Example of a Chaotic System: The Logistic Map

Consider the iterative map:

$$x_{n+1} = R x_n (1 - x_n) \qquad 0 < R \leq 4$$

which maps $[0,1]$ to $[0,1]$.

The above is a model of population dynamics (reproduction & starvation):

- For small x_n, the population increases like $R x_n$.

  $R = 2 \Rightarrow$ 2 kids per couple every cycle!
  Hence population doubles every cycle.

- Growth rate decreases at a rate proportional to $(1 - x_n)$, the "carrying capacity" of the environment.

  Small x_n: "Optimal" capacity of nearly 1

  Large x_n: Capacity goes to zero

The character of the solution depends on R:

1. $R \leq 1$: Steady state $x^* = 0$ is stable.
2. $1 \leq R < 3$: Steady state $x^* = 1 - 1/R$ is stable.
3. $3 < R < \lambda_1 = 1 + \sqrt{6} \approx 3.449$: Stable limit cycle of period 2.
4. $\lambda_1 < R < \lambda_2$: Stable limit cycle of period 4.
5. $\lambda_{n-1} < R < \lambda_n$: Stable limit cycle of period 2^n.
6. $R = \lambda_\infty \approx 3.56995$: Onset of chaos.

Period Doubling

The above transitions are displayed in images and movies on the D2L website under:

Content → Course Materials → Logistic Map

Of particular interest is the image:

Logistic_Bifurcation_map_High_Resolution

which displays a <u>bifurcation</u> <u>diagram</u> for the logistic map. On the horizontal axes are values for R and on the vertical axes are possible <u>long-term</u> values for x.

Note for $1 < R < 3$, only one long-term value for x is attained.

At $R = 3$, the curve <u>bifurcates</u>, indicating there are two long-term values. These correspond to a limit cycle of ~~perid~~ period 2.

As R increases further, the curve further bifurcates, indicating the period doubling cascade.

This cascade is self-similar – future bifurcations behave like past ones. In fact:

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \rightarrow 4.66920\ldots \text{ (Feigenbaum's constant)}$$

as $n \rightarrow \infty$. This constant pops up regularly in chaos theory.

Past $R = \lambda_\infty$ is the onset of chaos where infinitely many long term values of x are expected. Hence, past $R = \lambda_\infty$, there is a lack of predictability for the system.

However, past $R = \lambda_\infty$, there are occasional islands of stability. For example, for:

$$R > 1 + \sqrt{8} \approx 3.82843$$

there is a small range of R which show oscillation among three values. For slightly larger R, these branches bifurcate themselves in self-similar behavior.

For the interval:

$$\lambda_\infty < R < 1 + \sqrt{8}$$

the so-called Pomeau-Manneville scenario exists. In this range, there are periodic (laminar) phases interrupted by bursts of aperiodic behavior. This type of intermittency is also seen in turbulent fluid flow where there might be irregular dissipation of kinetic energy due to turbulent spots and bursts.

Analysis for $R = 4$:

Let $x_n = \dfrac{1 - \cos(2\pi\theta_n)}{2}$.

Then:

$$\begin{aligned} x_{n+1} &= 4(x_n - x_n^2) \\ &= 4\left(\frac{1 - \cos(2\pi\theta_n)}{2} - \frac{1 - 2\cos(2\pi\theta_n) + \cos^2(2\pi\theta_n)}{4}\right) \\ &= 1 - \cos^2(2\pi\theta_n) \\ &= \sin^2(2\pi\theta_n) \\ &= \frac{1 - \cos(2\pi(2\theta_n))}{2} \end{aligned}$$

Trig. identity !!

So: $\theta_{n+1} = 2\theta_n$

Since cosine is periodic w/ period 2π, only the fractional part of θ is important.

Suppose: $\theta_0 = 0.100110100001101\ldots$ (in binary)

Then: $\theta_1 = 1.00110100001101\ldots$

$\theta_2 = 10.0110100001101\ldots$

$\theta_3 = 100.110100001101\ldots$

$\vdots$

$\theta_{14} = 100110100001101.\ldots$

Hence, if we only have k digits of binary accuracy at the zeroth step, we lose all notion of predictability by the k^{th} step!

Moreover, noise will also pollute our solution in finite time!

Computer w/ 24 bit mantissa (32 bit machine):

All accuracy is lost after 24 steps!

Measurement w/ 5 (base 10) digits of accuracy:

All accuracy is lost after ~ 17 steps!

The above model demonstrates:

* 1. Sensitivity to initial conditions results in a lack of predictability in chaotic systems.
* 2. Systems can undergo a large series of bifurcations leading to chaos.
* 3. Chaotic systems can explore large regions of state space.