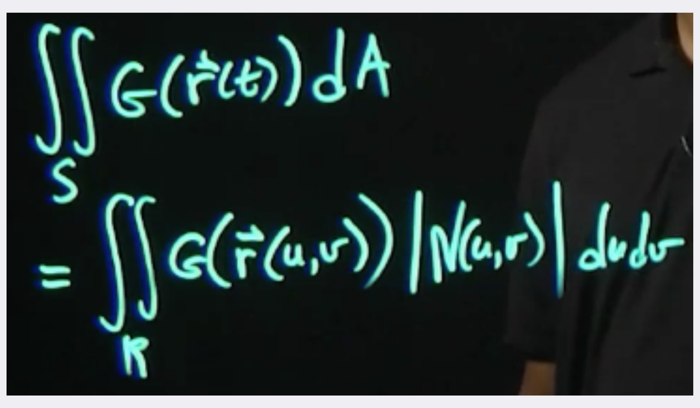
W6 - Gauss and Stokes Theorems

6A: Evaluation of Surface Integrals and Orientation

- $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ and unit normal vector $\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}$
- $$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v)\, du\, dv.$$
- $$\mathbf{n}\, dA = \mathbf{n}|\mathbf{N}|\, du\, dv = \mathbf{N}\, du\, dv.$$
- 

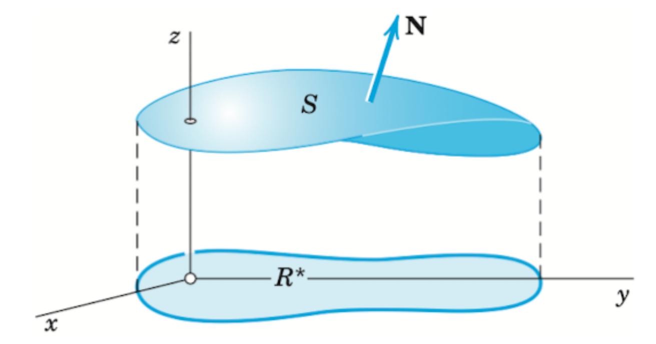
  no n in quesion
- Orientation of piecewise smooth surfaces
  - $\mathbf{n}$ *by* $-\mathbf{n}$ (*hence of* $\mathbf{N}$ *by* $-\mathbf{N}$)
  - 

    Fig. 250. Formula (11)
    z=f(x,y), r=[u,v,f(u,v)]
- Examples
  - **Flux through a surface**
  - **Surface Integral**
  - Change of orientation in a surface integral
  - Area of sphere
  - Torus surface: Representation and area
  - Moment of inertia of a surface

6B: Triple Integrals and the Divergence Theorem of Gauss

- $$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
- $$\iiint_T \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dA.$$
- $$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$
- Examples
  - **Evaluation of a Surface Integral by the divergence theorem**
  - **Verification of divergence theorem**
  - Fluid flow. Physical interpretation of the divergence
  - Modeling of heat flow. Heat or diffusion equation
  - A basic property of solutions of Laplace’s equation
  - **Green’s Theorems**
  - Uniqueness of solution of Laplace’s equation

6C: Stokes's Theorem

- $$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$
- $$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s)\, ds.$$
- (2*) $$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) N_3\right] du\, dv = \oint_C (F_1\, dx + F_2\, dy + F_3\, dz).$$
- Examples
  - **Verification of Stoke’s Theorem**
  - **Green’s theorem in the plane as a special case of stoke’s theoream**
  - **Evaluation of a line integral by stocke’s theorem**
  - Physical meaning of the curl in Fluid Motion. Circulation
  - Work done in the displacement around a closed curve