Homogeneous Isotropic Turbulence: Dynamics of the Spectrum Tensor

By simply Fourier transforming the evolution equation for R_{ij}, one obtains the evolution equation for the spectrum tensor. Defining:

$$\Phi_{ij}(\vec{K},t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} R_{ij}(\vec{r},t)\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}$$

$$\Gamma_{ij}(\vec{K},t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} T_{ij}(\vec{r},t)\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}$$

we have:

$$\boxed{\frac{\partial \Phi_{ij}}{\partial t} = \Gamma_{ij} - 2\nu K^2 \Phi_{ij} \qquad \text{where} \quad K = K_i K_i}$$

We can reduce the equation above by recognizing in isotropic turbulence, the spectrum tensor is isotropic:

$$\Phi_{ij}(\vec{K}) = A(K)\,\delta_{ij} + B(K)\,K_i K_j$$

In Fact, it can be shown that:

$$\Phi_{ij}(\vec{K}) = \frac{E(K)}{4\pi K^4}\left(K^2\delta_{ij} - K_i K_j\right)$$

where $E(K)$ is the energy-spectrum tensor:

$$E(K) \equiv \iiint_{-\infty}^{\infty} \tfrac{1}{2}\,\Phi_{ii}(\vec{K})\,\delta(|\vec{K}|-K)\,\delta\vec{K}$$

which is the kinetic energy per unit wave number. Consequently,

$$\Phi_{ii}(\vec{K}) = \frac{E(K)}{2\pi K^2}$$

and defining

only depends on K due to isotropy

$$T(K) = 2\pi K^2\, \Gamma_{ii}(\vec{K})$$

we have the following equation for the energy spectrum:

$$\boxed{\frac{\partial E(K)}{\partial t} = T(K) - 2\nu K^2 E(K)}$$

Transfer of energy to wavenumber K from all other wavenumbers

Dissipation spectrum

By definition, $k = \int_0^\infty E(K)\,dK$, and for isotropic turbulence, $\frac{dk}{dt} = -\varepsilon$. This implies:

$$\int_0^\infty T(K)\,dK = 0 \qquad \text{and} \qquad \varepsilon = 2\nu \int_0^\infty K^2 E(K)$$

Thus, the transfer $T(K)$ has no effect on the turbulent kinetic energy. It is just the net nonlinear transfer of energy to wavenumber K from all other wave numbers. The quantity $2\nu K^2 E(K)$ is called the dissipation spectrum since it integrates to ε and is often given the symbol $D(K)$:

$$D(K) = 2\nu K^2 E(K)$$

In the inertial subrange, $E(K) \sim K^{-5/3}$. If $E(K) \to 0$ slower than K^{-2} in general, then:

$$E(K_1) \gg E(K_2)$$
$$D(K_1) \ll D(K_2)$$

for $K_1 \ll K_2$, consistent with our assumption that dissipation occurs at (large K) small scales and energy exists at large scales.

Define $\Upsilon(K)$ be the <u>rate</u> at which energy is transferred across wave number K:

$$T(K) = -\frac{d}{dK}\Upsilon(K) \Rightarrow \Upsilon(K) = -\int_0^K T(\alpha)\,d\alpha$$

$\Upsilon(K)$ is called the trans<u>fer</u> spec<u>trum</u>. In the inertial subrange, all the energy exists for wavenumber smaller than K and all the dissipation occurs for wavenumber larger than K. Thus:

$$\underbrace{\frac{\partial}{\partial t}\int_0^K E(K')\,dK'}_{\sim\, -\varepsilon} = \underbrace{\int_0^K T(K')\,dK'}_{-\Upsilon(K)} - \underbrace{2\nu\int_0^K K'^2 E(K')\,dK'}_{0}$$

for K in the inertial subrange, giving:

$$\Upsilon(K) = \varepsilon \qquad \text{for } K \text{ in the inertial subrange}$$

This is consistent with Richardson's view of turbulence.