Blind Deconvolutional Phase Retrieval via Convex Programming

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Abstract

- **Problem:** Consider the task of recovering two real or complex L-vectors from phaseless Fourier measurements of their circular convolution
- **Novel Solution:** We propose a novel convex relaxation that is based on a lifted matrix recovery formulation that allows a nontrivial convex relaxation of the bilinear measurements from convolution
- **Main Results:** We prove that if the two signals belong to known random subspaces of dimensions K and N, then they can be recovered up to the inherent scaling ambiguity with $L \gtrsim (K+N)\log^2 L$ phaseless measurements
- Our method provides the first theoretical recovery guarantee for this problem by a computationally efficient algorithm and does not require a solution estimate to be computed for initialization
- **Analysis:** Our proof is based Rademacher complexity estimates
- **Numerics:** Additionally, we provide an ADMM implementation of the method and provide numerical experiments that verify the theory

Problem Formulation

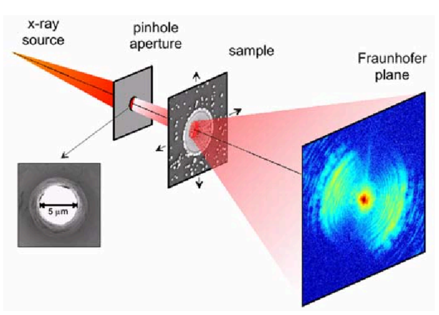
Want to recover two real or complex L-vectors from the phaseless Fourier measurements of their circular convolution.

Observation Model: Observe the magnitude measurements of the Fourier transform of the circular convolution of two vectors $\boldsymbol{w}$, and $\boldsymbol{x}$

$$\boldsymbol{y} = |\boldsymbol{F}(\boldsymbol{w} \circledast \boldsymbol{x})|, \tag{1}$$

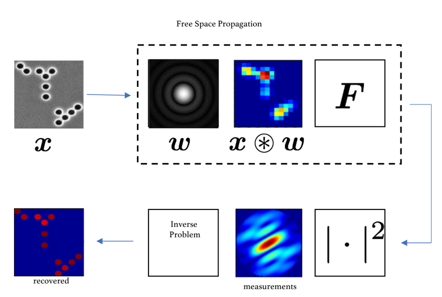
where $\boldsymbol{F}$ is the DFT matrix with entries

$$F[\omega, t] = \frac{1}{\sqrt{L}} e^{-j2\pi\omega t/L}, \ 1 \le \omega, t \le L.$$

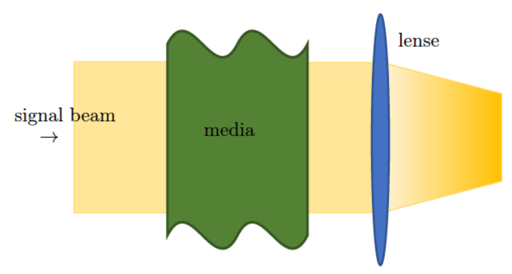


Xray Crystallography

Fourier Ptychography



Coherent Light Imaging



Visible Light Communications

Objective: Recover $\boldsymbol{w} \in \mathbb{C}^L$, and $\boldsymbol{x} \in \mathbb{C}^L$ from $\boldsymbol{y} \in \mathbb{C}^L$

Challenge: Highly ill-posed; combinations of two difficult problems, namely, blind deconvolution, and phase retrieval

- Easy to see in the frequency domain, define $\hat{\boldsymbol{w}} = \sqrt{L}\boldsymbol{F}\boldsymbol{w}$, and $\hat{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x}$. Measurements:

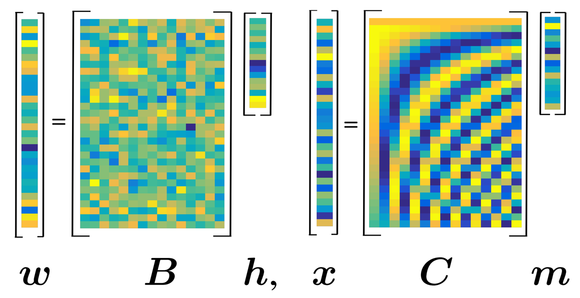
$$(y[\ell])^2 = |\hat{w}[\ell]|^2 |\hat{x}[\ell]|^2, \ \ell = 1,2,3,\ldots,L$$

- Want to recover two numbers by observing only their product!
- Not only that the phase of the two numbers is also missing!
- Impossible to solve without further information.

Structural Assumptions: $\boldsymbol{w}$, and $\boldsymbol{x}$ are members of known subspaces.

- The unknowns $\boldsymbol{w}$, and $\boldsymbol{x}$ reside in known K, and N-dimensional subspaces of $\mathbb{C}^L$, respectively. This means, we can write

$$\boldsymbol{w} = \boldsymbol{B}\boldsymbol{h}, \ \boldsymbol{x} = \boldsymbol{C}\boldsymbol{m}, \ \text{where } \boldsymbol{B} \in \mathbb{C}^{L\times K}, \text{ and } \boldsymbol{C} \in \mathbb{C}^{L\times N}.$$



Subspace Constraints

- Matrices $\boldsymbol{B}$, and $\boldsymbol{C}$ are known
- Corresponding expansion coefficients $\boldsymbol{h} \in \mathbb{C}^K$, and $\boldsymbol{m} \in \mathbb{C}^N$ are unknown.
- # of unknowns = $K + N$, # of observations = L

Observations in Matrix Form: Incorporating the subspace constraints, the observations become

$$\boldsymbol{y} = \sqrt{L}|\boldsymbol{F}\boldsymbol{B}\boldsymbol{h} \odot \boldsymbol{F}\boldsymbol{C}\boldsymbol{m}|,$$

where $\odot$ represents the Hadamard product. Let $\boldsymbol{b}_\ell^*$ and $\boldsymbol{c}_\ell^*$ be the rows of $\sqrt{L}\boldsymbol{F}\boldsymbol{B}$, and $\boldsymbol{F}\boldsymbol{C}$, respectively. The entries of the measurements $\boldsymbol{y}$ can be expressed as

$$y^2[\ell] = |\langle \boldsymbol{b}_\ell, \boldsymbol{h}\rangle\langle \boldsymbol{c}_\ell, \boldsymbol{m}\rangle|^2, \ \ell = 1,2,3,\ldots,L.$$

- Observations are non-linear in $\boldsymbol{h}$, and $\boldsymbol{m}$. Lift the unknown vectors to rank-1 matrices

$$y^2[\ell] = \langle \boldsymbol{b}_\ell\boldsymbol{b}_\ell^*, \boldsymbol{h}\boldsymbol{h}^*\rangle\langle \boldsymbol{c}_\ell\boldsymbol{c}_\ell^*, \boldsymbol{m}\boldsymbol{m}^*\rangle, \ \ell = 1,2,3,\ldots,L.$$

New lifted unknowns: $\boldsymbol{X}_1 = \boldsymbol{h}\boldsymbol{h}^*$, and $\boldsymbol{X}_2 = \boldsymbol{m}\boldsymbol{m}^*$.

- Observations are only bilinear in the lifted unknowns $\boldsymbol{X}_1$, and $\boldsymbol{X}_2$

Novel Convex Relaxation

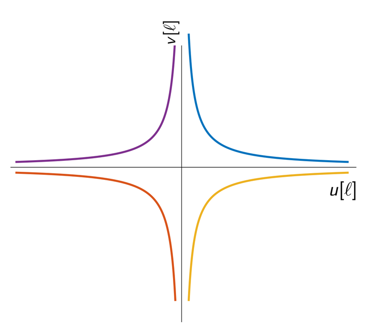
- Formulation as a non-convex matrix feasibility problem

Observe: $y^2[\ell] = |\boldsymbol{b}_\ell^*\boldsymbol{h}|^2 \cdot |\boldsymbol{c}_\ell^*\boldsymbol{m}|^2$
$\boldsymbol{b}_\ell^*$ is ℓth row of $\boldsymbol{F}\boldsymbol{B}$
$\boldsymbol{c}_\ell^*$ is ℓth row of $\boldsymbol{F}\boldsymbol{C}$

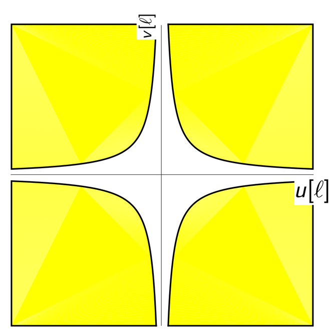
Find: $\boldsymbol{h} \in \mathbb{R}^K, \boldsymbol{m} \in \mathbb{R}^N$

Solve:
$$\begin{aligned} &\underset{\boldsymbol{h},\boldsymbol{m}}{\text{minimize}} \ \|\boldsymbol{h}\|^2 + \|\boldsymbol{m}\|^2 \\ &\text{subject to } \langle \boldsymbol{b}_\ell\boldsymbol{b}_\ell^*, \boldsymbol{X}_1\rangle\langle \boldsymbol{c}_\ell\boldsymbol{c}_\ell^*, \boldsymbol{X}_2\rangle = y^2[\ell] \\ &\qquad \boldsymbol{X}_1 = \boldsymbol{h}\boldsymbol{h}^*, \boldsymbol{X}_2 = \boldsymbol{m}\boldsymbol{m}^* \end{aligned}$$

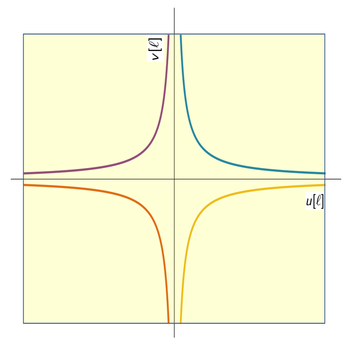
- Denote $u[\ell] = \langle \boldsymbol{b}_\ell\boldsymbol{b}_\ell^*, \boldsymbol{X}_1\rangle$, and $v[\ell] = \langle \boldsymbol{c}_\ell\boldsymbol{c}_\ell^*, \boldsymbol{X}_2\rangle$.
- The non-convex hyperbolic feasible set $u[\ell]v[\ell] = y^2[\ell]$, and its convex relaxation



Feasible Set



Convex hull?

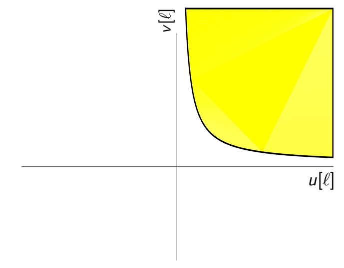


Convex hull, not good!

- The unknowns $\boldsymbol{X}_1$, and $\boldsymbol{X}_2$ are positive-semidefinite (PSD) matrices; we have not yet accounted for this structure!

$$\boldsymbol{X}_1 \succcurlyeq \boldsymbol{0} \triangleq \boldsymbol{z}^*\boldsymbol{X}_1\boldsymbol{z}^* \ge 0, \forall \boldsymbol{z} \implies u[\ell] \ge 0$$
$$\boldsymbol{X}_2 \succcurlyeq \boldsymbol{0} \triangleq \boldsymbol{z}^*\boldsymbol{X}_2\boldsymbol{z}^* \ge 0, \forall \boldsymbol{z} \implies v[\ell] \ge 0$$

- Incorporating the $u[\ell] \ge 0$, and $v[\ell] \ge 0$ restricts the hyperbola to one of the first quadrant, which yields a non-trivial convex relaxation

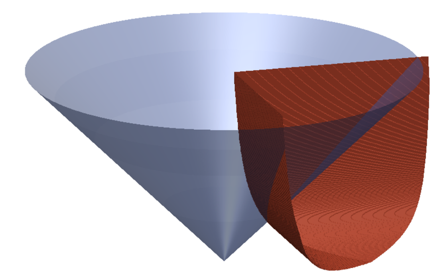


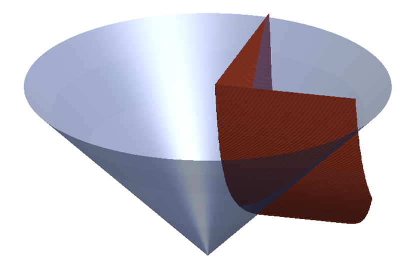
Convex hull of the feasible set including the PSD constraints

- We propose solving the following convex program then

$$\begin{aligned} &\underset{\boldsymbol{X}_1,\boldsymbol{X}_2}{\text{minimize}} \ \ \operatorname{trace}(\boldsymbol{X}_1) + \operatorname{trace}(\boldsymbol{X}_2) \\ &\text{subject to } \ \langle \boldsymbol{b}_\ell\boldsymbol{b}_\ell^*, \boldsymbol{X}_1\rangle\langle \boldsymbol{c}_\ell\boldsymbol{c}_\ell^*, \boldsymbol{X}_2\rangle \ge y^2[\ell] \leftarrow \text{BranchHull} \\ &\qquad \boldsymbol{X}_1 \succcurlyeq \boldsymbol{0}, \boldsymbol{X}_2 \succcurlyeq \boldsymbol{0}. \end{aligned}$$

Cartoon of the BranchHull Geometry





Blue: PSD Cone, Red: Boundary of Hyperbolic Constraint

Point in intersection with smallest trace lives along the ridge, where $L = 2$ hyperbolic constraints are satisfied with equalities.

Main Result: Exact Recovery

- Convex program for Blind Deconvolutional Phase Retrieval

$$\begin{aligned} &\underset{\boldsymbol{X}_1,\boldsymbol{X}_2}{\text{minimize}} \ \operatorname{trace}(\boldsymbol{X}_1) + \operatorname{trace}(\boldsymbol{X}_2) \\ &\text{subject to } \langle \boldsymbol{b}_\ell\boldsymbol{b}_\ell^*, \boldsymbol{X}_1\rangle\langle \boldsymbol{c}_\ell\boldsymbol{c}_\ell^*, \boldsymbol{X}_2\rangle \ge y^2[\ell] \\ &\qquad \boldsymbol{X}_1 \succcurlyeq \boldsymbol{0}, \boldsymbol{X}_2 \succcurlyeq \boldsymbol{0}. \end{aligned}$$

- **Theorem** [Ahmed, Aghasi, Hand]: Choose $\boldsymbol{B}$ and $\boldsymbol{C}$ to have i.i.d. standard normal entries. Then, $\boldsymbol{h} \in \mathbb{R}^K$ and $\boldsymbol{m} \in \mathbb{R}^N$ can be exactly recovered (up to global rescaling) with high probability if $L \gtrsim (K+N)\log^2 L$.

ADMM Implementation

- Convex program:

$$\begin{aligned} &\underset{\boldsymbol{X}_1,\boldsymbol{X}_2}{\text{minimize}} \ \operatorname{trace}(\boldsymbol{X}_1) + \operatorname{trace}(\boldsymbol{X}_2) \\ &\text{subject to } \langle \boldsymbol{a}_{1,\ell}\boldsymbol{a}_{1,\ell}^*, \boldsymbol{X}_1\rangle\langle \boldsymbol{a}_{2,\ell}\boldsymbol{a}_{2,\ell}^*, \boldsymbol{X}_2\rangle \ge y^2[\ell], \ \ell = 1,2,3,\ldots,L \\ &\qquad \boldsymbol{X}_1 \succcurlyeq \boldsymbol{0}, \boldsymbol{X}_2 \succcurlyeq \boldsymbol{0}. \end{aligned}$$

- Define

$$\mathcal{C} := \{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{R}^L \times \mathbb{R}^L : u[\ell]v[\ell] \ge y^2[\ell] > 0, u[\ell] \ge 0\}$$
$$\mathbb{I}_{\mathcal{C}}(\boldsymbol{u},\boldsymbol{v}) = 0 \text{ when } (\boldsymbol{u},\boldsymbol{v}) \in \mathcal{C}, \text{ and } +\infty \text{otherwise.}$$
$$\mathbb{I}_+(\boldsymbol{Z}) = 0 \text{ when } \boldsymbol{Z} \succcurlyeq \boldsymbol{0}, \text{ and } +\infty \text{ otherwise.}$$

- Reformulation:

$$\begin{aligned} &\underset{\{\boldsymbol{X}_j,\boldsymbol{Z}_j,\boldsymbol{q}_j\}_{j=1,2}}{\text{minimize}} \ \mathbb{I}_{\mathcal{C}}(\boldsymbol{q}_1,\boldsymbol{q}_2) + \sum_{j=1}^{2}\operatorname{trace}(\boldsymbol{X}_j) + \mathbb{I}_+(\boldsymbol{Z}_j) \\ &\text{subject to } \ q_{j,\ell} = \langle \boldsymbol{a}_{j,\ell}\boldsymbol{a}_{j,\ell}^*, \boldsymbol{X}_j\rangle, \ \ell = 1,2,3,\ldots,L, \ j = 1,2. \\ &\qquad \boldsymbol{Z}_j = \boldsymbol{X}_j, \ j = 1,2. \end{aligned}$$

- Lagrangian:

$$\mathcal{L}(\{\boldsymbol{X}_j,\boldsymbol{Z}_j,\boldsymbol{P}_j,\boldsymbol{q}_j,\boldsymbol{\alpha}_j\}_{j=1,2}) := \mathbb{I}_{\mathcal{C}}(\boldsymbol{q}_1,\boldsymbol{q}_2) + \sum_{j=1}^{2}\operatorname{trace}(\boldsymbol{X}_j) + \mathbb{I}_+(\boldsymbol{Z}_j)$$
$$+ \frac{\rho_1}{2}\sum_{j=1}^{2}\sum_{\ell=1}^{L}(q_{j,\ell} - \langle \boldsymbol{a}_{j,\ell}\boldsymbol{a}_{j,\ell}^*, \boldsymbol{X}_j\rangle + \alpha_{j,\ell})^2 + \sum_{j=1}^{2}\|\boldsymbol{X}_j - \boldsymbol{Z}_j + \boldsymbol{P}_j\|_F^2$$

- Primal updates:

$$\boldsymbol{X}_j^{(k+1)} = \underset{\boldsymbol{X}_j}{\operatorname{argmin}} \ \operatorname{trace}(\boldsymbol{X}_j) + \frac{\rho_1}{2}\sum_{\ell=1}^{L}\left(\langle \boldsymbol{a}_{j,\ell}\boldsymbol{a}_{j,\ell}^*, \boldsymbol{X}_j\rangle - q_{j,\ell}^{(k)} - \alpha_{j,\ell}^{(k)}\right)^2 + \|\boldsymbol{X}_j - \boldsymbol{Z}_j^{(k)} - \boldsymbol{P}_j^{(k)}\|_F^2$$

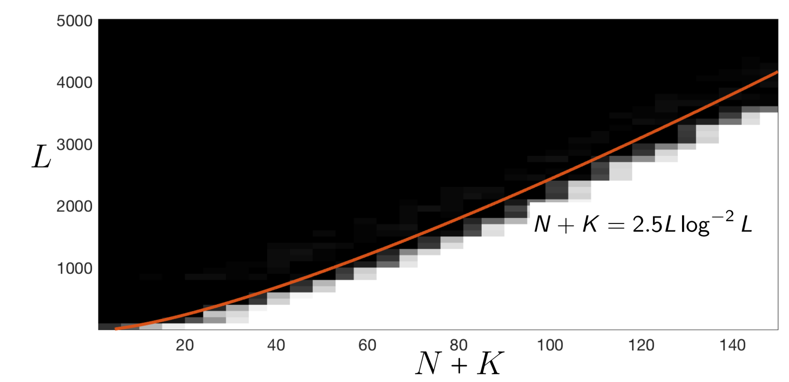
$$\boldsymbol{Z}_j^{(k+1)} = \underset{\boldsymbol{Z}_j}{\operatorname{argmin}} \ \frac{1}{2}\|\boldsymbol{Z}_j - \boldsymbol{X}_j^{(k+1)} - \boldsymbol{P}_j^{(k)}\|_F^2 + \mathbb{I}_+(\boldsymbol{Z}_j)$$

$$(\boldsymbol{q}_1^{(k+1)}, \boldsymbol{q}_2^{(k+1)}) = \underset{\boldsymbol{q}_1,\boldsymbol{q}_2}{\operatorname{argmin}} \ \frac{1}{2}\sum_{j=1}^{2}\sum_{\ell=1}^{L}\left(q_{j,\ell} - \langle \boldsymbol{a}_{j,\ell}\boldsymbol{a}_{j,\ell}^*, \boldsymbol{X}_j^{(k+1)}\rangle + \alpha_{j,\ell}^{(k)}\right)^2 + \mathbb{I}_{\mathcal{C}}(\boldsymbol{q}_1,\boldsymbol{q}_2)$$

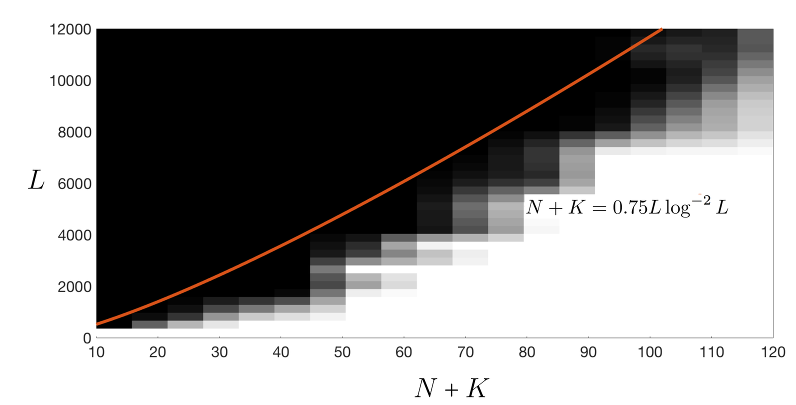
- Dual updates:

$$\alpha_{j,\ell}^{(k+1)} = \alpha_{j,\ell}^{(k)} + u_{j,\ell}^{(k+1)} - \langle \boldsymbol{a}_{j,\ell}\boldsymbol{a}_{j,\ell}^*, \boldsymbol{X}_j^{(k+1)}\rangle$$
$$\boldsymbol{P}_j^{(k+1)} = \boldsymbol{P}_j^{(k)} + \boldsymbol{X}_j^{(k+1)} - \boldsymbol{Z}_j^{(k+1)}$$

Phase Portrait for an ADMM Implementation



Black: Success, White: Failure. $\boldsymbol{B}$, and $\boldsymbol{C}$ are Gaussian matrices. Convex BDPR succeeds for reasonable constants in sample complexity.



Black: Success, White: Failure. $\boldsymbol{C}$ is Gaussian, and $\boldsymbol{B}$ is a subset of the columns of identity matrix. Convex BDPR succeeds for reasonable constants in sample complexity.

Acknowledgement

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References

[1] A. Aghasi, A. Ahmed, and P. Hand. Branchhull: Convex bilinear inversion from the entrywise product of signals with known signs. *arXiv preprint arXiv:1702.04342*, 2017.

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