Advanced Macroeconomics

Lecture 6 - Introduction to dynamic programming

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Two different approaches

consider the following problem:

$$\sup_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$, $k_{t+1} \geq 0$ for all t, k_0 given

- direct approach: set up the Lagrangian, find the two optimal infinite sequences $\{k_{t+1}\}_{t=0}^{\infty}$ and $\{c_t\}_{t=0}^{\infty}$
- dynamic programming approach: find a time-invariant policy function $h(\cdot)$ mapping wealth into optimal consumption, i.e. $c_t = h(k_t)$

  iterating together with $k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$ starting from k_0 gives the two optimal sequences

Potential advantages of dynamic programming

it is unclear that finding a policy function is easier than finding an infinite sequence, but it has three advantages:

1. sometimes we can find a closed-form solution for the policy function $h(\cdot)$
2. sometimes we can characterize theoretical properties of the policy function $h(\cdot)$
3. various numerical methods are available to solve dynamic programs

We will proceed by

2.1 setting up the Sequence problem

2.2 the Bellman equation

2.3 solving the Bellman equation

2.1 Setting up

- discrete time dynamic optimization
- infinite-horizon stationary problem

the sequence problem: find $v(x)$ such that

$$v(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to $x_{t+1} \in \Gamma(x_t)$ (or $x_{+1} \in \Gamma(x)$), with x_0 given

where

- x_t is the state vector at date t
- $F(x_t, x_{t+1})$ is the flow payoff at date t (F is 'stationary')
- β is the exponential discount factor, β^t is the exponential discount function

Example 1

optimal growth with log utility and Cobb-Douglas production function:

$$\sup_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints: $c_t, k_t > 0$ and $c_t + k_{t+1} = k_t^{\alpha}$, k_0 given

Question: how do you formulate this as a sequence problem?

[Hint: eliminate redundant variables, and introduce the constraint function Γ]

2.2 The Bellman equation

the Bellman equation expresses the value function as a combination of a flow payoff and a discounted continuation payoff

$$v(x) = \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta v(x_{+1})\} \quad \forall x$$

- flow payoff is $F(x, x_{+1})$
- current value function is $v(x)$, continuation value function is $v(x_{+1})$
- equation holds for all feasible values of x

$\rightarrow$ compare the sequence problem and the Bellman equation

- we call $v(\cdot)$ the solution to the Bellman equation
- not trivial to find
- we haven't even demonstrated yet that there exists a function $v(\cdot)$ that will satisfy the Bellman equation
- we will show that the (unique) value function defined by the sequence problem is also the unique solution to the Bellman equation
- once we determine the value function $v(x)$, we can solve for the optimal policy:

$$x_{+1}^* = \arg\max_{x_{+1}\in\Gamma(x)}\{F(x, x_{+1}) + \beta v(x_{+1})\}$$

$\rightarrow$ x_{+1}^* is a function of x: $h(x) \equiv x_{+1}^*$

a solution to the sequence problem is also a solution to the Bellman equation

$$
\begin{aligned}
v(x_0) &= \sup_{x_{+1} \in \Gamma(x)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
&= \sup_{x_{+1} \in \Gamma(x)} \left\{ F(x_0, x_1) + \sum_{t=1}^{\infty} \beta^t F(x_t, x_{t+1}) \right\} \\
&= \sup_{x_{+1} \in \Gamma(x)} \left\{ F(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}) \right\} \\
&= \sup_{x_1 \in \Gamma(x_0)} \left\{ F(x_0, x_1) + \beta \sup_{x_{+1} \in \Gamma(x)} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \right\} \\
&= \sup_{x_1 \in \Gamma(x_0)} \{ F(x_0, x_1) + \beta v(x_1) \}
\end{aligned}
$$

and vice versa: a solution to the Bellman equation is also a solution to the sequence problem

$$
\begin{aligned}
v(x_0) &= \sup_{x_1 \in \Gamma(x_0)} \{F(x_0, x_1) + \beta v(x_1)\} \\
&= \sup_{x_{+1} \in \Gamma(x)} \{F(x_0, x_1) + \beta [F(x_1, x_2) + \beta v(x_2)]\} \\
&= \sup_{x_{+1} \in \Gamma(x)} \left\{F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 [F(x_2, x_3) + \beta v(x_3)]\right\} \\
&\vdots \\
&= \sup_{x_{+1} \in \Gamma(x)} \left\{F(x_0, x_1) + \ldots + \beta^{n-1} F(x_{n-1}, x_n) + \beta^n v(x_n)\right\} \\
&= \sup_{x_{+1} \in \Gamma(x)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})
\end{aligned}
$$

sufficient condition: $\lim_{n\to\infty} \beta^n v(x_n) = 0$ for all feasible x sequences

$\Rightarrow$ put together: a solution of the Bellman equation will also be a solution to the sequence problem and vice versa

Example 2

optimal growth (log utility and Cobb-Douglas):

$$\sup_{\{k_{t+1} c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints: $c_t, k_t > 0$ and $c_t + k_{t+1} = k_t^{\alpha}$, k_0 given

sequence problem formulation

with Bellman equation notation

2.3 Solving the Bellman equation

- three methods

  1. guess a solution and verify
  2. iterate a functional operator analytically
  3. iterate a functional operator numerically

1. in practice: guess a function $v(x)$ and then check to see that this function satisfies the Bellman equation at all possible values of x

Example 3

- for the growth example guess that the solution takes the following form:

$$v(k) = A + B\ln(k)$$

  where A and B are constants for which we need to find solutions
- here the value function inherits the functional form of the utility function (ln)
- to solve for the constants rewrite the Bellman equation as

$$v(k) = \sup_{k_{+1}\in\Gamma(k)} \{\ln(k^\alpha - k_{+1}) + \beta v(k_{+1})\} \quad \forall k$$

$$A + B\ln(k) = \sup_{k_{+1}\in\Gamma(k)} \{\ln(k^\alpha - k_{+1}) + \beta(A + B\ln(k_{+1}))\} \quad \forall k$$

$\rightarrow$ verify that this functional form works and calculate A and B (Problem Set)

- to solve such a problem, we need to use the first order condition (on the right hand side) of the Bellman equation:

$$\frac{\partial F(x, x_{+1})}{\partial x_{+1}} + \beta v'(x_{+1}) = 0$$

- and the envelope theorem

$$v'(x) = \frac{\partial F(x, x_{+1})}{\partial x}$$

- heuristic demonstration of the ET

$$\begin{aligned}
v'(x) &= \frac{\partial F(x, x_{+1})}{\partial x} + \frac{\partial F(x, x_{+1})}{\partial x_{+1}} \frac{dx_{+1}}{dx} + \beta v'(x_{+1}) \frac{dx_{+1}}{dx} \\
&= \frac{\partial F(x, x_{+1})}{\partial x} + \underbrace{\left[\frac{\partial F(x, x_{+1})}{\partial x_{+1}} + \beta v'(x_{+1}) \right]}_{=0 \text{ from the FOC}} \frac{dx_{+1}}{dx} \\
&= \frac{\partial F(x, x_{+1})}{\partial x}
\end{aligned}$$

The FOC and the ET provide the Euler equation

- forwarding the ET by one period we get

$$v'(x_{+1}) = \frac{\partial F(x_{+1}, x_{+2})}{\partial x_{+1}}$$

- plug this into the FOC

$$\frac{\partial F(x, x_{+1})}{\partial x_{+1}} + \beta v'(x_{+1}) = 0$$
$$\frac{\partial F(x, x_{+1})}{\partial x_{+1}} + \beta \frac{\partial F(x_{+1}, x_{+2})}{\partial x_{+1}} = 0$$

- in our growth example $F(x, x_{+1}) = u(k^\alpha - k_{+1}) = u(c)$ so the above becomes

$$u'(c) = \beta \underbrace{\alpha k_{+1}^{\alpha-1}}_{=R_{+1}} u'(c_{+1})$$

Iterative solutions to the Bellman equation

the Bellman equation

$$v(x) = \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta v(x_{+1})\} \quad \forall x$$

the Bellman operator, operating on function w is defined as

$$(\mathbf{B}w)(x) = \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta w(x_{+1})\} \quad \forall x$$

- definition is expressed pointwise (for a value of x), but holds for all values of x
- operator $\mathbf{B}$ maps a function w to a new function $\mathbf{B}w$
- operator $\mathbf{B}$ maps functions, it is referred to as a functional operator
- the argument of $\mathbf{B}$, the function w, is not necessarily a solution to the Bellman equation

- let v be a solution to the Bellman equation
- if $w = v$, then $\mathbf{B}v = v$

$$
\begin{aligned}
(\mathbf{B}v)(x) &= \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta v(x_{+1})\} \quad \forall x \\
&= v(x) \quad \forall x
\end{aligned}
$$

- the function v is a fixed point of $\mathbf{B}$, i.e. $\mathbf{B}$ maps v to v

$\rightarrow$ the iterative solution methods for the Bellman equation pick some v_0 and analytically or numerically iterate $\mathbf{B}^n v_0$ until convergence, until finding a fixed point how do you iterate $\mathbf{B}^n w$?

$$
\begin{aligned}
(\mathbf{B}w)(x) &= \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta w(x_{+1})\} \\
(\mathbf{B}(\mathbf{B}w))(x) &= \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta (\mathbf{B}w)(x_{+1})\} \\
(\mathbf{B}(\mathbf{B}^2 w))(x) &= \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta (\mathbf{B}^2 w)(x_{+1})\} \\
&\vdots \\
(\mathbf{B}(\mathbf{B}^n w))(x) &= \sup_{x_{+1} \in \Gamma(x)} \{F(x, x_{+1}) + \beta (\mathbf{B}^n w)(x_{+1})\}
\end{aligned}
$$

what does it mean for functions to converge?

$$\lim_{n \to \infty} \mathbf{B}^n v_0 = v$$

- informally as n gets large, the set of remaining elements in the sequence of functions
$$\{\mathbf{B}^n v_0, \mathbf{B}^{n+1} v_0, \mathbf{B}^{n+2} v_0, \ldots\}$$
are getting closer and closer
- two functions are close if the maximum distance between the functions is bounded by ε

Contraction mapping and Blackwell's Theorem

- $\mathbf{B}^n w$ converges as $n \to \infty$ because $\mathbf{B}$ is a contraction mapping
- $\mathbf{B}$ is a contraction mapping if operating $\mathbf{B}$ on any two functions moves them closer together
- if $\mathbf{B}$ is a contraction mapping, then
  - $\mathbf{B}$ has exactly one fixed point v
  - for any v_0, $\lim \mathbf{B}^n v_o = v$
  - $\mathbf{B}^n v_0$ has an exponential convergence rate
- Blackwell's Theorem gives sufficient (but not necessary) conditions for $\mathbf{B}$ to be a contraction mapping

$\Rightarrow$ if Blackwell's conditions are satisfied the Bellman equation can be solved using iterative methods