Chapter 9 Vector Differential Calculus. Grad, Div, Curl

P360 - Problem set 9.1

Problem 1-5

PS 9.1

1–5 COMPONENTS AND LENGTH

Find the components of the vector **v** with initial point P and terminal point Q. Find $|\mathbf{v}|$. Sketch $|\mathbf{v}|$. Find the unit vector **u** in the direction of **v**.

1. P: (1, 1, 0), Q: (6, 2, 0)
2. P: (1, 1, 1), Q: (2, 2, 0)
3. P: (−3.0, 4,0, −0.5), Q: (5.5, 0, 1.2)
4. P: (1, 4, 2), Q: (−1, −4, −2)
5. P: (0, 0, 0), Q: (2, 1, −2)

1. $\vec{V} = [5, 1, 0]$

$|V| = \sqrt{26}$

$\vec{u} = [\frac{5\sqrt{26}}{26}, \frac{\sqrt{26}}{26}, 0]$

2. $\vec{V} = [1, 1, -1]$

$|V| = \sqrt{3}$

$\vec{u} = [\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}]$

3. $\vec{V} = [8.5, -4, 1.7]$

$|V| = \sqrt{72.25 + 16 + 2.89}$

$= \sqrt{91.14}$

$\vec{u} = [\frac{8.5}{\sqrt{91.14}}, \frac{-4}{\sqrt{91.14}}, \frac{1.7}{\sqrt{91.14}}]$

4) $\vec{V} = [-2, -8, -4]$

$|V| = \sqrt{4 + 64 + 16}$

$= \sqrt{84}$

$\vec{u} = [-\frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}]$

5) $\vec{V} = [2, 1, -2]$

$|V| = \sqrt{4 + 1 + 4} = 3$

$\vec{u} = [\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}]$

6–10 **Find the terminal point Q** of the vector $\mathbf{v}$ with components as given and initial point P. Find $|\mathbf{v}|$.

6. $4, 0, 0;\quad P\colon (0, 2, 13)$

7. $\frac{1}{2}, 3, -\frac{1}{4};\quad P\colon (\frac{7}{2}, -3, \frac{3}{4})$

8. $13.1, 0.8, -2.0;\quad P\colon (0, 0, 0)$

9. $6, 1, -4;\quad P\colon (-6, -1, -4)$

10. $0, -3, 3;\quad P\colon (0, 3, -3)$

Latex

Mistake

11–18 **ADDITION, SCALAR MULTIPLICATION**

Let $\mathbf{a} = [3, 2, 0] = 3\mathbf{i} + 2\mathbf{j}$; $\mathbf{b} = [-4, 6, 0] = 4\mathbf{i} + 6\mathbf{j}$, $\mathbf{c} = [5, -1, 8] = 5\mathbf{i} - \mathbf{j} + 8\mathbf{k}$, $\mathbf{d} = [0, 0, 4] = 4\mathbf{k}$. Find:

6. Q=[4, 2, 13], |v|=4

7. Q=[4, 0. 1/2], |v|=$\sqrt{149}/4$

8. Q=[13.1, 0.8, -2.0], |v|=$\sqrt{171.61+0.64+4}$=$\sqrt{176.25}$
9. Q=[0, 0, 0], |v|=$\sqrt{53}$
10. Q=[0, 0, 0], |v|=$3\sqrt{2}$
11. 2a=[6, 4, 0], 1/2a=[3/2, 1, 0], -a = [-3, -2, 0]
12. (a+b)+c=a+(b+c)=[4, 7, 8] $b\ is\ not\ consistant.$
13. b+c=c+b = [1, 5, 8]
14. 3c-6d=3(c-2d)=[15, -3, 0]
15. 7(c-b)=7c-7b=7*[9, -7, 8]=[63, -49, 56]
16. $\frac{9}{2}a-3c$=$9(\frac{1}{2}a-\frac{1}{3}c)$=[-3/2, 12, -24]
17. (7-3)a=7a-3a=4a=[12, 8, 0]
18. 4a+3b = [0, 26, 0], -4a-3b=-(4a+3b)=[0, -26, 0]
19. 12-associative, 13-commutative, 14-16 scalar multiplication is distributive.
20. $a+b=[a_1,a_2,...,a_n]+[b_1,b_2,...,b_n]=[a_1+b_1,a_2+b_2,...,a_n+b_n]=[b_1+a_1,b_2+a_2,...,b_n+a_n]=b+a$

    The rest can be approved in a simliar way.

21–25 **FORCES, RESULTANT**

Find the resultant in terms of components and its magnitude.

21. $\mathbf{p} = [2, 3, 0]$, $\mathbf{q} = [0, 6, 1]$, $\mathbf{u} = [2, 0, -4]$

22. $\mathbf{p} = [1, -2, 3]$, $\mathbf{q} = [3, 21, -16]$, $\mathbf{u} = [-4, -19, 13]$

23. $\mathbf{u} = [8, -1, 0]$, $\mathbf{v} = [\frac{1}{2}, 0, \frac{4}{3}]$, $\mathbf{w} = [-\frac{17}{2}, 1, \frac{11}{3}]$

24. $\mathbf{p} = [-1, 2, -3]$, $\mathbf{q} = [1, 1, 1]$, $\mathbf{u} = [1, -2, 2]$

25. $\mathbf{u} = [3, 1, -6]$, $\mathbf{v} = [0, 2, 5]$, $\mathbf{w} = [3, -1, -13]$

21) $\vec{r} = \vec{p} + \vec{q} + \vec{u}$
$= [4, 9, -3]$
$|\vec{r}| = \sqrt{16+81+9} = \sqrt{106}$

22) $\vec{r} = \vec{p} + \vec{q} + \vec{u}$
$= [0, 0, 0]$
$|\vec{r}| = 0$

23) $\vec{r} = \vec{u} + \vec{v} + \vec{w}$
$= [0, 0, 5]$
$|\vec{r}| = 5$

24) $\vec{r} = \vec{p} + \vec{q} + \vec{u}$
$= [1, 1, 0]$
$|\vec{r}| = \sqrt{2}$

25) $\vec{r} = \vec{u} + \vec{v} + \vec{w}$
$= [6, 2, -14]$
$|\vec{r}| = \sqrt{36+4+196} = \sqrt{236}$

26–37 **FORCES, VELOCITIES**

26. Equilibrium. Find **v** such that **p**, **q**, **u** in Prob. 21 and **v** are in equilibrium.

27. Find **p** such that **u**, **v**, **w** in Prob. 23 and **p** are in equilibrium.

28. Unit vector. Find the unit vector in the direction of the resultant in Prob. 24.

26) $\vec{v} = 0 - \vec{r} = [-4, -9, 3]$

27) $\vec{p} = 0 - \vec{r} = [0, 0, -5]$

28) $\vec{u} = \frac{\vec{r}}{|\vec{r}|} = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0]$

29. Restricted resultant. Find all **v** such that the resultant of **v**, **p**, **q**, **u** with **p**, **q**, **u** as in Prob. 21 is parallel to the *xy*-plane.

30. Find **v** such that the resultant of **p**, **q**, **u**, **v** with **p**, **q**, **u** as in Prob. 24 has no components in *x*- and *y*-directions.

31. For what *k* is the resultant of [2, 0, −7], [1, 2, −3], and [0, 3, *k*] parallel to the *xy*-plane?

32. If $|\mathbf{p}| = 6$ and $|\mathbf{q}| = 4$, what can you say about the magnitude and direction of the resultant? Can you think of an application to robotics?

33. Same question as in Prob. 32 if $|\mathbf{p}| = 9$, $|\mathbf{q}| = 6$, $|\mathbf{u}| = 3$.

29 Vector start from origin. parallel to x-y means no component in z-direction.

Let result

$\vec{z} = \vec{v} + \vec{p} + \vec{q} + \vec{u} = [x, y, 0]$

So x and y are arbitrary

$\vec{v} = \vec{z} - [4, 9, -3]$

$= [x-4, y-9, 3]$

Let $v_1 = x-4$, $v_2 = y-9$.

So $\vec{v} = [v_1, v_2, 3]$

where v_1 and v_2 are arbitrary

30. $\vec{v} = [0, 0, c] - [1, 1, 0]$

$= [-1, -1, c]$ where c is a constant.

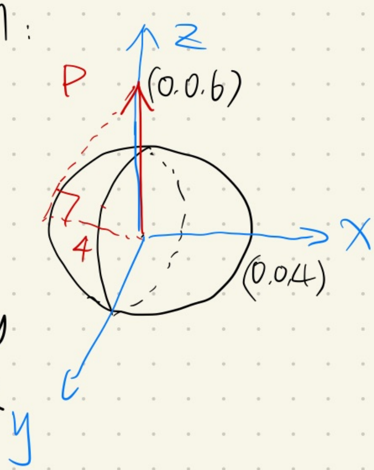
31 $-7-3+k=0$

$k=10$

32. magnitude: min=2, max=10

direction:

A sphere with R=10 and with a r=2 empty sphere inside



Robotic: Vision, the area where Robot can see/detect or scan?

33. magnitude: min=0, max=18

direction

enhanced ice cream without blind spot?

A solid sphere with R=18, No blind point.

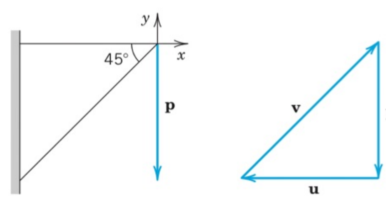
$|p+q+u| \le 18$

34. Relative velocity. If airplanes A and B are moving southwest with speed $|\mathbf{v}_A| = 550$ mph, and northwest with speed $|\mathbf{v}_B| = 450$ mph, respectively, what is the relative velocity $\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$ of B with respect to A?

35. Same question as in Prob. 34 for two ships moving northeast with speed $|\mathbf{v}_A| = 22$ knots and west with speed $|\mathbf{v}_B| = 19$ knots.

36. Reflection. If a ray of light is reflected once in each of two mutually perpendicular mirrors, what can you say about the reflected ray?

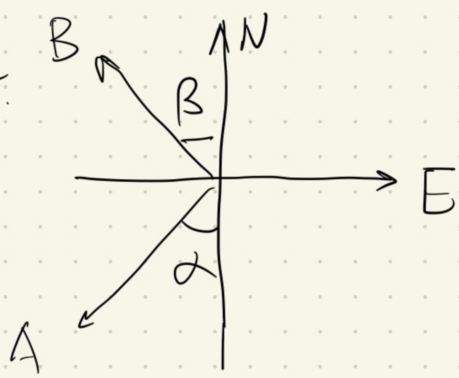
37. Force polygon. Truss. Find the forces in the system of two rods (*truss*) in the figure, where $|\mathbf{p}| = 1000$ nt. *Hint.* Forces in equilibrium form a polygon, the *force polygon*.



Truss   Force polygon

Problem 37

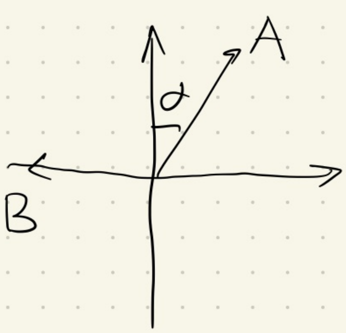
34.



$V = V_B - V_A$ on north east.

$[550 \sin\alpha - 450 \sin\beta,$
$550 \cos\alpha - 450 \cos\beta]$

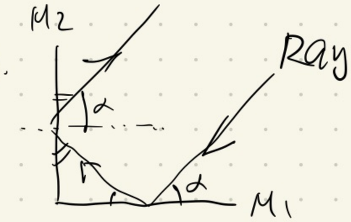
35



$V = V_B - V_A$ on northeast

$[22 \sin\alpha - 19 \quad 22\cos\alpha]$

36.



Reverse the direction.

37. $P: [0, -1000]$

$U: [-1000, 0]$

$V: [1000, 1000]$

34. 35: it seems assume α β are 45°
maybe this is by default

38. TEAM PROJECT. Geometric Applications. To increase your skill in dealing with vectors, use vectors to prove the following (see the figures).

(a) The diagonals of a parallelogram bisect each other.

(b) The line through the midpoints of adjacent sides of a parallelogram bisects one of the diagonals in the ratio 1 : 3.

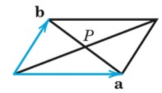
(c) Obtain (b) from (a).

(d) The three medians of a triangle (the segments from a vertex to the midpoint of the opposite side) meet at a single point, which divides the medians in the ratio 2 : 1.

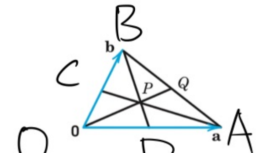
(e) The quadrilateral whose vertices are the midpoints of the sides of an arbitrary quadrilateral is a parallelogram.

(f) The four space diagonals of a parallelepiped meet and bisect each other.

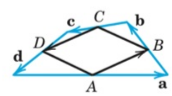
(g) The sum of the vectors drawn from the center of a regular polygon to its vertices is the zero vector.



Team Project 38(a)



Team Project 38(d)



Team Project 38(e)

$\overrightarrow{OC} = \frac{1}{2}b$

$\overrightarrow{OD} = \frac{1}{2}a$

$\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AB}$

Vector way ?

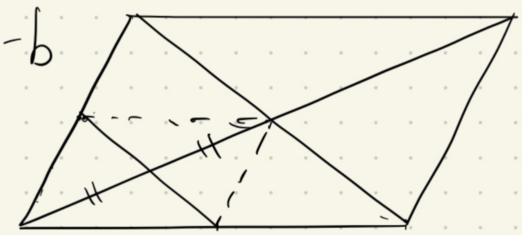
$\triangle CDP \sim \triangle ABP$

$\overrightarrow{DP} = \frac{1}{2}\overrightarrow{PB}$

38-a $\frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}(a+b)$

$\frac{1}{2}a - \frac{1}{2}b = \frac{1}{2}(a-b)$

38-b



$\frac{1}{4}a - \frac{1}{4}b = \frac{1}{4}(a-b)$

$\frac{1}{4} : \frac{3}{4} = 1:3$

38-c : obvious.

38-d. $\overrightarrow{AB} = a - b$

$\overrightarrow{AQ} = \frac{1}{2}(a-b)$

$\overrightarrow{QO} = a + \frac{1}{2}(a-b)$

38-e $a - d = c - b$

$\frac{1}{2}(a-d) = \frac{1}{2}(c-b)$

$\overrightarrow{AD} = \overrightarrow{BC}$

same $\Rightarrow$ $\overrightarrow{AB} = \overrightarrow{DC}$

38-f.

38-g.

P367 - Problem set 9.2

PS 9-2

1–10 INNER PRODUCT

Let $\mathbf{a} = [1, -3, 5]$, $\mathbf{b} = [4, 0, 8]$, $\mathbf{c} = [-2, 9, 1]$. Find:

1. $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{b} \cdot \mathbf{a}$, $\mathbf{b} \cdot \mathbf{c}$
2. $(-3\mathbf{a} + 5\mathbf{c}) \cdot \mathbf{b}$, $15(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b}$
3. $|\mathbf{a}|$, $|2\mathbf{b}|$, $|-\mathbf{c}|$
4. $|\mathbf{a} + \mathbf{b}|$, $|\mathbf{a}| + |\mathbf{b}|$
5. $|\mathbf{b} + \mathbf{c}|$, $|\mathbf{b}| + |\mathbf{c}|$
6. $|\mathbf{a} + \mathbf{c}|^2 + |\mathbf{a} - \mathbf{c}|^2 - 2(|\mathbf{a}|^2 + |\mathbf{c}|^2)$
7. $|\mathbf{a} \cdot \mathbf{c}|$, $|\mathbf{a}||\mathbf{c}|$
8. $5\mathbf{a} \cdot 13\mathbf{b}$, $65\mathbf{a} \cdot \mathbf{b}$
9. $15\mathbf{a} \cdot \mathbf{b} + 15\mathbf{a} \cdot \mathbf{c}$, $15\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
10. $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c})$, $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c}$

1. $a \cdot b = 4+0+40 = 44$

$b \cdot a = 44$

$b \cdot c = -8+0+8 = 0$

2. $(-3a+5c) \cdot b = (-13, 54, -10) \cdot b$

$= -52+0-80 = -132$

$15(a-c) b = 15(-3, -12, 4) \cdot b$

$= 15 \cdot (-12+0+32) = 300$

3. $|a| = \sqrt{1+9+25} = \sqrt{35}$

$|2b| = 2 \cdot \sqrt{16+64} = 8\sqrt{5}$

$|-c| = \sqrt{4+81+1} = \sqrt{86}$

4. $|a+b| = \sqrt{25+9+169} = \sqrt{203}$

$|a|+|b| = \sqrt{35} + \sqrt{16+64} = \sqrt{35} + 4\sqrt{5}$

5. $|b+c| = \sqrt{2^2+9^2+9^2} = \sqrt{166}$

$|b|+|c| = 4\sqrt{5} + \sqrt{86}$

6. $(1+36+36)+(9+144+16)$

$-2(35+86) = 0$

7. $|a \cdot c| = |-2-27+5| = 24$

$|a||c| = \sqrt{35} \cdot \sqrt{86}$

8. $5a \cdot 13b = 65a \cdot b = 2860$

9. $15a \cdot b + 15a \cdot c = 15a(b+c)$

$= 15 \cdot a \cdot (2, 9, 9) = 300$

10. $a \cdot (b-c) = a \cdot (6, -9, 7)$

$= 68$

$(a-b) \cdot c = (-3, -3, -3) c$

$= 6-27-3 = -24$

11–16 GENERAL PROBLEMS

11. What laws do Probs. 1 and 4–7 illustrate?

12. What does $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ imply if $\mathbf{u} = \mathbf{0}$? If $\mathbf{u} \neq \mathbf{0}$?

13. Prove the Cauchy–Schwarz inequality.

14. Verify the Cauchy–Schwarz and triangle inequalities for the above **a** and **b**.

15. Prove the parallelogram equality. Explain its name.

16. **Triangle inequality.** Prove Eq. (7). *Hint.* Use Eq. (3) for $|\mathbf{a} + \mathbf{b}|$ and Eq. (6) to prove the square of Eq. (7), then take roots.

11-1 $a \cdot b = b \cdot a$

$ab = 0$. orthogonality

4-5) $|\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}|$ **(Triangle inequality)**.

6) $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$ *(Parallelogram equality)*.

7) $|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}||\mathbf{b}|$ (*Cauchy–Schwarz inequality*).

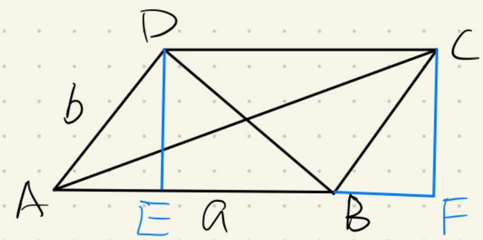
12. if $u=0$, We know nothing about $u.w$

if $u \neq 0$, $P = \dfrac{a \cdot b}{|b|}$. So $v.w$ belongs to a set B that the projections of u on the vector muliple the magnitude of the vector is a constant

13. projection $\le$ itself. $(\cos\theta \le 1)$

14. 4-5, and 7

15



1) $AE = BF$, $DE = CF$.

2) $AC^2 + BD^2 = (a + BF)^2 + CF^2$

$+ (a - AE)^2 + DE^2$

$= 2(a^2 + b^2)$

16. $|a+b|^2 = (a+b)(a+b)$

$= a^2 + b^2 + 2ab$

$\le |a|^2 + |b|^2 + 2|a||b|$

$= (|a| + |b|)^2$

17–20 **WORK** Mistake

Find the work done by a force **p** acting on a body if the body is displaced along the straight segment $\overline{AB}$ from A to B. Sketch $\overline{AB}$ and **p**. Show the details.

17. $\mathbf{p} = [2, 5, 0]$, A: (1, 3, 3), B: (3, 5, 5)

18. $\mathbf{p} = [-1, -2, 4]$, A: (0, 0, 0), B: (6, 7, 5)

19. $\mathbf{p} = [0, 4, 3]$, A: (4, 5, −1), B: (1, 3, 0)

20. $\mathbf{p} = [6, -3, -3]$, A: (1, 5, 2), B: (3, 4, 1)

21. Resultant. Is the work done by the resultant of two forces in a displacement the sum of the work done by each of the forces separately? Give proof or counterexample.

22–30 **ANGLE BETWEEN VECTORS**

Let $\mathbf{a} = [1, 1, 0]$, $\mathbf{b} = [3, 2, 1]$, and $\mathbf{c} = [1, 0, 2]$. Find the angle between:

22. **a**, **b**

23. **b**, **c**

24. **a** + **c**, **b** + **c**

17. $W = P \cdot d = P \cdot (2, 2, 2) = 14$

18 $W = p \cdot (6, 7, 5) = 0$

19. $W = p \cdot (-3, -2, 1) = -5$

negative, because of direction.

20. $W = p \cdot (2, -1, -1) = 18$

21. Yes.

$W = P_1 \cdot d + P_2 \cdot d$

$= (P_1 + P_2) \cdot d$

22. $\cos\alpha = \frac{ab}{|a||b|} = \frac{3+2}{\sqrt{2} \cdot \sqrt{14}}$

$\alpha = \arccos\left(\frac{5\sqrt{7}}{14}\right)$

23. $\cos\alpha = \frac{bc}{|b||c|} = \frac{3+2}{\sqrt{14}\sqrt{5}}$

$\alpha = \arccos\left(\frac{\sqrt{70}}{14}\right)$

24. $\cos\alpha = \frac{(a+c)(b+c)}{|a+c| \cdot |b+c|}$

$= \frac{(4,3,1)(4,2,3)}{\sqrt{16+9+1} \cdot \sqrt{16+4+9}}$

$= \frac{16+6+3}{\sqrt{26} \cdot \sqrt{29}} = \frac{25}{\sqrt{26} \cdot \sqrt{29}}$

$\alpha = \arccos\left(\frac{25}{\sqrt{26} \cdot \sqrt{29}}\right)$

25. What will happen to the angle in Prob. 24 if we replace $\mathbf{c}$ by $n\mathbf{c}$ with larger and larger n?

26. Cosine law. Deduce the law of cosines by using vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - \mathbf{b}$.

27. Addition law. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha \sin\beta$. Obtain this by using $\mathbf{a} = [\cos\alpha, \sin\alpha]$, $\mathbf{b} = [\cos\beta, \sin\beta]$ where $0 \leqq \alpha \leqq \beta \leqq 2\pi$.

28. Triangle. Find the angles of the triangle with vertices A: (0, 0, 2), B: (3, 0, 2), and C: (1, 1, 1). Sketch the triangle.

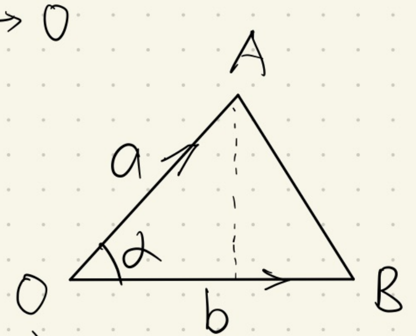
29. Parallelogram. Find the angles if the vertices are (0, 0), (6, 0), (8, 3), and (2, 3).

30. Distance. Find the distance of the point A: (1, 0, 2) from the plane P: $3x + y + z = 9$. Make a sketch.

25. $\lim_{n\to\infty} \dfrac{(a+nc)(b+nc)}{|a+nc|\cdot|b+nc|} = \dfrac{c\cdot c}{|c|^2} = 1$

$\alpha \to 0$

26.



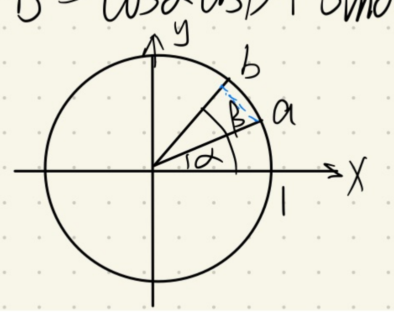
$(\overrightarrow{BA})^2 = (a-b)^2 =$

$(a\sin\alpha)^2 + (b - a\cos\alpha)^2$

$= a^2\sin^2\alpha + b^2 + a^2\cos^2\alpha - 2ab\cos\alpha$

$= a^2 + b^2 - 2ab\cos\alpha$.

27. $a\cdot b = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



the projection of a on b is

$p = a\cdot b = \cos(\beta - \alpha) = \cos(\alpha - \beta)$

28. $\overrightarrow{AB} = (3, 0, 0) = a$

$\overrightarrow{BC} = (-2, 1, -1) = b$

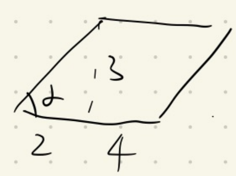
$\overrightarrow{CA} = (-1, -1, 1) = c$

$\cos\alpha = \dfrac{-ab}{|a||b|} = \dfrac{-6}{3\cdot\sqrt{6}} = \dfrac{\sqrt{6}}{3}$

$\cos\beta = \dfrac{-bc}{|b||c|} = \dfrac{2-1-1}{\sqrt{\ }\sqrt{\ }} = 0$

$\cos\gamma = \dfrac{-ca}{|c||a|} = \dfrac{+3}{3\cdot\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

29. $\tan\alpha = \dfrac{3}{2}$ $\alpha = \arctan\dfrac{3}{2}$



$\cos\alpha = \dfrac{(2,3)(6,0)}{\sqrt{4+9}\cdot 6}$

$= \dfrac{2}{\sqrt{13}}$

30. moping to $(x-1, y, z-2)$

$A = (0, 0, 0)$

$P = 3x - 3 + y + z - 2 = 9$

$3x + y + z = 14$

Per example 6,

$p = \dfrac{c}{|a|} = \dfrac{14}{\sqrt{11}}$

31–35 **ORTHOGONALITY** is particularly important, mainly because of orthogonal coordinates, such as ***Cartesian coordinates***, whose ***natural basis*** [Eq. (9), Sec. 9.1], consists of three orthogonal unit vectors.

31. For what values of a_1 are $[a_1, 4, 3]$ and $[3, -2, 12]$ orthogonal?

32. Planes. For what c are $3x + z = 5$ and $8x - y + cz = 9$ orthogonal?

33. Unit vectors. Find all unit vectors $\mathbf{a} = [a_1, a_2]$ in the plane orthogonal to $[4, 3]$.

34. Corner reflector. Find the angle between a light ray and its reflection in three orthogonal plane mirrors, known as *corner reflector*.

35. Parallelogram. When will the diagonals be orthogonal? Give a proof.

31. $3a_1 - 8 + 36 = 0$

$a_1 = -\frac{28}{3}$

32. $(3, 0, 1)\cdot(8, -1, c) = 0$

$24 + c = 0 \quad c = -24$

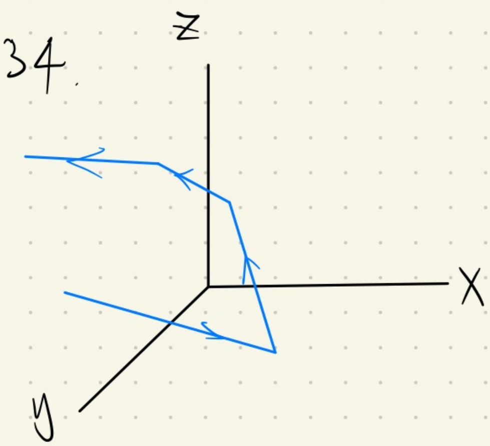
33. $4a_1 + 3a_2 = 0$

$a_2 = -\frac{4}{3}a_1$

$a_1^2 + \frac{16}{9}a_1^2 = 1$

$a_1 = \frac{3}{5}. \quad [\frac{3}{5}, -\frac{4}{5}]$

34.



$[a, b, c]$ Reflect on x-y plane.

$\Rightarrow [a, b, -c]$ Reflect on xz plane

$\Rightarrow [a, -b, -c]$ R on zy plane

$[-a, -b, -c]$

35. $(a+b)\cdot(a-b) = 0$

$a^2 - b^2 = 0 \quad a^2 = b^2$

diamond shape. answer is wrong.

36–40 **COMPONENT IN THE DIRECTION OF A VECTOR**

Find the component of **a** in the direction of **b**. Make a sketch.

36. $\mathbf{a} = [1, 1, 1]$, $\mathbf{b} = [2, 1, 3]$

37. $\mathbf{a} = [3, 4, 0]$, $\mathbf{b} = [4, -3, 2]$

38. $\mathbf{a} = [8, 2, 0]$, $\mathbf{b} = [-4, -1, 0]$

39. When will the component (the projection) of **a** in the direction of **b** be equal to the component (the projection) of **b** in the direction of **a**? First guess.

40. What happens to the component of **a** in the direction of **b** if you change the length of **b**?

36. $p = \frac{a \cdot b}{|b|} = \frac{2+1+3}{\sqrt{14}} = \frac{3\sqrt{14}}{7}$

37. $p = \frac{a \cdot b}{|b|} = \frac{12-12+0}{\sqrt{\ }} = 0$ perpendicular

38. $p = \frac{ab}{|b|} = \frac{-32-2}{\sqrt{17}} = -2\sqrt{17}$

On the opp direction.

39. $|a| = |b|$ or $ab = 0$

40. $p = \frac{a \cdot b}{|b|} = \frac{a \cdot kb}{k|b|}$

nothing change.

P374 - Problem set 9.3

PS 9.3

1–10 GENERAL PROBLEMS

1. Give the details of the proofs of Eqs. (4) and (5).
2. What does $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ with $\mathbf{a} \neq \mathbf{0}$ imply?
3. Give the details of the proofs of Eqs. (6) and (11).

1.

$$(4) \qquad (l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

$$(2^{**}) \qquad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$la \times b$

$$\begin{vmatrix} i & j & k \\ la_1 & la_2 & la_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (la_2b_3 - la_3b_2)i - (la_1b_3 - la_3b_1)j + (la_1b_2 - la_2b_1)k$$

$$= l\,[(a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

Extend the rest, Same.

$$(5) \qquad \begin{aligned} &(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}), \\ &(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}). \end{aligned}$$

$a \times (b+c)$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_1 & b_3+c_3 \end{vmatrix} = [a_2(b_3+c_3) - a_3(b_2+c_2)]\,i - [a_1(b_3+c_3) - a_3(b_1+c_1)]\,j + [a_1(b_2+b_3) - a_2(b_1+c_1)]\,k$$

$a \times b$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\,i - (a_1b_3 - a_3b_1)\,j + (a_1b_2 - a_2b_1)\,k$$

$a \times c$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{Reference above.}$$

$$a \times (b+c) = (a \times b) + (a \times c)$$

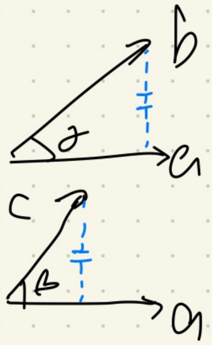
(β): Same

2. What does $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ with $\mathbf{a} \neq \mathbf{0}$ imply?
3. Give the details of the proofs of Eqs. (6) and (11).

② $|v| = |a \times b| = |a||b| \sin\alpha$

$= |a \times c| = |a||c| \sin\beta$

$|b| \sin\alpha = |c| \sin\beta$.



and the direction of the result is same.

① $a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ x & y & z \end{vmatrix} =$

$(0-y)i - (z-x)j + yk$

$b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \dots$

② $a = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = zi - xk$

$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \dots$

③

(6) $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$

$\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ and $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

exchange row, determinant $\times -1$

P295. Sec 7.7 Theorem 1.

(11) $(\mathbf{a} \ \ \mathbf{b} \ \ \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

$$b \times c = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)i - (b_1c_3 - b_3c_1)j + (b_1c_2 - b_2c_1)k$$

$$a \cdot (b \times c) = (a_1b_2c_3 - a_1b_3c_2) - (a_2b_1c_3 - a_2b_3c_1) + (a_3b_1c_2 - a_3b_2c_1)$$

$(a \times b) \cdot c \Rightarrow a \to b,\ b \to c,\ c \to a$

$$= c_1a_2b_3 - c_1a_3b_2 - (c_2a_1b_3 - c_2a_3b_1) + c_3a_1b_2 - c_3a_2b_1$$

Geo perspective, same volume.

4. Lagrange's identity for $|\mathbf{a} \times \mathbf{b}|$. Verify it for $\mathbf{a} = [3, 4, 2]$ and $\mathbf{b} = [1, 0, 2]$. Prove it, using $\sin^2 \gamma = 1 - \cos^2 \gamma$. The identity is

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2}. \qquad (12)$$

4. $\{3,4,2\}$ cross $\{1,0,2\} = [8, -4, -4]$

$|a \times b| = \sqrt{64+16+16} = \sqrt{96}$

$\sqrt{(a\cdot a)(b \cdot b) - (a\cdot b)^2}$

$= \sqrt{(9+16+4)(1+4) - (3+4)^2}$

$= \sqrt{29 \times 5 - 49} = \sqrt{96}$

Prove: $|a \times b| = |a||b| \sin\alpha$

$= \sqrt{a \cdot a}\sqrt{b \cdot b} \sin\alpha$

$\cos\alpha = \frac{a \cdot b}{|a||b|} = \frac{a \cdot b}{\sqrt{a\cdot a}\sqrt{b \cdot b}}$

$|a \times b| = \sqrt{a \cdot a}\sqrt{b\cdot b}\sqrt{1 - \cos^2\alpha}$

$= \sqrt{a \cdot a}\sqrt{b \cdot b} \cdot \sqrt{1 - \frac{(a\cdot b)^2}{(a \cdot a)\, b \cdot b}}$

$= \sqrt{(a\cdot a)(b \cdot b) - (a \cdot b)^2}$

5. What happens in Example 3 of the text if you replace $\mathbf{p}$ by $-\mathbf{p}$?

m change the direction.

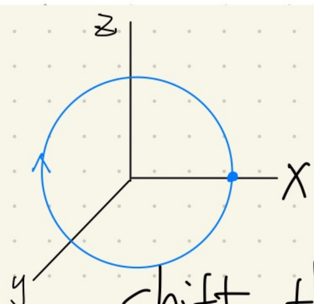
6. What happens in Example 5 if you choose a P at distance $2d$ from the axis of rotation?

W remains the same

$|r| \sin\alpha$ become doubled.

So v get doubled.

7. Rotation. A wheel is rotating about the y-axis with angular speed $\omega = 20\ \text{sec}^{-1}$. The rotation appears clockwise if one looks from the origin in the positive y-direction. Find the velocity and speed at the point $[8, 6, 0]$. Make a sketch.



shift the center of wheel to $[0, 0, 0]$.

So $[8, 6, 0]$ equals to $[8, 0, 0]$

$S = 8 \times 20 = 160 \quad v = [0, 0, -160]$

$v = [0, 20, 0] \times [8, 6, 0]$

$= [0, 0, -160]$

Not quit understand

8. Rotation. What are the velocity and speed in Prob. 7 at the point $(4, 2, -2)$ if the wheel rotates about the line $y = x, z = 0$ with $\omega = 10 \text{ sec}^{-1}$?

(8) $y^2 + x^2 = 10^2$

$x^2 + x^2 = 100$

$x = \sqrt{50}$

$[\sqrt{50}, \sqrt{50}, 0] \times [4, 2, -2]$

$$= \begin{vmatrix} i & j & k \\ \sqrt{50} & \sqrt{50} & 0 \\ 4 & 2 & -2 \end{vmatrix}$$

$= -2\sqrt{50}\, i + 2\sqrt{50}\, j - 2\sqrt{50}\, k$

9. Scalar triple product. What does $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = 0$ imply with respect to these vectors?

(9) $V = 0$.

1. a. b. c on the same plane
2. bc on the same line
3. one of a. b. c = 0.

(10.)

11–23 **VECTOR AND SCALAR TRIPLE PRODUCTS**

With respect to right-handed Cartesian coordinates, let $\mathbf{a} = [2, 1, 0]$, $\mathbf{b} = [-3, 2, 0]$, $\mathbf{c} = [1, 4, -2]$, and $\mathbf{d} = [5, -1, 3]$. Showing details, find:

11. $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{a}$, $\mathbf{a} \bullet \mathbf{b}$

12. $3\mathbf{c} \times 5\mathbf{d}$, $15\mathbf{d} \times \mathbf{c}$, $15\mathbf{d} \bullet \mathbf{c}$, $15\mathbf{c} \bullet \mathbf{d}$

$(0\ -2,0)\times$
$(-2\ 0\ 0)$

(11) $a\times b = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix}$

$= 0i - 0j + 7k$

$= 7k$

$b\times a = -a\times b = -7k$

$a\cdot b = -6+2+0 = -4$

(12) $3c\times 5d = 15\cdot c\times d$

$= 15\begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix}$

$= 15(10i - 13j - 21k)$

$15d\times c = -15\, c\times d$

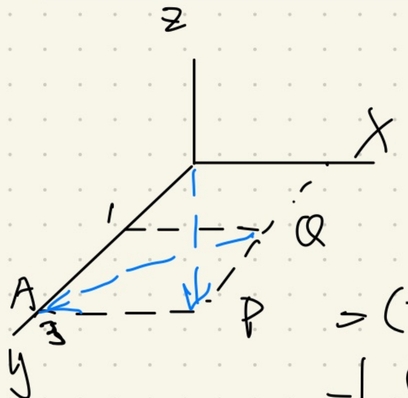
$= 15(-10i + 13j + 21k)$

$15\, d\cdot c = 15\cdot(5-4-6)$

$= -75 = 15\, c\cdot d$

25–35 **APPLICATIONS**

25. Moment *m* of a force p. Find the moment vector **m** and m of $\mathbf{p} = [2, 3, 0]$ about Q: $(2, 1, 0)$ acting on a line through A: $(0, 3, 0)$. Make a sketch.



$m = r\times p$

$= \overrightarrow{QA}\times\vec{P}$

$= (-2, 2, 0)\times(2, 3, 0)$

$= \begin{vmatrix} i & j & k \\ -2 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix}$

$= -10k$

26. Moment. Solve Prob. 25 if $\mathbf{p} = [1, 0, 3]$, Q: $(2, 0, 3)$, and A: $(4, 3, 5)$.

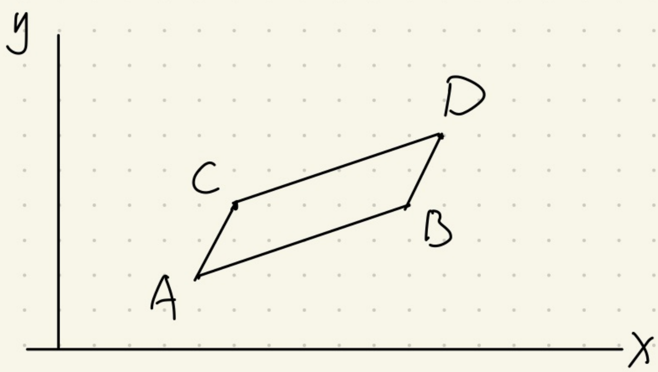
$m = r\times p$

$= \overrightarrow{QA}\times\vec{P}$

$= [2, 3, 2]\times[1, 0, 3]$

$= \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 9i - 4j - 3k$

27. Parallelogram. Find the area if the vertices are (4, 2, 0), (10, 4, 0), (5, 4, 0), and (11, 6, 0). Make a sketch.



$$\text{Area} = |\vec{AB} \times \vec{AC}|$$

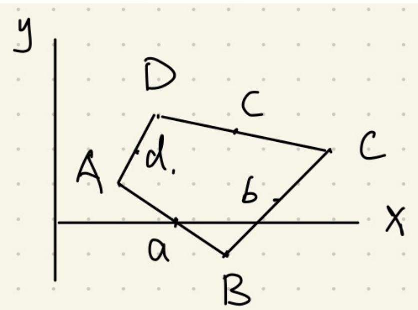
$$= \left| \begin{vmatrix} \vec{i} & \vec{j} & k \\ 6 & 2 & 0 \\ 1 & 2 & 0 \end{vmatrix} \right|$$

$$= |10k| = 10$$

answer is wrong.

Area is a scalar.

28. A remarkable parallelogram. Find the area of the quadrangle Q whose vertices are the midpoints of the sides of the quadrangle P with vertices A: (2, 1, 0), B: (5, −1. 0), C: (8, 2, 0), and D: (4, 3, 0). Verify that Q is a parallelogram.



set a. b. c. d as the middle point, so we get

a: (3.5, 0. 0)

b: (6.5, 0.5, 0)

c: (6, 2.5, 0)

d: (3, 2, 0)

$$\vec{ab} = (3, 0.5, 0) = \vec{dc}$$

$$\vec{ad} = (-0.5, 2, 0) = \vec{bc}$$

$\Rightarrow$ Q is a parallelogram.

29. Triangle. Find the area if the vertices are (0, 0, 1), (2, 0, 5), and (2, 3, 4).

$$\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} \left| \begin{vmatrix} i & j & k \\ 2 & 0 & 4 \\ 2 & 3 & 3 \end{vmatrix} \right|$$

$$= \frac{1}{2}|-12i + 2j + 6k|$$

$$= \frac{1}{2}\sqrt{12^2 + 2^2 + 6^2}$$

$$= \sqrt{46}$$

30. Plane. Find the plane through the points A: $(1, 2, \frac{1}{4})$, B: $(4, 2, -2)$, and C: $(0, 8, 4)$.

For $ax+by+cz+d\cdot 1=0$

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & \frac{1}{4} & 1 \\ 4 & 2 & -2 & 1 \\ 0 & 8 & 4 & 1 \end{vmatrix} = 0$$

$$x(-4+8+2+16-\tfrac{1}{2}-8)$$
$$-y(-2+16+0-0-4-1)$$
$$+z(2+32+0-0-8-8)$$
$$-1\cdot(8+8+0-0+16-32)=0$$

$$13.5x-9y+18z=0$$
$$27x-18y+36z=0$$
$$3x-2y+4z=0$$

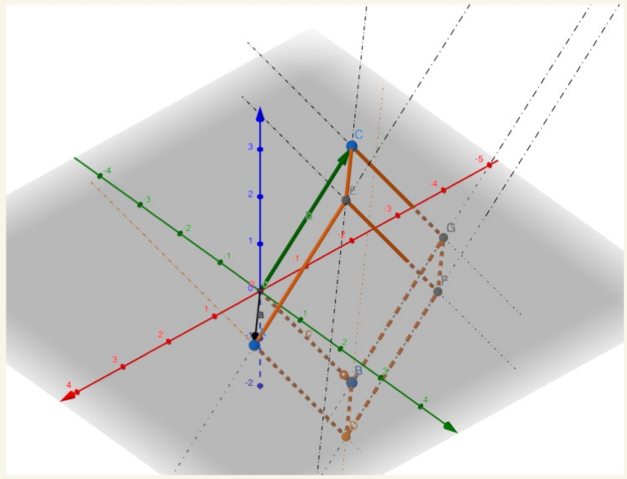
31. Plane. Find the plane through $(1, 3, 4)$, $(1, -2, 6)$, and $(4, 0, 7)$.

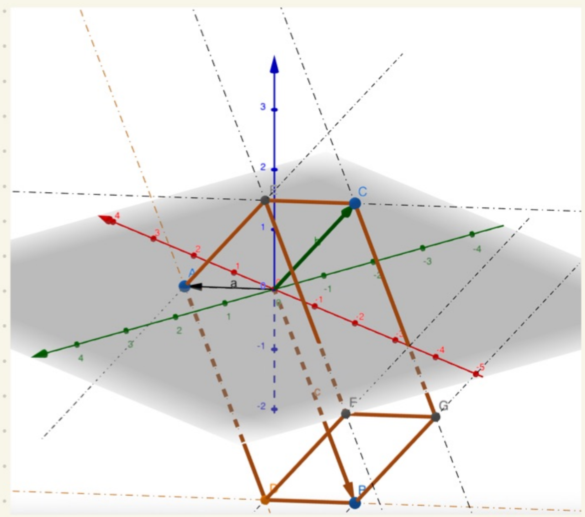
Reference 30

$$3x-2y-5z+23=0$$

answer is wrong.

32. Parallelepiped. Find the volume if the edge vectors are $\mathbf{i} + \mathbf{j}$, $-2\mathbf{i} + 2\mathbf{k}$, and $-2\mathbf{i} - 3\mathbf{k}$. Make a sketch.





$$(a, b, c) = a\cdot(b\times c)$$
$$=(i+j)\cdot\begin{vmatrix} i & j & k \\ -2 & 0 & 2 \\ -2 & 0 & -3 \end{vmatrix}$$
$$=(i+j)\cdot[-j(6+4)]=-10$$
$$V=|(a, b, c)|=10$$

33. Tetrahedron. Find the volume if the vertices are (1, 1, 1), (5, −7, 3), (7, 4, 8), and (10, 7, 4).

Per Wolfram or Wiki:

$$V = \frac{1}{3!}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \frac{|(A-D)\cdot[(B-D)\times(C-D)]|}{6} = \frac{1}{3!}|(a,b,c)|$$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & -7 & 3 & 1 \\ 7 & 4 & 8 & 1 \\ 10 & 7 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{6}\cdot 474 = 79$$

$$= \frac{1}{6}\left|(4,-8,2)\cdot[(6,3,7)\times(9,6,3)]\right|$$

$$= \frac{1}{6}\left|(4,-8,2)\cdot(-33,45,9)\right|$$

$$= \frac{1}{6}\cdot 474 = 79$$

This is intuitive.

Why?

34. Tetrahedron. Find the volume if the vertices are (1, 3, 6), (3, 7, 12), (8, 8, 9), and (2, 2, 8).

$$V = \frac{1}{6}\cdot 90 = 15$$

35.

P380 - Problem set 9.4

P.S. 9.4

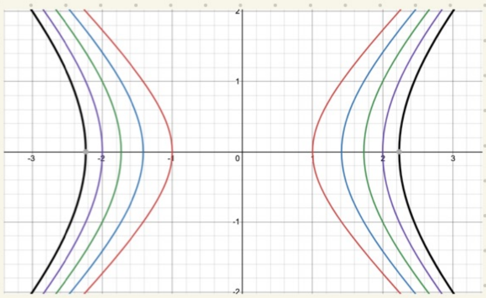
1–8 SCALAR FIELDS IN THE PLANE

Let the temperature T in a body be independent of z so that it is given by a scalar function $T = T(x, t)$. Identify the isotherms $T(x, y) = \text{const}$. Sketch some of them.

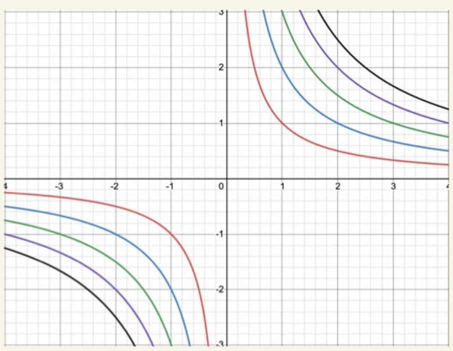
1. $T = x^2 - y^2$
2. $T = xy$
3. $T = 3x - 4y$
4. $T = \arctan(y/x)$
5. $T = y/(x^2 + y^2)$
6. $T = x/(x^2 + y^2)$
7. $T = 9x^2 + 4y^2$

Red = 1. black = 5.

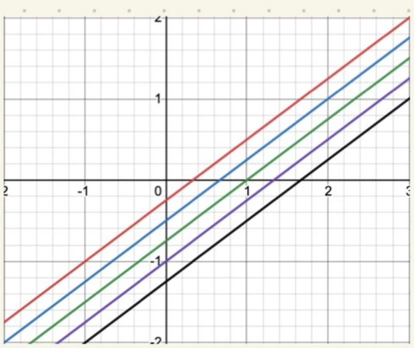
1. $T = x^2 - y^2 = C$



2. $T = xy = C$



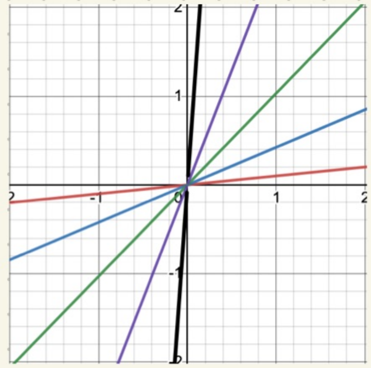
3. $T = 3x - 4y = C$



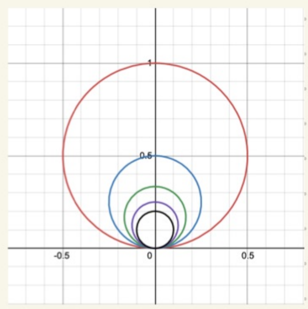
4. $T = \arctan(y/x) = C$

$C \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

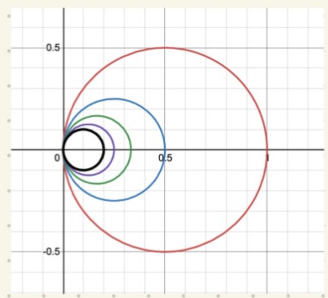
R: 0.1. B: 0.4 G 0.8 P 1.2. B 1.5



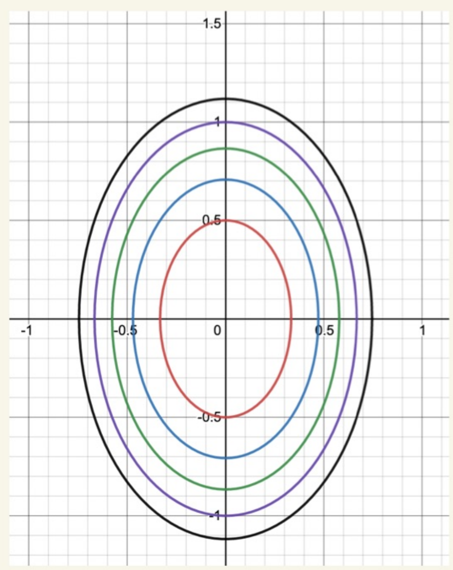
5. $T = y/(x^2+y^2) = C$



6. $T = \frac{x}{x^2+y^2} = C$



7. $T = 9x^2 + 4y^2 = C$



8. CAS PROJECT. Scalar Fields in the Plane. Sketch or graph isotherms of the following fields and describe what they look like.

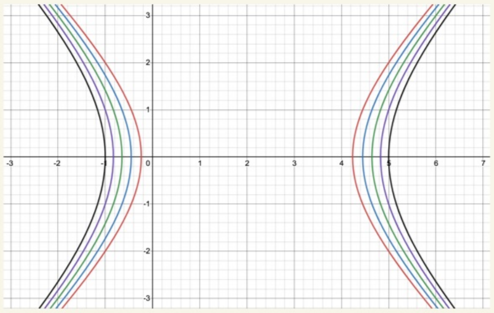
(a) $x^2 - 4x - y^2$ **(b)** $x^2y - y^3/3$

(c) $\cos x \sinh y$ **(d)** $\sin x \sinh y$

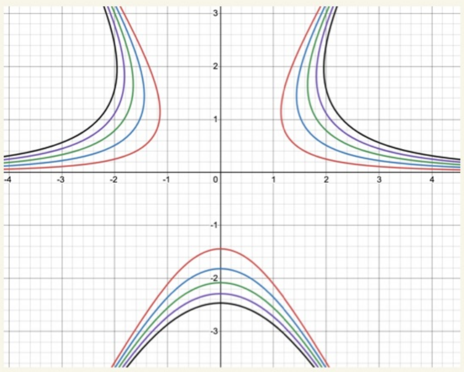
(e) $e^x \sin y$ **(f)** $e^{2x} \cos 2y$

(g) $x^4 - 6x^2y^2 + y^4$ **(h)** $x^2 - 2x - y^2$

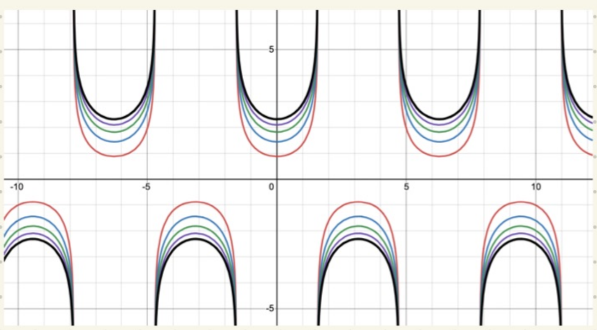
a.



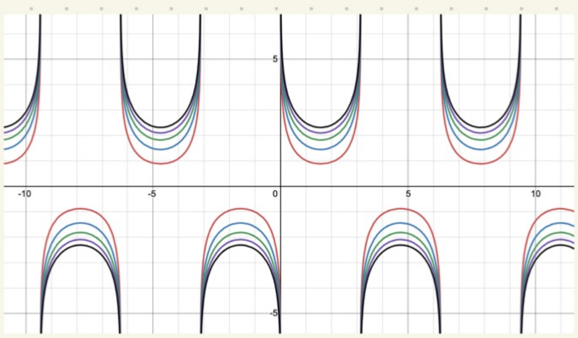
b.



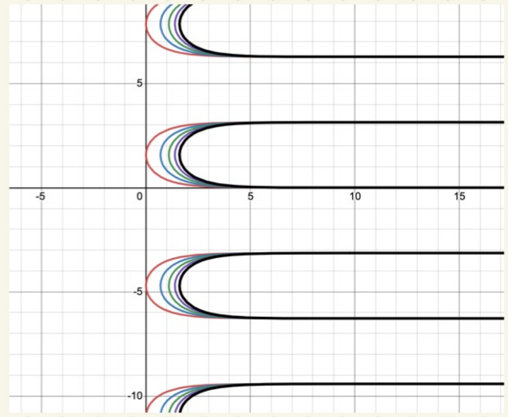
c.



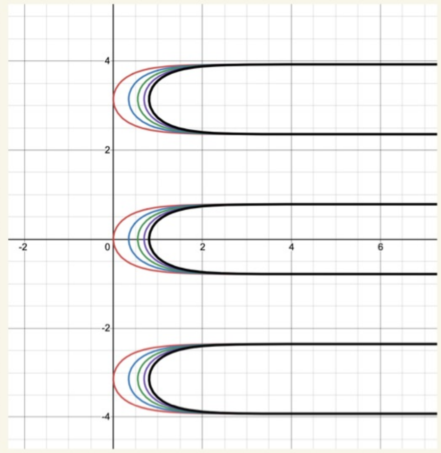
d.



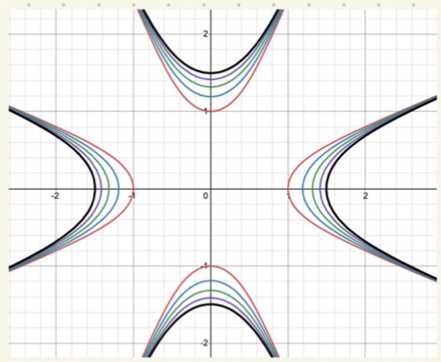
e.



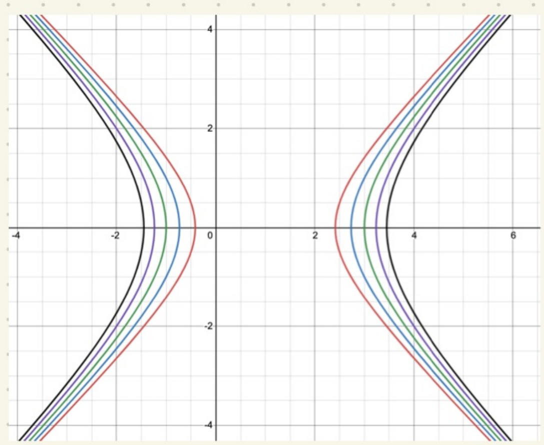
f.



g.



h.



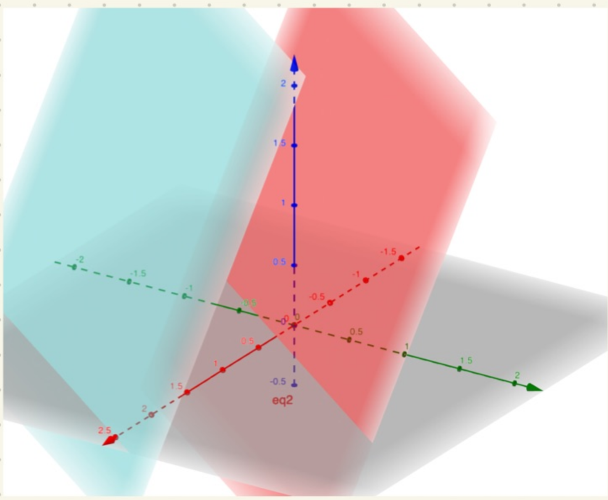
9–14 SCALAR FIELDS IN SPACE

What kind of surfaces are the **level surfaces** $f(x, y, z) =$ const?

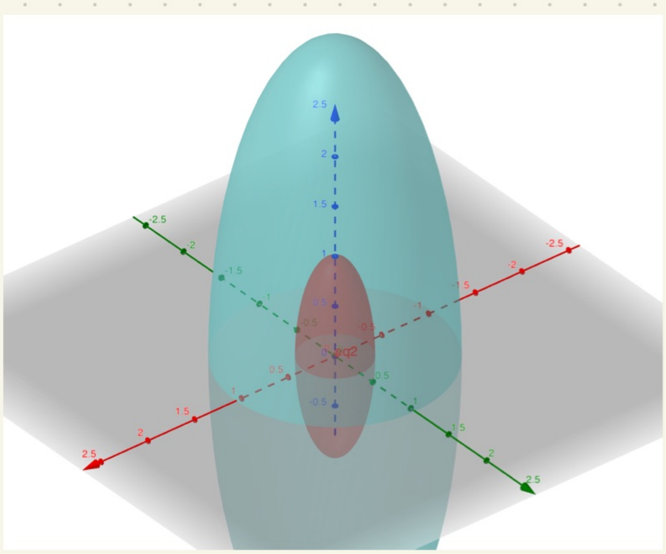
9. $f = 4x - 3y + 2z$ **10.** $f = 9(x^2 + y^2) + z^2$

11. $f = 5x^2 + 2y^2$ **12.** $f = z - \sqrt{x^2 + y^2}$

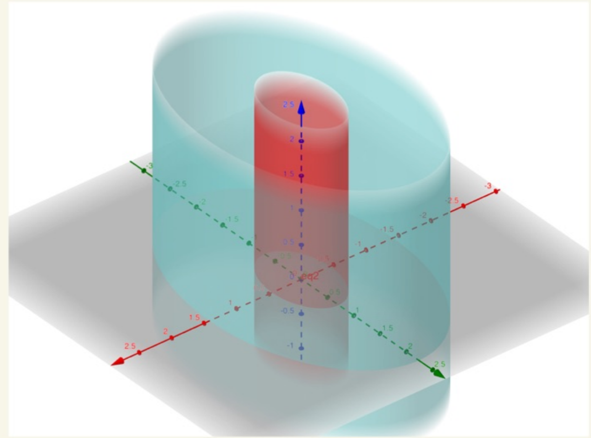
9. Teal: $f=10$ Red: $f=1$



10.

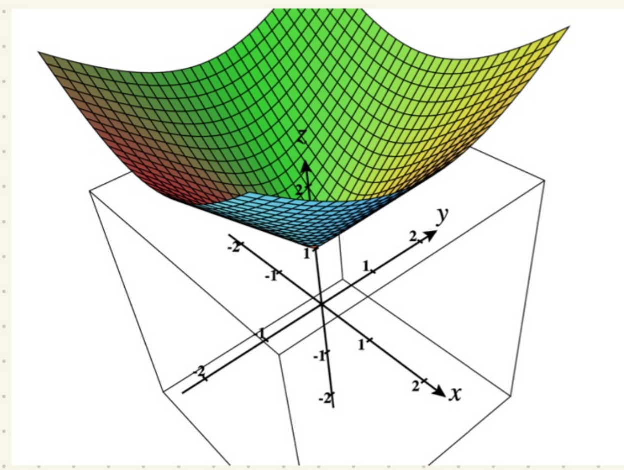


11.



12. quit a lot of tools can not plot this.

$f=1$



9–14 SCALAR FIELDS IN SPACE

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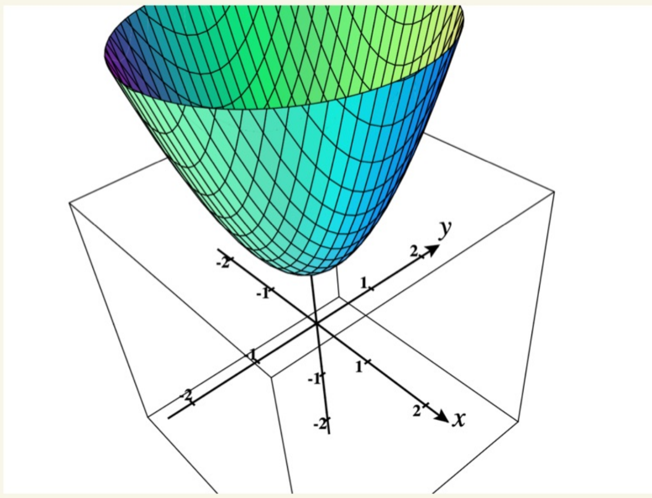
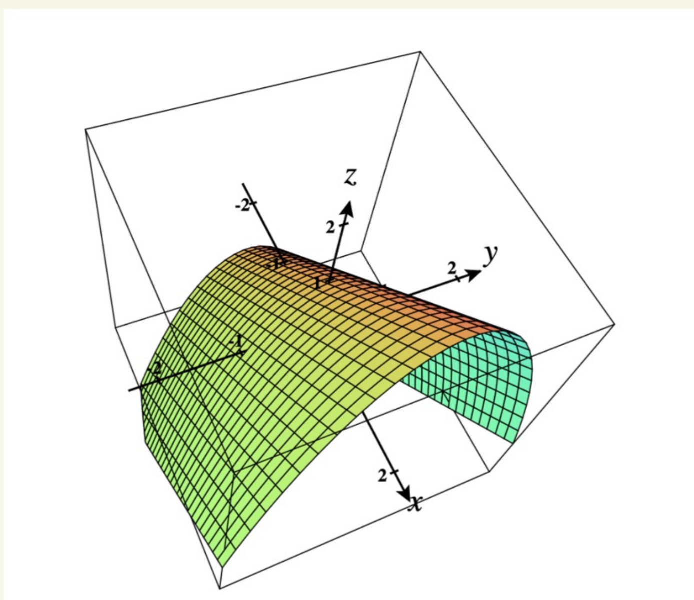
11. $f = 5x^2 + 2y^2$
12. $f = z - \sqrt{x^2 + y^2}$
13. $f = z - (x^2 + y^2)$
14. $f = x - y^2$

https://c3d.libretexts.org/CalcPlot3D/index.html

13.



14.

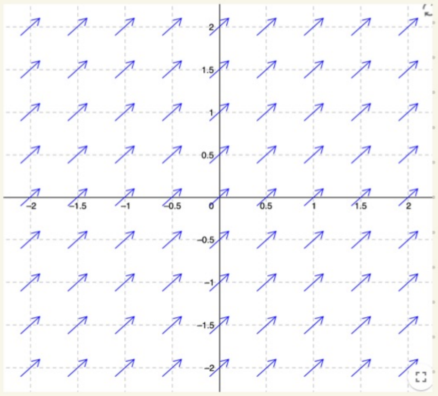


15–20 **VECTOR FIELDS**

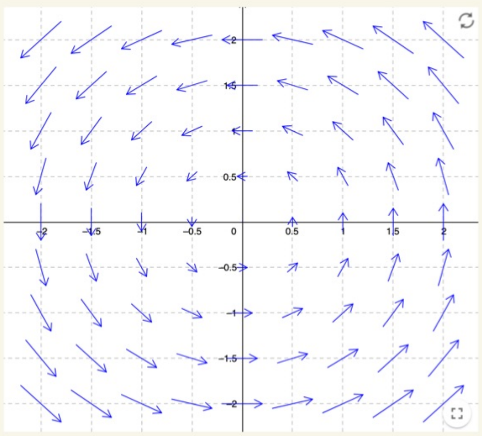
Sketch figures similar to Fig. 198. Try to interpet the field of **v** as a velocity field.

15. $\mathbf{v} = \mathbf{i} + \mathbf{j}$	**16.** $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$
17. $\mathbf{v} = x\mathbf{j}$	**18.** $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$
19. $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$	**20.** $\mathbf{v} = y\mathbf{i} - x\mathbf{j}$

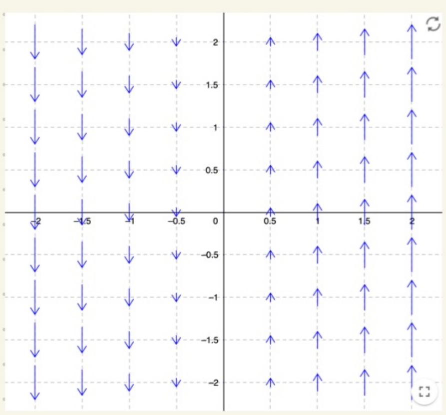
15.



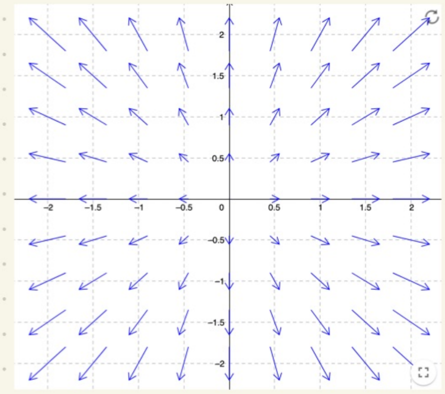
16



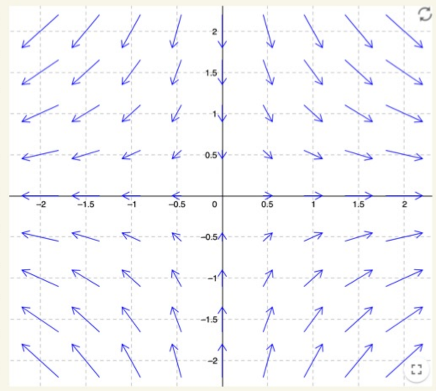
17.



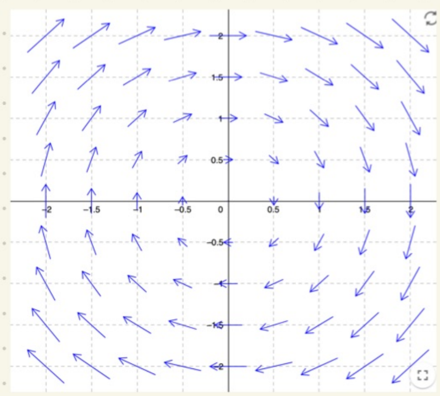
18.



19.



20.



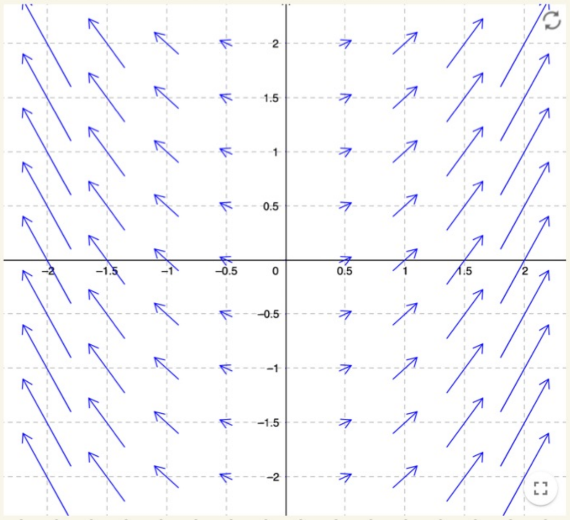
Geogebra.org

21. CAS PROJECT. Vector Fields. Plot by arrows:

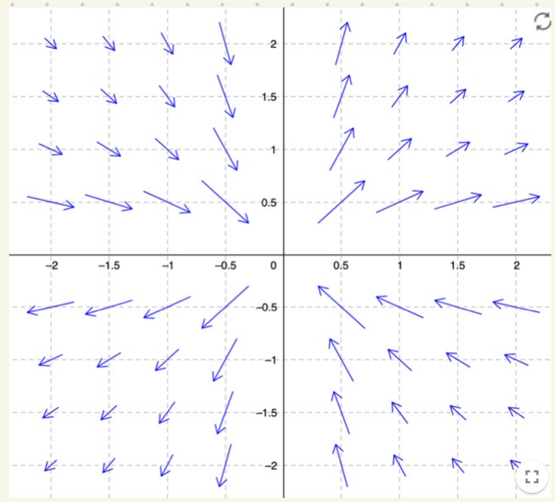
(a) $\mathbf{v} = [x, x^2]$ **(b)** $\mathbf{v} = [1/y, 1/x]$

(c) $\mathbf{v} = [\cos x, \sin x]$ **(d)** $\mathbf{v} = e^{-(x^2+y^2)}[x, -y]$

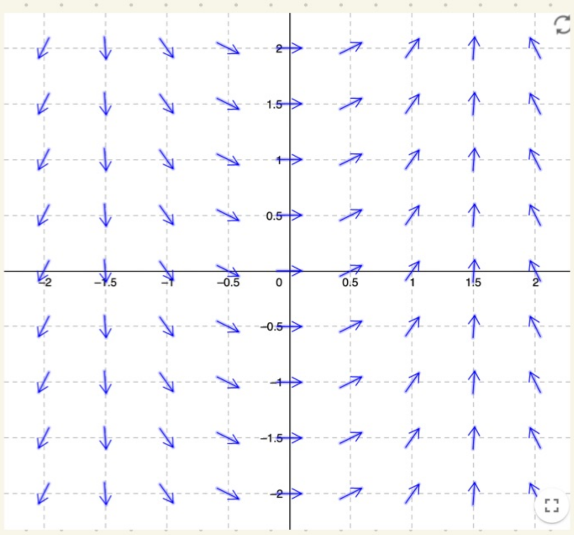
a)



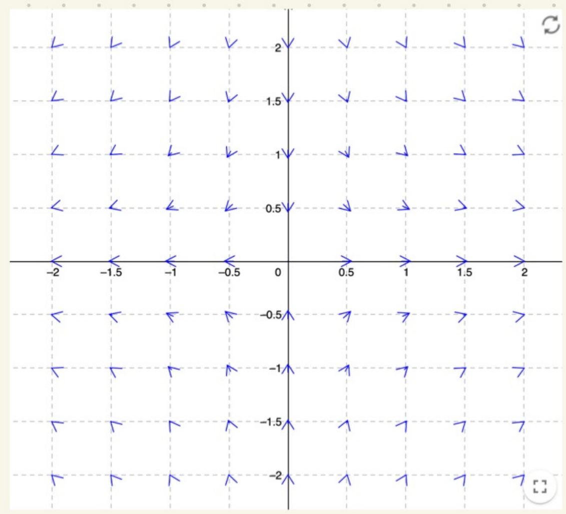
b)



c)



d)



22–25 DIFFERENTIATION

22. Find the first and second derivatives of $\mathbf{r} = [3\cos 2t, 3\sin 2t, 4t]$.

23. Prove (11)–(13). Give two typical examples for each formula.

24. Find the first partial derivatives of $\mathbf{v_1} = [e^x \cos y, e^x \sin y]$ and $\mathbf{v_2} = [\cos x \cosh y, -\sin x \sinh y]$.

25. WRITING PROJECT. Differentiation of Vector Functions. Summarize the essential ideas and facts and give examples of your own.

22. $r' = [-6\sin 2t,\ 6\cos 2t,\ 4]$

$r'' = [-12\cos 2t.\ -12\sin 2t,\ 0]$

23.

24. $\dfrac{\partial v_1}{\partial x} = [e^x \cdot \cos y,\ e^x \sin y]$

$\dfrac{\partial v_1}{\partial y} = [-e^x \sin y,\ e^x \cos y]$

$\dfrac{\partial v_2}{\partial x} = [-\sin x \cosh y,\ -\cos x \sinh y]$

$\dfrac{\partial v_2}{\partial y} = [\cos x \sinh y,\ -\sin x \cosh y]$

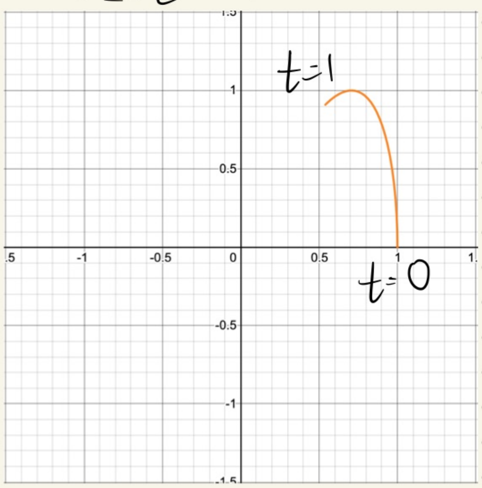
25.

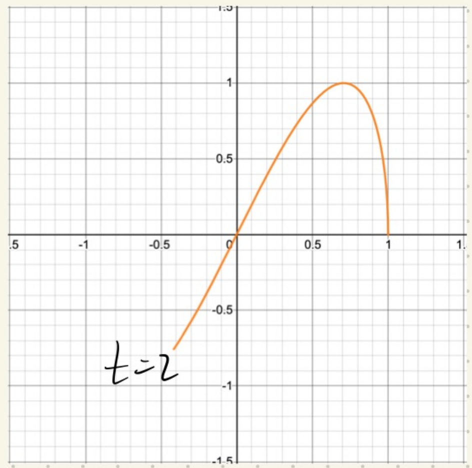
P390 - Problem set 9.5

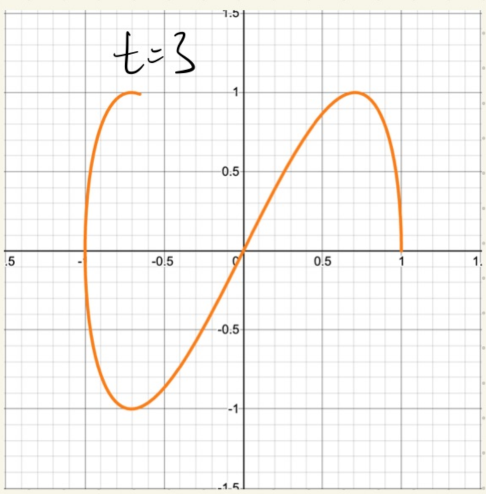
P.S. 9.5

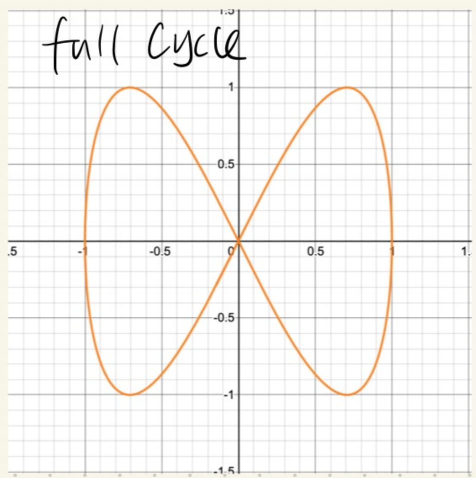
9. [cos t, sin $2t$, 0]

$0 \leq t \leq 1$









11–20 FIND A PARAMETRIC REPRESENTATION

11. Circle in the plane $z = 1$ with center (3, 2) and passing through the origin.

12. Circle in the yz-plane with center (4, 0) and passing through (0, 3). Sketch it.

13. Straight line through (2, 1, 3) in the direction of $\mathbf{i} + 2\mathbf{j}$.

13. $r(t) = a + tb$

$b = i + 2j$

$t=0$, on point $(2, 1, 3)$

$r(t) = [2, 1, 3] + t[1, 2, 0]$

$= [t+2, 2t+1, 3]$

24–28 TANGENT $u(t)$

Given a curve C: $\mathbf{r}(t)$, find a tangent vector $\mathbf{r}'(t)$, a unit tangent vector $\mathbf{u}'(t)$, and the tangent of C at P. Sketch curve and tangent.

24. $\mathbf{r}(t) = [t, \frac{1}{2}t^2, 1]$, P: (2, 2, 1)

25. $\mathbf{r}(t) = [10 \cos t, 1, 10 \sin t]$, P: (6, 1, 8)

26. $\mathbf{r}(t) = [\cos t, \sin t, 9t]$, P: $(1, 0, 18\pi)$

27. $\mathbf{r}(t) = [t, 1/t, 0]$, P: $(2, \frac{1}{2}, 0)$

27. $r'(t) = [1, -\frac{1}{t^2}, 0]$

$u(t) = \frac{1}{|r'|} \cdot r'$

$= \frac{1}{\sqrt{1 + \frac{1}{t^4}}} \cdot r'$

$= \frac{t^2}{t^4+1} \cdot r'$

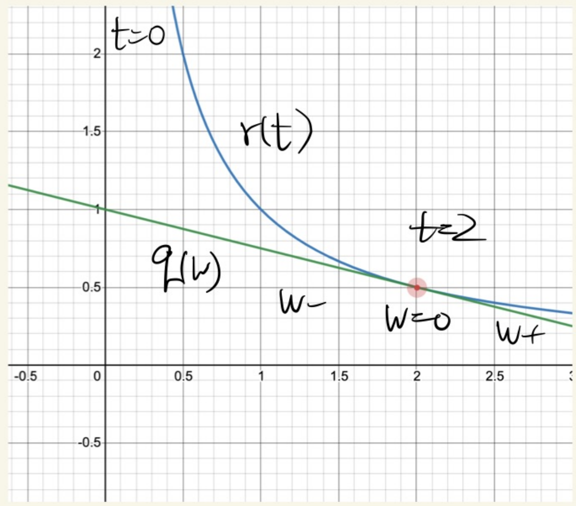
$= [\frac{t^2}{t^4+1}, -\frac{1}{t^4+1}, 0]$

$q(w) = r + wr'$

$= [t, \frac{1}{t}, 0] + [w, -\frac{w}{t^2}, 0]$

$= [2, \frac{1}{2}, 0] + [w, -\frac{w}{4}, 0]$

$= [2+w, \frac{1}{2} - \frac{w}{4}, 0]$



29–32 LENGTH

Find the length and sketch the curve.

31. Circle $\mathbf{r}(t) = [a \cos t, a \sin t]$ from $(a, 0)$ to $(0, a)$.

from $t=0$ to $t=\frac{\pi}{2}$

$$l = \int_0^{\frac{\pi}{2}} \sqrt{r' \cdot r'} \, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(-a\sin t, \ a\cos t) \cdot (-a\sin t, \ a\cos t)} \, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2 t + a^2\cos^2 t} \, dt$$

$$= \int_0^{\frac{\pi}{2}} |a| \cdot dt = |a| \cdot t \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} \cdot |a|$$

35–46 CURVES IN MECHANICS

Forces acting on moving objects (cars, airplanes, ships, etc.) require the engineer to know corresponding ***tangential*** and ***normal accelerations***. In Probs. 35–38 find them, along with the ***velocity*** and ***speed***. Sketch the path.

35. Parabola $\mathbf{r}(t) = [t, t^2, 0]$. Find $\mathbf{v}$ and $\mathbf{a}$.

$v = r'(t) = [1, 2t, 0]$

$a = v' = [0, 2, 0]$

speed $= \sqrt{1^2 + 4t^2}$

acceleration $= 2$.

47–55 CURVATURE AND TORSION

47. Circle. Show that a circle of radius a has curvature $1/a$.

$r(t) = [a\cos t, a\sin t]$, $a > 0$

$r(s) = [a\cos(\frac{s}{r}), a\sin(\frac{s}{r})]$

$K(s) = |r''(s)| = |[-\sin(\frac{s}{a}) \cdot \frac{1}{a}, \cos(\frac{s}{a}) \cdot \frac{1}{a}]'|$

$= |[-\sin\frac{s}{a}, \cos(\frac{s}{a})]'|$

$= |\frac{1}{a}[-\cos(\frac{s}{a}), -\sin\frac{s}{a}]|$

$= \frac{1}{a} \cdot 1 = \frac{1}{a}$