Homogeneous Isotropic Turbulence: Dynamics of the Two-Point Correlation Tensor

Now that we have "fully" characterized isotropic turbulence with zero mean, we may begin to discuss its dynamics. We begin by considering the dynamics of the two-point correlation tensor. Note that we may also obtain knowledge about the second-order structure function from this discussion since:

$$f(r,t) = 1 - \frac{1}{2\sigma^2} D_{LL}(r,t)$$

To begin our discussion, we recall the Navier-Stokes momentum equation:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i u_k) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \qquad (*)$$

Defining $\check{u}_j(\check{x}) = u_j(\vec{x}+\vec{r})$ where $\check{x} = \vec{x}+\vec{r}$, we also have:

$$\frac{\partial \check{u}_j}{\partial t} + \frac{\partial}{\partial \check{x}_k}(\check{u}_j \check{u}_k) = -\frac{1}{\rho}\frac{\partial \check{p}}{\partial \check{x}_j} + \nu \frac{\partial^2 \check{u}_j}{\partial \check{x}_k \partial \check{x}_k} \qquad (**)$$

To proceed forward, we must interpret the derivatives in (*) as being:

$$\frac{\partial}{\partial x_k} = \left.\frac{\partial}{\partial x_k}\right|_{\check{x}}$$

which means we evaluate the derivatives keeping $\check{x}$ fixed. Analogously:

$$\frac{\partial}{\partial \check{x}_k} = \left.\frac{\partial}{\partial \check{x}_k}\right|_{\vec{x}}$$

Hence, if we multiply (*) by $\check{u}_j$ and (**) by u_i, add the two resulting expressions, and then take the mean, we obtain:

$$\frac{\partial}{\partial t}\langle u_i \check{u}_j\rangle + \frac{\partial}{\partial x_k}\langle \check{u}_j u_i u_k\rangle + \frac{\partial}{\partial \check{x}_k}\langle u_i \check{u}_j \check{u}_k\rangle =$$
$$-\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle \check{u}_j p\rangle - \frac{1}{\rho}\frac{\partial}{\partial \check{x}_j}\langle u_i \check{p}\rangle$$
$$+ \nu\left[\frac{\partial^2}{\partial x_k \partial x_k}\langle u_i \check{u}_j\rangle + \frac{\partial^2}{\partial \check{x}_k \partial \check{x}_k}\langle u_i \check{u}_j\rangle\right]$$

To proceed, note that:

$$\left.\frac{\partial}{\partial x_k}\right|_{\check{x}} = -\frac{\partial}{\partial r_k} \qquad \text{and} \qquad \left.\frac{\partial}{\partial \check{x}_k}\right|_{\vec{x}} = \frac{\partial}{\partial r_k}$$

Moreover, we define:

$$S_{ijk}(\vec{r},t) \equiv \langle u_i(\vec{x},t)\, u_j(\vec{x},t)\, u_k(\vec{x}+\vec{r},t)\rangle \quad \text{Two-Pt. Three Velocity Correlation}$$

$$R_{pi}(\vec{r},t) \equiv \langle p(\vec{x},t)\, u_i(\vec{x}+\vec{r},t)\rangle \quad \text{Pressure-Two-Pt. Velocity Correlation}$$

By isotropy:

$$S_{ijk}(-\vec{r},t) = -S_{ijk}(\vec{r},t)$$
$$R_{pi}(-\vec{r},t) = -R_{pi}(\vec{r},t)$$

Consequently, we have that:

$$\frac{\partial}{\partial x_k}\langle \check{u}_j u_i u_k\rangle = -\frac{\partial}{\partial r_k} S_{ikj}(\vec{r},t)$$

$$\frac{\partial}{\partial \check{x}_k}\langle u_i \check{u}_j \check{u}_k\rangle = \frac{\partial}{\partial r_k} S_{jki}(-\vec{r},t) = -\frac{\partial}{\partial r_k} S_{jki}(\vec{r},t)$$

$$\frac{\partial}{\partial x_i}\langle \check{u}_j p\rangle = -\frac{\partial}{\partial r_i} R_{pj}(\vec{r},t)$$

$$\frac{\partial}{\partial \check{x}_j}\langle u_i \check{p}\rangle = \frac{\partial}{\partial r_j} R_{pi}(-\vec{r},t) = -\frac{\partial}{\partial r_j} R_{pi}(\vec{r},t)$$

$$\frac{\partial^2}{\partial x_k \partial x_k}\langle u_i \check{u}_j\rangle = \frac{\partial^2}{\partial r_k \partial r_k} R_{ij}(\vec{r},t)$$

$$\frac{\partial^2}{\partial \check{x}_k \partial \check{x}_k}\langle u_i \check{u}_j\rangle = \frac{\partial^2}{\partial r_k \partial r_k} R_{ij}(\vec{r},t)$$

Thus defining:

$$T_{ij} \equiv \frac{\partial}{\partial r_k} S_{ikj} + \frac{\partial}{\partial r_k} S_{jki}$$

$$P_{ij} \equiv \frac{\partial}{\partial r_i} R_{pj} + \frac{\partial}{\partial r_j} R_{pi}$$

we have:

pressure processes

$$\frac{\partial R_{ij}}{\partial t} = T_{ij} + P_{ij} + 2\nu \frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}$$

inertial processes — viscous processes

which is the equation governing the dynamics of the two-point correlation tensor. The above is complemented by the constraints:

$$\frac{\partial R_{ij}}{\partial r_i} = \frac{\partial R_{ij}}{\partial r_j} = 0$$

Simplifications can be made by considering the structure of R_{pi} and S_{ijk}. By isotropy:

$$R_{pi}(\vec{r},t) = a(r,t)\frac{r_i}{r}$$

Since $\partial u_i/\partial x_i = 0$, we have:

$$0 = \frac{\partial R_{pi}(\vec{r},t)}{\partial r_i} = \frac{\partial a(r,t)}{\partial r}\overset{r_i/r}{\cancel{\frac{\partial r}{\partial r_i}}}\frac{r_i}{r} + a(r,t)\overset{3}{\cancel{\frac{\partial r_i}{\partial r_i}}} + a(r,t)r_i\left(-\frac{1}{r^2}\right)\overset{r_i/r}{\cancel{\frac{\partial r}{\partial r_i}}}$$

$$= \frac{\partial a(r,t)}{\partial r} + 2a(r,t)/r$$

So:

$$a(r,t) = c(t)/r^2$$

However, we require that $a(r,t) < \infty$ for $r = 0$, so $a(r,t) \equiv 0$. Thus:

$$R_{pi}(\vec{r},t) = 0 \quad \Rightarrow \quad P_{ij} = 0$$

and velocity and pressure are uncorrelated. We also obtain the simplified equation:

$$\boxed{\frac{\partial R_{ij}}{\partial t} = T_{ij} + 2\nu \frac{\partial^2 R_{ij}}{\partial r_k \partial r_k}}$$

Two-Pt. Correlation Evolution Equation

which states the two-point correlation tensor evolves due to inertial and viscous processes only.

By isotropy, we also have:

$$S_{ijk}(\vec{r},t) = \left[\left(\bar{k}(r,t) - \bar{h}(r,t) - 2\bar{q}(r,t) \right) \frac{r_i r_j r_k}{r^2} + \delta_{ij} \bar{h}(r,t) \frac{r_k}{r} + \bar{q}(r,t) \left(\delta_{ik} \bar{h} \frac{r_k}{r} + \delta_{jk} \frac{r_i}{r} \right) \right] \sigma^3$$

and hence:

$$S_{111}(r\hat{e}_1,t) = \sigma^3 \bar{k}(r,t)$$
$$S_{221}(r\hat{e}_1,t) = \sigma^3 \bar{h}(r,t)$$
$$S_{212}(r\hat{e}_1,t) = \sigma^3 \bar{q}(r,t)$$

Due to incompressibility:

$$\frac{\partial}{\partial r_k} S_{ijk}(\vec{r},t) = 0$$

which gives:

$$\bar{q}(r,t) = \frac{1}{4r} \frac{\partial}{\partial r} \left(\bar{k}(r,t) r^2 \right)$$

$$\bar{h}(r,t) = -\tfrac{1}{2} \bar{k}(r,t)$$

Hence, S_{ijk} may be completely characterized by $\bar{k}$:

$$S_{ijk} = \sigma^3 \left[\left(\bar{k} - r \frac{\partial \bar{k}}{\partial r} \right) \frac{r_i r_j r_k}{r^3} - \frac{\bar{k}}{2} \frac{r_k}{r} \delta_{ij} + \frac{1}{4r} \frac{\partial}{\partial r}(r^2 \bar{k}) \left(\delta_{ik} \frac{r_j}{r} + \delta_{jk} \frac{r_i}{r} \right) \right]$$

Since R_{ij} is completely characterized by f and S_{ijk} by $\bar{k}$, we may compress the two-pt. correlation evolution equation into a single scalar equation for f:

$$\boxed{\frac{\partial}{\partial t}(\sigma^2 f) - \frac{\sigma^3}{r^4} \frac{\partial}{\partial r}(r^4 \bar{k}) = \frac{2\nu\sigma^2}{r^4} \frac{\partial}{\partial r}\left(r^4 \frac{\partial f}{\partial r} \right)}$$

Kármán–Howarth equation

A detailed derivation of the above equation is included below.

Derivation of the Kármán–Howarth Equation:

We begin by noting R_{ij} has the form:

$$R_{ij} = \sigma^2 \left[-\frac{1}{2r} \frac{\partial f}{\partial r} r_i r_j + \left(f + \frac{1}{2} r \frac{\partial f}{\partial r} \right) \delta_{ij} \right]$$

Contracting on i yields:

$$R_{ii} = \frac{\sigma^2}{r^2} \frac{\partial}{\partial r} \left[r^3 f \right]$$

So:

$$\frac{\partial}{\partial t} R_{ii}(\vec{r},t) = \frac{1}{r^2} \frac{\partial^2}{\partial r \partial t} \left[r^3 \sigma^2(t) f(r,t) \right]$$

Algebraic manipulations also reveal:

$$\frac{\partial^2}{\partial r_k r_k} R_{ii}(\vec{r},t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left[\frac{\overset{\sigma(t)^2}{\sigma^2}}{r^2}\frac{\partial}{\partial r}\left[r^3 \overset{f(r,t)}{f}\right]\right]\right]$$

By direct differentiation, we have:

$$\frac{\partial}{\partial r_j} S_{ijk}(\vec{r},t) = \sigma^3\left[\left(-\frac{1}{4r}\frac{\partial^2 \bar{k}(r,t)}{\partial r^2} - \frac{1}{r^2}\frac{\partial \bar{k}}{\partial r} + \frac{\bar{k}}{r^3}\right) r_i r_k + \left(\frac{r}{4}\frac{\partial^2 \bar{k}}{\partial r^2} + \frac{3}{2}\frac{\partial \bar{k}}{\partial r} + \frac{\bar{k}}{r}\right)\delta_{ik}\right]$$

Hence:

$$T_{ij}(\vec{r},t) = \sigma^3\left[\left(-\frac{1}{2r}\frac{\partial^2 \bar{k}(r,t)}{\partial r^2} - \frac{2}{r^2}\frac{\partial \bar{k}}{\partial r} + \frac{2\bar{k}}{r^3}\right) r_i r_j + \left(\frac{r}{2}\frac{\partial^2 \bar{k}}{\partial r^2} + 3\frac{\partial \bar{k}}{\partial r} + \frac{2\bar{k}}{r}\right)\delta_{ij}\right]$$

and contracting on i yields:

$$T_{ii} = \sigma^3\left(r\frac{\partial^2 \bar{k}}{\partial r^2} + 7\frac{\partial \bar{k}}{\partial r} + 8\frac{\bar{k}}{r}\right) = \sigma^3\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial \bar{k}}{\partial r} + 4r^2\bar{k}\right)$$

Combining the above with the two-pt. correlation evolution equation yields:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial t}\left(r^3\sigma^2 f\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\sigma^3\left(r^3\frac{\partial \bar{k}}{\partial r} + 4r^2\bar{k}\right)\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(2\nu r^2\frac{\partial}{\partial r}\left(\frac{\sigma^2}{r^2}\frac{\partial}{\partial r}\left(r^3 f\right)\right)\right)$$

Integrating in r yields:

$$\frac{\partial}{\partial t}\left(\sigma^2 f\right) = \sigma^3\underbrace{\left(\frac{\partial \bar{k}}{\partial r} + \frac{4}{r}\bar{k}\right)}_{\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\bar{k}\right)} + \frac{2\nu}{r}\sigma^2\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 f\right)\right)}_{\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)}$$

giving precisely:

$$\frac{\partial}{\partial t}\left(\sigma^2 f\right) - \frac{\sigma^3}{r^4}\frac{\partial}{\partial r}\left(r^4\bar{k}\right) = \frac{2\nu\sigma^2}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \qquad \square$$

A few remarks:

1.) There is a closure problem with the Kármán-Howarth equation. Namely, the single equation involves two unknowns, $f(r,t)$ and $\bar{k}(r,t)$.

2.) The terms in $\bar{k}$ and ν represent inertial and viscous processes, respectively.

3.) The function $\bar{k}(r,t)$ is necessarily odd in r and the continuity equation implies $\bar{k}'(0,t) = 0$. Hence:

$$\bar{k}(r,t) = \bar{k}''' r^3/3! + \bar{k}''''' r^5/5! \ \ldots$$

4.) At $r=0$, the term in $\bar{k}$ vanishes. Moreover, as $f(r,t)$ is even in r due to isotropy,

$$\left[\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)\right]_{r=0} = 5 f''(0,t) = -\frac{5}{\lambda_g(t)^2}$$

This implies the kinetic-energy equation $\frac{d}{dt}k = -\varepsilon$ for isotropic turbulence.

5.) In the Richardson-Kolmogorov view of the energy cascade at high Re, the transfer of energy in the inertial subrange is an inertial process. This transfer is accomplished by the term in $\underline{k}$ in the Kármán-Howarth equation.

6.) If $\vec{u}(\vec{x},t)$ were Gaussian, $\bar{k}$ would be zero as it is a third moment. Hence the energy cascade depends on non-Gaussian aspects of the flow field.

The Kármán-Howarth equation may be reexpressed in terms of the second-order structure function D_{LL} and the third-order structure function:

$$D_{LLL}(r,t) = \langle [u_1(\vec{x}+r\hat{e}_1,t) - u_1(\vec{x},t)]^2 \rangle$$

For stationary turbulence, the resulting equation is:

$$\frac{3}{r^4}\int_0^r s^4 \frac{\partial}{\partial t} \cancelto{0}{D_{LL}(s,t)}\,ds = 6\nu \frac{\partial D_{LL}}{\partial r} - D_{LLL} - \frac{4}{5}\varepsilon r$$

For $\eta \ll r$, the viscous term may be neglected, giving the Kolmogorov $\frac{4}{5}$ law:

$$D_{LLL}(r,t) = -\frac{4}{5}\varepsilon r$$

The above is a remarkable result, and the only instance I know of where turbulence scaling theory yields a relationship with $\underline{\text{no}}$ undetermined constants.

Finally, in the viscous range $(r \ll \eta)$, inertial processes are negligible. As isotropic turbulence decays, the Reynolds number decreases and eventually the entire flow's within the viscous range. For this final period of decay, Batchelor and Townsend showed in 1948 that:

$$f(r,t) = \exp[-r^2/(8\nu t)]$$

which is in excellent agreement with experimental data.