Joint Random Variables and Joint-Normal Distributions:

So far, we have discussed scalar-valued random variables. We can also discuss vector-valued random variables which are also called joint random variables.

For instance, the velocity $\vec{u} = (u_1, u_2, u_3)$ may be a joint random variable with associated sample-space variable $\vec{v} = (v_1, v_2, v_3)$.

The CDF of the joint random variable $\vec{u}$ is:

$$F(\vec{v}) \equiv P(u_1 < v_1,\ u_2 < v_2,\ u_3 < v_3)$$

which has the properties:

- $F((v_1 + \delta v_1, v_2 + \delta v_2, v_3 + \delta v_3)) \geq F((v_1, v_2, v_3))$
  $\delta v_1, \delta v_2, \delta v_3 \geq 0$
- $F((-\infty, v_2, v_3)) = F((v_1, -\infty, v_3)) = F((v_1, v_2, -\infty)) = 0$
- $F((\infty, v_2, v_3)) = P(u_2 < v_2, u_3 < v_3) = F_{23}(v_2, v_3)$
- $F((\infty, \infty, v_3)) = P(u_3 < v_3) = F_3(v_3)$

CDF of the single-random variable u_3!

Slight abuse of notation

Here, we have abused notation in the sense that F can denote the CDF of a single or joint random variable where the meaning is implicitly understood.

The joint PDF is defined as:

$$f(\vec{v}) \equiv \frac{\partial^3}{\partial v_1 \partial v_2 \partial v_3}\left(F(\vec{v})\right)$$

The joint PDF has the properties:

- $f(\vec{v}) \geq 0$

Marginal PDF of the single-random variable u_3!

- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{v})\, dv_1\, dv_2 = f_3(v_3)$
- $\int_{\Omega} f(\vec{v})\, d\vec{v} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{v})\, dv_1\, dv_2\, dv_3 = 1$
- $P(v_{1a} \leq u_1 < v_{1b},\ v_{2a} \leq u_2 < v_{2b},\ v_{3a} \leq u_3 < v_{3b})$

$$= \int_{v_{3a}}^{v_{3b}}\int_{v_{2a}}^{v_{2b}}\int_{v_{1a}}^{v_{1b}} f(\vec{v})\, d\vec{v}$$

Joint sample space!

If $Q(\vec{u})$ is a function of $\vec{u}$, its mean is defined by:

$$\langle Q(\vec{u}) \rangle = \int_{\Omega} Q(\vec{v})\, f(\vec{v})\, d\vec{v}$$

Means and variances of individual components can be computed using the above expression. For example:

$$\langle u_i \rangle = \int_{\Omega} v_i\, f(\vec{v})\, d\vec{v} \qquad i = 1,2,3$$

$$\langle (u_i')^2 \rangle = \int_{\Omega} (v_i - \langle u_i \rangle)^2 f(\vec{v})\, d\vec{v} \qquad i = 1,2,3$$

The mixed second moments of the joint random variable $\vec{u}$ are stored in the covariance tensor:

$$\Gamma_{ij} = \langle u_i' u_j' \rangle = \int_{\Omega} (v_i - \langle u_i \rangle)(v_j - \langle u_j \rangle)\, f(\vec{v})\, d\vec{v}$$

The covariance may also be written as:

$$\mathrm{cov}(u_i, u_j) = \Gamma_{ij}$$

Note that Γ_{11}, Γ_{22}, and Γ_{33} give the variances $\langle (u_1')^2 \rangle$, $\langle (u_2')^2 \rangle$, and $\langle (u_3')^2 \rangle$.

The correlation coefficient is defined as:

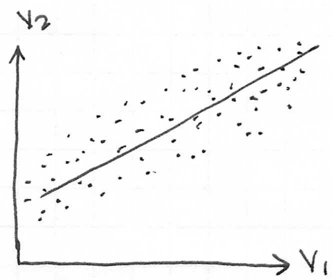
$$\rho_{ij} \equiv \frac{\langle u_i' u_j' \rangle}{\left[\langle (u_i')^2 \rangle \langle (u_j')^2 \rangle\right]^{1/2}} = \frac{\Gamma_{ij}}{\left[\Gamma_{ii}\, \Gamma_{jj}\right]^{1/2}}$$

A positive correlation coefficient arises when positive fluctuations for one random variable are preferentially associated with positive fluctuations for the other. It may be easily shown that:

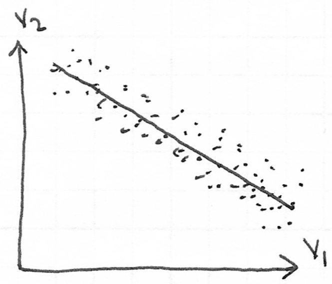
$$-1 \leq \rho_{ij} \leq 1$$

We have three extremes:

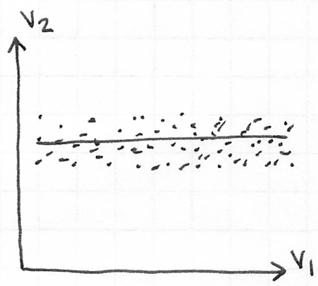
1) $\rho_{ij} = 0$: The random variables u_i and u_j are uncorrelated.
2) $\rho_{ij} = 1$: The random variables u_i and u_j are perfectly correlated.
3) $\rho_{ij} = -1$: The random variables u_i and u_j are perfectly negatively correlated.



Positive Correlation



Negative Correlation



No Correlation

The marginal PDFs are defined as:

$$f_{ij}(v_i, v_j) = \int_{-\infty}^{\infty} f(\vec{v})\, dv_k \qquad i \neq j,\ i \neq k,\ j \neq k$$

$$f_i(v_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{v})\, dv_j\, dv_k \qquad i \neq j,\ i \neq k,\ j \neq k$$

I will also denote:

$$f_{123}(v_1, v_2, v_3) = f(\vec{v})$$

The PDF of (u_2, u_3) conditional on $u_1 = v_1$ is:

$$f_{23|1}(v_2, v_3 \mid v_1) \equiv \frac{f(\vec{v})}{f_1(v_1)}$$

Note that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{23|1}(v_2, v_3 \mid v_1)\, dv_2\, dv_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{f(\vec{v})}{f_1(v_1)}\right) dv_2\, dv_3$$

$$= 1$$

by construction.

Consequently, the conditional PDF is a PDF in the standard sense. We can define other conditional PDFs analogously.

The conditional mean is defined as:

$$\langle Q(\vec{u}) \mid u_1 = v_1 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q((v_1, v_2, v_3))\, f_{23|1}(v_2, v_3 \mid v_1)\, dv_2\, dv_3$$

for the conditional PDF defined above. Note that $\langle Q(\vec{u}) \mid u_1 = v_1 \rangle$ is the mean of $Q(\vec{u})$ conditional on $u_1 = v_1$.

Two or more random variables are independent if knowledge of the value of any of them provides no information about the others. Consequently:

$$f_{23|1}(v_2, v_3 \mid v_1) = f_{23}(v_2, v_3) \quad \text{if } u_1, u_2, u_3 \text{ are independent}$$

and:

$$f(\vec{v}) = f_1(v_1)\, f_2(v_2)\, f_3(v_3) \quad \text{if } u_1, u_2, u_3 \text{ are independent}$$

Independent random variables are uncorrelated, but uncorrelated random variables are not necessarily independent.

The concept of a joint random variable extends to a general set of D random variables:

$$\vec{u} = (u_1, u_2, \ldots, u_D)$$

Let us define:

$$\vec{\mu} = \langle \vec{u} \rangle \quad \text{"The Mean"}$$

$$\vec{u}' = \vec{u} - \langle \vec{u} \rangle \quad \text{"The Fluctuation"}$$

The covariance matrix then takes the form:

$$\overset{\rightrightarrows}{C} = \langle (\vec{u}')(\vec{u}')^T \rangle$$

If $\vec{u}$ is joint-normally distributed, then its joint PDF is:

$$f(\vec{v}) = \left[(2\pi)^D \det(\overset{\rightrightarrows}{C}) \right]^{-1/2} \exp\left[-\tfrac{1}{2} (\vec{v}-\vec{\mu})^T \overset{\rightrightarrows}{C}^{-1} (\vec{v}-\vec{\mu}) \right]$$

Joint-normally distributed random variables have the following properties:

- Their marginal and conditional PDFs are Gaussian.
- If the components are uncorrelated, they are independent.

However, joint-normally distributed random variables have much more structure! In particular, it is symmetric and positive definite, meaning it admits the eigendecomposition:

$$\overset{\rightrightarrows}{C} = \overset{\rightrightarrows}{A}\, \overset{\rightrightarrows}{\Lambda}\, \overset{\rightrightarrows}{A}^T$$

Diagonal

Unitary: $\overset{\rightrightarrows}{A}^{-1} = \overset{\rightrightarrows}{A}^T$

The eigenvalues of $\overset{\rightrightarrows}{C}$ line the diagonal of $\overset{\rightrightarrows}{\Lambda}$ and the columns of $\overset{\rightrightarrows}{A}$ are the eigenvectors of $\overset{\rightrightarrows}{C}$.

The transformed random variable:

$$\hat{u}' \equiv \overset{\rightrightarrows}{A}^T (\vec{u}')$$

thus has a diagonal covariance matrix:

$$\begin{aligned} \hat{C} &= \langle (\hat{u}')(\hat{u}')^T \rangle \\ &= \langle \overset{\rightrightarrows}{A}^T (\vec{u}')(\vec{u}')^T \overset{\rightrightarrows}{A} \rangle \\ &= \overset{\rightrightarrows}{A}^T \overset{\rightrightarrows}{C} \overset{\rightrightarrows}{A} \\ &= \overset{\rightrightarrows}{\Lambda} \end{aligned}$$

Consequently:

(i) If $\vec{u}$ is the velocity vector, $\hat{u}'$ is the fluctuating velocity in the coordinate system defined by the eigenvectors of the covariance matrix.

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(ii) The fluctuating velocities $\{\hat{u}_1', \hat{u}_2', \ldots, \hat{u}_D'\}$ are uncorrelated.

(iii) The fluctuating velocities $\{\hat{u}_1', \hat{u}_2', \ldots, \hat{u}_D'\}$ are independent Gaussian random variables.

The third property above relied on the fact that $\vec{u}$ is joint normal.