Linear first-order difference equation,

$$x_t = a x_{t-1}.$$

For multivariate difference equation

$$\mathbf{x}_t = F(\mathbf{x}_{t-1})$$

for now we focus on F is linear

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1}$$

suppose $\mathbf{x}$ is K by 1. Then $\mathbf{A}$ needs to be K by K.

No AR(p)

We will *not* generalize this to AR(p)! Because any K-dimensional AR(p) process can be turned into Kp-dimensional AR(1) process.

For $K = 1$ and $p = 2$:

$$x_t = a_1 x_{t-1} + a_2 x_{t-2}$$

introduce a 2-dim vector

$$\mathbf{x}_t := \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix}$$

> We want to get
>
> $$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1}$$

$$Take the 2x2 matrix$$

\mathbf A :=
\begin{bmatrix}
a_1 & a_2\
1 & 0
\end{bmatrix}

$$and note that \$\$\mathbf{x}_{t-1} := \begin{pmatrix} x_{t-1} \\ x_{t-2} \end{pmatrix}$$

so that the second-order DE can be written as

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1}.$$

General KxK case

Iterate forward

$$\mathbf{x}_t = \mathbf{A}^t \mathbf{x}_0$$

how can we characterize this given A?

Do an eigenvalue decomposition. If A is a square matrix, there exists Lambda and invertible Q such that

$$\mathbf{A} = \mathbf{Q} \cdot \mathbf{\Lambda} \cdot \mathbf{Q}^{-1},$$

with $\mathbf{\Lambda}$ a diagonal matrix. Introduce new vector

$$\mathbf{v}_t = \mathbf{Q}^{-1}\mathbf{x}_t$$

Then

$$\mathbf{x}_t = \mathbf{Q}\mathbf{v}_t.$$

For this new vector,

$$\mathbf{Q}\mathbf{v}_t = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{v}_{t-1} = \mathbf{Q}\mathbf{\Lambda}\mathbf{v}_{t-1}.$$

Multiply by Q inverse,

$$\mathbf{v}_t = \mathbf{\Lambda}\mathbf{v}_{t-1}$$

and

$$\mathbf{v}_t = \mathbf{\Lambda}^t\mathbf{v}_0,$$

which is a diagonal matrix. The vector v is a set of independent AR(1) processes.

$$\mathbf{\Lambda}^t = \begin{bmatrix} \lambda_1^t & 0 & 0 & \dots \\ 0 & \lambda_2^t & 0 & \dots \\ 0 & 0 & \lambda_3^t & \dots \\ \dots & & & \end{bmatrix}$$

Λ is unique up to reordering its elements.

Eigenvalues

For an eigenvalue λ_i and corresponding eigenvector $\mathbf{v}_i$,

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

Every K by K matrix has K such vectors and values. finding them:

$$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}.$$

For the matrix in () to be singular

$$\det(\mathbf{A} - \lambda_i\mathbf{I}) = 0.$$

This is a order k polynomial equation in λ with k roots. These are the eigenvalues.

- the same lambda may appear multiple times
- lambda may be complex

Properties

1. Lambdas are unique up to ordering and multiplicity.
2. If $\mathbf{v}$ is an eigenvector, then so is $2\mathbf{v}$. they are only unique up to scaling.