 5A: Evaluation of Line Integrals and Double Integrals
  - A Line Integral
    - direction
    - Contour
      - Closed path
      - Smooth curve
        - Unique tangent
        - piecewise smooth
    - Definition and Evaluation of Line Integrals
      - Contour dot r
      - $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \qquad \mathbf{r}' = \frac{d\mathbf{r}}{dt}$
      - $$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt.$$
    - Application
      - Work done by a Force
  - Line Integral in Space
    - $\int_C \mathbf{F}(\mathbf{r})\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t))\, dt = \int_a^b [F_1(\mathbf{r}(t)), \quad F_2(\mathbf{r}(t)), \quad F_3(\mathbf{r}(t))]\, dt.$
    - Obviously, a special case of (7) is obtained by taking $F_1 = f, F_2 = F_3 = 0$. Then

      (8*) $\int_C f(\mathbf{r})\, dt = \int_a^b f(\mathbf{r}(t))\, dt$
  - Path Depdendence
  - Path Independence
    - T1: F = grad f
      - $\int_A^B (F_1\,dx + F_2\,dy + F_3\,dz) = f(B) - f(A) \qquad [\mathbf{F} = \operatorname{grad} f]$
    - T2: integration of a close curve always = 0
    - T3: curl F = **0**
    - Potential Theory
    - Application
      - Determination of a potential
      - Conservative
      - Exactness
        - $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$
      - Simply connected
  - Double Integration
    - sequence converge
    - properties
    - mean value theorem
    - Evaluate: by 2 sucessive integrations
      - $$\iint_R f(x, y)\, dx\, dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)\, dy\right] dx$$
      - $$\iint_R f(x, y)\, dx\, dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x, y)\, dx\right] dy$$
    - Application
      - Area
      - Volumne
        - Under surface f(x,y)
      - Total Mass:
        - Density f(x,y)
      - Center of gravity
      - Moments of inertia
      - Polar moment of inertia
    - Change variables, Jacobian
      - $\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv;$
        $\qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$
      - Polar cooridiates
        - $\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$
- 5B: Green's Theorem in the Plane
  - Green's Theoream in the Plane
    - $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_C (F_1\,dx + F_2\,dy). \qquad \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dx\, dy = \oint_C \mathbf{F} \cdot d\mathbf{r}.$
    - simplify the evaluation
    - Switch from one to another
  - Verify Green's Theorem
    - Reference P431, Example 2
    - Then P434 Example 1
  - Application
    - Area of a Plane Region
      - as a Line Integral Over the Boundary
        - $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$
      - in Polar Coordinates
        - $A = \frac{1}{2}\oint_C r^2\, d\theta.$
    - Transform of a double integral of the Laplacian of a function into a line integral of its Normal derivative
      - $\iint_R \nabla^2 w\, dx\, dy = \oint_C \frac{\partial w}{\partial n}\, ds.$
- 5C: Surfaces
  - Parapatric representation
    - $z = f(x, y) \quad \text{or} \quad g(x, y, z) = 0.$
    - $\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$
  - of surfaces
    - Cylinder
      - $\mathbf{r}(u, v) = [a\cos u, a\sin u, v] = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + v\mathbf{k}$
    - Sphere
      - $\mathbf{r}(u, v) = a\cos v\cos u\,\mathbf{i} + a\cos v\sin u\,\mathbf{j} + a\sin v\,\mathbf{k}$
      - $\mathbf{r}(u, v) = a\cos u\sin v\,\mathbf{i} + a\sin u\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$
    - Cone
      - $\mathbf{r}(u, v) = [u\cos v, u\sin v, u] = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\mathbf{k},$
  - Tangent Plane and Surface Normal
    - Normal Vector
      - $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}.$
    - Unit Normal Vector
      - $\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v.$
      - $\mathbf{n} = \frac{1}{|\operatorname{grad} g|}\operatorname{grad} g.$