Chapter 7 Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

P261 - Problem set 7.1

1. 2x2: $a_{11} \neq b_{11}, b_{12} \neq c_{12}$, 2x3: $d_{11} \neq e_{11}$
2. $a_{31} = 10, \ \ a_{13} = 81 \ \ , a_{26} = 96, \ \ a_{33} = 0$
3. E1: 3x3, 3x4,
   E2: 3x7,
   E3: 2x2, 2x2, 2x2, 2x3, 2x3,
   E5:3x2
4. 1A: 4, 0, 1
   3A: a_{11}, a_{22}
   3B: 4, -1
5. $B = \frac{1}{5}A$,
   $B = \frac{1}{10}A$
6. $B = \frac{1}{1.609}A$
7. No. No(1x1 as exception?). Yes. Maybe not in math (how about 1x1?) but OK in python. No.
8. $2A + 4B = 4B + 2A = \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix}$
   B
   $0.4B - 4.2A = \begin{bmatrix} 0 & -6.4 & -16 \\ -23.2 & -19.8 & -19.4 \\ -5 & 1.6 & 11.8 \end{bmatrix}$
9. $3A = \begin{bmatrix} 0 & 6 & 12 \\ 18 & 15 & 15 \\ 3 & 0 & -9 \end{bmatrix}$
   $0.5B = \begin{bmatrix} 0 & 2.5 & 1 \\ 2.5 & 1.5 & 2 \\ -1 & 2 & -1 \end{bmatrix}$
   $3A + 0.5B = \begin{bmatrix} 0 & 8.5 & 13 \\ 20.5 & 16.5 & 17 \\ 2 & 2 & -10 \end{bmatrix}$
   3A+0.5B+C is not defined.

10. $(4 \bullet 3)A = 4(3A) = \begin{bmatrix} 0 & 24 & 48 \\ 72 & 60 & 60 \\ 12 & 0 & -36 \end{bmatrix}$

$14B - 3B = 11B = \begin{bmatrix} 0 & 55 & 22 \\ 55 & 33 & 44 \\ -22 & 44 & -22 \end{bmatrix}$

11. $8C + 10D = 2(5D + 4C) = \begin{bmatrix} 0 & 26 \\ 34 & 32 \\ 28 & -10 \end{bmatrix}$

$0.6C - 0.6D = 0.6(C - D) = \begin{bmatrix} 5.4 & 0.6 \\ -4.2 & 2.4 \\ -0.6 & 0.6 \end{bmatrix}$

12. $(C + D) + E = (D + E) + C = \begin{bmatrix} 1 & 5 \\ 6 & 8 \\ 6 & -2 \end{bmatrix}$

$0(C - E) + 4D = 4D = \begin{bmatrix} -16 & 4 \\ 20 & 0 \\ 8 & -4 \end{bmatrix}$

$A - 0C$: 3x3 can not minus 3x2, not defined

13. $(2 \bullet 7)C = 2(7C) = \begin{bmatrix} 70 & 28 \\ -28 & 56 \\ 14 & 0 \end{bmatrix}$

$-D + 0E = -D = \begin{bmatrix} 4 & -1 \\ -5 & 0 \\ -2 & 1 \end{bmatrix}$

$E - D + C + u$: Since EDC are 3x2 but u is 3x1, not defined.

14. $(5u + 5v) - \frac{1}{2}w = \begin{bmatrix} 5 \\ 30 \\ -10 \end{bmatrix}$

$-20(u + v) + 2w = -4[(5u + 5v) - \frac{1}{2}w] = \begin{bmatrix} -20 \\ -120 \\ 40 \end{bmatrix}$

$E - (u + v)$: 3x2 can not minus 3x1, not defined

$10(u + v) + w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

15. $(u + v) - w = u + (v - w) = \begin{bmatrix} 5.5 \\ 33 \\ -11 \end{bmatrix}$

$C + 0w$: 3x2 can not minus 3x1, not defined

$0E + u - v$: 3x2 can not minus 3x1, not defined

16. $15v - 3w - 0u = -3w + 15v = \begin{bmatrix} 0 \\ 135 \\ 0 \end{bmatrix}$

$D - u + 3C$: 3x2 can not minus 3x1, not defined

$8.5w - 11.1u + 0.4v = \begin{bmatrix} 25.45 \\ 256.2 \\ 119.1 \end{bmatrix}$

17. $u + v + w = \begin{bmatrix} -4.5 \\ -27 \\ 9 \end{bmatrix}$

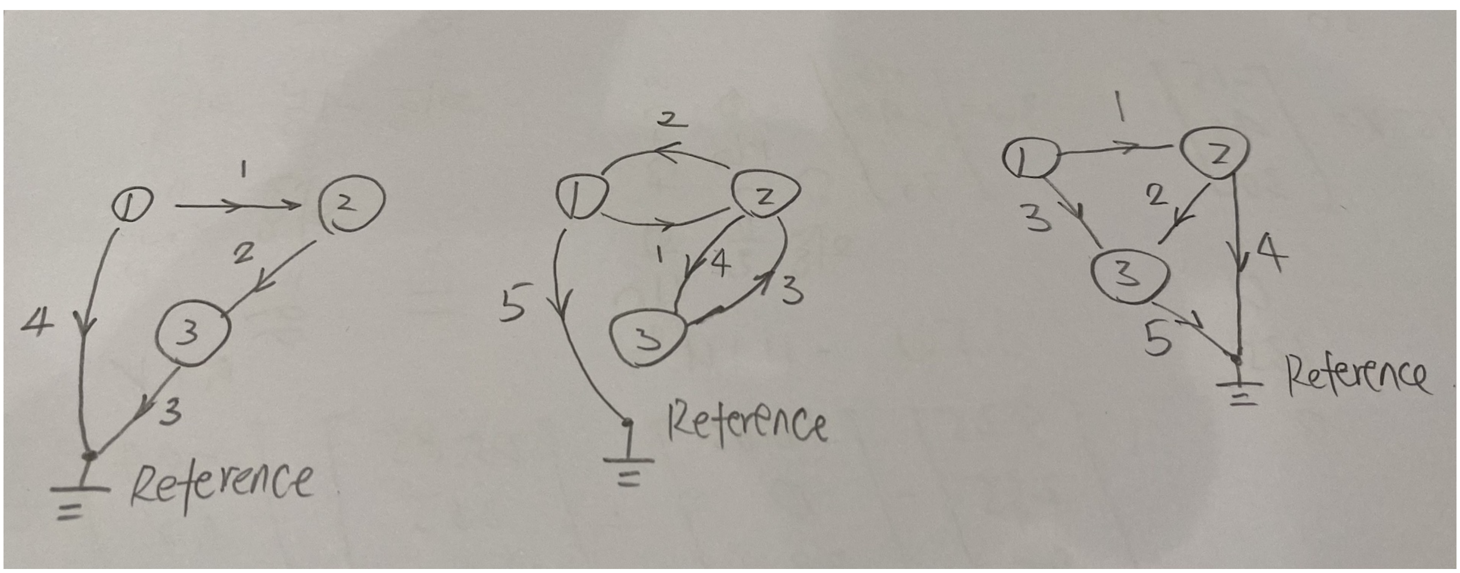
18. $p = 0 - u - v - w = \begin{bmatrix} 4.5 \\ 27 \\ -9 \end{bmatrix}$

19. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

20. b-1: $\begin{bmatrix} -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

b-2: $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

c:



P270 - Problem set 7.2

Example13. In the final stable situation(limit),

$I + C + R = 100$

$0.7C + 0.1I = C$
$0.2C + 0.9I + 0.2R = I$
$0.1C + 0.8R = R$
So we can get C=200/9, I=200/3, R=100/9.
Will revisit it after Sec. 8.2

1. Per definition, the number of the entries in the columns of the second matrix have to be same as the number of the entries in the rows of the first matrix. In short, if mxn matrix multiple pxq, then n=p. Or you won't be able to perform the dot product.
2. All entries or components are 0
3. No. All rows are proportional.
4. Min is 1 which is 0, and max is $n(n-1)+1$
   Take 3x3 as example, $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$
5. Min is 1 which is 0, and max is $\frac{n(n+1)}{2}$
   Take 3x3 as example, $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$
6. $U_1 + U_2, U_1U_2, U_1^2$ are upper triangular matrices.$L_1 + L_2$ is lower triangular.
7. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
8. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for any $m \geq 1, m \in N$. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ for any $m \geq 2, m \in N$.
9. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
10. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
11. $AB = AB^T = \begin{bmatrix} 10 & -14 & -6 \\ -5 & 7 & -12 \\ -5 & -1 & -4 \end{bmatrix}$

$$BA = B^TA = \begin{bmatrix} 10 & -5 & -15 \\ -14 & 7 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

12. $AA^T = \begin{bmatrix} 29 & 8 & 6 \\ 8 & 41 & 12 \\ 6 & 12 & 9 \end{bmatrix}, A^2 = \begin{bmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{bmatrix}, BB^T = B^2 = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

13. $CC^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 13 & -6 \\ 0 & -6 & 4 \end{bmatrix}$, $BC = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}$, CB not defined, $C^T B = \begin{bmatrix} -9 & 3 & 4 \\ -5 & -1 & 0 \end{bmatrix}$

14. $3A - 2B = \begin{bmatrix} 10 & 0 & 9 \\ 0 & 1 & 18 \\ 3 & 6 & 10 \end{bmatrix}$, $(3A-2B)^T = 3A^T - 2B^T = \begin{bmatrix} 10 & 0 & 3 \\ 0 & 1 & 6 \\ 9 & 18 & 10 \end{bmatrix}$,

$(3A-2B)^T a^T = \begin{bmatrix} 10 \\ -2 \\ -27 \end{bmatrix}$

15. Aa not defined, $Aa^T = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$, $(Ab)^T = b^T A^T = \begin{bmatrix} 7 & -11 & 3 \end{bmatrix}$

16. $BC = Problem 13.2 = \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}$, BC^T not defined, $Bb = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix}$, $b^T B =$

$\begin{bmatrix} 0 & -8 & 2 \end{bmatrix}$

17. $ABC = \begin{bmatrix} -30 & -18 \\ 45 & 9 \\ 5 & -7 \end{bmatrix}$, ABa not defined, $ABb = \begin{bmatrix} 22 \\ 4 \\ -12 \end{bmatrix}$, Ca^T= not defined.

18. $ab = 1$, $ba = \begin{bmatrix} 3 & -6 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$, $aA = \begin{bmatrix} 8 & -4 & -9 \end{bmatrix}$, $Bb = problem 16.3 = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix}$

19. $1.5a + 3.0b$ not defined. $1.5a^T + 3.0b = \begin{bmatrix} 4.5 \\ -2 \\ -1 \end{bmatrix}$, $(A - B)b = Ab - Bb = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$

20. $b^T Ab$=7, aBa^T=17, $aCC^T = \begin{bmatrix} -3 & -24 & 12 \end{bmatrix}$, $C^T ba = \begin{bmatrix} 5 & -10 & 0 \\ 5 & -10 & 0 \end{bmatrix}$

21. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

22. $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$, $AB = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$

23. $AB = A \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix}$

24. $AB = BA$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$2a_{11} + 3a_{12} = 2a_{11} + 3a_{21}$=> $a_{12} = a_{21}$

$3a_{11} + 4a_{12} = 2a_{12} + 3a_{22}$=> $3a_{11} + 2a_{12} = 3a_{22}$

$2a_{21} + 3a_{22} = 3a_{11} + 4a_{21}$,

$3a_{21} + 4a_{22} = 3a_{12} + 4a_{22}$

Let $A = \begin{bmatrix} x & y \\ y & \frac{3x+2y}{3} \end{bmatrix}$,

Check: $AB = BA = \begin{bmatrix} 2x+3y & 3x+4y \\ 3x+4y & 4x+5\frac{2}{3}y \end{bmatrix}$

25. a) Obvious.

b) $C = [c_{ij}], C^T = [c_{ji}]$

$D = C + C^T = [d_{ij}] = [c_{ij} + c_{ji}] = [c_{ji} + c_{ij}] = [d_{ji}]$, so D is symmetric

$E = C - C^T = [e_{ij}] = [c_{ij} - c_{ji}] = -[c_{ji} - c_{ij}] = -[e_{ji}]$, so E is skew-symmetric.

Let $S = \frac{1}{2}D, T = \frac{1}{2}E$

$S + T = \frac{1}{2}(D + E) = \frac{1}{2}(C + C^T + C - C^T) = C$

$A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}, A^T = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 6 & 2 \end{bmatrix},$

$S = \frac{1}{2}(A + A^T) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix}, T = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B^T = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$S = \frac{1}{2}(B + B^T) = B = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, T = \frac{1}{2}(B - B^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c) symmetric: $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}], ..., M = [m_{ij}] = [m_{ji}]$

$aA + bB + ... + mM = a[a_{ij}] + b[b_{ij}] + ... + m[m_{ij}] = a[a_{ji}] + b[b_{ji}] + +... + m[m_{ji}]$.

Skew-symmetric: $A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}], ..., M = [m_{ij}] = -[m_{ji}]$

$aA + bB + ... + mM = a[a_{ij}] + b[b_{ij}] + ... + m[m_{ij}] = -(a[a_{ji}] + b[b_{ji}] + +... + m[m_{ji}])$

.

d) $A = [a_{ij}] = [a_{ji}], B = [b_{ij}] = [b_{ji}]$

$AB = [a_p b_q]$, if AB is symmetric, then $AB = [a_p b_q] = [a_q b_p] = [b_p a_q] = BA$

vice verse.

e)$A = [a_{ij}] = -[a_{ji}], B = [b_{ij}] = -[b_{ji}]$

$AB = [a_p b_q]$, if AB is skew-symmetric, then $AB = [a_p b_q] = -[a_q b_p] = -[b_p a_q] = -BA$

vice verse.

26. First day, status = $\begin{bmatrix} N \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, stochastic matrix = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$

Second day = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$

Two days after today = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$

Three days after today = $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.722 \\ 0.278 \end{bmatrix}$

The limit of N is $\frac{5}{7}$

27. Reserve for future

28. Present = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 1200 \\ 98800 \end{bmatrix}$, stochastic matrix = $\begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix}$

After 1 season = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1200 \\ 98800 \end{bmatrix} = \begin{bmatrix} 1278 \\ 98722 \end{bmatrix}$, increase

After 2 seasons = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1278 \\ 98722 \end{bmatrix} = \begin{bmatrix} 1344 \\ 98656 \end{bmatrix}$, increase

After 3 seasons = $\begin{bmatrix} Subs. \\ Not \end{bmatrix} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix} \begin{bmatrix} 1344 \\ 98656 \end{bmatrix} = \begin{bmatrix} 1407 \\ 98593 \end{bmatrix}$, increase

29. $p = \begin{bmatrix} 35 \\ 62 \\ 30 \end{bmatrix}$

$v = Ap = \begin{bmatrix} 24,920 \\ 25,940 \end{bmatrix}$

30.$y = Ax$

$y_1 = x_1 \cos\theta - x_2 \sin\theta, y_2 = x_1 \sin\theta + x_2 \cos\theta$

$|y|^2 = (x_1 \cos\theta - x_2 \sin\theta)^2 + (x_1 \sin\theta + x_2 \cos\theta)^2 = x_1^2 + x_2^2 = |x|^2$

$\cos\alpha = \frac{x*y}{|x||y|} = \frac{x_1^2 \cos\theta + x_2^2 \cos\theta}{x_1^2 + x_2^2} = \cos\theta$

so x and y have the same length, and from x to y is counterclockwise rotate of θ

b)$AA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} =$
$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

c) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} =$
$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

d)$[x_1, x_2, x_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = [3x_1, x_2, \frac{1}{2}x_3]$

$[x_1, x_2, x_3] \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} = [cx_1, cx_2, cx_3]$, Scalar matrix will amplify or squeeze the picture by c.

e) $[x_1, x_2, x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = [x_1, x_2\cos\theta + x_3\sin\theta, -x_2\sin\theta + x_3\cos\theta]$

$x_1$remain the same. counterclockwise rotation of the Cartesian coordinate system x_2x_3 in the plane about the origin by angle of θ

$[x_1, x_2, x_3] \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} = [x_1\cos\varphi + x_3\sin\varphi, x_2, -x_1\sin\varphi + x_3\cos\varphi]$

$x_2$remain the same. counterclockwise rotation of the Cartesian coordinate system x_1x_3 in the plane about the origin by angle of φ

$[x_1, x_2, x_3] \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = [x_1\cos\psi + x_2\sin\psi, -x_1\sin\psi + x_2\cos\psi, x_3]$

$x_3$remain the same. counterclockwise rotation of the Cartesian coordinate system x_1x_2 in the plane about the origin by angle of ψ

P280 - Problem set 7.3

1. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{6}{5} \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -4 & 1 & -2 \\ 0 & 17 & -4 & 12 \\ 0 & -34 & 8 & -13 \end{bmatrix}$ = No solution

5. $\begin{bmatrix} 1 & 33 & -225 \\ 0 & 139 & -973 \\ 0 & -376 & 2632 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -7 \\ 0 & 1 & -7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -2 & 2 & 9 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -5 & -20 \end{bmatrix} = \begin{bmatrix} 2t+1 \\ t \\ 4 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 5 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3t \\ t \\ 2t \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 3 & 8 \\ 0 & 4 & 3 & 4 \end{bmatrix}$ = No solution

9. $\begin{bmatrix} 3 & 4 & -5 & 13 \\ 0 & 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 3t-1 \\ 4-t \\ t \end{bmatrix}$

10. $\begin{bmatrix} 5 & -7 & 3 & 17 \\ 5 & -7 & 3 & -50/3 \end{bmatrix}$ = No solution

11. $\begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so we can get $\begin{bmatrix} 1 \\ 2m-n \\ n \\ m \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & -1 & 2 & -5/3 & -5 \end{bmatrix}$, so we can get $\begin{bmatrix} n-2m \\ n \\ m \\ 3 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 10 & 4 & -2 & -4 \\ -3 & -17 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 10 & 4 & -2 & -4 \\ 0 & -14 & 4 & 2 & 20 \\ 0 & 21 & -4 & 5 & 22 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -12 & 9 & 30 \\ 0 & 0 & -41 & 32 & 110 \\ 0 & 0 & -31 & 23 & 76 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 2 & 3 & 1 & -11 & 1 \\ 5 & -2 & 5 & -4 & 5 \\ 3 & 4 & -7 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 3 & -10 & 11 & -10 \\ 0 & 7 & -16 & 11 & -16 \end{bmatrix} =$
$\begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$

so we can get $\begin{bmatrix} 0 \\ 3t \\ 1+2t \\ t \end{bmatrix}$

15. Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.
16. Reserve for future
17. $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 16 \\ 0 & 4 & 1 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 4 & 1 & 32 \\ 0 & 0 & 6 & 48 \end{bmatrix}$

    so we can get $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$
18. I am highly unsure about this one. Need a physical book

    $\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 4 & 1 & 9 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

    so we can get $\begin{bmatrix} 9/5 \\ 12/5 \\ -3/5 \end{bmatrix}$
19. $\begin{bmatrix} \frac{E_0}{R_2} + \frac{E_0}{R_1} \\ -\frac{E_0}{R_1} \\ \frac{E_0}{R_2} \end{bmatrix}$
20. $I_3 = I_x, I_1 = I_2$

    $I_1R_1 = I_xR_x, I_3R_3 = I_2R_2$, so we can get

    $R_x = R_3R_1/R_2$
21. $\begin{bmatrix} 1 & 1 & 0 & 0 & 1600 \\ 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 1 & 0 & 2200 \\ 0 & 0 & 1 & 1 & 1600 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1600 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 0 & 1 & 1 & 1600 \end{bmatrix}$

Rank(3) < N (4), So the solutin is not unique.

22. $40 - 2P_1 - P2 = 4P_1 - P_2 + 4, 6P_1 = 36, P_1 = 6$
$5P_1 - 2P_2 + 16 = 3P_2 - 4, P_1 = P_2 - 4, P_2 = 2$

23. $\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4 & -3 & 0 \end{bmatrix}$

so we can get $\begin{bmatrix} t \\ 5t \\ 3t \\ 4t \end{bmatrix}$

The smallest positive integers are 1, 5, 3, 4

24. a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \\ 10 & 11 & 12 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

$$B = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ -5a_{11} + a_{21} & -5a_{12} + a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11} & a_{12} \\ -5a_{11} + a_{31} & -5a_{12} + a_{32} \\ a_{21} & a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

So $B \neq C$

b)Natually.

Row switch: reference E1

Row multiplication: reference E3 (replace by c)

Row addition and subtraction: reference E2.

Expand metrics with entries a_{ij}, then follow the basic arithmetic rule.

P287 - Problem Set 7.4

1. $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, Rank=1, $\{[2, -1, 3]\}$

$$A^T = \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \{[2,-1]^T\}$$

2. $\begin{bmatrix} a & b \\ a & \frac{a^2}{b} \end{bmatrix}$,
   if $a = b = 0$, rank = 0, $\{0\}$, $\{0\}$
   if $b = \pm a$, rank = 1, $\{[1,-1]\}$, $\{[1,-1]^T\}$
   The rest, rank = 2, $\{[a,b],[b,a]\}$, $\{[a,b]^T,[b,a]^T\}$

3. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix}$, rank = 3, $\{[1,0,0],[0,1,0],[0,0,1]\}$,
   $\{[1,0,0]^T,[0,1,0]^T,[0,0,1]^T\}$,

4. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, rank = 3, $\{[1,0,0],[0,1,0],[0,0,1]\}$,
   $\{[1,0,0]^T,[0,1,0]^T,[0,0,1]^T\}$,

5. $\begin{bmatrix} 1 & 0 & -21 \\ 0 & 11 & -3 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -21 \\ 0 & 11 & -3 \\ 0 & 0 & 1 \end{bmatrix}$, rank = 3, $\{[1,0,0],[0,1,0],[0,0,1]\}$,
   $\{[1,0,0]^T,[0,1,0]^T,[0,0,1]^T\}$,

6. $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, rank = 2, $\{[1,1,4],[0,1,0]\}$,

$$A^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$\{[1,-1,4]^T,[0,1,0]^T,\}$,

7. $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, rank = 2, $\{[2,0,1,0],[0,1,0,2]\}$,

$$A^T = \begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \{[2,0,1]^T,[0,1,0]^T\},$$

8. $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 12 & 30 & 63 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 12 & 30 & 63 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 30 & 75 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rank = 4,

$\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$,

$\{[1,0,0,0]^T,[0,1,0,0]^T,[0,0,1,0]^T,[0,0,0,1]^T\}$

9. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 9 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 9 & 8 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, rank = 3, $\{[1,1,1,1],[0,9,8,9],[0,0,1,0]\}$,

$$A^T = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \{[9,0,1,0]^T,[0,0,1,0]^T,[1,1,1,1]^T\}$$

10. $\begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 5 & -2 & 1 & 0 \\ -2 & 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{bmatrix}$, rank = 3,

$\{[1,-4,-11,2],[0,1,2,0],[0,0,2,-1]\}$,

$$A^T = A, \{[1,-4,-11,2]^T,[0,1,2,0]^T,[0,0,2,-1]^T\}$$

11. New row 1 = row 2 - row 1 = [1, 1, ... , 1]

    Add new row 1 to row k will get row k+1. so rank = 2, base is row 1 and row 2.

    b) Same

    c) All rows similar to row 1, just matter of factor 2^k. So rank = 1

12. $Rank(AB) = Rank[(AB)^T] = Rank(B^T A^T)$

13. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

14. Let A is a $m \times n$ matrix, and assume $m > n$

    $Rank(A) \leq n < m$. so A is linearly dependent on the row vectors

    verse vise, L.D on the column vectors

15. n = Rank of row = rank of column

16. Matrix A, B, AB.

    Let A as the base of the vector space V(A), then V(AB) is the subset of V(A).

    $Rank(A) = dim[V(A)] \geq dim[V(AB)] = Rank(AB)$

    If B is nonsingular, then Rank(A)=Rank(AB)

    Vise verse on B.

17. $\begin{bmatrix} 1 & 16 & -12 & -22 \\ 3 & 4 & 0 & 2 \\ 2 & -1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 16 & -12 & -22 \\ 0 & 11 & -9 & -17 \\ 0 & 33 & -37 & -51 \end{bmatrix} = \begin{bmatrix} 1 & 16 & -12 & -22 \\ 0 & 11 & -9 & -17 \\ 0 & 0 & 10 & 0 \end{bmatrix}$

    Linear independent.

18. $\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 30 & 20 & 15 & 12 \\ 20 & 15 & 12 & 10 \\ 105 & 84 & 70 & 60 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1 & 1 & 0.9 \\ 0 & 15 & 16 & 15 \\ 0 & 126 & 140 & 135 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1 & 1 & 0.9 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 14 & 21.6 \end{bmatrix}$

    Rank = 4, Linear independent.

19. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

    Rank = 3, Linear independent.

20. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

    Rank = 2, Linear Dependent.

21. $\begin{bmatrix} 2 & 0 & 0 & 7 \\ 2 & 0 & 0 & 8 \\ 2 & 0 & 0 & 9 \\ 2 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

    Rank = 3, Linear Dependent.

22. V1 * 30/4 = V3, rank=1. Linear Dependent.

23. $\begin{bmatrix} 9 & 8 & 7 & 6 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

    Rank = 2, Linear independent.

24. 4 rows 3 column, Linear Dependent.

25. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 6 & 0 & -1 & 3 \\ 2 & 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 6 & 7 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

    Rank = 3, Linear independent.
26. $V_4 = 2V_1$, discard V_4

    $\begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -2 & -4 \end{bmatrix} discardV_3
27. Yes, dimension=2, {[-2, 0, 1], [0, 2, 1]}
28. k=0, Yes, dimension=2, {[1, 0, 0], [0, 1, -3]}

    if k!=0, No. 2*V is not in the set.
29. No. -1*V not in the set.
30. n = 2, dimension = 1, {0}.

    $n > 2$. dimension = 2, {[0, 0 ... 1, 0], [0, 0 ... 0, 1]}
31. No. -1 * V not include in the set.
32. Yes, dimension=1, {[-5/4, 1, -23/4]}
33. Yes, dimension=1, {[1, 10/3, 3]}
34. No. 2 * V not in the set.
35. Yes, dimension=1, {[1, 1/2, 1/3, 1/4]}

P300 - Problem set 7.7

Theorems 1-a: we change from right handed to the left handed, so we get -1?

1. Theorems 1-a) $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

    $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

Theorems 1-b) $\begin{vmatrix} 1 & 0 \\ c & 1 \end{vmatrix} = 1 - 0 = 1$

Theorems 1-c) $\begin{vmatrix} 1 & 0 \\ 0 & c \end{vmatrix} = c - 0 = c$

Theorems 2-a) $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

Theorems 2-b) $\begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

Theorems 2-c) $\begin{vmatrix} 1 & 0 \\ 0 & c \end{vmatrix} = c - 0 = c$

Theorems 2-d) In this example, $A = A^T = 1$

Theorems 2-e) $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$

Theorems 2-f) $\begin{vmatrix} 1 & 2 \\ a & 2a \end{vmatrix} = 2a - 2a = 0$

2. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$
   $= a_{11}|a_{22}| - a_{12}|a_{21}| = a_{11}a_{22} - a_{12}a_{21}$
   $= a_{11}|a_{22}| - a_{21}|a_{12}| = a_{11}a_{22} - a_{12}a_{21}$
   $= a_{22}|a_{11}| - a_{12}|a_{21}| = a_{11}a_{22} - a_{12}a_{21}$
   $= a_{22}|a_{11}| - a_{21}|a_{12}| = a_{11}a_{22} - a_{12}a_{21}$
3. My guess is Example 2 but not Theoream 2?
4. Not sure why we want to expand the high order determinant. Gauss elimination obviously a better option. It takes n^3 (I heard it can improve), which is way better than $n!$
5. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1, \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix} = k^2 - 0 = k^2$
6. My guess is Example 2 but not Example 1?
7. $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha + \beta)$
8. -7.87
9. $\cos(n\theta)\cos(n\theta) + \sin(n\theta)\sin(n\theta) = \cos(n\theta - n\theta) = 1$
10. $\cosh t \cosh t - \sinh t \sinh t = \cosh(t - t) = \frac{1}{2}(e^0 + e^{-0}) = 1$
11. 40
12. $a^3 + b^3 + c^3 - 3abc$
13. $0 \cdot (0 + 6 + -6 - 0 - 0 - 0) - 4 \cdot (0 + -15 + 2 - 0 - 0 - 4) + (-1) \cdot (0 + 0 + -4 - -30 - 0 - -8) - 5 \cdot (-12 + 0 + 6 - 45 - 0 - 0) = 0 - 4 \cdot (-17) + (-1) \cdot (34) - 5 \cdot (-51) = 289$
14. Question: I feel we can do it in the below way, with certain condition. Can not remember what exactly it is, and it does not apply for 13. The result is same while I expand the 4th order determinants.
    $\begin{vmatrix} 4 & 7 \\ 2 & 8 \end{vmatrix} \begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} = (32 - 14)(2 + 10) = 216$

P.S: it is called block matrices. for upper (lower) triangular block matrix, diagonal blocks $A_1, A_2..A_n$, and we will get $det = det(A_1)det(A_2)..det(A_n)$.

15. $\begin{vmatrix}1&2&0&0\\2&4&2&0\\0&2&9&2\\0&0&2&16\end{vmatrix} = \begin{vmatrix}1&2&0&0\\0&0&2&0\\0&2&9&2\\0&0&2&16\end{vmatrix} = -\begin{vmatrix}1&2&0&0\\0&2&9&2\\0&0&2&0\\0&0&2&16\end{vmatrix} = -\begin{vmatrix}1&2&0&0\\0&2&9&2\\0&0&2&0\\0&0&0&16\end{vmatrix} = -64$

16. $\begin{vmatrix}0&1\\1&0\end{vmatrix} = 0-1=-1$

$\begin{vmatrix}0&1&1\\1&0&1\\1&1&0\end{vmatrix} = 2$

$\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix} = -3$

So I would assume this special n order matrix have determinants $(-1)^{n-1}(n-1)$

Try to prove it by induction - Placeholder

Incidence Matrices ??

17. $\begin{vmatrix}4&9\\-8&-6\end{vmatrix} = -24+72 \neq 0$

$\begin{bmatrix}4&9\\0&12\\0&24\end{bmatrix}$, rank = 2

18. $\begin{vmatrix}4&4&4\\4&0&10\\-6&10&0\end{vmatrix} = 0+(-240)+(-240)-0-0-0>0$

$\begin{bmatrix}4&4&4\\4&0&10\\-6&10&0\end{bmatrix} = \begin{bmatrix}4&4&4\\0&4&-6\\0&16&6\end{bmatrix} = \begin{bmatrix}4&4&4\\0&4&-6\\0&0&30\end{bmatrix}$, rank=3

19. $\begin{vmatrix}1&5&2\\1&3&2\\4&0&8\end{vmatrix} = 24+40+0-24-40-0=0$

$\begin{vmatrix}5&2&2\\3&2&6\\0&8&48\end{vmatrix} = 480+0+48-0-48*5-48*6=0$

$\begin{vmatrix}1&5\\1&3\end{vmatrix} = 3-5=-2\neq 0$

$\begin{bmatrix}1&5&2&2\\1&3&2&6\\4&0&8&48\end{bmatrix} = \begin{bmatrix}1&5&2&2\\0&2&0&-4\\0&20&0&-40\end{bmatrix}$, rank=2

20. b) $\begin{cases} ax + by + cz + d * 1 = 0 \\ ax_1 + by_1 + cz_1 + d * 1 = 0 \\ ax_2 + by_2 + cz_2 + d * 1 = 0 \\ ax_3 + by_3 + cz_3 + d * 1 = 0 \end{cases}$

21. $D = \begin{vmatrix} 3 & -5 \\ 6 & 16 \end{vmatrix} = 78$

$x = \frac{1}{78} \begin{vmatrix} 15.5 & -5 \\ 5 & 16 \end{vmatrix} = 3.5$

$y = \frac{1}{78} \begin{vmatrix} 3 & 15.5 \\ 6 & 5 \end{vmatrix} = -1$

$\begin{bmatrix} 3 & -5 & 15.5 \\ 6 & 16 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 15.5 \\ 0 & 26 & -26 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3.5 \\ 0 & 1 & -1 \end{bmatrix}$

22. $D = \begin{vmatrix} 2 & -4 \\ 5 & 2 \end{vmatrix} = 24$

$x = \frac{1}{24} \begin{vmatrix} -24 & -4 \\ 0 & 2 \end{vmatrix} = -2$

$y = \frac{1}{24} \begin{vmatrix} 2 & -24 \\ 5 & 0 \end{vmatrix} = 5$

$\begin{bmatrix} 2 & -4 & -24 \\ 5 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -12 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

23. $D = \begin{vmatrix} 0 & 3 & -4 \\ 2 & -5 & 7 \\ -1 & 0 & -9 \end{vmatrix} = 0 - 21 + 0 - (-20) - 0 - (-54) = 53$

$x = \frac{1}{53} \begin{vmatrix} 16 & 3 & -4 \\ -27 & -5 & 7 \\ 9 & 0 & -9 \end{vmatrix} = \frac{1}{53}(80 * 9 + 21 * 9 + 0 - 20 * 9 - 0 - 81 * 9) = 0$

$y = \frac{1}{53} \begin{vmatrix} 0 & 16 & -4 \\ 2 & -27 & 7 \\ -1 & 9 & -9 \end{vmatrix} = \frac{1}{53}(0 - 112 - 72 + 108 - 0 + 288) = 212/53 = 4$

$z = \frac{1}{53} \begin{vmatrix} 0 & 3 & 16 \\ 2 & -5 & -27 \\ -1 & 0 & 9 \end{vmatrix} = \frac{1}{53}(0 - +81 + 0 - 80 - 0 - 54) = -1$

$\begin{bmatrix} 0 & 3 & -4 & 16 \\ 2 & -5 & 7 & -27 \\ -1 & 0 & -9 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 11 & 3 & 41 \\ 0 & 3 & -4 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 15 & -7 \\ 0 & 0 & 53 & -53 \end{bmatrix} =$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

24.

25.