Random Variables and Probability Spaces:

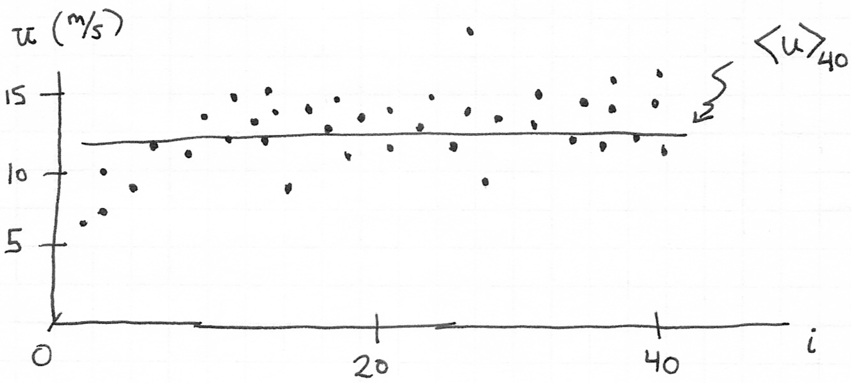
Let u be the specified component of velocity at a specified position and time as measured from an experiment.

For each experiment we conduct, we obtain a different value for u!

In this sense, u is a random variable.

If we conduct N experiments, we can average over these to obtain the ensemble average for u:

Ensemble Average: $\langle u \rangle_N = \frac{1}{N}\sum_{i=1}^{N} u^{(i)}$ ← i^{th} observation of u



We can define a set to denote the event that $u < 10\,m/s$:

$$A \equiv \{u < 10\,m/s\}$$

and we can easily define the probability that this event occurs as:

$$P(A) = \frac{1}{N}\sum_{i=1}^{N} I_A(u^{(i)}) < 1$$

where:

$$I_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{else} \end{cases}$$

To make this framework more formal, let us introduce:

$\Omega \equiv$ Sample space = set of all possible values of u

$v \in \Omega$ gives a random sample-space variable.

$\mathcal{A} \equiv$ set of all subsets (events!) of Ω

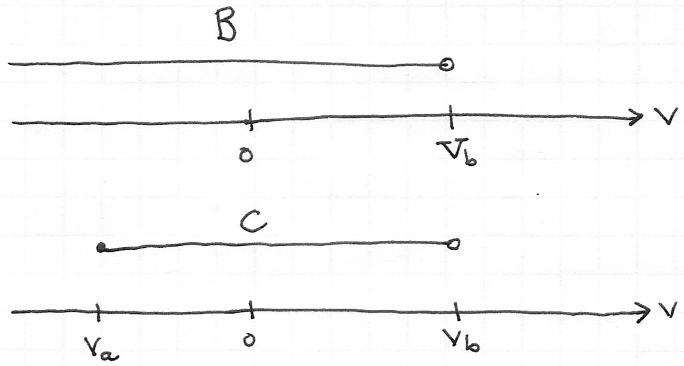
$A_i \in \mathcal{A}$ corresponds to an event.

For example, we may have the two events $B, C \in \mathcal{A}$ defined by:

$$B = \{ v < v_b \}$$

$$C = \{ v_a \leq v < v_b \}$$

which look like:



Define probability P to be a map from $\mathcal{A}$ to $[0,1]$ such that:

$$P(\Omega) = 1$$

and when $A_i \in \mathcal{A}$ are disjoint subsets of Ω, then:

$$P(\bigcup_i A_i) = \sum_i P(A_i)$$

For example, if B and C are defined as above and $A = \{ v < v_a \}$ then:

$$P(B) = P(A \cup C) = P(A) + P(C)$$

The three objects $(\Omega, \mathcal{A}, P)$ give a complete formal description of a probability space of random variables!

~~Given the probability, we can define the cumulative distribution function~~

The above framework works for both discrete and continuous sample spaces!

However, any discrete random variable can be represented as a continuous random variable using the notion of a delta-function distribution. See page 51 of Pope.

Without loss of generality, we now assume:

$$\Omega = (-\infty, \infty)$$

Given the probability P, we can define the cumulative distribution function (CDF) as:

$$F(v) = P(u < v)$$

The CDF has several important properties:

- $F(-\infty) = 0$
- $F(\infty) = 1$
- $F(v_b) - F(v_a) = P(v_a \le u < v_b) \ge 0$ for $v_b > v_a$

The third property listed above expresses the fact that the CDF is non-decreasing.

The probability density function (PDF) is the derivative of the CDF:

$$f(v) \equiv \frac{dF(v)}{dv}$$

The PDF also has several important properties, including:

- $f(v) \ge 0$
- $f(-\infty) = f(\infty) = 0$
- $\int_{\Omega} f(v)\,dv = 1$

Perhaps the most important property of the PDF is:

$$P(v_a \le u < v_b) = \int_{v_a}^{v_b} f(v)\,dv$$

which explicitly demonstrates the PDF is the probability per unit distance in the sample space.

probability density!

Two random variables that have the same PDF are said to be identically distributed or statistically identical.