Homogeneous Isotropic Turbulence: Characterization and Two-Point Correlations

So far, we have discussed the kinematic properties of a turbulent flow. We have discussed the stastics of velocity differences in the universal equilibrium range and remarked on the consequences of local homogeneity and isotropy. To study the dynamics of a turbulent flow, we must appeal to the Navier-Stokes equations. Before doing so, we present the two-point correlation tensor for a homogeneous isotropic turbulent flow and discuss how it may be used to characteize isotropic turbulence.

Consider a homogeneous isotropic turbulent flow with zero mean:

$$\bar{u} \equiv 0$$

and let $\sigma = \langle (u_1')^2 \rangle^{1/2} = \langle u_1^2 \rangle^{1/2}$ denote the root-mean-square velocity, which may depend on time (i.e., $\sigma = \sigma(t)$), $\varepsilon(t)$ denote the mean dissipation rate, and ν denote the kinematic viscosity. Since $\bar{u} = 0$, the two-point correlation tensor takes the form:

$$\begin{aligned} R_{ij}(\vec{r},t) &\equiv \langle u_i'(\vec{x}+\vec{r},t)\, u_j'(\vec{x},t) \rangle \\ &= \langle u_i(\vec{x}+\vec{r},t)\, u_j(\vec{x},t) \rangle \end{aligned}$$

and, because of isotropy:

$$R_{ij}(0,t) = \sigma^2 \delta_{ij}$$

In fact, as with the second-order structure function, we may decompose $R_{ij}(\vec{r},t)$ as:

$$R_{ij}(\vec{r},t) = \sigma^2 \left(\underset{\substack{\uparrow \\ \text{Tranverse Autocorrelation}}}{g(r,t)}\, \delta_{ij} + \left[\underset{\substack{\uparrow \\ \text{Longitudinal Autocorrelation}}}{f(r,t)} - g(r,t) \right] \frac{r_i r_j}{r^2} \right)$$

because of isotropy. For $\vec{r} = r\hat{e}_1$:

$$\begin{aligned} R_{11}/\sigma^2 &= f \\ R_{22}/\sigma^2 &= R_{33}/\sigma^2 = g \end{aligned} \qquad R_{ij} = 0 \quad i \neq j$$

so the interpretation of f and g as autocorrelation functions is clear. Also in analogy to the second-order structure function, the autocorrelation functions are related via:

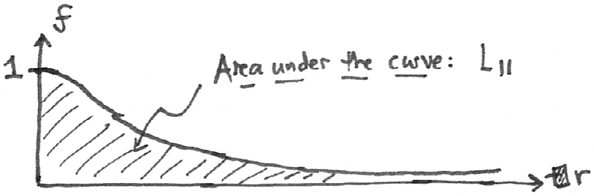
$$g(r,t) = f(r,t) + \frac{1}{2} r \frac{\partial}{\partial r} f(r,t)$$

From the autocorrelation functions, we may deduce length-scales for isotropic turbulence. The first are the integral length-scales:

Longitudinal Integral Scale: $L_{11}(t) \equiv \int_0^\infty f(r,t)\, dr$

Transverse Integral Scale: $L_{22}(t) \equiv \int_0^\infty g(r,t)\, dr$

Visually:

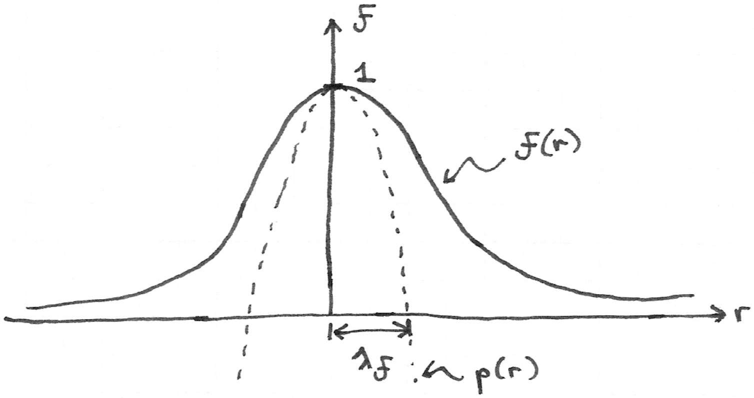


For grid turbulence, the integral length scales increase in time. This is a consequence of decaying turbulence. Indeed, the integral length scales measure how far one must travel before the velocity field at a point $\vec{x}+r$ and the velocity field at $\vec{x}$ are "roughly" uncorrelated.

Integral lengthscales are quite difficult to measure, but we may obtain a second notion of length-scale by examining the structure of the autocorrelation functions. In particular, consider $f(r,t)$ and note that, necessarily, $f(0)=1$, $f'(0)=0$, and $f''(0) \leq 0$. We may define a quadratic function $p(r,t)$ such that $p(0)=f(0)$, $p'(0)=f'(0)$, and $p''(0)=f''(0)$. Notably:

$$p(r,t) = 1 + \tfrac{1}{2} f''(0) r^2$$

and visually:



The location where $p(r,t)$ intercepts the r-axis is the longitudinal Taylor microscale:

$$\lambda_f(t) \equiv \left[-\tfrac{1}{2} f''(0,t)\right]^{-1/2}$$

$\lambda_f(t)$ is a measure of how rapidly the longitudinal two-point correlation separates from its zero-separation value, the covariance. A more physical interpretation is obtained by recognizing that:

$$\lambda_f^2(t) = -\frac{1}{1/2\, f''(0,t)}$$

$$= \frac{\sigma^2}{-1/2\,(R_{11}(0,t))_{,r_1 r_1}} \longrightarrow \frac{\partial^2 R_{11}(0,t)}{\partial r_1\, \partial r_1}$$

$$= \frac{2\langle (u_1)^2 \rangle}{\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle}$$

So $\lambda_f^2(t)$ is a measure of the ratio of the root-mean-square velocity and the root-mean-square normal strain rate. Moreover,

$$\varepsilon = 15\nu \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \quad \text{and} \quad \langle (u_1)^2 \rangle = \sigma^2$$

So:

$$\lambda_f^2(t) = \frac{30\nu\sigma^2}{\varepsilon}$$

is precisely characterized by the viscosity, root-mean-square velocity, and dissipation rate. The transverse Taylor microscale is defined analogously:

$$\lambda_g(t) \equiv \left[-\frac{1}{2}\, g''(0,t)\right]^{-1/2}$$

and it is related to $\lambda_f(t)$ as:

$$\lambda_g(t) = \lambda_f(t)/\sqrt{2}$$

Hence:

$$\varepsilon = 15\nu\sigma^2/\lambda_g^2$$

In a seminal paper marking the start of the study of isotropic turbulence, Taylor defined λ_g and obtained the above equation for ε. He then incorrectly surmised that λ_g may be regarded as a measure of the diameter of the smallest eddies responsible for dissipation of energy. To determine the relationship between the Taylor and Kolmogorov scales, define $L \equiv k^{3/2}/\varepsilon$ to be the lengthscale characterizing the large eddies and the turbulent Reynolds number as:

$$Re_L \equiv \frac{k^{1/2}L}{\nu} = \frac{k^2}{\varepsilon\nu} \quad \text{w/} \quad k = \frac{1}{2}\langle(u_i')^2\rangle = \text{turb. kinetic energy}$$

Then:

$$\lambda_g/L = \sqrt{10}\, Re_L^{-1/2}$$
$$\eta/L = Re_L^{-3/4}$$

and:

$$\lambda_g = \sqrt{10}\, \eta^{2/3} L^{1/3}$$

Clearly, for high-Re flows:

$$\eta \ll \lambda_g \ll L$$

so λ_g is a lengthscale associated with the inertial subrange. The Taylor microscale does not have a clear physical interpretation. Nonetheless, the Taylor-scale Reynolds number

$$Re_\lambda \equiv \frac{\sigma\lambda_g}{\nu} = \left(\frac{20}{3} Re_L\right)^{1/2}$$

is often used to characterize an isotropic turbulent flow such as grid turbulence. An isotropic turbulent flow has an inertial subrange if $Re_\lambda \gtrsim 90$. Finally, observe that the ratio

$$\lambda_g/\sigma = (15\nu/\varepsilon)^{1/2} = \sqrt{15}\, \tau_\eta$$

does correctly characterize the timescale of the smallest eddies, and this equivalence indeed misled Taylor in his interpretation of λ_g.