

[2024-1 Robotics]

Chapter 6. Inverse Kinematics

Gyunghoon Park

School of ECE, University of Seoul



서울시립대학교
UNIVERSITY OF SEOUL

Kinematics?: Revisited

6.0. Introduction to Chapter

Kinematics defines the relation between the joint coordinate θ and the pos./ori. of the end-effector frame, x_{EE} .

- ▶ **Forward kinematics** (in Chapter 4): $\theta \rightarrow x_{EE}$
- ▶ **Inverse kinematics** (in Chapter 6): $x_{EE} \rightarrow \theta$
- ▶ **Velocity kinematics** (in Chapter 5): $(\theta, \dot{\theta}) \rightarrow \dot{x}_{EE}$

Inverse kinematics problem: For given desired x_{EE}^* , find $\theta^* = \text{IK}(x_{EE}^*)$.

- ▶ Help to construct a **reference signal** for control of each joint.
- ▶ Example: <https://spotmicroai.readthedocs.io/en/latest/>

Inverse kinematics: A 2R planar robot case

6.0. Introduction to Chapter

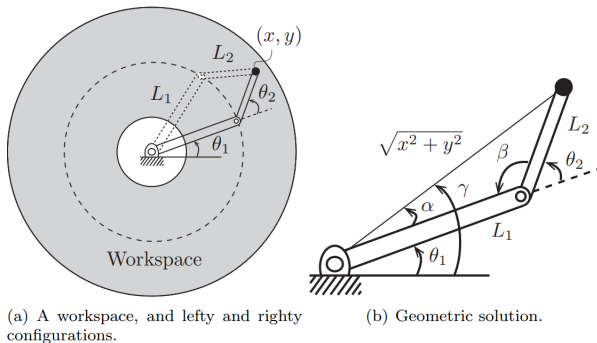


Figure 6.1: Inverse kinematics of a 2R planar open chain.

Note: The forward kinematics is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

(Review) atan2 function

6.0. Introduction to Chapter

The function $\text{atan2}(y, x)$ is a 2-arguments version of $\arctan(y/x)$.

Why required?

- ▶ (y, x) and $(-y, -x)$ gives the same value of \arctan .
- ▶ Yet, the actual $\theta \in (-\pi, \pi]$ are different.

$$\text{atan2}(y, x) = \begin{cases} \arctan(y/x), & \text{if } x > 0 \\ \arctan(y/x) + \pi, & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi, & \text{if } x < 0 \text{ and } y < 0 \\ +\pi/2, & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2, & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined}, & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

– atan2 in Wikipedia: <https://en.wikipedia.org/wiki/Atan2>

Inverse kinematics: A 2R planar robot case (Cont'd)

6.0. Introduction to Chapter

By the law of cosines,

$$\begin{aligned}L_2^2 &= x^2 + y^2 + L_1^2 - 2\sqrt{x^2 + y^2}L_1 \cos \alpha, \\x^2 + y^2 &= L_1^2 + L_2^2 - 2L_1L_2 \cos \beta,\end{aligned}$$

so that

$$\begin{aligned}\alpha &= \cos^{-1} \left(\frac{x^2 + y^2 + L_1^2 - L_2^2}{2L_1\sqrt{x^2 + y^2}} \right), \\ \beta &= \cos^{-1} \left(\frac{L_1^2 + L_2^2 - x^2 - y^2}{2L_1L_2} \right)\end{aligned}$$

and

$$\frac{y}{x} = \tan(\gamma) \quad \Rightarrow \quad \gamma = \text{atan2}(y, x).$$

Inverse kinematics: A 2R planar robot case (Cont'd)

6.0. Introduction to Chapter

Thus, a possible solution for the IK problem is

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \gamma - \alpha \\ \pi - \beta \end{bmatrix} \quad \text{where } \alpha, \beta, \gamma \text{ are given above (as functions of } (x, y)).$$

Note:

- ▶ IK may not have a unique solution, \therefore

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \gamma + \alpha \\ \beta - \pi \end{bmatrix}.$$

is also a solution of the IK problem.

- ▶ No solution exists for the case when $x^2 + y^2$ lies outside the range $[L_1 - L_2, L_1 + L_2]$.

Two ways of solving IK

6.1. Analytic Inverse Kinematics

One can express the forward kinematics as

$$T(\theta) = e^{[S_1]\theta_1} \dots e^{[S_6]\theta_6} M.$$

Then the IK problem can be understood as:

for given $X^* \in SE(3)$, find θ such that $T(\theta) = X^*$.

- ▶ **Analytic method:** Compute the inverse function T^{-1} by hand!
 - Not always possible
 - Only limited structure of the robot is available
- ▶ **Numerical method:** Solve it using computers
 - More general

Analytic IK solution for 6R PUMA-type arm

6.1. Analytic Inverse Kinematics

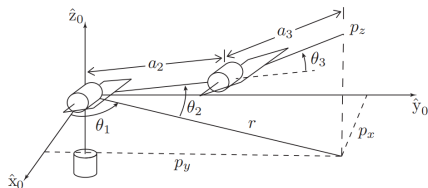


Figure 6.2: Inverse position kinematics of a 6R PUMA-type arm.

Features:

- ▶ Two shoulder joints axes intersect orthogonally at a common point
- ▶ Joint axis 3 lies in \hat{x}_0 - \hat{y}_0 plane and is aligned parallel with joint axis 2.
- ▶ Joint axes 4–6 intersect orthogonally at a common point.

This allows to decompose the IK problem as

- ▶ Inverse position kinematics (for joints 1–3)
- ▶ Inverse orientation kinematics (for joints 4–6)

(Cont'd)

6.1. Analytic Inverse Kinematics

We consider two cases: without and with the offset d_1 .

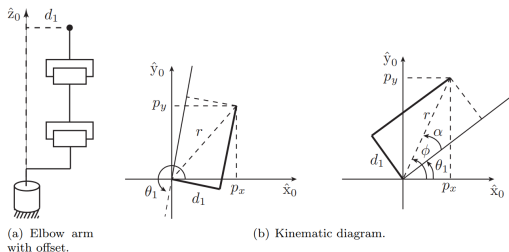


Figure 6.3: A 6R PUMA-type arm with a shoulder offset.

► **Case 1** ($d_1 = 0$): Projecting p onto the $\hat{x}_0 - \hat{y}_0$ -plane, it can be seen that

$$\theta_1 = \text{atan2}(p_x, p_y) \quad \text{or} \quad \theta_1 = \text{atan2}(p_x, p_y) + \pi.$$

Since $\text{atan2}(0, 0)$ is ill-defined, we have singularity at $p_x = p_y = 0$.

(Cont'd)

6.1. Analytic Inverse Kinematics

Singular configuration with zero-offset:

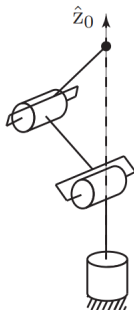


Figure 6.4: Singular configuration of the zero-offset 6R PUMA-type arm.

(Cont'd)

6.1. Analytic Inverse Kinematics

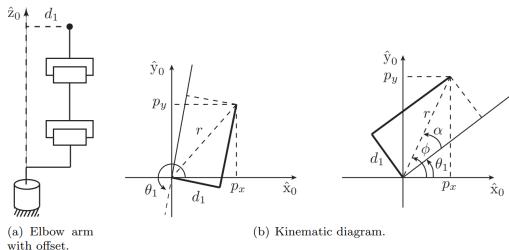


Figure 6.3: A 6R PUMA-type arm with a shoulder offset.

► **Case 2** ($d_1 \neq 0$): We have two general solutions:

$$\theta_1 = \phi - \alpha = \text{atan2}(p_y, p_x) - \text{atan2}(d_1, \sqrt{r^2 - d_1^2}), \quad \text{or}$$

$$\theta_1 = \pi + \text{atan2}(p_y, p_x) + \text{atan2}(-\sqrt{p_x^2 + p_y^2 - d_1^2}, d_1).$$

Note: Once θ_1 is determined, the remaining part can be viewed as a 2R planar robot (why?)

(Cont'd)

6.1. Analytic Inverse Kinematics

Four possible inverse kinematics solutions with shoulder offset:

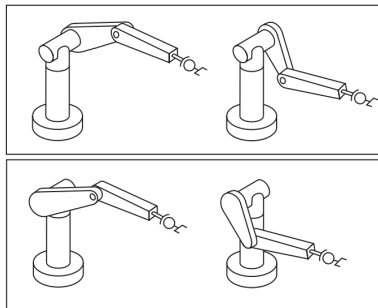


Figure 6.5: Four possible inverse kinematics solutions for the 6R PUMA-type arm with shoulder offset.

- ▶ \exists 2 cases in selecting θ_1 ,
- ▶ \exists 2 cases in selecting θ_2 and θ_3 .

Newton-Raphson method

6.2. Numerical Inverse Kinematics

We want to solve the equation, for a differentiable function g ,

$$g(\theta) = 0.$$

Note that the Taylor expansion of g at $\theta = \theta^0$ is given by

$$g(\theta) = g(\theta^0) + \frac{\partial g}{\partial \theta}(\theta^0)(\theta - \theta^0) + \text{H.O.T. (= Higher-order term)}$$

By truncating the H.O.T. and computing θ in an iterative fashion, we have an iterative algorithm (under the assumption that $\partial g / \partial \theta$ invertible)

$$\theta^{k+1} = \theta^k - \left(\frac{\partial g}{\partial \theta}(\theta^k) \right)^{-1} g(\theta^k)$$

whose solution θ^k is expected to converge to the real solution of $g(\theta) = 0$.

Numerical IK

6.2. Numerical Inverse Kinematics

In the IK problem, we have

$$g(\theta_d) = x_d - f(\theta_d) = 0$$

where x_d is given and θ_d needs to be computed.

Similar computations as above lead to

$$\begin{aligned} x_d = f(\theta_d) &= f(\theta^0) + \frac{\partial f}{\partial \theta}(\theta^0)(\theta_d - \theta^0) + \text{H.O.T.} \\ &= f(\theta^0) + J(\theta^0)\Delta\theta + \text{H.O.T.} \end{aligned}$$

We thus have, with the zero H.O.T.,

$$J(\theta^0)\Delta\theta = x_d - f(\theta^0) \quad \Rightarrow \quad \Delta\theta = J(\theta^0)^{-1} (x_d - f(\theta^0))$$

which can be solved by the Newton-Raphson method (when J is invertible!)

(Cont'd)

6.2. Numerical Inverse Kinematics

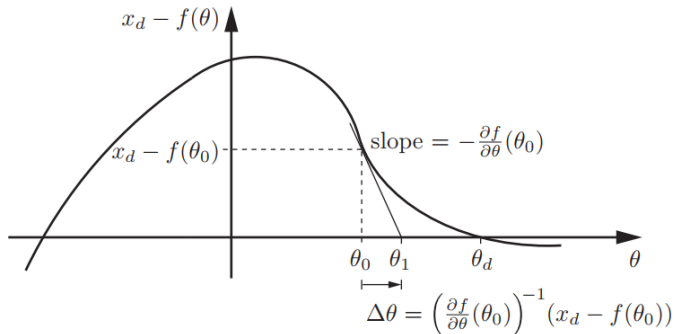


Figure 6.7: The first step of the Newton–Raphson method for nonlinear root-finding for a scalar x and θ . In the first step, the slope $-\partial f/\partial\theta$ is evaluated at the point $(\theta^0, x_d - f(\theta^0))$. In the second step, the slope is evaluated at the point $(\theta^1, x_d - f(\theta^1))$ and eventually the process converges to θ_d . Note that an initial guess to the left of the plateau of $x_d - f(\theta)$ would be likely to result in convergence to the other root of $x_d - f(\theta)$, and an initial guess at or near the plateau would result in a large initial $|\Delta\theta|$ and the iterative process might not converge at all.

When J is not invertible

6.2. Numerical Inverse Kinematics

We may have to use the (Moore-Penrose) pseudo-inverse J^\dagger instead of J^{-1}

► If J is fat and has full rank, then

- $J^\dagger = J^\top (JJ^\top)^{-1}$ (right-inverse)
- there may exist several solutions y^* for $Jy^* = z$
- among the candidates, $y^* = J^\dagger z$ has the minimum two-norm.

► If J is tall and has full rank, then

- $J^\dagger = (J^\top J)^{-1} J^\top$ (left-inverse)
- no solution exists for $Jy^* = z$
- among the candidates, $y^* = J^\dagger z$ minimizes the two-norm error $\|Jy^* - z\|$.

For both cases, we will employ

$$\Delta\theta = J(\theta^0)^\dagger (x_d - f(\theta^0))$$

in solving the IK numerically.

Newton-Raphson iterative algorithm

6.2. Numerical Inverse Kinematics

Algorithm:

- (a) (Initialization) Given x_d and initial guess θ^0 , set $i = 0$.
- (b) Set $e = x_d - f(\theta^i)$. While $\|e\| > \epsilon$ for some small ϵ :
 - Set $\theta^{i+1} = \theta^i + J^\dagger(\theta^i)e$, and
 - Increment i .

(A similar method can be derived for geometric Jacobian.)

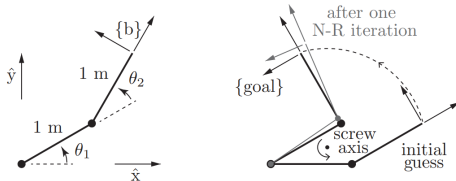


Figure 6.8: (Left) A 2R robot. (Right) The goal is to find the joint angles yielding the end-effector frame $\{goal\}$ corresponding to $\theta_1 = 30^\circ$ and $\theta_2 = 90^\circ$. The initial guess is $(0^\circ, 30^\circ)$. After one Newton-Raphson iteration, the calculated joint angles are $(34.23^\circ, 79.18^\circ)$. The screw axis that takes the initial frame to the goal frame (by means of the curved dashed line) is also indicated.

Closed-loop inverse kinematics

Supplementary material: (B. Siciliano *et al.*)

We now study another method for solving the IK with the **differential kinematics**.

In view of the differential kinematics, the IK problem can be understood as

For given x_d , find $\dot{\hat{\theta}}(t)$ such that

► $\hat{x}(t) := f(\hat{\theta}(t)) \rightarrow x_d$ as $t \rightarrow \infty$.

► $\hat{x}(t)$ satisfies $\dot{\hat{x}} = J_a(\hat{\theta})\dot{\hat{\theta}}$.

We now define the **operational space error**

$$e(t) := \hat{x}(t) - x_d$$

Then its time derivative is computed by

$$\dot{e} = J_a(\hat{\theta})\dot{\hat{\theta}} - \dot{x}_d$$

where $\dot{\hat{\theta}}$ is viewed as a virtual input to stabilize e .

Method 1: Jacobian (psuedo-)inverse

Supplementary material: (B. Siciliano *et al.*)

Suppose that $J_a(\hat{\theta})$ has full rank.

Then the **Jacobian (psuedo-)inverse-based IK algorithm** is given by

$$\dot{\hat{\theta}} = \begin{cases} J_a^{-1}(\dot{x}_d + He), & \text{if } J_a \text{ is square,} \\ J_a^\dagger(\dot{x}_d + He) + (I - J_a^\dagger J_a)\dot{\hat{\theta}}_{\text{red}}, & \text{if } J_a \text{ is fat} \end{cases}$$

where $\dot{\hat{\theta}}_{\text{red}}$ can be any vector with a proper dimension.

In both cases, we have

$$\dot{e} = He.$$

\therefore If H is Hurwitz, then $e(t) \rightarrow 0$ as $t \rightarrow \infty$.

(For more details, we should study the **Lyapunov stability analysis**.)

Method 2: Jacobian transpose

Supplementary material: (B. Siciliano *et al.*)

Now suppose that $\dot{x}_d = 0$.

The **Jacobian transpose-based IK algorithm** is given by

$$\dot{\hat{\theta}} = J_a^\top(\hat{\theta})Pe$$

where P is a positive definite matrix.

It then follows that

$$V(e) := \frac{1}{2}e^\top Pe \quad \Rightarrow \quad \dot{V} = -e^\top PJ_a(\hat{\theta})J_a(\hat{\theta})^\top Pe \leq 0.$$

\therefore Then the error $e(t)$ converges to 0 IF $\mathcal{N}(J_a^\top) = \emptyset$.

Note:

- ▶ Jacobian transpose-based algorithm does not require the inverse of J .
- ▶ If $Pe(t)$ eventually enters $\mathcal{N}(J_a^\top)$ (with $e(t_0) \neq 0$), then $\hat{\theta}$ gets stuck (so that $e(t) \neq 0$) for all $t \geq t_0$.

Additional issues on closed-loop inverse kinematics

Supplementary material: (B. Siciliano *et al.*)

- ▶ The idea above can be clearly applied to the **analytic Jacobian**.
- ▶ We have to discuss more for the **geometric Jacobian**.
- ▶ The main reason is that, the definition of e is not trivial.
- ▶ Nonetheless, a similar conclusion can be found at least for rotation matrices and unit quaternion.
(See Chapter 3 of (B. Siciliano *et al.*).)

Inverse differential (or velocity) kinematics

Supplementary material: (B. Siciliano *et al.*)

The problem of **inverse differential kinematics** is that,
for given θ_d and \dot{x}_d , find a solution $\dot{\theta}_d$ of the matrix equation $\dot{x}_d = J(\theta_d)\dot{\theta}_d$.

- If J is square and nonsingular, the solution is uniquely determined as

$$\dot{\theta}_d = J^{-1}\dot{x}_d.$$

- If J is fat or is singular, then the matrix equation may have infinitely many solutions.

In this case, one can choose the **optimal solution** among candidates, which is the solution of the following **minimization problem**:

$$\begin{aligned} \min_{\dot{\theta}_d \in \mathbb{R}^n} \quad & \frac{1}{2} \dot{\theta}_d^\top W \dot{\theta}_d \\ \text{subject to} \quad & \dot{x}_d = J(\theta_d)\dot{\theta}_d \end{aligned}$$

where W is positive definite.

The method of Lagrange multiplier

Supplementary material: (B. Siciliano *et al.*)

Consider a general equality-constrained minimization problem

$$\begin{aligned} \min_{\chi \in \mathbb{R}^n} \quad & f(\chi) \\ \text{subject to} \quad & g(\chi) = 0. \end{aligned}$$

The problem is in fact equivalent to an **unconstrained** minimization problem

$$\min_{(\chi, \lambda) \in \mathbb{R}^{n+m}} \mathcal{L}(\chi, \lambda) := f(\chi) + \lambda^\top g(\chi)$$

where \mathcal{L} is called **the Lagrangian**, and the variable λ is called **the Lagrange multiplier**.

Note that the solution of the second problem must satisfy

$$\frac{\partial \mathcal{L}}{\partial \chi}(\chi^*, \lambda^*) = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda}(\chi^*, \lambda^*) = g(\chi^*) = 0.$$

By the latter, one can conclude that minimizing \mathcal{L} is the same as that of f . 23 / 24

(Cont'd)

Supplementary material: (B. Siciliano *et al.*)

In our problem, the Lagrangian has the form

$$\mathcal{L}(\dot{\theta}_d, \lambda) = \frac{1}{2} \dot{\theta}_d^\top W \dot{\theta}_d + \lambda^\top (J(\theta_d) \dot{\theta}_d - x_d).$$

The necessary condition for optimality is then given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_d}^\top &= W \dot{\theta}_d + J(\theta_d)^\top \lambda = 0 \quad \Rightarrow \quad \dot{\theta}_d^* = -W^{-1} J(\theta_d)^\top \lambda^*, \\ \frac{\partial \mathcal{L}}{\partial \lambda}^\top &= J(\theta_d) \dot{\theta}_d - x_d = 0 \quad \Rightarrow \quad x_d = J \dot{\theta}_d^* = -J W^{-1} J^\top \lambda^*. \end{aligned}$$

If J (is fat but) has full rank, then

$$\lambda^* = -(J W^{-1} J^\top)^{-1} x_d \quad \Rightarrow \quad \dot{\theta}_d^* = W^{-1} J^\top (J W^{-1} J^\top)^{-1} x_d.$$

Note: The solution for the special case with $W = I$ is

$$\dot{\theta}_d^* = J^\top (J J^\top)^{-1} x_d = J^\dagger x_d \quad \text{where } J^\dagger \text{ is Penrose-Moore psuedo-inverse.}$$