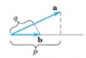
W3

C9: Vector Differential Calculus, Vector Algebra

- Vectors in 2-Space and 3-Space
  - Examples
    - Scalar
    - Vector
      - Initial and terminal
      - Length/Norm
        - Unit vector
      - Direction
      - Equality of Vectors
  - Components of a Vector
    - Cartesian Coordinate System
  - Vector Addition
    - Parallelogram Law
      - Resultant of two forces
      - Relative of velocity
    - Commutativity: a+b=b+a
    - Associativity: (u+v)+w=u+(v+w)
    - Difference of vectors: a-b = a+(-b)
  - Scalar Multiplication
  - Unit Vectors
    - Standard basis for R^3
- Inner Product. Dot
  - Orthogonality
    - Angle between vectors
      - $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma$ if $\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$
      - $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.
    - CS inequality
      - $|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}||\mathbf{b}|$
    - Triangle Inequality
      - $|\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}|$
    - Parallelogram equality
      - $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$
  - Applications
    - Work done by a force
      - $W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d},$
    - Force in a rope
    - Component of a Force
      - $p = |\mathbf{a}| \cos \gamma.$
      - $p = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$ $(\mathbf{b} \neq \mathbf{0}).$
      - $p = \mathbf{a} \cdot \mathbf{b}$ $(|\mathbf{b}| = 1).$
      - 

        Projections p of $\mathbf{a}$ on $\mathbf{b}$ and q of $\mathbf{b}$ on $\mathbf{a}$
    - Orthogonal Straight Lines
      - Normal vector
    - Normal Vector to a plane
      - Unit normal vector
- Vector Product. Cross
  - Geometry: Area
    - Right Handed Triple
      - Right-handed cartesian coordinate system
    - Vector product
      - $|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma,$
    - Anti-commutativity
      - $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$
    - In general, not associative
      - $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
  - Applications
    - Moment of a Force
      - $\mathbf{m} = \mathbf{r} \times \mathbf{p}$
    - Velocity of a Rotating Body
      - Angular speed
      - $\omega d = |\mathbf{w}||\mathbf{r}| \sin \gamma = |\mathbf{w} \times \mathbf{r}|.$
      - $\mathbf{v} = \mathbf{w} \times \mathbf{r}.$
  - Scalar Triple Product
    - Geometric interpretation: Volume of the parallelepiped
      - $(\mathbf{a} \;\; \mathbf{b} \;\; \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$
      - dot and cross are interchangable
        - $(\mathbf{a} \;\; \mathbf{b} \;\; \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$
    - Linear independence: if and only if (a, b, c) not zero
    - LaGrange Identity
      - $|\mathbf{a} \times \mathbf{b}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2}.$
    - Tetrahedron

C9: Vector Differential Calculus, Vector Calculus

- Vector and Scalar Functions and Their Fields
  - Functions and Field
    - Vector function
    - Scalar function
      - f(P) - independent to coordinate system
      - Direction depends on coordinate system
    - Vector Field
      - Field of tangent vectors of a curve
      - Field of normal vectors of a surface
      - Velocity Field
      - Field of force, Gravitational field
      - Components depends on coordinate system, but vector field depdending only on P(point in space)
  - Vector Calculus
    - Convergence
    - Continuity
    - Differentiable
    - $\mathbf{v}'(t) = [v_1'(t), \quad v_2'(t), \quad v_3'(t)].$
    - $$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$ $$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$ $$(\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}').$$
    - The derivative of a vector function v(t) of constant length is either the zero vector or is perpendicular to v(t).
  - Demo
    - Isotherms
    - Derivativie
  - Partial Derivatives of a Vector Function
    - $$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$
- Curves. Arc Lenth. Curvature. Torsion
  - Definition
    - Differential geometry
    - Parametric representations
      - xyz all play an equal role
      - Induces an orientation on C
        - Positive sense
        - Negative sense
      - Application
        - Circle
        - Ellipse
        - Straight Line
        - Plane curve
        - Twisted curve
          - Circular Helix
        - Simple Curve
          - Arc
  - Tangent to a Curve
    - Tangent vector
      - Unit tangent vector
  - Length of a Curve
  - Arc Length s of a Curve
  - Curves in Mechanics
  - Curvature and Torsion
- Calculus Review
  - Chain Rules
  - Mean Value Theorems

C9

Basic Properties of Scalar Multiplication. From the definitions we obtain directly

$$\text{(6)}\quad \begin{aligned} &\text{(a)} && c(\mathbf{a}+\mathbf{b}) = c\mathbf{a} + c\mathbf{b} \\ &\text{(b)} && (c+k)\mathbf{a} = c\mathbf{a} + k\mathbf{a} \\ &\text{(c)} && c(k\mathbf{a}) = (ck)\mathbf{a} \qquad \text{(written } ck\mathbf{a}\text{)} \\ &\text{(d)} && 1\mathbf{a} = \mathbf{a}. \end{aligned}$$

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Basic Properties of Vector Addition. Familiar laws for real numbers give immediately

$$\text{(4)}\quad \begin{aligned} &\text{(a)} && \mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a} && \textit{(Commutativity)} \\ &\text{(b)} && (\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}) && \textit{(Associativity)} \\ &\text{(c)} && \mathbf{a}+\mathbf{0} = \mathbf{0}+\mathbf{a} = \mathbf{a} \\ &\text{(d)} && \mathbf{a}+(-\mathbf{a}) = \mathbf{0}. \end{aligned}$$

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\gamma && \text{if} \quad \mathbf{a}\neq\mathbf{0}, \mathbf{b}\neq\mathbf{0} \\ \mathbf{a}\cdot\mathbf{b} &= 0 && \text{if} \quad \mathbf{a}=\mathbf{0} \text{ or } \mathbf{b}=\mathbf{0}. \end{aligned}$$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

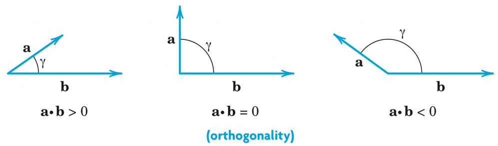


Fig. 178. Angle between vectors and value of inner product

THEOREM 1

Orthogonality Criterion

The inner product of two nonzero vectors is 0 *if and only if these vectors are perpendicular.*

Length and Angle. Equation (1) with $\mathbf{b} = \mathbf{a}$ gives $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$. Hence

$$\text{(3)}\qquad |\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}.$$

From (3) and (1) we obtain for the angle γ between two nonzero vectors

$$\text{(4)}\qquad \cos\gamma = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\sqrt{\mathbf{a}\cdot\mathbf{a}}\sqrt{\mathbf{b}\cdot\mathbf{b}}}.$$

$$
\text{(5)}\quad
\begin{array}{lll}
\text{(a)} & (q_1\mathbf{a} + q_2\mathbf{b}) \bullet \mathbf{c} = q_1\mathbf{a} \bullet \mathbf{c} + q_1\mathbf{b} \bullet \mathbf{c} & \textit{(Linearity)} \\
\text{(b)} & \mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} & \textit{(Symmetry)} \\
\text{(c)} & \left.\begin{array}{c} \mathbf{a} \bullet \mathbf{a} \geqq 0 \\ \mathbf{a} \bullet \mathbf{a} = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{array}\right\} & \textit{(Positive-definiteness).}
\end{array}
$$

Hence *dot multiplication is commutative* as shown by (5b). Furthermore, it is *distributive with respect to vector addition.* This follows from (5a) with $q_1 = 1$ and $q_2 = 1$:

$$\text{(5a*)}\qquad (\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c} \qquad \textit{(Distributivity).}$$

Furthermore, from (1) and $|\cos\gamma| \leqq 1$ we see that

$$\text{(6)}\qquad |\mathbf{a} \bullet \mathbf{b}| \leqq |\mathbf{a}||\mathbf{b}| \qquad \textit{(Cauchy–Schwarz inequality).}$$

Using this and (3), you may prove (see Prob. 16)

$$\text{(7)}\qquad |\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}| \qquad \textbf{(Triangle inequality).}$$

Geometrically, (7) with $<$ says that one side of a triangle must be shorter than the other two sides together; this motivates the name of (7).

A simple direct calculation with inner products shows that

$$\text{(8)}\qquad |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad \textit{(Parallelogram equality).}$$

Work Done by a Force Expressed as an Inner Product

This is a major application. It concerns a body on which a *constant* force $\mathbf{p}$ acts. (For a *variable* force, see Sec. 10.1.) Let the body be given a displacement $\mathbf{d}$. Then the work done by $\mathbf{p}$ in the displacement is defined as

$$\text{(9)}\qquad W = |\mathbf{p}||\mathbf{d}| \cos\alpha = \mathbf{p} \bullet \mathbf{d},$$

Example 3 is typical of applications that deal with the **component** or **projection** *of a vector* $\mathbf{a}$ *in the direction of a vector* $\mathbf{b}\,(\neq \mathbf{0})$. If we denote by p the length of the orthogonal projection of $\mathbf{a}$ on a straight line l parallel to $\mathbf{b}$ as shown in Fig. 181, then

$$\text{(10)}\qquad p = |\mathbf{a}| \cos\gamma.$$

Here p is taken with the plus sign if $p\mathbf{b}$ has the direction of $\mathbf{b}$ and with the minus sign if $p\mathbf{b}$ has the direction opposite to $\mathbf{b}$.

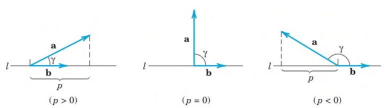


Fig. 181. Component of a vector **a** in the direction of a vector **b**

Multiplying (10) by $|\mathbf{b}|/|\mathbf{b}| = 1$, we have $\mathbf{a} \bullet \mathbf{b}$ in the numerator and thus

$$\text{(11)}\qquad p = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|} \qquad (\mathbf{b} \neq \mathbf{0}).$$

If $\mathbf{b}$ is a unit vector, as it is often used for fixing a direction, then (11) simply gives

$$\text{(12)}\qquad p = \mathbf{a} \bullet \mathbf{b} \qquad (|\mathbf{b}| = 1).$$

Figure 182 shows the projection p of $\mathbf{a}$ in the direction of $\mathbf{b}$ (as in Fig. 181) and the projection $q = |\mathbf{b}| \cos\gamma$ of $\mathbf{b}$ in the direction of $\mathbf{a}$.

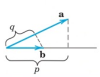


Fig. 182. Projections p of **a** on **b** and q of **b** on **a**

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma,$$

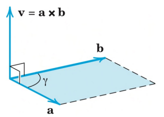


Fig. 185. Vector product

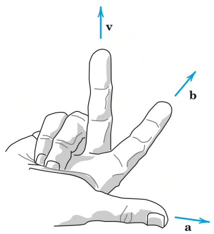


Fig. 186. Right-handed triple of vectors **a**, **b**, **v**

$$(2^{**}) \qquad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

General Properties of Vector Products

(a) *For every scalar l,*

$$(4) \qquad (l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) *Cross multiplication is distributive with respect to vector addition; that is,*

$$(5) \qquad \begin{aligned} &(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}), \\ &(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}). \end{aligned}$$

(c) *Cross multiplication is* ***not commutative*** *but* ***anticommutative;*** *that is,*

$$\textbf{(6)} \qquad \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \qquad \text{(Fig. 189).}$$

(d) *Cross multiplication is* ***not associative****; that is, in general,*

$$(7) \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

so that the parentheses cannot be omitted.

$$(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

THEOREM 2

Properties and Applications of Scalar Triple Products

(a) *In* (10) *the dot and cross can be interchanged:*

$$(11) \qquad (\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

(b) Geometric interpretation. *The absolute value* $|(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c})|$ *of* (10) *is the volume of the parallelepiped* (oblique box) *with* **a**, **b**, **c** *as edge vectors* (Fig. 193).

(c) Linear independence. *Three vectors in* R^3 *are linearly independent if and only if their scalar triple product is not zero.*

$$(c\mathbf{v})' = c\mathbf{v}' \qquad (c \text{ constant}),$$
$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

$$(\mathbf{u} \bullet \mathbf{v})' = \mathbf{u}' \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{v}'$$
$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$
$$(\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}').$$

$$l = \int_a^b \sqrt{\mathbf{r}' \bullet \mathbf{r}'}\, dt \qquad \left(\mathbf{r}' = \frac{d\mathbf{r}}{dt}\right).$$

Constant

the *arc length function* or simply the **arc length** of C. Thus

$$\text{(11)} \qquad s(t) = \int_a^t \sqrt{\mathbf{r}' \bullet \mathbf{r}'}\, d\tilde{t} \qquad \left(\mathbf{r}' = \frac{d\mathbf{r}}{d\tilde{t}}\right).$$

$$\text{(13)} \qquad ds^2 = d\mathbf{r} \bullet d\mathbf{r} = dx^2 + dy^2 + dz^2.$$

ds is called the **linear element** of C.

$$\mathbf{u}(s) = \mathbf{r}'(s).$$

$$\kappa(s) = |\mathbf{u}'(s)| = |\mathbf{r}''(s)| \qquad ({}' = d/ds).$$

$$|\tau(s)| = |\mathbf{b}'(s)|.$$

$$\tau(s) = -\mathbf{p}(s) \bullet \mathbf{b}'(s).$$

W4

W4 - Gradient Divergence and Curl

- Gradient of a Scalar Field
  - Gradient
    - Input: Scalar Field
    - Output: Vector Field
    - $\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$
  - Directional Derivative
    - Input: f(x, y, z), p, a
    - Output: Scalar
    - Calculation
      1. get grad f
      2. Substitute point to get grad f(P)
      3. b = Normalize of a
      4. b*grad f(P)
  - Gradient is a vector. Maximum increase
  - Gradient as Surface Normal Vector
  - Vector Fields that are gradients of scalar fields (Potentials)
    - Conservative
      - no energy lost or gain
- Divergence of a Vector Field
  - Divergence
    - Input: Vector Field
    - Output: Scalar Field
    - $\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v}$ $= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$
    - Calculation
      1. Get div v
      2. Substitute P(x, y, z), get div at p
    - Measures outflow minus inflow
  - Invariance of the divergence
    - $\operatorname{div} \mathbf{v} = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*}.$
  - Laplacian
    - The divergence of the gradient is the Laplacian
      - $\operatorname{div} \mathbf{v} = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$
      - $\operatorname{div}(\operatorname{grad} f) = \nabla^2 f.$
  - Application
    - Gravitational Force
      - Laplacian Equation = 0
    - Flow of a compressible fluid
      - incompressibility: div v = 0
- Curl of a Vector Field
  - 3D
    - Input: Vector Field
    - Output: Vector Field
    - Need review 2D. Wiki might not agree with Khan?
      - Input: Vector Field
      - Output: Scalar Field
  - $\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$
  - T1: Rotating body
    - Direction: the axis of the rotation
    - Magnitude: Twice the angular speed of the rotation
  - T2:
    - Gradient fields are irrotational
    - $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}.$
      - 0 vector
    - $\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0.$
      - 0 Scalar
    - irrotational: curl v = 0
  - T3: Invariance of the curl

9.6

chain rule

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

9.7. Grad.

$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

$$D_{\mathbf{b}} f = \frac{df}{ds} = \mathbf{b} \bullet \operatorname{grad} f \qquad (|\mathbf{b}| = 1).$$

$$D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|}\mathbf{a} \bullet \operatorname{grad} f.$$

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{p} = \frac{k}{r^3}\mathbf{r}$$ **(Coulomb's law[6]).**

All vectors of the form $\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ constitute the **real vector space** R^3 with componentwise vector addition

$$(1) \qquad [a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

and componentwise scalar multiplication (c a scalar, a real number)

$$(2) \qquad c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3] \qquad \text{(Sec. 9.1).}$$

For instance, the *resultant* of forces $\mathbf{a}$ and $\mathbf{b}$ is the sum $\mathbf{a} + \mathbf{b}$.

The **inner product** or **dot product** of two vectors is defined by

$$(3) \qquad \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(Sec. 9.2)}$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This gives for the **norm** or **length** $|\mathbf{a}|$ of $\mathbf{a}$

$$(4) \qquad |\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

as well as a formula for γ. If $\mathbf{a} \bullet \mathbf{b} = 0$, we call $\mathbf{a}$ and $\mathbf{b}$ **orthogonal**. The dot product is suggested by the *work* $W = \mathbf{p} \bullet \mathbf{d}$ done by a force $\mathbf{p}$ in a displacement $\mathbf{d}$.

The **vector product** or **cross product** $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is a vector of length

$$(5) \qquad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma \qquad \text{(Sec. 9.3)}$$

and perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$ form a *right-handed* triple. In terms of components with respect to right-handed coordinates,

$$(6) \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{(Sec. 9.3).}$$

The vector product is suggested, for instance, by moments of forces or by rotations. CAUTION! This multiplication is *anti*commutative, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and is *not* associative.

An (oblique) box with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ has volume equal to the absolute value of the **scalar triple product**

$$(7) \qquad (\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Sections 9.4–9.9 extend differential calculus to vector functions

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$$

and to vector functions of more than one variable (see below). The derivative of $\mathbf{v}(t)$ is

$$(8) \qquad \mathbf{v}' = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = [v_1', v_2', v_3'] = v_1'\mathbf{i} + v_2'\mathbf{j} + v_3'\mathbf{k}.$$

Differentiation rules are as in calculus. They imply (Sec. 9.4)

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \qquad (\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'.$$

Curves C in space represented by the position vector $\mathbf{r}(t)$ have $\mathbf{r}'(t)$ as a **tangent vector** (the **velocity** in mechanics when t is time), $\mathbf{r}'(s)$ (s arc length, Sec. 9.5) as the *unit tangent vector,* and $|\mathbf{r}''(s)| = \kappa$ as the *curvature* (the *acceleration* in mechanics).

Vector functions $\mathbf{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$ represent vector fields in space. Partial derivatives with respect to the Cartesian coordinates x, y, z are obtained componentwise, for instance,

$$\frac{\partial \mathbf{v}}{\partial x} = \left[\frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial x}, \frac{\partial v_3}{\partial x}\right] = \frac{\partial v_1}{\partial x}\mathbf{i} + \frac{\partial v_2}{\partial x}\mathbf{j} + \frac{\partial v_3}{\partial x}\mathbf{k} \qquad \text{(Sec. 9.6).}$$

The **gradient** of a scalar function f is

$$(9) \qquad \operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \qquad \text{(Sec. 9.7).}$$

The **directional derivative** of f in the direction of a vector $\mathbf{a}$ is

$$(10) \qquad D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \nabla f \qquad \text{(Sec. 9.7).}$$

The **divergence** of a vector function $\mathbf{v}$ is

$$(11) \qquad \operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}. \qquad \text{(Sec. 9.8).}$$

The **curl** of **v** is

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{12}$$ (Sec. 9.9)

or minus the determinant if the coordinates are left-handed.

Some basic formulas for grad, div, curl are (Secs. 9.7–9.9)

(13)
$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$$

(14)
$$\text{div}\,(f\mathbf{v}) = f\,\text{div}\,\mathbf{v} + \mathbf{v} \cdot \nabla f$$
$$\text{div}\,(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$$

(15)
$$\nabla^2 f = \text{div}\,(\nabla f)$$
$$\nabla^2(fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g$$

(16)
$$\text{curl}\,(f\mathbf{v}) = \nabla f \times \mathbf{v} + f\,\text{curl}\,\mathbf{v}$$
$$\text{div}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$$

(17)
$$\text{curl}\,(\nabla f) = \mathbf{0}$$
$$\text{div}\,(\text{curl}\,\mathbf{v}) = 0.$$

For grad, div, curl, and ∇^2 in **curvilinear coordinates** see App. A3.4.