Chapter 15 - Power Series, Taylor Series

Selected Problem set 15.1

15.1 1. 5, 9, 17, 19, 23, 25

bound.
Convergent
Limit

Convergent, hence bounded

1–10 SEQUENCES

Is the given sequence $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find its limit points. Show your work in detail.

1. $z_n = (1+i)^{2n}/2^n$ **2.** $z_n = (3+4i)^n/n!$

1. $$\frac{z_{n+1}}{z_n} = \frac{(1+i)^{2(n+1)}}{2^{n+1}} \cdot \frac{2^n}{(1+i)^{2n}} = \frac{(1+i)^2}{2} = \frac{1+i^2+2i}{2} = i$$

$z_1 = \frac{(1+i)^2}{2} = i$, $z_2 = -1$, $z_3 = -i$, $z_4 = 1$, $z_5 = z_1$

bounded.

Theorem 7. $\left|\frac{z_{n+1}}{z_n}\right| = 1 \geq 1$ diverges

5. $z_n = (-1)^n + 10i$

$z_1 = -1+10i$ $z_2 = 1+10i$ $z_3 = z_1$ $z_4 = z_2$ ...

bounded.

for $t = 1, 2, \ldots$

$$\left|\frac{z_{2t+1}}{z_{2t}}\right| = \left|\frac{(1+10i)^2}{(-1+10i)(1+10i)}\right| = \left|\frac{1-100+20i}{-100-1}\right| = \left|+\frac{99}{101} - \frac{20}{101}i\right| \geq 1$$

diverges.

9. $z_n = (3+3i)^{-n}$

$|z_1| = \left|\frac{1}{6}(1-i)\right| = \frac{\sqrt{2}}{6}$ $\qquad |z_2| = \left|-\frac{i}{18}\right| = \frac{1}{18}$

$|z_3| = \left|-\frac{1}{108}(1+i)\right| = \frac{\sqrt{2}}{108}$

$|z_{n+1}| = \left|z_n \cdot \left(\frac{1}{3+3i}\right)\right| = |z_n| \cdot \frac{\sqrt{2}}{6} \le |z_1| = \frac{\sqrt{2}}{6}$ bounded.

$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(3+3i)^{-(n+1)}}{(3+3i)^{-n}}\right| = \left|(3+3i)^{-1}\right| = \frac{\sqrt{2}}{6} < 1$ converges

16–25 SERIES

Is the given series convergent or divergent? Give a reason. Show details.

16. $\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$ $\qquad$ **17.** $\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$

17. $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{(-1)^n}\right| = \left|(-1) \cdot \frac{\ln n}{\ln(n+1)}\right| = \left|\frac{\ln n}{\ln(n+1)}\right|$

$n^{n+1} > (n+1)^n$ for $n \ge 3$. $\Rightarrow \frac{\ln n}{\ln(n+1)} > \frac{n}{n+1}$

$\Rightarrow \left|\frac{z_{n+1}}{z_n}\right| > \left|\frac{n}{n+1}\right|$ diverges

19. $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{(n+1)^2 - i} \cdot \frac{n^2 - i}{i^n}\right| = \left| i \cdot \frac{n^2 - i}{(n+1)^2 - i}\right|$$

$$= \left|\frac{(n^2 - i)[(n+1)^2 + i]}{[(n+1)^2 - i][(n+1)^2 + i]}\right|$$

$$= \left|\frac{(n^2 - i)(n^2 + 2n + 1 + i)}{(n+1)^4 + 1}\right|$$

$$= \left|\frac{n^4 + 2n^3 + n^2 + \cancel{in^2} - \cancel{in^2} - 2ni - i + 1}{(n+1)^4 + 1}\right|$$

$$= \left|\frac{1 + \frac{2}{n} + \frac{1}{n^2} - \frac{2i}{n^3} + \frac{1-i}{n^4}}{\left(1 + \frac{1}{n}\right)^4 + \frac{1}{n}}\right|$$

$\lim \to 1 \quad \Rightarrow$ don't know

$$|z_n|^{\frac{1}{n}} = \left|\frac{i}{\sqrt[n]{n^2 - i}}\right| = \left|\frac{i}{(n^2 + 1)^{\frac{1}{n}}}\right| < \left|\frac{i}{n^{\frac{2}{n}}}\right| = 1$$

converge absolutely

23. $\sum_{n=0}^{\infty} \frac{(-1)^n(1+i)^{2n}}{(2n)!}$

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}(1+i)^{2n+2}}{(2n+2)!} \cdot \frac{2n!}{(-1)^n(1+i)^{2n}}\right|$$

$$= \left|\frac{-1\cdot(1+i)^2}{(2n+2)(2n+1)}\right|$$

$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0 < 1$ converge absolutely

25. $\sum_{n=1}^{\infty} \frac{i^n}{n}$

$\lim \Rightarrow 1$

$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{n+1}\cdot\frac{n}{i^n}\right| = \left|\frac{ni}{n+1}\right| = \left|\frac{n}{n+1}\right|$ not sure

$|z_n^{\frac{1}{n}}| = \left|\frac{1}{\sqrt[n]{n}}\right| \Rightarrow \lim_{n\to\infty}|z_n|^{\frac{1}{n}} = 1$ not sure

Per example 3. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots$ conditionally converge.

so $1, -1, -i, 1.$

$\{z_{4t+1}+z_{4t+3}\}$ t is even $\Rightarrow$ converge. x_i

$\{z_{4t+2}+z_{4t+4}\}$ t is odd $\Rightarrow$ converge y

$\Rightarrow S_n$ jumping from x_i to y, diverge?

Selected Problem set 15.2

15.2 7. 9. 11. 13. 17

6–18 RADIUS OF CONVERGENCE

Find the center and the radius of convergence.

6. $\sum_{n=0}^{\infty} 4^n(z+1)^n$ 7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z-\frac{1}{2}\pi\right)^{2n}$

$$\sqrt{\left|\frac{a_{n+1}\left(z-\frac{\pi}{2}\right)^{2n+2}}{a_n\left(z-\frac{\pi}{2}\right)^{2n}}\right|} = \sqrt{\left|\frac{a_{n+1}}{a_n}\right| \cdot \left|\left(z-\frac{\pi}{2}\right)^2\right|}$$

$$= \sqrt{\left|\frac{(-1)^{n+1}}{(2n+2)!} \cdot \frac{2n!}{(-1)^n}\right|} \cdot \left|\left(z-\frac{\pi}{2}\right)\right|$$

$$= \sqrt{\left|\frac{1}{(2n+2)(2n+1)}\right|} \cdot \left|\left(z-\frac{\pi}{2}\right)\right|$$

$L = L^* \cdot \left(z-\frac{\pi}{2}\right)$. $\lim_{n\to\infty} L^* = 0$, $\lim_{n\to+\infty} L = 0$

Converge for all z by the ratio test

9. $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n}$

$$\sqrt{\left|\frac{a_{n+1}(z-i)^{2n+2}}{a_n(z-i)^{2n}}\right|} = \sqrt{\frac{a_{n+1}}{a_n}} \cdot |z-i|$$

$$= \sqrt{\frac{(n+1)\cdot n}{3^{n+1}} \cdot \frac{3^n}{n(n-}} \cdot |z-i|$$

$$= \sqrt{\frac{n+1}{n-1} \cdot \frac{1}{3}} \quad |z-i|$$

$L = L^* \; |z-i|$. $\lim_{n\to\infty} L^* = \frac{\sqrt{3}}{3}$. $\lim_{n\to+\infty} L = \sqrt{3}$ Center $= i$

11. $\sum_{n=0}^{\infty}\left(\frac{2-i}{1+5i}\right)z^n$

$|z|\geq 1$: divergence.

$|z|<1$ Converge

$$S_n=\left(\frac{2-i}{1+5i}\right)\cdot z^0+\left(\frac{2-i}{1+5i}\right)\cdot z+\cdots\left(\frac{2-i}{1+5i}\right)z^n$$

$$zS_n=\left(\frac{2-i}{1+5i}\right)z^1+\cdots\left(\frac{2-i}{1+5i}\right)\cdot z^{n+1}$$

$$(1-z)S_n=\frac{2-i}{1+5i}\cdot 1-\left(\frac{2-i}{1+5i}\right)\cdot z^{n+1}$$

$n\to+\infty$ $z^{n+1}\to 0$

$$S_n=\frac{2-i}{(1-z)(1+5i)}$$

Center: 0

Radiu: 1

13. $\sum_{n=0}^{\infty}16^n(z+i)^{4n}$

$$\sqrt[4]{\left|\frac{a_{n+1}(z+i)^{4n+4}}{a_n(z+i)^{4n}}\right|}=\sqrt[4]{\left|\frac{a_{n+1}}{a_n}\right|}\cdot|z+i|$$

$$=\sqrt[4]{16}\cdot|z+i|=2|z+i|$$

$\lim_{n\to\infty}L^*=2$ $\lim_{n\to\infty}L=\frac{1}{2}$ Center $-i$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n(n+1)} z^{2n+1}$

$$\sqrt{\left|\frac{a_{n+1}\cdot z^{2n+3}}{a_n \cdot z^{2n+1}}\right|} = \sqrt{\frac{a_{n+1}}{a_n}}\cdot|z|$$

$$= \sqrt{\frac{2^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{2^n}}\ |z|$$

$$= \sqrt{\frac{2n}{n+2}}\cdot|z|$$

$\lim\limits_{n\to\infty} L^{*} = \sqrt{2}$ $\quad$ $\lim\limits_{n\to\infty} L = \frac{\sqrt{2}}{2}$ $\quad$ Center : 0

Selected Problem set 15.3

15.3 5, 7, 9, 11, 15

5–15 RADIUS OF CONVERGENCE BY DIFFERENTIATION OR INTEGRATION

Find the radius of convergence in two ways: **(a)** directly by the Cauchy–Hadamard formula in Sec. 15.2, and **(b)** from a series of simpler terms by using Theorem 3 or Theorem 4.

5. $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}(z-2i)^n$ **6.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1}$

5. (a) $R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ **(Cauchy–Hadamard formula**[1]**).**

$$R = \lim_{n\to\infty}\left|\frac{n(n-1)}{2^n}\cdot\frac{2^{n+1}}{(n+1)\cdot n}\right| = \lim_{n\to\infty}\left|\frac{2(n-1)}{n+1}\right| = 2$$

(b) $$f(z) = \sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}(z-2i)^n = \sum_{n=2}^{\infty} n(n-1)\cdot\left(\frac{z-2i}{2}\right)^n$$

$$g(x) = \sum_{n=2}^{\infty} x^n \quad \Rightarrow R=1$$

$$g^{(2)}(x) = \sum_{n=2}^{\infty} n(n-1)\cdot x^{n-2} \quad \Rightarrow R=1.$$

$$\left|\frac{z-2i}{2}\right| < 1 \qquad |z-2i| < 2 \quad \Rightarrow R=2$$

$$7.\ \sum_{n=1}^{\infty} \frac{n}{3^n}(z+2i)^{2n}$$

a) per P683.

$$\left| \frac{\frac{n+1}{3^{n+1}}(z+2i)^{2(n+1)}}{\frac{n}{3^n}(z+2i)^{2n}} \right| = \left| \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} \right| \cdot \left| (z+2i)^2 \right| = \left| \frac{n+1}{3n} \right| \cdot \left| (z+2i)^2 \right|$$

The limit is $L = L^* \ |z+2i| = \frac{\sqrt{3}}{3} |z+2i|$

$R = \frac{1}{L^*} = \sqrt{3}$ or $\frac{\sqrt{3}}{3} |z+2i| < 1$, $|z+2i| < \sqrt{3}$.

b) $f(x) = 3 \cdot \left(\frac{x}{3}\right)^n = \frac{1}{3^{n-1}} \cdot x^n$

$f'(x) = \frac{n}{3^{n-1}} \cdot x^{n-1}$

$f'(x) \cdot \frac{x}{3} = \frac{n}{3^n} x^n$

$f(x)$: $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{3^n}{3^{n-1}} \right| = 3$

$f'(x)$: $R = 3$ $\quad f'(x) \cdot \frac{x}{3}$: $R = 3$.

$\Rightarrow R = \sqrt{3}$

9. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n(n+1)(n+2)} z^{2n}$

a) $\left| \frac{(-2)^{n+1} \cdot z^{2(n+1)}}{(n+1)(n+2)(n+3)} \cdot \frac{n(n+1)(n+2)}{(-2)^n \cdot z^{2n}} \right| = \left| \frac{n}{n+3} \cdot (-2) \cdot z^2 \right|$

$= \left| \frac{2n}{n+3} \right| \cdot |z^2|$

The lim is $2|z^2| < 1$, $|z| < \frac{\sqrt{2}}{2}$, $R = \frac{\sqrt{2}}{2}$

b). $f(x) = x^{n+2}$ $|x| < 1$

$f^{(3)}(x) = n(n+1)(n+2)x^n$ $|x| < 1$

$|-2z^2| < 1$ $|z| < \frac{\sqrt{2}}{2}$ $R = \frac{\sqrt{2}}{2}$

11. $\sum_{n=1}^{\infty} \frac{3^n n(n+1)}{7^n} (z+2)^{2n}$

a) $\left| \frac{3^{n+1}(n+1)(n+2)\cdot(z+2)^{2n+2}}{7^{n+1}} \cdot \frac{7^n}{(z+2)^{2n}\cdot 3^n \cdot n(n+1)} \right|$

$= \left| \frac{3(n+2)}{7n} \right| \cdot \left| (z+2)^2 \right|$

the lim is $\frac{3}{7} \cdot \left| (z+2)^2 \right| < 1$ $\qquad R: \sqrt{\frac{7}{3}}$

b). $f(x) = \left(\frac{3}{7}x\right)^{n+1} = \left(\frac{3}{7}\right)^{n+1} \cdot x^{n+1}$ $\qquad \left|\frac{3}{7}x\right| < 1$

$f^{(2)}(x) = n(n+1)\left(\frac{3}{7}\right)^{n+1} \cdot x^{n-1} = \frac{3^{n+1}\cdot n(n+1)}{7^{n+1}} \cdot x^{n-1}$

$f^{(2)}(x) \cdot \frac{7}{3} \cdot x = \frac{3^n n(n+1)}{7^n} \cdot x^n$

$\left| (z+2)^2 \cdot \frac{3}{7} \right| < 1 \qquad \left| (z+2)^2 \right| < \frac{7}{3} \qquad R: \sqrt{\frac{7}{3}}$

15. $\sum_{n=2}^{\infty} \frac{4^n n(n-1)}{3^n}(z-i)^n$

a) $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to+\infty} \left| \frac{4^n n(n-1)}{3^n} \cdot \frac{3^{n+1}}{4^{n+1}(n+1)\cdot n} \right|$

$= \lim_{n\to+\infty} \left|\frac{3(n-1)}{4(n+1)}\right| = \frac{3}{4}$

b) $f(x) = \left(\frac{4}{3}x\right)^n = \left(\frac{4}{3}\right)^n \cdot x^n$ $\qquad \left|\frac{4}{3}x\right| < 1$

$f^{(2)}(x) = \left(\frac{4}{3}\right)^n \cdot n(n-1) \cdot x^{n-2}$

$f^{(2)}(x) \cdot x^2 = \frac{4^n n(n-1)}{3^n} \cdot x^n$

$\left|\frac{4}{3}(z-i)\right| < 1$ $\qquad R = \frac{3}{4}$

Selected Problem set 15.4