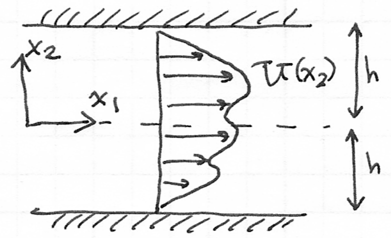
Stability and Transition to Turbulence:

In the dynamical systems we have studied thus far, the solution became chaotic after a nonchaotic system became unstable.

Similarly, turbulence often arises when a laminar flow is unstable!

Consider the following simple shear flow:

$$\vec{u} = \begin{bmatrix} U(x_2) \\ 0 \\ 0 \end{bmatrix}$$



$$Re = \frac{U_0 h}{\nu}$$

where:

$U(x_2)$ = Solution to Navier-Stokes
U_0 = Characteristic speed
h = Half-channel height
ν = Kinematic viscosity

Now suppose we introduce a small, 2-D perturbation:

$$\vec{u}^* = \vec{u} + \delta\vec{u} \qquad \text{where:} \qquad \delta\vec{u} = \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ 0 \end{bmatrix}$$

$$u_i^* = \delta_{i1} U(x_2) + \delta u_i$$

Plugging the above velocity field into the incompressible Navier-Stokes equations, recognizing the laminar base flow $\vec{u}$ is a solution, and ignoring higher-order terms in $\delta\vec{u}$, we obtain the system:

$$\frac{\partial}{\partial t}(\delta u_i) + U \frac{\partial}{\partial x_1}(\delta u_i) + \delta u_2 \, \delta_{i1} \frac{\partial}{\partial x_2}(U) = -\frac{\partial}{\partial x_i}(\delta P) + \frac{1}{Re} \frac{\partial^2}{\partial x_j^2}(\delta u_i)$$

which is a linearized set of equations in terms of the velocity perturbations $\delta\vec{u}$ and the pressure perturbation δp.

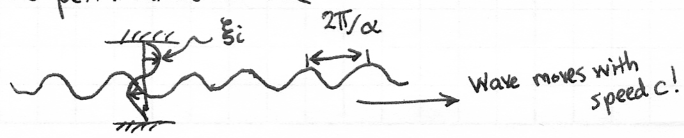
To proceed further, we assume the perturbations take the form:

$$\delta u_i = \xi_i(x_2)\, e^{i\alpha(x_1 - ct)}$$

where:

ξ_i = Amplitude
α = Angular Frequency
c = Wave Speed

So if c is real, the perturbations look like



If c has positive imaginary part, the perturbation wave will grow exponentially (instability), while if it has negative imaginary part, the wave will decay exponentially (asymptotic stability).

Recall the continuity equation:

$$\vec{\nabla}\cdot\vec{u}^* = \cancelto{0}{\vec{\nabla}\cdot\vec{u}} + \vec{\nabla}\cdot\delta\vec{u} = 0$$

Plugging in our form for the perturbation, one obtains:

$$\frac{\partial}{\partial x_1}\left(\xi_1(x_2)\, e^{i\alpha(x_1-ct)}\right) + \frac{\partial}{\partial x_2}\left(\xi_2(x_2)\, e^{i\alpha(x_1-ct)}\right) = 0$$

$$\Rightarrow \quad i\alpha\,\xi_1\,\cancel{e^{i\alpha(x_1-ct)}} + \frac{\partial\xi_2}{\partial x_2}\,\cancel{e^{i\alpha(x_1-ct)}} = 0$$

$$\Rightarrow \quad \frac{d\xi_2}{dx_2} + i\alpha\,\xi_1 = 0$$

which gives the amplitude ξ_1 in terms of ξ_2.

To proceed further, we take the curl of the momentum equation twice, consider its x_2 component, substitute in our expression for the perturbations, and exploit the continuity equation to obtain the equations (left as an exercise):

$$(U-c)(\xi_2'' - \alpha^2\xi_2) - U''\xi_2 + \frac{i}{Re\cdot\alpha}\left(\xi_2'''' - 2\alpha^2\xi_2'' + \alpha^4\xi_2\right) = 0$$

which is the famed Orr-Sommerfeld equation. For a given $U(x_2)$ and angular frequency α, the above is an eigenproblem for the (complex) wave speed c and associated eigenfunction ξ_2. To better see this, let us write the above as:

$$\mathcal{L}_{left}\,\xi_2 = c\,\mathcal{L}_{right}\,\xi_2$$

where $\mathcal{L}_{left}$ and $\mathcal{L}_{right}$ are the operators:

$$\begin{aligned}\mathcal{L}_{left}\,\xi_2 = & -U(\xi_2'' - \alpha^2\xi_2) \\ & + U''\xi_2 \\ & - \frac{i}{Re\cdot\alpha}\left(\xi_2'''' - 2\alpha^2\xi_2'' + \alpha^4\xi_2\right)\end{aligned}$$

$$\mathcal{L}_{right}\,\xi_2 = -(\xi_2'' - \alpha^2\xi_2)$$

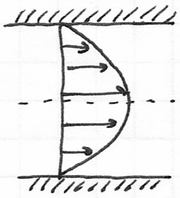
In all but the most trivial configurations, the above eigenproblem cannot be solved analytically. Instead, numerics must be employed. For example, finite differences have been employed to discretize ξ_2, as has the spectral method using Chebyshev polynomials. It should be mentioned the following BC's must be employed:

$$\underbrace{\xi_2(\pm h) = 0}_{\text{No penetration}}, \qquad \underbrace{\frac{d\xi_2}{dx_2}(\pm h) = 0}_{\text{No slip}}$$

Example: Channel Flow

$$U(x_2) = U_0\left(1 - \left(\frac{x_2}{h}\right)^2\right)$$

$$Re = \frac{U_0 h}{\nu}$$



Numerical results indicate that for $Re > \sim 5770$, there exists an eigenvalue (for some α) with positive imaginary part so the flow becomes unstable.

Using a Chebyshev spectral method, Steven Orszag computed the following values for the critical Reynolds number and angular frequency (1972):

$$Re_{cr.} = 5772.22$$
$$\alpha_{cr.} = 1.02056$$

The reference detailing Orszag's methodology is available on D2L.

Here are some critical Reynolds numbers for other flows of interest:

Blasius Boundary Layer Flow: $Re_{\delta^*} \approx 520$ $\quad \delta^*$ = Displacement thickness

Plane Jet: $Re_{cr.} \approx 5$

Mixing Layer: Unstable for every Re

Pipe Flow: Stable for any Re

Remark: If one conducts a linear stability analysis in the limit of $Re \to \infty$, one finds that channel flow and B.L. flow are stable!

Thus, viscosity acts as a destabilizing mechanism for these flows!

The above analysis is useful, but linear stability analysis has many shortcomings:

1. Disturbances are usually three-dimensional.
2. Large fluctuations are nonlinear and become unstable to other disturbances.
   Ex: 2D disturbances may become unstable to 3D disturbances.
   $\Rightarrow$ Secondary instability
3. Turbulence can occur at a lower Re than linear stability analysis predicts.
   Ex: For channel flow, turbulence may occur for $Re > \sim 1300$!
4. Some flows are always linearly stable and yet turbulence still occurs.
   Ex: For pipe flow, turbulence may occur for $Re > \sim 2000$!
5. Flows can undergo a series of instabilities before the flow becomes chaotic and then turbulent.
   Ex: Taylor-Couette Flow
   Flow Over a Cylinder

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6. Transition scenarios can depend on the disturbance environment.

   Ex: In a standard environment, a boundary layer evolves as follows:

   1. Stable boundary layer
   2. Growth of two-dimensional unstable wave (the so-called Tollmien-Schlichting wave)
   3. Secondary instability to three-dimensional disturbances (Crossflow vortices)
   4. Nonlinear breakdown of three-dimensional disturbances
   5. Formation of turbulent spots
   6. Onset of fully turbulent flow

   In a noisy environment, the boundary layer may instead experience "bypass transition" directly to turbulence.

In this class, we will not discuss stability and transition in detail, but both of these topics are of critical importance when studying how to predict the onset of turbulence and how to control turbulence.

Reference on Stability: "Hydrodynamic Stability" by Drazin and Reid