

## Similarity Analysis: Jet with Coflow and Wake:

Note: This is in the far field!

For a jet with coflow or wake, we have that  $u_s \ll u_{\text{offset}}$ . If we thus ignore higher-order terms in  $u_s$  (except the Reynolds stress), the shear layer equation becomes:

$$\left[ u_{\text{offset}} \frac{du_s}{dx} \right] \frac{dF}{d\eta} - \left[ \frac{\left( \frac{d\delta}{dx} \right) u_s u_{\text{offset}}}{\delta} \right] \eta \frac{d^2 F}{d\eta^2} = - \left[ \frac{u_s^2}{\delta} \right] \frac{dg}{d\eta}$$

The terms in brackets must be proportional, which requires that:

$$u_s \propto \left( \frac{d\delta}{dx} \right) u_{\text{offset}} \quad \left( \frac{du_s}{dx} \right) \propto \left( \frac{d\delta}{dx} \right) \frac{u_s}{\delta}$$

$\uparrow$  const.

To proceed forward, we need more information. If we integrate the momentum equation in the frame moving at  $u_{\text{offset}}$ :

$$\frac{\partial}{\partial x} [\bar{u} (u_{\text{offset}} - \bar{u})] + \frac{\partial}{\partial y} [\bar{v} (u_{\text{offset}} - \bar{u})] = \frac{\partial}{\partial y} \langle u'v' \rangle$$

From  $y = -\infty$  to  $y = +\infty$ , we find the momentum deficit flow rate is conserved:

$$\text{Constant} = \int_{-\infty}^{\infty} \rho \bar{u} (u_{\text{offset}} - \bar{u}) dy = \rho u_{\text{offset}} u_s(x) \delta(x) \int_{-\infty}^{\infty} \left( 1 - \frac{u_s}{u_{\text{offset}}} f(\xi) \right) f(\xi) d\xi$$

$u_s \ll u_{\text{offset}}$   
 $\underbrace{\int_{-\infty}^{\infty} f(\xi) f(\xi) d\xi}_{\approx 0}$   
independent of  $x$ !

Thus,  $u_s(x) \delta(x) = \text{const.}$ , which implies that:

$$u_s \propto \delta^{-1}$$

The scaling above coupled with the previously obtained scalings dictate that:

$$\frac{du_s}{dx} \propto u_s^3$$

which requires:

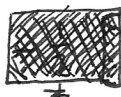
$$u_s \propto x^{-1/2} \Rightarrow \delta \propto x^{1/2}$$

We set:

$$u_s = C (x - x_0)^{-1/2}$$

$$\delta = 2 \left( \frac{u_s}{u_{\text{off.}}} \right) S (x - x_0) = 2 \frac{SC}{u_{\text{off.}}} (x - x_0)^{1/2} \quad \left\{ \begin{array}{l} \frac{d\delta}{dx} = \left( \frac{u_s}{u_{\text{off.}}} \right) S \\ \uparrow \\ \text{spreading rate!} \end{array} \right.$$

Then the shear layer equation becomes:



$$S \frac{d}{d\eta} \left( \eta \frac{dF}{d\eta} \right) = \frac{dg}{d\eta}$$

Integrating the above expression coupled with the condition  $g(0)=0$  (due to symmetry) gives:

$$S \eta \frac{dF}{d\eta} = g$$

Consequently, the Reynolds stress is a direct function of the mean axial velocity and the spreading rate. If one employs a turbulent viscosity of the form:

$$\nu_T = \hat{\nu}_T u_s(x) \delta(x) \quad \text{w/ } \hat{\nu}_T \equiv \text{const.}$$

then the shear equation further simplifies to:

$$S \eta \frac{dF}{d\eta} = -\hat{\nu}_T \frac{d^2 F}{d\eta^2}$$

To proceed forward, we need boundary conditions. Notably:

$$\begin{aligned} \bar{u}(x,0) = u_0 &\rightarrow \frac{dF}{d\eta}(0) = 1 \quad \text{for a coflowing jet} \\ &\frac{dF}{d\eta}(0) = -1 \quad \text{for a wake} \end{aligned}$$

$$\begin{aligned} \bar{v}(x,0) = 0 &\rightarrow F(0) = 0 \\ \bar{u}(x,\pm\infty) = u_\infty &\rightarrow \frac{dF}{d\eta}(\pm\infty) = 0 \\ |\bar{v}(x,\pm\infty)| \neq \infty &\rightarrow |F(\pm\infty)| \neq \infty \end{aligned}$$

Thus we have:

$$F(\eta) = \begin{cases} \frac{1}{\sqrt{\alpha}} \frac{\sqrt{\pi}}{2} \operatorname{erf}(\sqrt{\alpha} \eta) & \text{for a coflowing jet} \\ -\frac{1}{\sqrt{\alpha}} \frac{\sqrt{\pi}}{2} \operatorname{erf}(\sqrt{\alpha} \eta) & \text{for a wake} \end{cases}$$

where  $\alpha = \frac{S}{2\hat{\nu}_T}$

and:

$$\bar{f}(\eta) = \frac{dF}{d\eta}(\eta) = \begin{cases} \exp(-\alpha \eta^2) & \text{for a coflowing jet} \\ -\exp(-\alpha \eta^2) & \text{for a wake} \end{cases}$$

Now that both  $F$  and  $\frac{dF}{d\eta}$  are known, we can derive expressions for our mean fields:

$$\frac{\bar{u}(x,y) - u_\infty}{u_s(x)} = \begin{cases} +\exp(-\alpha \eta^2) & \text{for a coflowing jet} \\ -\exp(-\alpha \eta^2) & \text{for a wake} \end{cases}$$

$$\frac{\bar{v}(x,y)}{u_s^2(x)} = \frac{S}{u_{\text{offset}}^2} \times \begin{cases} +\eta \exp(-\alpha \eta^2) & \text{for a coflowing jet} \\ -\eta \exp(-\alpha \eta^2) & \text{for a wake} \end{cases}$$

$$\frac{\langle u'v' \rangle}{u_s^2(x)} = \hat{\nu}_T \times \begin{cases} +2\alpha \eta \exp(-\alpha \eta^2) & \text{for a coflowing jet} \\ -2\alpha \eta \exp(-\alpha \eta^2) & \text{for a wake} \end{cases}$$

The value for  $\alpha$  can be determined by further specifying  $\delta$ . Namely, if we choose  $\delta$  to be the half-width  $y_{1/2}$  defined as:

$$\bar{u}(x, y_{1/2}(x)) = \frac{1}{2} u_{\infty} + \frac{1}{2} u_0(x)$$

$\nwarrow$  free stream velocity       $\nwarrow$  centerline velocity

Then:

$$\alpha = \ln(2) \approx 0.693$$

With  $\alpha$  known, one can find  $\hat{\gamma}_T$  once the spreading rate  $S$  is known. The spreading rate depends on the geometrical configuration. For the wake behind a flat plate,  $S = 0.073$ . For the wake behind a cylinder,  $S = 0.083$ . For the wake behind an airfoil,  $S = 0.103$ . Thus, for the wake behind an airfoil:

$$\hat{\gamma}_T = \frac{1}{2} \frac{S}{\alpha} \approx \frac{1}{2} \frac{0.103}{0.693} \approx 0.0743$$

Plots of the mean fields are included in a PDF on D2L at the link:

2.2 - Similarity Analysis - Jet with Coflow and Wake → Plots.pdf

Moreover, Fig. 5.26 from Pope has been included as well. In this figure, the uniform turbulent-viscosity solution above are compared with experimental data of Wygnanski et al. (1986). The agreement between theory and experiment is excellent except at the edge of the jet/wake as before.

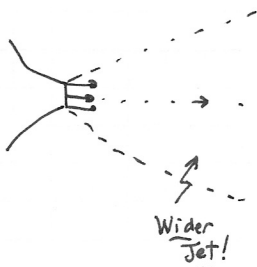
The flow behind an airfoil is considered!

Note that the lateral velocity field  $\bar{v}(x, y)$  is nonzero only within the jet/wake. This means no fluid outside the jet/wake is being entrained!

We finish here by remarking that noncoflowing jets and coflowing jets behave radically differently.

### Noncoflowing Jet

1. Convective velocity goes to zero as  $x \rightarrow \infty$ .
2. There is no homogeneity.
3. The jet grows like  $x$ .
4. Ambient fluid is entrained into the jet.



### Coflowing Jet

1. Convective velocity goes to  $u_{\infty}$  as  $x \rightarrow \infty$ .
2. The flow approaches a streamwise homogeneous parallel flow as  $x \rightarrow \infty$ .
3. The jet grows like  $x^{1/2}$ .
4. There is no entrainment.

