Similarity Analysis: Jet with Coflow and Wake:

Note: This is in the far field!

For a jet with coflow or wake, we have that $u_s \ll u_{offset}$. If we thus ignore higher-order terms in u_s (except the Reynolds stress), the shear layer equation becomes:

$$\left[u_{offset} \frac{du_s}{dx}\right] \frac{dF}{d\eta} - \left[\frac{\left(\frac{d\delta}{dx}\right) u_s\, u_{offset}}{\delta}\right] \eta \frac{d^2F}{d\eta^2} = -\left[\frac{u_s^2}{\delta}\right] \frac{dg}{d\eta}$$

The terms in brackets must be proportional, which requires that:

$$u_s \sim \underbrace{\left(\frac{d\delta}{dx}\right) u_{offset}}_{\text{Const.}} \qquad \frac{\left(\frac{du_s}{dx}\right)}{u_s} \sim \frac{\left(\frac{d\delta}{dx}\right)}{\delta}$$

To proceed forward, we need more information. If we integrate the momentum equation in the frame moving at U_{offset}:

$$\frac{\partial}{\partial x}\left[\bar{u}\left(u_{offset} - \bar{u}\right)\right] + \frac{\partial}{\partial y}\left[\bar{v}\left(u_{offset} - \bar{u}\right)\right] = \frac{\partial}{\partial y}\langle u'v' \rangle$$

from $y = -\infty$ to $y = +\infty$, we find the momentum deficit flow rate is conserved:

$u_s \ll u_{offset}$!

$$\text{Constant} = \int_{-\infty}^{\infty} \rho\, \bar{u}\left(u_{offset} - \bar{u}\right) dy = \rho\, U_{offset}\, u_s(x)\, \delta(x) \underbrace{\int_{-\infty}^{\infty} \Big(1 - \underbrace{\frac{u_s}{u_{offset}}}_{\sim 0} f(\xi)\Big) f(\xi)}_{\text{independent of } x!}$$

Thus, $u_s(x)\, \delta(x) = \text{const.}$, which implies that:

$$u_s \sim \delta^{-1}$$

The scaling above coupled with the previously obtained scalings dicate that:

$$\frac{du_s}{dx} \sim u_s^3$$

which requires:

$$u_s \sim x^{-1/2} \Rightarrow \delta \sim x^{1/2}$$

We set:

$$u_s = C\,(x - x_0)^{-1/2}$$

$$\left.\begin{aligned} \delta &= 2\left(\frac{u_s}{u_{off.}}\right) S\,(x - x_0) \\ &= 2\,\frac{SC}{u_{off.}}\,(x - x_0)^{1/2} \end{aligned}\right\} \quad \frac{d\delta}{dx} = \left(\frac{u_s}{u_{off.}}\right) \underbrace{S}_{\text{spreading rate!}}$$

Then the shear layer equation becomes:

$$S \frac{d}{d\eta}\left(\eta \frac{dF}{d\eta}\right) = \frac{dg}{d\eta}$$

Integrating the above expression coupled with the condition $g(0)=0$ (due to symmetry) gives:

$$S\eta \frac{dF}{d\eta} = g$$

Consequently, the Reynolds stress is a direct function of the mean axial velocity and the spreading rate. If one employs a turbulent viscosity of the form:

$$\nu_T = \hat{\nu}_T \, u_s(x) \, \delta(x) \quad \text{w/} \quad \hat{\nu}_T \equiv \text{const.}$$

then the shear equation further simplifies to:

$$S\eta \frac{dF}{d\eta} = -\hat{\nu}_T \frac{d^2F}{d\eta^2}$$

To proceed forward, we need boundary conditions. Notably:

$$\bar{u}(x,0) = u_0 \longrightarrow \begin{array}{ll} \frac{dF}{d\eta}(0) = 1 & \text{for a coflowing jet} \\ \frac{dF}{d\eta}(0) = -1 & \text{for a wake} \end{array}$$

$$\begin{array}{lcl} \bar{v}(x,0) = 0 & \longrightarrow & F(0) = 0 \\ \bar{u}(x,\pm\infty) = u_\infty & \longrightarrow & dF/d\eta(\pm\infty) = 0 \\ |\bar{v}(x,\pm\infty)| \neq \infty & \longrightarrow & |F(\pm\infty)| \neq \infty \end{array}$$

Thus we have:

$$F(\eta) = \begin{cases} \frac{1}{\sqrt{\alpha}}\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\sqrt{\alpha}\,\eta) & \text{for a coflowing jet} \\ -\frac{1}{\sqrt{\alpha}}\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\sqrt{\alpha}\,\eta) & \text{for a wake} \end{cases}$$

where $\alpha = \frac{S}{2\hat{\nu}_T}$

and:

$$\bar{f}(\eta) = \frac{dF}{d\eta}(\eta) = \begin{cases} \exp(-\alpha\eta^2) & \text{for a coflowing jet} \\ -\exp(-\alpha\eta^2) & \text{for a wake} \end{cases}$$

Now that both F and $\frac{dF}{d\eta}$ are known, we can derive expressions for our mean fields:

$$\frac{\bar{u}(x,y) - u_\infty}{u_s(x)} = \begin{cases} +\exp(-\alpha\eta^2) & \text{for a coflowing jet} \\ -\exp(-\alpha\eta^2) & \text{for a wake} \end{cases}$$

$$\frac{\bar{v}(x,y)}{u_s^2(x)} = \frac{S}{u_{offset}} \times \begin{cases} +\eta\exp(-\alpha\eta^2) & \text{for a coflowing jet} \\ -\eta\exp(-\alpha\eta^2) & \text{for a wake} \end{cases}$$

$$\frac{\langle u'v' \rangle}{u_s^2(x)} = \hat{\nu}_T \times \begin{cases} +2\alpha\eta\exp(-\alpha\eta^2) & \text{for a coflowing jet} \\ -2\alpha\eta\exp(-\alpha\eta^2) & \text{for a wake} \end{cases}$$

The value for α can be determined by further specifying δ. Namely, if we choose δ to be the half-width $y_{1/2}$ defined as:

$$\bar{u}(x, y_{1/2}(x)) = \frac{1}{2} \underbrace{U_\infty}_{\text{free stream velocity}} + \frac{1}{2} \underbrace{U_0(x)}_{\text{centerline velocity}}$$

Then:

$$\alpha = \ln(2) \approx 0.693$$

With α known, one can find $\hat{\nu}_T$ once the spreading rate S is known. The spreading rate depends on the geometrical configuration. For the wake behind a flat plate, $S = 0.073$. For the wake behind a cylinder, $S = 0.083$. For the wake behind an airfoil, $S = 0.103$. Thus, for the wake behind an airfoil:

$$\hat{\nu}_T = \frac{1}{2}\frac{S}{\alpha} \approx \frac{1}{2}\frac{0.103}{0.693} \approx 0.0743$$

Plots of the mean fields are included in a PDF on D2L at the link:

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Moreover, Fig. 5.26 from Pope has been included as well. In this figure, the uniform turbulent-viscosity solution above are compared with experimental data of Wygnanski et al. (1986). The agreement between theory and experiment is excellent except at the edge of the jet/wake as before.

The flow behind an airfoil is considered!

Note that the lateral velocity field $\bar{v}(x,y)$ is nonzero only within the jet/wake. This means no fluid outside the jet/wake is being entrained!

We finish here by remarking that noncoflowing jets and coflowing jets behave radically differently:

Noncoflowing Jet	Coflowing Jet
1. Convective velocity goes to zero as $x \to \infty$.	1. Convective velocity goes to U_∞ as $x \to \infty$.
2. There is no homogeneity.	2. The flow approaches a streamwise homogeneous parallel flow as $x \to \infty$.
3. The jet grows like x.	3. The jet grows like $x^{1/2}$.
4. Ambient fluid is entrained into the jet.	4. There is no entrainment.