Chapter 7 Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

P261 - Problem set 7.1

1. 2x2: $a_{11} \neq b_{11}, b_{12} \neq c_{12}$, 2x3: $d_{11} \neq e_{11}$
2. $a_{31} = 10,\ \ a_{13} = 81\ \ , a_{26} = 96,\ \ a_{33} = 0$
3. E1: 3x3, 3x4,
   E2: 3x7,
   E3: 2x2, 2x2, 2x2, 2x3, 2x3,
   E5:3x2
4. 1A: 4, 0, 1
   3A: a_{11}, a_{22}
   3B: 4, -1
5. $B = \frac{1}{5}A$,
   $B = \frac{1}{10}A$
6. $B = \frac{1}{1.609}A$
7. No. No(1x1 as exception?). Yes. Maybe not in math (how about 1x1?) but OK in python. No.
8. $2A + 4B = 4B + 2A = \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix}$
   B
   $0.4B - 4.2A = \begin{bmatrix} 0 & -6.4 & -16 \\ -23.2 & -19.8 & -19.4 \\ -5 & 1.6 & 11.8 \end{bmatrix}$
9. $3A = \begin{bmatrix} 0 & 6 & 12 \\ 18 & 15 & 15 \\ 3 & 0 & -9 \end{bmatrix}$
   $0.5B = \begin{bmatrix} 0 & 2.5 & 1 \\ 2.5 & 1.5 & 2 \\ -1 & 2 & -1 \end{bmatrix}$
   $3A + 0.5B = \begin{bmatrix} 0 & 8.5 & 13 \\ 20.5 & 16.5 & 17 \\ 2 & 2 & -10 \end{bmatrix}$
   3A+0.5B+C is not defined.

10. $(4 \bullet 3)A = 4(3A) = \begin{bmatrix} 0 & 24 & 48 \\ 72 & 60 & 60 \\ 12 & 0 & -36 \end{bmatrix}$

$14B - 3B = 11B = \begin{bmatrix} 0 & 55 & 22 \\ 55 & 33 & 44 \\ -22 & 44 & -22 \end{bmatrix}$

11. $8C + 10D = 2(5D + 4C) = \begin{bmatrix} 0 & 26 \\ 34 & 32 \\ 28 & -10 \end{bmatrix}$

$0.6C - 0.6D = 0.6(C - D) = \begin{bmatrix} 5.4 & 0.6 \\ -4.2 & 2.4 \\ -0.6 & 0.6 \end{bmatrix}$

12. $(C + D) + E = (D + E) + C = \begin{bmatrix} 1 & 5 \\ 6 & 8 \\ 6 & -2 \end{bmatrix}$

$0(C - E) + 4D = 4D = \begin{bmatrix} -16 & 4 \\ 20 & 0 \\ 8 & -4 \end{bmatrix}$

$A - 0C$: 3x3 can not minus 3x2, not defined

13. $(2 \bullet 7)C = 2(7C) = \begin{bmatrix} 70 & 28 \\ -28 & 56 \\ 14 & 0 \end{bmatrix}$

$-D + 0E = -D = \begin{bmatrix} 4 & -1 \\ -5 & 0 \\ -2 & 1 \end{bmatrix}$

$E - D + C + u$: Since EDC are 3x2 but u is 3x1, not defined.

14. $(5u + 5v) - \frac{1}{2}w = \begin{bmatrix} 5 \\ 30 \\ -10 \end{bmatrix}$

$-20(u + v) + 2w = -4[(5u + 5v) - \frac{1}{2}w] = \begin{bmatrix} -20 \\ -120 \\ 40 \end{bmatrix}$

$E - (u + v)$: 3x2 can not minus 3x1, not defined

$10(u + v) + w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

15. $(u + v) - w = u + (v - w) = \begin{bmatrix} 5.5 \\ 33 \\ -11 \end{bmatrix}$

$C + 0w$: 3x2 can not minus 3x1, not defined

$0E + u - v$: 3x2 can not minus 3x1, not defined

16. $15v - 3w - 0u = -3w + 15v = \begin{bmatrix} 0 \\ 135 \\ 0 \end{bmatrix}$

$D - u + 3C$: 3x2 can not minus 3x1, not defined

$8.5w - 11.1u + 0.4v = \begin{bmatrix} 25.45 \\ 256.2 \\ 119.1 \end{bmatrix}$

17. $u + v + w = \begin{bmatrix} -4.5 \\ -27 \\ 9 \end{bmatrix}$

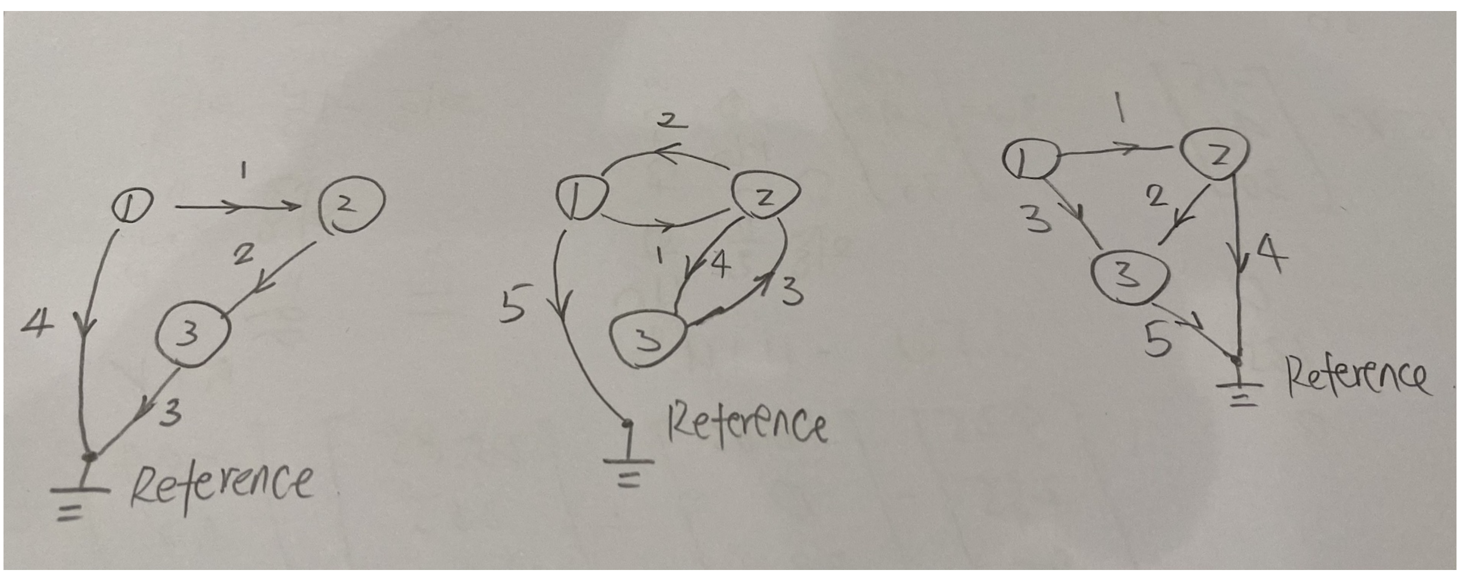
18. $p = 0 - u - v - w = \begin{bmatrix} 4.5 \\ 27 \\ -9 \end{bmatrix}$

19. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

20. b-1: $\begin{bmatrix} -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

b-2: $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

c:



P270 - Problem set 7.2

Example13. In the final stable situation(limit),

$I + C + R = 100$

$0.7C + 0.1I = C$
$0.2C + 0.9I + 0.2R = I$
$0.1C + 0.8R = R$
So we can get C=200/9, I=200/3, R=100/9.
Will revisit it after Sec. 8.2

1. Per definition, the number of the entries in the columns of the second matrix have to be same as the number of the entries in the rows of the first matrix. In short, if mxn matrix multiple pxq, then n=p. Or you won't be able to perform the dot product.
2. All entries or components are 0
3. No. All rows are proportional.
4. Min is 1 which is 0, and max is $n(n-1)+1$
   Take 3x3 as example, $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$
5. Min is 1 which is 0, and max is $\frac{n(n+1)}{2}$
   Take 3x3 as example, $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$
6. $U_1 + U_2, U_1U_2, U_1^2$ are upper triangular matrices.$L_1 + L_2$ is lower triangular.
7. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
8. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for any $m \geq 1, m \in N$. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ for any $m \geq 2, m \in N$.
9. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
10. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
11. $AB = AB^T = \begin{bmatrix} 10 & -14 & -6 \\ -5 & 7 & -12 \\ -5 & -1 & -4 \end{bmatrix}$

    $BA = B^TA = \begin{bmatrix} 10 & -5 & -15 \\ -14 & 7 & -3 \\ -2 & -4 & -4 \end{bmatrix}$
12. $AA^T = \begin{bmatrix} 29 & 8 & 6 \\ 8 & 41 & 12 \\ 6 & 12 & 9 \end{bmatrix}$, $A^2 = \begin{bmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{bmatrix}$, $BB^T = B^2 = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix}$
13. $CC^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 13 & -6 \\ 0 & -6 & 4 \end{bmatrix}$, $BC = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}$, CB not defined, $C^TB = \begin{bmatrix} -9 & 3 & 4 \\ -5 & -1 & 0 \end{bmatrix}$

14. $3A-2B=\begin{bmatrix}10&0&9\\0&1&18\\3&6&10\end{bmatrix}$, $(3A-2B)^T=3A^T-2B^T=\begin{bmatrix}10&0&3\\0&1&6\\9&18&10\end{bmatrix}$,

$(3A-2B)^Ta^T=\begin{bmatrix}10\\-2\\-27\end{bmatrix}$

15. Aa not defined, $Aa^T=\begin{bmatrix}8\\-4\\-3\end{bmatrix}$, $(Ab)^T=b^TA^T=\begin{bmatrix}7&-11&3\end{bmatrix}$

16. $BC=Problem13.2=\begin{bmatrix}-9&-5\\3&-1\\4&0\end{bmatrix}$, BC^T not defined, $Bb=\begin{bmatrix}0\\-8\\2\end{bmatrix}$, $b^TB=$ $\begin{bmatrix}0&-8&2\end{bmatrix}$

17. $ABC=\begin{bmatrix}-30&-18\\45&9\\5&-7\end{bmatrix}$, ABa not defined, $ABb=\begin{bmatrix}22\\4\\-12\end{bmatrix}$, Ca^T= not defined.

18. $ab=1$, $ba=\begin{bmatrix}3&-6&0\\1&-2&0\\-1&2&0\end{bmatrix}$, $aA=\begin{bmatrix}8&-4&-9\end{bmatrix}$, $Bb=problem16.3=\begin{bmatrix}0\\-8\\2\end{bmatrix}$

19. $1.5a+3.0b$ not defined. $1.5a^T+3.0b=\begin{bmatrix}4.5\\-2\\-1\end{bmatrix}$, $(A-B)b=Ab-Bb=\begin{bmatrix}7\\-3\\1\end{bmatrix}$

20. b^TAb=7, aBa^T=17, $aCC^T=\begin{bmatrix}-3&-24&12\end{bmatrix}$, $C^Tba=\begin{bmatrix}5&-10&0\\5&-10&0\end{bmatrix}$

21. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

22. $A=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$, $B=\begin{bmatrix}b_1&b_2&b_3\end{bmatrix}$, $AB=\begin{bmatrix}a_1b_1&a_1b_2&a_1b_3\\a_2b_1&a_2b_2&a_2b_3\\a_3b_1&a_3b_2&a_3b_3\end{bmatrix}$

23. $AB=A\begin{bmatrix}b_1&b_2&b_3\end{bmatrix}=\begin{bmatrix}Ab_1&Ab_2&Ab_3\end{bmatrix}$

24. $AB=BA$, $\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}2&3\\3&4\end{bmatrix}=\begin{bmatrix}2&3\\3&4\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$,

$2a_{11}+3a_{12}=2a_{11}+3a_{21}\Rightarrow a_{12}=a_{21}$

$3a_{11}+4a_{12}=2a_{12}+3a_{22}\Rightarrow 3a_{11}+2a_{12}=3a_{22}$

$2a_{21}+3a_{22}=3a_{11}+4a_{21}$,

$3a_{21}+4a_{22}=3a_{12}+4a_{22}$

Let $A=\begin{bmatrix}x&y\\y&\frac{3x+2y}{3}\end{bmatrix}$,

Check: $AB=BA=\begin{bmatrix}2x+3y&3x+4y\\3x+4y&4x+5\frac{2}{3}y\end{bmatrix}$

25. a) Obvious.

b) $C=[c_{ij}]$, $C^T=[c_{ji}]$

$D = C + C^T = [d_{ij}] = [c_{ij} + c_{ji}] = [c_{ji} + c_{ij}] = [d_{ji}]$, so D is symmetric
$E = C - C^T = [e_{ij}] = [c_{ij} - c_{ji}] = -[c_{ji} - c_{ij}] = -[e_{ji}]$, so E is skew-symmetric.
Let $S = \frac{1}{2}D, T = \frac{1}{2}E$
$S + T = \frac{1}{2}(D + E) = \frac{1}{2}(C + C^T + C - C^T) = C$

$$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}, A^T = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 6 & 2 \end{bmatrix},$$

$$S = \frac{1}{2}(A + A^T) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix}, T = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B^T = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$S = \frac{1}{2}(B + B^T) = B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, T = \frac{1}{2}(B - B^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) symmetric: $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}], ..., M = [m_{ij}] = [m_{ji}]$
$aA + bB + ... + mM = a[a_{ij}] + b[b_{ij}] + ... + m[m_{ij}] = a[a_{ji}] + b[b_{ji}] + +... + m[m_{ji}]$.
Skew-symmetric: $A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}], ..., M = [m_{ij}] = -[m_{ji}]$
$aA + bB + ... + mM = a[a_{ij}] + b[b_{ij}] + ... + m[m_{ij}] = -(a[a_{ji}] + b[b_{ji}] + +... + m[m_{ji}])$
.

d) $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}]$
$AB = [a_p b_q]$, if AB is symmetric, then $AB = [a_p b_q] = [a_q b_p] = [b_p a_q] = BA$
vice verse.

e)$A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}]$
$AB = [a_p b_q]$, if AB is skew-symmetric, then $AB = [a_p b_q] = -[a_q b_p] = -[b_p a_q] = -BA$
vice verse.

26. First day, status = $\begin{bmatrix} N \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, stochastic matrix = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$

    Second day = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$

    Two days after today = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$

    Three days after today = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.722 \\ 0.278 \end{bmatrix}$

    The limit of N is $\frac{5}{7}$
27. Leave for future

28. Present = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 1200 \\ 98800 \end{bmatrix}$, stochastic matrix = $\begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix}$

After 1 season = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1200 \\ 98800 \end{bmatrix} = \begin{bmatrix} 1278 \\ 98722 \end{bmatrix}$, increase

After 2 seasons = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1278 \\ 98722 \end{bmatrix} = \begin{bmatrix} 1344 \\ 98656 \end{bmatrix}$, increase

After 3 seasons = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1344 \\ 98656 \end{bmatrix} = \begin{bmatrix} 1407 \\ 98593 \end{bmatrix}$, increase

29. $p = \begin{bmatrix} 35 \\ 62 \\ 30 \end{bmatrix}$

$v = Ap = \begin{bmatrix} 24,920 \\ 25,940 \end{bmatrix}$

30.$y = Ax$

$y_1 = x_1 \cos\theta - x_2 \sin\theta, y_2 = x_1 \sin\theta + x_2 \cos\theta$

$|y|^2 = (x_1 \cos\theta - x_2 \sin\theta)^2 + (x_1 \sin\theta + x_2 \cos\theta)^2 = x_1^2 + x_2^2 = |x|^2$

$\cos\alpha = \frac{x*y}{|x||y|} = \frac{x_1^2 \cos\theta + x_2^2 \cos\theta}{x_1^2 + x_2^2} = \cos\theta$

so x and y have the same length, and from x to y is counterclockwise rotate of θ

b)$AA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} =$
$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

c)$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} =$
$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

d)$[x_1, x_2, x_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = [3x_1, x_2, \frac{1}{2}x_3]$

$[x_1, x_2, x_3] \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} = [cx_1, cx_2, cx_3]$, Scalar matrix will amplify or squeeze the picture by c.

e) $[x_1, x_2, x_3]\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix} = [x_1, x_2\cos\theta + x_3\sin\theta, -x_2\sin\theta + x_3\cos\theta]$

$x_1$remain the same. counterclockwise rotation of the Cartesian coordinate system x_2x_3 in the plane about the origin by angle of θ

$$[x_1, x_2, x_3]\begin{bmatrix}\cos\varphi & 0 & -\sin\varphi\\ 0 & 1 & 0\\ \sin\varphi & 0 & \cos\varphi\end{bmatrix} = [x_1\cos\varphi + x_3\sin\varphi, x_2, -x_1\sin\varphi + x_3\cos\varphi]$$

$x_2$remain the same. counterclockwise rotation of the Cartesian coordinate system x_1x_3 in the plane about the origin by angle of φ

$$[x_1, x_2, x_3]\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix} = [x_1\cos\psi + x_2\sin\psi, -x_1\sin\psi + x_2\cos\psi, x_3]$$

$x_3$remain the same. counterclockwise rotation of the Cartesian coordinate system x_1x_2 in the plane about the origin by angle of ψ