The Normal Distribution:

Of fundamental importance in probability theory is the normal or Gaussian distribution. If u is normally distributed with mean μ and standard deviation σ, then the PDF of u is:

$$f(v) = \mathcal{N}(v;\mu,\sigma^2) \equiv \frac{1}{(2\sigma^2\pi)^{1/2}} \exp\left[-\frac{1}{2}\frac{(v-\mu)^2}{\sigma^2}\right]$$

The characteristic function is given by:

$$\Psi(s) = \exp[i\mu s]\exp[-\tfrac{1}{2}\sigma^2 s^2]$$

Gaussian random variables have important properties, and it is common to compare turbulence statistics to those of a Gaussian random variable (i.e., how close to Gaussian is a particular statistic?).

If u is equal in distribution to a normal random variable, we write:

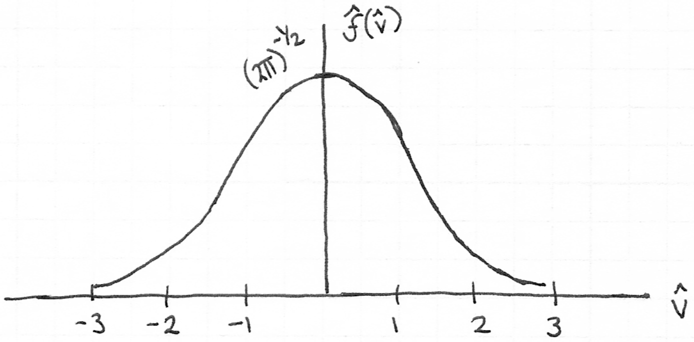
$$u \overset{D}{=} \mathcal{N}(\mu,\sigma^2) \equiv \mathcal{N}(v;\mu,\sigma^2)$$

and:

$$\hat{u} \equiv \frac{(u-\mu)}{\sigma}$$

is a standardized Gaussian random variable with PDF:

$$\hat{f}(\hat{v}) = \mathcal{N}(\hat{v};0,1) = \frac{1}{\sqrt{2\pi}}e^{-\hat{v}^2/2}$$



The moments of u take the form:

$$\mu_{2k} = \frac{(2k)!}{2^k k!}\sigma^{2k} \qquad k=0,1,2,\ldots$$

$$\mu_{2k+1} = 0 \qquad k=0,1,2,\ldots$$

So:

$$\mu_0 = 1,\ \mu_1 = 0,\ \mu_2 = \sigma^2,\ \mu_3 = 0,\ \mu_4 = 3\sigma^4$$

and the skewness and flatness of a normally distributed random variable u are:

$$S(u) = 0 \quad \& \quad K(u) = 3$$

The normal distribution plays an important role when discussing ensemble averages.
Let $u^{(i)}$ denote the i^{th} observation of u, a component of velocity at a particular position and time. Each observation is itself a random variable.

Suppose the observations $\{u^{(1)}, u^{(2)}, u^{(3)}, \ldots\}$ are independent and have the same distribution. That is, they are independent and identically distributed (i.i.d.). The ensemble average,

$$\langle u \rangle_N = \frac{1}{N} \sum_{i=1}^{N} u^{(i)},$$

is then itself a random variable with mean and variance:

$$\langle \langle u \rangle_N \rangle = \langle u \rangle$$

$$\text{var}(\langle u \rangle_N) = \frac{1}{N} \text{var}(u)$$

Note as $N \to \infty$, $\text{var}(\langle u \rangle_N) \to 0$. Consequently, the ensemble average converges to the mean with zero variance.

Define:

$$\hat{u} = \frac{[\langle u \rangle_N - \langle u \rangle] N^{1/2}}{\sigma_u}$$

Note that:

$$\langle \hat{u} \rangle = 0$$

$$\langle ((\hat{u})')^2 \rangle = 1$$

As $N \to \infty$, the PDF of $\hat{u}$ tends to the standardized normal distribution. This is the central limit theorem.

In other words, the computed values of the ensemble average will be distributed according to the normal distribution for a sufficiently large number of observations.