Basics of Probability

Bayes Algorithm

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Probability and Statistics

- Probability $\neq$ Statistics
- Probability: Known distributions $\Rightarrow$ what are the outcomes?
- Statistics: Known outcomes $\Rightarrow$ what are the distributions?

Probability Theory

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- **In statistics, using actual observed data, we make an inference of the model that is used to generate this set of data.**

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- *Example*: Assume there are 1 dog and 2 cats in a room. You pick an animal randomly. What is the probability P_1 that this is a dog? If you pick two animals randomly, what is the probability P_2 that these are a dog and a cat?
- *Answer*: You have the possible outcomes: (D), (C1), (C2) so

$$P_1 = \frac{\#\ \text{"successful" events}}{\#\ \text{events}} = \frac{\#\ \text{dogs}}{\#\ \text{dogs} + \#\ \text{cats}} = \frac{1}{3}.$$

  To compute P_2, you can think of all the possible events: (D,C1), (D,C2), (C1,C2) so

$$P_2 = \frac{\#\ \text{"successful" events}}{\#\ \text{events}} = \frac{2}{3}.$$

Sample Space

Definition

The *sample space* S of an experiment (whose outcome is uncertain) is the set of all possible outcomes of the experiment.

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$$S = \{\text{all } 12! \text{ permutations of } (1, 2, 3, ..., 11, 12)\}.$$

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- *Example* (lifetime): If the experiment consists of measuring the lifetime (in years) of your pet then the sample space consists of all nonnegative real numbers: $S = \{x; 0 \leq x < \infty\}$.

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- If the outcome of the experiment is in E, then we say that E has occurred.

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- *Example* (coins): The event $E = \{(H, T), (T, T)\}$ is the event that a tail appears on the second coin.
- *Example* (lifetime): The event $E = \{x : 3 \leq x \leq 15\}$ is the event that your pet will live more than 3 years but won't live more than 15 years.

Union of Events

Given events E and F, $E \cup F$ is the set of all outcomes *either* in E or F or in *both* E and F.

$E \cup F$ occurs if *either* E or F occurs.

$E \cup F$ is the **union** of events E and F

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- *Example* (horse race): If we have $E = \{\text{all outcomes in } S \text{ starting with a } 7\}$ and $F = \{\text{all outcomes in } S \text{ finishing with a } 3\}$ then $E \cup F$ is the event that the race was won by horse 7 and/or the last horse was horse 3.

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- *Example* (lifetime): If $E = \{x : 0 \leq x \leq 10\}$ and $F = \{x : 15 \leq x < \infty\}$ then $E \cup F$ is the event that your pet will die before 10 or will die after 15.

Intersection of Events

Given events E and F, $E \cap F$ is the set of all outcomes which are *both* in E and F.

$E \cap F$ is also denoted as EF.

Intersection of Events

- *Example* (coins): If we have $E = \{(H, H), (H, T), (T, H\}$ (event that one H at least occurs) and $F = \{(H, T), (T, H), (T, T)\}$ (even that one T at least occurs) then $E \cap F = \{(H, T), (T, H)\}$ is the event that one H and one T occur.

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  $E = \{\text{all outcomes in } S \text{ starting with a } 7\}$ and
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- *Example* (lifetime): If we have $E = \{x : 0 \leq x \leq 15\}$ and $F = \{x : 10 \leq x < 15\}$ then $E \cap F = \{x : 10 \leq x \leq 15\}$ is the event that your pet will die between 10 and 15.

Notations and Properties

- For any event E, E^c denote the *complement* set of all outcomes in S which are not in E.
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  Hence we have $E \cup E^c = S$ and $E \cap E^c = \varnothing$.
- For any two events E and F, we write $E \subset F$ is all the outcomes of E are in F.

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- **Axiom 1**
$$0 \leq P(E) \leq 1$$
- **Axiom 2**
$$P(S) = 1$$
- **Axiom 3**. For any sequence of mutually exclusive events $\{E_i\}_{i \geq 1}$, i.e. $E_i \cap E_j = \varnothing$ when $i \neq j$, then
$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

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- **Proposition**: We have $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Conditional Probabilities

- **Conditional Probability**. Consider an experiment with sample space S. Let E and F be two events, then the conditional probability of E given F is denoted by $P(E|F)$ and satisfies if $P(F) > 0$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

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- **Intuition**: If F has occured, then, in order for E to occur, it is necessary that the occurence be both in E and F, hence it must be in $E \cap F$. Once F has occured, F is the new sample space.

Conditional Probabilities

- *Equally likely outcomes*. In this case, we have

$$
\begin{aligned}
P(E|F) &= \frac{\#\text{ outcomes in } E \cap F}{\#\text{ outcomes in } F} \\
&= \underbrace{\frac{\#\text{ outcomes in } E \cap F}{\#\text{ outcomes in } S}}_{P(E\cap F)} \Big/ \underbrace{\left(\frac{\#\text{ outcomes in } F}{\#\text{ outcomes in } S}\right)}_{P(F)}.
\end{aligned}
$$

Independence

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- Equivalent to $P(A|B) = P(A)$
- One event occurring does not effect the probability of another occurring

The Multiplication Rule

Let $E_1, E_2, \ldots, E_n$ be a sequence of events, then we have

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)\, P(E_2 | E_1) \times$$
$$\times P(E_3 | E_1 \cap E_2) \cdots P(E_n | E_1 \cap \cdots \cap E_{n-1})$$

Example

- **Example**: You have a box with 3 blue marbles, 2 red marbles, and 4 yellow marbles. You are going to pull out one marble, record its color, put it back in the box and draw another marble. What is the probability of pulling out a red followed by a blue?

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- **Example**: Consider the same box of marbles. However, we are going to pull out the first marble, leave it out and then pull out the second marble. What is the probability of pulling out a red marble followed by a blue marble?

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- Range of R is the real line

Random Variables

Example: **Discrete Random Variable**
Experiment: flip 10 coins
Desired outcome: the number of heads
We care about: the number of heads that appear among 10 tosses (not the probability of getting a particular sequence of heads and tails)

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Experiment: flip 10 coins

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Probability of a random variable R taking on some specific value k is: $P(R = k) = P(\{s : R(s) = k\})$, with $R(s)$ - number of heads occuring after s tosses

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$R(s)$ - random variable indicating the amount of time it takes for a fast food burger to decay

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Probability that R takes on a value between two real constants a and b is:
$P(a \leq R \leq b) = P(\{s : a \leq R(s) \leq b\})$

Probability Distribution

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- **Mean** is the arithmetical average value of the data.
- **Median** is the middle value of the data.
- **Mode** is the most frequently occurring value of the data.
- **Expected value** of some a random variable `X` with respect to a distribution `P(X=x)` is the mean value of `X` when `x` is drawn from `P`.
- **Variance** is the measure of variability in the data from the mean value.

Probability Distribution

Binomial: the random variable can have only two outcomes.

```
import numpy as np
n=100 # number of trials
p=0.5 # probability of success
s=1000 # size
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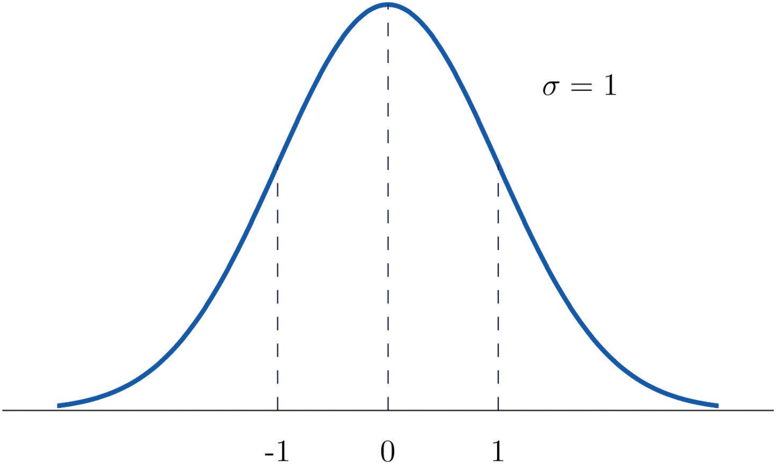
Uniform: equal likelihood.

```
import numpy as np
np.random.uniform(low=1, high=10,size=100)
```

Probability Distribution

Normal (Gaussian): most common.



Bayes's theorem

Bayesian approach provides mathematical rule explaining how you should change your existing beliefs in the light of new evidence.

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- $P(R = r|e) = \frac{P(e|R=r)P(R=r)}{P(e)}$
- $P(R = r|e)$: probability that random variable R has value r given evidence e

Bayes's theorem

- $posterior = \frac{\text{likelihood * prior}}{\text{marginal likelihood}}$
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- $P(R = r|e)$: probability that random variable R has value r given evidence e
- The denominator is just a normalizing constant (called *marginal likelihood*) that ensures the posterior adds up to 1; it can be computed by summing up the numerator over all possible values of R, i.e.,
  $P(e) = P(R = 0, e) + P(R = 1, e) + ... =$
  $\sum_r P(e|R = r)P(R = r)$

Naive Bayes Algorithm

- Simple ("naive") classification method based on Bayes rule.
- It is fast and robust.

Given features $X_1, ..., X_n$, predict label Y

- $arg\ max_Y P(Y|X_1, ..., X_N)$

Classification Overview

Classification Methods:

1. Model a classification rule directly
   - Examples: k-NN, decision trees, perceptron, SVM
2. Model the probability of class memberships given input data
   - Example: multi-layered neural networks
3. Make a probabilistic model of data within each class
   - Examples: naive Bayes, model-based classifiers