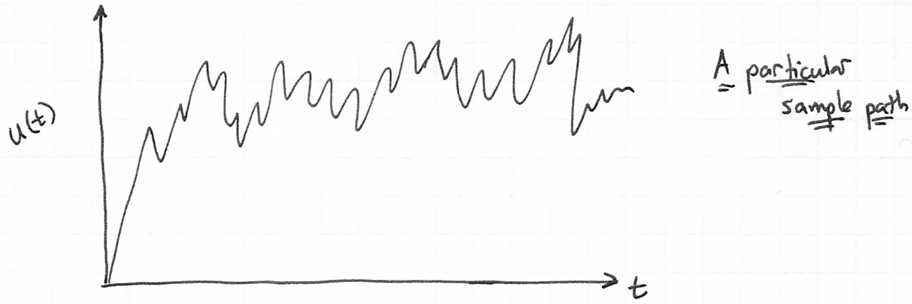
Random Processes and Random Fields:

realizations!

So far, we have considered random variables associated with measurements at a particular location and time. However, measurements may vary with space and time. A time-dependent random variable is called a random process and a space-and-time-dependent random variable is called a random field.

Let $u(t)$ be a random process. By taking a set of samples at time instances (via an experiment), we obtain a sample path:



At each point in time, $u(t)$ may be interpreted as a random variable and has a one-time CDF:

$$F(v;t) \equiv P(u(t) < v)$$

and one-time PDF:

$$f(v;t) \equiv \frac{\partial F(v;t)}{\partial v}$$

If we have a selection of time-instances $t_1, t_2, \ldots, t_N$, then the variables $u(t_1), u(t_2), \ldots, u(t_n)$ are all random variables with joint CDF:

N-time Joint CDF: $$F_N(v_1,t_1; v_2,t_2; \ldots; v_N,t_N) = P(u(t_1)<v_1, u(t_2)<v_2, \ldots, u(t_N)<v_N)$$

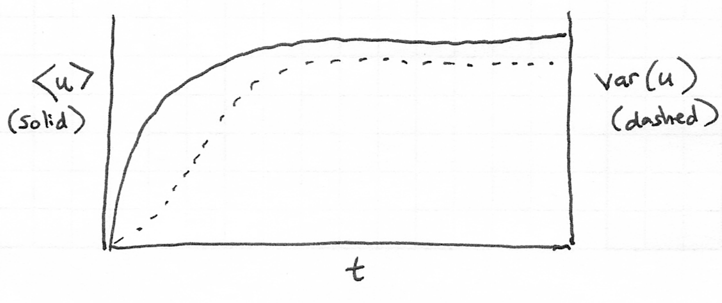
and joint PDF $f_N(v_1,t_1; v_2,t_2; \ldots, v_N,t_N)$. The N-time joint CDF contains joint information about the process $u(t)$ at two or more times.

To completely characterize $u(t)$, we would need to know the above joint CDF for all instances of time, an impossible task.

A statistically stationary process is one in which all multi-time statistics are invariant under a shift in time, that is,

$$f(v_1,t_1+T; v_2,t_2+T; \ldots, v_N,t_N+T) = f(v_1,t_1; v_2,t_2; \ldots; v_N,t_N)$$

where $T > 0$ is arbitrary. Many turbulent flows reach a statistically stationary state after an initial transient period.



For a statistically stationary process, we define the autocovariance as:

$$R(s) \equiv \langle u'(t)\, u'(t+s) \rangle \qquad \text{(independent of } t\text{)}$$

and the autocorrelation function as:

$$\rho(s) \equiv \frac{\langle u'(t)\, u'(t+s) \rangle}{\langle (u'(t))^2 \rangle}$$

By construction:

- $\rho(0) = 1$
- $|\rho(s)| \leq 1$
- $\rho(s) = \rho(-s)$

The integral time-scale of the process is:

Note: This time-scale may be equal to infinity!

$$\bar{\tau} \equiv \int_0^\infty \rho(s)\, ds$$

Note if $\rho(\tau) = 0$, then the process $u(t)$ is uncorrelated at times t and $t+\tau$. In turbulence, we expect $\rho(s) \to 0$ as $s \to \infty$.

On the D2L website, figures have been uploaded illustrating sample paths of five statistically stationary random processes and their autocorrelation functions.

The frequency spectrum is the Fourier transform of the autocovariance (up to scaling):

$$E(\omega) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} R(s)\, e^{-i\omega s}\, ds = \frac{2}{\pi} \int_0^\infty R(s) \cos(\omega s)\, ds$$

The integral:

$$\int_{\omega_a}^{\omega_b} E(\omega)\, d\omega$$

gives the contribution of all modes in the frequency range $\omega_b \leq \omega < \omega_a$ to the variance $\langle (u'(t))^2 \rangle$. This is because:

$$\langle (u'(t))^2 \rangle = R(0) = \int_0^\infty E(\omega)\, d\omega$$

Consequently:

$$\bar{\tau} = \frac{\pi\, E(0)}{2\, R(0)}$$

Spectra are very commonly utilized to study turbulent flows. On the D2L website, a figure has been uploaded to visualize the spectra associated with the aforementioned random processes.

A last comment should be made that the one-time PDF and autocorrelation function do not completely characterize a random process.

A Gaussian process is a random process for which every N-time PDF is joint normal. A Gaussian process is completely characterized by its mean, variance, and autocorrelation function if it is statistically stationary.

Addendum: Consider the autocovariance:

$$R(s) \equiv \langle u'(t)\, u'(t+s) \rangle$$

Because it is a function of the random process at two-times t and $t+s$, the mean is taken with respect to the two-time joint PDF:

$$R(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_1 - \langle u(t) \rangle)(v_2 - \langle \cancel{u(t)} \rangle)\, f_2(v_1, t\,; v_2, t+s)\, dv_1\, dv_2$$

because

$$\langle u(t) \rangle \equiv \langle u(t+s) \rangle$$

Due to stationarity:

$$R(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_1 - \langle u \rangle)(v_2 - \langle u \rangle)\, f_2(v_1, 0\,; v_2, s)\, dv_1\, dv_2$$

Independent of time!

Contains correlation information!

Now, let $\vec{u}(\vec{x},t) = (u_1(\vec{x},t), u_2(\vec{x},t), u_3(\vec{x},t))$ be a time-dependent random vector field representing the velocity of a fluid flow. For every location $\vec{x}$ and time t we have a CDF for the velocity field:

$$\text{One-point, one-time CDF:} \quad F(\vec{v},\vec{x},t) = P(u_i(\vec{x},t) < v_i, \; i=1,2,3)$$

and associated PDF:

$$\text{One-point, one-time PDF:} \quad f(\vec{v};\vec{x},t) = \frac{\partial^3 F(\vec{v},\vec{x},t)}{\partial v_1 \, \partial v_2 \, \partial v_3}$$

At each location and time, the PDF characterizes the random velocity vector, but it does not contain joint information at two or more times or locations. The mean velocity at $(\vec{x},t)$ is:

$$\langle \vec{u}(\vec{x},t) \rangle \equiv \int_{\Omega} \vec{v} \, f(\vec{v};\vec{x},t) \, d\vec{v}$$

and the fluctuating velocity field is:

$$\vec{u}'(\vec{x},t) \equiv \vec{u}(\vec{x},t) - \langle \vec{u}(\vec{x},t) \rangle$$

The one-point, one-time covariances,

$$\langle u_i'(\vec{x},t) \, u_j'(\vec{x},t) \rangle$$

are referred to as Reynolds stresses for reasons that will be explained later.

As before, we can define a joint CDF & PDF associated with a specified set of positions and times:

$$f_N(\vec{v}^{(1)}, \vec{x}^{(1)}, t^{(1)}; \ldots; \vec{v}^{(N)}, \vec{x}^{(N)}, t^{(N)}) \quad \text{N-point PDF}$$

The random field $\vec{u}(\vec{x},t)$ is statistically stationary if all statistics are invariant under a shift in time, and the field is statistically homogeneous if all statistics are invariant under a shift in position. Homogeneous turbulence occurs when the velocity fluctuations are statistically homogeneous.

Turbulent flows can be statistically one-dimensional, two-dimensional, or axisymmetric in analogous fashion.

If a statistically homogeneous vector field is invariant under a rotation or reflection, then it is (statistically) isotropic. Much of turbulence theory centers on homogeneous, isotropic turbulence.

The simplest statistic containing information on the spatial structure of the random field is the two-point, one-time autocovariance:

$$R_{ij}(\vec{r},\vec{x},t) \equiv \langle u_i'(\vec{x},t) \, u_j'(\vec{x}+\vec{r},t) \rangle$$

which is often referred to as the two-point correlation. For homogeneous turbulence, the two-point correlation is independent of $\vec{x}$, so we write:

$$R_{ij}(\vec{r},t) \equiv R_{ij}(\vec{r},\vec{x},t)$$

As the correlation is independent of $\vec{x}$, we can express it in terms of the wavenumber spectrum. The velocity spectrum tensor Φ_{ij} is the Fourier transform of R_{ij}:

$$\Phi_{ij}(\vec{K},t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-i\vec{K}\cdot\vec{r}} R_{ij}(\vec{r},t)\, d\vec{r}$$

where $\vec{K}$ is the wavenumber vector. One has that:

$$R_{ij}(\vec{r},t) = \iiint_{-\infty}^{\infty} e^{i\vec{K}\cdot\vec{r}} \Phi_{ij}(\vec{K},t)\, d\vec{K}$$

as well. Note that:

$$\langle u_i' u_j' \rangle = R_{ij}(0,t) = \iiint_{-\infty}^{\infty} \Phi_{ij}(\vec{K},t)\, d\vec{K}$$

so $\Phi_{ij}(\vec{K},t)$ represents the contribution to the covariance $\langle u_i' u_j' \rangle$ (which is one of the components of the Reynolds stress tensor) of velocity modes with wave number $\vec{K}$.

The energy spectrum function is defined as:

$$\underset{\text{scalar}}{E(K,t)} \equiv \iiint_{-\infty}^{\infty} \frac{1}{2}\Phi_{ii}(\vec{K},t)\, \delta(|\vec{K}|-K)\, d\vec{K}$$

Dirac delta distribution

Note that:

$$\frac{1}{2}\langle (u_i')^2 \rangle = \int_0^{\infty} E(K,t)\, dK$$

so $E(K,t)\,dK$ represents the contribution to the turbulent kinetic energy from all modes with $|\vec{K}|$ in the range $K \le |\vec{K}| \le K + dK$. For isotropic turbulence, $\Phi_{ij}(\vec{K})$ can be completely determined by $E(K)$.

The two-point correlation and velocity spectrum tensors contain a wealth of information about the statistics of a random vector field. They contain information about the directional dependence of correlation and the directions of the vector field itself.