Algebraic Stress and Nonlinear Viscosity Models:

By introducing an approximation for the acceleration and transport terms in a Reynolds stress model, a set of differential equations is reduced to a set of algebraic equations. These equations form an algebraic stress model (ASM), which implicitly defines the Reynolds stresses as functions of k, ε, and the mean velocity gradient. Algebraic stress models are inherently less accurate than Reynolds stress transport models, but they are simpler and less expensive.

A standard model Reynolds stress transport equation is:

$$D_{ij} \equiv \frac{\bar{D}\langle u_i'u_j'\rangle}{\bar{D}t} - \frac{\partial}{\partial x_k}\left(\left(\nu\delta_{kl} + \left(\frac{C_s k}{\varepsilon}\right)\langle u_k'u_l'\rangle\right)\frac{\partial}{\partial x_l}\langle u_i'u_j'\rangle\right)$$

$$= P_{ij} + R_{ij} - \tfrac{2}{3}\varepsilon\delta_{ij}$$

In algebraic stress models, the transport terms on the left-hand-side are approximated by an algebraic expression. To this effect, Rodi (1972) introduced the weak-equilibrium assumption. The Reynolds stress can be decomposed as:

$$\langle u_i'u_j'\rangle = k\,\frac{\langle u_i'u_j'\rangle}{k} = k\left(2b_{ij} + \tfrac{2}{3}\delta_{ij}\right)$$

In the weak-equilibrium assumption, variations in b_{ij} are neglected but variations in k are maintained. This results in the approximation:

$$D_{ij} \approx \frac{\langle u_i'u_j'\rangle}{k}\,\frac{1}{2}D_{ll} = \frac{\langle u_i'u_j'\rangle}{k}(P-\varepsilon)$$

Plugging this approximation into the Reynolds stress transport equation yields the algebraic stress model:

$$\frac{\langle u_i'u_j'\rangle}{k}(P-\varepsilon) = P_{ij} + R_{ij} - \tfrac{2}{3}\varepsilon\delta_{ij} \qquad \text{Algebraic Stress Model}$$

The algebraic stress model is closed by choosing an appropriate model for R_{ij}. The above comprises five independent algebraic equations (since the trace contains no information) which can be used to determine $\langle u_i'u_j'\rangle/k$ in terms of k, ε, and $\partial\bar{u}_i/\partial x_j$.

The algebraic stress model is an implicit equation for the Reynolds stresses, or equivalently for the anisotropy b_{ij}. Clearly, it would be beneficial to have an explicit relation of the form:

$$b_{ij} = B_{ij}(\hat{S},\hat{\Omega}) \qquad \text{Nonlinear Viscosity Model}$$

where $\hat{S}$ and $\hat{\Omega}$ are the normalized mean rate-of-strain and rotation tensors. Such explicit models are referred to as nonlinear viscosity models. The most general form of a nonlinear viscosity model is:

$$B_{ij}(\hat{S},\hat{\Omega}) = \sum_{n=1}^{10} G^{(n)}\hat{T}_{ij}^{(n)}$$

where:

$$\hat{T}^{(1)} = \hat{S}$$
$$\hat{T}^{(2)} = \hat{S}\hat{\Omega} - \hat{\Omega}\hat{S}$$

$$
\begin{aligned}
\hat{T}^{(3)} &= \hat{S}^2 - \tfrac{1}{3}\,\mathrm{tr}(\hat{S}^2)\,I \\
\hat{T}^{(4)} &= \hat{\Omega}^2 - \tfrac{1}{3}\,\mathrm{tr}(\hat{\Omega}^2)\,I \\
\hat{T}^{(5)} &= \hat{\Omega}\hat{S}^2 - \hat{S}^2\hat{\Omega} \\
\hat{T}^{(6)} &= \hat{\Omega}^2\hat{S} + \hat{S}\hat{\Omega}^2 - \tfrac{2}{3}\,\mathrm{tr}(\hat{S}\hat{\Omega}^2)\,I \\
\hat{T}^{(7)} &= \hat{\Omega}\hat{S}\hat{\Omega}^2 - \hat{\Omega}^2\hat{S}\hat{\Omega} \\
\hat{T}^{(8)} &= \hat{S}\hat{\Omega}\hat{S}^2 - \hat{S}^2\hat{\Omega}\hat{S} \\
\hat{T}^{(9)} &= \hat{\Omega}^2\hat{S}^2 + \hat{S}^2\hat{\Omega}^2 - \tfrac{2}{3}\,\mathrm{tr}(\hat{S}^2\hat{\Omega}^2)\,I \\
\hat{T}^{(10)} &= \hat{\Omega}\hat{S}^2\hat{\Omega}^2 - \hat{\Omega}^2\hat{S}^2\hat{\Omega}
\end{aligned}
$$

Like b_{ij}, each tensor $\hat{T}^{(n)}$ is non-dimensional, symmetric, and deviatoric, and in fact every symmetric deviatoric second-order tensor formed from $\hat{S}$ and $\hat{\Omega}$ can be expressed as a linear combination of these ten tensors. This is a result of the Cayley-Hamilton theorem.

If $G^{(1)} = -C_\mu$ and $G^{(n)} = 0$ for $n > 1$, the linear k-ε turbulent viscosity formula is recovered. For flows that are statistically two-dimensional, only the tensors $\hat{T}^{(1)}$, $\hat{T}^{(2)}$, and $\hat{T}^{(3)}$ are required, resulting in a dramatically simpler model. Furthermore, the term in $\hat{T}^{(3)}$ can be absorbed into a modified pressure term. This leads to the simplified model:

$$\langle u_i' u_j' \rangle - \tfrac{2}{3} k \delta_{ij} = 2 G^{(1)} \frac{k^2}{\varepsilon} \bar{S}_{ij} + 2 G^{(2)} \frac{k^3}{\varepsilon^2} \left(\bar{S}_{ik} \bar{\Omega}_{kj} - \bar{\Omega}_{ik} \bar{S}_{kj} \right)$$

One way to obtain a suitable specification of the coefficients $G^{(n)}$ is from an algebraic stress model. For example, for statistically two-dimensional flows, the coefficients associated with the basic model are:

$$G^{(1)} = -C_\mu, \quad G^{(2)} = -\lambda C_\mu, \quad G^{(3)} = 2\lambda C_\mu, \quad G^{(4)} - G^{(10)} = 0$$

where:

$$\lambda \equiv \frac{1 - C_2}{C_R - 1 + P/\varepsilon}, \quad C_\mu \equiv \frac{\tfrac{2}{3}\lambda}{1 - \tfrac{2}{3}\lambda^2 \hat{S}_{ii}^2 - 2\lambda^2 \hat{\Omega}_{ii}^2}$$

This nonlinear viscosity model is not completely explicit, but Girimaji (1996) devised an explicit scheme by solving the implied equation for λ. Taulbee (1992) extended this approach to three-dimensional flows where, in general, all ten coefficients $G^{(n)}$ are nonzero.

Additional nonlinear viscosity models have been proposed by Yoshizawa (1984), Speziale (1987), Rubinstein and Barton (1990), Craft, Launder, and Suga (1996), and others. Nonlinear viscosity models have received considerable interest as they offer better predictions than linear models at a much cheaper cost than Reynolds stress transport models.