Similarity Analysis for Free Shear Flows:

As was noted earlier, it has been observed in many experimental studies that turbulent free shear flows evolve toward self-similar states in their far-fields. Indeed, much of our understanding of free shear flows is based on similarity analysis. However, similarity analysis of turbulent shear layers is more subtle than that for laminar flows. The goal here is to highlight some of the subtleties.

We begin with the shear layer axial mean momentum equation and neglect viscous forces (what does this imply?):

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial}{\partial y}\langle u'v'\rangle$$

As the flow is incompressible, there exists a streamfunction $\bar{\Psi}(x,y)$ such that:

$$\bar{u} = \frac{\partial \bar{\Psi}}{\partial y}, \quad \bar{v} = -\frac{\partial \bar{\Psi}}{\partial x}$$

We now assume the streamfunction has the form:

$$\bar{\Psi}(x,y) = \underbrace{\overset{\text{constant!}}{U_{offset}}\, y}_{\text{Free stream part}} + \underbrace{U_s(x)\,\delta(x)\,F(\eta)}_{\text{Similarity part}} \qquad \eta = \frac{y}{\delta(x)}$$

where η is our similarity variable and $F(\eta)$ is an unknown, dimensionless function. Differentiating the above expression yields:

$$\bar{u}(x,y) = U_{offset} + U_s(x)\frac{dF}{d\eta}(\eta)$$

$$\bar{v}(x,y) = -\frac{dU_s}{dx}(x)\,\delta(x)\,F(\eta) - U_s(x)\frac{d\delta}{dx}(x)\,F(\eta) + U_s(x)\frac{d\delta}{dx}(x)\,\eta\,\frac{dF}{d\eta}(\eta)$$

Analyzing the above, we see that:

$$\bar{f}(\eta) = \frac{dF}{d\eta}(\eta)$$

is the similarity function for the axial velocity perturbation $\delta\bar{u} = U_s(x)\frac{dF}{d\eta}(\eta)$.

The value of U_{offset} is chosen such that we may expect $\delta\bar{u}$ to be self-similar. Hence:

$$U_{offset} = U_\infty \quad \text{for planes \& wakes}$$
$$U_{offset} = U_c \quad \text{for mixing layers}$$

For the turbulence to be self-similar, it is clear it must maintain a constant scaling relationship with the mean velocity. This suggests the decomposition:

$$\langle u'v'\rangle(x,y) = U_s^2(x)\, g(\eta)$$

Similarity function for the Reynolds stress

To proceed forward, we need to insert the preceding relations into the shear layer equation. The following equivalences will aid in our derivation:

$$\frac{\partial \bar{u}}{\partial x} = \left(\frac{dU_s}{dx}\right)\frac{dF}{d\eta} - \left(\frac{\left(\frac{d\delta}{dx}\right)U_s}{\delta}\right)\eta\,\frac{d^2F}{d\eta^2}$$

$$\frac{\partial \bar{u}}{\partial y} = \left(\frac{U_s}{\delta}\right)\frac{d^2F}{d\eta^2}, \qquad \frac{\partial}{\partial y}\langle u'v'\rangle = \left(\frac{U_s^2}{\delta}\right)\frac{dg}{d\eta}$$

So:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \left(U_{offset} + U_s\frac{dF}{d\eta}\right)\left(\left(\frac{dU_s}{dx}\right)\frac{dF}{d\eta} - \left(\frac{\left(\frac{d\delta}{dx}\right)U_s}{\delta}\right)\eta\,\frac{d^2F}{d\eta^2}\right)$$

$$+\left(-\left(\frac{dU_s}{dx}\,\delta\right)F - \left(U_s\frac{d\delta}{dx}\right)F + \left(U_s\frac{d\delta}{dx}\right)\eta\,\frac{dF}{d\eta}\right)\left(\left(\frac{U_s}{\delta}\right)\frac{d^2F}{d\eta^2}\right)$$

$$= \left[U_{offset}\frac{dU_s}{dx}\right]\frac{dF}{d\eta} - \left[\frac{\left(\frac{d\delta}{dx}\right)U_s U_{offset}}{\delta}\right]\eta\,\frac{d^2F}{d\eta^2}$$

$$+\left[U_s\frac{dU_s}{dx}\right]\left(\left(\frac{dF}{d\eta}\right)^2 - F\frac{d^2F}{d\eta^2}\right) - \left[\frac{\left(\frac{d\delta}{dx}\right)U_s^2}{\delta}\right]F\,\frac{d^2F}{d\eta^2}$$

Hence the shear layer equation is:

$$\left[U_{offset}\frac{dU_s}{dx}\right]\frac{dF}{d\eta} - \left[\frac{\left(\frac{d\delta}{dx}\right)U_s U_{offset}}{\delta}\right]\eta\,\frac{d^2F}{d\eta^2} + \left[U_s\frac{dU_s}{dx}\right]\left(\left(\frac{dF}{d\eta}\right)^2 - F\frac{d^2F}{d\eta^2}\right)$$

$$- \left[\frac{\left(\frac{d\delta}{dx}\right)U_s^2}{\delta}\right]F\,\frac{d^2F}{d\eta^2} = -\left[\frac{U_s^2}{\delta}\right]\frac{dg}{d\eta}$$

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In order for the similarity assumption to hold, all the terms in box brackets above must have the same x-dependence. Immediately note that this requires:

$$U_{off.} \sim U_s$$

However, $U_{off.}$ is a constant while U_s in general is not. To resolve this, we must acknowledge necessary conditions for a similarity solution to exist. The condition is that one of the following three conditions apply:

1. $U_{off.} = 0$ as in a jet with no coflow
2. $U_s = \text{const.}$ as in a mixing layer
3. $U_s = 0$ (unphysical) or $U_s \ll U_{off.}$ as in a coflowing jet or wake.

We handle each of these cases in kind.