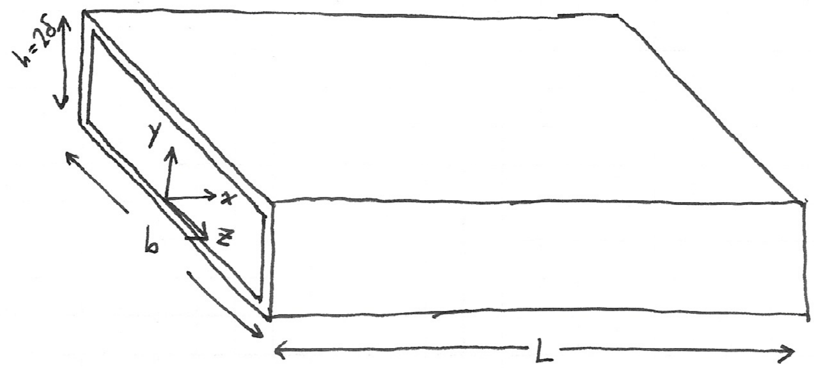
Turbulent Channel Flow – Characterization and Mean-Flow Equations:

The first wall-bounded shear flow we consider is channel flow, the flow through a rectangular duct of height $h = 2\delta$, length L, and width b. The situation is depicted below:



The velocity field in the duct is $\vec{u} = (u, v, w)$. We assume:

1. The duct is long ($L/\delta \gg 1$).
2. The duct is wide ($b/\delta \gg 1$).
3. The mean flow is predominantly in the axial (x) direction, with the mean velocity varying mainly in the cross-stream (y) direction.

The extent of the channel in the spanwise (z) direction is large compared with δ so that (remote from the end walls) the flow is statistically independent of z and the mean cross-stream velocity $\bar{w}$ is zero. Thus:

$$\bar{u} = \bar{u}(y) \qquad \langle (u')^2 \rangle = \langle (u')^2 \rangle (y)$$
$$\bar{v} = \bar{v}(y) \qquad \langle (v')^2 \rangle = \langle (v')^2 \rangle (y)$$
$$\bar{w} = 0 \qquad \langle (w')^2 \rangle = \langle (w')^2 \rangle (y)$$
$$\langle u'v' \rangle = \langle u'v' \rangle (y)$$

Note that the bottom and top walls are at $y = 0$ and $y = 2\delta$, respectively, with the mid-plane being $y = \delta$.

Near the entry of the duct ($x = 0$), there is a flow-development region, but we confine our attention to the fully developed region (large x) in which velocity statistics no longer vary with x. Hence the fully developed channel is statistically stationary and statistically one-dimensional. Experiments also indicate the flow is statistically symmetric about the mid-plane.

Turbulent channel flow is characterized by two velocities, the centerline velocity and the bulk velocity, and associated with these velocities are two Reynolds numbers:

Centerline Velocity: $\quad u_0 \equiv \bar{u}\big|_{y=\delta} \qquad Re_0 \equiv \dfrac{u_0 \delta}{\nu}$

Bulk Velocity: $\quad \bar{U} \equiv \dfrac{1}{\delta} \displaystyle\int_0^\delta \bar{u}\, dy \qquad Re \equiv \dfrac{2\bar{U}\delta}{\nu}$

Channel flow is laminar for $Re < 1{,}350$ and fully turbulent for $Re > 1{,}800$. Transitional effects are felt up to $Re = 3{,}000$.

The mean-flow equations for a channel flow are:

Mean Continuity: $$\frac{d\bar{v}}{dy} = 0$$

Lateral Mean-Momentum: $$\bar{v}\frac{d\bar{v}}{dy} = -\frac{d}{dy}\langle (v')^2\rangle - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \nu\frac{d^2\bar{v}}{dy^2}$$

Axial Mean-Momentum: $$\bar{v}\frac{d\bar{u}}{dy} = -\frac{d}{dy}\langle u'v'\rangle - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\frac{d^2\bar{u}}{dy^2}$$

Due to the no-penetration BC at the bottom and top walls ($\bar{v}=0$ at $y=0$ and $y=2\delta$), the mean continuity equation implies:

$$\bar{v} \equiv 0$$

Thus, the lateral mean-momentum equation simplifies to:

$$0 = -\frac{d}{dy}\langle (v')^2\rangle - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}$$

Integrating in y with the BC $\langle (v')^2\rangle = 0$ at $y=0$ gives:

$$\langle (v')^2\rangle + \bar{p}/\rho = p_w(x)/\rho$$

where p_w is the mean pressure on the bottom wall ($p_w = \bar{p}(x,0,0)$). Differentiating in x then yields:

$$\frac{\partial \bar{p}}{\partial x} = \frac{dp_w}{dx}$$

implying that the mean axial pressure gradient is uniform across the flow. All of the above implies that the axial mean-momentum equation simplifies to:

$$0 = -\frac{d}{dy}\langle u'v'\rangle - \frac{1}{\rho}\frac{dp_w}{dx} + \nu\frac{d^2\bar{u}}{dy^2}$$

or:

$$\frac{d\tau}{dy} = \frac{dp_w}{dx}$$

where τ is the total <u>shear</u> stress:

$$\tau(y) = \underbrace{\mu}_{\rho\nu}\underbrace{\frac{d\bar{u}(y)}{dy}}_{\text{Viscous Stress}} - \underbrace{\rho\langle u'v'\rangle(y)}_{\text{Reynolds Stress}}$$

Since τ is a function of y only and p_w a function of x only:

$$\frac{d\tau}{dy} = \frac{dp_w}{dx} = \text{const.}$$

The above equation amounts to a <u>balance of forces</u>: the axial normal stress gradient is balanced by the cross-stream shear stress gradient. We write the solution τ of the above equation in terms of the <u>wall shear stress</u>:

$$\tau_w \equiv \tau(0)$$

As the flow field is statistically symmetric, the total shear stress is antisymmetric. Consequently, $\tau(\delta) = 0$ and $\tau(2\delta) = -\tau_w$. This gives:

$$\boxed{\tau(y) = \tau\left(1 - \frac{y}{\delta}\right)}$$

Moreover:

$$\boxed{-\frac{dp_w}{dx} = \frac{\tau_w}{\delta}}$$

Note that while we have a complete description for the total shear stress, we do not have a complete description for the mean velocity field. This is because we do not know the Reynolds stress $\langle u'v' \rangle$. Nonetheless, the above analysis reveals quite a bit of information about turbulent channel flow. The flow is driven by the drop in pressure between the entrance and exit of the channel. In the fully developed region, there is a constant mean pressure gradient which is balanced by the total shear stress gradient. The total shear stress is a linear function of y and is composed of the viscous stress and the Reynolds stress. Consequently, in a laminar channel, the viscous stress is a linear function of y, which is consistent with experiment.

The wall shear stress is often normalized by a reference velocity, giving the skin-friction coefficient:

$$c_f \equiv \tau_w / \left(\tfrac{1}{2} \rho U_o^2\right)$$

$$C_f \equiv \tau_w / \left(\tfrac{1}{2} \rho \bar{U}^2\right)$$

The skin-friction coefficient is not known *a priori* but may be determined by experiment. For example, if a flow is defined by ρ, ν, δ, and dp_w/dx, one may measure U_o and $\bar{U}$ via experiment and compute the skin-friction coefficient as:

$$c_f \equiv -\left(\frac{dp_w}{dx}\right)\delta / \left(\tfrac{1}{2} \rho U_o^2\right)$$

$$C_f \equiv -\left(\frac{dp_w}{dx}\right)\delta / \left(\tfrac{1}{2} \rho \bar{U}^2\right)$$

For a laminar flow:

$$c_f = \frac{4}{Re_o} = \frac{16}{3\,Re}$$

$$C_f = \frac{9}{Re_o} = \frac{12}{Re}$$