The Scales of Turbulent Motion – Structure Functions:

To illustrate the power of the Kolmogorov hypotheses, we consider structure functions which are moments of the increment between the velocity field at two points. The second-order structure function is the covariance of the difference in the velocity between two points $\vec{x}+\vec{r}$ and $\vec{r}$:

$$D_{ij}(\vec{r},\vec{x},t) \equiv \left\langle \left[u_i(\vec{x}+\vec{r},t) - u_i(\vec{x},t)\right]\left[u_j(\vec{x}+\vec{r},t) - u_j(\vec{x},t)\right]\right\rangle$$

As a consequence of the hypothesis of local isotropy, we have that:

1. $D_{ij}(\vec{r},\vec{x},t)$ is independent of $\vec{x}$ for $|\vec{r}| \ll \mathcal{L}$. (Local Homogeneity)

$$\Rightarrow \quad D_{ij}(\vec{r},\vec{x},t) = D_{ij}(\vec{r},t) \quad \text{for } |\vec{r}| \ll \mathcal{L}$$

2. $D_{ij}(\vec{r},t)$ is an isotropic function of $\vec{r}$. (Local Isotropy) (for $|\vec{r}| \ll \mathcal{L}$)

The second hypothesis consequence above indicates that the second-order structure function has the form:

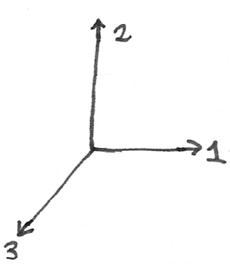
$$D_{ij}(\vec{r},t) = \underbrace{D_{NN}(r,t)}_{\substack{\text{Transverse} \\ \cancel{\text{Longitudinal}} \\ \text{Structure} \\ \text{Function}}}\,\delta_{ij} + \left[\underbrace{D_{LL}(r,t)}_{\substack{\text{Longitudinal} \\ \cancel{\text{Transverse}} \\ \text{Structure} \\ \text{Function}}} - D_{NN}(r,t)\right]\frac{r_i r_j}{\underbrace{r^2}_{\text{magnitude of } |\vec{r}|}}$$

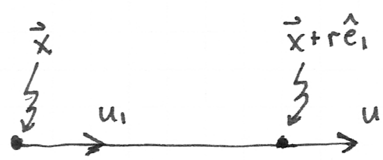
Now, let us choose the coordinate system such that $\vec{r} = r\hat{e}_1$ where $\hat{e}_1$ is the unit vector in the x_1-direction. Then, by construction:

$$D_{11} = D_{LL} \qquad\qquad D_{ij} = 0 \quad \text{for } i \neq j$$

$$D_{22} = D_{33} = D_{NN}$$

Visually:



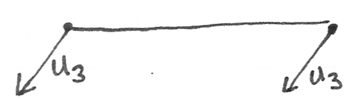


$$D_{11}(r,t) = \left\langle \left(u_1(\vec{x}+r\hat{e}_1) - u_1(\vec{x})\right)^2\right\rangle$$

$$= D_{LL} \qquad \text{Longitudinal S.F.}$$

$$D_{22}(r,t) = \left\langle \left(u_2(\vec{x}+r\hat{e}_1) - u_2(\vec{x})\right)^2\right\rangle$$

$$= D_{NN} \qquad \text{Transverse S.F.}$$



$$D_{33} = \left\langle \left(u_3(\vec{x}+r\hat{e}_1) - u_3(\vec{x})\right)^2\right\rangle$$

$$= D_{NN}$$

The interpretation of D_{LL} and D_{NN} as longitudinal and transverse structure functions is made clear by the illustrations above.

Now, suppose we have a homogeneous turbulent flow with $\bar{u} = 0$. A simple computation reveals:

$$\frac{\partial D_{ij}(\vec{r},t)}{\partial r_i} = 0$$

Then:

$$0 = \frac{\partial D_{ij}}{\partial r_i} = \frac{\partial D_{NN}}{\partial r}\frac{\partial r}{\partial r_i}\delta_{ij} + \frac{\partial}{\partial r}\left[D_{LL} - D_{NN}\right]\frac{\partial r}{\partial r_i}\frac{r_i r_j}{r^2}$$
$$+ \left[D_{LL} - D_{NN}\right]\frac{\partial}{\partial r_i}\left(\frac{r_i r_j}{r^2}\right)$$

To proceed, note:

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r}$$

$$\frac{\partial}{\partial r_i}\left(\frac{r_i r_j}{r^2}\right) = \underbrace{\frac{\partial r_i}{\partial r_i}}_{\delta_{ii}=3}\frac{r_j}{r^2} + \underbrace{\frac{\partial r_j}{\partial r_i}}_{\delta_{ij}}\frac{r_i}{r^2} - 2\frac{r_i r_j}{r^3}\underbrace{\frac{\partial r}{\partial r_i}}_{\frac{r_i}{r}}$$
$$= 3\frac{r_j}{r^2} + \delta_{ij}\frac{r_i}{r^2} - 2\frac{\cancel{r_i r_i}}{\cancel{r^2}}\frac{r_j}{r^2}$$
$$= 3\frac{r_j}{r^2} + \frac{r_j}{r^2} - 2\frac{r_j}{r^2}$$
$$= 2\frac{r_j}{r^2}$$

So:

$$0 = \frac{\partial D_{ij}}{\partial r_i} = \frac{\partial D_{NN}}{\partial r}\frac{r_i}{r}\delta_{ij} + \frac{\partial}{\partial r}\left[D_{LL} - D_{NN}\right]\frac{\cancel{r_i r_i}}{\cancel{r^2}}\frac{r_j}{r}$$
$$+ 2\left[D_{LL} - D_{NN}\right]\frac{r_j}{r^2}$$
$$= \frac{\partial D_{NN}}{\partial r}\frac{r_j}{r} + \frac{\partial}{\partial r}\left[D_{LL} - D_{NN}\right]\frac{r_j}{r} + 2\left[D_{LL} - D_{NN}\right]\frac{r_j}{r^2}$$
$$= \frac{\partial D_{LL}}{\partial r}\frac{r_j}{r} + 2\left[D_{LL} - D_{NN}\right]\frac{r_j}{r^2}$$

Therefore:

$$D_{NN}(\vec{r},t) = D_{LL}(\vec{r},t) + \frac{1}{2}r\frac{\partial}{\partial r}D_{LL}(\vec{r},t)$$

and hence $D_{ij}(\vec{r},t)$ is determined solely by the longitudinal structure function! This equation also holds for locally homogeneous flows, which is true for general flows if $|\vec{r}| \ll \mathcal{L}$.

Now, according to the first similarity hypothesis, given $\vec{r}$ with $|\vec{r}| \ll \mathcal{L}$, D_{ij} is uniquely determined by ε and ν. Dimensional analysis then dictates that:

$$D_{LL}(r,t) = (\varepsilon r)^{2/3}\hat{D}_{LL}(r/\eta) \quad \text{for } |\vec{r}| \ll \mathcal{L}$$

where $\hat{D}_{LL}$ is a universal, non-dimensional function. According to the second similarity hypothesis, given $\vec{r}$ with $\eta \ll |\vec{r}| \ll \mathcal{L}$, D_{LL} is independent of ν and hence η. Thus:

$$D_{LL}(r,t) = C_2 (\varepsilon r)^{2/3} \quad \text{for } \eta \ll |\vec{r}| \ll \mathcal{L}$$

where C_2 is a universal constant. We may then directly compute:

$$D_{NN}(r,t) = \tfrac{4}{3} C_2 (\varepsilon r)^{2/3} \quad \text{for } \eta \ll |\vec{r}| \ll \mathcal{L}$$

and:

$$D_{ij}(\vec{r},t) = C_2 (\varepsilon r)^{2/3} \left(\frac{4}{3}\delta_{ij} - \frac{1}{3}\frac{r_i r_j}{r^2} \right)$$

$$\text{for } \eta \ll |\vec{r}| \ll \mathcal{L}$$

Thus, in the inertial subrange, we may compute the second-order structure function solely in terms of ε, r, and C_2!

A selection of plots comparing the above predicted value for $D_{ij}(\vec{r},t)$ with experiment is included in a pdf on D2L at the link:

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Taking a value of $C_2 = 2.0$, we observe that:

- For $7{,}000\,\eta \approx \frac{1}{2}\mathcal{L} > r > 20\eta$, $D_{LL}/(\varepsilon r)^{2/3}$ is within $\pm 15\%$ of C_2.
- For $1{,}200\,\eta \approx \frac{1}{10}\mathcal{L} > r > 12\eta$, $D_{NN}/(\varepsilon r)^{2/3}$ is within $\pm 15\%$ of C_2.

Consequently, there is great experimental support for Kolmogorov's hypotheses. We may also conduct the above analyses for other structure functions to find:

$$\underbrace{\left\langle \left(u_i(\vec{x} + r\hat{e}_i, t) - u_i(\vec{x},t) \right)^p \right\rangle}_{p^{th}\text{-order Longitudinal structure function}} \sim (\varepsilon r)^{p/3}$$

but departures from this estimate have been observed for high p. This departure is due to small-scale intermittency, which is not captured by Kolmogorov's theory.