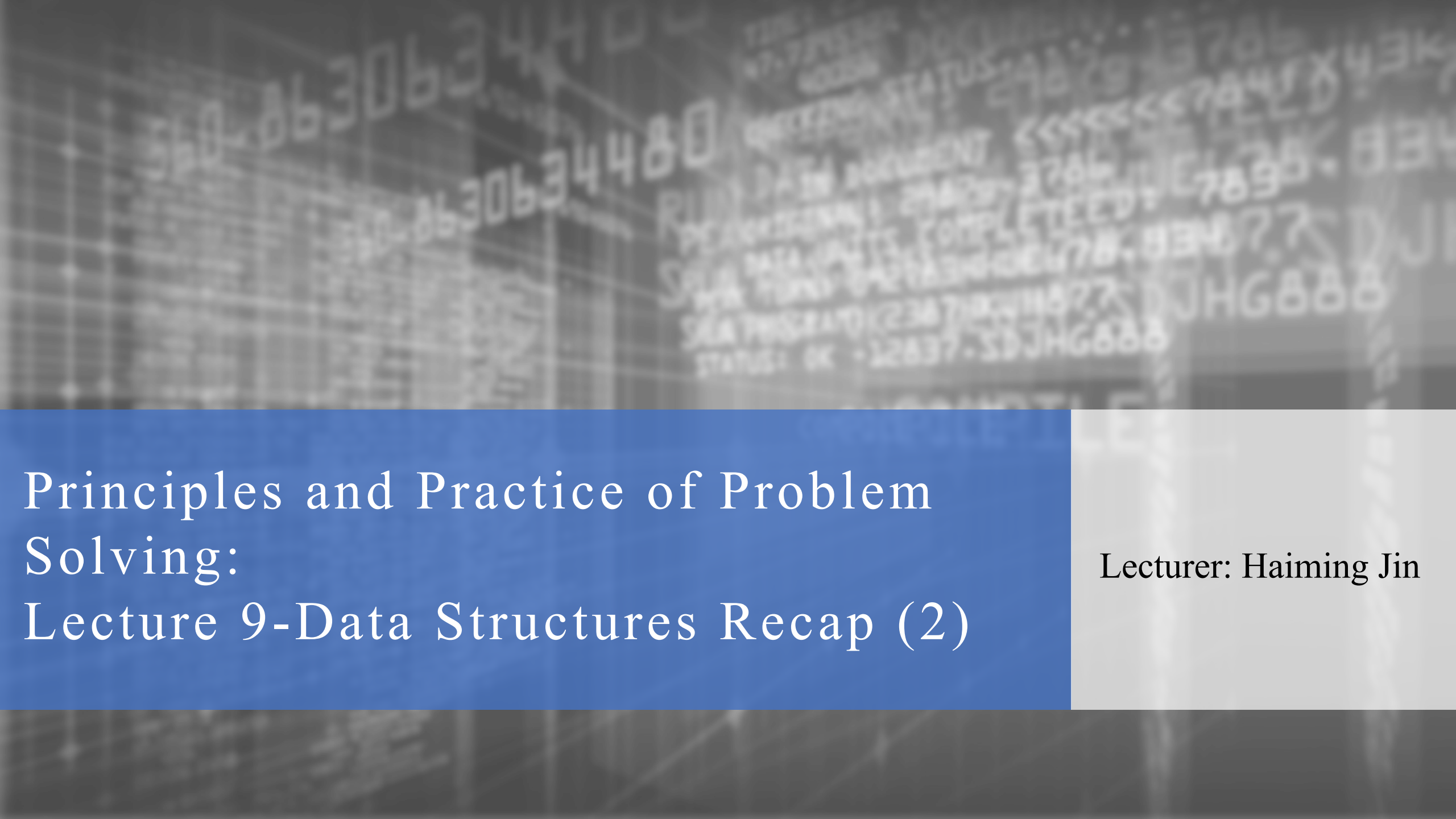


Principles and Practice of Problem Solving: Lecture 9-Data Structures Recap (2)

Lecturer: Haiming Jin

Outline

- Linked List
- Stack
- Queue
- Sort
- Linear-Time Selection
- Hashing
- Bloom Filter

Hashing

- Hashing Basics
- Hash Function
- Collision Resolution
- Hash Table Size and Rehashing
- Applications of Hashing

Hashing: High-Level Idea

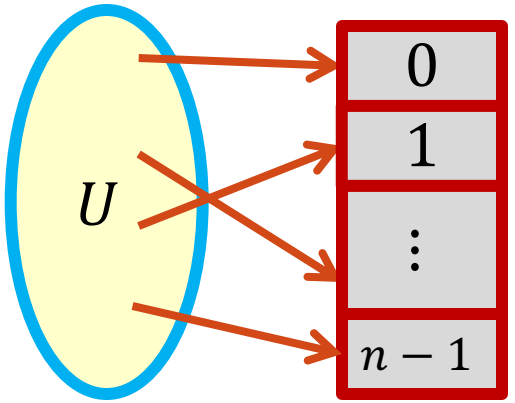
- **Setup**: A universe U of objects
  - E.g., All names, all IP addresses, etc.
  - Generally, very BIG!
- **Goal**: Want to maintain an evolving set $S \subseteq U$
  - E.g., 200 students, 500 IP addresses
  - Generally, of reasonable size.
- Naïve solutions

1. Array-based solution (index by $u \in U$)
   - $\Theta(1)$ operation time, BUT $\Theta(|U|)$ space.
2. Linked list-based solution:
   - $\Theta(|S|)$ space, BUT $\Theta(|S|)$ operation time.

Can we get the best of both solutions?

Hashing: High-Level Idea

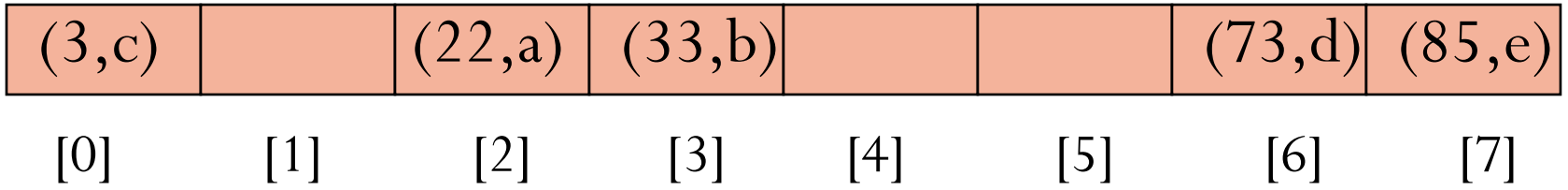
- Solution:
  - Pick an array A of n buckets.
    - $n = c|S|$: a small multiple of $|S|$.
  - Choose a hash function $h: U \rightarrow \{0,1, \ldots, n-1\}$
    - h is fast to compute.
    - The same key is always mapped to the **same** location.
  - Store item k in A[h(k)]



- The array is called **hash table**
  - An array of **buckets**, where each bucket contains items as assigned by a hash function.
  - h[k] is called the **home bucket** of key k.

Hashing Example

- Pairs are: (22,a), (33,b), (3,c), (73,d), (85,e)
- Hash table is **`A[0:7]`** and table size is **`M = 8`**
- Hash function is **`h[key] = key/11`**
- Every item with **`key`** is stored in the bucket **`A[h(key)]`**



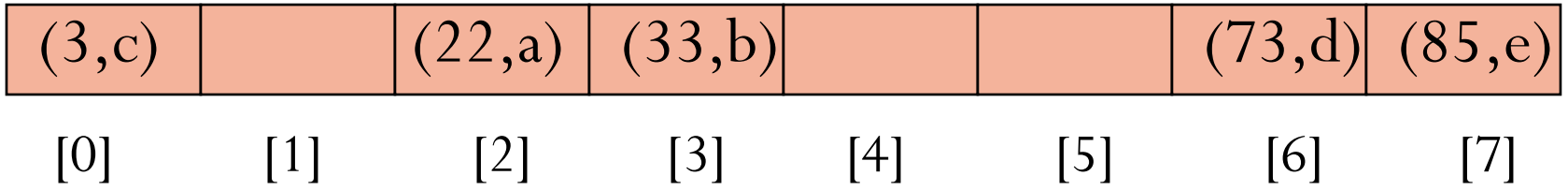
Question: What is the time complexity for **`find()`**, **`insert()`**, and **`remove()`**?

Methods

- **`Value find(Key k)`**: Return the value whose key is **`k`**. Return **`Null`** if none.
- **`void insert(Key k, Value v)`**: Insert a pair **`(k, v)`** into the Hash Table.
- **`Value remove(Key k)`**: Remove the pair with key as **`k`** from the Hash Table and return its value. Return **`Null`** if none.

Hashing Example

- Pairs are: (22,a), (33,b), (3,c), (73,d), (85,e)
- Hash table is **`A[0:7]`** and table size is **`M = 8`**
- Hash function is **`h[key] = key/11`**
- Every item with **`key`** is stored in the bucket **`A[h(key)]`**

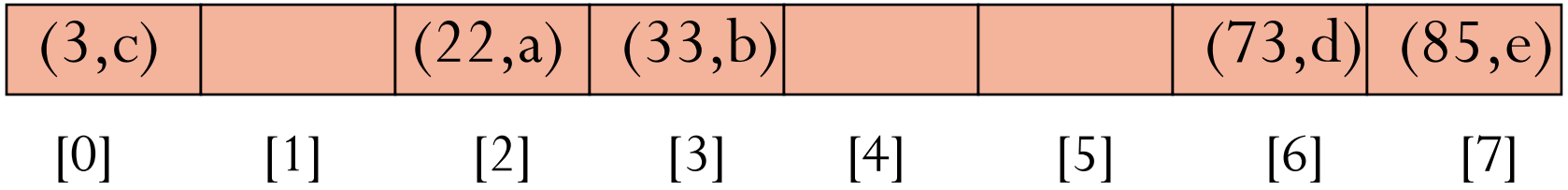


Question: What is the time complexity for **`find()`**, **`insert()`**, and **`remove()`**?

O(1)

Achieves both fast runtime and efficient memory usage

What Can Go Wrong?



- Where does (35, g) go?
- Problem: The home bucket for (35, g) is already occupied!
  - This is a "**collision**".

Collision and Collision Resolution

- Collision occurs when the hash function maps two or more items—all having **different** search keys—into the **same** bucket.

- What to do when there is a collision?
  - **Collision-resolution scheme**: assigns distinct locations in the hash table to items involved in a collision.

- Two major schemes:
  - Separate chaining
  - Open addressing

Hash

- Hashing Basics
- Hash Function
- Collision Resolution
- Hash Table Size and Rehashing
- Applications of Hashing

Hash Function Design Criteria

- Must compute a bucket for every key in the universe.
- Must compute the same bucket for the same key.
- Should be easy and quick to compute.
- Minimizes collision ← The hardest criterion
  - Spread keys out evenly in hash table
  - **Gold standard**: **completely random hashing**
    - The probability that a randomly selected key has bucket i as its home bucket is $1/n$, $0 \leq i < n$.
    - Completely random hashing **minimizes** the likelihood of an collision when keys are selected at random.
    - However, completely random hashing is **infeasible** due to the need to remember the random bucket.

Bad Hash Functions

- Example: keys = phone number in China (11 digits)
  - $|U| = 10^{11}$
  - **Terrible** hash function: $h(key)$ = first 3 digits of key, i.e., area code
    - The keys are not spread out evenly. Buckets 010, 021 may have a lot of keys mapped to them, while some buckets have no keys.
  - **Mediocre** hash function: $h(key)$ = last 3 digits of key.
    - Still vulnerable to patterns in last 3 digits.

Hash Functions

- Hash function ($h(key)$) maps key to buckets in two steps:

1. Convert key into an integer in case the key is not an integer.
   - A function $t(key)$ which returns an integer value, known as **hash code**.
2. **Compression map**: Map an integer (hash code) into a home bucket.
   - A function $c(hashcode)$ which gives an integer in the range $[0, n-1]$, where n is the number of buckets in the table.

- In summary, $h(key) = c(t(key))$, which gives an index in the table.

Map Non-integers into Hash Code

- String: use the ASCII (or UTF-8) encoding of each char and then perform arithmetic on them.

- Floating-point number: treat it as a string of bits.

- Images, (viral) code snippets, (malicious) Web site URLs: in general, treat the representation as a bit-string, using all of it or **extracting** parts of it (i.e., www.**sjtu**.**edu**.**cn**).

Strings to Integers

- Simple scheme: adds up all the ASCII codes for all the chars in the string.
  - Example: t(“He”) = 72 + 101 = 173.

- Not good. Why?
  - Consider English words “post”, “pots”, “spot”, “stop”, “tops”.

Strings to Integers

- A better strategy: Polynomial hash code taking **positional** info into account.

$$t(s[]) = s[0]a^{k-1} + s[1]a^{k-2} + \cdots + s[k-2]a + s[k-1]$$

where a is a constant.

- If $a = 33$, the hash codes for "post" and "stop" are

$$t(\text{post}) = 112 \cdot 33^3 + 111 \cdot 33^2 + 115 \cdot 33 + 116 = 4149734$$
$$t(\text{stop}) = 115 \cdot 33^3 + 116 \cdot 33^2 + 111 \cdot 33 + 112 = 4262854$$

Strings to Integers

$$t(s[]) = s[0]a^{k-1} + s[1]a^{k-2} + \cdots + s[k-2]a + s[k-1]$$

- Good choice of a for English words: 31, 33, 37, 39, 41
  - What does it mean for a to be a **good** choice? Why are these particular values **good**?
  - Answer: according to statistics on 50,000 English words, each of these constants will produce less than 7 collisions.
- In Java, its **`string`** class has a built-in **`hashCode()`** function. It takes $a = 31$. Why?
  - Multiplication by 31 can be replaced by a shift and a subtraction for **better performance**: `31*i == (i << 5) - i`

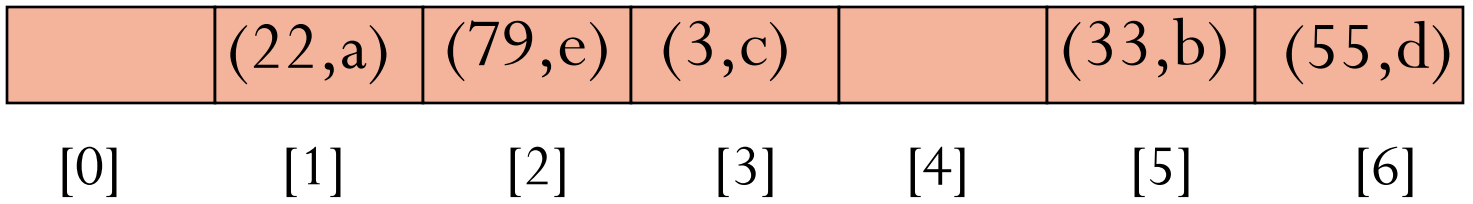
Hash function criteria: Should be easy and quick to compute.

Compression Map

- Map an integer (hash code) into a home bucket.
- The most common method is by **modulo arithmetic**.

  `homeBucket = c(hashcode) = hashcode % n`

  where n is the **number of buckets** in the hash table.
- Example: Pairs are (22,a), (33,b), (3,c), (55,d), (79,e). Hash table size is 7.



Hashing by Modulo

- In practice, keys of an application tend to have a specific pattern
  - For example, memory address in computer is multiple of 4.

- The choice of the hash table size n will affect the distribution of home buckets.

Hashing by Modulo

- Suppose the keys of an application are more likely to be mapped into even integers.
  - E.g., memory address is always a multiple of 4.

- When the hash table size n is an **even** number, **even** integers are hashed into **even** home buckets.
  - E.g., n = 14: 20%14 = 6, 30%14 = 2, 8%14 = 8

- The bias in the keys results in a bias toward the **even** home buckets.
  - All **odd** buckets are **guaranteed** to be empty.
  - The distribution of home buckets is not uniform!

Hashing by Modulo

- However, when the hash table size n is **odd**, even (or odd) integers may be hashed into both odd and even home buckets.
  - E.g., n = 15: 20%15 = 5, 30%15 = 0, 8%15 = 8
    15%15 = 0, 3%15 = 3, 23%15 = 8
- The bias in the keys does not result in a bias toward either the odd or even home buckets.
  - Better chance of uniform distribution of home buckets.
- So **do not** use an even hash table size n.

Hashing by Modulo

- Similar **biased** distribution of home buckets happens in practice when the hash table size n is a multiple of small prime numbers.

- The effect of each prime divisor p of n **decreases** as p gets **larger**.

- Ideally, choose the hash table size n as a **large prime number**.

Hash

- Hashing Basics
- Hash Function
- Collision Resolution
- Hash Table Size and Rehashing
- Applications of Hashing

Outline

- Collision Resolution: Separate Chaining

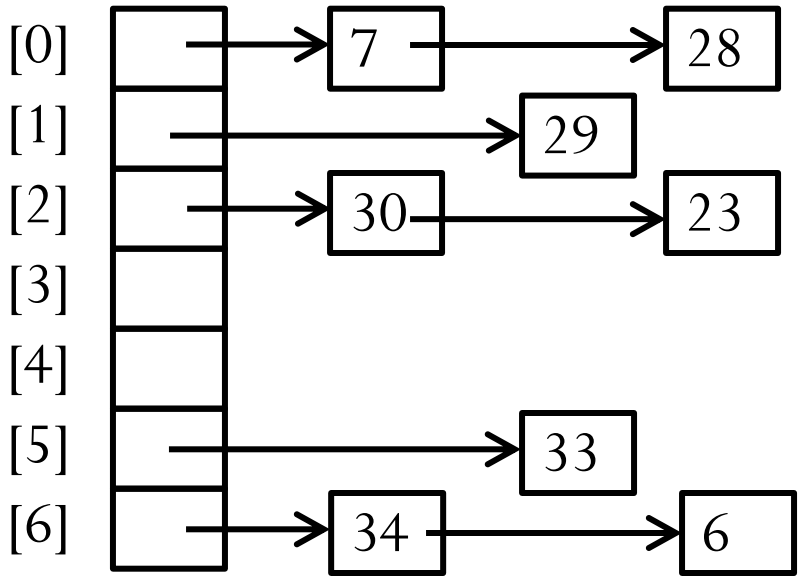
- Collision Resolution: Open Addressing
  - Linear Probing
  - Quadratic Probing and Double Hashing
  - Performance of Open Addressing

Collision Resolution Scheme

- **Collision-resolution scheme**: assigns distinct locations in the hash table to items involved in a collision.
- Two major scheme:
  - Separate chaining
  - Open addressing

Separate Chaining

- Each bucket keeps a **linked list** of all items whose home buckets are that bucket.
- Example: Put pairs whose keys are 6, 23, 34, 28, 29, 7, 33, 30 into a hash table with $n = 7$ buckets.
  - **`homeBucket = key % 7`**
  - **Note**: we insert object at the beginning of a linked list.



Separate Chaining

- **`Value find(Key key)`**
  - Compute **`k = h(key)`**
  - Search in the linked list located at the k-th bucket (e.g., check every entry) with the key.

- **`void insert(Key key, Value value)`**
  - Compute **`k = h(key)`**
  - Search in the linked list located at the k-th bucket. If found, update its value; otherwise, insert the pair at the beginning of the linked list in O(1) time.

Separate Chaining

- **`Value remove(Key key)`**
  - Compute **`k = h(key)`**
  - Search in the linked list located at the k-th bucket. If found, remove that pair.

Outline

- Collision Resolution: Separate Chaining

- Collision Resolution: Open Addressing
  - Linear Probing
  - Quadratic Probing and Double Hashing
  - Performance of Open Addressing

Open Addressing

- Reuse empty space in the hash table to hold colliding items.

- To do so, search the hash table in some systematic way for a bucket that is empty.
  - Idea: we use a sequence of hash functions h_0, h_1, h_2, . . . to probe the hash table until we find an empty slot.
    - I.e., we **probe** the hash table buckets mapped by $h_0(key)$, $h_1(key)$, …, in sequence, until we find an empty slot.
    - Generally, we could define $h_i(x) = h(x) + f(i)$

Open Addressing

- Three methods:
  - Linear probing:
    $h_i(x) = (h(x) + i) \% n$
  - Quadratic probing:
    $h_i(x) = (h(x) + i^2) \% n$
  - Double hashing:
    $h_i(x) = (h(x) + i*g(x)) \% n$

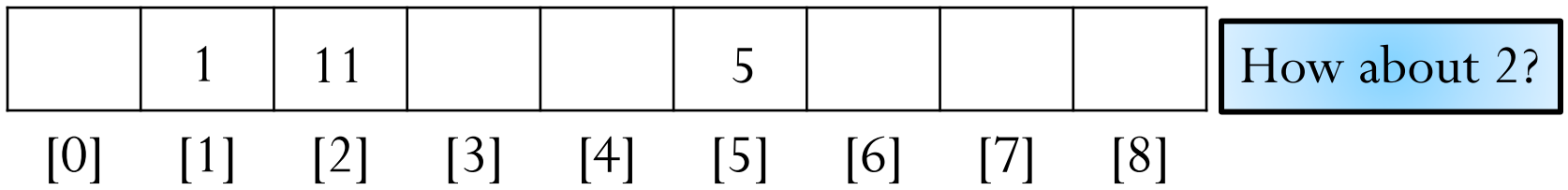
n is the hash table size

Linear Probing

$$h_i(key) = (h(key)+i) \% n$$

- Apply hash function h_0, h_1, …, in sequence until we find an empty slot.
  - This is equivalent to doing a linear search from **`h(key)`** until we find an empty slot.

- Example: Hash table size **`n = 9`**, **`h(key) = key%9`**
  - Thus **h_i`(key) = (key%9+i)%9`**
  - Suppose we insert 1, 5, 11, 2, 17, 21, 31 in sequence



Linear Probing

Example

- Hash table size **`n = 9`**, **`h(key) = key%9`**
  - Thus **h_i`(key) = (key%9+i)%9`**
  - Suppose we insert 1, 5, 11, 2, 17, 21, 31 in sequence.

	1	11	2	21	5	31		17
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

  - **h_0`(2) = 2.`** Not empty!
  - So we try **h_1`(2) = 3.`** It is empty, so we insert there!
  - **h_0`(21) = 3.`** Not empty!
  - **h_1`(21) = 4.`** It is empty, so we insert there!
  - **h_0`(31) = 4.`** Not empty!
  - **h_1`(31) = 5.`** Not empty!
  - **h_2`(31) = 6.`** It is empty, so we insert there!

Linear Probing

find()

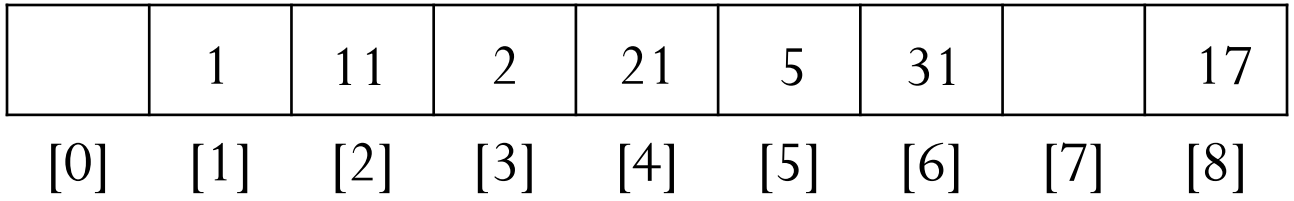
	1	11	2	21	5	31		17
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

- With linear probing **`h_i(key) = (key%9+i)%9`**
  - How will you **search** an item with key = 31?
  - How will you **search** an item with key = 10?

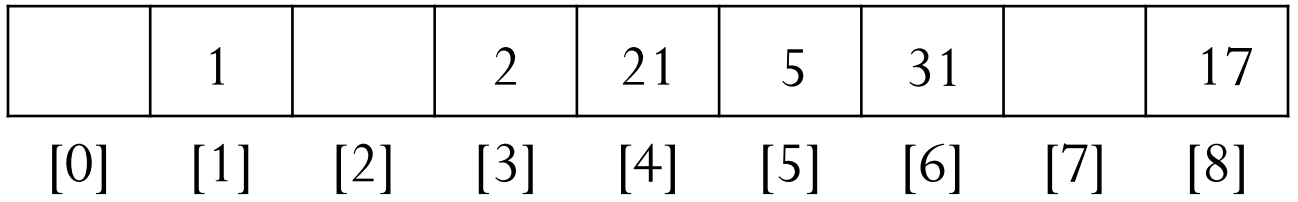
- Procedure: probe in the buckets given by $h_0(key)$, $h_1(key)$, …, in sequence **until**
  - we find the key,
  - or we find an empty slot, which means the key is not found.

Linear Probing

remove()



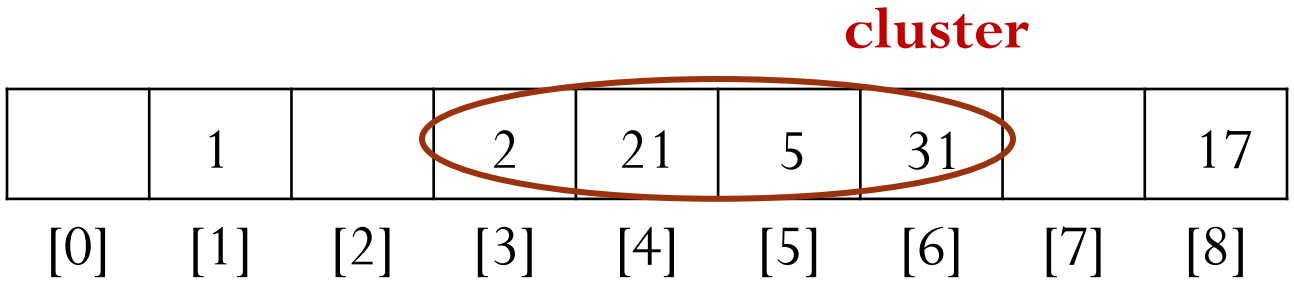
- With linear probing h_i**(key) = (key%9+i)%9**
  - How will you **remove** an item with key = 11?
  - If we just find 11 and delete it, will this work?



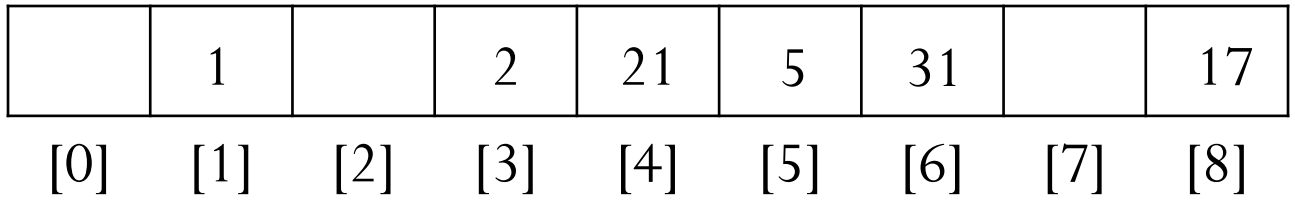
What is the result for searching key = 2 with the above hash table?

Linear Probing

remove()

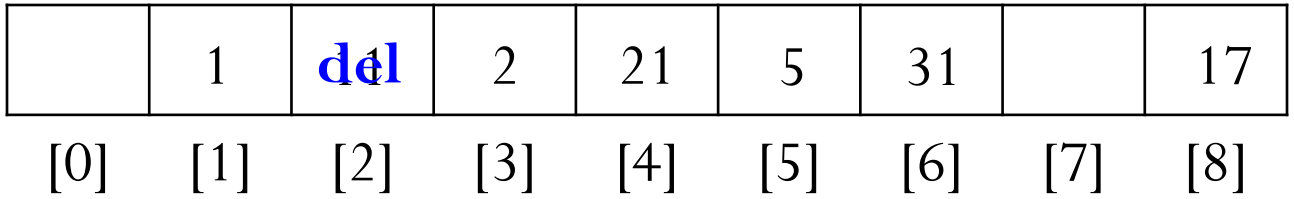


- After deleting 11, we need to **rehash** the following "cluster" to fill the vacated bucket.
- However, we cannot move an item **beyond** its **actual** hash position. In this example, 5 cannot be moved ahead.



Linear Probing

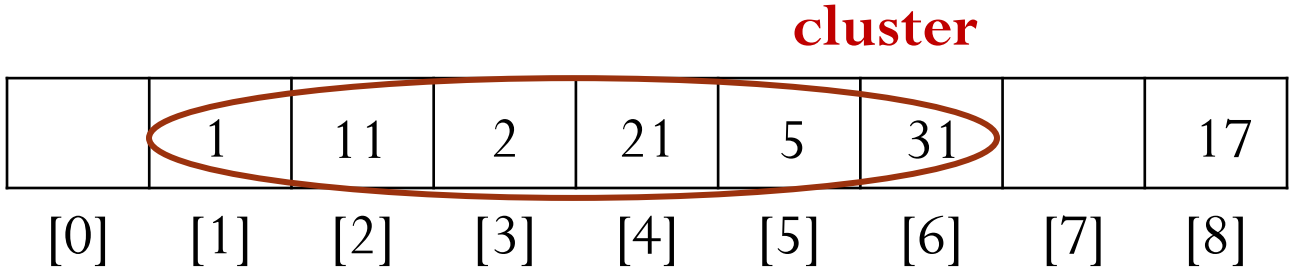
Alternative implementation of remove()



- **Lazy deletion**: we mark deleted entry as "**deleted**".
  - "deleted" is not the same as "empty".
  - Now each bucket has three states: "occupied", "empty", and "deleted".
- We can overwrite the "deleted" entry when inserting.
- When we **search**, we will keep looking if we encounter a "deleted" entry.

Linear Probing

Clustering Problem



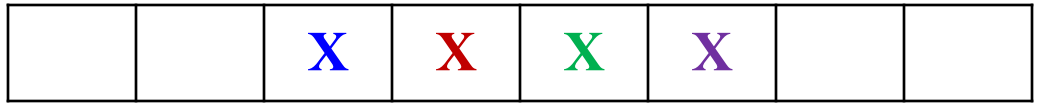
- Clustering: when **contiguous** buckets are all occupied.
- **Claim**: Any hash value inside the cluster adds to **the end** of that cluster.
- Problems with a **large** cluster:
  - It becomes more likely that the next hash value will collide with the cluster.
  - Collisions in the cluster get more expensive to resolve.

Linear Probing

Clustering Problem

- Assuming input size N, table size 2N:
  - What is the best-case cluster distribution?

  - What is the worst-case cluster distribution?



Which Statements Are Correct?

- Assuming input size N, table size 2N. Analyze the average number of probes to find an empty slot for best-case and worst-case clusters. Select all the correct answers.

A. The average number for best-case cluster is 1.5.

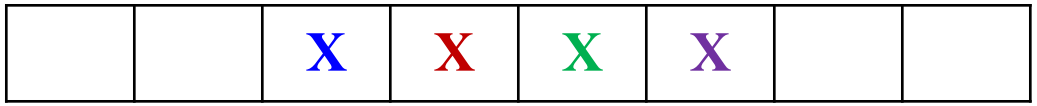
B. The average number for best-case cluster is 1.

C. The average number for worst-case cluster is roughly $\frac{1}{4}N$.

D. The average number for worst-case cluster is roughly $\frac{1}{2}N$.

Best Case

Worst Case



Outline

- Collision Resolution: Separate Chaining

- Collision Resolution: Open Addressing
  - Linear Probing
  - Quadratic Probing and Double Hashing
  - Performance of Open Addressing

Quadratic Probing

$$h_i(key) = (h(key)+i^2) \% n$$

- It is less likely to form large clusters.
- Example: Hash table size $n = 7$, $h(key) = key\%7$
  - Thus $h_i(key) = (key\%7+i^2)\%7$
  - Suppose we insert 9, 16, 11, 2 in sequence.

		9	16	11		2
[0]	[1]	[2]	[3]	[4]	[5]	[6]

  - $h_0(16) = 2$. Not empty!
  - $h_1(16) = 3$. It is empty, so we insert there.
  - $h_0(2) = 2$. Not empty!
  - $h_1(2) = 3$. Not empty!
  - $h_2(2) = 6$. It is empty, so we insert there.

Problem of Quadratic Probing

- However, sometimes we will never find an empty slot even if the table isn't full!
- Luckily, if the **load factor** $L \leq 0.5$, we are guaranteed to find an empty slot.
  - Definition: given a hash table with n buckets that stores m objects, its **load factor** is

$$L = \frac{m}{n} = \frac{\#\text{objects in hash table}}{\#\text{buckets in hash table}}$$

More on Load Factor of Hash Table

- Question: which collision resolution strategy is feasible for load factor larger than 1?
  - Answer: separate chaining.
  - Note: for open addressing, we require $L \leq 1$.
- Claim: $L = O(1)$ is a necessary condition for operations to run in constant time.

Double Hashing

$$h_i(x) = (h(x) + i*g(x)) \% n$$

- Uses 2 distinct hash functions.

- Increment **differently** depending on the key.
  - If h(x) = 13, g(x) = 17, the probe sequence is 13, 30, 47, 64, …
  - If h(x) = 19, g(x) = 7, the probe sequence is 19, 26, 33, 40, …
  - For linear and quadratic probing, the incremental probing patterns are **the same** for all the keys.

Double Hashing

Example

- Hash table size **`n = 7, h(key) = key%7,`**
  `g(key) = (5-key)%5`
  - Thus **h_i`(key) = (key%7+(5-key)%5*i)%7`**
  - Suppose we insert 9, 16, 11, 2 in sequence.

		9		11	2	16
[0]	[1]	[2]	[3]	[4]	[5]	[6]

  - **h_0`(16) = 2.`** Not empty!
  - **h_1`(16) = 6.`** It is empty, so we insert there.
  - **h_0`(2) = 2.`** Not empty!
  - **h_1`(2) = 5.`** It is empty, so we insert there.

Outline

- Collision Resolution: Separate Chaining

- Collision Resolution: Open Addressing
  - Linear Probing
  - Quadratic Probing and Double Hashing
  - Performance of Open Addressing

Performance of Open Addressing

- Hard to analyze rigorously.
- The runtime is dominated by the number of comparisons.
- The number of comparisons depends on the load factor L.
- Define the expected number of comparisons in an **unsuccessful search** as $U(L)$.
- Define the expected number of comparisons in a **successful search** as $S(L)$.

Expected Number of Comparisons

- Linear probing

$$U(L) = \frac{1}{2}\left[1 + \left(\frac{1}{1-L}\right)^2\right]$$

$$S(L) = \frac{1}{2}\left[1 + \frac{1}{1-L}\right]$$

L	$U(L)$	$S(L)$
0.5	2.5	1.5
0.75	8.5	2.5
0.9	50.5	5.5

$L \leq 0.75$ is recommended.

Expected Number of Comparisons

- Quadratic probing and double hashing

$$U(L) = \frac{1}{1-L}$$

$$S(L) = \frac{1}{L}\ln\frac{1}{1-L}$$

L	$U(L)$	$S(L)$
0.5	2	1.4
0.75	4	1.8
0.9	10	2.6

Which Strategy to Use?

- Both separate chaining and open addressing are used in real applications.

- Some basic guidelines:
  - If space is important, better to use open addressing.
  - If need removing items, better to use separate chaining.
    - `remove()` is tricky in open addressing.
  - In mission critical application, prototype both and compare.

Hash

- Hashing Basics
- Hash Function
- Collision Resolution
- Hash Table Size and Rehashing
- Applications of Hashing

Determine Hash Table Size

- First, given **performance** requirements, determine the maximum permissible **load factor**.

- Example: we want to design a hash table based on **linear probing** so that on average
  - An **unsuccessful** search requires no more than 13 compares.
  - A **successful** search requires no more than 10 compares.

$$U(L) = \frac{1}{2}\left[1 + \left(\frac{1}{1-L}\right)^2\right] \leq 13 \quad \Rightarrow L \leq \frac{4}{5}$$

$$S(L) = \frac{1}{2}\left[1 + \frac{1}{1-L}\right] \leq 10 \quad \Rightarrow L \leq \frac{18}{19}$$

$$L \leq \frac{4}{5}$$

Determine Hash Table Size

- For a fixed table size, estimate maximum number of items that will be inserted.

- Example: no more than 1000 items.
  - For load factor $L = \frac{|S|}{n} \leq \frac{4}{5}$, table size

$$n \geq \frac{5}{4} \cdot 1000 = 1250$$

  - Pick n as a **prime** number. For example, $n = 1259$.

However, sometimes there is no limit on the number of items to be inserted.

Rehashing

Motivation

- With more items inserted, the load factor increases. At some point, it will exceed the threshold (4/5 in the previous example) determined by the performance requirement.

- For the separate chaining scheme, the hash table becomes inefficient when load factor *L* is too high.
  - If the size of the hash table is fixed, search performance deteriorates with more items inserted.

- Even worse, for the open addressing scheme, when the hash table becomes full, we **cannot** insert a new item.

Rehashing

- To solve these problems, we need to **rehash**:
  - Create a **larger** table, scan the current table, and then insert items into new table using the new hash function.
  - **Note**: The order is from the beginning to the end of the current table. Not original insertion order.

- We can approximately double the size of the current table.

- Observation: The single operation of rehashing is time-consuming. However, it does not occur frequently.
  - How should we justify the time complexity of rehashing?

Amortized Analysis

- **Amortized analysis**: A method of analyzing algorithms that considers the entire sequence of operations of the program.
  - The idea is that while certain operations may be costly, they don't occur frequently; the less costly operations are much more than the costly ones in the long run.
  - Therefore, the cost of those expensive operations is **averaged** over a sequence of operations.
  - In contrast, our previous complexity analysis only considers a single operation, e.g., insert, find, etc.

Amortized Analysis of Rehashing

- Suppose the threshold of the load factor is 0.5. We will double the table size after reaching the threshold.
- Suppose we start from an empty hash table of size $2M$.

- Assume $O(1)$ operation to insert up to M items.
  - Total cost of inserting the first M items: $O(M)$
- For the $(M+1)$-th item, create a new hash table of size $4M$.
  - Cost: $O(1)$
- Rehash all M items into the new table. Cost: $O(M)$
- Insert new item. Cost: $O(1)$

Total cost for inserting $M+1$ items is $2O(M) + 2O(1) = O(M)$.

Amortized Analysis of Rehashing

Total cost for inserting $M + 1$ items is $O(M)$.

- The average cost to insert $M + 1$ items is O(1).
  - Rehashing cost is **amortized** over individual inserts.

Hash

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Application: De-Duplication

- Given: a stream of objects
  - Linear scan through a huge file
  - Or, objects arriving in real time

- Goal: remove duplicates (i.e., keep track of unique objects)
  - E.g., report unique visitors to website
  - Or, avoid duplicates in search result

- Solution: when new object x arrives,
  - Look x in hash table H
  - If not found, insert x into H

Application: 2-SUM Problem

- Given: an unsorted array A of n integers. Target sum t.
- Goal: determine whether or not there are two numbers x and y in A with

$$x + y = t$$

1. Naïve solution: exhaustive search of pairs of number
   - Time: $\Theta(n^2)$
2. Better solution: 1) Sort A; 2) For each x in A, look for $t - x$ in A via binary search.
   - Time: $\Theta(n \log n)$
3. Best: 1) Insert elements of A into hash table H; 2) For each x in A, search for $t - x$.
   - Time: $\Theta(n)$

Further Immediate Application

- Spellchecker
- Database

Hash Table

Summary

- Choice of the hash function.
- Collision resolution scheme.
- Hash table size and rehashing.

- Time complexity of hash table versus sorted array
  - insert(): $O(1)$ versus $O(n)$
  - find(): $O(1)$ versus $O(\log n)$

- When **NOT** to use hash?
  - **Rank search**: return the k-th largest item.
  - **Sort**: return the values in order.

Outline

- Linked List
- Stack
- Queue
- Sort
- Linear-Time Selection
- Hashing
- Bloom Filter

Bloom Filter

- Invented by Burton Bloom in 1970
- Supports **fast insert** and **find**
- Comparison to hash tables:
  - Pros: more space efficient
  - Cons:
    1. Can't store an associated object
    2. No deletion (There are variations support deletion, but this operation is complicated)
    3. Small **false positive** probability: may say x has been inserted even if it hasn't been
       - But no false negative (x is inserted, but says not inserted)

Bloom Filter Applications

- When to use bloom filter?
  - If the false positive is not a concern, no deletion, and you look for space efficiency
- Original application: spell checker
  - 40 years ago, space is a big concern, it's OK to tolerate some error
- Canonical application: list of forbidden passwords
  - Don't care about the false positive issue
- Modern applications: network routers
  - Limited memory, need to be fast
  - Applications include keeping track of blocked IP address, keeping track of contents of caches, etc.

Bloom Filter Implementation: Components

- An array of n **bits**. Each bit 0 or 1
  - $n = b|S|$, where b is small real number. For example, $b \approx 8$ for 32-bit IP address (That's why it is space efficient)

- k hash functions $h_1, \ldots, h_k$, each mapping inside $\{0,1, \ldots, n-1\}$.
  - k usually small.

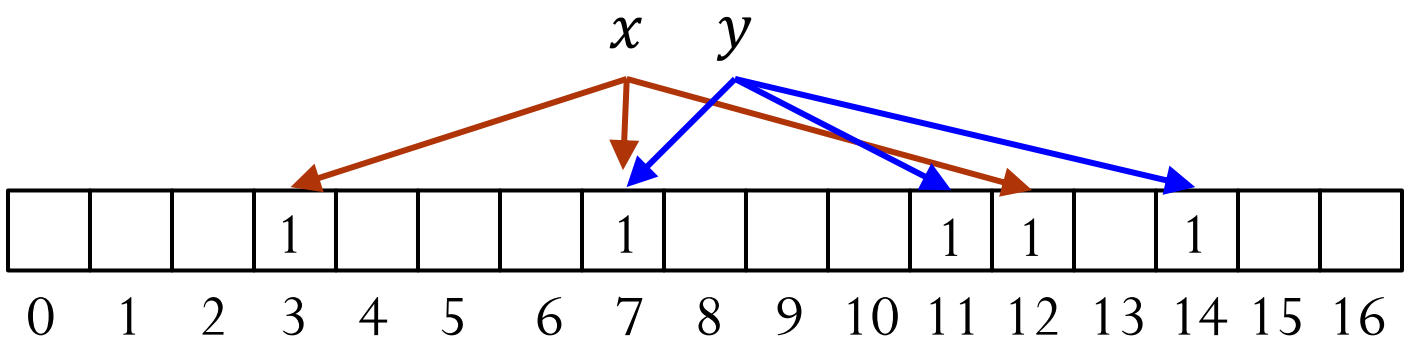
Bloom Filter Insert

- Initially, the array is all-zero.
- Insert x: For $i = 1,2, \ldots, k$, set $A[h_i(x)] = 1$
  - No matter whether the bit is 0 or 1 before

Example: $n = 17$, 3 hash functions

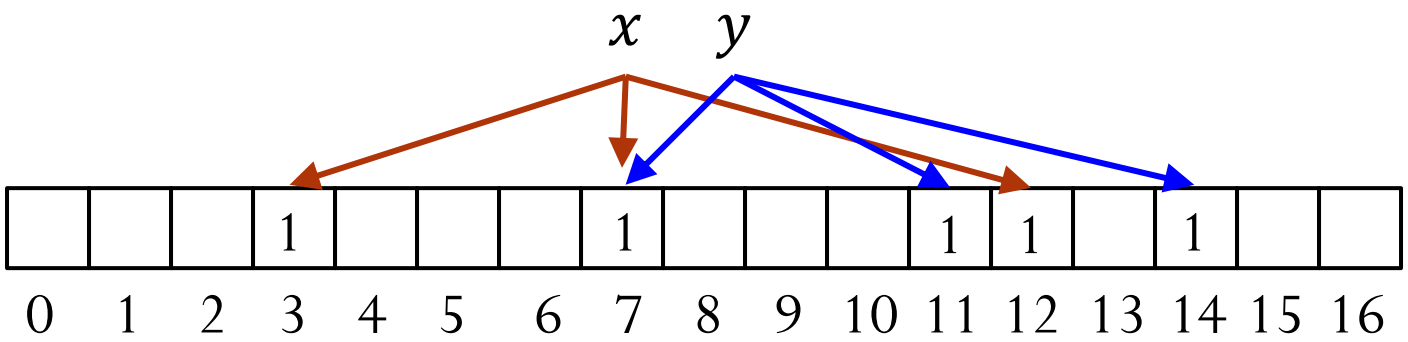
$h_1(x) = 7, h_2(x) = 3, h_3(x) = 12$

$h_1(y) = 11, h_2(y) = 14, h_3(y) = 7$



Bloom Filter Find

- Find x: return true if and only if $A[h_i(x)] = 1, \forall i = 1, \ldots, k$



Suppose $h_1(x) = 7, h_2(x) = 3, h_3(x) = 12$. Find x? Yes!

Suppose $h_1(z) = 3, h_2(z) = 11, h_3(z) = 5$. Find z? No!

- No false negative: if x was inserted, find(x) guaranteed to return true
- False positive possible: consider $h_1(w) = 11, h_2(w) = 12, h_3(w) = 7$ in the above example

Heuristic Analysis of Error Probability

- Intuition: should be a trade-off between space (array size) and false positive probability
  - Array size decreases, more reuse of bits, false positive probability increases
- Goal: analyze the false positive probability
- Setup: Insert data set S into the Bloom filter, use k hash functions, array has n bits
- Assumption: All k hash functions map keys uniformly random and these hash functions are independent

Probability of a Slot Being 1

- For an arbitrary slot j in the array, what's the probability that the slot is 1?

- Consider when slot j is 0
  - Happens when $h_i(x) \neq j$ for all $i = 1, \ldots, k$ and $x \in S$
  - $\Pr(h_i(x) \neq j) = 1 - \frac{1}{n}$
  - $\Pr(A[j] = 0) = \left(1 - \frac{1}{n}\right)^{k|S|} \approx e^{-\frac{k|S|}{n}} = e^{-\frac{k}{b}}$
    - $b = \frac{n}{|S|}$ denotes # of bits per object
- $\Pr(A[j] = 1) \approx 1 - e^{-\frac{k}{b}}$

False Positive Probability

- For x not in S, the false positive probability happens when all $A[h_i(x)] = 1$ for all $i = 1, \ldots, k$
  - The probability is $\epsilon \approx \left(1 - e^{-\frac{k}{b}}\right)^k$
- For a fixed b, ϵ is minimized when $k = (\ln 2) \cdot b$
- The minimal error probability is $\epsilon \approx \left(\frac{1}{2}\right)^{\ln 2 \cdot b} \approx 0.6185^b$
  - Error probability decreases exponentially with b
- Example: $b = 8$, could choose k as 5 or 6. Min error probability $\approx 2\%$