The Scales of Turbulent Motion - The Kolmogorov Hypotheses and Energy Spectrum:

Several fundamental questions remain unanswered:

1. What is the size of the smallest eddies which dissipate energy?
2. As ℓ decreases, how do $u(\ell)$ and $\tau(\ell)$ scale?

These questions were answered in 1941 by Kolmogorov who expanded and quantified Richardson's theory of turbulence. Specifically, Kolmogorov provided three hypotheses regarding the statistics of small-scale turbulent motion. These hypotheses invoke the following concepts on the local statistics of a turbulent flow:

Local Homogeneity: The turbulence located in the domain $\mathcal{G}$ is locally homogeneous if every N-point PDF:

$$f_N(\vec{v}^{(1)}, \vec{y}^{(1)}; \dots; \vec{v}^{(N)}, \vec{x}^{(N)}) \qquad \vec{y}^{(i)} = \vec{x}^{(i)} - \vec{x}^{(0)}$$

where $\vec{x}^{(i)} \in \mathcal{G}$ for the velocity difference $\vec{u}(\vec{x},t) - \vec{u}(\vec{x}^{(0)},t)$ is independent of $\vec{x}^{(0)} \in \mathcal{G}$ and $\vec{u}(\vec{x}^{(0)},t)$.

Local Isotropy: The turbulence located in the domain $\mathcal{G}$ is locally isotropic if it is locally homogeneous and every N-point PDF for the velocity difference $\vec{u}(\vec{x},t) - \vec{u}(\vec{x}^{(0)},t)$ with $\vec{x}^{(0)} \in \mathcal{G}$ is invariant with respect to rotations and reflections of the coordinate axes (of $\vec{y}$).

Note the above two conditions are much weaker than the standard notions of statistical homogeneity and isotropy. Not only are they local definitions, but they involve velocity differences rather than the velocity itself. By working with velocity differences, Kolmogorov was able to postulate hypotheses which apply to any turbulent flow rather than only those which are statistically homogeneous (at least in one direction).

Kolmogorov's first hypothesis concerns the isotropy of small-scale turbulent motions. In general, the largest eddies are anisotropic and are affected by walls and other boundary conditions. Kolmogorov argued directional biases are lost in the chaotic scale-reduction process, by which energy is transferred to smaller and smaller scales.

Kolmogorov's Hypothesis of Local Isotropy: At sufficiently high Re, the small-scale turbulent motions ($\ell \ll \ell_0$) are statistically isotropic.

More precisely, at sufficiently high Re, the turbulence in any sufficiently small domain $\mathcal{G}$ (with $|\mathcal{G}| \ll \mathcal{L}$) located away from the boundary of the flow is locally isotropic.

We introduce a length scale ℓ_{EI} to mark the transition between anisotropic large eddies ($\ell > \ell_{EI}$) and isotropic small eddies ($\ell < \ell_{EI}$). Computations and experiments suggest $\ell_{EI} \approx \frac{1}{6} \ell_0$.

Note that Kolmogorov's hypothesis of local isotropy implies all information about the geometry of the large eddies is lost in the small scales. Consequently, the statistics of the small-scale motions are in a sense universal.

Kolmogorov's next hypothesis is predicated on the notion that interactions at scale $\ell < \ell_{EI}$ with scales much larger than ℓ are weak, and the two dominant processes for such scales are the transfer of energy to successively smaller scales and viscous dissipation. We shall later see the energy transfer rate is roughly the dissipation rate, which motivates the following:

Kolmogorov's First Similarity Hypothesis: At sufficiently high Re, the statistics of small-scale turbulent motions ($\ell < \ell_{EI}$) have a universal form that is uniquely determined by ν and ε.

More precisely, at sufficiently high Re, the turbulence in any sufficiently small domain (with $|\mathcal{G}| < \ell_{EI}$) is such that every N-point PDF for the velocity difference $\vec{u}(\vec{x},t) - \vec{u}(\vec{x}^{(0)},t)$ with $\vec{x}^{(0)} \in \mathcal{G}$ is uniquely determined by ν and ε.

The size range $\ell < \ell_{EI}$ is named the universal equilibrium range, and small eddies in this range can quickly adapt to maintain a dynamic equilibrium with the energy-transfer rate T_{EI} imposed by the large eddies.

There are unique scales that can be formed given ε and ν. These are the Kolmogorov scales:

$$\eta \equiv \left(\nu^3/\varepsilon\right)^{1/4}$$

$$u_\eta \equiv (\varepsilon\nu)^{1/4}$$

$$\tau_\eta \equiv (\nu/\varepsilon)^{1/2}$$

Immediately note that $Re_\eta \equiv \frac{\eta u_\eta}{\nu} = 1$ and $\varepsilon = \frac{\nu}{\tau_\eta^2}$, so the Kolmogorov scales by construction characterize the smallest, dissipative eddies.

Note that it is not possible to form a non-dimensional parameter from ν and ε alone. Consequently, by dimensional analysis, it follows that on the small scales, all high Re flows are statistically similar when they are scaled by the Kolmogorov scales.

Additionally note that:

$$\eta/\ell_0 \sim Re^{-3/4}$$

$$u_\eta/u_0 \sim Re^{-1/4}$$

$$\tau_\eta/\tau_0 \sim Re^{-1/2}$$

so obviously at high Re there is a clear separation between the largest scales and the Kolmogorov scales. Consequently, there is some ℓ such that $\ell \gg \eta$ and $\ell \ll \ell_0$. Since $\ell \gg \eta$, we expect that eddies of size ℓ are unaffected by viscosity. As $\ell \ll \ell_0$, the first similarity hypothesis of Kolmogorov applies. This yields Kolmogorov's second similarity hypothesis.

Kolmogorov's Second Similarity Hypothesis: At sufficiently high Re, the statistics of small-scale turbulent motions with $\eta \ll \ell \ll \ell_0$ have a universal form that is uniquely determined by ε.

More precisely, at sufficiently high Re, every N-point PDF:

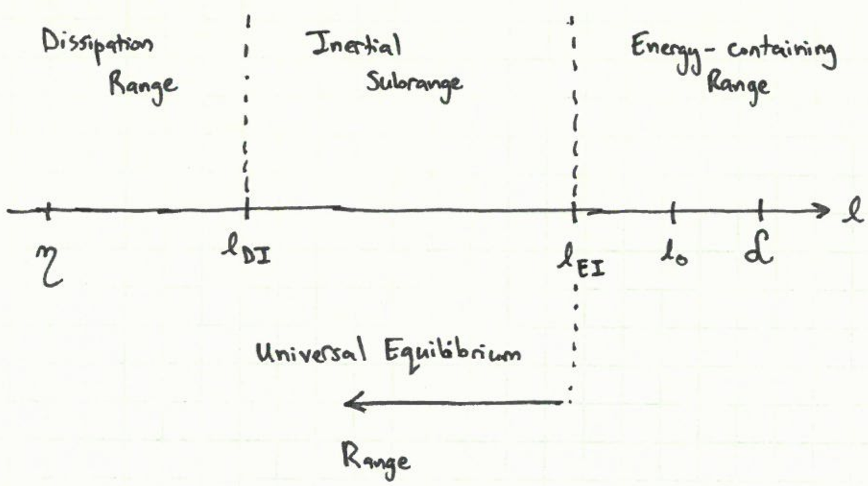
$$f_N\left(\vec{v}^{(1)}, \vec{y}^{(1)}; \dots; \vec{v}^{(N)}, \vec{y}^{(N)}\right) \qquad \vec{y}^{(i)} = \vec{x}^{(i)} - \vec{x}^{(0)}$$

for the velocity difference $\vec{v}(\vec{x},t) - \vec{u}(\vec{x}^{(0)},t)$, with $\vec{x}^{(0)}$ and $\vec{x}^{(i)}$ located in a domain $\mathcal{G}$ of size $|\mathcal{G}| < \ell_{EI}$, is independent of ν and a universal function of ε if:

$$|\vec{x}^{(i)} - \vec{x}^{(j)}| \gg \eta \qquad i \neq j$$

We introduce a length scale ℓ_{DI} to mark the transition between the regions where Kolmogorov's first similarity hypothesis only applies and where both similarity hypotheses apply. Computations and experiments suggest $\ell_{DI} \approx 60\eta$.

The length scale ℓ_{DI} splits the universal equilibrium range into the inertial subrange ($\ell_{DI} < \ell < \ell_{EI}$) and the dissipation range ($\ell < \ell_{DI}$). Motions in the inertial range are determined by inviscid inertial effects whereas motions in the dissipative range are responsible for virtually all of the dissipation of kinetic energy. The bulk of the energy is contained in the energy-containing range ($\ell > \ell_{EI}$).



Given ε and length scale ℓ, we can postulate velocity and time-scales for eddies in the inertial subrange:

$$u(\ell) = (\varepsilon\ell)^{1/3} \sim u_0 (\ell/\ell_0)^{1/3}$$

$$\tau(\ell) = (\ell^2/\varepsilon)^{1/3} \sim \tau_0 (\ell/\ell_0)^{2/3}$$

So the velocity and time-scales $u(\ell)$ and $\tau(\ell)$ decrease with ℓ. Moreover, the rate at which energy is transferred from eddies larger than ℓ to eddies smaller than ℓ is proportional to the kinetic energy $u(\ell)^2$ over the time $\tau(\ell)$, or mathematically,

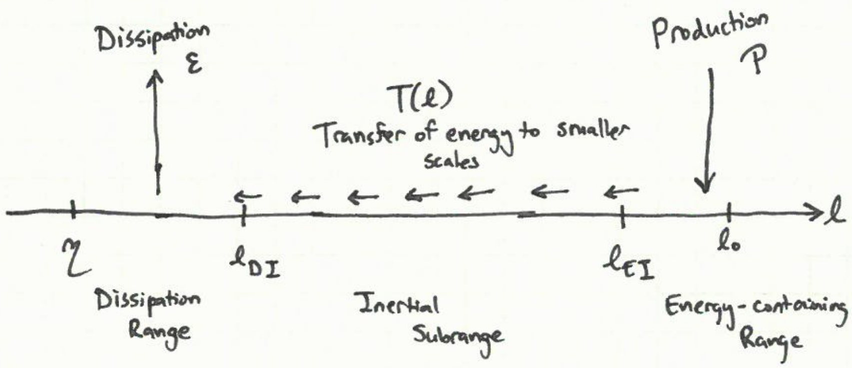
$$\text{Rate of Energy Transfer} = T(\ell) \sim u(\ell)^2/\tau(\ell) = \varepsilon$$

Consequently, $T(\ell)$ is independent of ℓ and proportional to the dissipation. In fact,

$T(\ell) = \varepsilon$. This gives:

$$T_{EI} \equiv T(\ell_{EI}) = T(\ell) = T_{DI} \equiv T(\ell_{DI}) = \varepsilon$$

So, the rate of transfer of energy from the large-scales, T_{EI}, determines the constant rate of energy transfer through the inertial range, $T(\ell)$, and hence the rate at which the energy leaves the inertial range and enters the dissipation range, T_{DI}, and these all equal the dissipation rate ε.



We finish here by discussing how the turbulent kinetic energy is distributed among eddies of different sizes. To do so, we employ the energy spectrum function $E(K,t)$ which is the energy per unit wave number. We define the energy spectrum as before:

$$R_{ij}(\vec{r},t) \equiv \langle u_i'(\vec{x},t)\, u_j'(\vec{x}+\vec{r},t) \rangle$$

$$\Phi_{ij}(\vec{K},t) \equiv \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-i\vec{K}\cdot\vec{r}} R_{ij}(\vec{r},t)\, d\vec{r}$$

$$E(K,t) \equiv \iiint_{-\infty}^{\infty} \frac{1}{2} \Phi_{ii}(\vec{K},t)\, \delta(|\vec{K}| - \underbrace{K}_{\text{scalar!}})\, d\vec{K}$$

and so:

$$\underbrace{k}_{\text{turbulent K.E.}} \equiv \int_0^{\infty} \underbrace{E(K,t)}_{\text{energy density}}\, dK$$

If a flow is statistically stationary, $E(K,t) = E(K)$, and from Kolmogorov's second similarity hypothesis, the spectrum has the following form in the inertial subrange (it can only depend on K and ε):

$$E(K) = C\, \varepsilon^{2/3} K^{-5/3}$$

where C is a universal constant. This is the famous Kolmogorov $-5/3$ spectrum. Unfortunately, we have no explicit form for the energy-containing and viscous/dissipation ranges, and in fact we expect none for the energy-containing range as it does not have a universal form.