Our problem is to minimise $\sum_n \sigma_n^2/N_n$ (the *objective function*) subject to $\sum_n N_n = N_T$ (the *constraint function*).[1] We can do this using the method of Lagrange multipliers. First we define the Lagrangian function

$$\mathcal{L} = \underbrace{\sum_n \frac{\sigma_n^2}{N_n}}_{\text{objective function}} \underbrace{-\lambda\left(-N_T + \sum_n N_n\right)}_{\text{constraint function}}. \tag{1}$$

We then set the partial derivatives of $\mathcal{L}$ with respect to $N_n$ to be zero for each index choice $n$; i.e.

$$\frac{\partial \mathcal{L}}{\partial N_n} = -\frac{\sigma_n^2}{N_n^2} + \lambda \equiv 0 \Longrightarrow N_n = \sigma_n/\sqrt{\lambda}. \tag{2}$$

We then satisfy the constraint, which is to say we enforce $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$. Thus we require

$$\sum_n \frac{\sigma_n}{\sqrt{\lambda}} = N_T \Longrightarrow \lambda = \left(\frac{\sigma_T}{N_T}\right)^2, \tag{3}$$

where we define $\sigma_T := \sum_n \sigma_n$. Hence the Lagrangian function $\mathcal{L}$ is extremised when

$$\frac{N_n}{N_T} = \frac{\sigma_n}{\sigma_T}, \tag{4}$$

which is to say that our optimal splitting would allocate a fraction of measurements to outcome $n$ that is roughly equal to the ratio $\sigma_n/\sigma_T$. This makes sense: we want to allocate more measurements to outcomes that are more uncertain.

It is then easy to check that the choice $N_n = \sigma_n/\sigma_T \times N_T$ satisfies Eq. (3) from `Instructions.pdf`. Simply substitute our choice of $N_n$ into the objective function to find

$$\sum_n \frac{\sigma_n^2}{N_n} = \sum_n \sigma_n \times \frac{\sigma_T}{N_T} = \frac{\sigma_T^2}{N_T}. \tag{5}$$

In other words, our choice saturates the inequality of Eq. (2) from `Instructions.pdf` and hence satisfies Eq. (3)[2].

[1] To simplify notation, we write $\sigma_n$ in place of $\sigma_{H_n}$.

[2] Eq. (3) is probably a mistake. We suspect it should be the same as Eq. (2) except with the inequality replaced with an equality.