1. Simulate realizations of MC [X]

Break until 14:28

2. More general results about convergence

Remember

$$\pi_t = \mathbf{P}'\pi_{t-1}$$

Stability properties?

1. Steady state: start there, stay there. This always exists, because every stochastic P has the eigenvalue 1 at least once.

$$\pi_* = \mathbf{P}'\pi_*$$

2. Convergence:

$$\pi_\infty = \lim_{t\to\infty} (\mathbf{P}')^t \pi_0$$

```
- does this exist?
- is it independent of $\pi_0$?
- is it the same as $\pi_*$?
```

If so, we call it the ergodic distribution of P. **Not every MC is ergodic.**

> See here for more precise definition.

Two sufficient (but not necessary) conditions for convergence to a unique steady state, $\pi_\infty = \pi_*$.

1. $P_{ij} > 0$ for all i, j.
2. For some $T > 1$, $[\mathbf{P}'^T]_{ij} > 0$ for all i, j

Counterexample:

The identity matrix is not ergodic, $\pi_\infty = \pi_0$.

Eigenvalues

If all eigenvalues are $|\lambda_i| < 1$ for $i = 2, \ldots, K$ then the MC is ergodic and $\pi_\infty = \pi_*$.

3. Parametric MCs

Take x_t: number of emails in your inbox on day t. Given x_t, you may get a new email, or you may delete an email from your inbox with some probability.

$$\Pr(x_t = j | x_{t-1} = i) = \begin{cases} \lambda & \text{if } j = i + 1 \\ \mu & \text{if } j = i - 1 \\ 1 - \lambda - \mu & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

(plus some lower bound if you don't have emails)

$$\Pr(x_t = j | x_{t-1} = 0) = \begin{cases} \lambda & j = 1 \\ 1 - \lambda & j = 0 \\ 0 & j > 1 \end{cases}$$

States: $k = 0, 1, 2, \ldots$ Transition matrix

$$\mathbf{P} = \begin{bmatrix} 1-\lambda & \lambda & 0 & 0 & \ldots \\ \mu & 1-\lambda-\mu & \lambda & 0 & \ldots \\ 0 & \mu & 1-\lambda-\mu & \lambda & \ldots \\ \ldots & & & & \end{bmatrix}$$

What is the steady state of this process?

$$\pi_*' = \pi_*' \mathbf{P}$$

$$\pi_{*0} = (1-\lambda)\pi_{*0} + \mu \pi_{*1}$$

$$\pi_{*1} = \lambda \pi_{*0} + (1 - \lambda - \mu)\pi_{*1} + \mu \pi_{*2}$$

resulting in a second-order difference equation for π_{*n}. I can solve for this recursively.

Birth and death process: this one with a fixed birth rate and death rate.

Steady state PMF is geometrically distributed. For convergence, we need $\lambda/\mu < 1$.

4. Start with dynamic programming