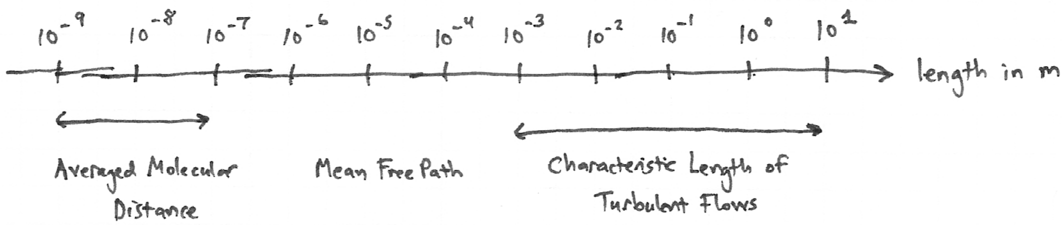
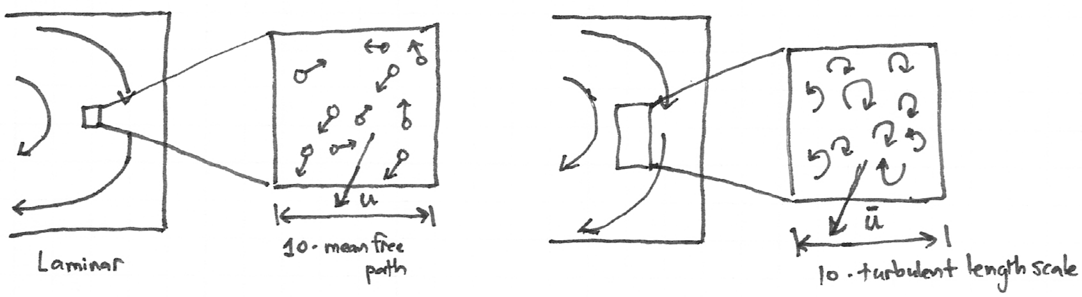
Turbulent-Viscosity Models – The Turbulent-Viscosity Hypothesis

The simplest model for the Reynolds stress tensor is the turbulent-viscosity model, which is posited on the assumption that the turbulent-viscosity hypothesis is valid. According to the hypothesis, the Reynolds stresses are given by:

$$\langle u_i' u_j' \rangle = \frac{2}{3} k \delta_{ij} - 2\nu_T \bar{S}_{ij} \quad \text{or,} \quad \underset{\text{Reynolds-stress anisotropy}}{a_{ij}} = -2\nu_T \bar{S}_{ij}$$

where ν_T is the so-called turbulent viscosity. Prandtl and von Karman first developed the turbulent-viscosity hypothesis by drawing an analogy between molecular-based laminar momentum and scalar transport, and eddy-based turbulent motion and scalar transport. The analogy to molecular processes, as well as characteristic length scales of turbulent flows under normal conditions, are illustrated below where the left-hand-picture shows the continuum mechanical velocity as the average of the molecular motion, while the right-hand-picture sketches the mean velocity as the average of the turbulent instantaneous velocity. In each case, the velocity is taken as an average and a relevant length scale is observed, namely the mean free path or a turbulent length scale.



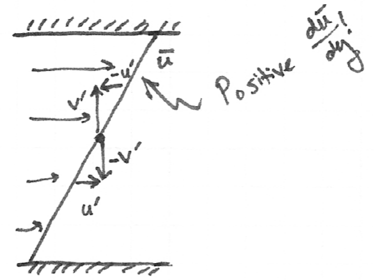


Compared with the molecular scales, the straining is very weak and consequently it produces a very small departure from isotropy. Thus, there is every reason to expect that the anisotropic viscous stresses to depend linearly on the velocity gradients. By contrast, the rate of straining might be relatively large compared with the turbulent ~~length~~ time scales, leading to relatively large anisotropies. Consequently, there is no expectation that the turbulent viscosity is constant.

The constant-viscosity hypothesis is perhaps best illustrated in the context of a simple shear flow. Here, the hypothesis reduces to:

$$\langle u'v' \rangle = -\nu_T \frac{\partial \bar{u}}{\partial y}$$

and ν_T is uniquely defined by the above equation. In what follows, we reference the below figure:



If a fluid particle is displaced by a velocity downward in direction, then it will move faster to the right than the surrounding fluid particles, yielding a negative Reynolds stress. Analogously, if a fluid particle is dispaced by a velocity upward in direction, then it will move slower to the right than the surrounding fluid particles, also yielding a negative Reynolds stress. Thus, we do indeed expect the relationship:

$$\langle u'v' \rangle = -\nu_T \frac{\partial \bar{u}}{\partial y}$$

where the turbulent viscosity ~~hypothesis~~ ν_T is positive.

It should be recognized that the turbulent-viscosity hypothesis has limited applicability. For example, there are certain problems in which the rate of straining is so large compared with the turbulent time scales that we cannot expect the Reynolds stress anisotropies to be determined by the local mean rate of strain. Axisymmetric contraction is an example of such a problem. For these problems, non-local transport processes are predominantly responsible for the evolution of the Reynolds stress anisotropies.

There is also little reason to expect the Reynolds-stress anisotropy to be related to the mean rate-of-strain via a ~~scalar~~ scalar turbulent viscosity. This assumption demands that the principal axes of a_{ij} are aligned with those of $\bar{S}_{ij}$. For several flows, this is indeed not the case. Examples are strongly swirling flows, flows with significant streamline curvature, and fully-developed flows in ducts of non-circular cross-section. For these flows, a possible nonlinear turbulent viscosity hypothesis is:

$$a_{ij} = -2\nu_{T1}\bar{S}_{ij} + \nu_{T2}\left(\bar{S}_{ik}\bar{\Omega}_{kj} - \bar{\Omega}_{ik}\bar{S}_{kj}\right) + \nu_{T3}\left(\bar{S}_{ik}\bar{S}_{kj} - \tfrac{1}{3}\bar{S}_{kk}^{2}\delta_{ij}\right)$$

$$\bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right) = \text{rate-of-rotation tensor}$$

where $\nu_{T1}, \nu_{T2}, \nu_{T3}$ may depend on the invariants of the mean-velocity-gradient tensor. Rational means of obtaining nonlinear turbulent viscosity laws have been developed, but they will not be discussed further here.

According to the turbulent-viscosity hypothesis put forth by Prandtl and von Karman, the turbulent viscosity only depends on the mean velocity gradients. This dependence can be further restricted by recognizing the Reynolds stresses should be frame invariant. Consequently, the turbulent viscosity may only depend on invariants of the mean velocity gradient.

Before proceeding, it is useful to verify the Reynolds stress approximation given by the turbulent-viscosity hypothesis satisfy the Reynolds stress constraints discussed earlier.

(i) Frame Invariance: The Reynolds stress approximation:

$$r_{ij} = -2\nu_T \bar{S}_{ij} + \frac{2}{3} k \delta_{ij}$$

indeed transforms as a tensor as $\bar{S}_{ij}$ and δ_{ij} are tensors and ν_T and k are frame invariant.

(ii) Realizability: The Reynold stress approximation:

$$r_{ij} = -2\nu_T \bar{S}_{ij} + \frac{2}{3} k \delta_{ij}$$

satisfies:

$$r_{ii} = -2\nu_T \overbrace{\cancel{\bar{S}_{ii}}}^{0\ \text{(Cons. of Mass)}} + \frac{2}{3} k \overbrace{\cancel{\delta_{ii}}}^{3} = 2k \geq 0$$

(iii) The Reynolds stress approximation $r_{ij} = -2\nu_T \bar{S}_{ij} + \frac{2}{3} k \delta_{ij}$ is invariant under Galilean transformations as ν_T, $\bar{S}_{ij}$, k, and δ_{ij} are.