Similarity Analysis: Jet with No Coflow:

For a jet with no coflow, we have that $U_{off} = 0$. Thus the shear layer equation reduces to:

$$\left[U_s \frac{dU_s}{dx}\right]\left(\left(\frac{dF}{d\eta}\right)^2 - F\frac{d^2F}{d\eta^2}\right) - \left[\frac{\left(\frac{d\delta}{dx}\right)U_s^2}{\delta}\right]F\frac{d^2F}{d\eta^2} = -\left[\frac{U_s^2}{\delta}\right]\frac{dg}{d\eta}$$

The terms in box brackets must still be proportional, which requires that:

$$\frac{d\delta}{dx} = \text{Const.} = S = \text{Spreading rate}$$

This is consistent with experimental observation, where it has been shown that $S \simeq 0.1$. We write:

$$\delta(x) = S(x - x_o)$$

where x_o is the integration constant, which corresponds to a virtual origin for the shear layer. With δ defined as above, the last condition we need to satisfy to guarantee similarity is:

$$\frac{\left(\frac{dU_s}{dx}\right)}{U_s} \sim \frac{\left(\frac{d\delta}{dx}\right)}{\delta}$$

which suggests that $U_s \sim x^r$ for some value r. To find an exact relationship for the shear velocity, recall that the momentum flow rate $\dot{M}$ is independent of x for a jet. Thus:

$$\text{Const.} = \int_{-\infty}^{\infty} \left(\bar{u}(x,y)\right)^2 dy = U_s^2(x)\,\delta(x) \underbrace{\int_{-\infty}^{\infty} \left(\frac{dF}{d\eta}\right)^2 d\eta}_{\text{independent of } x}$$

Consequently, we have that:

$$U_s \sim \delta^{-1/2}$$

We write:

$$U_s(x) = C\,(x - x_o)^{-1/2}$$

With the above observations and definitions, we compute:

$$\left[U_s \frac{dU_s}{dx}\right] = -\frac{C^2}{2}(x - x_o)^{-2}$$

$$\left[\frac{\left(\frac{d\delta}{dx}\right)U_s^2}{\delta}\right] = C^2 (x - x_o)^{-2}$$

$$\left[\frac{U_s^2}{\delta}\right] = \frac{C^2}{S}(x - x_o)^{-2}$$

Thus, the shear layer equations simplify further to:

$$\boxed{\frac{1}{2} S \left(\left(\frac{dF}{d\eta}\right)^2 + F \frac{d^2F}{d\eta} \right) = \frac{dg}{d\eta}}$$

Note that we are still left with a closure problem. Notably, we need a separate equation for g. Experimental observation suggests we employ a turbulent viscosity:

$$\langle u'v' \rangle (x,y) = -\nu_T(x,y) \frac{d\bar{u}}{dy}(x,y)$$

Comparing the above expression with our similarity form shows:

$$U_s^2(x)\, g(\eta) = -\nu_T(x,y) \frac{d\bar{u}}{dy}(x,y) = -\nu_T(x,y) \left(\frac{U_s(x)}{\delta(x)} \right) \frac{d^2F}{d\eta^2}(\eta)$$

which implies the self-similar form of the viscosity:

$$\nu_T(x,y) = U_s(x)\, \delta(x)\, \hat{\nu}_T(\eta)$$

Since $U_s \delta \sim x^{1/2}$, the turbulent viscosity for the jet increases as $x^{1/2}$, which acts to dissipate energy in the mean. Moreover, the Reynolds number:

$$Re_\delta(x) = \frac{U_s(x)\, \delta(x)}{\nu}$$

also increases as $x^{1/2}$, justifying our assumption that the viscous terms may be neglected. On the other hand, the turbulent Reynolds number:

$$R_T = \frac{U_s(x)\, \delta(x)}{\nu_T(x,y)}$$

is independent of x and hence turbulent effects are felt for all x.

A dramatic simplification is to assume $\hat{\nu}_T \equiv$ const. Then, we have:

$$g(\eta) = -\hat{\nu}_T \frac{d^2F}{d\eta^2}(\eta)$$

and hence:

$$\frac{1}{2} S \left(\left(\frac{dF}{d\eta} \right)^2 + F \frac{d^2F}{d\eta^2} \right) = -\hat{\nu}_T \frac{d^3F}{d\eta^3}$$

Note that:

$$\left(\frac{dF}{d\eta} \right)^2 + F \frac{d^2F}{d\eta^2} = \frac{d}{d\eta}\left(F \frac{dF}{d\eta} \right) = \frac{1}{2} \frac{d^2}{d\eta^2}(F^2)$$

Thus:

$$\frac{1}{2} \cdot \frac{1}{2} S \frac{d^2}{d\eta^2}(F^2) = -\hat{\nu}_T \frac{d^3F}{d\eta^3}$$

Integrating twice yields:

$$\frac{1}{4} S F^2 = -\hat{\nu}_T \frac{dF}{d\eta} + a + b\eta$$

where a and b are integration constants. To proceed forward, we need boundary conditions. Notably:

$$\begin{aligned}
\bar{U}(x,0) &= U_s(x) &\Rightarrow&\quad \frac{dF}{d\eta}(0) = 1 \\
\bar{V}(x,0) &= 0 &\Rightarrow&\quad F(0) = 0 \\
\bar{u}(x,\pm\infty) &= 0 &\Rightarrow&\quad \frac{dF}{d\eta}(\pm\infty) = 0 \\
|\bar{v}(x,\pm\infty)| &\neq \infty &\Rightarrow&\quad |F(\pm\infty)| \neq \infty
\end{aligned}$$

The last two conditions imply:

$$b = 0$$

while the first two conditions imply:

$$a = \hat{\nu}_T$$

So:

$$\frac{dF}{d\eta} = 1 - (\alpha F)^2 \qquad \text{where} \qquad \alpha = \sqrt{\frac{S}{4\hat{\nu}_T}}$$

Integrating once more yields:

$$\boxed{F(\eta) = \frac{1}{\alpha} \tanh(\alpha\eta)}$$

and so:

$$\boxed{\bar{f}(\eta) = \frac{dF}{d\eta}(\eta) = \operatorname{sech}^2(\alpha\eta)}$$

The precise value for α can be found by further specifying $\delta(x)$. Notably, let us define $\delta(x)$ to be the jet's half-width. Then:

$$\delta(x) = y_{1/2}(x)$$

where:

$$u(x, y_{1/2}(x)) = \tfrac{1}{2} U_s(x)$$

This gives:

$$\bar{f}(1) = \frac{1}{2} = \operatorname{sech}^2(\alpha \overset{1}{\eta})$$

$$\Rightarrow \boxed{\alpha = \frac{1}{2} \ln\left(1+\sqrt{2}\right)^2 \approx 0.88}$$

Thus, once we know the spreading rate, we know the turbulent viscosity and vice versa. Moreover, once we know the spreading rate, we know the turbulent Reynolds number:

$$R_T = \frac{1}{\hat{\nu}_T} = \frac{\left[\ln\left(1+\sqrt{2}\right)^2\right]^2}{S}$$

For $S \approx 0.1$, $R_T \approx \underline{31}$!

Given the above information, we see that:

$$\frac{\bar{u}(x,y)}{U_s(x)} = \operatorname{sech}^2(\alpha\eta)$$

$$\frac{\bar{v}(x,y)}{U_s(x)} = S\left(-\frac{1}{2\alpha}\tanh(\alpha\eta) + \eta\operatorname{sech}^2(\alpha\eta)\right)$$

$$\frac{\langle u'v' \rangle}{U_s^2(x)} = 2\hat{\nu}_T\, \alpha \tanh(\alpha\eta)\operatorname{sech}^2(\alpha\eta)$$

where:

$$\alpha \approx 0.88\ ,\quad S \approx 0.1\ ,\quad \hat{\nu}_T \approx \frac{1}{31} \approx 0.0323$$

Plots of the above are included in a PDF on D2L at the link:

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Moreover, Fig. 5.19 from Pope has been included as well. In this figure, the uniform-turbulent-viscosity solution above is compared with experimental data of Heskestad (1965). The agreement between theory and experiment is excellent except at the edge of the shear layer, where the experimental results tend to zero more rapidly than the uniform turbulent viscosity result. This is because the physical turbulent viscosity $\hat{\nu}_T$ tends to zero as $\eta \to \pm\infty$.

Note also that the lateral velocity field $\bar{v}(x,y)$ is negative for sufficiently large η, leading to entrainment of ambient fluid. This causes mixing between the jet and ambient fluid.