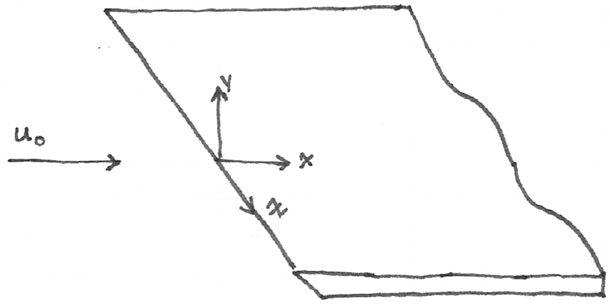
Turbulent Boundary Layer - Characterization and Mean-Flow Equations:

The second wall-bounded shear flow we consider is a boundary layer which is formed when a uniform-velocity stream flows over a smooth flat plate as depicted below:



Compared with fully developed channel flow, a boundary layer has the following primary differences:

(i) The boundary layer develops continuously in x and the boundary layer thickness $\delta(x)$ increases with x.

(ii) The wall shear stress $\tau_w(x)$ is not known *a priori*.

(iii) The outer part of the flow is intermittent.

However, the inner layer of boundary layer flow is essentially the same as channel flow. In the defect layer, differences are more significant.

The coordinate system for boundary layer flow is essentially the same as channel flow. The surface of the plate is at $y=0$, with the leading edge at $x=y=0$. The mean flow is primarily in the x-direction, with the incoming free-stream velocity denoted as u_0. Statistics vary primarily in y and are independent of z. Unlike channel flow, statistics depend on both x and y. The velocity is denoted as $\vec{u} = (u, v, w)$ with $\bar{w} = 0$.

We assume throughout that the pressure gradient is zero, which follows from Bernoulli's equation in the free-stream:

$$-\frac{dp_0}{dx} = \rho u_0 \cancelto{0}{\frac{du_0}{dx}}$$

Nonzero pressure gradients greatly impact the structure of boundary layer flow. Namely, if the pressure-gradient is positive (i.e., adverse), corresponding to decelarating flow, separation of the boundary layer may occur.

The boundary layer thickness may be defined in a number of ways. For example, it may be defined as $\delta_{99}(x)$, the value of y at which $\bar{u}(x,y) = 0.99\, u_0$. However, this quantity is very sensitive to velocity differences. Alternatively, one may use the displacement thickness:

$$\delta^*(x) \equiv \int_0^\infty \left(1 - \frac{\bar{u}}{u_0}\right) dy$$

or the momentum thickness:

$$\theta(x) \equiv \int_0^\infty \frac{\bar{u}}{u_0}\left(1 - \frac{\bar{u}}{u_0}\right) dy$$

Based on the above, we may define various Reynolds numbers for channel flow:

$$Re_x \equiv \frac{u_0 x}{\nu}, \quad Re_{\delta_{99}} \equiv \frac{u_0 \delta_{99}}{\nu}, \quad Re_{\delta^*} \equiv \frac{u_0 \delta^*}{\nu}, \quad Re_\theta \equiv \frac{u_0 \theta}{\nu}$$

Re_x serves as a local Reynolds number. In a zero-pressure-gradient boundary layer, the flow is laminar from the leading edge ($x = 0$) until $Re_x \approx 10^6$ which marks the start of transition to turbulence.

For boundary layer flow, the boundary layer equations apply:

Mean Continuity: $$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

Mean Axial Momentum: $$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y}\langle u'v'\rangle - \frac{1}{\rho}\frac{dp_0}{dx}^{\nearrow 0}$$
$$= \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

Above, $\tau(x,y)$ is the total shear stress as with a channel flow:

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho\langle u'v'\rangle$$

At the wall, $\bar{u} = \bar{v} = 0$, giving:

$$\frac{1}{\rho}\left(\frac{\partial \tau}{\partial y}\right)_{y=0} = \nu\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)_{y=0} = 0$$

since $\langle u'v'\rangle = 0$ at $y = 0$. Integrating the mean axial momentum equation from $y=0$ to $y=\infty$ yields the von Kármán integral momentum equation:

$$\tau_w = \frac{d}{dx}\left(\rho u_0^2 \theta\right) = \rho u_0^2 \frac{d\theta}{dx}$$

or equivalently:

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho u_0^2} = 2\frac{d\theta}{dx}$$

The above quantifies the decrease in the momentum-flow rate of the stream caused by the wall shear.

For a laminar boundary layer subject to zero pressure gradient, similarity analysis may be used to solve the boundary layer equations. This procedure is due to Blasius, and the solution is known as the Blasius boundary layer:

$$u(x,y) = u_0 \frac{df}{d\eta}(\eta) \qquad \eta = y\left(\frac{u_0}{\nu x}\right)^{1/2}$$

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0, \quad f(0) = \frac{df}{d\eta}(0) = 0, \quad \frac{df}{d\eta} \to 1 \text{ as } \eta \to \infty$$

The thicknesses associated with the Blasius boundary layer are:

$$\frac{\delta_{99}}{x} \approx 4.9\, Re_x^{-1/2}, \quad \frac{\delta^*}{\delta} \approx 0.35, \quad \frac{\theta}{\delta} \approx 0.14$$

and the skin-friction coefficient is:

$$C_f \approx 0.664\, Re_x^{-1/2}$$