The Reynolds Stresses

Evidently the Reynolds stresses $\langle u_i' u_j' \rangle$ play a critical role in the evolution of the mean velocity field $\overline{u}_i$ and mean pressure field $\overline{p}$. Consequently, it is important to understand the properties of the Reynolds stresses.

First and foremost, the Reynolds stresses are indeed stresses. The Reynolds stresses are the components of a second-order tensor, which is obviously symmetric. The diagonal components are normal stresses while the off-diagonal components are shear stresses.

The turbulent kinetic energy $k(\vec{x},t)$ is defined to be half the trace of the Reynolds stress tensor:

$$k \equiv \frac{1}{2} \langle \vec{u}' \cdot \vec{u}' \rangle = \frac{1}{2} \langle u_i' u_i' \rangle$$

It is the mean kinetic energy per unit mass in the fluctuating velocity field. Note that:

$$\frac{1}{2} \langle \vec{u} \cdot \vec{u} \rangle = \frac{1}{2} \langle \overline{u} \cdot \overline{u} \rangle + \frac{1}{2} \langle \vec{u}' \cdot \vec{u}' \rangle$$

$$= \underbrace{\frac{1}{2}\, \overline{u} \cdot \overline{u}}_{\text{M.K.E. in mean}} + \underbrace{\frac{1}{2} \langle \vec{u}' \cdot \vec{u}' \rangle}_{\text{M.K.E. in fluctuations}}$$

The Reynolds stress tensor is also positive-semi-definite. That is:

$$\underline{x}^T \underbrace{\underline{\underline{\Gamma}}}_{\text{Matrix of Reynolds Stresses}} \underline{x} = x_i \langle u_i' u_j' \rangle x_j = \langle x_i u_i' \, x_j u_j' \rangle = \langle (x_i u_i')^2 \rangle \geq 0 \quad \forall \underline{x}$$

As a consequence, the eigenvalues of the Reynolds stress tensor are all positive or zero! Moreover, the normal stresses are always compressive. In general, all eigenvalues of the tensor are positive.

The Reynolds stress tensor may be decomposed as follows:

$$\langle u_i' u_j' \rangle = \underbrace{\frac{2}{3} k \delta_{ij}}_{\text{Isotropic Stress}} + \underbrace{\left(\langle u_i' u_j' \rangle - \frac{2}{3} k \delta_{ij} \right)}_{\text{Anisotropic Stress}}$$

Isotropic Stress (Pressure-like Term) — Does not affect mean flow!

Anisotropic Stress (Responsible for Transporting Momentum)

We denote the anisotropy (anisotropic) stress components by a_{ij}, and define the normalized anisotropy tensor as:

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\langle u_i' u_j' \rangle}{\langle u_l' u_l' \rangle} - \frac{1}{3} \delta_{ij}$$

In the event that the flow is irrotational, then:

$$\frac{\partial u_i'}{\partial x_j} - \frac{\partial u_j'}{\partial x_i} = 0$$

As a consequence:

$$\frac{\partial}{\partial x_j}\left(\frac{1}{2}\langle u_i' u_i'\rangle\right) - \frac{\partial}{\partial x_i}\left(\langle u_i' u_j'\rangle\right) = \left\langle u_i'\left(\frac{\partial u_i'}{\partial x_j} - \frac{\partial u_j'}{\partial x_i}\right)\right\rangle = 0$$

This gives rise to the Corrsin-Kistler equation:

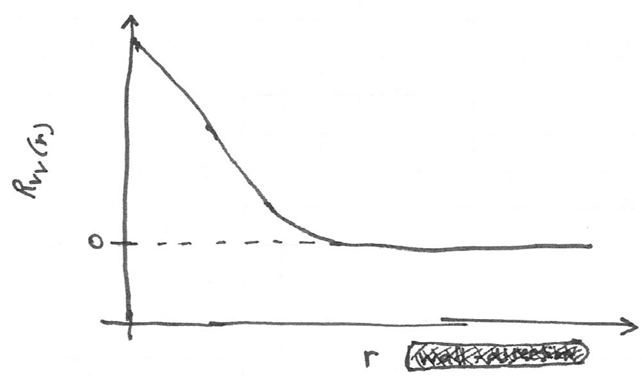
$$\frac{\partial}{\partial x_i}\left(\langle u_i' u_j'\rangle\right) = \frac{\partial k}{\partial x_j}$$

The above implies that the Reynolds stress is a pressure-like term for irrotational flow and has absolutely no effect on the mean velocity field. Therefore, a flow is turbulent only if it is rotational!

The Reynolds stress tensor is the zero-separation value of the two-point correlation:

$$R_{ij}(\vec{r},\vec{x},t) = \langle u_i'(\vec{x},t)\, u_j'(\vec{x}+\vec{r},t)\rangle$$

The two-point correlation is a commonly studied statistical quantity, both because of its relation to the Reynolds stress tensor and because of its association with turbulent structures. For turbulent flows, two-point correlations generally decay smoothly to zero for large separation, with finite correlations for small separation.



Representative plot of two-point correlation of the wall-normal velocity component for turbulent channel flow for separations in the streamwise direction.

Finally, for some flows, symmetries in the flow geometry determine properties of the Reynolds stress tensor. See page 89 of Pope for more details.