## Similarity Analysis: Mixing Layer:

For a mixing layer, we have that $U_S$ = const. Thus the shear layer equation reduces to:

$$-\left[\frac{\left(\frac{d\delta}{dx}\right)U_S\, U_{offset}}{\delta}\right]\eta\frac{d^2F}{d\eta^2}-\left[\frac{\left(\frac{d\delta}{dx}\right)U_S^2}{\delta}\right]F\frac{d^2F}{d\eta^2}=-\left[\frac{U_S^2}{\delta}\right]\frac{dg}{d\eta}$$

The terms in box brackets must be proportional, which requires that:

$$\frac{d\delta}{dx}=\text{Const.}=\left(\frac{U_S}{U_c}\right)S$$

and, analogous to the non-coflowing jet case, we set:

$$\delta(x)=\left(\frac{U_S}{U_c}\right)S\,(x-x_0)$$

The constant $S$ appearing above is approximately independent of the velocity ratio, with experiments yielding values from $S \approx 0.06$ to $S \approx 0.11$. Noting the above form for $\delta$, the shear layer equation becomes:

$$S\left(\eta\frac{d^2F}{d\eta^2}+2r\frac{d^2F}{d\eta^2}F\right)=\frac{dg}{d\eta}$$

where:

$$r=\frac{U_2-U_1}{U_1+U_2}$$

Unfortunately, the above equation is not easy to solve in general. A dramatic simplification occurs in the limit $r\to 0$, when our equation becomes:

$$S\eta\frac{d^2F}{d\eta^2}=\frac{dg}{d\eta}$$

Note that the case $r \to 0$ corresponds to the setting in which both fluid streams are moving at the "same" speed. If one employs a turbulent viscosity of the form:

$$\nu_T=\hat{\nu}_T\,U_S\,\delta(x) \qquad w/\quad \hat{\nu}_T\equiv\text{Const.}$$

one is able to solve the aforementioned differential equation exactly subject to the boundary conditions:

$$\frac{dF}{d\eta}(\pm\infty)=\pm\frac{1}{2}$$

$$\frac{dF}{d\eta}(0)=0$$

to obtain:

$$\frac{dF}{d\eta}(\eta)=\frac{1}{2}\,\mathrm{erf}\left(\frac{\eta}{\sigma\sqrt{2}}\right)$$

where:

$$\sigma^2=\frac{\hat{\nu}_T}{S} \qquad \text{and} \qquad \mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

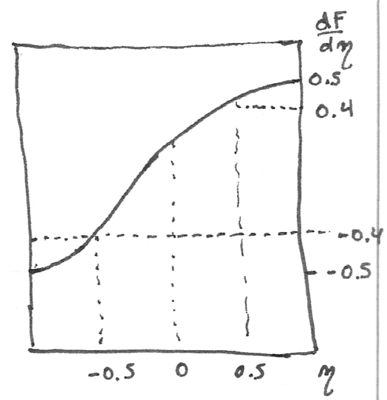
The value for $\sigma$ can be determined by further specifying $\delta$ and observing experimental data. Namely, if we choose:

$$\delta(x) = y_{0.9}(x) - y_{0.1}(x)$$

where:

$$\bar{u}(x, y_{0.9}(x)) = u_1 + 0.9\,(u_2 - u_1)$$

$$\bar{u}(x, y_{0.1}(x)) = u_1 + 0.1\,(u_2 - u_1)$$




we find that:

$$\frac{dF}{d\eta}\left(\pm\tfrac{1}{2}\right) \approx \pm 0.4$$

See, for example, Fig. 5.21 of Pope. This suggests:

$$\sigma \approx \left[2\sqrt{2}\,\mathrm{erf}^{-1}\left(\tfrac{4}{5}\right)\right]^{-1} \approx 0.3902$$

To find F itself, we need to apply an extra boundary condition. We assume the lateral velocity at $y = \infty$ is zero (since the velocities of the streams are "equal"), which gives:

$$\lim_{\eta \to \infty} \left(F(\eta) - \frac{\eta}{2}\right) = 0$$

Integrating our expression for $\frac{dF}{d\eta}$ with the above expression yields:

$$F(\eta) = \frac{1}{2}\,\eta\,\mathrm{erf}\left(\frac{\eta}{\sigma\sqrt{2}}\right) + \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{\eta^2}{2\sigma^2}}$$

Given both F and $\frac{dF}{d\eta}$, we can now derive expressions for our mean velocity fields and Reynolds stresses:

Velocity Difference

$$\frac{\bar{u}(x,y) - u_c}{u_s} = \frac{1}{2}\,\mathrm{erf}\left(\frac{\eta}{\sigma\sqrt{2}}\right)$$

$$\frac{\bar{v}(x,y)}{u_s^2/u_c} = -S\,\frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{\eta^2}{2\sigma^2}}$$

$$\frac{\langle u'v' \rangle}{u_s^2} = -\frac{\hat{\nu}_T}{\sigma\sqrt{2\pi}}\, e^{-\frac{\eta^2}{2\sigma^2}}$$

Similar trends! And in fact the same similarity form!!! (Show this)

where:

$$\sigma \approx 0.3902 \quad \text{and} \quad S \approx 0.1 \quad \text{and} \quad \hat{\nu}_T \approx 0.0152$$

Plots of the above are included in a PDF on D2L at the link:

22- Similarity Analysis – Mixing Layer → Plots.pdf

Note from the plots that the lateral velocity and Reynolds stress are only nonzero within the mixing layer. This indicates there is no fluid being entrained. Moreover, the lateral velocity is always negative, indicating there is a net velocity from the "faster" upper stream to the "slower" lower stream and consequently a positive transfer of mass from the upper stream to the lower stream.

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The axial velocity and Reynolds stress plots match reasonably well with experiment, and a collection of experimental plots are supplied at the end of Plots.pdf. These plots were obtained by Bell and Mehta in 1990 and correspond to an experiment where $u_1 = 0.6\, u_2$, so the assumption that $r \to 0$ is not necessarily a good one. Nonetheless, the theoretical mean axial velocity curve matches quite well with experiment, but the experimental curve tends faster to the free-stream velocity. The theoretical Reynolds stress curve qualitatively matches the experimental plot, but it is off by a constant factor. This is because the rate $S$ was observed to be 0.069 in the experiment while the theoretical plot was for $S = 0.1$.

The shear layer that forms for the limit as $r \to 0$ is approximately homogeneous in the streamwise direction. Indeed, if we observed the layer in a moving frame with velocity $U_c$ to the right, the resulting flow would be statistically one-dimensional and time-dependent. This layer is thus called the temporal mixing layer. Spatial mixing layers, for which $r \neq 0$, exhibit different properties.

The temporal mixing layer is symmetric and the free streams are parallel, while the spatial mixing layer is not symmetric. The spatial mixing layer spreads preferentially into the low-speed stream, and it entrains fluid. As a consequence, the free-streams are not exactly parallel. The fact that spatial mixing layers entrain ambient fluid is important for enhancing mixing processes.

As a final note, we remark that since $U_s$ is constant and $\delta$ varies linearly with $x$, the flow rate of turbulent kinetic energy:

$$\dot{K} = \int_{-\infty}^{\infty} \bar{u}\, k\, dy = \frac{1}{2}\int_{-\infty}^{\infty} \bar{u} \langle |\vec{u}'|^2 \rangle\, dy$$

also increases linearly with $x$ for a mixing layer. As a consequence, turbulent production must exceed dissipation in a mixing layer. By contrast, the flow rate of turbulent energy decreases with $x$ for jets and wakes.

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