The mean velocity equations!

The mean velocity equations for a stationary plane free shear flow are as follows:

Axial Mean Momentum:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \frac{\partial}{\partial x}\langle (u')^2 \rangle + \frac{\partial}{\partial y}\langle u'v' \rangle = -\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right)$$

Lateral Mean Momentum:

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \frac{\partial}{\partial x}\langle u'v' \rangle + \frac{\partial}{\partial y}\langle (v')^2 \rangle = -\frac{\partial \bar{p}}{\partial y} + \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right)$$

Mean Mass:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

We simplify the above equations by comparing the relative size of each term. This is similar to what is done in deriving the boundary layer equations. In what follows, denote:

U_s = Shear velocity
U_c = Convection velocity
δ = Shear layer thickness
L = Length scale characteristicing the variation of statistics in the streamwise direction
q = Velocity scale characterizing turbulent fluctuations

I know, bad English!

The shear layer assumption is that:

$$\frac{\delta}{L} \ll 1$$

With the above notation, note that the derivative operations scale like:

$$\frac{\partial}{\partial x} \sim \frac{1}{L} \qquad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

Moreover:

$$\bar{u} \sim U_c$$

However, we do not immediately know a scale for $\bar{v}$. To find one, note that:

$$\frac{\partial \bar{u}}{\partial x} \sim \frac{U_s}{L}$$ Not U_c because we want the change in $\bar{u}$!

By conservation of mass:

$$\frac{\partial \bar{v}}{\partial y} \sim \frac{\partial \bar{u}}{\partial x} \sim \frac{U_s}{L}$$

Thus we must have:

$$\bar{v} \sim U_s\left(\frac{\delta}{L}\right)$$

This gives us the following scalings for our statistical quantities:

$$\bar{u} \sim U_c \ , \quad \bar{v} \sim U_s\left(\frac{\delta}{L}\right) , \quad \langle (u')^2 \rangle \sim \langle (v')^2 \rangle \sim \langle u'v' \rangle \sim q^2$$

and their derivatives:

$$\frac{\partial \bar{u}}{\partial x} \sim \frac{U_s}{L} , \quad \frac{\partial \bar{u}}{\partial y} \sim \frac{U_s}{\delta} , \quad \frac{\partial^2 \bar{u}}{\partial x^2} \sim \frac{U_s}{L^2} , \quad \frac{\partial^2 \bar{u}}{\partial y^2} \sim \frac{U_s}{\delta^2}$$

$$\frac{\partial \bar{v}}{\partial x} \sim \frac{U_s}{L}\left(\frac{\delta}{L}\right) , \quad \frac{\partial \bar{v}}{\partial y} \sim \frac{U_s}{L} , \quad \frac{\partial^2 \bar{v}}{\partial x^2} \sim \frac{U_s}{L^2}\left(\frac{\delta}{L}\right) , \quad \frac{\partial^2 \bar{v}}{\partial y^2} \sim \frac{U_s}{\delta L}$$

$$\frac{\partial}{\partial x} \langle (u')^2 \rangle \sim \frac{\partial}{\partial x} \langle (v')^2 \rangle \sim \frac{\partial}{\partial x} \langle u'v' \rangle \sim \frac{q^2}{L}$$

$$\frac{\partial}{\partial y} \langle (u')^2 \rangle \sim \frac{\partial}{\partial y} \langle (v')^2 \rangle \sim \frac{\partial}{\partial y} \langle u'v' \rangle \sim \frac{q^2}{\delta}$$

A pressure scaling will naturally fall out of our proceeding analysis. Using the above, we find the components of the lateral mean momentum equation scale like:

$$\underset{\left[\frac{U_c U_s}{L}\left(\frac{\delta}{L}\right)\right]}{\bar{u}\frac{\partial \bar{v}}{\partial x}} + \underset{\left[\frac{U_s^2}{L}\left(\frac{\delta}{L}\right)\right]}{\bar{v}\frac{\partial \bar{v}}{\partial y}} + \underset{\left[\frac{q^2}{L}\right]}{\frac{\partial}{\partial x}\langle u'v' \rangle} + \underset{\left[\frac{q^2}{L}\left(\frac{\delta}{L}\right)^{-1}\right]}{\frac{\partial}{\partial y}\langle (v')^2 \rangle} =$$

$$\underset{\left[\frac{\hat{p}}{L}\left(\frac{\delta}{L}\right)^{-1}\right]}{-\frac{\partial \bar{p}}{\partial y}} + \underset{\left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_s^2}{L}\right)\left(\frac{\delta}{L}\right)^2\right]}{\nu\frac{\partial^2 \bar{v}}{\partial x^2}} + \underset{\left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_s^2}{L}\right)\right]}{\nu\frac{\partial^2 \bar{v}}{\partial y^2}}$$

unknown pressure scaling!

where:

$$Re_\delta = \frac{U_s \delta}{\nu} = \text{Reynolds number of the turbulence}$$

Since: $\delta/L \ll 1$, the terms above scaling like $\left(\frac{\delta}{L}\right)^{-1}$ dominate. Neglecting the smaller terms yields the equation:

$$\frac{\partial}{\partial y}\langle (v')^2 \rangle = -\frac{\partial \bar{p}}{\partial y}$$

Shear layer Lateral Mean Momentum

Note this implies that the mean pressure scales like q^2!

If we integrate the lateral mean momentum equation in y and differentiate in x, we obtain:

$$\frac{\partial}{\partial x} \langle (v')^2 \rangle = -\frac{\partial \bar{p}}{\partial x}$$

This allows us to remove pressure from the axial mean momentum equation:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) - \frac{\partial}{\partial x}\left(\langle (u')^2\rangle - \langle (v')^2\rangle\right) - \frac{\partial}{\partial y}\langle u'v'\rangle$$

Now, we compute the scales for each term in the axial mean momentum equation:

$$\underset{\left[\frac{U_c U_s}{L}\right]}{\underset{\updownarrow}{\bar{u}\frac{\partial \bar{u}}{\partial x}}} + \underset{\left[\frac{U_s^2}{L}\right]}{\underset{\updownarrow}{\bar{v}\frac{\partial \bar{u}}{\partial y}}} = \underset{\left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_s^2}{L}\right)\left(\frac{\delta}{L}\right)\right]}{\underset{\updownarrow}{\nu\frac{\partial^2 \bar{u}}{\partial x^2}}} + \underset{\left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_s^2}{L}\right)\left(\frac{\delta}{L}\right)^{-1}\right]}{\underset{\updownarrow}{\nu\frac{\partial^2 \bar{u}}{\partial y^2}}}$$

$$- \frac{\partial}{\partial x}\underset{\left[\frac{q^2}{L}\right]}{\underset{\updownarrow}{\left(\langle (u')^2\rangle - \langle (v')^2\rangle\right)}}$$

$$- \frac{\partial}{\partial y}\underset{\left[\frac{q^2}{L}\left(\frac{\delta}{L}\right)^{-1}\right]}{\underset{\updownarrow}{\langle u'v'\rangle}}$$

We must be careful when proceeding forward. If we just keep the $\left(\frac{\delta}{L}\right)^{-1}$ terms, only a viscous and Reynolds stress term remains. We know this is incorrect, so we must look further into what is happening above. It helps to consider two scenarios:

Scenario #1: $U_c \gg U_s$ — Far Field — $\boxed{\text{Wake or Coflowing Jet}}$ $\boxed{\text{Mixing Layer with } r \ll 1}$

In this scenario, convection of turbulence is much faster than its evolution so that:

$$q \sim U_s$$

Furthermore, the evolution of turbulence is on a time scale of δ / U_s, which implies the statistics evolve on a length scale of:

$$L \sim \frac{U_c \delta}{U_s} \leftrightarrow \frac{U_c}{U_s} \sim \frac{L}{\delta}$$

Consequently:

$$\left[\frac{U_s^2}{L}\right] \sim \left[\frac{U_c^2}{L}\left(\frac{\delta}{L}\right)^2\right], \quad \left[\frac{U_c U_s}{L}\right] \sim \left[\frac{U_c^2}{L}\left(\frac{\delta}{L}\right)\right]$$

$$\left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_s^2}{L}\right)\left(\frac{\delta}{L}\right)\right] \sim \left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_c^2}{L}\right)\left(\frac{\delta}{L}\right)^3\right]$$

$$\left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_s^2}{L}\right)\left(\frac{\delta}{L}\right)^{-1}\right] \sim \left[\left(\frac{1}{Re_\delta}\right)\left(\frac{U_c^2}{L}\right)\left(\frac{\delta}{L}\right)\right]$$

$$\left[\frac{q^2}{L}\right] \sim \left[\frac{U_s^2}{L}\right] \sim \left[\left(\frac{U_c^2}{L}\right)\left(\frac{\delta}{L}\right)^2\right]$$

$$\left[\frac{q^2}{\delta}\right] \sim \left[\frac{U_s^2}{\delta}\right] \sim \left[\left(\frac{U_c^2}{L}\right)\left(\frac{\delta}{L}\right)\right]$$

With these new scalings, we see that every term scales like $\left(\frac{U_c^2}{L}\right)\left(\frac{\delta}{L}\right)^k$ for some power k. Hence, we have a proper scaling, and the terms scaling like $\left(\frac{\delta}{L}\right)$ dominate. Neglecting the smaller terms yields:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} = \nu \cancel{\frac{\partial^2 \bar{u}}{\partial y^2}} - \frac{\partial}{\partial y}\langle u'v' \rangle$$

which is the proper axial mean momentum equation for a shear layer in a wake or coflowing jet.

Scenario # 2: $U_c \sim U_s$ $\boxed{\text{Non-coflowing Jet}}$ $\boxed{\text{Mixing Layer with } r \sim 1}$

In this scenario, the mean-convection terms must balance the Reynolds stress term. Thus:

$$\frac{q^2}{U_s^2} \sim \frac{\delta}{L} \Rightarrow \left[\frac{q^2}{L}\right] \sim \left[\frac{U_s^2}{L}\left(\frac{\delta}{L}\right)\right], \left[\frac{q^2}{\delta}\right] \sim \left[\frac{U_s^2}{L}\right]$$

Moreover, the viscous terms should be balanced by the other terms. This gives:

$$Re_\delta \gtrsim \frac{L}{\delta}$$

$$\left(\frac{U_s^2}{L}\right) \simeq \left(\frac{U_s U_c}{L}\right)$$

With the above scalings, we see that every term scales like $\left(\frac{U_c^2}{L}\right)\left(\frac{\delta}{L}\right)^k$ for some power k. As before, the terms scaling like $\left(\frac{\delta}{L}\right)^0$ dominate. Neglecting the smaller terms yields:

$O(1)$

$$\boxed{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y}\langle u'v' \rangle}$$

Shear layer axial mean momentum

which is the same as before except with an additional convection term.

Discussion on Viscous Stresses:

- Scenario # 1: If $Re_\delta \gg 1$, all viscous terms may be ignored.
- Scenario # 2: If $Re_\delta \gg \frac{L}{\delta}$, all viscous terms may be ignored.

Exercise: Show this!