CEU – Advanced Macroeconomics
Problem Set 6 – Solutions

Question 1 – Dynamic Programming Fundamentals

a. The sequence problem formulation is:

$$v(k_0) = \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(Rk_t - k_{t+1})$$

subject to:

$$0 \le k_{t+1} \le Rk_t \text{ for all } t \ge 0$$

$$k_0 \text{ given}$$

b. The Bellman equation is:

$$v(k) = \sup_{0 \le k_{+1} \le Rk} \{\ln(Rk - k_{+1}) + \beta v(k_{+1})\}$$

c. Using the guess $v(k) = A + B\ln(k)$:

i. The first-order condition is:

$$-\frac{1}{Rk - k_{+1}} + \beta \frac{B}{k_{+1}} = 0$$

ii. The envelope condition is:

$$v'(k) = \frac{B}{k} = \frac{R}{Rk - k_{+1}}$$

iii. From the FOC, we can solve for k_{+1}:

$$k_{+1} = \frac{\beta B}{1 + \beta B} Rk$$

Substituting this back into the Bellman equation:

$$A + B\ln(k) = \ln\left(Rk\left(1 - \frac{\beta B}{1+\beta B}\right)\right) + \beta\left(A + B\ln\left(\frac{\beta B}{1+\beta B}Rk\right)\right)$$

Collecting terms:

$$A + B\ln(k) = \ln(R) + \ln(k) + \ln\left(\frac{1}{1+\beta B}\right) + \beta A + \beta B\ln(\beta B) + \beta B\ln(R) + \beta B\ln(k) - \beta B\ln(1+\beta B)$$

Matching coefficients:

$$B = 1 + \beta B \implies B = \frac{1}{1-\beta}$$

$$A = \ln(R) + \ln\left(\frac{1}{1+\beta B}\right) + \beta A + \beta B\ln(\beta B) + \beta B\ln(R) - \beta B\ln(1+\beta B)$$

iv. The optimal policy function for consumption is:

$$c = (1-\beta)Rk$$

d. To verify the transversality condition:

$$\lim_{t\to\infty}\beta^t v'(k_t)k_t = \lim_{t\to\infty}\beta^t\frac{B}{k_t}k_t = \lim_{t\to\infty}\beta^t\frac{1}{1-\beta} = 0$$

since $\beta < 1$.

Question 2 – Properties of the Bellman Operator

a. Let $w_1(x) \leq w_2(x)$ for all x. Then:

$$(Bw_1)(x) = \sup_{x_{+1}\in\Gamma(x)}\{F(x,x_{+1}) + \beta w_1(x_{+1})\} \leq \sup_{x_{+1}\in\Gamma(x)}\{F(x,x_{+1}) + \beta w_2(x_{+1})\} = (Bw_2)(x)$$

The inequality follows because for any x_{+1}, $w_1(x_{+1}) \leq w_2(x_{+1})$, so the supremum must also be smaller.

b. For any constant a:

$$\begin{aligned} B(w+a)(x) &= \sup_{x_{+1}\in\Gamma(x)} \{F(x,x_{+1}) + \beta(w(x_{+1})+a)\} \\ &= \sup_{x_{+1}\in\Gamma(x)} \{F(x,x_{+1}) + \beta w(x_{+1})\} + \beta a \\ &= (Bw)(x) + \beta a \end{aligned}$$

c. These properties are important because:

- Monotonicity ensures that if we start with a function that's below the true value function, iterations will increase it, and if we start above, iterations will decrease it.
- Discounting ensures that the distance between successive iterations shrinks by at least factor β, guaranteeing convergence.

Question 3 – Analytical Value Function Iteration

a. Starting with $v_0(k) = 0$:

i. $(Bv_0)(k) = \sup_{0\le k_{+1}\le k}\{\ln(k-k_{+1}) + \beta\cdot 0\} = \ln(k)$

ii. $(B^2 v_0)(k) = \sup_{0\le k_{+1}\le k}\{\ln(k-k_{+1}) + \beta\ln(k_{+1})\}$

$$= \ln(k) + \beta\ln(\beta k)$$

iii. $(B^3 v_0)(k) = \ln(k) + \beta\ln(\beta k) + \beta^2\ln(\beta^2 k)$

b. The pattern appears to be:

$$(B^n v_0)(k) = \sum_{i=0}^{n-1} \beta^i \ln(\beta^i k)$$

c. As $n\to\infty$, the limit function should be:

$$v(k) = \sum_{i=0}^{\infty} \beta^i \ln(\beta^i k) = \frac{\ln(k)}{1-\beta} + \text{constant}$$

To verify this satisfies the Bellman equation, substitute it back and solve the maximization problem:

$$\frac{\partial}{\partial k_{+1}}\left[\ln(k-k_{+1}) + \beta\left(\frac{\ln(k_{+1})}{1-\beta} + \text{constant}\right)\right] = 0$$

This gives the optimal policy $k_{+1} = \beta k$, which is consistent with our iteration results.