Reynolds Stress Transport Models: Elliptic Relaxation:

Elliptic relaxation is a rather different approach to incorporating wall effects in the redistribution tensor. It uses a higher level of closure to introduce non-locality. The motivation for elliptic relaxation stems from the Poisson problem for the fluctuating pressure field:

$$\nabla^2 p'(\vec{x}) = S(\vec{x})$$

Formally, we may solve the above problem using a Green's function. Assuming for convenience the absence of walls:

$$p'(\vec{x}) = -\frac{1}{4\pi}\iiint_{-\infty}^{\infty} S(\vec{y}) \frac{d\vec{y}}{|\vec{x}-\vec{y}|}$$

Now suppose $\phi(\vec{x})$ is a random field. The correlation between p' and ϕ is then:

$$\langle p'(\vec{x})\phi(\vec{x})\rangle = \iiint_{-\infty}^{\infty} \langle S(\vec{y})\phi(\vec{x})\rangle \left(-\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}\right) d\vec{y}$$

We make the strong assumption that there is a correlation length scale L_D such that:

$$\langle S(\vec{y})\phi(\vec{x})\rangle = \langle S(\vec{y})\phi(\vec{y})\rangle e^{-|\vec{x}-\vec{y}|/L_D}$$

and hence:

$$\langle p'(\vec{x})\phi(\vec{x})\rangle = -\frac{1}{4\pi}\iiint_{-\infty}^{\infty} \langle S(\vec{y})\phi(\vec{y})\rangle \frac{e^{-|\vec{x}-\vec{y}|/L_D}}{|\vec{x}-\vec{y}|} d\vec{y}$$

If $L_D \equiv$ const., then:

$$(\nabla^2 - L_D^{-2})\langle p'(\vec{x})\phi(\vec{x})\rangle = \langle S(\vec{x})\phi(\vec{x})\rangle$$

This suggests that the pressure-rate-of-strain tensor may be better approximated through the introduction of the differential operator $(\nabla^2 - L_D^{-2})$. Ultimately, the motivation for the elliptic relaxation model is to enable boundary conditions and anisotropic wall effects to be introduced into the Reynolds stress transport model in a flexible and geometry-independent manner.

The starting point for Durbin's original elliptic relaxation model (1995) is the exact Reynolds stress transport model written as:

$$\frac{\bar{D}\langle u_i'u_j'\rangle}{\bar{D}t} + \frac{\partial}{\partial x_k}\left(T_{kij}^{(\nu)} + T_{kij}^{(p')} + T_{kij}^{(u)}\right) = P_{ij} + R_{ij}^{(e)} - \frac{\langle u_i'u_j'\rangle}{k}\varepsilon$$

where $T_{kij}^{(p')} \equiv 2/3\, \delta_{ij}\langle u_k p'\rangle/\rho$ and:

$$R_{ij}^{(e)} \equiv \left(\Pi_{ij} - \frac{1}{3}\Pi_{kk}\delta_{ij}\right) - \left(\varepsilon_{ij} - \frac{\langle u_i'u_j'\rangle}{k}\varepsilon\right)$$

$$\Pi_{ij} \equiv -\frac{1}{\rho}\left\langle u_i'\frac{\partial p'}{\partial x_j} + u_j'\frac{\partial p'}{\partial x_i}\right\rangle \quad \text{Velocity-Pressure-Gradient Tensor}$$

Like R_{ij}, $R_{ij}^{(e)}$ is a redistribution tensor, but it is also zero at the walls. This aids in the imposition of boundary conditions. The turbulent-transport-term is modeled as:

$$\frac{\partial}{\partial x_k}\left(T_{kij}^{(\nu)} + T_{kij}^{(p')} + T_{kij}^{(u)}\right) = -\frac{\partial}{\partial x_k}\left(\left(\nu\delta_{kl} + C_s T\langle u_k' u_l'\rangle\right)\frac{\partial\langle u_i' u_j'\rangle}{\partial x_l}\right)$$

where $T = \max(k/\varepsilon, C\tau_\eta)$. The intermediate quantity $\bar{R}_{ij}$ is defined via the basic LRR-IP model for R_{ij}:

$$\bar{R}_{ij} = -\frac{(C_R - 1)}{T}\left(\langle u_i' u_j'\rangle - \tfrac{2}{3}k\delta_{ij}\right) - C_2\left(P_{ij} - \tfrac{2}{3}P\delta_{ij}\right)$$

except that $1/T$ is used in place of ε/k and $C_R - 1$ is used in place of C_R to account for the presence of anisotropic diffusion. Finally, motivated by our prior analysis, we model $R_{ij}^{(e)}$ as:

$$R_{ij}^{(e)} = k f_{ij}$$

where:

$$(I - L_D^2\nabla^2) f_{ij} = \frac{\bar{R}_{ij}}{k}$$

where the lengthscale L_D is taken as:

$$L_D = \max(C_L L, C_L C_\eta \eta), \quad L = \frac{k^{3/2}}{\varepsilon}, \quad \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

In practice, $C_L = 0.2$ and $C_\eta = 80$. In homogeneous turbulence, $\nabla^2 f_{ij}$ is zero and hence the basic model is recovered. Otherwise, f_{ij} is determined nonlocally by $\bar{R}_{ij}$ and the boundary conditions imposed on f_{ij}.

Perhaps the most appealing property of the introduction of the elliptic relaxation equation is that it allows for additional boundary conditions to be applied. In a standard Reynolds stress model, only one boundary condition per Reynolds stress may be applied at the wall, and the asymptotic variation of the Reynolds stress with distance is determined by the model. In an elliptic relaxation model, two boundary conditions can be applied to each $\langle u_i' u_j'\rangle$ and f_{ij} pair, allowing more control of the asymptotic Reynolds stress variation.

Reynolds stress models using elliptic relaxation have been used with success to calculate boundary layers with adverse pressure gradients and convex curvatures, and for channel flow, the Reynolds stress budgets are reproduced quite accurately.