Turbulent Channel Flow – Reynolds Number Dependencies:

Having characterized the structure of the mean velocity profile for a fully-developed channel, we may now determine the dependence of statistical quantities on the Reynolds number of the flow. We begin by establishing a relationship between the characteristic velocities u_0 (the centerline velocity) and $\overline{U}$ (the bulk velocity) and the friction velocity u_τ.

Recall that the log-law does a great job of approximating the mean velocity $\bar{u}$ throughout the outer layer and that the viscous sublayer is very thin for a high Re flow. Thus, for sufficiently high Re,

$$\frac{u_0 - \bar{u}}{u_\tau} \approx -\frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + B_1$$

Moreover, B_1 is quite small and may be neglected. Integrating the above from $y = 0$ to $y = \delta$ and setting $B_1 = 0$ then yields:

$$\frac{u_0 - \overline{U}}{u_\tau} \approx \frac{1}{\kappa} \approx 2.4$$

Thus, the difference between the centerline and bulk velocities, normalized by the friction velocity, is roughly independent of Re. This estimate agrees well with experimental data which gives values between 2 and 3 (Dean 1978).

In the overlap region, we have two equivalent expressions for the mean velocity field:

Inner Layer: $$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y}{\delta_\nu}\right) + B$$

Outer Layer: $$\frac{u_0 - \bar{u}}{u_\tau} = -\frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + B_1$$

Adding the above yields:

$$\frac{u_0}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{\delta}{\delta_\nu}\right) + B + B_1$$

$$= \frac{1}{\kappa}\ln\left(Re_\tau\right) + B + B_1$$

Consequently, the ratio of the centerline and friction velocities depends on Re only through the friction Reynolds number, which is:

$$Re_\tau = \frac{u_\tau \delta}{\nu}$$

and consequently it depends on how the friction velocity depends on Re. A plot of Re_τ vs. Re is included on D2L in a PDF at the link:

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and from this plot we find:

$$Re_{\tau} \approx 0.09\, Re^{0.88}$$

For high Re, this implies that δ_ν is quite small compared with δ, which is expected. It also implies that:

$$\boxed{\frac{u_0}{u_\tau} \approx 5 \log_{10} Re}$$

and:

$$\boxed{\frac{\overline{U}}{u_\tau} \approx 5 \log_{10} Re - 2.4}$$

A plot comparing the above expressions is also included in Plots.pdf. Note that the velocity ratios increase very slowly with Re.

Now that we know the Reynolds number dependencies of u_0, $\overline{U}$, and u_τ, we may determine the Reynolds number dependence of the wall shear stress τ_w. We express this dependence through the skin-friction coefficient:

$$c_f \equiv \frac{\tau_w}{\left(\frac{1}{2}\rho u_0^2\right)}$$

Namely:

$$c_f = 2\left(\frac{u_\tau}{u_0}\right)^2$$

So:

$$\boxed{c_f \approx \frac{2}{25}\left(\log_{10} Re\right)^{-2}}$$

A plot comparing the above approximation with the skin-friction coefficient obtained from experimental data (Dean 1978) is included in Plots.pdf. It is noted that a good match is obtained for Re > 3,000.

Finally, a figure is included in Plots.pdf which shows the Reynolds number dependencies of the various regions and layers in a turbulent channel flow. From the figure, it is observed that a log-law region only exists for Re > 3,000 which is consistent with experiment (Patel and Head 1969). Moreover, a Reynolds number greater than 20,000 is required for there to be an overlap region.