Turbulent Viscosity Models – The k-ε Model

To avoid specifying a mixing-length scale, we can recognize that the turbulence quantities k and ε naturally form a length-scale:

$$\ell^* \sim k^{3/2}/\varepsilon$$

This inspires the turbulent viscosity model:

$$\nu_T = C_\mu \, k^2/\varepsilon$$

where C_μ is a model constant. However, we find ourselves again in the position of needing to specify the quantity ε either by a priori knowledge or a model. The premise of the two-equation k-ε model is to introduce a second model transport equation for ε in addition to the previously introduced model equation for k.

To motivate the model equation for ε, we appeal to the exact transport equation for ε. The form of this equation is quite complicated (see "Turbulence Modeling for CFD" by Wilcox, for example), so we alternately present it in the compact form:

$$\frac{\bar{D}\varepsilon}{\bar{D}t} = \underbrace{\vec{\nabla}\cdot(\nu\vec{\nabla}\varepsilon)}_{\text{Viscous Diffusion of Dissipation}} + \underbrace{D_\varepsilon}_{\text{Transport of Dissipation}} + \underbrace{P_\varepsilon}_{\text{Production of Dissipation}} - \underbrace{\Phi_\varepsilon}_{\text{Dissipation of Dissipation}}$$

The transport of dissipation D_ε represents transport processes due to turbulent convection and pressure transport. Hence, to model it, we turn to the gradient-diffusion hypothesis, obtaining:

$$D_\varepsilon = \vec{\nabla}\cdot\left(\frac{\nu_T}{\sigma_\varepsilon}\vec{\nabla}\varepsilon\right)$$

where σ_ε is a dimensionless constant analogous to the turbulent Prandtl number. The production of dissipation P_ε is more difficult to model. To begin, we ask the question: what should the production depend on? The standard answer to this question is given by:

$$P_\varepsilon = P_\varepsilon\left(a_{ij}, \frac{\partial \bar{u}_i}{\partial x_j}, k, \varepsilon\right)$$

where we have explicitly separated the effects of anisotropy of the Reynolds stress and the turbulent kinetic energy. Appealing dimensional analysis and requiring P_ε to be an invariant, we obtain:

$$P_\varepsilon = -C_{\varepsilon 1}\frac{\varepsilon}{k}a_{ij}\frac{\partial \bar{u}_i}{\partial x_j}$$

$$\xrightarrow{\text{Assumed Constant}} \; = C_{\varepsilon 1}\frac{P\varepsilon}{k} \quad \text{(via the turbulent viscosity hypothesis)}$$

The dissipation (or destruction) of dissipation

$$\Phi_\varepsilon = (2\nu)^2\left\langle \frac{\partial}{\partial x_k}(S_{ij}')\frac{\partial}{\partial x_k}(S_{ij}')\right\rangle$$

is also tricky to model. Recognizing that dissipation occurs in the dissipation range, we write:

$$\Phi_\varepsilon = \Phi_\varepsilon(k, \varepsilon)$$

Appealing to dimensional analysis again yields:

$$\Phi_\varepsilon = C_{\varepsilon 2} \, \varepsilon^2/k$$

where $C_{\varepsilon 2}$ is yet another dimensionless constant. Introducing the model constants proposed by Launder and Sharma in 1974, the full k-ε model is then:

Reynolds Stress Model: $\langle u_i' u_j' \rangle = -2\nu_T \bar{S}_{ij} + \frac{2}{3} k \delta_{ij}$

Turbulent Viscosity: $\nu_T = C_\mu k^2/\varepsilon$

Model Equation for k: $\bar{D}k/\bar{D}t = \vec{\nabla}\cdot\left(\left(\nu + \frac{\nu_T}{\sigma_k}\right)\vec{\nabla}k\right) + P - \varepsilon$

Model Equation for ε: $\bar{D}\varepsilon/\bar{D}t = \vec{\nabla}\cdot\left(\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\vec{\nabla}\varepsilon\right) + C_{\varepsilon 1}\frac{P\varepsilon}{k} - C_{\varepsilon 2}\,\varepsilon^2/k$

Constants: $C_\mu = 0.09$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$
(Launder and Sharma, 1974)

The two-equation k-ε model is complete. There are no flow-dependent specifications such as the mixing-length. In fact, the k-ε model is arguably the simplest complete turbulence model and hence it has the broadest range of applicability. For this reason, the k-ε model is the most widely used turbulence model, and it is incorporated in most commercial CFD codes.

The values of the standard k-ε constants represent a compromise. For any particular flow, the accuracy may be improved by adjusting these constants. In what follows, we examine the veracity of the k-ε model and the standard k-ε constants by studying various flows.

Decaying Homogeneous Turbulence:

We begin by studying decaying homogeneous turbulence. In this setting, the production is zero and there is no convection or transport. Hence the k-ε model reduces to:

$$\frac{dk}{dt} = -\varepsilon$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

The above system has the solution:

$$k(t) = k_0 \left(t/t_0\right)^{-n}$$

$$\varepsilon(t) = \varepsilon_0 \left(t/t_0\right)^{-(n+1)}$$

where:

$$t_0 = n\, k/\varepsilon_0$$

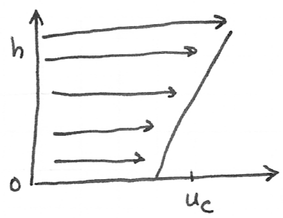
$$n = 1/(C_{\varepsilon 2} - 1)$$

This power-law decay is consistent with that observed in grid turbulence. Assuming that $C_{\varepsilon 2} = 1.92$, the k-ε model predicts that $n = 1.09$, which is nearly in the range

$n \in [1.15, 1.45]$ obtained from experimental observations.

Homogeneous Shear Flow:

The second case we consider is homogeneous shear flow:



In this setting, the flow becomes self-similar after a development time and the non-dimensional parameters Sk/ε and P/ε become constant. This suggests that the turbulence time scale $\tau = k/\varepsilon$ also becomes constant since S is fixed. As the statistics of homogeneous shear flow are self-similar, we obtain:

$$0 = \frac{d}{dt}\left(k/\varepsilon\right) = (C_{\varepsilon 2}-1) - (C_{\varepsilon 1}-1)\left(\frac{P}{\varepsilon}\right)$$

from the k-ε model. Assuming that $C_{\varepsilon 1} = 1.44$ and $C_{\varepsilon 2} = 1.92$, the k-ε model predicts that $P/\varepsilon = 2.1$ which is in the ballpark of the value $P/\varepsilon \approx 1.7$ observed in experiments and DNS.

Channel Flow:

The third case we consider is channel flow. For high-Re, fully-developed turbulent channel flow the statistical quantities only depend on y, so the k-ε model reduces to:

$$0 = \frac{d}{dy}\left(\frac{\nu_T}{\sigma_k}\frac{dk}{dy}\right) + P - \varepsilon$$

$$0 = \frac{d}{dy}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{d\varepsilon}{dy}\right) + C_{\varepsilon 1}\frac{P\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

if we ignore the viscosity ν, which is valid for high-Re outside of the viscous sublayer and buffer layer. In the log-law region, production and dissipation balance, and in fact:

$$\frac{u_\tau^3}{\kappa y} = -\langle u'v'\rangle \frac{d\bar{u}}{dy} = P = \varepsilon$$

since the Reynolds stress equals the total stress in the log-law region. The k-ε model then implies that k is uniform in the log-law region, which is approximately correct. The turbulent viscosity model implies:

$$\langle u'v'\rangle = -\nu_T \frac{d\bar{u}}{dy} = -C_\mu \frac{k^2}{\varepsilon}\frac{d\bar{u}}{dy}$$

and hence comparing with above:

$$\frac{u_\tau^2}{k} = C_\mu^{1/2}$$

Assuming that $C_\mu = 0.09$, the k-ε model predicts that $\frac{u_\tau^2}{k} = 0.3$ which is consistent with experiment. In the ε-equation, the equality of P and ε leads to a net sink equal to $-(C_{\varepsilon 2} - C_{\varepsilon 1})\varepsilon^2/k$ that varies as y^{-2}. This net sink is balanced by the diffusion of ε away from the wall. Substituting the above relations directly into the ε-equation yields:

$$-(C_{\varepsilon 2}-C_{\varepsilon 1})\,\varepsilon^2/k + \frac{d}{dy}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{d\varepsilon}{dy}\right) = 0$$

$$\Rightarrow -(C_{\varepsilon 2}-C_{\varepsilon 1})\frac{u_\tau^6}{\kappa^2 y^2 k} + \frac{u_\tau^4}{\sigma_\varepsilon y^2} = 0$$

Solving for κ yields:

$$\kappa^2 = \sigma_\varepsilon C_\mu^{1/2}(C_{\varepsilon 2}-C_{\varepsilon 1})$$

The standard k-ε constants then yield $\kappa = 0.43$, which is close to the commonly accepted value of $\kappa = 0.41$.

To complete the k-ε model, we need to introduce boundary conditions. There are actually two boundary conditions that naturally apply on k at a no-slip wall:

$$k|_{wall} = \frac{\partial k}{\partial n}\Big|_{wall} = 0$$

These two conditions turn out to be enough for the coupled k and ε equations. It is also common to note that the k-equation evaluated at the wall implies:

$$\varepsilon|_{wall} = \nu \frac{\partial^2 k}{\partial n^2}\Big|_{wall}$$

Inlet boundary conditions present more of a challenge. At the inlet it is more natural to apply a boundary condition to/on the turbulence intensity:

$$I = \frac{\sigma}{\bar{u}} = \frac{\sqrt{2/3\,k}}{\bar{u}}$$

or the turbulent length scale $l^* = C_\mu k^{3/2}/\varepsilon$. For example, for fully-developed pipe flow:

$$\begin{cases} I \approx 0.16\, Re_{d_h}^{-1/8} \\ l^* \approx 0.038\, d_h \end{cases}$$

where d_h is the hydraulic diameter and Re_{d_h} is the Reynolds number based on d_h.

For all its sophistication, the k-ε model suffers from a variety of difficulties. One problem, of course, is that it is still an eddy viscosity model and hence not sensitive to the state of rotation of the mean. A much more severe problem is that the k-ε model does not perform well near walls. This is not surprising since the model is really only valid for mildly inhomogeneous and anisotropic turbulence. For this reason, there has been considerable effort invested in designing damping strategies to fix up the model near walls.

It should again be emphasized that the values of the standard k-ε model constants represent a compromise. For any particular flow, it is likely that the accuracy of the model calculations can be improved by adjusting the constants. For decaying turbulence, $C_{\varepsilon 2} = 1.77$ is more suitable than $C_{\varepsilon 2} = 1.92$ to get the correct rate of decay. A well-known deficiency of the k-ε model is that it significantly overpredicts the rate of spreading for the round jet. This problem again can be remedied by adjusting $C_{\varepsilon 1}$ and/or $C_{\varepsilon 2}$. For a complete, generally applicable model, a single specification of the model constants is required, and the standard values are chosen (with subjective judgment) to give the 'best performance' over a range of flows.