[2024-1 Robotics]
Chapter 2. Configuration Space

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Overview of this chapter

2.0. Introduction to Chapter

- Basic components of the robot
  - joints
  - links
- Configuration space or C-space
  - its dimension = degree of freedoms
  - its shape = topology
- Constraints
  - Holonomic
  - Nonholonomic
- Task space and workspace

Basic components of the robot

2.0. Introduction to Chapter

A robot is mechanically constructed from links that are connected by various types of joints

Figure: Franka Emika - Panda

`https://wego-robotics.com/franka-emika-panda/`

- Link ≈ 뼈
- Joint ≈ 관절
- End-effector ≈ 손, 발

Basic components of the robot: Link

- The links are usually modeled as rigid bodies.
- Sometimes, they are constructed with soft materials for some purposes.

Figure: Valkyrie from NASA

https://www.nasa.gov/feature/r5/

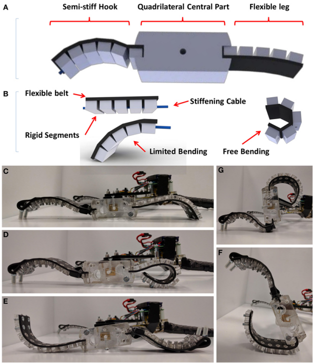


Figure: Example of robot with soft link

https://en.wikipedia.org/wiki/Soft_robotics

Basic components of the robot: Joint

2.0. Introduction to Chapter

- A (active) joint of a robot is commonly actuated by a motor.
- Some robots are driven by hydraulic(유압) actuators
  - Power $\uparrow$, accuracy $\downarrow$ compared with motors
  - https://www.youtube.com/watch?v=_sBBaNYex3E
- or tendon mechanism

Figure: Atlas with hydraulic actuators

https://en.wikipedia.org/wiki/Atlas_(robot)

Figure: AMBIDEX with tendon mechanism

https://en.yna.co.kr/view/PYH20190109121200320

Basic components of the robot: Joint (Cont'd)

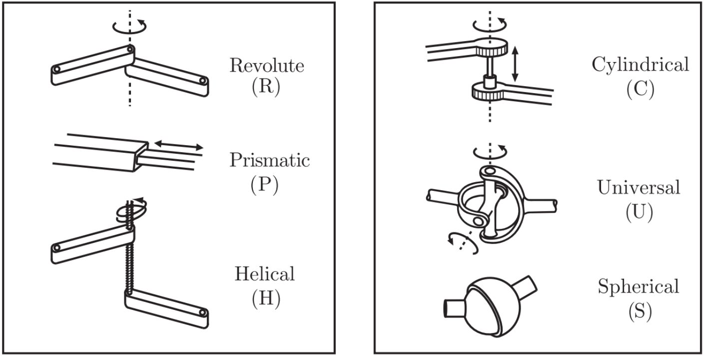


Figure 2.3: Typical robot joints.

Basic components of the robot: End-effector

- Motor-driven gripper (w/ single motor)
  `https://www.youtube.com/watch?v=zzL7GO0eRXY`
- Robot hand
  - Allegro hand
    https://www.youtube.com/watch?v=PnS0vXohzDs
  - FLLEX hand
    https://www.youtube.com/watch?v=cZuzXdMyJsA
- Vacuum gripper
  `https://www.youtube.com/watch?v=h7MpTfmNCAo`

Which variables we have to consider in performing a task

- Angle θ_i of the i-th joint
- Position (위치) & orientation (자세) of the end-effector

Figure: Franka Emika - Panda

https://wego-robotics.com/franka-emika-panda/

How many variables are needed to represent each point?

2.1. Degrees of Freedom of a Rigid Body

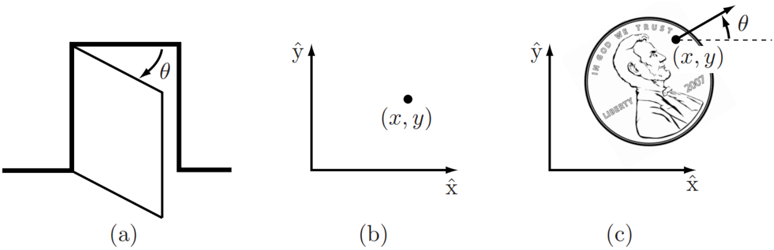


Figure 2.1: (a) The configuration of a door is described by the angle θ. (b) The configuration of a point in a plane is described by coordinates (x, y). (c) The configuration of a coin on a table is described by (x, y, θ), where θ defines the direction in which Abraham Lincoln is looking.

Configuration and degree of freedom

2.1. Degrees of Freedom of a Rigid Body

Definition 2.1:

- Configuration (형상) of a robot
  = A complete specification of the position of every point of the robot.
- The number of degrees of freedom (dof, 자유도) of a robot
  = The minimum number n of real-valued coordinates needed to represent the configuration.
- Configuration space (C-space, 형상공간)
  = The n-dimensional space containing all possible configurations of the robot

of dof of the coin on the table?

2.1. Degrees of Freedom of a Rigid Body

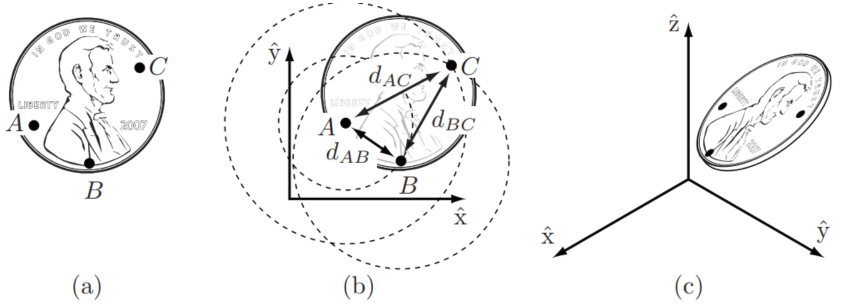


Figure 2.2: (a) Choosing three points fixed to the coin. (b) Once the location of A is chosen, B must lie on a circle of radius d_{AB} centered at A. Once the location of B is chosen, C must lie at the intersection of circles centered at A and B. Only one of these two intersections corresponds to the "heads up" configuration. (c) The configuration of a coin in three-dimensional space is given by the three coordinates of A, two angles to the point B on the sphere of radius d_{AB} centered at A, and one angle to the point C on the circle defined by the intersection of the a sphere centered at A and a sphere centered at B.

(Cont'd)

2.1. Degrees of Freedom of a Rigid Body

- Position of each point: (x_A, y_A), (x_B, y_B), (x_C, y_C)
- Rigidity of the coin $\Rightarrow$ 3 equality constraints

$$\begin{aligned} d(A,B) &= d_{AB} \quad \text{(1st constant)}, \\ d(B,C) &= d_{BC} \quad \text{(2nd constant)}, \\ d(A,C) &= d_{AC} \quad \text{(3rd constant)} \end{aligned}$$

  where $d(a,b) := \|a-b\|$: distance between a and b
- dof of the coin on the table (on 2D)

$$\begin{aligned} \text{dof} &= 2 \text{ (for point } A) + 1 \text{ (for point } B) + 0 \text{ (for point } C) \\ &= 6 \text{ (for total number of free variables)} - 3 \text{ (for number of constraints)} \end{aligned}$$

- Adding one point D to the coin does not give any info.
  $\Rightarrow d(A,D) = d_{AD}, \ldots$ is redundant.

dof in general rigid body

Generalizing the above concept, we have

$$
\begin{aligned}
&\text{dof} \\
&= (\text{sum of freedoms of the points}) - (\text{number of independent constraints}) \\
&= (\text{number of variables}) - (\text{number of independent equations}) \\
&= (\text{sum of freedoms of the bodies}) - (\text{number of independent constraints})
\end{aligned}
$$

Types of joints and their dofs

2.2. Degrees of Freedom of a Robot

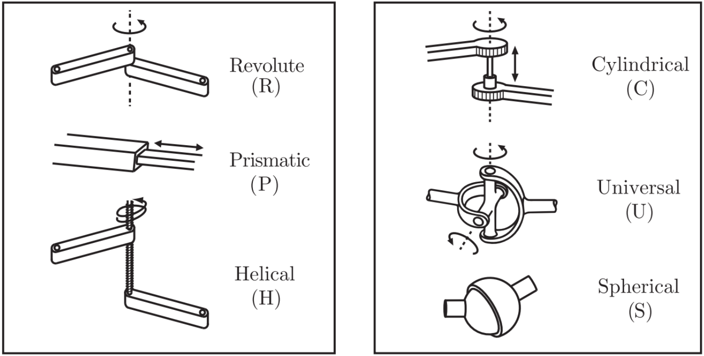


Figure 2.3: Typical robot joints.

- 1 dof: Revolute, prismatic, helical
- 2 dofs: Cylindrical, universal
- 3 dofs: Spherical (≈ Human's shoulder)

(Cont'd)

2.2. Degrees of Freedom of a Robot

Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Table 2.1: The number of degrees of freedom f and constraints c provided by common joints.

Grübler's formula

2.2. Degrees of Freedom of a Robot

Proposition 2.2 (Grübler's formula):

$$\begin{aligned}\text{dof} &= m(N-1) - \sum_{i=1}^{J} c_i = m(N-1) - \sum_{i=1}^{J} (m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i\end{aligned}$$

where

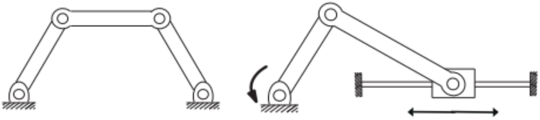
- N: number of links (w/ ground as an another link)
- J: number of joints
- m: number of dof of a rigid body (3 for planar (=2D), 6 for spatial (=3D))
- f_i: number of freedoms provided by joint i
- c_i: number of constraints provided by joint i

Types of joints and their dofs

2.2. Degrees of Freedom of a Robot

Example 2.3:

- dof of the four-bar linkage (planar) = 1
- dof of the slider-crank mechanism (planar) = 1



(a) (b)

Figure 2.4: (a) Four-bar linkage. (b) Slider–crank mechanism.

Note:

- Open-chain mechanism = Any mechanism without a closed loop;
- Closed-chain mechanism = Any mechanism that has a closed loop.

Topology

2.3. Configuration Space: Topology and Representation

- Topology
  $\approx$ A shape of surfaces or the C-space, such as sphere, line, circle $\cdots$
  (Note: dof = Dimension of the C-space)
- Two spaces are topologically equivalent
  IF one can be deformed into another without gluing or cutting.

Figure 2.9: An open interval of the real line, denoted (a, b), can be deformed to an open semicircle. This open semicircle can then be deformed to the real line by the mapping illustrated: beginning from a point at the center of the semicircle, draw a ray that intersects the semicircle and then a line above the semicircle. These rays show that every point of the semicircle can be stretched to exactly one point on the line, and vice versa. Thus an open interval can be continuously deformed to a line, so an open interval and a line are topologically equivalent.

Topologically distinct 1-dim. spaces and their combinations

Usual types of topologically 1-dimensional spaces:

- A circle: S or S^1 $(= \{\theta : -\pi \leq \theta < \pi\})$
- A line: $\mathbb{E}$ or $\mathbb{R}$
- A closed line $[a, b] \subset \mathbb{R}$

Their combination gives the C-space of a robot.

- The C-space of a rigid body: $\mathbb{R}^2 \times S^1$ (WHY? Think about the coin!)
- The C-space of a PR robot arm: $\mathbb{R}^1 \times S^1$
- The C-space of a 2R robot arm: $S^1 \times S^1$ (or say T^2, torus)

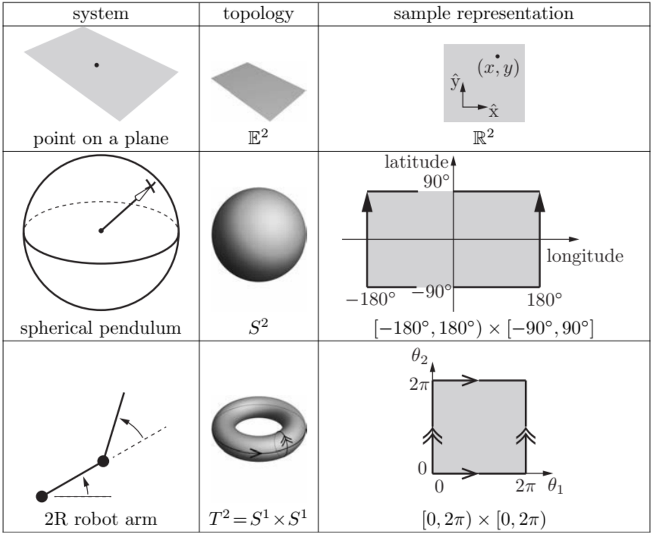
where

- "P" means a prismatic joint, and
- "R" means a revolute joint.

Note: S^n represents n-dim. surface of a sphere in $(n+1)$-dim. space.

Representations of 2-dim. C-spaces

2.3. Configuration Space: Topology and Representation



Loop-closure equations: Four-bar linkage case

2.3. Configuration Space: Topology and Representation

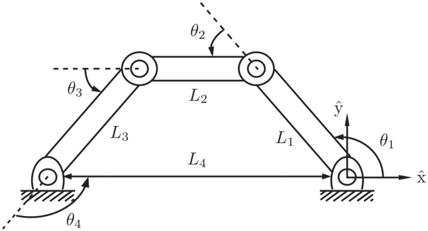


Figure 2.10: The four-bar linkage.

The four-bar linkage satisfies the 3 equality constraints:

$$
\begin{aligned}
L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + \cdots + L_4 \cos(\theta_1 + \cdots + \theta_4) &= 0. \\
L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + \cdots + L_4 \sin(\theta_1 + \cdots + \theta_4) &= 0, \\
\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi &= 0.
\end{aligned}
$$

Note: dof $= 4$ (number of variables) - 3 (number of constraints)

Holonomic constraint

2.3. Configuration Space: Topology and Representation

For a configuration $\theta = (\theta_1, \cdots, \theta_n)$, the loop-closure equations have the form

$$g(\theta) = \begin{bmatrix} g_1(\theta_1, \ldots, \theta_n) \\ \vdots \\ g_k(\theta_1, \ldots, \theta_n) \end{bmatrix} = 0.$$

which is called holonomic constraints.

Note: The holonomic constraints should be satisfied over time!

$$\text{(Holonomic constraints)} \quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \quad \frac{\mathrm{d}}{\mathrm{d}t} g(\theta(t)) \equiv 0.$$

This leads to a velocity constraint (called Pfaffian constraint)

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\theta(t)) = A(\theta)\dot{\theta} = 0 \quad \text{where} \quad A(\theta) = \frac{\partial g(\theta)}{\partial \theta} \quad \text{(called Jacobian matrix)}$$

Holonomic constraint vs. nonholonomic constraint

A velocity or Pfaffian constraint $A(\theta)\dot{\theta} = 0$ is called

- holonomic (= integrable) IF $\exists\ g(\theta)$ such that $A(\theta)$ can be represented as

$$A(\theta) = \frac{\partial g(\theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} & \cdots & \frac{\partial g_1}{\partial \theta_n} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} & \cdots & \frac{\partial g_2}{\partial \theta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial \theta_1} & \frac{\partial g_k}{\partial \theta_2} & \cdots & \frac{\partial g_k}{\partial \theta_n} \end{bmatrix} \in \mathbb{R}^{k \times n}$$

- nonholonimic (= nonintegrable) IF it is NOT holonomic.

$\therefore$

- A holonomic constraint can be represented only with θ; while
- A nonholonomic constraint cannot be handled without $\dot{\theta}$.

Example of nonholonomic constraint: No-slip rolling motion

2.4. Configuration and Velocity Constraint

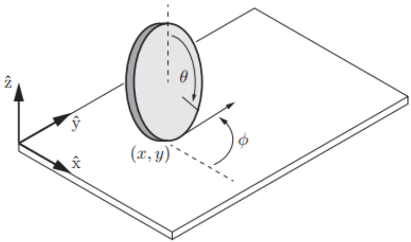


Figure 2.11: A coin rolling on a plane without slipping.

The coin must always roll in the fixed direction indicated by $(\cos\phi, \sin\phi)$ with forward speed $r\dot{\theta}$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = r\dot{\theta} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

Let $q = (x, y, \phi, \theta)$ be a coordinate of C-space.

Then, the above equations can be represented as $A(q)\dot{q} = 0$.

Task space and workspace

2.5. Task Space and Workspace

- The task space of a robot
  = A space in which the robot's task can be naturally expressed.
- The workspace of a robot
  = A specification of the configuration that the end-effector of the robot can reach.

Roughly speaking, IF (task space) $\subset$ (workspace),
THEN the robot can complete the task (with a proper control).

Examples

2.5. Task Space and Workspace

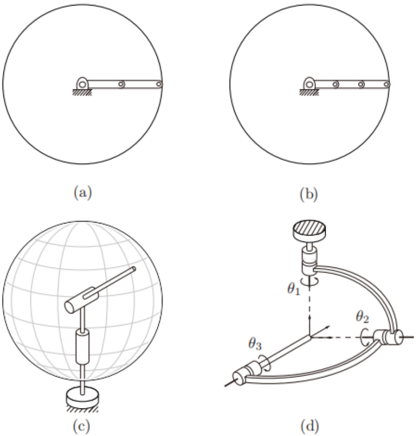


Figure 2.12: Examples of workspaces for various robots: (a) a planar 2R open chain; (b) a planar 3R open chain; (c) a spherical 2R open chain; (d) a 3R orienting mechanism.