The Reynolds Equations:

Given the statistical tools introduced earlier, we are able to derive equations for the statistics of a turbulent flow. The heart of this derivation is the Reynolds decomposition:

$$\vec{u}(\vec{x},t) = \langle \vec{u}(\vec{x},t) \rangle + \vec{u}'(\vec{x},t)$$

$$= \overline{u}(\vec{x},t) + \vec{u}'(\vec{x},t)$$

Indicial:

$$u_i = \overline{u}_i + u_i'$$

in which the velocity is decomposed into its mean (or expected value) and the fluctuation.

In many situations of engineering interest, knowledge of the mean velocity of a turbulent flow is enough to make a prediction. For example, knowledge of the mean allows one to determine the "average" lift and drag on an airfoil or the pressure drop in a pipe. If the mean temperature or chemical concentration is also known, one could determine rates of mass and heat transfer to a surface. Consequently, much effort has been directed towards determining expected values of turbulent flow fields.

In order to derive an equation for the mean velocity, one must start with the Navier-Stokes equations:

$$\frac{\partial}{\partial t}(u_i) + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial}{\partial x_i}(p) + \frac{1}{Re}\frac{\partial^2}{\partial x_j^2}(u_i)$$

Now we take the mean of the above equations. Each term then becomes:

$$\left\langle \frac{\partial}{\partial t}(u_i) \right\rangle = \frac{\partial}{\partial t}(\langle u_i \rangle) = \frac{\partial}{\partial t}(\overline{u}_i)$$

$$\left\langle \frac{\partial}{\partial x_j}(u_i u_j) \right\rangle = \frac{\partial}{\partial x_j}(\langle u_i u_j \rangle)$$

$$\left\langle \frac{\partial}{\partial x_i}(p) \right\rangle = \frac{\partial}{\partial x_i}(\langle p \rangle) = \frac{\partial}{\partial x_i}(\overline{p})$$

$$\left\langle \frac{1}{Re}\frac{\partial^2}{\partial x_j^2}(u_i) \right\rangle = \frac{1}{Re}\frac{\partial^2}{\partial x_j^2}(\langle u_i \rangle) = \frac{1}{Re}\frac{\partial^2}{\partial x_j^2}(\overline{u}_i)$$

To complete our derivation, notice that:

$$\begin{aligned}
\langle u_i u_j \rangle &= \langle (\overline{u}_i + u_i')(\overline{u}_j + u_j') \rangle \\
&= \langle \overline{u}_i \overline{u}_j \rangle + \langle u_i' \overline{u}_j \rangle + \langle \overline{u}_i u_j' \rangle + \langle u_i' u_j' \rangle \\
&= \overline{u}_i \overline{u}_j + \overline{u}_j \cancelto{0}{\langle u_i' \rangle} + \overline{u}_i \cancelto{0}{\langle u_j' \rangle} + \langle u_i' u_j' \rangle \\
&= \overline{u}_i \overline{u}_j + \langle u_i' u_j' \rangle
\end{aligned}$$

where we have exploited the fact that the mean of the fluctuation is zero.

Consequently, we are left with the following equations for the means:

Mean-Momentum Equations

$$\boxed{\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(-\langle u_i' u_j' \rangle + \frac{1}{Re}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right)}$$

where we have exploited the fact that:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(\bar{u}_j)\right) = \left\langle \frac{\partial^2 u_j}{\partial x_j \partial x_i} \right\rangle$$

$$= \left\langle \frac{\partial}{\partial x_i}\left(\cancel{\frac{\partial u_j}{\partial x_j}}^{\,0}\right)\right\rangle$$

$$= 0$$

since $\vec{\nabla} \cdot \vec{u} = 0$ (conservation of mass). The mean-momentum equations, also referred to as the Reynolds equations, looks nearly identical to the Navier-Stokes equations except for the term involving $-\langle u_i' u_j' \rangle$, which is referred to as the Reynolds stress or the turbulent stress. The mean-momentum equations are complemented by the mean-mass equation:

Mean-Mass Equation

$$\boxed{\frac{\partial \bar{u}_j}{\partial x_j} = 0}$$

which gives a constraint for the mean velocity field.

The mean-momentum equations and mean-mass equation comprise a system of four equations for the four mean quantities $\bar{u}_i$ and $\bar{p}$. Unfortunately, they also involve six additional unknown parameters $\langle u_i' u_j' \rangle$. To be able to solve for the mean velocity and pressure fields, one must either explicitly know the Reynolds stresses or introduce an additional set of equations.

The above is a manifestation of the closure problem. We have a set of equations with more unknowns than equations – that is, we have an unclosed system. We could introduce six more equations for the evolution of the Reynolds stresses, but these would involve third-order moments of the velocity field. In general, the set of equations governing the evolution of statistical quantities (obtained from Navier-Stokes) will always be unclosed!

As a consequence of the closure problem, modeling of some statistical quantities (e.g., the Reynolds stresses) is always required. One major turbulence research field is turbulence modeling, which has the primary goal of approximating the Reynolds stress.

Remarks:

1. The mean-momentum equations as written above are in conservation form. Noting that:

$$\frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) = \bar{u}_i \cancel{\frac{\partial \bar{u}_j}{\partial x_j}}^{\,0} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}$$

we can also write them in advective form:

$$\frac{\bar{D} \bar{u}_i}{\bar{D} t} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(-\langle u_i' u_j' \rangle + \frac{1}{Re}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right)$$

where

$$\frac{\bar{D}}{\bar{D}t} \equiv \frac{\partial}{\partial t} + \bar{u} \cdot \vec{\nabla}$$

is the mean substantial derivative.

2. If a turbulent flow field is statistically stationary, then:

$$\frac{\partial}{\partial t}\left(\bar{u}_i\right) = 0$$

which implies we are left with a steady set of PDEs. Note that the mean is not taken to be the time-average above! Instead, we simply have a flow with stationary statistics.