[2024-1 Robotics]
Chapter 3. Rigid-Body Motions

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Purpose of this chapter

3.0. Introduction to Chapter

In the robotics, one may need to have various representations of

- position (위치) of a rigid body (p)
- orientation (자세) of a rigid body ($(\hat{\mathrm{x}}_{\mathrm{b}}, \hat{\mathrm{y}}_{\mathrm{b}}, \hat{\mathrm{z}}_{\mathrm{b}})$)

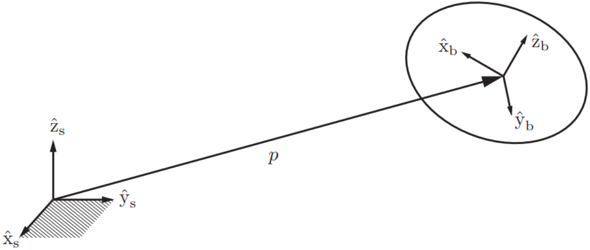
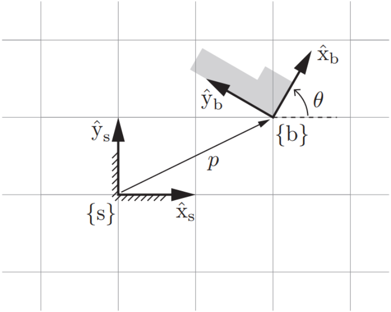


Figure 3.6: Mathematical description of position and orientation.

- Rigid-body motion in 2D plane (= Planar case) $\leftarrow$ 3 dof
- Rigid-body motion in 3D plane (= Spatial case) $\leftarrow$ 6 dof

Notations that we will use

- $\{\mathrm{s}\}$: Fixed frame or space frame
- $\{\mathrm{b}\}$: Body frame (attached to a part of robot, or not)
- v (in *Italic* font): Coordinate-dependent vector
- v (in Roman font): Coordinate-free vector
- $\hat{(\cdot)}$: Unit vector



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3.0. Introduction to Chapter

When representing a frame or rotation, we follow the right-handed rules.

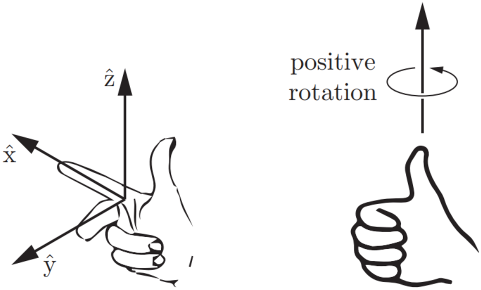
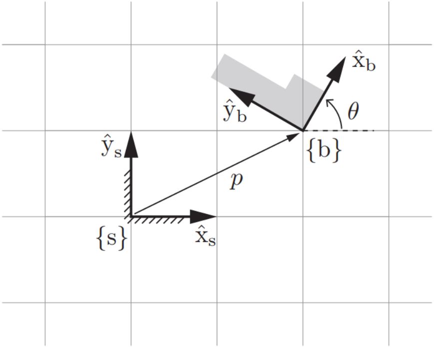


Figure 3.2: (Left) The $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes of a right-handed reference frame are aligned with the index finger, middle finger, and thumb of the right hand, respectively. (Right) A positive rotation about an axis is in the direction in which the fingers of the right hand curl when the thumb is pointed along the axis.

Position of $\{\mathrm{b}\}$ in terms of $\{\mathrm{s}\}$: Planar case

3.1. Rigid-body Motions in the Plane

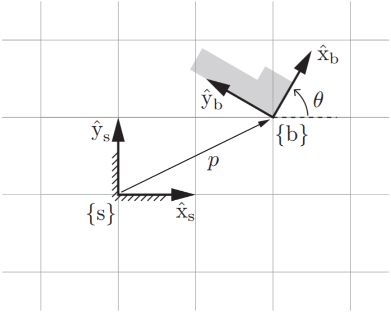


The body-frame origin (= position of the body frame) expressed in terms of coordinate axes of $\{\mathrm{s}\}$:

$$p = p_x \hat{\mathrm{x}}_\mathrm{s} + p_y \hat{\mathrm{y}}_\mathrm{s}$$

Orientation of $\{b\}$ in terms of $\{s\}$: Planar case

3.1. Rigid-body Motions in the Plane



The orientation of $\{b\}$ can be specified by the angle $\theta \in \mathbb{R}$.
$\Rightarrow$ One can represent the coordinate axes $\{b\}$ in terms of those of $\{s\}$ as

$$\hat{x}_b = \cos\theta \hat{x}_s + \sin\theta \hat{y}_s,$$
$$\hat{y}_b = -\sin\theta \hat{x}_s + \cos\theta \hat{y}_s,$$

$\therefore$ A matrix-vector pair gives a configuration of a frame.

Thus, in the planar case, we express in terms of $\{\mathrm{s}\}$

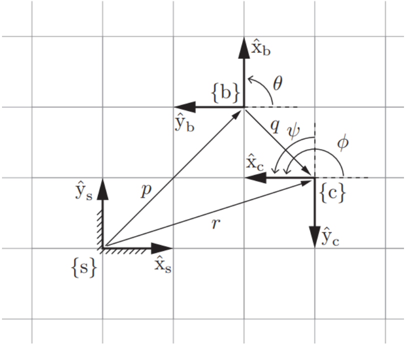
- Position of $\{\mathrm{b}\}$ $\rightarrow p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \in \mathbb{R}^2$
- Orientation of $\{\mathrm{b}\}$ $\rightarrow P = \begin{bmatrix} \hat{\mathrm{x}}_\mathrm{b} & \hat{\mathrm{y}}_\mathrm{b} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathbb{R}^{2\times 2}$

  (where such a P is often called rotation matrix (회전 행렬).)

The matrix-vector pair (P, p) represents a configuration of $\{\mathrm{b}\}$ in $\{\mathrm{s}\}$.

How can we represent pos./ori. with another frame

3.1. Rigid-body Motions in the Plane



For another frame $\{\mathrm{c}\}$, you may express, in terms of $\{\mathrm{s}\}$,

- Position of $\{\mathrm{c}\}$ $\rightarrow r = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$
- Orientation of $\{\mathrm{c}\}$ $\rightarrow R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

(Cont'd)

3.1. Rigid-body Motions in the Plane

Let

- (P, p) represents the configuration of $\{\mathrm{b}\}$ expressed in $\{\mathrm{s}\}$;
- (Q, q) represents the configuration of $\{\mathrm{c}\}$ expressed in $\{\mathrm{b}\}$;
- (R, r) represents the configuration of $\{\mathrm{c}\}$ expressed in $\{\mathrm{s}\}$.

$\Rightarrow$ the pair (Q, q) can be represented with (P, p) and (R, r) as follows:

$$R = PQ \qquad \text{(convert } Q \text{ to the } \{\mathrm{s}\} \text{ frame)}$$
$$r = Pq + p \qquad \text{(convert } q \text{ to the } \{\mathrm{s}\} \text{ frame and vector-sum with } p\text{)}$$

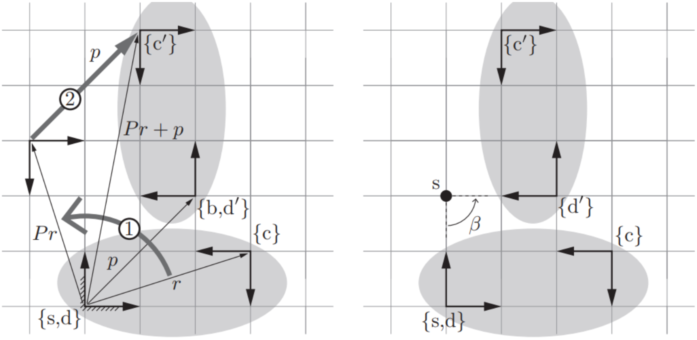
Lesson: The matrix-vector pair also changes the reference frame.

Understanding a rigid-body motion: Approach 1

A rigid-body motion can be represented by "rotation → translation" (Fig. (a)).

The transformation of the rigid body (from $\{\mathrm{c}\}$ to $\{\mathrm{c}'\}$)

1. rotates $\{\mathrm{c}\}$ according to the rotation matrix P, and
2. translates Pr by p in $\{\mathrm{s}\}$.



(a) (b)

Approach 2 = Screw motion (in a planar case)

$$\begin{aligned}\text{Rigid-body motion} &= \text{Rotation} \rightarrow \text{translation} \quad \text{(Fig. (a))}\\ &= \text{Rotation of the body about s by an angle } \beta \text{ (Fig. (b))}\\ &= \text{Screw motion}\\ &\quad \text{(that can be represented by a screw coordinate } (\beta, s_x, s_y)\text{)}\end{aligned}$$

- Exponential coordinates $\mathcal{S}\theta$ with $\theta = \dfrac{\pi}{2}$ and representation of screw axis

$$\mathcal{S} = \begin{bmatrix} \omega \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \text{unit angular velocity} \\ \text{linear velocity of the origin of the } \{s\} \text{ frame} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

- Twist = the velocity of $\mathcal{S}\theta$

$$\mathcal{V} = \mathcal{S}\dot{\theta}.$$

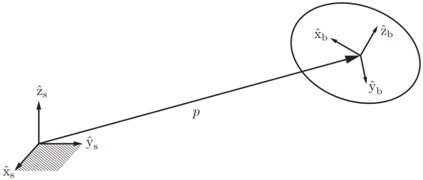


Figure 3.6: Mathematical description of position and orientation.

The origin of the $\{\mathrm{b}\}$ frame can be represented as

- (position) $p = p_1\hat{\mathrm{x}}_{\mathrm{s}} + p_2\hat{\mathrm{y}}_{\mathrm{s}} + p_3\hat{\mathrm{z}}_{\mathrm{s}}$,
- (orientation)

$$\begin{aligned} \hat{\mathrm{x}}_{\mathrm{b}} &= r_{11}\hat{\mathrm{x}}_{\mathrm{s}} + r_{21}\hat{\mathrm{y}}_{\mathrm{s}} + r_{31}\hat{\mathrm{z}}_{\mathrm{s}} \\ \hat{\mathrm{y}}_{\mathrm{b}} &= r_{21}\hat{\mathrm{x}}_{\mathrm{s}} + r_{22}\hat{\mathrm{y}}_{\mathrm{s}} + r_{23}\hat{\mathrm{z}}_{\mathrm{s}} \\ \hat{\mathrm{z}}_{\mathrm{b}} &= r_{13}\hat{\mathrm{x}}_{\mathrm{s}} + r_{23}\hat{\mathrm{y}}_{\mathrm{s}} + r_{33}\hat{\mathrm{z}}_{\mathrm{s}} \end{aligned} \quad \Rightarrow \begin{bmatrix} \hat{\mathrm{x}}_{\mathrm{b}} & \hat{\mathrm{y}}_{\mathrm{b}} & \hat{\mathrm{z}}_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} =: R$$

Rotation matrix (회전 행렬)

The matrix R in the previous slide is often called a rotation matrix.

Note that the body-frame unit axes satisfy

- (Unit norm) $\|\hat{\mathrm{x}}_{\mathrm{b}}\| = \|\hat{\mathrm{y}}_{\mathrm{b}}\| = \|\hat{\mathrm{z}}_{\mathrm{b}}\| = 1$
- (Orthogonality) $\hat{\mathrm{x}_{\mathrm{b}}} \cdot \hat{\mathrm{y}}_{\mathrm{b}} = 0, \cdots$
- (Right-handed rules) $\hat{\mathrm{x}}_{\mathrm{b}} \times \hat{\mathrm{y}}_{\mathrm{b}} = \hat{\mathrm{z}}_{\mathrm{b}}$

Definitions 3.1–3.2: For an integer n,
the special orthogonal group $SO(n)$ is the set of $n \times n$ matrices R such that

- $R^\top R = I$ ($\because$ (Unit norm) + (Orthogonality))
- $\det(R) = 1$ ($\because$ (Right-handed rules)).

Note: An element of $SO(n)$ is a rotation matrix.

- $SO(2) \subset \mathbb{R}^{2\times 2}$?
- $SO(3) \subset \mathbb{R}^{3\times 3}$?

Examples and properties of rotation matrix

Example:

- In the planar (2D) case

$$R = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \quad \Rightarrow \quad Rx?$$

- In the spatial (3D) case

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/4) & -\sin(\pi/4) \\ 0 & \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \quad \Rightarrow \quad Rx?$$

Properties: For rotation matrices R_1, R_2, R_3,

1. $R_1^{-1} = R_1^\top$ is a rotation matrix.
2. $R_1 R_2$ is a rotation matrix.
3. $(R_1 R_2) R_3 = R_1 (R_2 R_3)$, but $R_1 R_2 \neq R_2 R_1$ (in $SO(3)$).
4. $\|Rx\| = \|x\|$ for any x.

Use of R (1/3): Representing an orientation

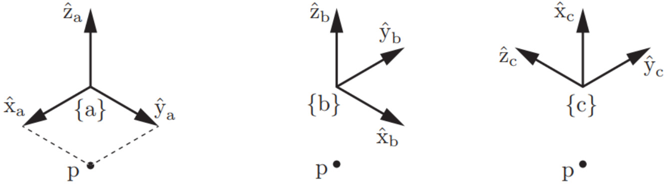


Figure 3.7: The same space and the same point p represented in three different frames with different orientations.

The orientation of the $\{a\}$, $\{b\}$, $\{c\}$ frames with respect to $\{s\} = \{a\}$ can be represented by the following rotation matrices:

$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

Use of R (2/3): Changing the reference frame

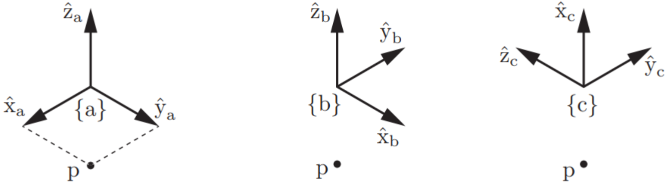


Figure 3.7: The same space and the same point p represented in three different frames with different orientations.

- R_{ab}: The rotation matrix associated with the orientation of $\{b\}$ w.r.t. $\{a\}$
- p_a: The vector p represented in the frame $\{a\}$, $\cdots$

We then have

$$R_{ab}R_{bc} = R_{ac}, \quad R_{ab}p_b = p_a.$$

Use of R (3/3): Rotating a vector or a frame

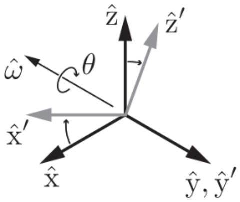


Figure 3.8: A coordinate frame with axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is rotated by θ about a unit axis $\hat{\omega}$ (which is aligned with $-\hat{y}$ in this figure). The orientation of the final frame, with axes $\{\hat{x}', \hat{y}', \hat{z}'\}$, is written as R relative to the original frame.

IF we rotate a frame about a unit axis $\hat{\omega}$ by an amount θ,
THEN the associated rotation matrix R can be represented as

$$R = \mathrm{Rot}(\hat{\omega}, \theta)$$

(Cont'd)

3.2. Rotations and Angular Velocities

Some basic forms of the rotation matrices are:

$$\mathrm{Rot}(\hat{\mathrm{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathrm{Rot}(\hat{\mathrm{y}}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\mathrm{z}}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A general form of $\mathrm{Rot}(\hat{\omega}, \theta)$ with the axis $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3))$ is

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} \mathrm{c}_\theta + \hat{\omega}_1^2(1-\mathrm{c}_\theta) & \hat{\omega}_1\hat{\omega}_2(1-\mathrm{c}_\theta) - \hat{\omega}_3\mathrm{s}_\theta & \hat{\omega}_1\hat{\omega}_3(1-\mathrm{c}_\theta) + \hat{\omega}_2\mathrm{s}_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-\mathrm{c}_\theta) + \hat{\omega}_3\mathrm{s}_\theta & \mathrm{c}_\theta + \hat{\omega}_2^2(1-\mathrm{c}_\theta) & \hat{\omega}_2\hat{\omega}_3(1-\mathrm{c}_\theta) + \hat{\omega}_2\mathrm{s}_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-\mathrm{c}_\theta) - \hat{\omega}_2\mathrm{s}_\theta & \hat{\omega}_2\hat{\omega}_3(1-\mathrm{c}_\theta) + \hat{\omega}_1\mathrm{s}_\theta & \mathrm{c}_\theta + \hat{\omega}_3^2(1-\mathrm{c}_\theta) \end{bmatrix}$$

where $c_\theta := \cos\theta$ and $s_\theta := \sin\theta$.

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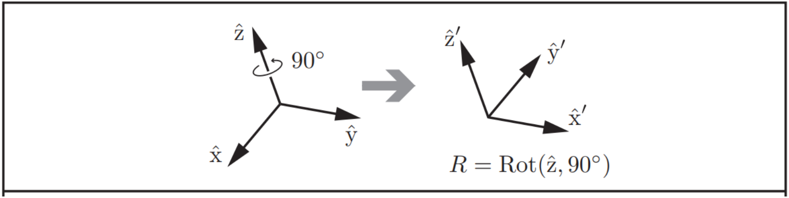

For a given frame,
the rotation matrix associated with the $(\pi/2)$-rotation along with z-axis:

$$R = \mathrm{Rot}(\hat{z}, \pi/2).$$

The meaning of R may differ when we specify the frame related to the z-axis:)

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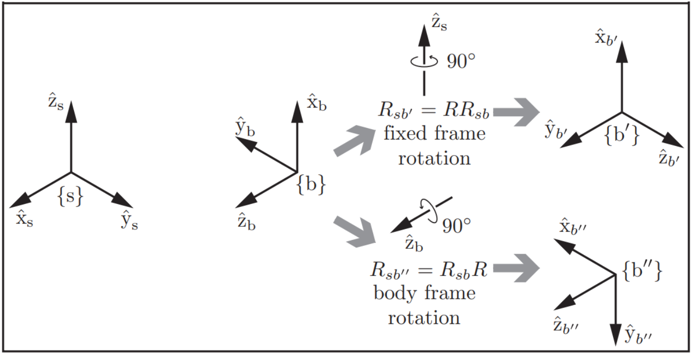
3.2. Rotations and Angular Velocities



For a fixed frame $\{\mathrm{s}\}$ and a body frame $\{\mathrm{b}\}$, IF we rotate

- the fixed frame $\{\mathrm{s}\}$, THEN $R_{sb'} = RR_{sb}$ (pre-multiplication).
- the body frame $\{\mathrm{b}\}$, THEN $R_{sb''} = R_{sb}R$ (post-multiplication).

Note: The order of multiplication is important!

Angular velocity (각속도)

3.2. Rotations and Angular Velocities

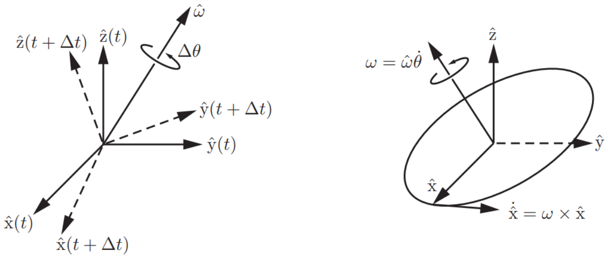


Figure 3.10: (Left) The instantaneous angular velocity vector. (Right) Calculating $\dot{\hat{x}}$.

- $\hat{\mathrm{w}} \in \mathbb{R}^3$: Instantaneous axis of rotation
- $\dot{\theta} = \lim(\Delta\theta/\Delta t) \in \mathbb{R}$: Rate of rotation

Definition: Angular velocity w of the rotating frame is defined as

$$\mathrm{w} := \hat{\mathrm{w}}\dot{\theta}.$$

Time derivative $\dot{R}$ of the rotation matrix

The time derivative of each coordinate axis can be represented as the result of the cross-product terms

$$\begin{aligned} \dot{\hat{\mathrm{x}}} &= \mathrm{w} \times \hat{\mathrm{x}}, \\ \dot{\hat{\mathrm{y}}} &= \mathrm{w} \times \hat{\mathrm{y}}, \\ \dot{\hat{\mathrm{z}}} &= \mathrm{w} \times \hat{\mathrm{z}} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \dot{\hat{x}}_s &= \omega_s \times \hat{x}_s, \\ \dot{\hat{y}}_s &= \omega_s \times \hat{y}_s, \\ \dot{\hat{z}}_s &= \omega_s \times \hat{z}_s \end{aligned}$$

- ω_s: the angular velocity w expressed in $\{\mathrm{s}\}$.
- $\hat{x}_s$: the unit vector $\hat{\mathrm{x}}$ expressed in $\{\mathrm{s}\}$.

Remind that, by definition,

$$\begin{aligned} R(t) &= \begin{bmatrix} r_1(t) & r_2(t) & r_3(t) \end{bmatrix} = \begin{bmatrix} \hat{x}_s(t) & \hat{y}_s(t) & \hat{z}_s(t) \end{bmatrix} \\ \Rightarrow \quad \dot{R}(t) &= \begin{bmatrix} \omega_s \times r_1 & \omega_s \times r_2 & \omega_s \times r_3 \end{bmatrix} = \omega_s(t) \times R(t). \end{aligned}$$

Matrix representation of $x \times y$

We now want to represent $\dot{R} = \omega_s \times R = (\star)R$ with some matrix $(\star)$.

Definition 3.7: For a vector $x = (x_1, x_2, x_3)$, $[x]$ is a 3×3 matrix defined by

$$[x] := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Note:

- $[x]$ is skew-symmetric: that is, $[x] = -[x]^\top$.
- This notation introduces an expression $x \times y = [x]y$.
- $[\omega]^3 = -[\omega]$, and $R[\omega]R^\top = [R\omega]$ (Proposition 3.8)
- The set of all 3×3 real skew-symmetric matrices is called $so(3)$.

$\therefore$ We have the relation between ω_s and R as

$$\dot{R} = \omega_s \times R = [\omega_s]R \quad \Rightarrow \quad [\omega_s] = \dot{R}R^{-1}.$$ (The order is important.)

Properties of $[\omega]$

3.2. Rotations and Angular Velocities

Notice that

- $R = R_{sb}$,
- ω_s represents w expressed in $\{\mathrm{s}\}$, and
- ω_b represents w expressed in $\{\mathrm{b}\}$.

$\therefore$ We have the relation between ω_b and ω_s as

$$\omega_s = R_{sb}\omega_b \quad \Rightarrow \quad \omega_b = R_{sb}^{-1}\omega_s = R^\top \omega_s.$$

Proposition 3.9: Consider the rotation matrix $R(t)$ that represents the orientation of the rotating $\{\mathrm{b}\}$ seen from the fixed frame $\{\mathrm{s}\}$. THEN

$$[\omega_s] = \dot{R}R^{-1}, \quad [\omega_b] = R^{-1}\dot{R}$$

where

- ω_s: Representation of w in the fixed frame $\{\mathrm{s}\}$
- ω_b: Representation of w in the body frame $\{\mathrm{b}\}$

Exponential coordinates of rotations

3.2. Rotations and Angular Velocities

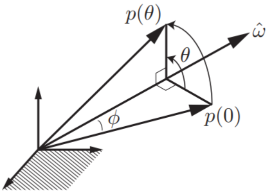


Figure 3.11: The vector $p(0)$ is rotated by an angle θ about the axis $\hat{\omega}$, to $p(\theta)$.

The derivative $p(t)$ along with t is computed by

$$\dot{p} = \hat{\omega} \times p = [\hat{\omega}]p.$$

Note: The solution $p(t)$ with a constant matrix $[\hat{\omega}]$ has the form

$$p(t) = e^{[\hat{\omega}]t} p(0), \quad p(0)\text{: Initial condition of } p(t)$$

(Cont'd)

By replacing t (time) with θ (angle), one has

$$p(\theta) = e^{[\hat{\omega}]\theta} p(0).$$

This $e^{[\hat{\omega}]\theta}$ must be the rotation matrix. (Why?)

Proposition 3.11 (= Derivation of the equation on $\mathrm{Rot}(\hat{\omega}, \theta)$):

$$\begin{aligned} \mathrm{Rot}(\hat{\omega}, \theta) &= e^{[\hat{\omega}]\theta} \\ &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \right) [\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots \right) [\hat{\omega}]^2 \\ &= I + \sin\theta [\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3) \end{aligned}$$

Note: The explicit form of $\mathrm{Rot}(\hat{\omega}, \theta)$ is given in 18 pages.

Matrix logarithm of rotation

Relation between $so(3)$ and $SO(3)$?

$$
\begin{aligned}
[\hat{\omega}]\theta \in so(3) &\quad \xrightarrow{\text{exp.}} \quad e^{[\hat{\omega}]\theta} \in SO(3), \\
R \in SO(3) &\quad \xrightarrow{\text{log.}} \quad [\hat{\omega}]\theta \in so(3).
\end{aligned}
$$

- Problem 1 (that we already know how to solve):
  Find $R = e^{[\hat{\omega}]\theta}$ for given $\hat{\omega}$ and θ
- Problem 2 (that we have to solve in the following slides):
  Find $\hat{\omega}$ and θ for given $R \in SO(3)$

Note: By the closed form of $R = e^{[\hat{\omega}]\theta}$,

$$
\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta.
$$

Compute $\hat{\omega}$ and θ for given $R = e^{[\hat{\omega}]\theta}$: Singular cases

- (Case 1-a) $\theta = 0, \pm 2\pi, \pm 4\pi, \cdots$:

$$R = e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 = I.$$

(where $\hat{\omega}$ is undefined.)

- (Case 1-b) $\theta = \pm\pi, \pm 3\pi, \cdots$ (when $\mathrm{tr}(R) = -1$):

$$R = e^{[\hat{\omega}]\theta} = I + 2[\hat{\omega}]^2$$

$\Rightarrow$ We have 4 constraints for 3 variables $\hat{\omega}_i$:

$$\hat{\omega}_i = \pm\sqrt{\frac{r_{ii} + 1}{2}}, \quad i = 1, 2, 3.$$

$$2\hat{\omega}_1\hat{\omega}_2 = r_{12}, \quad 2\hat{\omega}_2\hat{\omega}_3 = r_{23}, \quad 2\hat{\omega}_1\hat{\omega}_3 = r_{13}$$

(Cont'd)

3.2. Rotations and Angular Velocities

- (Cont'd) In summary, $\hat{\omega}$ could be one of the following:

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}, \quad \text{or}$$

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{33} \end{bmatrix} \quad \text{or}$$

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

Compute $\hat{\omega}$ and θ for given $R = e^{[\hat{\omega}]\theta}$: Non-trivial cases

- (Case 2) $\theta \neq \pm k\pi$: By the closed form of $R = e^{[\omega]\theta}$, one has

$$\begin{cases} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta, \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta, \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta \end{cases} \quad \Rightarrow \quad \begin{cases} \hat{\omega}_1 &= \frac{1}{2\sin\theta}(r_{32} - r_{23}), \\ \hat{\omega}_2 &= \frac{1}{2\sin\theta}(r_{13} - r_{31}), \\ \hat{\omega}_3 &= \frac{1}{2\sin\theta}(r_{21} - r_{12}). \end{cases}$$

Thus, the skew-symmetric matrix $[\hat{\omega}]$ is computed by

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^\top),$$

and the angle θ is given by

$$\theta = \cos^{-1}\left(\frac{\mathrm{tr}(R) - 1}{2}\right).$$

The orientation can also be expressed by

Appendix B. Other Representations of Rotations

- Rotation matrix and exponential coordinates $R = e^{[\hat{\omega}]\theta}$ (as above)
- Euler angles
- Roll-pitch-yaw angles
- Unit quaternions

ZYX Euler angle

Appendix B. Other Representations of Rotations

The ZYX Euler angle $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ means

1. Rotate the body by α about $\hat{\mathrm{z}}_{\mathrm{b}}$-axis
2. Rotate the body by β about $\hat{\mathrm{y}}_{\mathrm{b}}$-axis
3. Rotate the body by γ about $\hat{\mathrm{x}}_{\mathrm{b}}$-axis

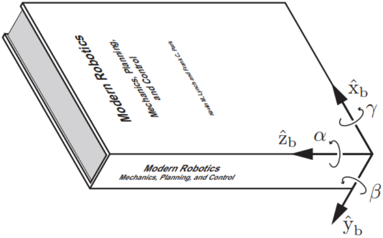


Figure B.1: To understand the ZYX Euler angles, use the corner of a box or a book as the body frame. The ZYX Euler angles correspond to successive rotations of the body about the $\hat{\mathrm{z}}_{\mathrm{b}}$-axis by α, the $\hat{\mathrm{y}}_{\mathrm{b}}$-axis by β, and the $\hat{\mathrm{x}}_{\mathrm{b}}$-axis by γ.

(Cont'd)

The final rotation matrix R is computed by

$$R(\alpha, \beta, \gamma) = \mathrm{Rot}(\hat{\mathrm{z}}, \alpha)\mathrm{Rot}(\hat{\mathrm{y}}, \beta)\mathrm{Rot}(\hat{\mathrm{x}}, \gamma)$$

$$= \begin{bmatrix} \mathrm{c}_\alpha \mathrm{c}_\beta & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma - \mathrm{s}_\alpha \mathrm{c}_\gamma & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma + \mathrm{s}_\alpha \mathrm{s}_\gamma \\ \mathrm{s}_\alpha \mathrm{c}_\beta & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma + \mathrm{c}_\alpha \mathrm{c}_\gamma & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma - \mathrm{c}_\alpha \mathrm{s}_\gamma \\ -\mathrm{s}_\beta & \mathrm{c}_\beta \mathrm{s}_\gamma & \mathrm{c}_\beta \mathrm{c}_\gamma \end{bmatrix}$$

where $\mathrm{c}_\theta = \cos(\theta)$ and $\mathrm{s}_\theta = \sin(\theta)$. (Why post-multiplication?)

Inverse problem: For given rotation matrix R,
we can find (α, β, γ) satisfying the above eq. (for most cases except $\beta \neq \pm\pi/2$)

Note: We sometimes use
the two-argument arctangent function $\mathrm{atan2}(y, x)$ to avoid confusion.

(Cont'd)

Appendix B. Other Representations of Rotations

Consider the case when $\beta \neq \pm\pi/2$.

Since $r_{11}^2 + r_{22}^2 = \cos^2\beta$, we have two candidates for β:

$$\beta = \operatorname{atan2}(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2})$$

where $+$ for $\beta \in [-\pi/2, \pi/2]$ and $-$ for $\beta \in [\pi/2, 3\pi/2]$.

This leads to a possible solution (α, β, γ) with

$$\alpha = \operatorname{atan2}(r_{21}, r_{11}),$$
$$\gamma = \operatorname{atan2}(r_{32}, r_{33})$$

Note: When $\beta = \pm\pi/2$, a family of solutions exists.

Algorithm for computing ZYX Euler angles

- (Case 1) $r_{31} \neq \pm 1$ (so that $\beta \neq \pm\pi/2$):
  (α, β, γ) is determined as above.
- (Case 2) $r_{31} = -1$ (so that $\beta = \pi/2$):
  one candidate among infinitely many others is
  $$\alpha = 0, \quad \gamma = \operatorname{atan2}(r_{12}, r_{22})$$
- (Case 3) $r_{31} = 1$ (so that $\beta = -\pi/2$):
  one candidate among infinitely many others is
  $$\alpha = 0, \quad \gamma = -\operatorname{atan2}(r_{12}, r_{22})$$

Illustrating ZYX Euler angles

The following wrist mechanism illustrates how ZYX Euler angle works.

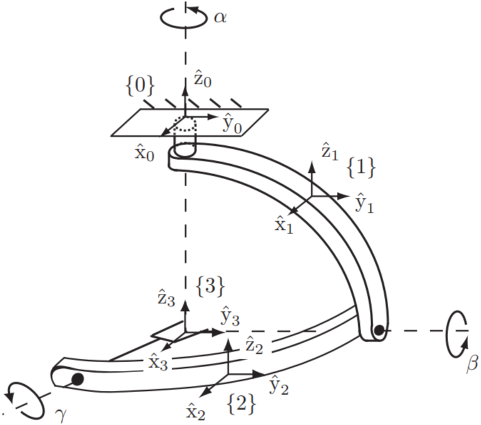


Figure B.2: Wrist mechanism illustrating the ZYX Euler angles.

Other Euler angle representations

Appendix B. Other Representations of Rotations

- ZYX Euler angle
- ZYZ Euler angle
- Generalization? $\mathrm{Rot}(\text{axis } 1, \alpha)\mathrm{Rot}(\text{axis } 2, \beta)\mathrm{Rot}(\text{axis } 3, \gamma)$

Note also that

- (ZYX) Euler angle $I\mathrm{Rot}(\hat{\mathrm{z}}, \alpha)\mathrm{Rot}(\hat{\mathrm{y}}, \beta)\mathrm{Rot}(\hat{\mathrm{x}}, \gamma)$
  = The angle associated with the rotation of a body frame
- (XYZ) Roll-pitch-yaw angles $I\mathrm{Rot}(\hat{\mathrm{z}}, \alpha)\mathrm{Rot}(\hat{\mathrm{y}}, \beta)\mathrm{Rot}(\hat{\mathrm{x}}, \gamma)$
  = The angle associated with the rotation of the space frame.

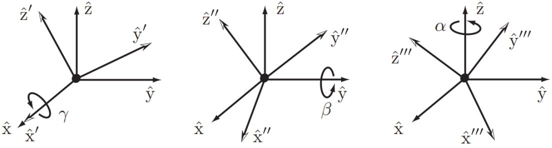


Figure B.4: Illustration of XYZ roll–pitch–yaw angles.

Unit quaternion

Appendix B. Other Representations of Rotations

Note: Other methods may suffer from the singularity issue.

For example,

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^\top) \quad \text{whose size may be too large if } \theta \text{ is small}$$

The unit quaternion q

- is an alternative that alleviates this singularity;
- associated with $e^{[\hat{\omega}]\theta}$ is given by

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \hat{\omega}\sin(\theta/2) \end{bmatrix} \in \mathbb{R}^{1+3}$$

where $\|q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$ (for the unit property).

Rotation matrix $R \leftrightarrow$ Unit quaternion q

- Rotation matrix $\to$ Unit quaternion

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}, \quad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

  where we use $1 + 2\cos\theta = \mathrm{tr}(R)$, and $\cos 2\phi = 2\cos^2\phi - 1$.

- Unit quaternion $\to$ Rotation matrix

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2(q_0 q_3 + q_1 q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Note: The unit-quaternion representation of $R_q R_p$ can be obtained by multiplying the 2×2 complex matrices $Q = \begin{bmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{bmatrix}$, $\cdots$

Homogeneous transformation

3.3. Rigid-body Motions and Twists

Definition 3.13: The special Euclidean group $SE(3)$
= The group of rigid-body motions
= The group of homogeneous transformation matrices of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$.

Definition 3.14: The special Euclidean group $SE(2)$
= The group of homogeneous transformation matrcies of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $R \in SO(2)$ and $p \in \mathbb{R}^3$.

Properties of the homogeneous transformation

- Property 1 (Proposition 3.15):

$$T \in SE(3) \quad \Rightarrow \quad T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix}$$

- Property 2 (Proposition 3.16):

$$T_1 \in SE(3), \quad T_2 \in SE(3) \quad \Rightarrow \quad T_1 T_2 \in SE(3)$$

- Property 3 (Proposition 3.17):

$$(T_1 T_2) T_3 = T_1 (T_2 T_3), \quad \text{and} \quad T_1 T_2 \neq T_2 T_1 \quad \text{in general}$$

Properties of the homogeneous transformation

Note that

$$T \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

- The vector $(x, 1)$ is the rep. of x in homogeneous coordinates.
- We abuse the notation by writing Tx for $Rx + p$.

Proposition 3.18:

- $\|Tx - Ty\| = \|x - y\|$.
- $(Tx - Tz)^\top (Ty - Tz) = (x - z)^\top (y - z)$

Lesson: T can be regarded as a transformation on points on $\mathbb{R}^3$.

Uses of HT 1: Representing the configuration

3.3. Rigid-body Motions and Twists

Consider 3 frames $\{a\} = \{s\}$, $\{b\}$, and $\{c\}$,

represented by $T_{sa} = (R_{sa}, p_{sa})$, $T_{sb} = (R_{sb}, p_{sb})$, and $T_{sc} = (R_{sc}, p_{sc})$.

The frame $\{b\}$ can be represented in terms of $\{s\}$ with

$$\text{(Orientation)} \quad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{(Position)} \quad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

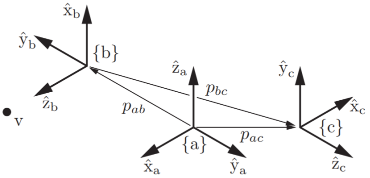


Figure 3.14: Three reference frames in space, and a point v that can be represented in $\{b\}$ as $v_b = (0, 0, 1.5)$.

Uses of HT 2: Changing the reference frame

3.3. Rigid-body Motions and Twists

In the example above,

$$T_{ab}T_{bc} = T_{ac}, \quad T_{ab}v_b = v_a$$

where

- v_b: The vector v expressed in $\{\mathrm{b}\}$
- v_b: The vector v expressed in $\{\mathrm{a}\}$

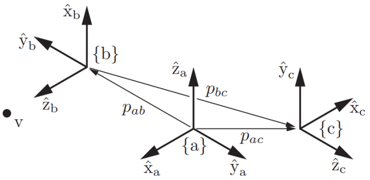


Figure 3.14: Three reference frames in space, and a point v that can be represented in $\{\mathrm{b}\}$ as $v_b = (0, 0, 1.5)$.

Uses of HT 3: Displacing a vector or a frame

Rotation and translation of a frame are expressed in the HT as

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathrm{Trans}(p) = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

- The fixed-frame transformation $T_{sb'}$ = Pre-multiply T_{sb} by T
- The body-frame transformation $T_{sb''}$ = Post-multiply T_{sb} by T

$$T_{sb'} = TT_{sb} = \mathrm{Trans}(p)\mathrm{Rot}(\hat{\omega}, \theta)T_{sb} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix},$$

$$T_{sb''} = T_{sb}T = T_{sb}\mathrm{Trans}(p)\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix}$$

(Cont'd)

3.3. Rigid-body Motions and Twists

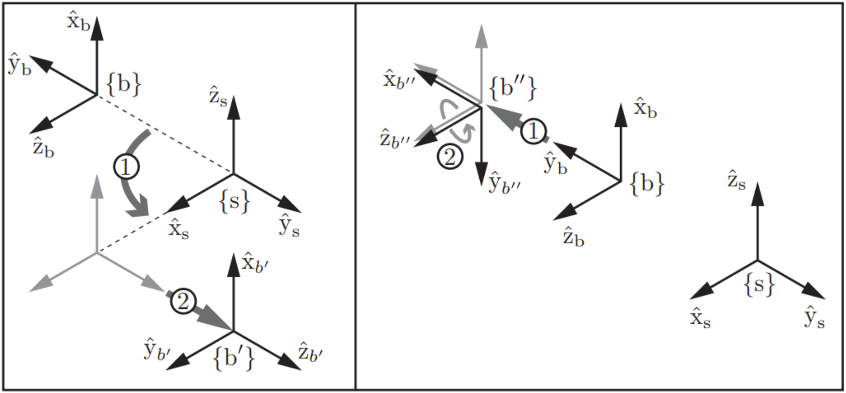


Figure 3.15: Fixed-frame and body-frame transformations corresponding to $\hat{\omega} = (0, 0, 1)$, $\theta = 90^\circ$, and $p = (0, 2, 0)$. (Left) The frame {b} is rotated by 90° about $\hat{z}_s$ and then translated by two units in $\hat{y}_s$, resulting in the new frame {b′}. (Right) The frame {b} is translated by two units in $\hat{y}_b$ and then rotated by 90° about its $\hat{z}$ axis, resulting in the new frame {b″}.

Homogeneous transformation of a moving frame in itself

When we have the fixed space frame $\{\mathrm{s}\}$ and a moving body frame $\{\mathrm{b}\}$, the homogeneous transformation T is dependent of t: i.e.,

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

which means that $\dot{T}(t) \neq 0$.

Remind: For $R \in SO(3)$, $[\omega_s] = \dot{R}R^{-1}$ and $[\omega_b] = R^{-1}\dot{R}$.

In a similar point of view, one computes $T^{-1}\dot{T}$ as

$$\begin{aligned} T^{-1}\dot{T} &= \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^\top \dot{R} & R^\top \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Body twist

3.3. Rigid-body Motions and Twists

- $\dot{p}$: The linear velocity of the origin of $\{\mathrm{b}\}$ expressed in $\{\mathrm{s}\}$
- $R^\top \dot{p} = R^{-1}\dot{p}$: The linear velocity of the origin of $\{\mathrm{b}\}$ expressed in $\{\mathrm{b}\}$
- $R^\top \dot{R}$: The skew-symmetric matrix representation of the angular velocity in $\{\mathrm{b}\}$ (that is, $[\omega_b]$)

This introduces the spatial velocity in the body frame, or the body twist:

$$\mathcal{V}_b = \begin{bmatrix} \text{angular velocity in } \{\mathrm{b}\} \\ \text{linear velocity in } \{\mathrm{b}\} \end{bmatrix} = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$$

so that we stretch the notation $[\cdot]$ for $\mathcal{V}_b$ as follows:

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

Spatial twist

3.3. Rigid-body Motions and Twists

On the other hand, $\dot{T}T^{-1}$ is computed as follows:

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^\top & \dot{p} - \dot{R}R^\top p \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}$$

- $\dot{R}R^\top = \dot{R}R^{-1}$: The skew-symmetric matrix representation of the angular velocity in $\{\mathrm{s}\}$ (that is, $[\omega_s]$)
- $\dot{p} - \dot{R}R^\top p$ satisfying

$$\dot{p} - \dot{R}R^\top p = \dot{p} - [\omega_s]p = \dot{p} + \omega_s \times (-p)$$

The instantaneous velocity of the point on the (infinitely large) moving body currently at the origin of $\{\mathrm{s}\}$, expressed in $\{\mathrm{s}\}$.

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3.3. Rigid-body Motions and Twists

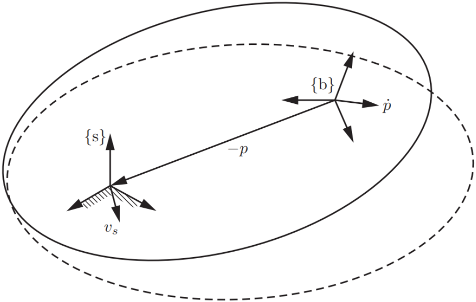

Figure 3.17: Physical interpretation of v_s. The initial (solid line) and displaced (dashed line) configurations of a rigid body.

The spatial twist = The spatial velocity in the space frame {s}:

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6, \quad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3).$$

Understanding the relation btw the body and spatial twists

- ω_b: The angular velocity in $\{\mathrm{b}\}$
- ω_s: The angular velocity in $\{\mathrm{s}\}$
- v_b: The linear velocity of a point at the origin of $\{\mathrm{b}\}$ in $\{\mathrm{b}\}$
- v_s: The linear velocity of a point at the origin of $\{\mathrm{s}\}$ in $\{\mathrm{s}\}$

We have (with $R[\omega]R^\top = [R\omega]$ and $[\omega]p = -[p]\omega$)

$$[\mathcal{V}_b] = T^{-1}\dot{T} = T^{-1}[\mathcal{V}_s]T = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix},$$

$$\begin{aligned}
[\mathcal{V}_s] &= T[\mathcal{V}_b]T^{-1} \\
&= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R[\omega_b]R^\top & -R[\omega_b]R^\top p + Rv_b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [R\omega_b] & [p]R\omega_b + Rv_b \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Adjoint representation

3.3. Rigid-body Motions and Twists

Definition 3.20: For given $T = (R, p) \in SE(3)$,
the adjoint representation of T is defined as

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$\Rightarrow$ In the previous slide, we derive the relations

$$\omega_s = R\omega_b, \quad v_s = [p]R\omega_b + Rv_b$$

which can be simply represented as follows:

$$\mathcal{V}_s = [\mathrm{Ad}_T]\mathcal{V}_b$$

which is called adjoint map associated with T.

Screw interpretation of a twist

The twist $\mathcal{V} = (\omega, v)$ can be rewritten by $\{\dot{\theta}, \hat{s}, h\}$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$$

- $\dot{\theta}$: The rate of rotation about $\hat{s}$.
- $\hat{s}$: A unit vector in the direction of the rotation axis
- h: The screw pitch
  = the ratio of the linear velocity $h\hat{s}\dot{\theta}$ and the angular velocity $\hat{s}\dot{\theta}$.

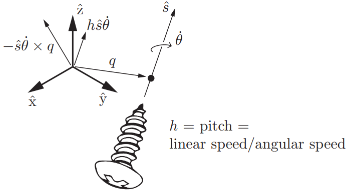

Figure 3.19: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch h.

Screw axis $\mathcal{S}$

The screw axis $\mathcal{S}$ is a normalized version of $\mathcal{V}$ in the sense that

1. If $\omega \neq 0$, then $\mathcal{S} := \mathcal{V}/\|\omega\| = \mathcal{V}/\dot{\theta}$
2. If $\omega = 0$, then $\mathcal{S} := \mathcal{V}/\|v\|$

In both cases, we can say that $\mathcal{S}\dot{\theta} = \mathcal{V}$.

Definition 3.24: A screw axis $\mathcal{S}$ is defined as

$$\mathcal{S} = \begin{bmatrix} \overline{\omega} \\ \overline{v} \end{bmatrix} \in \mathbb{R}^6$$

where (a) $\|\overline{\omega}\| = 1$, or (b) $\overline{\omega} = 0$ and $\|\overline{v}\| = 1$.

As the screw axis is just a normalized twist, it follows the language of the twist.

Exponential coordinate representation of a rigid-body motion

We define the exponential coordinates of H.T. T as $\mathcal{S}\theta \in \mathbb{R}^6$

$$
\begin{aligned}
[\mathcal{S}]\theta \in se(3) \quad &\xrightarrow{\text{exp}} \quad T \in SE(3) \\
T \in SE(3) \quad &\xrightarrow{\text{log}} \quad [\mathcal{S}]\theta \in se(3)
\end{aligned}
$$

This means that (Proposition 3.25),

$$
\begin{aligned}
T = e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2 \frac{\theta^2}{2!} + \cdots \\
&= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Moment generated by a force

3.4. Wrenches

- f_a: A force acting on a rigid body at a point in $\{\mathrm{a}\}$.
- r_a: The point represented in $\{\mathrm{a}\}$

A torque or moment $m_a \in \mathbb{R}^3$ in $\{\mathrm{a}\}$ generated by f_a:

$$m_a = r_a \times f_a$$

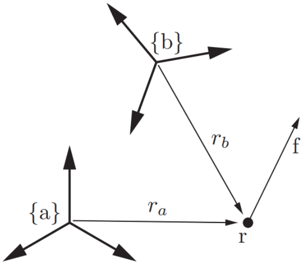

Figure 3.21: Relation between wrench representations $\mathcal{F}_a$ and $\mathcal{F}_b$.

Spatial force (= wrench)

The spatial force or wrench $\mathcal{F}_a$ expressed in $\{\mathrm{a}\}$ frame:

$$\mathcal{F}_a = \begin{bmatrix} \text{moment in } \{\mathrm{a}\} \\ \text{linear force in } \{\mathrm{a}\} \end{bmatrix} = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6$$

Note: The power (=velocity $\times$ force) $\mathcal{V}^\top \mathcal{F}$ is a coordinate-free quantity: i.e.,

$$\mathcal{V}_b^\top \mathcal{F}_b = \mathcal{V}_a^\top \mathcal{F}_a = \mathcal{V}_b^\top [\mathrm{Ad}_{T_{ab}}]^\top \mathcal{F}_a$$

where we use $\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$.

Proposition 3.27:

$$\mathcal{F}_b = \text{Body wrench} = [\mathrm{Ad}_{T_{sb}}]^\top \mathcal{F}_s,$$
$$\mathcal{F}_s = \text{Spatial wrench} = [\mathrm{Ad}_{T_{bs}}]^\top \mathcal{F}_b$$

Example on wrench

3.4. Wrenches

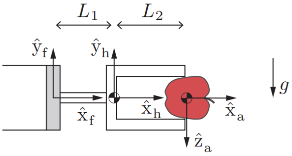

Figure 3.22: A robot hand holding an apple subject to gravity.

- $\{f\}$: The frame at the force-torque sensor.
- $\{h\}$: The frame at the center of mass of the hand.
- $\{a\}$: The frame of the center of mass of the apple.

Gravitational wrench on the hand in $\{h\}$:

$$\mathcal{F}_h = (0, 0, 0, 0, -5, 0)$$

Gravitational wrench on the apple in $\{a\}$:

$$\mathcal{F}_a = (0, 0, 0, 0, 0, 1)$$

(Cont'd)

3.4. Wrenches

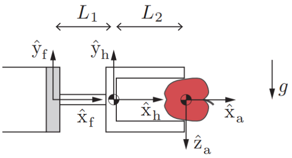

Figure 3.22: A robot hand holding an apple subject to gravity.

Given $L_1 = 0.1$ m, $L_2 = 0.15$ m, the transformation matrices

$$T_{hf} = \begin{bmatrix} 1 & 0 & 0 & -0.1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{af} = \begin{bmatrix} 1 & 0 & 0 & -0.25 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the wrench measured by the force-torque sensor:

$$\mathcal{F}_f = [\mathrm{Ad}_{T_{hf}}]^\top \mathcal{F}_h + [\mathrm{Ad}_{T_{af}}]^\top \mathcal{F}_a$$