

PG No. 13 C_{4v} $4mm$ [tetragonal] (polar, internal polar dipole)

* Harmonics for rank 0

$$\bar{Q}_0^{(1,1)}[q](A_1)$$

** symmetry

$$1$$

** expression

$$\frac{\sqrt{3}Q_x x}{3} + \frac{\sqrt{3}Q_y y}{3} + \frac{\sqrt{3}Q_z z}{3}$$

* Harmonics for rank 1

$$\bar{Q}_1^{(1,-1)}[q](A_1)$$

** symmetry

$$z$$

** expression

$$Q_z$$

$$\bar{Q}_1^{(1,1)}[q](A_1)$$

** symmetry

$$z$$

** expression

$$\frac{3\sqrt{10}Q_x xz}{10} + \frac{3\sqrt{10}Q_y yz}{10} - \frac{\sqrt{10}Q_z (x^2 + y^2 - 2z^2)}{10}$$

$$\bar{Q}_{1,1}^{(1,-1)}[q](E), \bar{Q}_{1,2}^{(1,-1)}[q](E)$$

** symmetry

$$x$$

$$y$$

** expression

$$Q_x$$

$$Q_y$$

$$\bar{Q}_{1,1}^{(1,1)}[q](E), \bar{Q}_{1,2}^{(1,1)}[q](E)$$

** symmetry

$$x$$

$$y$$

** expression

$$\frac{\sqrt{10}Q_x (2x^2 - y^2 - z^2)}{10} + \frac{3\sqrt{10}Q_y xy}{10} + \frac{3\sqrt{10}Q_z xz}{10}$$

$$\frac{3\sqrt{10}Q_x xy}{10} - \frac{\sqrt{10}Q_y (x^2 - 2y^2 + z^2)}{10} + \frac{3\sqrt{10}Q_z yz}{10}$$

* Harmonics for rank 2

$$\bar{Q}_2^{(1,-1)}[q](A_1)$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{6}Q_x x}{6} - \frac{\sqrt{6}Q_y y}{6} + \frac{\sqrt{6}Q_z z}{3}$$

$$\bar{Q}_2^{(1,1)}[q](A_1)$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{21}Q_x x (x^2 + y^2 - 4z^2)}{14} - \frac{\sqrt{21}Q_y y (x^2 + y^2 - 4z^2)}{14} - \frac{\sqrt{21}Q_z z (3x^2 + 3y^2 - 2z^2)}{14}$$

$$\tilde{\mathbb{Q}}_2^{(1,-1)}[q](B_1)$$

** symmetry

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{2}Q_x x}{2} - \frac{\sqrt{2}Q_y y}{2}$$

$$\tilde{\mathbb{Q}}_2^{(1,1)}[q](B_1)$$

** symmetry

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{7}Q_x x (3x^2 - 7y^2 - 2z^2)}{14} + \frac{\sqrt{7}Q_y y (7x^2 - 3y^2 + 2z^2)}{14} + \frac{5\sqrt{7}Q_z z (x-y)(x+y)}{14}$$

$$\tilde{\mathbb{Q}}_2^{(1,-1)}[q](B_2)$$

** symmetry

$$\sqrt{3}xy$$

** expression

$$\frac{\sqrt{2}Q_x y}{2} + \frac{\sqrt{2}Q_y x}{2}$$

$$\tilde{\mathbb{Q}}_2^{(1,1)}[q](B_2)$$

** symmetry

$$\sqrt{3}xy$$

** expression

$$\frac{\sqrt{7}Q_x y (4x^2 - y^2 - z^2)}{7} - \frac{\sqrt{7}Q_y x (x^2 - 4y^2 + z^2)}{7} + \frac{5\sqrt{7}Q_z x y z}{7}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,-1)}[q](E), \tilde{\mathbb{Q}}_{2,2}^{(1,-1)}[q](E)$$

** symmetry

$$\sqrt{3}xz$$

$$\sqrt{3}yz$$

** expression

$$\frac{\sqrt{2}Q_x z}{2} + \frac{\sqrt{2}Q_z x}{2}$$

$$\frac{\sqrt{2}Q_y z}{2} + \frac{\sqrt{2}Q_z y}{2}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,1)}[q](E), \tilde{\mathbb{Q}}_{2,2}^{(1,1)}[q](E)$$

** symmetry

$$\sqrt{3}xz$$

$$\sqrt{3}yz$$

** expression

$$\frac{\sqrt{7}Q_x z (4x^2 - y^2 - z^2)}{7} + \frac{5\sqrt{7}Q_y x y z}{7} - \frac{\sqrt{7}Q_z x (x^2 + y^2 - 4z^2)}{7}$$

$$\frac{5\sqrt{7}Q_x x y z}{7} - \frac{\sqrt{7}Q_y z (x^2 - 4y^2 + z^2)}{7} - \frac{\sqrt{7}Q_z y (x^2 + y^2 - 4z^2)}{7}$$

* Harmonics for rank 3

$$\bar{Q}_3^{(1,-1)}[q](A_1)$$

** symmetry

$$-\frac{z(3x^2+3y^2-2z^2)}{2}$$

** expression

$$-\frac{\sqrt{15}Q_x xz}{5} - \frac{\sqrt{15}Q_y yz}{5} - \frac{\sqrt{15}Q_z(x^2+y^2-2z^2)}{10}$$

$$\bar{Q}_3^{(1,1)}[q](A_1)$$

** symmetry

$$-\frac{z(3x^2+3y^2-2z^2)}{2}$$

** expression

$$-\frac{5Q_x xz(3x^2+3y^2-4z^2)}{12} - \frac{5Q_y yz(3x^2+3y^2-4z^2)}{12} + \frac{Q_z(3x^4+6x^2y^2-24x^2z^2+3y^4-24y^2z^2+8z^4)}{12}$$

$$\bar{Q}_3^{(1,-1)}[q](B_1)$$

** symmetry

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$Q_x xz - Q_y yz + \frac{Q_z(x-y)(x+y)}{2}$$

$$\bar{Q}_3^{(1,1)}[q](B_1)$$

** symmetry

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{15}Q_x xz(5x^2-9y^2-2z^2)}{12} + \frac{\sqrt{15}Q_y yz(9x^2-5y^2+2z^2)}{12} - \frac{\sqrt{15}Q_z(x-y)(x+y)(x^2+y^2-6z^2)}{12}$$

$$\bar{Q}_3^{(1,-1)}[q](B_2)$$

** symmetry

$$\sqrt{15}xyz$$

** expression

$$Q_x yz + Q_y xz + Q_z xy$$

$$\bar{Q}_3^{(1,1)}[q](B_2)$$

** symmetry

$$\sqrt{15}xyz$$

** expression

$$\frac{\sqrt{15}Q_x yz(6x^2-y^2-z^2)}{6} - \frac{\sqrt{15}Q_y xz(x^2-6y^2+z^2)}{6} - \frac{\sqrt{15}Q_z xy(x^2+y^2-6z^2)}{6}$$

$$\bar{Q}_{3,1}^{(1,-1)}[q](E,1), \bar{Q}_{3,2}^{(1,-1)}[q](E,1)$$

** symmetry

$$\frac{x(2x^2-3y^2-3z^2)}{2}$$

$$-\frac{y(3x^2-2y^2+3z^2)}{2}$$

** expression

$$\frac{\sqrt{15}Q_x(2x^2-y^2-z^2)}{10} - \frac{\sqrt{15}Q_y xy}{5} - \frac{\sqrt{15}Q_z xz}{5}$$

$$-\frac{\sqrt{15}Q_x xy}{5} - \frac{\sqrt{15}Q_y(x^2-2y^2+z^2)}{10} - \frac{\sqrt{15}Q_z yz}{5}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,-1)}[q](E, 2), \tilde{\mathbb{Q}}_{3,2}^{(1,-1)}[q](E, 2)$$

** symmetry

$$\frac{\sqrt{15}x(y-z)(y+z)}{2}$$

$$\frac{\sqrt{15}y(x-z)(x+z)}{2}$$

** expression

$$\frac{Q_x(y-z)(y+z)}{2} + Q_yxy - Q_zxz$$

$$Q_xxy + \frac{Q_y(x-z)(x+z)}{2} - Q_zyz$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,1)}[q](E, 1), \tilde{\mathbb{Q}}_{3,2}^{(1,1)}[q](E, 1)$$

** symmetry

$$\frac{x(2x^2 - 3y^2 - 3z^2)}{2}$$

$$-\frac{y(3x^2 - 2y^2 + 3z^2)}{2}$$

** expression

$$\begin{aligned} & \frac{Q_x(8x^4 - 24x^2y^2 - 24x^2z^2 + 3y^4 + 6y^2z^2 + 3z^4)}{12} + \frac{5Q_yxy(4x^2 - 3y^2 - 3z^2)}{12} + \frac{5Q_zxz(4x^2 - 3y^2 - 3z^2)}{12} \\ & - \frac{5Q_xxy(3x^2 - 4y^2 + 3z^2)}{12} + \frac{Q_y(3x^4 - 24x^2y^2 + 6x^2z^2 + 8y^4 - 24y^2z^2 + 3z^4)}{12} - \frac{5Q_zyz(3x^2 - 4y^2 + 3z^2)}{12} \end{aligned}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,1)}[q](E, 2), \tilde{\mathbb{Q}}_{3,2}^{(1,1)}[q](E, 2)$$

** symmetry

$$\frac{\sqrt{15}x(y-z)(y+z)}{2}$$

$$\frac{\sqrt{15}y(x-z)(x+z)}{2}$$

** expression

$$\begin{aligned} & \frac{\sqrt{15}Q_x(y-z)(y+z)(6x^2 - y^2 - z^2)}{12} - \frac{\sqrt{15}Q_yxy(2x^2 - 5y^2 + 9z^2)}{12} + \frac{\sqrt{15}Q_zxz(2x^2 + 9y^2 - 5z^2)}{12} \\ & \frac{\sqrt{15}Q_xxy(5x^2 - 2y^2 - 9z^2)}{12} - \frac{\sqrt{15}Q_y(x-z)(x+z)(x^2 - 6y^2 + z^2)}{12} + \frac{\sqrt{15}Q_zyz(9x^2 + 2y^2 - 5z^2)}{12} \end{aligned}$$

* Harmonics for rank 4

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](A_1, 1)$$

** symmetry

$$\frac{\sqrt{21}(x^4 - 3x^2y^2 - 3x^2z^2 + y^4 - 3y^2z^2 + z^4)}{6}$$

** expression

$$\frac{\sqrt{3}Q_xx(2x^2 - 3y^2 - 3z^2)}{6} - \frac{\sqrt{3}Q_yy(3x^2 - 2y^2 + 3z^2)}{6} - \frac{\sqrt{3}Q_zz(3x^2 + 3y^2 - 2z^2)}{6}$$

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](A_1, 2)$$

** symmetry

$$-\frac{\sqrt{15}(x^4 - 12x^2y^2 + 6x^2z^2 + y^4 + 6y^2z^2 - 2z^4)}{12}$$

** expression

$$-\frac{\sqrt{105}Q_xx(x^2 - 6y^2 + 3z^2)}{42} + \frac{\sqrt{105}Q_yy(6x^2 - y^2 - 3z^2)}{42} - \frac{\sqrt{105}Q_zz(3x^2 + 3y^2 - 2z^2)}{42}$$

$$\tilde{\mathbb{Q}}_4^{(1,1)}[q](A_1, 1)$$

** symmetry

$$\frac{\sqrt{21} (x^4 - 3x^2y^2 - 3x^2z^2 + y^4 - 3y^2z^2 + z^4)}{6}$$

** expression

$$\frac{\sqrt{1155}Q_{xx} (x^4 - 5x^2y^2 - 5x^2z^2 + 3y^4 - 3y^2z^2 + 3z^4)}{66} + \frac{\sqrt{1155}Q_{yy} (3x^4 - 5x^2y^2 - 3x^2z^2 + y^4 - 5y^2z^2 + 3z^4)}{66} + \frac{\sqrt{1155}Q_{zz} (3x^4 - 3x^2y^2 - 5x^2z^2 + 3y^4 - 5y^2z^2 + z^4)}{66}$$

$$\tilde{\mathbb{Q}}_4^{(1,1)}[q](A_1, 2)$$

** symmetry

$$-\frac{\sqrt{15} (x^4 - 12x^2y^2 + 6x^2z^2 + y^4 + 6y^2z^2 - 2z^4)}{12}$$

** expression

$$-\frac{\sqrt{33}Q_{xx} (5x^4 - 88x^2y^2 + 38x^2z^2 + 33y^4 + 66y^2z^2 - 30z^4)}{132} - \frac{\sqrt{33}Q_{yy} (33x^4 - 88x^2y^2 + 66x^2z^2 + 5y^4 + 38y^2z^2 - 30z^4)}{132} + \frac{\sqrt{33}Q_{zz} (3x^4 + 132x^2y^2 - 50x^2z^2 + 3y^4 - 50y^2z^2 + 10z^4)}{132}$$

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](A_2)$$

** symmetry

$$\frac{\sqrt{35}xy (x - y) (x + y)}{2}$$

** expression

$$\frac{\sqrt{5}Q_{xy} (3x^2 - y^2)}{4} + \frac{\sqrt{5}Q_{yx} (x^2 - 3y^2)}{4}$$

$$\tilde{\mathbb{Q}}_4^{(1,1)}[q](A_2)$$

** symmetry

$$\frac{\sqrt{35}xy (x - y) (x + y)}{2}$$

** expression

$$\frac{\sqrt{77}Q_{xy} (6x^4 - 11x^2y^2 - 3x^2z^2 + y^4 + y^2z^2)}{22} - \frac{\sqrt{77}Q_{yx} (x^4 - 11x^2y^2 + x^2z^2 + 6y^4 - 3y^2z^2)}{22} + \frac{9\sqrt{77}Q_{zz}xyz (x - y) (x + y)}{22}$$

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](B_1)$$

** symmetry

$$\frac{\sqrt{5} (x - y) (x + y) (x^2 + y^2 - 6z^2)}{4}$$

** expression

$$\frac{\sqrt{35}Q_{xx} (x^2 - 3z^2)}{14} - \frac{\sqrt{35}Q_{yy} (y^2 - 3z^2)}{14} - \frac{3\sqrt{35}Q_{zz} (x - y) (x + y)}{14}$$

$$\tilde{\mathbb{Q}}_4^{(1,1)}[q](B_1)$$

** symmetry

$$\frac{\sqrt{5} (x - y) (x + y) (x^2 + y^2 - 6z^2)}{4}$$

** expression

$$\frac{\sqrt{11}Q_{xx} (5x^4 - 4x^2y^2 - 46x^2z^2 - 9y^4 + 66y^2z^2 + 12z^4)}{44} + \frac{\sqrt{11}Q_{yy} (9x^4 + 4x^2y^2 - 66x^2z^2 - 5y^4 + 46y^2z^2 - 12z^4)}{44} + \frac{21\sqrt{11}Q_{zz} (x - y) (x + y) (x^2 + y^2 - 2z^2)}{44}$$

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](B_2)$$

** symmetry

$$-\frac{\sqrt{5}xy (x^2 + y^2 - 6z^2)}{2}$$

** expression

$$-\frac{\sqrt{35}Q_{xy} (3x^2 + y^2 - 6z^2)}{28} - \frac{\sqrt{35}Q_{yx} (x^2 + 3y^2 - 6z^2)}{28} + \frac{3\sqrt{35}Q_{zz}xyz}{7}$$

$$\tilde{\mathbb{Q}}_4^{(1,1)}[q](B_2)$$

** symmetry

$$-\frac{\sqrt{5}xy(x^2+y^2-6z^2)}{2}$$

** expression

$$-\frac{\sqrt{11}Q_{xx}(6x^4+5x^2y^2-51x^2z^2-y^4+5y^2z^2+6z^4)}{22}+\frac{\sqrt{11}Q_{yy}(x^4-5x^2y^2-5x^2z^2-6y^4+51y^2z^2-6z^4)}{22}-\frac{21\sqrt{11}Q_{zz}xyz(x^2+y^2-2z^2)}{22}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E,1), \tilde{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E,1)$$

** symmetry

$$\frac{\sqrt{35}xz(x-z)(x+z)}{2}$$

$$\frac{\sqrt{35}yz(y-z)(y+z)}{2}$$

** expression

$$\frac{\sqrt{5}Q_xz(3x^2-z^2)}{4}+\frac{\sqrt{5}Q_zx(x^2-3z^2)}{4}$$

$$\frac{\sqrt{5}Q_yz(3y^2-z^2)}{4}+\frac{\sqrt{5}Q_zy(y^2-3z^2)}{4}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E,2), \tilde{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E,2)$$

** symmetry

$$-\frac{\sqrt{5}xz(x^2-6y^2+z^2)}{2}$$

$$\frac{\sqrt{5}yz(6x^2-y^2-z^2)}{2}$$

** expression

$$-\frac{\sqrt{35}Q_xz(3x^2-6y^2+z^2)}{28}+\frac{3\sqrt{35}Q_yxyz}{7}-\frac{\sqrt{35}Q_zx(x^2-6y^2+3z^2)}{28}$$

$$\frac{3\sqrt{35}Q_xxyz}{7}+\frac{\sqrt{35}Q_yz(6x^2-3y^2-z^2)}{28}+\frac{\sqrt{35}Q_zy(6x^2-y^2-3z^2)}{28}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,1)}[q](E,1), \tilde{\mathbb{Q}}_{4,2}^{(1,1)}[q](E,1)$$

** symmetry

$$\frac{\sqrt{35}xz(x-z)(x+z)}{2}$$

$$\frac{\sqrt{35}yz(y-z)(y+z)}{2}$$

** expression

$$\frac{\sqrt{77}Q_xz(6x^4-3x^2y^2-11x^2z^2+y^2z^2+z^4)}{22}+\frac{9\sqrt{77}Q_yxyz(x-z)(x+z)}{22}-\frac{\sqrt{77}Q_zx(x^4+x^2y^2-11x^2z^2-3y^2z^2+6z^4)}{22}$$

$$\frac{9\sqrt{77}Q_xxyz(y-z)(y+z)}{22}-\frac{\sqrt{77}Q_yz(3x^2y^2-x^2z^2-6y^4+11y^2z^2-z^4)}{22}-\frac{\sqrt{77}Q_zy(x^2y^2-3x^2z^2+y^4-11y^2z^2+6z^4)}{22}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,1)}[q](E,2), \tilde{\mathbb{Q}}_{4,2}^{(1,1)}[q](E,2)$$

** symmetry

$$-\frac{\sqrt{5}xz(x^2-6y^2+z^2)}{2}$$

$$\frac{\sqrt{5}yz(6x^2-y^2-z^2)}{2}$$

** expression

$$-\frac{\sqrt{11}Q_xz(6x^4-51x^2y^2+5x^2z^2+6y^4+5y^2z^2-z^4)}{22}-\frac{21\sqrt{11}Q_yxyz(x^2-2y^2+z^2)}{22}+\frac{\sqrt{11}Q_zx(x^4-5x^2y^2-5x^2z^2-6y^4+51y^2z^2-6z^4)}{22}$$

$$\frac{21\sqrt{11}Q_xxyz(2x^2-y^2-z^2)}{22}-\frac{\sqrt{11}Q_yz(6x^4-51x^2y^2+5x^2z^2+6y^4+5y^2z^2-z^4)}{22}-\frac{\sqrt{11}Q_zy(6x^4+5x^2y^2-51x^2z^2-y^4+5y^2z^2+6z^4)}{22}$$