Homogeneous Isotropic Turbulence: Dynamics of Fourier Modes

Further insight into the dynamics of homogeneous isotropic turbulence can be gained by considering the dynamics of individual modes. Consider a cube $\Omega = (0, \mathcal{L})^3$ in physical space, where the length scale $\mathcal{L}$ is large compared with the turbulent integral scale. Assuming the velocity field is periodic, i.e.,

$$\vec{u}(\vec{x} + \vec{N}\mathcal{L}, t) = \vec{u}(\vec{x}, t) \qquad \vec{N} = \text{integer vector}$$

then we may decompose it as:

$$\vec{u}(\vec{x}, t) = \sum_{\vec{K}} e^{i\vec{K}\cdot\vec{x}}\, \hat{u}(\vec{K}, t) = \sum_{K_1=-\infty}^{\infty} \sum_{K_2=-\infty}^{\infty} \sum_{K_3=-\infty}^{\infty} e^{i\vec{K}\cdot\vec{x}}\, \hat{u}(\vec{K}, t)$$

where:

$$\hat{u}_i(\vec{K}, t) = \underbrace{\mathcal{F}_{\vec{K}}}_{\text{Fourier transform operator}} \{u_i(\vec{x}, t)\} \frac{1}{\mathcal{L}^3} \int_{\Omega} u_i(\vec{x}, t)\, e^{-i\vec{K}\cdot\vec{x}}\, d\vec{x}$$

To obtain an evolution equation for the modes, we need to apply the Fourier transform operator to the Navier-Stokes equations. Note, by the properties of Fourier transforms:

$$\mathcal{F}_{\vec{K}}\left\{\frac{\partial u_j}{\partial t}\right\} = \frac{\partial \hat{u}_j}{\partial t}$$

$$\mathcal{F}_{\vec{K}}\left\{\nu \frac{\partial^2 u_j}{\partial x_k \partial x_k}\right\} = -\nu K^2 \hat{u}_j$$

$$\mathcal{F}_{\vec{K}}\left\{-\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right\} = -i K_j \hat{p}$$

$$\mathcal{F}_{\vec{K}}\left\{\frac{\partial}{\partial x_k}(u_j u_k)\right\} = \hat{G}_j(\vec{K}, t) \quad \text{Definition}$$

Thus:

$$\frac{\partial \hat{u}_j}{\partial t} + \nu K^2 \hat{u}_j = -i K_j \hat{p} - \hat{G}_j \qquad \text{Momentum Balance for Each Mode}$$

Continuity gives:

$$0 = \mathcal{F}_{\vec{K}}\left\{\frac{\partial u_j}{\partial x_j}\right\} = i K_j u_j$$

So:

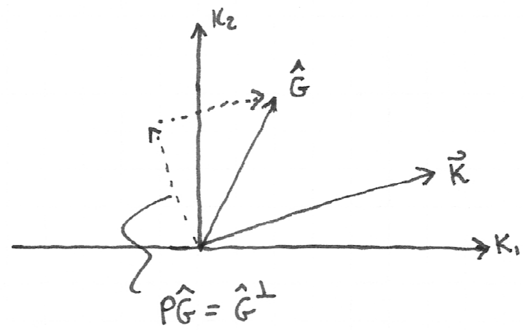
$$0 = -iK_j\left(\frac{\partial \hat{u}_j}{\partial t} + \nu K^2 \hat{u}_j + iK_j\hat{p} + \hat{G}_j\right) = K^2\hat{p} - iK_j\hat{G}_j$$

$$\Rightarrow \hat{p} = iK_j\hat{G}_j$$

Thus:

$$\frac{\partial \hat{u}_j}{\partial t} + \nu K^2 \hat{u}_j = -\underbrace{\left(\delta_{jk} - \frac{K_j K_k}{K^2}\right)}_{P_{jk} = \text{Projection Tensor}} \hat{G}_k = -P_{jk}\hat{G}_k = -\hat{G}_k^{\perp}$$

The projection tensor projects a vector onto the plane normal to $\vec{K}$, as illustrated below:



Note that $u_j = P_{jk} u_k$ due to continuity. Hence, P_{jk} projects onto a divergence-free space in Fourier space.

Note that:

$$\hat{G}_j(\vec{K},t) \equiv \mathcal{F}_{\vec{K}}\left\{ \frac{\partial}{\partial x_k}(u_j u_k) \right\}$$

$$= i K_k \, \mathcal{F}_{\vec{K}} \{ u_j u_k \}$$

$$= i K_k \, \mathcal{F}_{\vec{K}} \left\{ \left(\sum_{\vec{K}'} \hat{u}_j(\vec{K}') e^{i\vec{K}'\cdot\vec{x}} \right) \left(\sum_{\vec{K}''} \hat{u}_k(\vec{K}'') e^{i\vec{K}''\cdot\vec{x}} \right) \right\}$$

$$= i K_k \sum_{\vec{K}'} \sum_{\vec{K}''} \hat{u}_j(\vec{K}') \hat{u}_k(\vec{K}'') \, \delta_{\vec{K},\vec{K}'+\vec{K}''}$$

$$\delta_{\vec{K},\vec{K}'+\vec{K}''} = \begin{cases} 1 & \text{if } \vec{K}'+\vec{K}''=\vec{K} \\ 0 & \text{otherwise} \end{cases}$$

$$= i K_k \sum_{\vec{K}'} \hat{u}_j(\vec{K}') \hat{u}_k(\vec{K}-\vec{K}')$$

$$= i K_k \underbrace{(\hat{u}_j * \hat{u}_k)}_{\text{Discrete convolution}}(\vec{K})$$

So:

$$\boxed{\left(\frac{\partial}{\partial t} + \nu K^2 \right) \hat{u}_j(\vec{K},t) = - i K_\ell \, P_{jk}(\vec{K}) \, (\hat{u}_k * \hat{u}_\ell)(\vec{K},t)}$$

Evolution of Fourier Modes

The left-hand-side above only involves $\hat{u}$ at $\vec{K}$. In contrast, the right-hand-side involves $\hat{u}$ at $\vec{K}'$ and $\vec{K}''$ such that $\vec{K}'+\vec{K}''=\vec{K}$. Thus, in wavenumber space, the convection term is nonlinear and nonlocal, involving the interaction of wavenumber triads $\vec{K}$, $\vec{K}'$, and $\vec{K}''$ s.t. the first is equal to the sum of the second and third.

The covariance of two Fourier coefficients is one of the simplest statistics in Fourier space:

$$\langle \hat{u}_i(\vec{K}',t) \hat{u}_j(\vec{K},t) \rangle$$

A direct computation gives:

$$\langle \hat{u}_i(\vec{K}',t) \hat{u}_j(\vec{K},t) \rangle = \langle \mathcal{F}_{\vec{K}'}\{u_i(\vec{x},t)\} \, \mathcal{F}_{\vec{K}}\{u_j(\vec{x},t)\} \rangle$$

$$= \frac{1}{\mathcal{L}^6} \int_{\Omega}\int_{\Omega} \langle u_i(\vec{x}',t)\, u_j(\vec{x},t)\rangle\, e^{-i(\vec{K}'\cdot\vec{x}' + \vec{K}\cdot\vec{x})}\, d\vec{x}\, d\vec{x}'$$

$\vec{x} = \vec{x}' + \vec{r}$

$$= \frac{1}{\mathcal{L}^6} \int_{\Omega}\int_{\Omega} R_{ij}(\vec{r},t)\, e^{-i\vec{K}\cdot\vec{r}}\, e^{-i\vec{x}'\cdot(\vec{K}+\vec{K}')}\, d\vec{r}\, d\vec{x}'$$

$$= \mathcal{F}_{\vec{K}}\{R_{ij}(\vec{r},t)\}\, \delta_{\vec{K},-\vec{K}'}$$

Moreover:

$$k(t) = \tfrac{1}{2}\langle u_i u_i\rangle$$

$$= \tfrac{1}{2} R_{ii}(0,t)$$

$$= \tfrac{1}{2} \sum_{\vec{K}}{}' \mathcal{F}_{\vec{K}}\{R_{ii}(\vec{r},t)\}$$

$$= \sum_{\vec{K}} \tfrac{1}{2} \langle \hat{u}_i(-\vec{K},t)\, \hat{u}_i(\vec{K},t)\rangle$$

complex symmetry due to isotropy

$$= \sum_{\vec{K}}{}' \underbrace{\tfrac{1}{2} \langle \hat{u}_i^*(\vec{K},t)\, \hat{u}_i(\vec{K},t)\rangle}_{\hat{E}(\vec{K},t) \,=\, \text{energy density per Fourier mode}}$$

$$= \sum_{\vec{K}}{}' \hat{E}(\vec{K},t)$$

Consequently, the kinetic energy associated with each mode evolves like:

$$\boxed{\frac{\partial}{\partial t}\hat{E}(\vec{K},t) = \hat{T}(\vec{K},t) - 2\nu K^2 \hat{E}(\vec{K},t)}$$

inertial processes   viscous processes

where:

$$\hat{T}(\hat{K},t) = K_\ell P_{jk}\, \mathcal{R}\Big\{ i \underbrace{\sum_{\vec{K}'} \langle \hat{u}_j(\vec{K})\, \hat{u}_k^*(\vec{K}')\, \hat{u}_\ell^*(\vec{K}-\vec{K}')\rangle}_{\langle \hat{u}_j (\hat{u}_k * \hat{u}_\ell)\rangle} \Big\}$$

represents the transfer of energy between modes. It is easily shown that:

$$\sum_{\vec{K}}{}' 2\nu K^2 \hat{E}(\vec{K},t) = \varepsilon \qquad \text{Dissipation}$$

so:

$$\sum_{\vec{K}} \hat{T}(\vec{K},t) = 0$$

There is a clear parallel between the evolution equation for $\hat{E}$ and the evolution equation for E. In fact, the two are related via Fourier transforms. The advantage of the formulation in terms of Fourier modes is it provides a clear quantification of the energy at different scales and an explicit expression for the energy-transfer rate is obtained.

There are two types of interactions between wavenumber triads: one is interactions between wavenumbers of the same order. Such interactions are called local. The other type of triad interaction is a distant interaction. Distant interactions couple large-scale modes with small-scales, yet Kolmogorov's hypotheses suggest small scales should be unaffected by large scales. Distant interactions appear to have the possibility of small-scale anisotropy, but there is little computational evidence of this.