Turbulent Viscosity Models - Turbulent Kinetic Energy Models

With the turbulent viscosity written as

$$\nu_T = l^* u^*,$$

we note that $l^* = l_m$ and $u^* = l_m \left| \frac{\partial \bar{u}}{\partial y} \right|$ in the mixing-length model. The implication is that the turbulent velocity is locally determined in both space and time by the mean velocity gradient. Consequently, the mixing-length model has no "history effects," and the turbulence is assumed to be in "equilibrium" with the local strain. However, there are many flows for which history effects are important. One example is decaying grid turbulence. For this flow, the mean velocity gradient is zero and hence so is the turbulent velocity scale. This is patently incorrect.

To address the fact that the mixing-length model has no history effects, we note that we can derive an alternate turbulent velocity scale from the turbulent kinetic energy. This notion was first put forth by Kolmogorov (1942) and Prandtl (1945) who suggested the scaling:

$$u^* = c k^{1/2}$$

where c is a constant. If the lengthscale is again taken to be the mixing-length, then the turbulent viscosity becomes:

$$\nu_T = c k^{1/2} l_m$$

The value of the constant $c \approx 0.55$ yields the correct behavior in the log-law region.

In order for our new model to be applied, the value of k must be known or estimated. Kolmogorov and Prandtl suggested achieving this by solving a model transport equation for k, resulting in a one-equation model for the unknown turbulence quantity k.

Our starting point in designing a model transport equation for k is the exact equation for k:

$$\frac{\bar{D}k}{\bar{D}t} = -\vec{\nabla} \cdot \vec{T}' + \mathcal{P} - \varepsilon \qquad \frac{\bar{D}}{\bar{D}t} \equiv \frac{\partial}{\partial t} + \bar{u} \cdot \vec{\nabla}$$

where :

$$\mathcal{P} = -\langle u_i' u_j' \rangle \frac{\partial \bar{u}_i}{\partial x_j} = 2\nu_T \bar{S}_{ij} \bar{S}_{ij} = \nu_T \bar{S}^2 \qquad \text{Production}$$

$$\varepsilon = 2\nu \langle S_{ij}' S_{ij}' \rangle \qquad S_{ij}' = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right) \qquad \text{Dissipation}$$

$$T_i' = \frac{1}{2} \langle u_i' u_j' u_j' \rangle + \frac{1}{\rho} \langle u_i' p' \rangle - 2\nu \langle u_j' S_{ij}' \rangle \qquad \text{Flux}$$

Above, $\frac{\bar{D}k}{\bar{D}t}$ and $\mathcal{P}$ are in closed form, but ε and T_i' involve additional unknowns and hence must be modeled using closure approximations.

At high Re, the dissipation rate ε scales like u_0^3/l_0 where u_0 and l_0 are the velocity scale and lengthscale of the energy-containing motions. This inspires the model:

$$\varepsilon = C_D k^{3/2} / l_m$$

where C_D is a model constant. Again, an examination of the log-law region suggests $C_D \approx c^3$.

Instead of modeling the energy flux $\vec{T}'$ immediately, we first decompose the turbulent transport term into four parts:

$$\vec{\nabla}\cdot\vec{T}' = \frac{\partial T_i'}{\partial x_i}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{1}{2}\langle u_i' u_j' u_j'\rangle\right) \quad \text{Turbulent Convection}$$

$$+ \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\langle u_i' p'\rangle\right) \quad \text{Pressure Transport}$$

$$- \frac{\partial}{\partial x_i}\left(\nu \frac{\partial k}{\partial x_i}\right) \quad \text{Viscous Diffusion}$$

$$+ \left(\hat{\varepsilon} - \varepsilon\right) \quad \text{Difference Between Pseudo-Dissipation and Dissipation}$$

where $\hat{\varepsilon} = \nu\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle$ is the pseudo-dissipation. In practice, the difference between the pseudo-dissipation and dissipation is small, so the fourth term above may be neglected. The third term, the viscous diffusion, is already in closed form. To model the turbulent convection and pressure transport, we invoke the gradient-diffusion hypothesis which states the combined effect of velocity and pressure fluctuations results in a net flux from regions of high energy to regions of low energy. This yields the model:

$$\vec{\nabla}\cdot\vec{T}' = -\vec{\nabla}\cdot\left(\left(\nu + \nu_T/\sigma_k\right)\vec{\nabla}k\right)$$

where σ_k is a constant known as the 'turbulent Prandtl number' for kinetic energy and is generally assumed to be one. Mathematically, the gradient-diffusion hypothesis ensures that the resulting transport equation for k yields smooth solutions, and that a boundary condition can be imposed on k everywhere on the boundaries of the solution domain.

In summary, the one-equation model for k takes the following form:

Reynolds Stress Model:	$\langle u_i' u_j'\rangle = -2\nu_T \bar{S}_{ij} + \frac{2}{3}k\delta_{ij}$
Turbulent Viscosity:	$\nu_T = c\,k^{1/2}\ell_m$
Model Equation for k:	$\bar{D}k/\bar{D}t = \vec{\nabla}\cdot\left(\left(\nu + \frac{\nu_T}{\sigma_k}\right)\vec{\nabla}k\right) + P - \varepsilon$, $\quad \varepsilon = C_D k^{3/2}/\ell_m$
Constants:	$c = 0.55$, $C_D = c^3$

A comparison of model predictions with experimental data (Wilcox 1993) reveals that the above one-equation model has a modest advantage in accuracy over mixing-length models, but the major issue of incompleteness remains: the length scale $\ell_m(\vec{x})$ must be specified.