Chapter 15 - Power Series, Taylor Series

Selected Problem set 15.1

15.1 1. 5, 9, 17. 19. 23. 25

bound.
Convergent
limit

Convergent, hence bounded

1–10 SEQUENCES

Is the given sequence $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find its limit points. Show your work in detail.

1. $z_n = (1+i)^{2n}/2^n$ **2.** $z_n = (3+4i)^n/n!$

1. $\frac{z_{n+1}}{z_n} = \frac{(1+i)^{2(n+1)}}{2^{n+1}} \cdot \frac{2^n}{(1+i)^{2n}} = \frac{(1+i)^2}{2} = \frac{1+i^2+2i}{2} = i$

$z_1 = \frac{(1+i)^2}{2} = i$, $z_2 = -1$, $z_3 = -i$, $z_4 = 1$, $z_5 = z_1$

bounded.

Theorem 7. $\left|\frac{z_{n+1}}{z_n}\right| = 1 \geq 1$ diverges

5. $z_n = (-1)^n + 10i$

$z_1 = -1+10i$ $z_2 = 1+10i$ $z_3 = z_1$ $z_4 = z_2$ ...

bounded.

for $t = 1, 2, \cdots$

$$\left|\frac{z_{2t+1}}{z_{2t}}\right| = \left|\frac{(1+10i)^2}{(-1+10i)(1+10i)}\right| = \left|\frac{1-100+20i}{-100-1}\right| = \left|+\frac{99}{101} - \frac{20}{101}i\right|$$

≥ 1

diverges.

9. $z_n = (3+3i)^{-n}$

$|z_1| = \left|\frac{1}{6}(1-i)\right| = \frac{\sqrt{2}}{6}$ $\quad |z_2| = \left|-\frac{i}{18}\right| = \frac{1}{18}$

$|z_3| = \left|-\frac{1}{108}(1+i)\right| = \frac{\sqrt{2}}{108}$

$|z_{n+1}| = \left|z_n \cdot \left(\frac{1}{3+3i}\right)\right| = |z_n| \cdot \frac{\sqrt{2}}{6} \le |z_1| = \frac{\sqrt{2}}{6}$ bounded.

$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(3+3i)^{-(n+1)}}{(3+3i)^{-n}}\right| = \left|(3+3i)^{-1}\right| = \frac{\sqrt{2}}{6} < 1$ converges

16–25 SERIES

Is the given series convergent or divergent? Give a reason. Show details.

16. $\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$ **17.** $\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$

17. $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{(-1)^n}\right| = \left|(-1) \cdot \frac{\ln n}{\ln(n+1)}\right| = \left|\frac{\ln n}{\ln(n+1)}\right|$

$n^{n+1} > (n+1)^n$ for $n \ge 3$. $\Rightarrow \frac{\ln n}{\ln(n+1)} > \frac{n}{n+1}$

$\Rightarrow \left|\frac{z_{n+1}}{z_n}\right| > \left|\frac{n}{n+1}\right|$ diverges

19. $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{(n+1)^2 - i} \cdot \frac{n^2 - i}{i^n}\right| = \left| i \cdot \frac{n^2-i}{(n+1)^2 - i}\right|$$

$$= \left|\frac{(n^2-i)[(n+1)^2+i]}{[(n+1)^2-i][(n+1)^2+i]}\right|$$

$$= \left|\frac{(n^2-i)(n^2+2n+1+i)}{(n+1)^4+1}\right|$$

$$= \left|\frac{n^4+2n^3+n^2+\cancel{in^2}-\cancel{in^2}-2ni-i+1}{(n+1)^4+1}\right|$$

$$= \left|\frac{1+\frac{2}{n}+\frac{1}{n^2}-\frac{2i}{n^3}+\frac{1-i}{n^4}}{(1+\frac{1}{n})^4+\frac{1}{n}}\right|$$

$\lim \to 1 \quad \Rightarrow$ don't know

$$|z_n|^{\frac{1}{n}} = \left|\frac{i}{\sqrt[n]{n^2-i}}\right| = \left|\frac{i}{(n^2+1)^{\frac{1}{n}}}\right| < \left|\frac{i}{n^{\frac{2}{n}}}\right| = 1$$

converge absolutely

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1+i)^{2n}}{(2n)!}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}(1+i)^{2n+2}}{(2n+2)!} \cdot \frac{2n!}{(-1)^n(1+i)^{2n}}\right|$$

$$= \left|\frac{-1\cdot(1+i)^2}{(2n+2)(2n+1)}\right|$$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0 < 1$$ Converge absolutely

25. $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{n+1}\cdot\frac{n}{i^n}\right| = \left|\frac{n\,i}{n+1}\right| = \left|\frac{n}{n+1}\right|$$ $\lim \Rightarrow 1$ not sure

$$|z_n^{\frac{1}{n}}| = \left|\frac{1}{\sqrt[n]{n}}\right| \Rightarrow \lim_{n\to\infty}|z_n|^{\frac{1}{n}} = 1$$ not sure

Per example 3. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ conditionally converge.

So $i, -1, -i, 1$.

$\{z_{4t+1} + z_{4t+3}\}$ t is even $\Rightarrow$ converge. x_i

$\{z_{4t+2} + z_{4t+4}\}$ t is odd $\Rightarrow$ converge y

$\Rightarrow S_n$ jumping from x_i to y, diverge?

Selected Problem set 15.2

15.2 7. 9. 11. 13. 17

6–18 RADIUS OF CONVERGENCE

Find the center and the radius of convergence.

6. $\sum_{n=0}^{\infty} 4^n (z+1)^n$ **7.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z - \frac{1}{2}\pi\right)^{2n}$

$$\sqrt{\left|\frac{a_{n+1}(z-\frac{\pi}{2})^{2n+2}}{a_n(z-\frac{\pi}{2})^{2n}}\right|} = \sqrt{\left|\frac{a_{n+1}}{a_n}\right| \cdot \left|(z-\frac{\pi}{2})^2\right|}$$

$$= \sqrt{\left|\frac{(-1)^{n+1}}{(2n+2)!} \cdot \frac{2n!}{(-1)^n}\right|} \cdot \left|(z-\frac{\pi}{2})\right|$$

$$= \sqrt{\left|\frac{1}{(2n+2)(2n+1)}\right|} \cdot \left|(z-\frac{\pi}{2})\right|$$

$L = L^* \cdot (z-\frac{\pi}{2})$. $\lim_{n\to\infty} L^* = 0$, $\lim_{n\to+\infty} L = 0$

Converge for all z by the ratio test

9. $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n}$

$$\sqrt{\left|\frac{a_{n+1}(z-i)^{2n+2}}{a_n(z-i)^{2n}}\right|} = \sqrt{\frac{a_{n+1}}{a_n}} \cdot |z-i|$$

$$= \sqrt{\frac{(n+1)\cdot n}{3^{n+1}} \cdot \frac{3^n}{n(n-}} \cdot |z-i|$$

$$= \sqrt{\frac{n+1}{n-1} \cdot \frac{1}{3}} \quad |z-i|$$

$L = L^* \ |z-i|$. $\lim_{n\to\infty} L^* = \frac{\sqrt{3}}{3}$. $\lim_{n\to+\infty} L = \sqrt{3}$ Center $= i$

11. $\sum_{n=0}^{\infty} \left(\frac{2-i}{1+5i}\right) z^n$

$|z| \geq 1$: divergence.

$|z| < 1$ Converge

$$S_n = \left(\frac{2-i}{1+5i}\right) \cdot z^0 + \left(\frac{2-i}{1+5i}\right) \cdot z + \cdots \left(\frac{2-i}{1+5i}\right) z^n$$

$$z S_n = \qquad \left(\frac{2-i}{1+5i}\right) z^1 + \cdots \left(\frac{2-i}{1+5i}\right) \cdot z^{n+1}$$

$$(1-z) S_n = \frac{2-i}{1+5i} \cdot 1 - \left(\frac{2-i}{1+5i}\right) \cdot z^{n+1}$$

$n \to +\infty \quad z^{n+1} \to 0$

$$S_n = \frac{2-i}{(1-z)(1+5i)}$$

Center: 0

Radiu: 1

13. $\sum_{n=0}^{\infty} 16^n (z+i)^{4n}$

$$\sqrt[4]{\left|\frac{a_{n+1}(z+i)^{4n+4}}{a_n (z+i)^{4n}}\right|} = \sqrt[4]{\left|\frac{a_{n+1}}{a_n}\right|} \cdot |z+i|$$

$$= \sqrt[4]{16} \cdot |z+i| = 2|z+i|$$

$\lim_{n\to\infty} L^* = 2 \quad \lim_{n\to\infty} L = \frac{1}{2} \quad$ Center $-i$

17. $\sum_{n=1}^{\infty} \frac{2^n}{n(n+1)} z^{2n+1}$

$$\sqrt{\left|\frac{a_{n+1} \cdot z^{2n+3}}{a_n \cdot z^{2n+1}}\right|} = \sqrt{\frac{a_{n+1}}{a_n}} \cdot |z|$$

$$= \sqrt{\frac{2^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{2^n}} \; |z|$$

$$= \sqrt{\frac{2n}{n+2}} \cdot |z|$$

$\lim\limits_{n \to r} L^* = \sqrt{2}$ $\quad$ $\lim\limits_{n \to \infty} L = \frac{\sqrt{2}}{2}$ $\quad$ center : 0

Selected Problem set 15.3