[2024-1 Robotics]
Chapter 5. Velocity Kinematics and Statics

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Kinematics in position and velocity levels

- Forward kinematics:

$$x(t) = f(\theta(t))$$

- Velocity kinematics:

$$\dot{x} = \frac{\partial f}{\partial \theta} \frac{d\theta(t)}{dt} =: J(\theta)\dot{\theta}$$

where $J(\theta)$ is called the Jacobian matrix that has the form

$$J(\theta) = \frac{\partial f}{\partial \theta} = \begin{bmatrix} \partial f_1/\partial\theta_1 & \partial f_1/\partial\theta_2 & \cdots & \partial f_1/\partial\theta_n \\ \partial f_2/\partial\theta_1 & \partial f_2/\partial\theta_2 & \cdots & \partial f_2/\partial\theta_n \\ \vdots & & \ddots & \\ \partial f_m/\partial\theta_1 & \partial f_m/\partial\theta_2 & \cdots & \partial f_m/\partial\theta_n \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Velocity kinematics: 2R manipulator

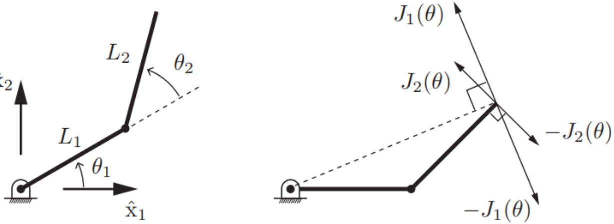


Figure 5.1: (Left) A 2R robot arm. (Right) Columns 1 and 2 of the Jacobian correspond to the endpoint velocity when $\dot{\theta}_1 = 1$ (and $\dot{\theta}_2 = 0$) and when $\dot{\theta}_2 = 1$ (and $\dot{\theta}_1 = 0$), respectively.

After some computations, we have the forward kinematics of the 2R manipulator

$$x_1 = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$x_2 = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

where (x_1, x_2) is the position of the end-effector.

Differentiating x_1 and x_2 in time, one has

$$\dot{x}_1 = -L_1\dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2)$$
$$\dot{x}_2 = L_1\dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2)$$

which can be represented in a simpler form

$$\begin{aligned} v_{\text{tip}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ &= J(\theta)\dot{\theta}. \end{aligned}$$

which is the velocity kinematics of the robot.

Note:

- $J(\theta)$ is invertible for most $\theta \in \mathbb{R}^2$, but not all $\theta \in \mathbb{R}^2$.
- At $\theta_2 = 0$ or $\theta_2 = \pi$, $J(\theta)$ becomes singular.

Singular configuration

5.0. Introduction to Chapter

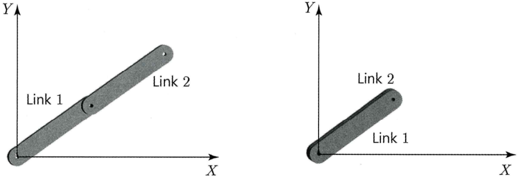


그림 4.3 특이형상 위치

The Jacobian matrices at singular configurations are computed by

$$
J(\theta)|_{\theta=(\theta_1,0)} = \begin{bmatrix} -(L_1+L_2)\sin(\theta_1) & -L_2\sin(\theta_1) \\ (L_1+L_2)\cos(\theta_1) & L_2\cos(\theta_1) \end{bmatrix},
$$

$$
J(\theta)|_{\theta=(\theta_1,\pi)} = \begin{bmatrix} -(L_1-L_2)\sin(\theta_1) & L_2\sin(\theta_1) \\ (L_1-L_2)\cos(\theta_1) & -L_2\cos(\theta_1) \end{bmatrix}
$$

Mapping from $\dot{\theta}$ to $\dot{x}$ at a specific posture

By definition, one has

$$\dot{x} = v_{\text{tip}} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2.$$

$\therefore$ $J_i(\theta)$ represents where the EE moves under variation of θ_i.

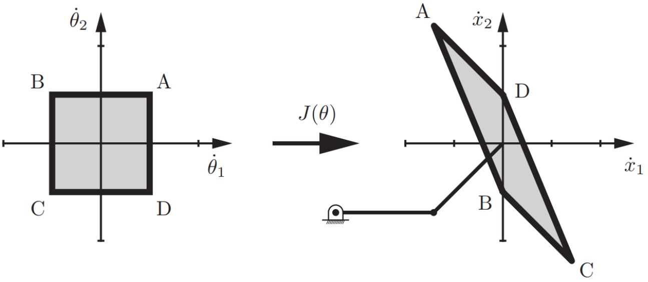


Figure 5.2: Mapping the set of possible joint velocities, represented as a square in the $\dot{\theta}_1$–$\dot{\theta}_2$ space, through the Jacobian to find the parallelogram of possible end-effector velocities. The extreme points A, B, C, and D in the joint velocity space map to the extreme points A, B, C, and D in the end-effector velocity space.

Manipulability ellipsoid

Manipulability ellipsoid $=$ The mapping from $\dot{\theta}$ satisfying $\|\dot{\theta}\| = 1$ to $\dot{x}$

$$= \{J(\theta)\dot{\theta} : \|\dot{\theta}\| = 1\} \subset \mathbb{R}^n.$$

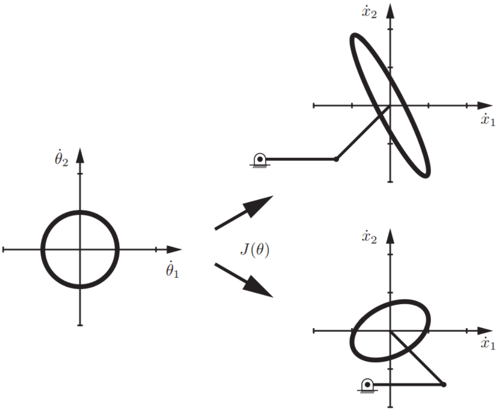


Figure 5.3: Manipulability ellipsoids for two different postures of the 2R planar open chain.

Statics

5.0. Introduction to Chapter

The principle of power conservation:

(power at the joints) = (power to move the robot) + (power at the end-effector).

Assume for now that (power to move the robot) ≈ 0.

We then always have, for any possible $\dot{\theta} \in \mathbb{R}^n$,

(power at the end-effector) $= f_{\mathrm{tip}}^\top v_{\mathrm{tip}} = f_{\mathrm{tip}}^\top J(\theta)\dot{\theta} = \tau^\top \dot{\theta} =$ (power at the joints).

This implies that

$$\tau = J^\top(\theta) f_{\mathrm{tip}}, \quad \text{or} \quad f_{\mathrm{tip}} = J^{-\top}(\theta)\tau.$$

where the latter can be obtained at nonsingular configurations.

Mapping from τ to f_{tip} at specific postures

5.0. Introduction to Chapter

$$\begin{aligned}\text{Force ellipsoid} &= \text{The mapping from } \tau \text{ satisfying } \|\tau\| = 1 \text{ to } f \\ &= \{J^{-\top}(\theta)\tau : \|\dot{\theta}\| = 1\} \subset \mathbb{R}^n.\end{aligned}$$

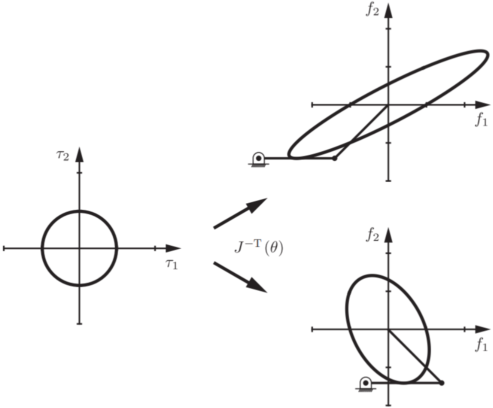


Figure 5.5: Force ellipsoids for two different postures of the 2R planar open chain.

Manipulability ellipsoid and force ellipsoid

5.0. Introduction to Chapter

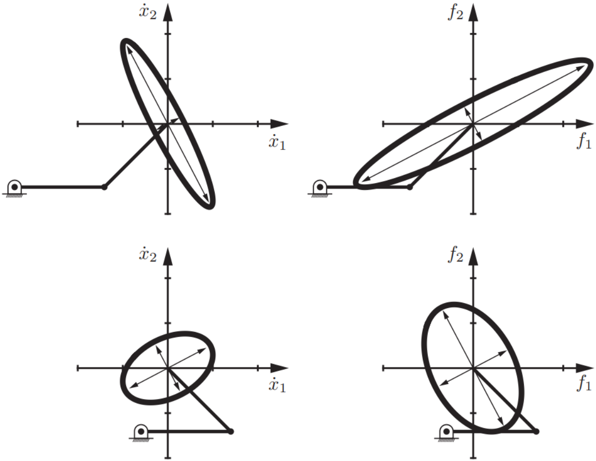


Figure 5.6: Left-hand column: Manipulability ellipsoids at two different arm configurations. Right-hand column: The force ellipsoids for the same two arm configurations.

Three different types of Jacobians

- Analytic Jacobian

$$\dot{x} = \text{ Velocity of the EE's position} = J_a(\theta)\dot{\theta}$$

- Space Jacobian

$$\mathcal{V}_s = \text{Spatial twist} = J_s(\theta)\dot{\theta}$$

- Body Jacobian

$$\mathcal{V}_b = \text{Body twist} = J_b(\theta)\dot{\theta}$$

Note: The latter two Jacobians are called geometric Jacobian.

Forward kinematics by product of exponentials (Revisited)

Consider

- $\mathcal{S}_i$: The screw axis for the i-th joint in $\{\mathrm{s}\}$
- $\mathcal{B}_i$: The screw axis for the i-th joint in $\{\mathrm{b}\}$

Then the homogeneous transformation T of $\{\mathrm{b}\}$ can be represented in

$$\begin{aligned} T(\theta_1, \ldots, \theta_n) &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M && \text{(represented in } \{\mathrm{s}\}) \\ &= M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n} && \text{(represented in } \{\mathrm{b}\}) \end{aligned}$$

Also note that,

$$[\mathcal{V}_s] = \dot{T}T^{-1}, \quad [\mathcal{V}_b] = T^{-1}\dot{T}$$

Space Jacobian $J_s(\theta)$

With the 1st representation of T, one can compute

$$\begin{aligned}\dot{T} &= [\mathcal{S}_1]\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]\dot{\theta}_2 e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots, \\ T^{-1} &= M^{-1} e^{-[\mathcal{S}_n]\theta_n} \cdots e^{-[\mathcal{S}_1]\theta_1}.\end{aligned}$$

Therefore, $[\mathcal{V}_s]$ is derived by

$$[\mathcal{V}_s] = \dot{T}T^{-1} = [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 + \cdots$$

which implies that

$$\begin{aligned}\mathcal{V}_s &= \mathcal{S}_1\dot{\theta}_1 + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)\dot{\theta}_2 + \cdots \qquad (\because \mathrm{Ad}_T(\mathcal{V}) = T[\mathcal{V}]T^{-1}) \\ &= \begin{bmatrix}\mathcal{S}_1 & \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2) & \cdots\end{bmatrix}\dot{\theta}.\end{aligned}$$

(Cont'd)

Definition 5.1.: A space Jacobian $J_s(\theta) \in \mathbb{R}^{6\times n}$ is the matrix that relates the joint rate vector $\dot{\theta}$ to the spatial twist $\mathcal{V}_s$ via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}.$$

The space Jacobian is computed by

$$J_s(\theta) = \begin{bmatrix} J_{s1} & J_{s2} & \cdots & J_{sn} \end{bmatrix}, \quad J_{si} = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$$

What does J_{si} imply?

It is the screw axis $\mathcal{S}_i$ that describes the rotation of the i-th joint, after the robot moves the first $i-1$ joints from $\theta_i = 0$ to θ_1, $\cdots$, θ_{i-1}.

Example 5.2

5.1. Manipulator Jacobian

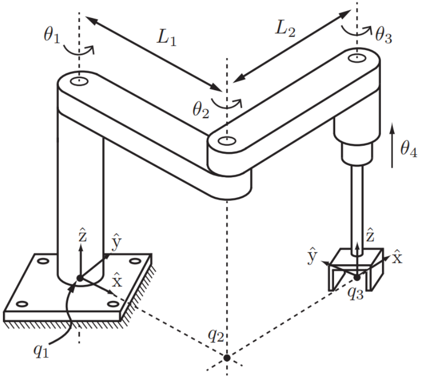


Figure 5.7: Space Jacobian for a spatial RRRP chain.

Example 5.3

5.1. Manipulator Jacobian

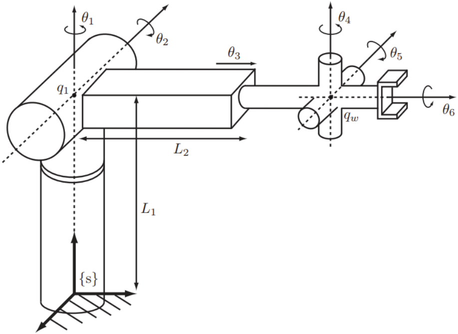


Figure 5.8: Space Jacobian for the spatial RRPRRR chain.

Another perspective based on DH convention

Supplementary material: (B. Siciliano *et al.*)

In the following, we derive the space Jacobian $J_s(\theta)$ in a different way.

For this, we consider link i of a manipulator in the open chain.

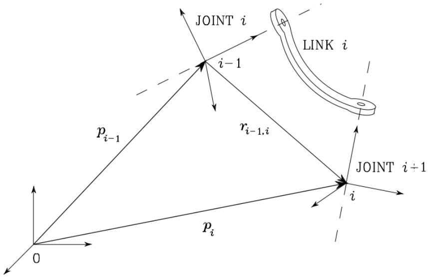


Fig. 3.1. Characterization of generic Link i of a manipulator

- p^{i-1}: Position vector of $\{i-1\}$ w.r.t. $\{0\}$ expressed in $\{0\}$
- p^{i}: Position vector of $\{i-1\}$ w.r.t. $\{0\}$ expressed in $\{0\}$
- $r_{i-1}^{i-1,i}$: Position vector of $\{i\}$ w.r.t. $\{i-1\}$ expressed in $\{i-1\}$

Linear velocity $\dot{p}^i$ of $\{i\}$

Supplementary material: (B. Siciliano *et al.*)

Then, the position vector p^i can be represented as

$$p^i = p^{i-1} + R_{i-1} r_{i-1}^{i-1,i}.$$

Differentiating both sides of the equation above leads to

$$\begin{aligned}
\dot{p}^i &= \dot{p}^{i-1} + R_{i-1}\dot{r}_{i-1}^{i-1,i} + \dot{R}_{i-1} r_{i-1}^{i-1,i} \\
&= \dot{p}^{i-1} + R_{i-1}\dot{r}_{i-1}^{i-1,i} + \omega^{i-1} \times (R_{i-1} r_{i-1}^{i-1,i}) \qquad (\leftarrow \quad \dot{R} = [\omega]R) \\
&= \dot{p}^{i-1} + v^{i-1,i} + \omega^{i-1} \times r^{i-1,i}
\end{aligned}$$

where

- $v^{i-1,i} := R_{i-1}\dot{r}_{i-1}^{i-1,i}$: The linear velocity of $\{i\}$ w.r.t. $\{i-1\}$ expressed in $\{0\}$
- ω^{i-1}: The angular velocity related to R_{i-1} expressed in $\{0\}$

Angular velocity ω^i of $\{i\}$

Supplementary material: (B. Siciliano *et al.*)

By definition, the rotation matrix R_i can be decomposed by

$$R_i = R_{i-1} R_{i-1,i}.$$

Then its time derivative is computed by

$$\begin{aligned}
\dot{R}_i &= [\omega^i] R_i \\
&= [\omega^{i-1}] R_{i-1} R_{i-1,i} + R_{i-1} [\omega_{i-1}^{i-1,i}] R_{i-1,i} \\
&= [\omega^{i-1}] R_i + R_{i-1} [\omega_{i-1}^{i-1,i}] R_{i-1}^T R_{i-1} R_{i-1,i} \\
&= [\omega^{i-1}] R_i + [R_{i-1} \omega_{i-1}^{i-1,i}] R_i && (\leftarrow \quad R[\omega] R^T = [R\omega]) \\
&= [\omega^{i-1}] R_i + [\omega^{i-1,i}] R_i
\end{aligned}$$

From invertibility of R_i, it follows that

$$\omega^i = \omega^{i-1} + \omega^{i-1,i}$$

Linear/angular velocity of $\{i\}$ for prismatic/revolute joints

Supplementary material: (B. Siciliano *et al.*)

Note again that

- $v^{i-1,i}$: Linear velocity of $\{i\}$ w.r.t. $\{i-1\}$ expressed in $\{0\}$
- $\omega^{i-1,i}$: Angular velocity of $\{i\}$ w.r.t. $\{i-1\}$ expressed in $\{0\}$.

Suppose that Joint i is either prismatic or revolute.

Then for each case, we obtain the following with DH convention:

- Prismatic joint:

$$\begin{aligned} \omega^{i-1,i} &= 0 \\ v^{i-1,i} &= \dot{\theta}_i \hat{z}_{i-1} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \omega^{i} &= \omega^{i-1}, \\ \dot{p}^{i} &= \dot{p}^{i-1} + \dot{\theta}_i \hat{z}_{i-1} + \omega^{i} \times r^{i-1,i} \end{aligned}$$

- Revolute joint:

$$\begin{aligned} \omega^{i-1,i} &= \dot{\theta}_i \hat{z}_{i-1} \\ v^{i-1,i} &= \omega^{i-1,i} \times r^{i-1,i} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \omega^{i} &= \omega^{i-1} + \dot{\theta}_i \hat{z}_{i-1}, \\ \dot{p}^{i} &= \dot{p}^{i-1} + \omega^{i} \times r^{i-1,i} \end{aligned}$$

Spatial manipulator Jacobian with DH convention

Supplementary material: (B. Siciliano *et al.*)

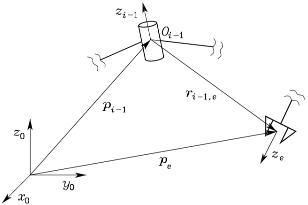


Fig. 3.2. Representation of vectors needed for the computation of the velocity contribution of a revolute joint to the end-effector linear velocity

- p^{e} be the position vector of $\{\mathrm{e}\}$, and
- w^{e} be the angular velocity vector of $\{\mathrm{e}\}$.

We now derive another geometric Jacobian $J_m(\theta)$ such that

$$\begin{bmatrix} \omega^{\mathrm{e}} \\ \dot{p}^{\mathrm{e}} \end{bmatrix} = \begin{bmatrix} \text{angular velocity of } \{\mathrm{e}\} \text{ expressed in } \{0\} \text{ (or } \{\mathrm{s}\}) \\ \text{linear velocity of } \{\mathrm{e}\} \text{ expressed in } \{0\} \text{ (or } \{\mathrm{s}\}) \end{bmatrix}$$

$$= J_m(\theta)\dot{\theta} = \begin{bmatrix} J_{m,\omega}(\theta) \\ J_{m,v}(\theta) \end{bmatrix} \dot{\theta}.$$

Derivation of $J_m(\theta)$: Angular part

Supplementary material: (B. Siciliano *et al.*)

Note again that $\omega^i = \omega^{i-1} + \omega^{i-1,i}$, from which

$$\omega^j = \sum_{i=1}^{j} \omega^{i-1,i} = \sum_{i=1}^{j} J_{\omega,i}(\theta)\dot{\theta}_i \quad \forall j \quad \Rightarrow \quad \omega^{i-1,i} = J_{m,\omega,i}(\theta)\dot{\theta}_i$$

- IF Joint i is prismatic, THEN

$$J_{m,\omega,i}(\theta)\dot{\theta}_i = \omega^{i-1,i} = 0 \quad \Rightarrow \quad J_{m,\omega,i}(\theta) = 0.$$

- IF Joint i is revolute, THEN

$$J_{m,\omega,i}(\theta)\dot{\theta}_i = \omega^{i-1,i} = \dot{\theta}_i \hat{z}_{i-1} \quad \Rightarrow \quad J_{m,\omega,i}(\theta) = \hat{z}_{i-1}.$$

where $\hat{z}_{i-1}$ represents the z-axis-related column of R_{i-1}: i.e.,

$$\hat{z}_{i-1} = R_{i-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_{01}(\theta_1) R_{12}(\theta_2) \cdots R_{i-2,i-1}(\theta_{i-1}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Derivation of $J_m(\theta)$: Linear part

Supplementary material: (B. Siciliano *et al.*)

In a similar way, we have

$$\dot{p}^{\mathrm{e}} = \sum_{i=1}^{n} \frac{\partial p^{\mathrm{e}}}{\partial \theta_i} \dot{\theta}_i \quad \Rightarrow \quad \frac{\partial p^{\mathrm{e}}}{\partial \theta_i} \dot{\theta} = J_{m,v,i}(\theta)$$

- IF Joint i is prismatic, THEN

$$J_{m,v,i}(\theta)\dot{\theta}_i = \dot{\theta}_i \hat{z}_{i-1} \quad \Rightarrow \quad J_{m,v,i}(\theta) = \hat{z}_{i-1}.$$

- IF Joint i is revolute, THEN

$$J_{m,v,i}(\theta)\dot{\theta}_i = \hat{z}_{i-1}\dot{\theta}_i \times (p^{\mathrm{e}} - p^{i-1}) \quad \Rightarrow \quad J_{m,v,i}(\theta) = \hat{z}_{i-1} \times (p^{\mathrm{e}} - p^{i-1})$$

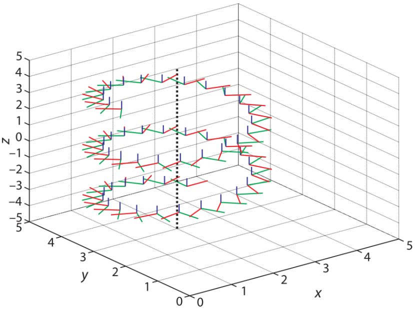
where p^{i-1} can be computed by $\begin{bmatrix} p^{i-1} \\ 1 \end{bmatrix} = T_{0,\mathrm{i-1}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = T_{01} \cdots T_{i-1,i-2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

Spatial twist $\mathcal{V}_s$ vs. spatial velocity $(v^{\mathrm{e}}, \omega^{\mathrm{e}})$

Supplementary material: (P. Corke)

Basically, the spatial twist is not the same as the spatial velocity.

- Spatial velocity $(\omega^{\mathrm{e}}, v^{\mathrm{e}})$ with
  - ω^{e}: The angular velocity of the origin of $\{\mathrm{e}\}$ expressed in $\{\mathrm{s}\}$.
  - v^{e}: The linear velocity of the origin of $\{\mathrm{e}\}$ expressed in $\{\mathrm{s}\}$.
- Spatial twist $\mathcal{V}_s = (\omega_s, v_s)$ with
  - ω_s: The direction vector of the screw axis
  - v_s: The linear velocity of the origin of $\{\mathrm{e}\}$ w.r.t. an auxiliary point on the screw axis



Screw interpretation of a twist: Revisited

Main Textbook

The twist $\mathcal{V} = (\omega, v)$ can be rewritten by $\{\dot{\theta}, \hat{s}, h\}$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ \hat{s}\dot{\theta} \times (-q) + h\hat{s}\dot{\theta} \end{bmatrix}$$

- $\dot{\theta}$: The rate of rotation about $\hat{s}$.
- $\hat{s}$: A unit vector in the direction of the rotation axis
- h: The screw pitch $(= \omega^T v)$
  $=$ the ratio of the linear velocity $h\hat{s}\dot{\theta}$ and the angular velocity $\hat{s}\dot{\theta}$.

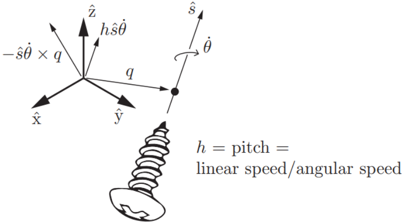


Figure 3.19: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch

$\mathcal{V}_s$ vs. $\mathcal{V}_b$ vs. $(v^{\mathrm{e}}, \omega^{\mathrm{e}})$

Supplementary material: (P. Corke)

Fact 1: $(\omega^{\mathrm{e}}, v^{\mathrm{e}})$ is indeed a different expression of the body twist $\mathcal{V}_b$.

- ω_b: The angular velocity in $\{\mathrm{b}\}$

  $\Rightarrow$ The angular velocity of $\{\mathrm{b}\}$ in $\{\mathrm{s}\}$ = $R\omega_b = \omega^{\mathrm{e}}$.

- v_b: The linear velocity of $\{\mathrm{b}\}$ in $\{\mathrm{b}\}$

  $\Rightarrow$ The angular velocity of $\{\mathrm{b}\}$ in $\{\mathrm{s}\}$ = $Rv_b = v^{\mathrm{e}}$

Fact 2: The spatial twist $\mathcal{V}_s$ and the body twist $\mathcal{V}_b$ are interconnected with

$$\omega_s = R\omega_b, \quad v_s = [p]R\omega_b + Rv_b.$$

Combining all together, we have

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R\omega_b \\ [p]R\omega_b + Rv_b \end{bmatrix} = \begin{bmatrix} \omega^{\mathrm{e}} \\ [p]\omega^{\mathrm{e}} + v^{\mathrm{e}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ [p] & I \end{bmatrix} \begin{bmatrix} \omega^{\mathrm{e}} \\ v^{\mathrm{e}} \end{bmatrix}.$$

Body Jacobian $J_b(\theta)$

5.1. Manipulator Jacobian

With $\mathcal{B}_i$ in hand, we compute $\dot{T}$ and T^{-1} in a different form:

$$\begin{aligned}\dot{T} &= Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_n]\theta_n}[\mathcal{B}_n]\dot{\theta}_n \\ &\quad + Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + \cdots ,\\ T^{-1} &= e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_1]\theta_1}M^{-1}.\end{aligned}$$

Evaluating $T^{-1}\dot{T}$, one has

$$T^{-1}\dot{T} = [\mathcal{V}_b] = [\mathcal{B}_n]\dot{\theta}_n + e^{-[\mathcal{B}_n]\theta_n}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + \cdots$$

and therefore,

$$\mathcal{V}_b = \mathcal{B}_n\dot{\theta}_n + \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1})\dot{\theta}_{n-1} + \cdots$$

(Cont'd)

5.1. Manipulator Jacobian

Definition 5.4.: A body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ is the matrix that relates the joint rate vector $\dot{\theta}$ to the body twist $\mathcal{V}_b$ via

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}.$$

The body Jacobian is computed by

$$J_b(\theta) = \begin{bmatrix} J_{b1} & J_{b2} & \cdots & J_{bn} \end{bmatrix}, \quad J_{bi} = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i)$$

What does J_{bi} imply?

It is the screw axis $\mathcal{B}_i$ that describes the rotation of the i-th joint expressed in $\{\mathrm{b}\}$.

Visualization of geometric Jacobians

5.1. Manipulator Jacobian

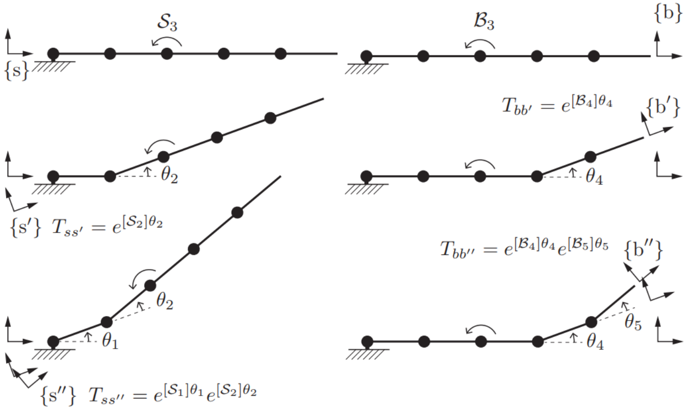


Figure 5.9: A 5R robot. (Left-hand column) Derivation of J_{s3}, the third column of the space Jacobian. (Right-hand column) Derivation of J_{b3}, the third column of the body Jacobian.

Relation between space and body Jacobians

Remind that the relation btw. two twists are given by

$$[\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1} \quad \Rightarrow \quad \mathcal{V}_s = \mathrm{Ad}_T(\mathcal{V}_b)$$

which then brings

$$\mathcal{V}_s = \mathrm{Ad}_T(\mathcal{V}_b) = J_s(\theta)\dot{\theta}.$$

Similar computations lead to

$$J_b(\theta)\dot{\theta} = \mathcal{V}_b = \mathrm{Ad}_{T^{-1}}(\mathcal{V}_s) = \mathrm{Ad}_{T^{-1}}(\mathrm{Ad}_T(\mathcal{V}_b)) = [\mathrm{Ad}_{T^{-1}}]J_s(\theta)\dot{\theta}.$$

$\therefore$ The two Jacobians are correlated with each other as follows:

$$J_b(\theta) = [\mathrm{Ad}_{T^{-1}}]J_s(\theta), \quad J_s(\theta) = [\mathrm{Ad}_T]J_b(\theta)$$

Analytic Jacobian

5.1. Manipulator Jacobian

The two Jacobians are correlated with each other as follows:

$$\dot{q} = J_a(\theta)\dot{\theta}$$

where q is a minimum set of coordinates, such as

- $q = q_{\mathrm{ZYZ}} := (\varphi, \vartheta, \psi, p)$ with ZYZ Euler angle, or
- $q = q_{\exp} := (\hat{\omega}\epsilon, p)$ with exponential coordinate

Note: Integral of ω does not admit a clear physical interpretation.

Example 3.3 of (B. Siciliano *et al.*) Consider two time profiles of ω:

$$\omega_1(t) = \begin{cases} (\pi/2, 0, 0), & 0 \le t \le 1, \\ (0, \pi/2, 0), & 1 \le t \le 2. \end{cases} \quad \omega_2(t) = \begin{cases} (0, \pi/2, 0), & 0 \le t \le 1, \\ (\pi/2, 0, 0), & 1 \le t \le 2 \end{cases}$$

whose integrals over time are the same.

(Cont'd)

5.1. Manipulator Jacobian

Example 3.3 of (B. Siciliano *et al.*) (Cont'd) However, the results of rotation is quite different!

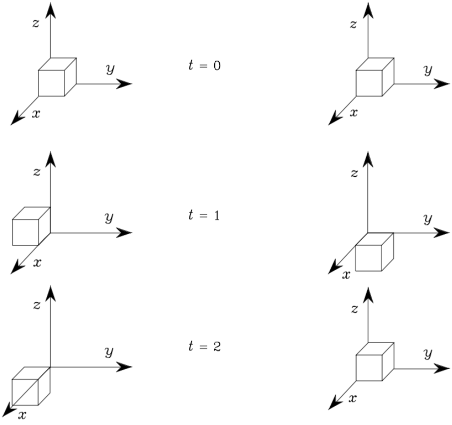


Fig. 3.10. Nonuniqueness of orientation computed as the integral of angular velocity

Analytic Jacobian 1: ZYZ Euler angle

5.1. Manipulator Jacobian

The ZYZ angle represents the angular velocity ω^{e} as follows:

$$\omega^{\mathrm{e}} = \omega_{\varphi} + \omega_{\vartheta} + \omega_{\psi}$$

where $\omega_{\cdot}$ is a part of ω due to variation of $\cdot$:

- $\omega_{\varphi} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\varphi}$
- $\omega_{\vartheta} = \mathrm{Rot}(\hat{\mathrm{z}}, \varphi) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \dot{\vartheta} = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} \dot{\vartheta}$
- $\omega_{\psi} = \mathrm{Rot}(\hat{\mathrm{z}}, \varphi)\mathrm{Rot}(\hat{\mathrm{y}}, \vartheta) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\psi} = \begin{bmatrix} \cos\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta \\ \cos\vartheta \end{bmatrix} \dot{\psi}$:

(Cont'd)

Supplementary material: (B. Siciliano *et al.*)

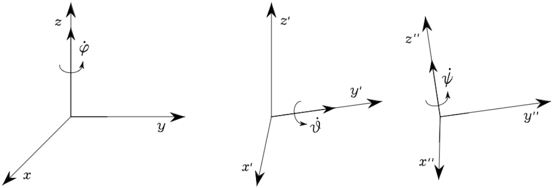


Fig. 3.8. Rotational velocities of Euler angles ZYZ in current frame

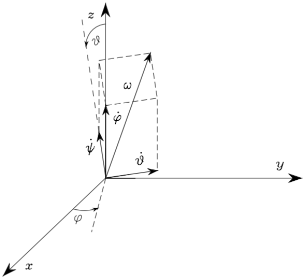


Fig. 3.9. Composition of elementary rotational velocities for computing angular velocity

(Cont'd)

Supplementary material: (B. Siciliano *et al.*)

One can represent the equations above in a compact form:

$$\omega^{\mathrm{e}} = \begin{bmatrix} 0 & -\sin\varphi & \cos\varphi\sin\vartheta \\ 0 & \cos\varphi & \sin\varphi\sin\vartheta \\ 1 & 0 & \cos\vartheta \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix}.$$

Analytic Jacobian with respect to the ZYZ Euler angle is related to $J_s(\theta)$ as

$$J_m(\theta) = \begin{bmatrix} \begin{bmatrix} 0 & -\sin\varphi & \cos\varphi\sin\vartheta \\ 0 & \cos\varphi & \sin\varphi\sin\vartheta \\ 1 & 0 & \cos\vartheta \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} J_{a,\mathrm{ZYZ}}(\theta)$$

Note: The determinant of $\begin{bmatrix} 0 & -\sin\varphi & \cos\varphi\sin\vartheta \\ 0 & \cos\varphi & \sin\varphi\sin\vartheta \\ 1 & 0 & \cos\vartheta \end{bmatrix}$ is $-\sin\vartheta$.

$\Rightarrow \omega$ cannot be represented by ZYZ Euler angle at $\vartheta = 0$ (rep. singularity).

Analytic Jacobian 2: Exponential coordinates

Consider another candidate for q:

$$q_{\text{exp}} = \begin{bmatrix} r \\ p \end{bmatrix} = \begin{bmatrix} \hat{\omega}\epsilon \\ p \end{bmatrix} = \begin{bmatrix} \text{angular position of EE in exponential coordinate} \\ \text{linear position of EE} \end{bmatrix}$$

We now show that how the time derivative of q and the body twist

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = J_b(\theta)\dot{\theta} = \begin{bmatrix} J_\omega(\theta) \\ J_v(\theta) \end{bmatrix} \dot{\theta}$$

are interconnected with each other.

- Linear motion

$$\dot{x} = Rv_b = RJ_v(\theta)\dot{\theta}$$

where $R = e^{[\hat{\omega}]\epsilon}$

- Angular motion (using $R(t) = e^{[r(t)]}$ and $[\omega_b] = R^\top \dot{R}$)

$$\begin{aligned} \omega_b &= \left(I - \frac{1-\cos\|r\|}{\|r\|^2}[r] + \frac{\|r\| - \sin\|r\|}{\|r\|^3}[r]^2 \right)\dot{r} = A(r)\dot{r} \\ \Rightarrow \quad \dot{r} &= A^{-1}(r)\omega_b = A^{-1}(r)J_\omega(\theta)\dot{\theta}. \end{aligned}$$

Putting all together,

$$\dot{q} = \begin{bmatrix} A^{-1}(r) & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \quad \Rightarrow \quad J_a(\theta) = \begin{bmatrix} A^{-1}(r) & 0 \\ 0 & R \end{bmatrix} J_b(\theta)$$

Statics

5.1. Manipulator Jacobian

The principle of power conservation:

(power at the joints) $=$ (power to move the robot) $+$ (power at the end-effector).

If we assume that the robot is at the "equillibrium",
then (power to move the robot) $= 0$, and thus,

$$\tau^\top \dot{\theta} = \mathcal{F}_b^\top \mathcal{V}_b = \mathcal{F}_s^\top \mathcal{V}_s = \cdots$$

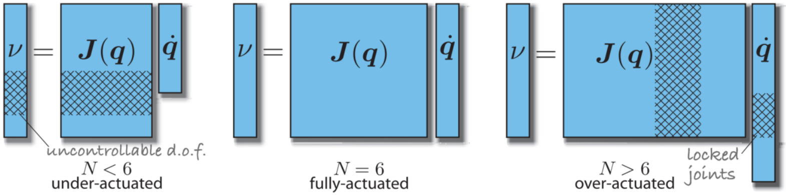
holds for any possible $\dot{\theta}$.

Thus the torque that generates the wrench $\mathcal{F}$ at the EE:

$$\tau = J^\top(\theta)\mathcal{F} \qquad \text{(that is frame-independent)}$$

Over- and under-actuation of robots and Jacobian

Supplementary material: (P. Corke)



Suppose that $J(\theta)$ has full rank. Then the robot is

- under-actuated if $N < 6$ (A part of v cannot be assigned freely);
- fully-actuated if $N = 6$;
- over-actuated (or redundant) if $N > 6$ (A part of joints need not be move)

Kinematic singularity of robot

Note again that

- At most of the configurations $J(\theta)$ has full rank.
- In some postures $J(\theta)$ turns out to have the rank less than n.

For the latter, the robot's EE loses the ability to move instantaneously in a direction.

(Kinematic) singularity is

- the posture where $J(\theta)$ has the rank $< n$;
- independent of the Jacobian's type:

$$\because \quad J_s(\theta) = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} J_b(\theta).$$

- independent of the frame.

Ability of robots to move: In case of full rank of J

Suppose that $J \in \mathbb{R}^{6\times n}$.

- Case 1: If $n \geq 6$ (i.e., redundant), then for any given $\mathcal{F}_{\text{des}}$,

$$\tau = J^\top(\theta)\mathcal{F}_{\text{des}} \quad \Rightarrow \quad \mathcal{F} = \mathcal{F}_{\text{des}} = (J^\top)^\dagger(\theta)\tau$$

  - The EE can move in any direction instantaneously.
  - Additional internal motion may be generated.

- Case 2: If $n < 6$ (i.e., lack of freedom), then

$$\mathrm{Null}(J^\top(\theta)) = \{\mathcal{F} : J^\top(\theta)\mathcal{F} = 0\} \neq \emptyset$$

  - Any actuation cannot force the robot to move in a specific direction (related to the nullspace of $J^\top$).

Manipulatability ellipsoid

$$\begin{aligned}\|\dot\theta\| = 1 \quad \Rightarrow \quad 1 &= \dot\theta^\top \dot\theta \\ &= (J^{-1}\dot q)^\top (J^{-1}\dot q) = \dot q^\top J^{-\top} J^{-1} \dot q \\ &= \dot q^\top (JJ^\top)^{-1} \dot q = \dot q^\top A^{-1} \dot q, \qquad (A := JJ^\top)\end{aligned}$$

Definition: The manipulability ellipsoid is defined as

$$(\text{Manipulability ellipsoid}) = \{\dot q : \dot q^\top A^{-1} \dot q = 1\}$$

(whose volume $\propto \sqrt{\det(A)} = \sqrt{\det(JJ^\top)}$)

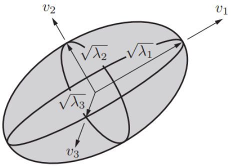


Figure 5.13: An ellipsoid visualization of $\dot q^\mathrm{T} A^{-1} \dot q = 1$ in the $\dot q$ space $\mathbb{R}^3$, where the principal semi-axis lengths are the square roots of the eigenvalues λ_i of A and the directions of the principal semi-axes are the eigenvectors v_i.

Kineto-statics duality

Supplementary material: (B. Siciliano *et al.*)

Consider the velocity relation

$$\dot{q} = J(\theta)\dot{\theta}$$

- $\mathcal{R}(J)$ (The range space of J): The set of end-effector velocities that can be generated by the joint motion.
- $\mathcal{N}(J)$ (The nullspace of J): The set of joint velocities that do not produce any end-effector velocities.

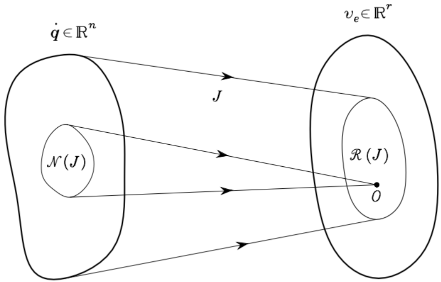


Fig. 3.7. Mapping between the joint velocity space and the end-effector velocity space

(Cont'd)

Supplementary material: (B. Siciliano *et al.*)

Now consider the force relation

$$\tau = J^\top(\theta)\mathcal{F}$$

- $\mathcal{R}(J^\top)$ (The range space of $J^\top$): The set of joint torques that can balance the end-effector forces.
- $\mathcal{N}(J^\top)$ (The nullspace of $J^\top$): The set of end-effector forces that do not require any balancing joint torque.

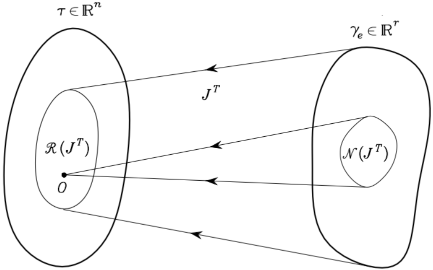


Fig. 3.21. Mapping between the end-effector force space and the joint torque space

(Cont'd)

Supplementary material: (B. Siciliano *et al.*)

The fundamental theorem of linear algebra says that

$$\mathcal{R}(J^\top) = \mathcal{N}(J)^\perp, \quad \mathcal{N}(J^\top) = \mathcal{R}(J)^\perp$$

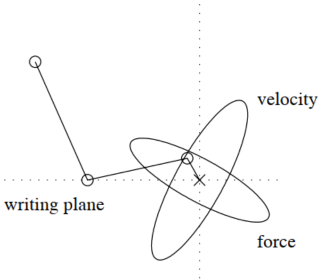


Fig. 3.26. Velocity and force manipulability ellipses for a 3-link planar arm in a typical configuration for a task of controlling force and velocity