Reynolds Stress Transport Models - An Introduction

The limitations of the k-ε and k-ω models largely stem from two issues: (i) the turbulence is represented by the scalar k and (ii) the Reynolds stress tensor is represented using a linear turbulent viscosity model. The former does not correctly represent turbulent anisotropy. The latter incorrectly assumes an instantaneous equilibrium between the Reynolds stress tensor and the mean rate of strain.

A shortcoming to representing the turbulent velocity fluctuations solely by k is that sometimes an external force acts on one component more strongly than the others, producing very different component energies (i.e., $\langle (u')^2\rangle \neq \langle (v')^2\rangle \neq \langle (w')^2\rangle$). This is referred to as normal stress anisotropy. Normal stress anisotropy occurs in the viscous wall region of a turbulent boundary layer, for example, where $\langle (u')^2\rangle \sim \langle (w')^2\rangle \sim y^2$ while $\langle (v')^2\rangle \sim y^4$.

Anisotropy exists in all real flows. In planar free shear flows, the dominant anisotropy is the shear stress and hence the turbulent viscosity hypothesis is reasonable. For other flows, however, the normal stress anisotropy is more significant. In Reynolds stress transport models, also known as second order closures, model transport equations are solved for the individual Reynolds stresses $\langle u_i' u_j'\rangle$ and for the dissipation ε. Consequently, the turbulence is not assumed to be represented by k and the turbulent viscosity hypothesis need not be invoked. Reynolds stress transport models introduce seven new equations in addition to the mean equations, and this of course introduces a host of new terms to be modeled.

The exact transport equation for the Reynolds stresses is the following:

$$\frac{\bar{D}\langle u_i' u_j'\rangle}{\bar{D}t} = -\frac{\partial}{\partial x_k} T_{kij} + P_{ij} + R_{ij} - \varepsilon_{ij}$$

where the Reynolds stress flux T_{kij} is:

$$T_{kij} \equiv T_{kij}^{(u)} + T_{kij}^{(p)} + T_{kij}^{(\nu)}$$

with:

$$T_{kij}^{(u)} \equiv \langle u_i' u_j' u_k'\rangle \qquad \text{Turbulent Convection}$$

$$T_{kij}^{(p)} \equiv \frac{1}{\rho}\langle u_i' p'\rangle \delta_{jk} + \frac{1}{\rho}\langle u_j' p'\rangle \delta_{ik} \qquad \text{Pressure Transport}$$

$$T_{kij}^{(\nu)} \equiv -\nu \frac{\partial \langle u_i' u_j'\rangle}{\partial x_k} \qquad \text{Viscous Diffusion}$$

the production tensor P_{ij} is:

$$P_{ij} \equiv -\langle u_i' u_k'\rangle \frac{\partial \bar{u}_j}{\partial x_k} - \langle u_j' u_k'\rangle \frac{\partial \bar{u}_i}{\partial x_k}$$

the pressure-rate-of-strain tensor R_{ij} is:

$$R_{ij} \equiv \left\langle \frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\right\rangle$$

and the dissipation tensor ε_{ij} is:

$$\varepsilon_{ij} \equiv 2\nu \left\langle \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \right\rangle$$

The most difficult term to model is the pressure-rate-of-strain tensor. The trace of R_{ij} is zero, and hence the term does not appear in the kinetic-energy equation. Instead, the term serves to redistribute the turbulent kinetic energy among the Reynolds stresses. Before we introduce models for the various terms above, we will discuss the behavior of the pressure-rate-of-strain tensor in the context of homogeneous turbulence. This will yield valuable insights regarding the form of R_{ij}.