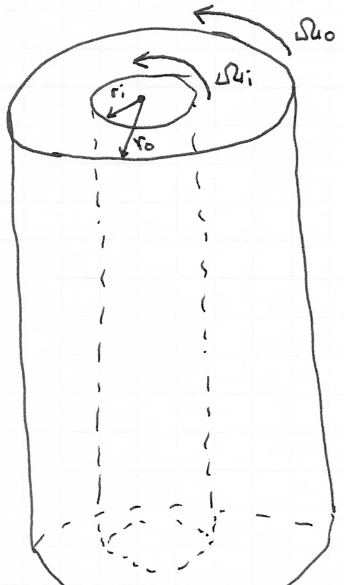
Dynamical Systems and Turbulence:

Previously, we had discussed at length how simple dynamical systems can exhibit chaotic behavior. But how do these relate back to turbulence?

As a thought experiment, suppose you discretize the 3-D incompressible Navier-Stokes equations using a finite difference method with N grid points. Then you would have $3N$ ordinary differential equations with N constraints! The ODEs have quadratic nonlinearity, just like the Lorenz equations, and hence these ODEs may also have chaotic solutions!

As $N \to \infty$, we expect convergence to the real solution (provided it exists) in an ∞-dimensional phase space. Thus, while there are similarities between turbulence and chaos, turbulence may be much more complicated! The mathematical tools needed to study the incompressible Navier-Stokes equations lie within the field of nonlinear functional analysis, which we will not touch upon in this class.

To see how a flow field might exhibit the properties discussed previously, consider Taylor-Couette flow, the flow between concentric cylinders driven by the rotation of the inner (and possibly outer) cylinder. The system has perfectly rigid boundaries, constant cylinder speeds, and no inflow or outflow, so it is a closed system.



Outer Reynolds Number: $R_o = \dfrac{(r_o - r_i)\Omega_i r_i}{\nu}$

Inner Reynolds Number: $R_i = \dfrac{(r_o - r_i)\Omega_o r_o}{\nu}$

If $R_o = 0$, Taylor-Couette flow undergoes a series of transitions (bifurcations) with increasing R_i:

increasing R_i ↓

1. Laminar, 1-D flow ($U_\theta(r)$)
2. Taylor vortices (axially symmetric azimuthal vortices)
3. Wavy vortices
4. Modulated wavy vortices
5. Weakly chaotic flow *
6. Turbulent Taylor vortices (Fully turbulent)

By carefully examining the flow at stages 1-6, we see it appears to evolve on an attractor. For almost all initial conditions, the system goes to the same generic behavior, but the details may be unpredictable.

Taylor-Couette flow can exhibit different gross behaviors depending on how the system is started (suddenly turned on, gradually turned on), so it has multiple attractors. The region of phase space in which trajectories all evolve to a given attractor is the basin of attraction.

Phase-space details are available on the D2L website!

* Weakly chaotic flow is characterized by subexponential instabilities. That is, perturbations diverge at a rate less than exponential. This may be precisely defined using the Lyapunov exponent.

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