

## Turbulent - Viscosity Models - Near-Wall Treatments

In a turbulent flow, the presence of a wall causes a number of different effects. For example, for a boundary layer, these effects are:

- (i) Low Re: The turbulence Reynolds number  $Re_L \equiv k^2 / \varepsilon y$  tends to zero as the wall is approached.
- (ii) High Shear Rate: The highest mean shear rate  $\frac{\partial \bar{u}}{\partial y}$  occurs at the wall.
- (iii) Two-Component Turbulence: For small  $y$ ,  $\langle (v')^2 \rangle$  varies as  $y^4$  while  $\langle (u')^2 \rangle$  and  $\langle (w')^2 \rangle$  vary as  $y^2$ . This leads to significant anisotropy.
- (iv) Wall Blocking: Through the pressure field, the impermeability condition affects the flow up to an integral scale from the wall.

These effects necessitate modifications to the basic  $k-\varepsilon$  turbulence model. Near-wall treatment has been an area of significant research interest for the last couple of decades and continues to be today.

Since the turbulence Reynolds number goes to zero at the wall, the turbulent time scale  $T = k/\varepsilon$  also goes to zero at the wall. Consequently, the production and dissipation model terms in the model transport equation for  $\varepsilon$ , namely:

$$C_{\varepsilon 1} \frac{P\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} = \frac{1}{T} (C_{\varepsilon 1} P - C_{\varepsilon 2} \varepsilon)$$

become singular at the wall, causing all sorts of problems. The underlying issue is the assumption of local high Reynolds number and the resulting assumption that viscosity is not important. However, there is a viscous time scale:

$$\tau_v = \sqrt{\nu/k}$$

that becomes important near the wall. This inspites the alternate model time scale:

$$T = \max \left( \frac{k}{\varepsilon}, C_T \tau_v \right)$$

where  $\tau_v = \sqrt{\nu/k}$  and  $C_T$  is a model constant often taken as  $C_T = 6.0$ .

In HW4 Part 1, it was shown that the mixing-length specification  $l_m = Ky$  is too large in the near-wall region (say  $y^+ < 30$ ), and a much improved specification is obtained from the van Driest damping function. It is perhaps of little surprise then that the  $k-\varepsilon$  specification  $\nu_T = C_M k^2/\varepsilon$  is also too large in the near-wall region, and the figure:

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demonstrates this succinctly. The solid line in the figure is the viscosity constructed from DNS data (Moser 1999) by evaluating the exact definition  $\nu_T = -\langle u'v' \rangle \frac{\partial \bar{u}}{\partial y}$ . It is compared with curves constructed by inserting the exact  $k$  and  $\varepsilon$  into the model  $\nu_T = C_M k^2/\varepsilon$ . The model formula is found to be grossly inaccurate for  $y^+ < 50$  for both  $C_M = 0.09$  and the alternate choice  $C_M = 0.08$  which results in an improved fit far from the wall.

A physical understanding of the failure is that  $k$  is the wrong velocity scale for transport to and from the wall. Near the surface,  $k \approx \frac{1}{2} \langle (u')^2 + (w')^2 \rangle$  since  $\langle (v')^2 \rangle \ll y^4$ . However, turbulent transport to or from the surface is more closely related to the normal

Component  $\langle (v')^2 \rangle$ . We will later discuss the  $v^2 - f$  model which exploits this very fact. One device to fix the problem at hand is simply to damp the viscosity as in the Van Driest mixing-length formula. To this end, we write:

$$\nu_T = f_\mu C_\mu \frac{k^2}{\epsilon}$$

where  $f_\mu$  is a damping function satisfying  $f_\mu \approx 1$  far away from the wall. Typically,  $f_\mu$  is a function of  $y k / v$  or  $k^2 / (\epsilon v)$ . The original model of Jones and Launder (1972) was:

$$f_\mu = \exp \left[ \frac{-3.4}{(1 + Re_L/50)^2} \right]$$

As if the situation was not complicated enough, it is not sufficient to only damp the eddy viscosity. All low  $Re$   $k-\epsilon$  models also modify the  $\epsilon$  equation in one way or another, usually through the addition of ad hoc terms.

The original damping model of Jones and Launder is actually quite inaccurate. A large number of alternatives have been proposed since, mostly by fitting  $f_\mu$  to data. In 1993, Rodi and Mansour evaluated these proposals and instead suggested the relation:

$$f_\mu = 1 - \exp(-0.0002 y^+ - 0.00065 y^{+2})$$

purely from empirical considerations. Since this relation has no connection to physics, there is no reason to suppose it is accurate for other flows, for example those near a separation or reattachment point.

Another method to circumvent the erroneous predictions in the near-wall region is to entirely abandon the  $k-\epsilon$  equations in a zone near the wall and impose boundary conditions at the top of the zone. This is the "wall function method." Conceptually, the wall function is used in the law-of-the-wall region and the  $k-\epsilon$  model predicts the flow field farther from the surface. For high  $Re$ , zero-pressure-gradient boundary layers, the log-law is:

$$\bar{u} = u_{\tau} \left( \frac{1}{k} \ln y^+ + B \right)$$

and the balance of production and dissipation yields:

$$\epsilon = \frac{u_{\tau}^3}{k y}$$

Then, the  $k-\epsilon$  expression for the turbulent viscosity gives:

$$-\langle u'v' \rangle = u_{\tau}^2 = C_\mu \frac{y}{k}$$

We define the nominal friction velocity as:

$$u_{\tau}^* \equiv C_\mu^{1/4} k^{1/2}$$

where  $k_p$  is the turbulent kinetic energy at a location  $y = y_p$  in the log-law region (e.g., where the dimensionless  $y^+$  is around 50). The wall-function approach consists in applying boundary conditions at  $y^+$ . From above, our estimate for  $y_p^+$  satisfies:

$$y_p^* = \frac{y_p u_{\tau}^*}{\nu} \approx \frac{y_p u_{\tau}}{\nu} = y_p^+$$

The nominal mean velocity is then:

$$\bar{u}_p^* = u_p^* \left( \frac{1}{k} \ln y_p^* + B \right)$$

However, applying the above boundary condition is not a robust approach, especially in the presence of a separation or reattachment point where  $\bar{u}_p^* \rightarrow 0$ . So, instead, the shear stress boundary condition:  $\hookrightarrow$  but  $u_p^* \neq 0$ !

$$-\langle u'v' \rangle_p = u_p^{*2} \frac{\bar{u}_p}{\bar{u}_p^*}$$

is applied in practice. This is a robust condition in that the shear stress is of the opposite sign to the velocity  $\bar{u}_p$ , everything is well-defined at a separation or reattachment point, and if  $\bar{u}_p > \bar{u}_p^*$ , then  $-\langle u'v' \rangle_p > u_p^{*2}$  and hence provides a "restoring force." The usual boundary conditions for  $\varepsilon$  and  $k$  are:

$$\varepsilon_p = \frac{u_p^{*2}}{K y_p} \quad (\text{inspired by } \varepsilon = \frac{u_p^3}{K y})$$

$$\frac{dk}{dy} = 0$$

In practice,  $y_p$  is taken to be the first grid node away from the wall.

The simplifications and savings in computational cost provided by wall functions are very attractive, and hence they are widely used in the application of commercial CFD to complex turbulent flows. However, under many flow configurations - Strong pressure gradients, separated and impinging flows - their physical basis is uncertain and their accuracy is poor.

As a practical manner, it is sometimes impossible to ensure that the first point of a computational grid lies in the log layer, if a log layer exists at all. The definition of  $y_p^*$  requires  $\bar{u}_p^*$ , but that is computed as part of our flow solution. It is not possible *a priori* to generate a mesh that will ensure conformity with the log-law requirement. If the first node is too close to the wall, then the  $k-\varepsilon$  model will severely overpredict  $\varepsilon$ . In boundary-layer flows, the tendency will then be to overpredict surface skin friction. Accurate computation may require *a posteriori* modification of the mesh to achieve a

A compromise between the assertion of a universal wall layer, as made by wall function methods, and a full simulation of the wall adjacent region is to formulate a simplified model for that region and patch it on to the  $k-\varepsilon$  model. The full model is solved in the outer region. The  $k-l$  formulation has been used to this end, in an approach known as the "two-layer  $k-\varepsilon$  model".

The  $k-l$  model uses the  $k$ -equation but replaces the  $\varepsilon$ -equation with:

$$\varepsilon = k^{3/2}/l_\varepsilon$$

where the dissipation length  $l_\varepsilon$  must be prescribed. Most simply, it can be made analogous to the mixing length:

$$l_\varepsilon = C_l y \left( 1 - e^{-y \sqrt{k}/(\nu A_\varepsilon)} \right)$$

The  $k$  equation near a surface then becomes:

$$\nu \frac{d^2 k}{dy^2} = \varepsilon = \frac{k^{3/2}}{l_\varepsilon} \approx \frac{\nu A_\varepsilon k}{C_l y^2} \quad (\text{Taylor series})$$

The correct behavior follows from the boundary condition:

$$\begin{aligned} \varepsilon|_{\text{wall}} &= \nu \frac{\partial^2 k}{\partial y^2}|_{\text{wall}} \\ \Rightarrow k &= A + B y + \frac{\varepsilon|_{\text{wall}} y^2}{2\nu} \quad \text{Near the Wall} \\ \text{as } &\quad \text{as } \\ k|_{\text{wall}} = 0 & \quad \frac{\partial k}{\partial y}|_{\text{wall}} = 0 \\ \therefore k &\equiv \frac{\varepsilon|_{\text{wall}}}{2\nu} y^2 \quad \text{Near the Wall!} \end{aligned}$$

Thus, for the  $k-l$  equation to be consistent with the boundary conditions near a surface:

$$A_\varepsilon = 2C_l =$$

It remains to specify the turbulent viscosity. Unfortunately, the turbulent viscosity will not have the right damping if we directly use  $\nu_T = C_\mu k^{3/2}/\varepsilon$  since we would obtain  $\nu_T = \sqrt{k} l_\nu$ . Therefore we use a separate length in the formula:

$$l_\nu = C_l y \left(1 - e^{-y\sqrt{k}/(\nu A_\nu)}\right)$$

Suggested the turbulent viscosity:

$$\nu_T = C_\mu \sqrt{k} l_\nu$$

In the log-law region,  $k^+ = 1/\sqrt{C_\mu}$  and  $\nu_T^+ = K y^+$ . Hence, for consistency, it is required that:

$$C_l = K/C_\mu^{3/4}$$

when the limit  $y\sqrt{k}/\nu \rightarrow 1$  is invoked. Given the von Karman constant  $K = 0.41$  and assuming  $C_\mu = 0.09$ , we find  $C_l = 2.50$ . Consequently, the only new empirical constant is  $A_\nu$ . This can be found by selecting a value that yields  $B \approx 5$  for the additive constant in the log law. Alternatively, we may choose the value of  $A_\nu$  that mostly closely reproduces the curve of skin friction versus  $Re$  in a flat-plate boundary layer. The value  $A_\nu = 62.5$  gives a good fit in the latter case and is a common selection.

In summary, the  $k-l$  model is:

$$\text{Reynolds Stress Model: } \langle u'_i u'_j \rangle = -2\nu_T \bar{S}_{ij} + \frac{2}{3} k \delta_{ij}$$

$$\text{Turbulent Viscosity: } \nu_T = C_\mu \sqrt{k} l_\nu$$

$$\text{Model Equation for } k: \quad \frac{Dk}{Dt} = \vec{\nabla} \cdot ((\nu + \frac{\nu_T}{C_k}) \vec{\nabla} k) + P - \varepsilon$$

$$\text{Model Equation for } \varepsilon: \quad \varepsilon = k^{3/2}/l_\varepsilon$$

$$\begin{aligned} \text{Length Scales:} \quad l_\nu &= C_l y \left(1 - \exp(-y\sqrt{k}/(\nu A_\nu))\right) \\ l_\varepsilon &= C_\varepsilon y \left(1 - \exp(-y\sqrt{k}/(\nu A_\varepsilon))\right) \end{aligned}$$

$$\text{Constants: } C_l = 2.50, C_\mu = 0.09, A_\nu = 62.5, A_\varepsilon = 2C_l$$

The two-layer  $k-\varepsilon$  model consists of the above  $k-l$  model near the wall and the standard  $k-\varepsilon$  model farther away. The model transitions at a specific location  $y_{sw}$ . For example, the model may be chosen to transition when  $1 - \exp(-y\sqrt{k}/(\nu A_y)) \approx 0.95$ . For a turbulent flat-plate boundary layer, this occurs when:

$$\text{or} \\ \text{other turbulent flow!} \quad y_{sw} = \log(20) \frac{A_y \nu}{\sqrt{k(y_{sw})}} \text{ is Function of Re}$$

Succinctly, the two-layer model solves the  $k$  equation at all points in the flow, but the  $\varepsilon$  equation is instead represented by:

$$\mathcal{L}[\varepsilon] = \phi$$

where:

$$\mathcal{L} = \begin{cases} \frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \left( (\nu + \frac{\nu_T}{C_\varepsilon}) \frac{\partial \varepsilon}{\partial x_j} \right) & \text{for } y > y_{sw} \\ 1 & \text{for } y \leq y_{sw} \end{cases}$$

and:

$$\phi = \begin{cases} \frac{C_{\varepsilon_1} P - C_{\varepsilon_2} \varepsilon}{T} & \text{for } y > y_{sw} \\ \frac{k^{3/2}}{l_\varepsilon} & \text{for } y \leq y_{sw} \end{cases}$$

Similarly, the turbulent viscosity is represented by:

$$\nu_T = \begin{cases} C_\mu \frac{k^2}{\varepsilon} & \text{for } y > y_{sw} \\ C_\mu \sqrt{k} l_\varepsilon & \text{for } y \leq y_{sw} \end{cases}$$

This two-layer formulation has proved quite effective in practical computations and usually gives better performance and predictive power than wall models. In the figure:

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We have compared the accuracy of the two-layer  $k-\varepsilon$  model and the low Re  $k-\varepsilon$  model of Jones and Launder (1972). The two models have been applied to the flow over a backward-facing step at a Reynolds number of  $Re_H = 35,700$  where  $Re_H$  is the Reynolds number based on the step height  $H$ . The two-layer model is in rather good accord with the data. While the low Re  $k-\varepsilon$  model predicts the correct location of the reattachment point, the magnitude of the minimum skin friction  $C_f$  is vastly overpredicted and the recovery of the wall shear stress following reattachment is much too rapid.

The one primary drawback of the  $k-\varepsilon$  model as compared to the wall function approach is that it requires a fine grid near to walls and so it is more computationally expensive.