[2024-1 Robotics]
Chapter 4. Forward Kinematics

Gyunghoon Park

School of ECE, University of Seoul

Kinematics?

4.0. Introduction to Chapter

Kinematics defines the relation between

- the joint coordinate θ, and
- the position/orientation of the end-effector frame, x_{EE}.

Types of kinematics?

- Forward kinematics (FK) (in this chapter): $\theta \to x_{\mathrm{EE}}$
- Inverse kinematics (IK) (in Chapter 6): $x_{\mathrm{EE}} \to \theta$
- Velocity kinematics (in Chapter 5): $(\theta, \dot{\theta}) \to \dot{x}_{\mathrm{EE}}$

Kinematics delivers

- a structural information on the robot,
- not a force-acceleration relation.

Example: 3R manipulator

4.0. Introduction to Chapter

Note: The forward kinematics of the robot implies $\theta_i \rightarrow (x, y, \phi)$.

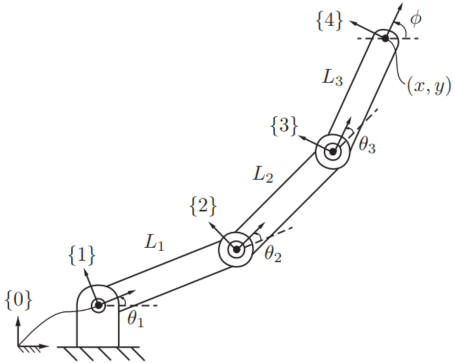


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

Forward kinematics of 3R manipulator

The forward kinematics of $3\mathrm{R}$ planar robot:

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + L_3 \cos(\theta_1+\theta_2+\theta_3),\\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + L_3 \sin(\theta_1+\theta_2+\theta_3),\\
\phi &= \theta_1+\theta_2+\theta_3
\end{aligned}
$$

We may want to express in a simpler form, such as

$$\text{homogeneous transformation} \;\; T_{04} = \begin{bmatrix} R_{04} & p_{04} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

Two ways of expressing the forward kinematics:

- Denavit-Hartenberg parameters (D-H parameters) = An usual approach
- Product of exponentials = Interest of this book

Method 1: Compute $T_{i-1,i}$

We may have T_{04} by computing

$$\begin{aligned} T_{04} &= \text{homogeneous transformation that represents } \{4\} \text{ in } \{0\} \\ &= T_{01}T_{12}T_{23}T_{34} \end{aligned}$$

where each homogeneous transformation $T_{i-1,i}$ is derived by

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{12} = \cdots$$

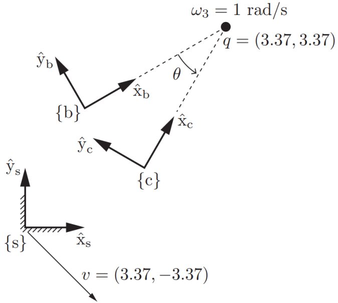
Note:

- Each $T_{i-1,i}$ is related only with θ_i
- Computation of $T_{i-1,i}$ is not that elegant, and also boring...

Method 2: A closer look at $T = e^{[\mathcal{S}]\theta}$

4.0. Introduction to Chapter

Consider the following example where $\{\mathrm{b}\}$ moves to $\{\mathrm{c}\}$:



$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Cont'd)

On the other hand, the screw axis $\mathcal{S} = (\omega, v)$ in $\{\mathrm{s}\}$ is computed by

- $\omega = (0, 0, 1)$: The angular velocity about $\hat{\mathrm{z}}_{\mathrm{s}}$
- $v = (v_1, v_2, 0)$: The linear velocity of a point currently at the origin of $\{\mathrm{s}\}$ in the $\{\mathrm{s}\}$ frame

$\Rightarrow$ the configuration of the final frame $\{\mathrm{c}\}$ can be represented as

$$T_{sc} = e^{[\mathcal{S}]\theta} T_{sb}, \quad \text{or} \quad e^{[\mathcal{S}]\theta} = T_{sc} T_{sb}^{-1}$$

Why pre-multiplication? (The order is important!)

Note:

- The screw axis $\mathcal{S}$ above is defined in $\{\mathrm{s}\}$.
- In the forward kinematics problem, T_{sc} will be of interest (Why?)

Method 2: Use the screw axis $\mathcal{S} = (\omega, v)$

4.0. Introduction to Chapter

Revisit the same example for the 3R manipulator:

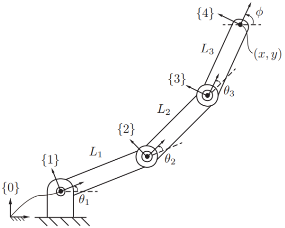


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

- **Step 1**: If $\theta_1 = \theta_2 = \theta_3 = 0$, then we have

$$T_{04} = M := \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{if } \theta_1 = \theta_2 = \theta_3 = 0$$

- Step 2: If $\theta_1 = \theta_2 = 0$ and Joint 3 is rotated (i.e., $\theta_3 \neq 0$), then the spatial twist $\mathcal{S}_3$ is given by

$$\mathcal{S}_3 = (\omega_3, v_3) = (0, 0, 1, 0, -(L_1 + L_2), 0)$$

  where

  - $\omega_3 = (0, 0, 1)$: The angular velocity in $\{0\}$
  - $v_3 = (0, -(L_1 + L_2), 0)$: The linear velocity of a point at the origin of $\{0\}$ in $\{0\}$

  $\therefore$ We have an homogeneous transformation in the constrained case

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M, \quad \text{if } \theta_1 = \theta_2 = 0$$

(Cont'd)

4.0. Introduction to Chapter

- Step 3: If $\theta_1 = 0$ and Joints 2 & 3 are rotated (i.e., $\theta_2 \neq 0$ and $\theta_3 \neq 0$), then the spatial twist $\mathcal{S}_2$ is given by

$$\mathcal{S}_2 = (\omega_3, v_3) = (0, 0, 1, 0, -L_1, 0) \quad \Rightarrow \quad T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M, \quad \text{if } \theta_1 = 0$$

- Step 4: If all the joints are rotated, then the spatial twist $\mathcal{S}_1$ is given by

$$\mathcal{S}_1 = (\omega_1, v_1) = (0, 0, 1, 0, 0, 0), \quad \Rightarrow \quad T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M.$$

Conclusion: After Step 4,

we have an homogeneous transformation T_{04} with no condition on θ_i:

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M.$$

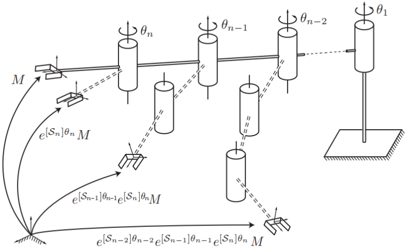
Generalize the screw-based method: Product of Exponentials

4.1. Product of Exponentials

The homogeneous transformation $T \in SE(3)$ from $\{\mathrm{s}\}$ to $\{\mathrm{b}\}$:

$$
\begin{aligned}
T &= e^{[\mathcal{S}_n]\theta_n} M, && \text{if } \theta_1 = \cdots = \theta_{n-2} = \theta_{n-1} = 0 \\
&= e^{[\mathcal{S}_{n-1}]\theta_{n-1}} (e^{[\mathcal{S}_n]\theta_n} M), && \text{if } \theta_1 = \cdots = \theta_{n-2} = 0 \\
&\vdots \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} (e^{[\mathcal{S}_n]\theta_n} M) && \text{with NO condition on } \theta_i!
\end{aligned}
$$

where the last term is the very homogeneous transformation we want to find.



Example 4.1: 3R spatial open chain

4.1. Product of Exponentials

Find the homogeneous transformation T from $\{\mathrm{s}\}$ to $\{\mathrm{b}\}$ in the following configuration:

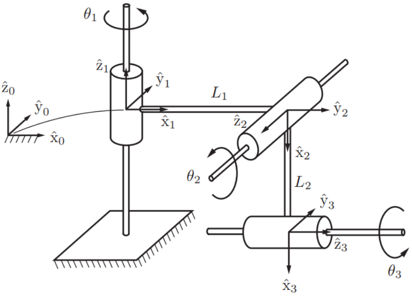


Figure 4.3: A 3R spatial open chain.

Note: The screw axis $\mathcal{S}_i$ in this example are given by

$$\mathcal{S}_1 = (0,0,1,0,0,0), \quad \mathcal{S}_2 = (0,-1,0,0,0,-L_1), \quad \mathcal{S}_3 = (1,0,0,0,-L_2,0)$$

Example 4.2: 3R planar open chain

4.1. Product of Exponentials

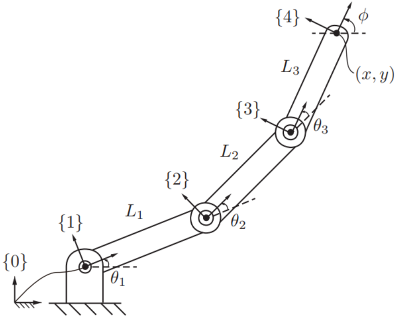
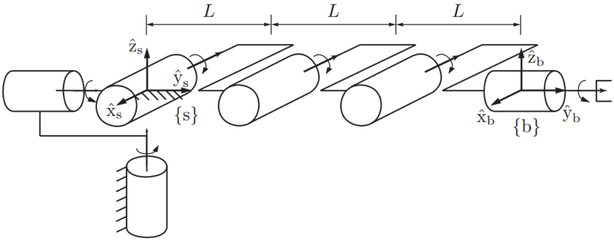


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, 0, 1)$	$(0, -L_1, 0)$
3	$(0, 0, 1)$	$(0, -(L_1 + L_2), 0)$

Example 4.3: 6R spatial open chain

4.1. Product of Exponentials



i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, 1, 0)$	$(0, 0, 0)$
3	$(-1, 0, 0)$	$(0, 0, 0)$
4	$(-1, 0, 0)$	$(0, 0, L)$
5	$(-1, 0, 0)$	$(0, 0, 2L)$
6	$(0, 1, 0)$	$(0, 0, 0)$

Introduction to D-H convention

Appendix C. Denavit-Hartenberg Parameters

Another way of finding T_{0n} is to post-multiply $T_{i-1,i}$ as

$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1) T_{12}(\theta_2) \cdots T_{n-1,n}(\theta_n)$$

where $T_{i-1,i}$ denotes the relative displacement btw. $\{i-1\}$ and $\{i\}$, related only with θ_i.

Note:

- There are many ways of determining the frame.
- $T_{i-1,i}$ would be dependent of selection of each frame.
- If we FIX the rule for selecting the frame, then?

A basic rule of assigning a frame on the joint

Supplementary material (정슬, 로봇공학)

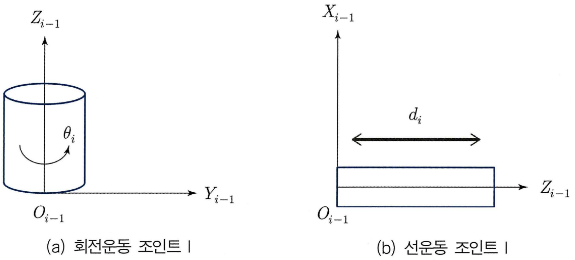


그림 2.35 한 축의 좌표 설정

정슬, *로봇공학* (5판), p. 101.

Each frame is selected such that the z-axis is aligned to the moving axis.

- Revolute joint (left)
- Prismatic joint (right)

(Cont'd)

Supplementary material (정슬, 로봇공학)

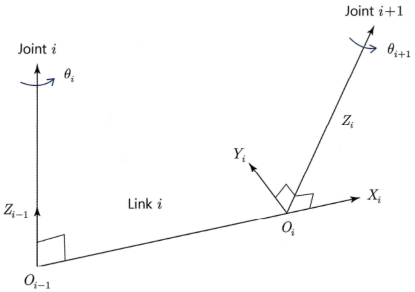


그림 2.36 한 축의 좌표 설정

정슬, *로봇공학* (5판), p. 102.

- z_i-axis: Set as the moving axis (related to θ_{i+1})
- x_i-axis: Perpendicular to both z_{i-1}- and z_i-axes
- y_i-axis: Taken by following the right-handed rule.

A basic rule for assigning a frame $\{i\}$ from $\{i-1\}$

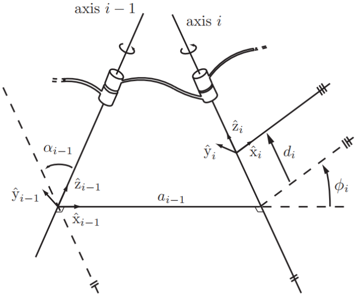


Figure C.1: Illustration of the Denavit–Hartenberg parameters.

- Rule 1: Set $\hat{z}_i$-axis to coincide with the joint axis i
- Rule 2: Find the line segment that orthogonally intersects both $\hat{z}_{i-1}$ and $\hat{z}_i$ axes.
- Rule 3: Connect the joint axes $i-1$ and i by a mutually perpendicular line
- Rule 4: The $\hat{x}$-axis is chosen to be in the direction of the mutually perpendicular line, pointing from the axis $i-1$ to the axis i.

D-H convention 1: Classical version (in Korean)

Supplementary Material: 정슬, 로봇공학

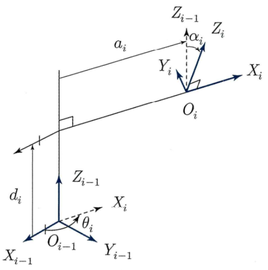


그림 2.47 두 좌표 간의 관계

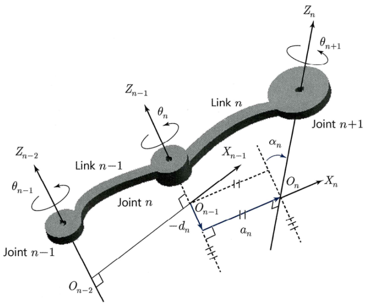


그림 2.48 좌표 간의 관계

- Link length a_i: z_i 축과 z_{i-1} 축 사이의 최단 거리
- Link twist α_i: {i} 좌표에서 z_i 축과 z_{i-1} 축 사이의 비틀림각
- Link offset d_i: x_i 축과 x_{i-1} 축 사이의 최단 거리.
- Joint angle θ_i: $\{i-1\}$ 좌표에서 z_{i-1} 축을 중심으로 회전한 각

Note: Be careful with indexing!

D-H convention 2: Modified version

Appendix C. Denavit-Hartenberg Parameters

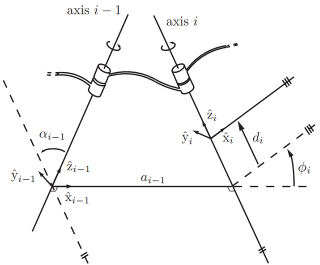


Figure C.1: Illustration of the Denavit–Hartenberg parameters.

- Link length a_{i-1} of link $i-1$: The length of the mutually perpendicular line
- Link twist α_{i-1}: The angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, measured about $\hat{x}_{i-1}$.
- Link offset d_i: The distance from the intersection of $\hat{x}_{i-1}$ and $\hat{z}_i$ to the origin of $\{i\}$.
- Joint angle ϕ_i: The angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, measured about $\hat{z}_i$-axis.

Note: Be careful with indexing!

Link frame transformation described by D-H parameters

Appendix C. Denavit-Hartenberg Parameters

- The textbook follows a modified D-H convention
  (where $\{i-1\}$ frame is related with the $(i-1)$-th axis), resulting in

$$T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$$

  (associated with the textbook) that means,
  1. Rotating the frame $\{i-1\}$ about its $\hat{\mathrm{x}}$-axis by α_{i-1}
  2. Translating the frame along its $\hat{\mathrm{x}}$-axis by a_{i-1}
  3. Translating the frame along its $\hat{\mathrm{z}}$-axis by d_i
  4. Rotating the frame about its $\hat{\mathrm{z}}$-axis by ϕ_i.
- The supplementary material follows a classical D-H convention
  (where $\{i-1\}$ frame is related with the i-th axis), with

$$T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{z}}, \theta_i)\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Trans}(\hat{\mathrm{x}}, a_i)\mathrm{Rot}(\hat{\mathrm{x}}, \alpha_i)$$

Example 2-21 (for classical D-H convention)

Supplementary Material: 정슬, 로봇공학

예제 2.21

다음 그림은 두 조인트가 선운동하는 조인트로 구성되어 있다. D-H 변수들을 구해 보자. 먼저 실제 변수는 d_1, d_2가 된다.

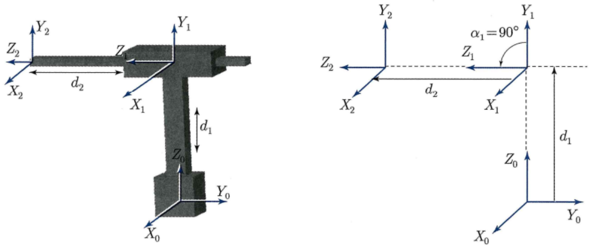


그림 2.50 2축 로봇의 좌표 설정 예 1

조인트	θ_i	α_i	d_i	a_i
1	0	90	d_1	0
2	0	0	d_2	0

Example 2-22 (for classical D-H convention)

Supplementary Material: 정슬, 로봇공학

예제 2.22

다음 그림은 조인트 1, 2가 회전운동하는 조인트로 구성되어 있다. D-H 변수들을 구해 보자. 먼저 실제 변수는 θ_1, θ_2가 된다.

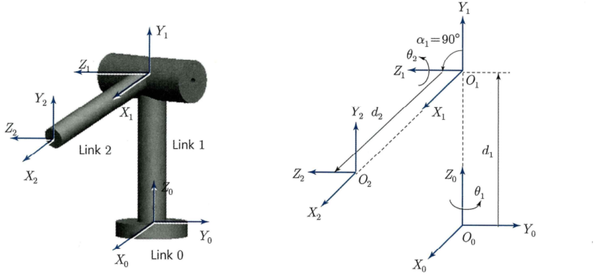


그림 2.51 2축 로봇의 좌표 설정 예 2

조인트	θ_i	α_i	d_i	a_i
1	θ_1	90	d_1	0
2	θ_2	0	0	d_2

Example C.1 (for modified D-H convention)

Appendix C. Denavit-Hartenberg Parameters

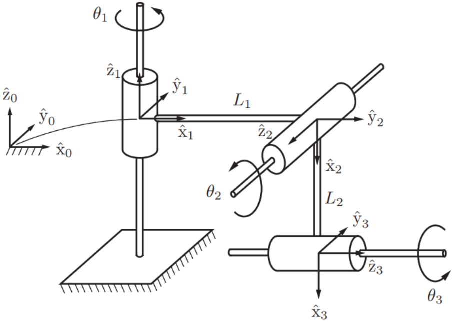
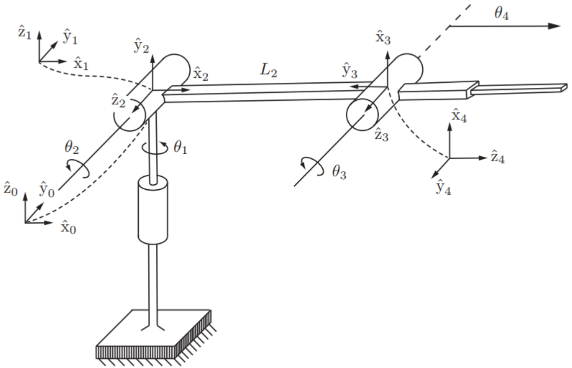


Figure 4.3: A 3R spatial open chain.

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	$90°$	L_1	0	$\theta_2 - 90°$
3	$-90°$	L_2	0	θ_3

Example C.2 (for modified D-H convention)

Appendix C. Denavit-Hartenberg Parameters



i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	$90°$	0	0	θ_2
3	0	L_2	0	$\theta_3 + 90°$
4	$90°$	0	θ_4	0

Universal Robot Description Format (URDF)?

4.2. The Universal Robot Description Format

- An XML file format that explains the kinematics/dynamics info. on a robot
- Usually used with the Robot Operating System (ROS)
- Applicable to any tree-structured robot
- Will be discussed later with URDF files...

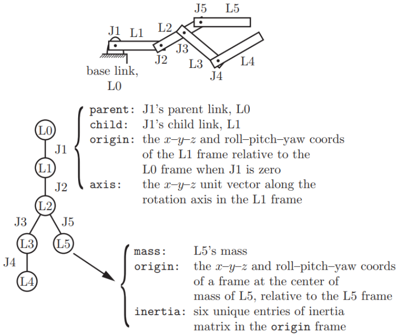


Figure 4.10: A five-link robot represented as a tree, where the nodes of the tree are the links and the edges of the tree are the joints.