Markov chain

Transition matrix

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \dots \\ p_{21} & p_{22} & \dots \\ \dots & & \end{bmatrix}$$

with $p_{ij} = \Pr(x_t = j | x_{t-1} = i)$. Each row is a PMF.

Example

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

p_{kk} are somewhat special in that they measure prob of **remaining** in state k.

Boundary condition

Where to start the system from?

Maybe some initial value x_0. Say, start from employment, $x_0 = 1$.

Instead, more generally, start from π_0, a PMF. This includes $\{1.0, 0.0\}$, but also $\{0.5, 0.5\}$.

One-period forecast

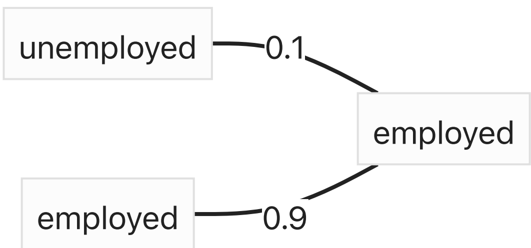
$$\Pr(x_1 = j) = ?$$

If initially I was employed, $x_0 = 1$, I can use conditional prob,

$$\Pr(x_1 = j) = \Pr(x_1 = j | x_0 = 1) = p_{1j}.$$

Enumerate the different ways in which x_1 can become 1:

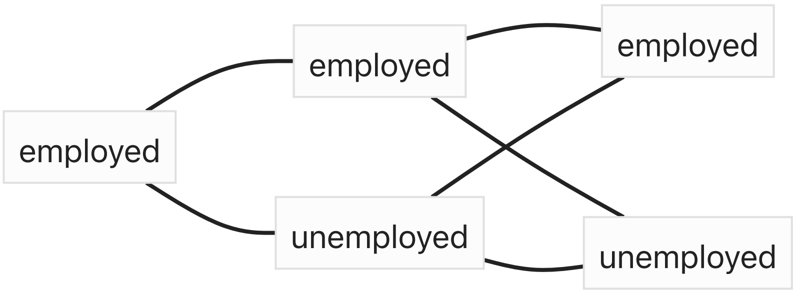
1. $x_0 = 1$ (employment) and stayed there
2. $x_0 = 2$ (unemployment) and switched to 1



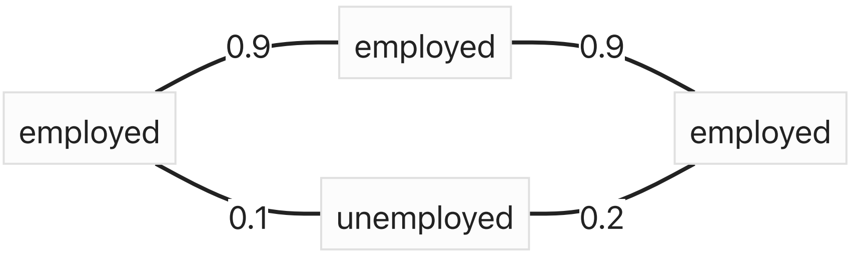
What is the prob of $x_0 = 1$? $\pi_{01} = 0.5$

Maybe less confusing

We do start from $x_0 = 1$. But! forecast for 2 periods:



What is the prob of $x_2 = 1$?



Add up two paths through which I can be employed in period 2:

1. $0.9 \times 0.9 = 0.81$
2. $0.1 \times 0.2 = 0.02$

$$\Pr(x_2 = 1 | x_0 = 1) = 0.81 + 0.02 = 0.83$$

$$\Pr(x_2 = 1 | x_0 = 1) = \sum_{k=1}^{K} \Pr(x_2 = 1 | x_1 = k) \Pr(x_1 = k | x_0 = 1)$$

$$= \sum_{k=1}^{K} p_{k1} \times p_{1k}$$

Back to q

Start from π_0.

$$\pi_{1j} := \Pr(x_1 = j) = \sum_{k=1}^{K} \Pr(x_1 = j | x_0 = k) \Pr(x_0 = k)$$

$$= \sum_{k=1}^{K} p_{kj} \pi_{0k} = [\mathbf{P}' \pi_0]_j.$$

In matrix notation

$$\pi_1 = \mathbf{P}' \pi_0,$$

or

$$\pi_t = \mathbf{P}' \pi_{t-1}.$$

> **Break until 14:34**

Check: if $x_0 = 1$ then $\pi_0 = (1, 0) = \mathbf{e}_1$ and

$$\pi_1 = \mathbf{P}'\mathbf{e}_1$$

is the first row if $\mathbf{P}$, the conditional probs for time 1.

Two-period forecast

$$\pi_{t+2} = \mathbf{P}'\mathbf{P}'\pi_t = (\mathbf{P}^2)'\pi_t$$