Turbulent Channel Flow – Mean Velocity Profiles:

Fully developed channel flow is completely specified by ρ, ν, δ, and dp_w/dx. Consequently, the velocity gradient $d\bar{u}/dy$, which is the dynamically important quantity in a turbulent channel flow, can be written as:

$$d\bar{u}/dy = \frac{u_\tau}{y}\,\Phi\left(\frac{y}{\delta_\nu}, \frac{y}{\delta}\right)$$

using dimensional analysis (the Buckingham Π theorem) where Φ is a universal non-dimensional function. δ_ν is the appropriate lengthscale in the viscous wall region while δ is the appropriate lengthscale in the outer layer.

In 1925, Prandtl hypothesized that at high Re, there is an inner layer close to the wall in which the mean velocity profile is completely determined by the viscous scales, independent of δ and U_0. Mathematically:

$$d\bar{u}/dy = \frac{u_\tau}{y}\,\Phi_I\left(\frac{y}{\delta_\nu}\right) \quad \text{for } y/\delta \ll 1$$

which implies that:

$$\Phi_I\left(\frac{y}{\delta_\nu}\right) = \lim_{y/\delta \to 0} \Phi\left(\frac{y}{\delta_\nu}, \frac{y}{\delta}\right)$$

Defining $u^+ \equiv \bar{u}/u_\tau$, we have that:

$$du^+/dy = \frac{1}{y^+}\,\Phi_I(y^+)$$

which upon integration gives:

$$\boxed{\begin{aligned} & u^+ = f_w(y^+) \\ \text{where: } & \\ & f_w(y^+) = \int_0^{y^+} \frac{1}{y'}\,\Phi_I(y')\,dy' \end{aligned}}$$

The law of the wall

The above is known as the law of the wall, and it implies that u^+ is solely a function of y^+ for $y/\delta \ll 1$.

There is abundant experimental verification that the function f_w is universal, not only for channel flow, but also for pipe flow and boundary layers. In fact, the form of f_w may be directly inferred for small and large values of y^+.

At the wall, the no-slip boundary condition gives $f_w(0) = 0$, while the viscous stress law at the wall gives:

$$\tau_w = \lim_{y^+ \to 0} \underbrace{\mu\left(\frac{u_\tau}{\delta_\nu}\right)}_{\tau_w} \frac{1}{y^+}\,\Phi_I(y^+)$$

$$\Rightarrow 1 = \lim_{y^+ \to 0} \frac{1}{y^+}\,\Phi_I(y^+)$$

which implies:

$$\frac{d f_w}{d y^+}(0) = 1$$

Thus, the Taylor-series expansion for $f_w(y^+)$ for small y^+ is:

$$f_w(y^+) = y^+ + O(y^{+2})$$

In fact, this is actually $O(y^{+4})$! Why?

This implies that close to the wall, the mean velocity profile is linear. Profiles of u^+ in the near-wall region obtained from direct numerical simulations are included in a pdf on D2L at the link:

29 - Turbulent Channel Flow - Mean Velocity Profiles → Plots.pdf

From these plots, it is apparent that depatures from the linear relation $u^+ = y^+$ are negligible for $y^+ < 5$. This region is known as the viscous sublayer since the Reynolds shear stress is negligible compared with the viscous stress in this region. For $y^+ > 12$, departures from the linear relation are significant (greater than 25%).

The inner layer is usually defined as $y/\delta < 0.1$. At high Re, the outer part of the inner layer corresponds to large y^+. As has already been discussed, for large y^+ the effects of viscosity are negligible. Consequently, the dependence of Φ_I on ν vanishes, giving:

$$\Phi_I(y^+) = \frac{1}{\kappa} \quad \text{for} \quad \frac{y}{\delta} \ll 1 \quad \text{and} \quad y^+ \gg 1$$

and hence:

$$\frac{du^+}{dy^+} = \frac{1}{\kappa y^+}$$

Integrating yields:

$$u^+(y^+) = \frac{1}{\kappa} \ln(y^+) + B$$

This is the celebrated logarithmic law of the wall due to von Kármán (1930), and κ is the von Kárman constant. The log-law constants are generally within 5% of:

$$\kappa = 0.41, \qquad B = 5.2$$

Comparisons between the log-law and DNS data in the inner part of the channel ($y/\delta < 1/4$) are provided in Plots.pdf where excellent agreement is seen for $y^+ > 30$.

The region between the viscous sublayer ($y^+ < 5$) and the log-law region ($y^+ > 30$) is called the buffer layer. It is the transition region between viscosity-dominated and the turbulence-dominated regions of the flow.

In the outer layer ($y^+ > 50$), the direct effect of viscosity on the mean flow is negligible. This means that Φ is independent of y/δ_ν for large y/δ_ν:

$$\lim_{y/\delta_\nu \to \infty} \Phi\left(y/\delta_\nu, y/\delta \right) = \Phi_o\left(y/\delta \right)$$

Integrating our mean velocity profile between y and δ then yields the velocity-defect law in the outer layer due to von Kármán (1930):

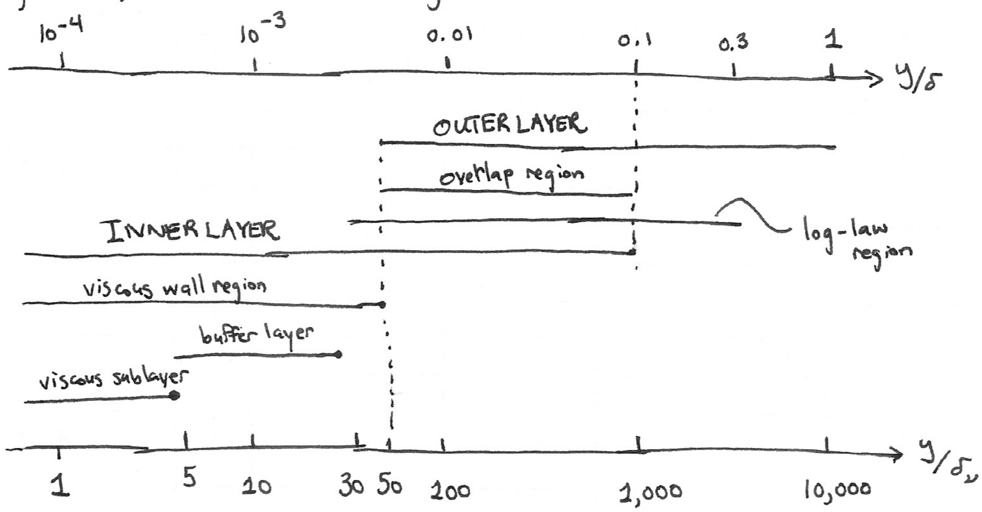
$$\boxed{\frac{u_o - \bar{u}}{u_\tau} = F_D\left(\frac{y}{\delta}\right)}$$

where:

$$F_D\left(\frac{y}{\delta}\right) = \int_{y/\delta}^{1} \frac{1}{y'} \Phi_o(y')\, dy'$$

By definition, the velocity defect is the difference between the centerline velocity and the mean axial velocity. The above law states that this defect, normalized by the friction velocity, only depends on y/δ. Unlike the law of the wall, there is no suggestion that F_D is universal: it is different for different flows.

For sufficiently high Re (approximately Re > 20,000), there is an overlap region between the inner layer ($y/\delta < 0.1$) and outer layer ($y/\delta_\nu > 50$).



Various Wall Regions for $Re_\tau = 10^4$

In the overlap region, both the law of the wall and the velocity-defect law hold:

$$\frac{y}{u_\tau}\frac{d\bar{u}}{dy} = \underbrace{\Phi_I\left(\frac{y}{\delta_\nu}\right)}_{\text{Not a function of } \delta} = \underbrace{\Phi_o\left(\frac{y}{\delta}\right)}_{\text{Not a function of } \delta_\nu} \quad \text{for } \delta_\nu \ll y \ll \delta$$

Thus:

$$\Phi_I\left(\frac{y}{\delta_\nu}\right) = \Phi_o\left(\frac{y}{\delta}\right) = \text{const.} \quad \text{for } \delta_\nu \ll y \ll \delta$$

which leads to:

$$\frac{y}{u_\tau}\frac{d\bar{u}}{dy} = \frac{1}{\kappa} \quad \text{for } \delta_\nu \ll y \ll \delta$$

Integrating yields the log-law in the inner layer and the form of the velocity-defect law for small y/δ in the outer layer:

$$\frac{u_0 - \bar{u}}{u_\tau} = F_D\left(\frac{y}{\delta}\right) = -\frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + B_1 \quad \text{for } \frac{y}{\delta} \ll 1$$

where B_1 is a flow-dependent constant! Note that the above provides an alternative derivation of the log-law, and it is due to Millikan (1938).

Comparisons between the log-law and the velocity defect obtained from DNS are provided in Plots.pdf where it is seen that the log-law is followed quite closely for $0.08 < y/\delta < 0.3$. Even in the central part of the channel, the so-called bulk region, the deviations from the log wall are quite small. The constant B_1 is quite difficult to measure. DNS indicates a value of $B_1 \approx 0.2$ but experiments suggest $B_1 \approx 0.7$. In any event, B_1 is quite small.

Below, the various regions and layers that are used to describe near-wall flows are summarized:

Region	Location	Characteristics
Inner Layer	$y/\delta < 0.1$	$\bar{u}$ determined solely by u_τ and y^+
Viscous Wall Region	$y^+ < 50$	The viscous shear stress is significant
Viscous Sublayer	$y^+ < 5$	The Reynolds shear stress is negligible
Outer Layer	$y^+ > 50$	Direct effects of viscosity on $\bar{u}$ are negligible
Overlap Region	$y^+ > 50$, $y/\delta < 0.1$	Region of overlap between inner and outer layers (at large Re)
Log-law Region	$y^+ > 30$, $y/\delta < 0.3$	The log-law holds
Buffer Layer	$5 < y^+ < 30$	The region between the viscous sublayer and log-law region