Example of a Chaotic System: The Lorenz Equations

Consider the following system of nonlinear ODEs:

$$\dot{x} = \sigma(y-x)$$

$$\dot{y} = -xz + Rx - y$$

$$\dot{z} = xy - bz$$

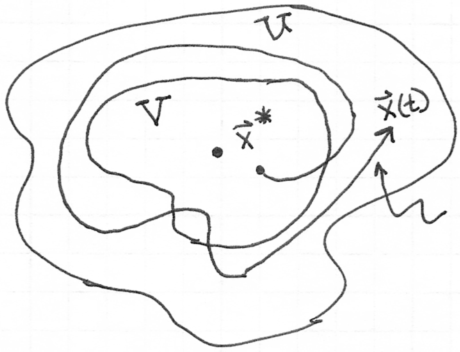
The above is a model of thermal convection where x, y, and z make up the system state, t is time, and:

$R > 0$: Scaled Rayleigh number (buoyancy vs. viscosity)
$\sigma > 0$: Scaled Prandtl number (momentum diffusivity vs. thermal diffusivity)
$b > 0$: Coupling constant

Aside: The Lorenz equations are derived from a spectral Galerkin approximation of the streamfunction and temperature fields appearing in the Boussinesq equations.

The Lorenz equations, first developed by Edward Lorenz in 1963, are notable for having chaotic solution values for certain parameter values. To proceed further, it will help to recall the definition of stability.

(Lyapunov) Stability: A fixed point $\vec{x}^*$ of a nonlinear system is (Lyapunov) stable if for every neighborhood $\mathcal{U}$ of $\vec{x}^*$ there is another neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of $\vec{x}^*$ such that every solution $\vec{x}(t)$ starting in $\mathcal{V}$ ($\vec{x}(0) \in \mathcal{V}$) remains in $\mathcal{U}$ for all time $t \geq 0$.



Initially within $\mathcal{U}$, the path of $\vec{x}(t)$ stays in $\mathcal{V}$!

In layman's terms, a fixed point is stable if all small perturbations result in trajectories that are close to the fixed point!

For the Lorenz equations, there are two classes of fixed points:

I. $x = y = z = 0$ (No Flow)

II. $x = y = \pm\sqrt{b(R-1)}$, $z = R-1$ (Steady Thermal Convection)

Note: $R > 1$

So the question is: Are the above fixed points stable?

We will begin to answer this question by considering the linearized Lorenz equations!

Let $\vec{X}^* = (x^*, y^*, z^*)$ be a fixed point and define $\vec{X} = \vec{X}^* + \delta\vec{X}$ where

$$\delta\vec{X} = (\delta x, \delta y, \delta z)$$

is a perturbation!

Plugging $\vec{X}$ into the Lorenz equations gives:

$$\frac{d}{dt}(x^* + \delta x) = \sigma(y^* + \delta y - x^* - \delta x)$$
$$\frac{d}{dt}(y^* + \delta y) = -(x^* + \delta x)(z^* + \delta z) + R(x^* + \delta x) - (y^* + \delta y)$$
$$\frac{d}{dt}(z^* + \delta z) = (x^* + \delta x)(y^* + \delta y) - b(z^* + \delta z)$$

Since $\vec{X}^*$ is a solution to Navier-Stokes, we have:

$$\frac{d}{dt}(\delta x) = \sigma(\delta y - \delta x)$$
$$\frac{d}{dt}(\delta y) = -x^*\delta z - \delta x\, z^* + \delta x \delta z + R\delta x - y^*$$
$$\frac{d}{dt}(\delta z) = x^*\delta y + \delta x\, y^* + \delta x\, \delta y - b\,\delta z$$

Ignoring higher-order terms in $\delta\vec{X}$ gives the linearized system:

$$\frac{d}{dt}\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ R - z^* & -1 & -x^* \\ y^* & x^* & -b \end{bmatrix}\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

or equivalently:

$$\frac{d}{dt}(\delta\vec{X}) = \vec{\vec{A}}\,\delta\vec{X}$$

Amplification Matrix

If $\vec{\vec{A}}$ is not defective (if it has three linearly independent eigenvectors), then it admits the eigen decomposition:

$$\vec{\vec{A}} = \vec{\vec{Q}}\,\vec{\vec{\Lambda}}\,\vec{\vec{Q}}^{-1}$$

where:

$$\vec{\vec{Q}} = \left[\begin{array}{c|c|c} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{array}\right] \qquad \vec{\vec{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Eigenvectors　　　　Eigenvalues

Note that we can write $\delta\vec{X}$ in terms of the eigenvectors:

$$\delta\vec{X} = c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3 \quad \Rightarrow \quad \delta\vec{X} = \vec{\vec{Q}}\,\vec{c} \qquad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

where $\vec{\vec{Q}}$ is independent of time and $\vec{c}$ consists of time-dependent parameters. Thus our linearized system takes the form:

$$\frac{d}{dt}\underbrace{(\delta\vec{X})}_{\vec{\vec{Q}}\vec{c}} = \underbrace{\vec{\vec{A}}}_{\vec{\vec{Q}}\vec{\vec{\Lambda}}\vec{\vec{Q}}^{-1}}\ \underbrace{\delta\vec{X}}_{\vec{\vec{Q}}\vec{c}}$$

$$\Rightarrow \quad \vec{\vec{Q}}\frac{d}{dt}(\vec{c}) = \vec{\vec{Q}}\vec{\vec{\Lambda}}\vec{c}$$

Pre-multiplying by $\vec{\vec{Q}}^{-1}$ yields:

$$\frac{d}{dt}(\vec{c}) = \vec{\vec{\Lambda}}\vec{c}$$

or equivalently:

$$\frac{d}{dt}(c_i) = \lambda_i c_i \qquad i = 1, 2, 3$$

Thus:

$$c_i(t) = c_i(0)\, e^{\lambda_i t}$$

and our solution takes the form:

$$\boxed{\delta\vec{X} = \sum_{i=0}^{3} c_i(0)\, e^{\lambda_i t}\, \vec{q}_i}$$

Note that <u>linear stability</u> depends <u>entirely</u> on the eigenvalues of $\vec{\vec{A}}$. In particular, if any of the eigenvalues have <u>positive real</u> part, the fixed point is unstable!

<u>Case I</u>: $x^* = y^* = z^* = 0$

$$\vec{\vec{A}} = \begin{bmatrix} -\sigma & \sigma & 0 \\ R & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

<u>Eigenvalues</u>:

$$\begin{cases} \lambda_1 = -b \\ \lambda_2 = -\dfrac{(1+\sigma) - \sqrt{(\sigma-1)^2 + 4R\sigma}}{2} \\ \lambda_3 = -\dfrac{(1+\sigma) + \sqrt{(\sigma-1)^2 + 4R\sigma}}{2} \end{cases}$$

All real!

The only eigenvalue which could possibly be ~~positive~~ non-negative is λ_2, and this happens if:

$$+(1+\sigma) \leq \sqrt{(\sigma-1)^2 + 4R\sigma}$$

$$(1+\sigma)^2 \leq (\sigma-1)^2 + 4R\sigma$$

$$\Rightarrow \quad R \geq \frac{(\sigma+1)^2 - (\sigma-1)^2}{4\sigma} = 1$$

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So, stability depends on R:

$0 \leq R < 1$: All eigenvalues are negative $\Leftrightarrow$ Fixed pt. is stable.
$R = 1$: $\lambda_1, \lambda_3 < 0$, $\lambda_2 = 0$ $\Leftrightarrow$ Marginal stability.
$R > 1$: $\lambda_1, \lambda_3 < 0$, $\lambda_2 > 0$ $\Leftrightarrow$ Fixed point is unstable!

Case II: $x^* = y^* = \pm\sqrt{b(R-1)}$, $z = R-1$

Necessarily, $R \geq 1$!

In this case, the amplification matrix becomes:

$$\overset{\rightrightarrows}{A} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(R-1)} \\ \pm\sqrt{b(R-1)} & \pm\sqrt{b(R-1)} & -b \end{bmatrix}$$

The eigenvalues satisfy the characteristic equation:

$$\det(\overset{\rightrightarrows}{A} - \lambda \overset{\rightrightarrows}{I}) = 0$$

which is equivalently:

$$-\lambda^3 + \mathrm{tr}(\overset{\rightrightarrows}{A})\lambda^2 - \tfrac{1}{2}\left((\mathrm{tr}(\overset{\rightrightarrows}{A}))^2 - \mathrm{tr}(\overset{\rightrightarrows}{A}^2)\right)\lambda + \det(\overset{\rightrightarrows}{A}) = 0$$

A few calculations show:

$$\det(\overset{\rightrightarrows}{A}) = -2\sigma b(R-1)$$

$$\mathrm{tr}(\overset{\rightrightarrows}{A}) = -(\sigma + b + 1)$$

$$\tfrac{1}{2}\left((\mathrm{tr}(\overset{\rightrightarrows}{A}))^2 - \mathrm{tr}(\overset{\rightrightarrows}{A}^2)\right) = b(R+\sigma)$$

So:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(R+\sigma)\lambda + 2\sigma b(R-1) = 0$$

Cubic equation for three eigenvalues!

The eigenvalues may be explicitly found (use Matlab's symbolic toolbox, for instance), but the expressions are quite nasty.

That being said, we find the following behavior:

Small R: $\lambda_1, \lambda_2, \lambda_3$ real & negative (stable)

Larger R: λ_2, λ_3 complex conjugate pair with negative real part

λ_1 real & negative (stable)

$R = R_t$: Real parts of λ_2, λ_3 become zero.

Critical Case! $\Rightarrow$ Marginal Stability!

When $R = R_t$:

$$\begin{aligned} \lambda_1 &= \lambda_1 \\ \lambda_2 &= i\lambda_0 \\ \lambda_3 &= -i\lambda_0 \end{aligned} \qquad \lambda_0, \lambda_1 \text{ real}$$

Due to invariance, we have:

$$\det(\vec{\vec{A}}) = \lambda_1\lambda_2\lambda_3 = \lambda_1\lambda_0^2$$

$$\mathrm{tr}(\vec{\vec{A}}) = \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1$$

$$\tfrac{1}{2}\left((\mathrm{tr}(\vec{\vec{A}}))^2 - \mathrm{tr}(\vec{\vec{A}}^2) \right) = \tfrac{1}{2}\left((\lambda_1+\lambda_2+\lambda_3)^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 \right) = \lambda_0^2$$

So:

$$\begin{aligned} \lambda_1\lambda_0^2 &= -2\sigma b(R_t - 1) \\ \lambda_1 &= -(\sigma + b + 1) \\ \lambda_0^2 &= b(R_t + \sigma) \end{aligned}$$

Thus:

$$\cancel{-}(\sigma+b+1)\cancel{b}(R_t+\sigma) = \cancel{-}2\sigma\cancel{b}(R_t - 1)$$

$$\Rightarrow \quad R_t(\sigma - b - 1) = \sigma(\sigma + b + 3)$$

$$\Rightarrow \quad \boxed{R_t = \frac{\sigma(\sigma+b+3)}{\sigma - b - 1}} \leftarrow \text{Critical } R!$$

For $R > R_t$: Real parts of λ_2, λ_3 are positive, indicating instability!

To get a better handle of solution behavior when λ_2, λ_3 have positive real parts, let us write out the following:

$\lambda_1 < 0$, $\lambda_0^r > 0$

Eigenvalues:

$$\begin{aligned} \lambda_1 &= \lambda_1 \\ \lambda_2 &= \lambda_0^r + i\lambda_0^i \\ \lambda_3 &= \lambda_0^r - i\lambda_0^i \end{aligned}$$

Eigenfunctions:

$$\begin{aligned} \vec{q}_1 &= \vec{\phi}_1 \\ \vec{q}_2 &= \vec{\phi}_0^r + i\vec{\phi}_0^i \\ \vec{q}_3 &= \vec{\phi}_0^r - i\vec{\phi}_0^i \end{aligned}$$

Then:

$$\delta\vec{X} = C_1(0)\,\vec{\phi}_1\, e^{\lambda_1 t} + C_2(0)\left(\vec{\phi}_0^r + i\vec{\phi}_0^i\right) e^{i\lambda_0^i t}\, e^{\lambda_0^r t}$$
$$+ C_3(0)\left(\vec{\phi}_0^r - i\vec{\phi}_0^i\right) e^{-i\lambda_0^i t}\, e^{\lambda_0^r t}$$

However:

$$e^{i\lambda_0^i t} = \cos(\lambda_0^i t) + i\sin(\lambda_0^i t)$$
$$e^{-i\lambda_0^i t} = \cos(\lambda_0^i t) - i\sin(\lambda_0^i t)$$

Recognizing that $\delta\vec{X}$ is real, we have that:

$$\boxed{\delta\vec{X} = e^{\lambda_1 t}\,\vec{\alpha}_1 + e^{\lambda_0^r t}\cos(\lambda_0^i t)\,\vec{\alpha}_c + e^{\lambda_0^r t}\sin(\lambda_0^i t)\,\vec{\alpha}_s}$$

where:

$$\vec{\alpha}_1 = C_1(0)\,\vec{\phi}_1$$
$$\vec{\alpha}_c = \left(\mathrm{Re}\left(C_2(0) + C_3(0)\right)\right)\vec{\phi}_0^r + \left(\mathrm{Im}\left(C_3(0) - C_2(0)\right)\right)\vec{\phi}_0^i$$
$$\vec{\alpha}_s = -\left(\mathrm{Re}\left(C_2(0) + C_3(0)\right)\right)\vec{\phi}_0^i + \left(\mathrm{Im}\left(C_3(0) - C_2(0)\right)\right)\vec{\phi}_0^r$$

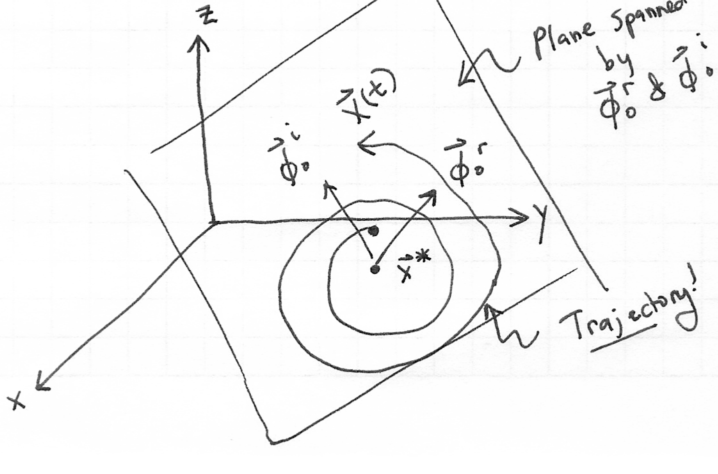
Independent of time!

So, as $t \to \infty$:

$$\delta\vec{X} \longrightarrow e^{\lambda_0^r t}\cos(\lambda_0^i t)\,\vec{\alpha}_c + e^{\lambda_0^r t}\sin(\lambda_0^i t)\,\vec{\alpha}_s$$

Thus, the solution converges onto the plane spanned by $\vec{\alpha}_c$ and $\vec{\alpha}_s$ (equivalently, the plane spanned by $\vec{\phi}_0^r$ and $\vec{\phi}_0^i$) as $t \to \infty$, rotating about the fixed point and spinning outward!

The Linear Stability Picture:



However, the above is just the linearized solution, and nonlinearities keep the solution from spinning out to ∞. Instead, the solution is pulled to a "strange attractor", a subset of solution (i.e., phase) space.

The attractor is stable, but the solution on the attractor is unstable. Perturbed solutions will remain on the attractor, but the perturbed solution trajectory will diverge exponentially from the unperturbed trajectory.

The above behavior is displayed in images on D2L under:

Content $\longrightarrow$ Course Materials $\longrightarrow$ Lorenz Equations

Some remarks:

1. Deterministic chaos results in unpredictability.
2. Evolution of trajectories on the attractor has predictable characteristics.
3. Solution instabilities are not completely random.
4. Chaos often occurs with increasing parameter.

References:

Non-technical: *Chaos* by James Gleick

Technical: *Nonlinear Dynamics and Chaos* by Steven Strogartz