

PG No. 22 $C_{3h} \bar{6}$ [hexagonal] (polar, internal polar dipole)

* Harmonics for rank 0

$$\bar{Q}_0^{(1,1)}[q](A')$$

** symmetry

$$1$$

** expression

$$\frac{\sqrt{3}Q_x x}{3} + \frac{\sqrt{3}Q_y y}{3} + \frac{\sqrt{3}Q_z z}{3}$$

* Harmonics for rank 1

$$\bar{Q}_1^{(1,-1)}[q](A'')$$

** symmetry

$$z$$

** expression

$$Q_z$$

$$\bar{Q}_1^{(1,1)}[q](A'')$$

** symmetry

$$z$$

** expression

$$\frac{3\sqrt{10}Q_x xz}{10} + \frac{3\sqrt{10}Q_y yz}{10} - \frac{\sqrt{10}Q_z (x^2 + y^2 - 2z^2)}{10}$$

$$\bar{Q}_{1,1}^{(1,-1)}[q](E'), \bar{Q}_{1,2}^{(1,-1)}[q](E')$$

** symmetry

$$x$$

$$y$$

** expression

$$Q_x$$

$$Q_y$$

$$\bar{Q}_{1,1}^{(1,1)}[q](E'), \bar{Q}_{1,2}^{(1,1)}[q](E')$$

** symmetry

$$x$$

$$y$$

** expression

$$\frac{\sqrt{10}Q_x (2x^2 - y^2 - z^2)}{10} + \frac{3\sqrt{10}Q_y xy}{10} + \frac{3\sqrt{10}Q_z xz}{10}$$

$$\frac{3\sqrt{10}Q_x xy}{10} - \frac{\sqrt{10}Q_y (x^2 - 2y^2 + z^2)}{10} + \frac{3\sqrt{10}Q_z yz}{10}$$

* Harmonics for rank 2

$$\bar{Q}_2^{(1,-1)}[q](A')$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{6}Q_x x}{6} - \frac{\sqrt{6}Q_y y}{6} + \frac{\sqrt{6}Q_z z}{3}$$

$$\bar{Q}_2^{(1,1)}[q](A')$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{21}Q_{xx}(x^2+y^2-4z^2)}{14}-\frac{\sqrt{21}Q_{yy}(x^2+y^2-4z^2)}{14}-\frac{\sqrt{21}Q_{zz}(3x^2+3y^2-2z^2)}{14}$$

$$\tilde{Q}_{2,1}^{(1,-1)}[q](E'), \tilde{Q}_{2,2}^{(1,-1)}[q](E')$$

** symmetry

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

$$-\sqrt{3}xy$$

** expression

$$\frac{\sqrt{2}Q_{xx}}{2}-\frac{\sqrt{2}Q_{yy}}{2}$$

$$-\frac{\sqrt{2}Q_{xy}}{2}-\frac{\sqrt{2}Q_{yx}}{2}$$

$$\tilde{Q}_{2,1}^{(1,1)}[q](E'), \tilde{Q}_{2,2}^{(1,1)}[q](E')$$

** symmetry

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

$$-\sqrt{3}xy$$

** expression

$$\frac{\sqrt{7}Q_{xx}(3x^2-7y^2-2z^2)}{14}+\frac{\sqrt{7}Q_{yy}(7x^2-3y^2+2z^2)}{14}+\frac{5\sqrt{7}Q_{zz}(x-y)(x+y)}{14}$$

$$-\frac{\sqrt{7}Q_{xy}(4x^2-y^2-z^2)}{7}+\frac{\sqrt{7}Q_{yx}(x^2-4y^2+z^2)}{7}-\frac{5\sqrt{7}Q_{xyz}}{7}$$

$$\tilde{Q}_{2,1}^{(1,-1)}[q](E''), \tilde{Q}_{2,2}^{(1,-1)}[q](E'')$$

** symmetry

$$\sqrt{3}yz$$

$$-\sqrt{3}xz$$

** expression

$$\frac{\sqrt{2}Q_{yz}}{2}+\frac{\sqrt{2}Q_{zy}}{2}$$

$$-\frac{\sqrt{2}Q_{xz}}{2}-\frac{\sqrt{2}Q_{zx}}{2}$$

$$\tilde{Q}_{2,1}^{(1,1)}[q](E''), \tilde{Q}_{2,2}^{(1,1)}[q](E'')$$

** symmetry

$$\sqrt{3}yz$$

$$-\sqrt{3}xz$$

** expression

$$\frac{5\sqrt{7}Q_{xyz}}{7}-\frac{\sqrt{7}Q_{yz}(x^2-4y^2+z^2)}{7}-\frac{\sqrt{7}Q_{zy}(x^2+y^2-4z^2)}{7}$$

$$-\frac{\sqrt{7}Q_{xz}(4x^2-y^2-z^2)}{7}-\frac{5\sqrt{7}Q_{yxyz}}{7}+\frac{\sqrt{7}Q_{zx}(x^2+y^2-4z^2)}{7}$$

* Harmonics for rank 3

$$\tilde{Q}_3^{(1,-1)}[q](A', 1)$$

** symmetry

$$\frac{\sqrt{10}y(3x^2-y^2)}{4}$$

** expression

$$\frac{\sqrt{6}Q_xxy}{2} + \frac{\sqrt{6}Q_y(x-y)(x+y)}{4}$$

$$\tilde{\mathbb{Q}}_3^{(1,-1)}[q](A', 2)$$

** symmetry

$$\frac{\sqrt{10}x(x^2 - 3y^2)}{4}$$

** expression

$$\frac{\sqrt{6}Q_x(x-y)(x+y)}{4} - \frac{\sqrt{6}Q_yxy}{2}$$

$$\tilde{\mathbb{Q}}_3^{(1,1)}[q](A', 1)$$

** symmetry

$$\frac{\sqrt{10}y(3x^2 - y^2)}{4}$$

** expression

$$\frac{\sqrt{10}Q_xxy(15x^2 - 13y^2 - 6z^2)}{24} - \frac{\sqrt{10}Q_y(3x^4 - 21x^2y^2 + 3x^2z^2 + 4y^4 - 3y^2z^2)}{24} + \frac{7\sqrt{10}Q_zyz(3x^2 - y^2)}{24}$$

$$\tilde{\mathbb{Q}}_3^{(1,1)}[q](A', 2)$$

** symmetry

$$\frac{\sqrt{10}x(x^2 - 3y^2)}{4}$$

** expression

$$\frac{\sqrt{10}Q_x(4x^4 - 21x^2y^2 - 3x^2z^2 + 3y^4 + 3y^2z^2)}{24} + \frac{\sqrt{10}Q_yxy(13x^2 - 15y^2 + 6z^2)}{24} + \frac{7\sqrt{10}Q_zxz(x^2 - 3y^2)}{24}$$

$$\tilde{\mathbb{Q}}_3^{(1,-1)}[q](A'')$$

** symmetry

$$-\frac{z(3x^2 + 3y^2 - 2z^2)}{2}$$

** expression

$$-\frac{\sqrt{15}Q_xxz}{5} - \frac{\sqrt{15}Q_yyz}{5} - \frac{\sqrt{15}Q_z(x^2 + y^2 - 2z^2)}{10}$$

$$\tilde{\mathbb{Q}}_3^{(1,1)}[q](A'')$$

** symmetry

$$-\frac{z(3x^2 + 3y^2 - 2z^2)}{2}$$

** expression

$$-\frac{5Q_xxz(3x^2 + 3y^2 - 4z^2)}{12} - \frac{5Q_yyz(3x^2 + 3y^2 - 4z^2)}{12} + \frac{Q_z(3x^4 + 6x^2y^2 - 24x^2z^2 + 3y^4 - 24y^2z^2 + 8z^4)}{12}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,-1)}[q](E'), \tilde{\mathbb{Q}}_{3,2}^{(1,-1)}[q](E')$$

** symmetry

$$-\frac{\sqrt{6}x(x^2 + y^2 - 4z^2)}{4}$$

$$-\frac{\sqrt{6}y(x^2 + y^2 - 4z^2)}{4}$$

** expression

$$-\frac{\sqrt{10}Q_x(3x^2 + y^2 - 4z^2)}{20} - \frac{\sqrt{10}Q_yxy}{10} + \frac{2\sqrt{10}Q_zxz}{5}$$

$$-\frac{\sqrt{10}Q_xxy}{10} - \frac{\sqrt{10}Q_y(x^2 + 3y^2 - 4z^2)}{20} + \frac{2\sqrt{10}Q_zyz}{5}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,1)}[q](E'), \tilde{\mathbb{Q}}_{3,2}^{(1,1)}[q](E')$$

** symmetry

$$-\frac{\sqrt{6}x(x^2+y^2-4z^2)}{4}$$

$$-\frac{\sqrt{6}y(x^2+y^2-4z^2)}{4}$$

** expression

$$-\frac{\sqrt{6}Q_x(4x^4+3x^2y^2-27x^2z^2-y^4+3y^2z^2+4z^4)}{24}-\frac{5\sqrt{6}Q_yxy(x^2+y^2-6z^2)}{24}-\frac{5\sqrt{6}Q_zxz(3x^2+3y^2-4z^2)}{24}$$

$$-\frac{5\sqrt{6}Q_xxy(x^2+y^2-6z^2)}{24}+\frac{\sqrt{6}Q_y(x^4-3x^2y^2-3x^2z^2-4y^4+27y^2z^2-4z^4)}{24}-\frac{5\sqrt{6}Q_zyz(3x^2+3y^2-4z^2)}{24}$$

$$\tilde{Q}_{3,1}^{(1,-1)}[q](E''), \tilde{Q}_{3,2}^{(1,-1)}[q](E'')$$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$Q_xyz + Q_yxz + Q_zxy$$

$$Q_xxz - Q_yyz + \frac{Q_z(x-y)(x+y)}{2}$$

$$\tilde{Q}_{3,1}^{(1,1)}[q](E''), \tilde{Q}_{3,2}^{(1,1)}[q](E'')$$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{15}Q_xyz(6x^2-y^2-z^2)}{6}-\frac{\sqrt{15}Q_yxz(x^2-6y^2+z^2)}{6}-\frac{\sqrt{15}Q_zxy(x^2+y^2-6z^2)}{6}$$

$$\frac{\sqrt{15}Q_xxz(5x^2-9y^2-2z^2)}{12}+\frac{\sqrt{15}Q_yyz(9x^2-5y^2+2z^2)}{12}-\frac{\sqrt{15}Q_z(x-y)(x+y)(x^2+y^2-6z^2)}{12}$$

* Harmonics for rank 4

$$\tilde{Q}_4^{(1,-1)}[q](A')$$

** symmetry

$$\frac{3x^4}{8} + \frac{3x^2y^2}{4} - 3x^2z^2 + \frac{3y^4}{8} - 3y^2z^2 + z^4$$

** expression

$$\frac{3\sqrt{7}Q_xx(x^2+y^2-4z^2)}{28} + \frac{3\sqrt{7}Q_yy(x^2+y^2-4z^2)}{28} - \frac{\sqrt{7}Q_zz(3x^2+3y^2-2z^2)}{7}$$

$$\tilde{Q}_4^{(1,1)}[q](A')$$

** symmetry

$$\frac{3x^4}{8} + \frac{3x^2y^2}{4} - 3x^2z^2 + \frac{3y^4}{8} - 3y^2z^2 + z^4$$

** expression

$$\frac{3\sqrt{55}Q_xx(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4)}{88} + \frac{3\sqrt{55}Q_yy(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4)}{88}$$

$$+ \frac{\sqrt{55}Q_zz(15x^4+30x^2y^2-40x^2z^2+15y^4-40y^2z^2+8z^4)}{88}$$

$$\tilde{Q}_4^{(1,-1)}[q](A'', 1)$$

** symmetry

$$\frac{\sqrt{70}xz(x^2-3y^2)}{4}$$

** expression

$$\frac{3\sqrt{10}Q_xz(x-y)(x+y)}{8} - \frac{3\sqrt{10}Q_yxyz}{4} + \frac{\sqrt{10}Q_zx(x^2-3y^2)}{8}$$

$$\tilde{Q}_4^{(1,-1)}[q](A'', 2)$$

** symmetry

$$\frac{\sqrt{70}yz(3x^2-y^2)}{4}$$

** expression

$$\frac{3\sqrt{10}Q_xxyz}{4} + \frac{3\sqrt{10}Q_yz(x-y)(x+y)}{8} + \frac{\sqrt{10}Q_zy(3x^2-y^2)}{8}$$

$$\tilde{Q}_4^{(1,1)}[q](A'', 1)$$

** symmetry

$$\frac{\sqrt{70}xz(x^2-3y^2)}{4}$$

** expression

$$\frac{3\sqrt{154}Q_xz(2x^4-9x^2y^2-x^2z^2+y^4+y^2z^2)}{44} + \frac{3\sqrt{154}Q_yxyz(5x^2-7y^2+2z^2)}{44} - \frac{\sqrt{154}Q_zx(x^2-3y^2)(x^2+y^2-8z^2)}{44}$$

$$\tilde{Q}_4^{(1,1)}[q](A'', 2)$$

** symmetry

$$\frac{\sqrt{70}yz(3x^2-y^2)}{4}$$

** expression

$$\frac{3\sqrt{154}Q_xxyz(7x^2-5y^2-2z^2)}{44} - \frac{3\sqrt{154}Q_yz(x^4-9x^2y^2+x^2z^2+2y^4-y^2z^2)}{44} - \frac{\sqrt{154}Q_zy(3x^2-y^2)(x^2+y^2-8z^2)}{44}$$

$$\tilde{Q}_{4,1}^{(1,-1)}[q](E', 1), \tilde{Q}_{4,2}^{(1,-1)}[q](E', 1)$$

** symmetry

$$\frac{\sqrt{35}(x^2-2xy-y^2)(x^2+2xy-y^2)}{8}$$

$$\frac{\sqrt{35}xy(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{5}Q_xx(x^2-3y^2)}{4} - \frac{\sqrt{5}Q_yy(3x^2-y^2)}{4}$$

$$\frac{\sqrt{5}Q_xy(3x^2-y^2)}{4} + \frac{\sqrt{5}Q_yx(x^2-3y^2)}{4}$$

$$\tilde{Q}_{4,1}^{(1,-1)}[q](E', 2), \tilde{Q}_{4,2}^{(1,-1)}[q](E', 2)$$

** symmetry

$$-\frac{\sqrt{5}(x-y)(x+y)(x^2+y^2-6z^2)}{4}$$

$$\frac{\sqrt{5}xy(x^2+y^2-6z^2)}{2}$$

** expression

$$-\frac{\sqrt{35}Q_xx(x^2-3z^2)}{14} + \frac{\sqrt{35}Q_yy(y^2-3z^2)}{14} + \frac{3\sqrt{35}Q_zz(x-y)(x+y)}{14}$$

$$\frac{\sqrt{35}Q_xy(3x^2+y^2-6z^2)}{28} + \frac{\sqrt{35}Q_yx(x^2+3y^2-6z^2)}{28} - \frac{3\sqrt{35}Q_zxyz}{7}$$

$$\tilde{Q}_{4,1}^{(1,1)}[q](E', 1), \tilde{Q}_{4,2}^{(1,1)}[q](E', 1)$$

** symmetry

$$\frac{\sqrt{35}(x^2-2xy-y^2)(x^2+2xy-y^2)}{8}$$

$$\frac{\sqrt{35}xy(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{77}Q_{xx}(5x^4 - 46x^2y^2 - 4x^2z^2 + 21y^4 + 12y^2z^2)}{88} + \frac{\sqrt{77}Q_{yy}(21x^4 - 46x^2y^2 + 12x^2z^2 + 5y^4 - 4y^2z^2)}{88} + \frac{9\sqrt{77}Q_{zz}(x^2 - 2xy - y^2)(x^2 + 2xy - y^2)}{88}$$

$$\frac{\sqrt{77}Q_{xy}(6x^4 - 11x^2y^2 - 3x^2z^2 + y^4 + y^2z^2)}{22} - \frac{\sqrt{77}Q_{yx}(x^4 - 11x^2y^2 + x^2z^2 + 6y^4 - 3y^2z^2)}{22} + \frac{9\sqrt{77}Q_{zxy}(x-y)(x+y)}{22}$$

$$\tilde{Q}_{4,1}^{(1,1)}[q](E', 2), \tilde{Q}_{4,2}^{(1,1)}[q](E', 2)$$

** symmetry

$$-\frac{\sqrt{5}(x-y)(x+y)(x^2+y^2-6z^2)}{4}$$

$$\frac{\sqrt{5}xy(x^2+y^2-6z^2)}{2}$$

** expression

$$\begin{aligned} & -\frac{\sqrt{11}Q_{xx}(5x^4 - 4x^2y^2 - 46x^2z^2 - 9y^4 + 66y^2z^2 + 12z^4)}{44} \\ & -\frac{\sqrt{11}Q_{yy}(9x^4 + 4x^2y^2 - 66x^2z^2 - 5y^4 + 46y^2z^2 - 12z^4)}{44} - \frac{21\sqrt{11}Q_{zz}(x-y)(x+y)(x^2+y^2-2z^2)}{44} \\ & \frac{\sqrt{11}Q_{xy}(6x^4 + 5x^2y^2 - 51x^2z^2 - y^4 + 5y^2z^2 + 6z^4)}{22} - \frac{\sqrt{11}Q_{yx}(x^4 - 5x^2y^2 - 5x^2z^2 - 6y^4 + 51y^2z^2 - 6z^4)}{22} + \frac{21\sqrt{11}Q_{zxy}(x^2+y^2-2z^2)}{22} \end{aligned}$$

$$\tilde{Q}_{4,1}^{(1,-1)}[q](E''), \tilde{Q}_{4,2}^{(1,-1)}[q](E'')$$

** symmetry

$$-\frac{\sqrt{10}yz(3x^2+3y^2-4z^2)}{4}$$

$$\frac{\sqrt{10}xz(3x^2+3y^2-4z^2)}{4}$$

** expression

$$\begin{aligned} & -\frac{3\sqrt{70}Q_{xyz}}{28} - \frac{\sqrt{70}Q_yz(3x^2+9y^2-4z^2)}{56} - \frac{3\sqrt{70}Q_{zy}(x^2+y^2-4z^2)}{56} \\ & \frac{\sqrt{70}Q_{xz}(9x^2+3y^2-4z^2)}{56} + \frac{3\sqrt{70}Q_{yxy}}{28} + \frac{3\sqrt{70}Q_{zx}(x^2+y^2-4z^2)}{56} \end{aligned}$$

$$\tilde{Q}_{4,1}^{(1,1)}[q](E''), \tilde{Q}_{4,2}^{(1,1)}[q](E'')$$

** symmetry

$$-\frac{\sqrt{10}yz(3x^2+3y^2-4z^2)}{4}$$

$$\frac{\sqrt{10}xz(3x^2+3y^2-4z^2)}{4}$$

** expression

$$\begin{aligned} & -\frac{21\sqrt{22}Q_{xyz}(x^2+y^2-2z^2)}{44} + \frac{\sqrt{22}Q_yz(3x^4-15x^2y^2-x^2z^2-18y^4+41y^2z^2-4z^4)}{44} + \frac{3\sqrt{22}Q_{zy}(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4)}{44} \\ & \frac{\sqrt{22}Q_{xz}(18x^4+15x^2y^2-41x^2z^2-3y^4+y^2z^2+4z^4)}{44} + \frac{21\sqrt{22}Q_{yxy}(x^2+y^2-2z^2)}{44} - \frac{3\sqrt{22}Q_{zx}(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4)}{44} \end{aligned}$$