Reynolds Stress Transport Models – Models for Inhomogeneous Turbulence

Now that we have discussed homogeneous turbulence at length, it now makes sense to discuss Reynolds stress transport models in the context of inhomogeneous flows. In what follows, we will discuss:

(i) Reynolds stress flux modeling
(ii) Dissipation tensor modeling
(iii) Dissipation transport modeling
(iv) Pressure-rate-of-strain modeling

Reynolds Stress Flux Modeling:

We begin by discussing the Reynolds stress flux:

$$T_{kij} = \underbrace{T_{kij}^{(\nu)}}_{\text{Viscous Diffusion}} + \underbrace{T_{kij}^{(p)}}_{\text{Pressure Transport}} + \underbrace{T_{kij}^{(u)}}_{\text{Turbulent Convection}}$$

The viscous diffusion is already in closed form so it requires no further discussion. The gradient diffusion hypothesis is typically used to model the pressure transport and turbulent convection terms. The simplest gradient-diffusion model is isotropic and due to Shir (1973):

$$T_{kij}^{(p)} + T_{kij}^{(u)} = -C_s \frac{k^2}{\varepsilon} \frac{\partial \langle u_i' u_j' \rangle}{\partial x_k}$$

Above, C_s is a model constant. A more popular model is the model of Daly and Harlow (1970) which uses the Reynolds stress tensor to define an anisotropic diffusion tensor:

$$T_{kij}^{(p)} + T_{kij}^{(u)} = -C_s \frac{k}{\varepsilon} \langle u_k u_\ell \rangle \frac{\partial \langle u_i' u_j' \rangle}{\partial x_\ell}$$

For this model, a value of $C_s = 0.22$ was suggested by Launder (1990). Near to a wall, the dominant component of the gradient is in the wall-normal direction, y. Then, the turbulent transport becomes $\partial/\partial y \left(C_s T \langle (v')^2 \rangle \, \partial/\partial y \langle u_i' u_j' \rangle \right)$ and hence the dominant eddy viscosity is $\nu_T = C_s T \langle (v')^2 \rangle$. Since the vertical velocity fluctuations are responsible for transporting turbulence away from the wall, we see that Daly and Harlow's model is qualitatively correct. Finally, near the wall, k/ε is typically replaced by $T = \max(k/\varepsilon, C_T \tau_\eta)$.

in the Shir model!

Dissipation Tensor Modeling:

For high Re, a consequence of local isotropy is:

$$\varepsilon_{ij} = \tfrac{2}{3} \varepsilon \delta_{ij}$$

This is usually taken as the model for ε_{ij}. However, for moderate Re, this isotropic relation is not completely correct. Nonetheless, the anisotropic component $(\varepsilon_{ij} - \tfrac{2}{3}\varepsilon \delta_{ij})$ is a redistribution tensor and can be absorbed into the pressure-rate-of-strain model. Explicit modeling of the anisotropic component has also been proposed. For instance, Hanjalic and Launder (1976) proposed the model:

$$\varepsilon_{ij} - \tfrac{2}{3}\varepsilon \delta_{ij} = 2 \varepsilon f_s b_{ij}$$

where f_s is a function of the turbulent Reynolds number $k^2/\varepsilon\nu$.

Close to the walls, the dissipation tensor is highly anisotropic and different models are appropriate. At the wall, $(\partial v'/\partial y)_{y=0}$ since $(\partial u'/\partial x)_{y=0} = (\partial w'/\partial z)_{y=0} = 0$. Consequently, for small y:

$$u' = \left(\frac{\partial u'}{\partial y}\right)_{y=0} y + \text{H.O.T.}$$

$$v' = \frac{1}{2}\left(\frac{\partial^2 v'}{\partial y^2}\right)_{y=0} y^2 + \text{H.O.T.}$$

$$w' = \left(\frac{\partial w'}{\partial y}\right)_{y=0} y + \text{H.O.T.}$$

It is straightforward to show that, as the wall is approached $(y \to 0)$, ε_{ij} is given by:

$$\frac{\varepsilon_{ij}}{\varepsilon} = \frac{\langle u_i' u_j' \rangle}{k} \quad \text{for } i \neq 2,\ j \neq 2$$

$$\frac{\varepsilon_{i2}}{\varepsilon} = 2\,\frac{\langle u_i' u_2' \rangle}{k} \quad \text{for } i \neq 2$$

$$\frac{\varepsilon_{22}}{\varepsilon} = 4\,\frac{\langle (u_2')^2 \rangle}{k}$$

To obtain the correct behavior at the wall, Rotta (1951) proposed the model:

$$\varepsilon_{ij} = \frac{\langle u_i' u_j' \rangle}{k}\,\varepsilon$$

However, for small y, this model underestimates ε_{12} and ε_{22} by factors of 2 and 4, respectively. To provide a more accurate representation, Launder and Reynolds (1983) introduced the quantity:

$$\varepsilon_{ij}^* = \frac{\varepsilon\left(\langle u_i' u_j' \rangle + n_j n_\ell \langle u_\ell' u_i' \rangle + n_i n_\ell \langle u_\ell' u_j' \rangle + \delta_{ij} n_\ell n_m \langle u_\ell' u_m' \rangle\right)/k}{1 + \frac{5}{2} n_\ell n_m \langle u_\ell' u_m' \rangle / k}$$

where $\vec{n}$ is the unit normal. Near the wall, ε_{ij}^* exhibits the correct behavior, but away from walls, it does not. Hence, it is natural to introduce the blended model:

$$\varepsilon_{ij} = f_s\,\varepsilon_{ij}^* + (1 - f_s)\,\frac{2}{3}\,\varepsilon\,\delta_{ij}$$

where $f_s \to 1$ near the walls and $f_s \to 0$ away from walls. For this purpose, Lai and So (1990) proposed to use:

$$f_s = \exp\left[-\left(\frac{Re_L}{150}\right)^2\right]$$

Dissipation Transport Modeling:

The standard model equation for ε used in Reynolds stress models is due to Hanjalić and Launder (1950):

$$\frac{\overline{D}\varepsilon}{\overline{D}t} = \frac{\partial}{\partial x_i}\left(C_\varepsilon \frac{k}{\varepsilon} \langle u_i' u_j' \rangle \frac{\partial \varepsilon}{\partial x_j}\right) + C_{\varepsilon 1}\frac{P_\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

with $C_\varepsilon = 0.15$, $C_{\varepsilon 1} = 1.44$, and $C_{\varepsilon 2} = 1.92$. Note that the above contains an anisotropic diffusivity. Near the walls, the above is replaced by:

$$\frac{\bar{D}\varepsilon}{\bar{D}t} = \frac{\partial}{\partial x_i}\left(\left(\nu\delta_{ij} + C_\varepsilon \frac{k}{\varepsilon}\langle u_i' u_j'\rangle\right)\frac{\partial \varepsilon}{\partial x_j}\right) + \frac{C_{\varepsilon 1} P - C_{\varepsilon 2}\varepsilon}{T}$$

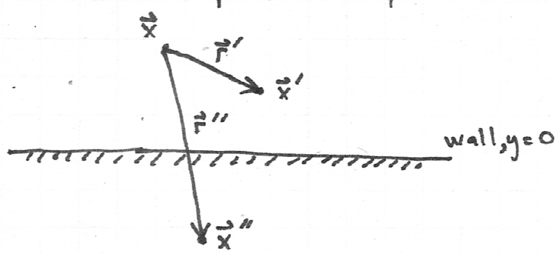
where $T = \max\left(k/\varepsilon,\ C_T \tau_\eta\right)$. Moreover, at the walls, $\varepsilon|_{wall} = \frac{\partial^2 k}{\partial y^2}\Big|_{wall}$ is applied.

Pressure - Rate - of - Strain Modeling:

Walls have an "elliptic effect" on the pressure field and hence also the pressure-rate-of-strain. Hence, there is a need to include non-local boundary effects in pressure-rate-of-strain modeling. Nonlocal effects are often referred to as "pressure reflections" or "pressure echoes" as they originate from the surface boundary condition:

$$\frac{\partial p'}{\partial n} = \frac{\partial p^{(r)}}{\partial n} + \frac{\partial p^{(s)}}{\partial n} + \frac{\partial p^{(h)}}{\partial n} = \nu \frac{\partial^2 v'}{\partial n^2} \approx 0$$

For large Re, the pressure BC is usually taken as zero and hence the harmonic pressure is neglected. Alternatively, we may consider the inertial pressure $p^{(i)} \equiv p^{(r)} + p^{(s)}$. Consider a turbulent flow above an infinite plane wall at $y=0$:



The inertial pressure is governed by:

$$\nabla^2 p^{(i)} = S \quad \text{for } y>0$$

$$\left(\frac{\partial p^{(i)}}{\partial y}\right)_{y=0} = 0$$

where S consists of rapid and slow sources. Let us extend S to the whole domain by reflecting it about the wall:

$$S(x,y,z,t) \equiv S(x,|y|,z,t) \quad \text{for } y<0$$

Since S is symmetric about $y=0$, the solution to the Poisson problem over all the domain is also symmetric about $y=0$ and satisfies the wall BC. Hence, we have:

$$p^{(i)}(\vec{x},t) = -\frac{1}{4\pi}\iiint_{-\infty}^{\infty} S(\vec{x}',t)\frac{d\vec{x}'}{|\vec{x}-\vec{x}'|}$$

For every point $\vec{x}' = (x', y', z')$ in the upper half plane there is an image point $\vec{x}'' = (x'', y'', z'')$ defined by:

$$(x'', y'', z'') = (x', -y', z')$$

and defining $\vec{r}' \equiv \vec{x}' - \vec{x}$ and $\vec{r}'' \equiv \vec{x}'' - \vec{x}$, we have:

$$p^{(i)}(\vec{x},t) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} S(\vec{x}',t)\left(\frac{1}{|\vec{r}'|} + \frac{1}{|\vec{r}''|}\right)d\vec{x}'$$

and the correlation at $\vec{x}$ between $p^{(i)}$ and a random field $\phi(\vec{x},t)$ is:

$$\langle p^{(i)}(\vec{x},t)\phi(\vec{x},t)\rangle = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\langle S(\vec{x}',t)\phi(\vec{x},t)\rangle\left(\frac{1}{|\vec{r}'|}+\frac{1}{|\vec{r}''|}\right)d\vec{x}'$$

Hence, pressure correlations have two contributions: one due to the free-space Green's function $|\vec{r}'|^{-1}$ and another due to the term $|\vec{r}''|^{-2}$. This second term is called the wall-reflection contribution or wall echo. The relative magnitude of the free-space and wall echo terms are in the ratio of L_s^{-1} to y^{-1}, where L_s is the characteristic correlation lengthscale of S and ϕ. Thus, remote from a wall, $L_s/y \ll 1$ and the wall reflection term is negligible. This is not necessarily the case close to a wall, though.

In some Reynolds-stress closures, additional redistribution terms are included to account for wall reflections. A simple example of a wall reflection term is the formula:

$$R_{ij}^{(w)} = -C_1^w\frac{\varepsilon}{k}\frac{L}{y}\left(\langle u_i'u_\ell'\rangle n_j n_\ell + \langle u_j'u_\ell'\rangle n_i n_\ell - \tfrac{2}{3}\langle u_\ell'u_m'\rangle n_\ell n_m\delta_{ij}\right)$$
$$- C_2^w\left(R_{i\ell}^{(r)} n_j n_\ell + R_{j\ell}^{(r)} n_i n_\ell - \tfrac{2}{3} R_{\ell m}^{(r)} n_\ell n_m \delta_{ij}\right)\frac{L}{y}$$

proposed by Gibson and Launder (1978). Here, $L = k^{3/2}/\varepsilon$ is the turbulence length scale. The factor L/y is consistent with the preceding Green's function analysis and it is effectively uniform in the log-law region, $L/y \approx 2.5$. The effect of this term is to reduce $\langle (v')^2\rangle$ and $\langle u'v'\rangle$ and to increase $\langle (u')^2\rangle$ and $\langle (w')^2\rangle$. The model constants $C_1^w = 0.3$ and $C_2^w = 0.18$ were suggested by Gibson and Launder.

Unfortunately, in a flow toward the wall, the wall correction enhances the normal component of intensity erroneously. In this setting, it is no easy matter to enforce this correctly. In fact, the wall echo must be adapted differently in each instance. For this reason, the additive wall echo methodology has been largely abandoned by the turbulence community in favor of alternative approaches such as elliptic relaxation.