Means, Moments, and the Characteristic Function:

The mean or expected value of the random variable u is defined by:

$$\langle u \rangle \equiv \int_{\Omega_u} v\, f(v)\, dv$$

(This is often written as $\bar{u}$.)

If $Q(u)$ is a function of the random variable u, then its mean is:

$$\langle Q(u) \rangle \equiv \int_{\Omega_u} Q(v)\, f(v)\, dv$$

The fluctuation of u is defined as:

$$u' \equiv u - \langle u \rangle = u - \bar{u}$$

The mean and fluctuation have several important properties:

- $\langle \langle u \rangle \rangle = \langle u \rangle$
- $\langle a u + b w \rangle = a \langle u \rangle + b \langle w \rangle$ if a & b are constants
- $\langle a Q(u) + b R(u) \rangle = a \langle Q(u) \rangle + b \langle R(u) \rangle$ if a & b are constants
- $\langle u' \rangle = 0$
- $\left\langle \frac{du(y)}{dy} \right\rangle = \frac{d \langle u(y) \rangle}{dy}$ if y is not a random variable!

The variance is defined to be the mean-square fluctuation:

$$\mathrm{var}(u) \equiv \langle (u')^2 \rangle = \int_{\Omega_u} (v - \langle u \rangle)^2 f(v)\, dv$$

and the standard deviation is the r.m.s. (root mean square) of u:

$$\mathrm{stdev}(u) \equiv \sqrt{\mathrm{var}(u)} = \langle (u')^2 \rangle^{1/2}$$

We will often denote the standard deviation as σ_u.

The n^{th} central moment is defined to be:

$$\mu_n \equiv \langle (u')^n \rangle = \int_{\Omega_u} (v - \langle u \rangle)^n f(v)\, dv$$

Note that $\mu_0 = 1$, $\mu_1 = 0$, and $\mu_2 = \mathrm{var}(u)$.

It is often convenient to work with standardized random variables, which by definition have zero mean and unit variance.

The standardized random variable $\hat{u}$ corresponding to u is:

$$\hat{u} \equiv \frac{u - \langle u \rangle}{\underbrace{\mathrm{stdev}(u)}_{\sigma_u}}$$

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The PDF of $\hat{u}$ - or equivalently, the standardized PDF of u, the original random variable - is:

$$\hat{f}(\hat{v}) = \underbrace{\text{stdev}(u)}_{\sigma_u} f\left(\langle u \rangle + \underbrace{\text{stdev}(u)}_{\sigma_u} \hat{v}\right)$$

The moments of $\hat{u}$ are:

$$\hat{\mu}_n = \frac{\langle (u')^n \rangle}{\sigma_u^n} = \frac{\mu_n}{\sigma_u^n} = \int_{\hat{\Omega}_u} \hat{v}^n \hat{f}(\hat{v})\, d\hat{v}$$

Evidently:

$$\hat{\mu}_0 = 1, \quad \hat{\mu}_1 = 0, \quad \hat{\mu}_2 = 1$$

The third and fourth moments of $\hat{u}$ give the skewness and flatness of u:

Skewness: $S(u) = \hat{\mu}_3 = \dfrac{\langle (u')^3 \rangle}{\sigma_u^3}$

Measures the asymmetry of the fluctuations about the mean.

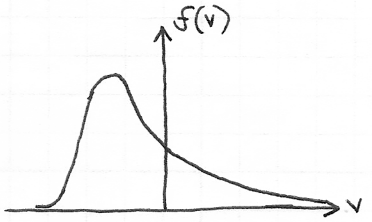
Flatness or Kurtosis: $K(u) = \hat{\mu}_4 = \dfrac{\langle (u')^4 \rangle}{\sigma_u^4}$

Measures the intermittency of the fluctuations.

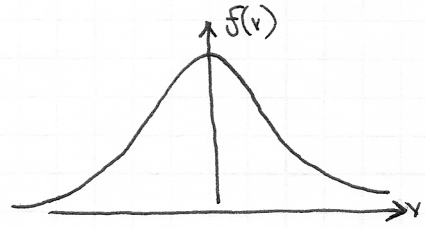
If $S(u) > 0$, then there is more likely to be large positive fluctuations than negative fluctuations.

If $K(u)$ is large, then most of the fluctuations occur in large rare excursions (i.e., most fluctuations are localized).

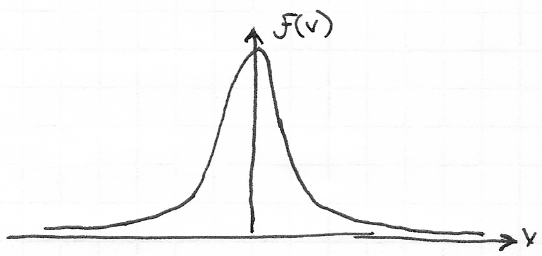
The skewness and kurtosis are quite valuable in characterizing the shape of the PDF.



Positive Skewness



Modest Flatness



Large flatness

The characteristic function of the random variable υ is defined as:

$$\Psi(s) \equiv \langle e^{i\upsilon s} \rangle = \int_{\Omega} f(v)\, e^{ivs}\, dv$$

Note that Ψ is the inverse Fourier transform of the PDF:

$$\Psi(s) = \mathcal{F}^{-1}(f(v))$$

$$f(v) = \mathcal{F}(\Psi(s))$$

Consequently, $\Psi(s)$ and $f(v)$ form a Fourier-transform pair and contain the same information.