Response Tensors up to 4th rank in C_i

— polar tensors —

$$C^{(0,Q)} = \left(C^{(0,Q)}\right)$$

$$C^{(0,Q)} = Q_0$$

$$S^{(2,Q)} = \begin{pmatrix} S_{xx}^{(2,Q)} & S_{xy}^{(2,Q)} & S_{xz}^{(2,Q)} \\ S_{xy}^{(2,Q)} & S_{yy}^{(2,Q)} & S_{yz}^{(2,Q)} \\ S_{xz}^{(2,Q)} & S_{yz}^{(2,Q)} & S_{zz}^{(2,Q)} \end{pmatrix}$$

$$\begin{aligned} S_{xx}^{(2,Q)} &= Q_0 - Q_u + Q_v \\ S_{xy}^{(2,Q)} &= Q_{xy} \\ S_{xz}^{(2,Q)} &= Q_{zx} \\ S_{yy}^{(2,Q)} &= Q_0 - Q_u - Q_v \\ S_{yz}^{(2,Q)} &= Q_{yz} \\ S_{zz}^{(2,Q)} &= Q_0 + 2Q_u \end{aligned}$$

$$A^{(2,Q)} = \begin{pmatrix} 0 & A_{xy}^{(2,Q)} & A_{xz}^{(2,Q)} \\ -A_{xy}^{(2,Q)} & 0 & A_{yz}^{(2,Q)} \\ -A_{xz}^{(2,Q)} & -A_{yz}^{(2,Q)} & 0 \end{pmatrix}$$

$$\begin{aligned} A_{xy}^{(2,Q)} &= G_z \\ A_{xz}^{(2,Q)} &= -G_y \\ A_{yz}^{(2,Q)} &= G_x \end{aligned}$$

$$S^{(4,Q)} = \begin{pmatrix} S_{11}^{(4,Q)} & S_{12}^{(4,Q)} & S_{13}^{(4,Q)} & S_{14}^{(4,Q)} & S_{15}^{(4,Q)} & S_{16}^{(4,Q)} \\ S_{12}^{(4,Q)} & S_{22}^{(4,Q)} & S_{23}^{(4,Q)} & S_{24}^{(4,Q)} & S_{25}^{(4,Q)} & S_{26}^{(4,Q)} \\ S_{13}^{(4,Q)} & S_{23}^{(4,Q)} & S_{33}^{(4,Q)} & S_{34}^{(4,Q)} & S_{35}^{(4,Q)} & S_{36}^{(4,Q)} \\ S_{14}^{(4,Q)} & S_{24}^{(4,Q)} & S_{34}^{(4,Q)} & S_{44}^{(4,Q)} & S_{45}^{(4,Q)} & S_{46}^{(4,Q)} \\ S_{15}^{(4,Q)} & S_{25}^{(4,Q)} & S_{35}^{(4,Q)} & S_{45}^{(4,Q)} & S_{55}^{(4,Q)} & S_{56}^{(4,Q)} \\ S_{16}^{(4,Q)} & S_{26}^{(4,Q)} & S_{36}^{(4,Q)} & S_{46}^{(4,Q)} & S_{56}^{(4,Q)} & S_{66}^{(4,Q)} \end{pmatrix}$$

$$
\begin{aligned}
S^{(4,Q)}_{11} &= Q_0[1] + 2Q_0[2] - Q_{4u} + Q_{4v} + 2Q_4 - 2Q_u[1] - 4Q_u[2] + 2Q_v[1] + 4Q_v[2] \\
S^{(4,Q)}_{12} &= Q_0[1] + 2Q_{4u} - Q_4 - 2Q_u[1] \\
S^{(4,Q)}_{13} &= Q_0[1] - Q_{4u} - Q_{4v} - Q_4 + Q_u[1] + Q_v[1] \\
S^{(4,Q)}_{14} &= 2Q^{\beta}_{4x} + Q_{yz}[1] \\
S^{(4,Q)}_{15} &= -Q^{\alpha}_{4y} - Q^{\beta}_{4y} + Q_{zx}[1] + 2Q_{zx}[2] \\
S^{(4,Q)}_{16} &= Q^{\alpha}_{4z} - Q^{\beta}_{4z} + Q_{xy}[1] + 2Q_{xy}[2] \\
S^{(4,Q)}_{22} &= Q_0[1] + 2Q_0[2] - Q_{4u} - Q_{4v} + 2Q_4 - 2Q_u[1] - 4Q_u[2] - 2Q_v[1] - 4Q_v[2] \\
S^{(4,Q)}_{23} &= Q_0[1] - Q_{4u} + Q_{4v} - Q_4 + Q_u[1] - Q_v[1] \\
S^{(4,Q)}_{24} &= Q^{\alpha}_{4x} - Q^{\beta}_{4x} + Q_{yz}[1] + 2Q_{yz}[2] \\
S^{(4,Q)}_{25} &= 2Q^{\beta}_{4y} + Q_{zx}[1] \\
S^{(4,Q)}_{26} &= -Q^{\alpha}_{4z} - Q^{\beta}_{4z} + Q_{xy}[1] + 2Q_{xy}[2] \\
S^{(4,Q)}_{33} &= Q_0[1] + 2Q_0[2] + 2Q_{4u} + 2Q_4 + 4Q_u[1] + 8Q_u[2] \\
S^{(4,Q)}_{34} &= -Q^{\alpha}_{4x} - Q^{\beta}_{4x} + Q_{yz}[1] + 2Q_{yz}[2] \\
S^{(4,Q)}_{35} &= Q^{\alpha}_{4y} - Q^{\beta}_{4y} + Q_{zx}[1] + 2Q_{zx}[2] \\
S^{(4,Q)}_{36} &= 2Q^{\beta}_{4z} + Q_{xy}[1] \\
S^{(4,Q)}_{44} &= Q_0[2] - Q_{4u} + Q_{4v} - Q_4 + Q_u[2] - Q_v[2] \\
S^{(4,Q)}_{45} &= 2Q^{\beta}_{4z} + Q_{xy}[2] \\
S^{(4,Q)}_{46} &= 2Q^{\beta}_{4y} + Q_{zx}[2] \\
S^{(4,Q)}_{55} &= Q_0[2] - Q_{4u} - Q_{4v} - Q_4 + Q_u[2] + Q_v[2] \\
S^{(4,Q)}_{56} &= 2Q^{\beta}_{4x} + Q_{yz}[2] \\
S^{(4,Q)}_{66} &= Q_0[2] + 2Q_{4u} - Q_4 - 2Q_u[2]
\end{aligned}
$$

$$
\bar{S}^{(4,Q)} = \begin{pmatrix}
0 & \bar{S}^{(4,Q)}_{12} & \bar{S}^{(4,Q)}_{13} & \bar{S}^{(4,Q)}_{14} & \bar{S}^{(4,Q)}_{15} & \bar{S}^{(4,Q)}_{16} \\
-\bar{S}^{(4,Q)}_{12} & 0 & \bar{S}^{(4,Q)}_{23} & \bar{S}^{(4,Q)}_{24} & \bar{S}^{(4,Q)}_{25} & \bar{S}^{(4,Q)}_{26} \\
-\bar{S}^{(4,Q)}_{13} & -\bar{S}^{(4,Q)}_{23} & 0 & \bar{S}^{(4,Q)}_{34} & \bar{S}^{(4,Q)}_{35} & \bar{S}^{(4,Q)}_{36} \\
-\bar{S}^{(4,Q)}_{14} & -\bar{S}^{(4,Q)}_{24} & -\bar{S}^{(4,Q)}_{34} & 0 & \bar{S}^{(4,Q)}_{45} & \bar{S}^{(4,Q)}_{46} \\
-\bar{S}^{(4,Q)}_{15} & -\bar{S}^{(4,Q)}_{25} & -\bar{S}^{(4,Q)}_{35} & -\bar{S}^{(4,Q)}_{45} & 0 & \bar{S}^{(4,Q)}_{56} \\
-\bar{S}^{(4,Q)}_{16} & -\bar{S}^{(4,Q)}_{26} & -\bar{S}^{(4,Q)}_{36} & -\bar{S}^{(4,Q)}_{46} & -\bar{S}^{(4,Q)}_{56} & 0
\end{pmatrix}
$$

$$
\begin{aligned}
\bar{S}_{12}^{(4,Q)} &= 4G_{xyz}[1] - 2Q_v[3] \\
\bar{S}_{13}^{(4,Q)} &= -4G_{xyz}[1] + 3Q_u[3] - Q_v[3] \\
\bar{S}_{14}^{(4,Q)} &= -4G_x^{\beta}[1] + Q_{yz}[3] \\
\bar{S}_{15}^{(4,Q)} &= -2G_y[1] + 2G_y^{\alpha}[1] + 2G_y^{\beta}[1] + Q_{zx}[3] \\
\bar{S}_{16}^{(4,Q)} &= 2G_z[1] - 2G_z^{\alpha}[1] + 2G_z^{\beta}[1] + Q_{xy}[3] \\
\bar{S}_{23}^{(4,Q)} &= 4G_{xyz}[1] + 3Q_u[3] + Q_v[3] \\
\bar{S}_{24}^{(4,Q)} &= 2G_x[1] - 2G_x^{\alpha}[1] + 2G_x^{\beta}[1] + Q_{yz}[3] \\
\bar{S}_{25}^{(4,Q)} &= -4G_y^{\beta}[1] + Q_{zx}[3] \\
\bar{S}_{26}^{(4,Q)} &= -2G_z[1] + 2G_z^{\alpha}[1] + 2G_z^{\beta}[1] + Q_{xy}[3] \\
\bar{S}_{34}^{(4,Q)} &= -2G_x[1] + 2G_x^{\alpha}[1] + 2G_x^{\beta}[1] + Q_{yz}[3] \\
\bar{S}_{35}^{(4,Q)} &= 2G_y[1] - 2G_y^{\alpha}[1] + 2G_y^{\beta}[1] + Q_{zx}[3] \\
\bar{S}_{36}^{(4,Q)} &= -4G_z^{\beta}[1] + Q_{xy}[3] \\
\bar{S}_{45}^{(4,Q)} &= -G_z[1] - 4G_z^{\alpha}[1] \\
\bar{S}_{46}^{(4,Q)} &= G_y[1] + 4G_y^{\alpha}[1] \\
\bar{S}_{56}^{(4,Q)} &= -G_x[1] - 4G_x^{\alpha}[1]
\end{aligned}
$$

$$
A^{(4,Q)} = \begin{pmatrix} A_{xx}^{(4,Q)} & A_{xy}^{(4,Q)} & A_{xz}^{(4,Q)} \\ A_{xy}^{(4,Q)} & A_{yy}^{(4,Q)} & A_{yz}^{(4,Q)} \\ A_{xz}^{(4,Q)} & A_{yz}^{(4,Q)} & A_{zz}^{(4,Q)} \end{pmatrix}
$$

$$
\begin{aligned}
A_{xx}^{(4,Q)} &= Q_0[3] - 2Q_u[6] + 2Q_v[6] \\
A_{xy}^{(4,Q)} &= 2Q_{xy}[6] \\
A_{xz}^{(4,Q)} &= 2Q_{zx}[6] \\
A_{yy}^{(4,Q)} &= Q_0[3] - 2Q_u[6] - 2Q_v[6] \\
A_{yz}^{(4,Q)} &= 2Q_{yz}[6] \\
A_{zz}^{(4,Q)} &= Q_0[3] + 4Q_u[6]
\end{aligned}
$$

$$
\bar{A}^{(4,Q)} = \begin{pmatrix} 0 & \bar{A}_{xy}^{(4,Q)} & \bar{A}_{xz}^{(4,Q)} \\ -\bar{A}_{xy}^{(4,Q)} & 0 & \bar{A}_{yz}^{(4,Q)} \\ -\bar{A}_{xz}^{(4,Q)} & -\bar{A}_{yz}^{(4,Q)} & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\bar{A}_{xy}^{(4,Q)} &= G_z[6] \\
\bar{A}_{xz}^{(4,Q)} &= -G_y[6] \\
\bar{A}_{yz}^{(4,Q)} &= G_x[6]
\end{aligned}
$$

$$
M^{(4,Q)} = \begin{pmatrix} M_{1x}^{(4,Q)} & M_{1y}^{(4,Q)} & M_{1z}^{(4,Q)} \\ M_{2x}^{(4,Q)} & M_{2y}^{(4,Q)} & M_{2z}^{(4,Q)} \\ M_{3x}^{(4,Q)} & M_{3y}^{(4,Q)} & M_{3z}^{(4,Q)} \\ M_{4x}^{(4,Q)} & M_{4y}^{(4,Q)} & M_{4z}^{(4,Q)} \\ M_{5x}^{(4,Q)} & M_{5y}^{(4,Q)} & M_{5z}^{(4,Q)} \\ M_{6x}^{(4,Q)} & M_{6y}^{(4,Q)} & M_{6z}^{(4,Q)} \end{pmatrix}
$$

$$
\begin{aligned}
M_{1x}^{(4,Q)} &= 2G_x[2] + G_x[3] + 2G_x^{\alpha}[2] \\
M_{1y}^{(4,Q)} &= G_y[3] - G_y^{\alpha}[2] - G_y^{\beta}[2] + 2Q_{zx}[4] \\
M_{1z}^{(4,Q)} &= G_z[3] - G_z^{\alpha}[2] + G_z^{\beta}[2] - 2Q_{xy}[4] \\
M_{2x}^{(4,Q)} &= G_x[3] - G_x^{\alpha}[2] + G_x^{\beta}[2] - 2Q_{yz}[4] \\
M_{2y}^{(4,Q)} &= 2G_y[2] + G_y[3] + 2G_y^{\alpha}[2] \\
M_{2z}^{(4,Q)} &= G_z[3] - G_z^{\alpha}[2] - G_z^{\beta}[2] + 2Q_{xy}[4] \\
M_{3x}^{(4,Q)} &= G_x[3] - G_x^{\alpha}[2] - G_x^{\beta}[2] + 2Q_{yz}[4] \\
M_{3y}^{(4,Q)} &= G_y[3] - G_y^{\alpha}[2] + G_y^{\beta}[2] - 2Q_{zx}[4] \\
M_{3z}^{(4,Q)} &= 2G_z[2] + G_z[3] + 2G_z^{\alpha}[2] \\
M_{4x}^{(4,Q)} &= G_{xyz}[2] - 3Q_u[4] - Q_v[4] \\
M_{4y}^{(4,Q)} &= G_z[2] - G_z^{\alpha}[2] - G_z^{\beta}[2] - Q_{xy}[4] \\
M_{4z}^{(4,Q)} &= G_y[2] - G_y^{\alpha}[2] + G_y^{\beta}[2] + Q_{zx}[4] \\
M_{5x}^{(4,Q)} &= G_z[2] - G_z^{\alpha}[2] + G_z^{\beta}[2] + Q_{xy}[4] \\
M_{5y}^{(4,Q)} &= G_{xyz}[2] + 3Q_u[4] - Q_v[4] \\
M_{5z}^{(4,Q)} &= G_x[2] - G_x^{\alpha}[2] - G_x^{\beta}[2] - Q_{yz}[4] \\
M_{6x}^{(4,Q)} &= G_y[2] - G_y^{\alpha}[2] - G_y^{\beta}[2] - Q_{zx}[4] \\
M_{6y}^{(4,Q)} &= G_x[2] - G_x^{\alpha}[2] + G_x^{\beta}[2] + Q_{yz}[4] \\
M_{6z}^{(4,Q)} &= G_{xyz}[2] + 2Q_v[4]
\end{aligned}
$$

$$
\bar{M}^{(4,Q)} = \begin{pmatrix}
\bar{M}_{x1}^{(4,Q)} & \bar{M}_{x2}^{(4,Q)} & \bar{M}_{x3}^{(4,Q)} & \bar{M}_{x4}^{(4,Q)} & \bar{M}_{x5}^{(4,Q)} & \bar{M}_{x6}^{(4,Q)} \\
\bar{M}_{y1}^{(4,Q)} & \bar{M}_{y2}^{(4,Q)} & \bar{M}_{y3}^{(4,Q)} & \bar{M}_{y4}^{(4,Q)} & \bar{M}_{y5}^{(4,Q)} & \bar{M}_{y6}^{(4,Q)} \\
\bar{M}_{z1}^{(4,Q)} & \bar{M}_{z2}^{(4,Q)} & \bar{M}_{z3}^{(4,Q)} & \bar{M}_{z4}^{(4,Q)} & \bar{M}_{z5}^{(4,Q)} & \bar{M}_{z6}^{(4,Q)}
\end{pmatrix}
$$

$$\bar{M}_{x1}^{(4,Q)} = 2G_x[4] + G_x[5] + 2G_x^{\alpha}[3]$$
$$\bar{M}_{x2}^{(4,Q)} = G_x[5] - G_x^{\alpha}[3] + G_x^{\beta}[3] - 2Q_{yz}[5]$$
$$\bar{M}_{x3}^{(4,Q)} = G_x[5] - G_x^{\alpha}[3] - G_x^{\beta}[3] + 2Q_{yz}[5]$$
$$\bar{M}_{x4}^{(4,Q)} = G_{xyz}[3] - 3Q_u[5] - Q_v[5]$$
$$\bar{M}_{x5}^{(4,Q)} = G_z[4] - G_z^{\alpha}[3] + G_z^{\beta}[3] + Q_{xy}[5]$$
$$\bar{M}_{x6}^{(4,Q)} = G_y[4] - G_y^{\alpha}[3] - G_y^{\beta}[3] - Q_{zx}[5]$$
$$\bar{M}_{y1}^{(4,Q)} = G_y[5] - G_y^{\alpha}[3] - G_y^{\beta}[3] + 2Q_{zx}[5]$$
$$\bar{M}_{y2}^{(4,Q)} = 2G_y[4] + G_y[5] + 2G_y^{\alpha}[3]$$
$$\bar{M}_{y3}^{(4,Q)} = G_y[5] - G_y^{\alpha}[3] + G_y^{\beta}[3] - 2Q_{zx}[5]$$
$$\bar{M}_{y4}^{(4,Q)} = G_z[4] - G_z^{\alpha}[3] - G_z^{\beta}[3] - Q_{xy}[5]$$
$$\bar{M}_{y5}^{(4,Q)} = G_{xyz}[3] + 3Q_u[5] - Q_v[5]$$
$$\bar{M}_{y6}^{(4,Q)} = G_x[4] - G_x^{\alpha}[3] + G_x^{\beta}[3] + Q_{yz}[5]$$
$$\bar{M}_{z1}^{(4,Q)} = G_z[5] - G_z^{\alpha}[3] + G_z^{\beta}[3] - 2Q_{xy}[5]$$
$$\bar{M}_{z2}^{(4,Q)} = G_z[5] - G_z^{\alpha}[3] - G_z^{\beta}[3] + 2Q_{xy}[5]$$
$$\bar{M}_{z3}^{(4,Q)} = 2G_z[4] + G_z[5] + 2G_z^{\alpha}[3]$$
$$\bar{M}_{z4}^{(4,Q)} = G_y[4] - G_y^{\alpha}[3] + G_y^{\beta}[3] + Q_{zx}[5]$$
$$\bar{M}_{z5}^{(4,Q)} = G_x[4] - G_x^{\alpha}[3] - G_x^{\beta}[3] - Q_{yz}[5]$$
$$\bar{M}_{z6}^{(4,Q)} = G_{xyz}[3] + 2Q_v[5]$$

— axial tensors —

$$C^{(1,G)} = \begin{pmatrix} C_x^{(1,G)} & C_y^{(1,G)} & C_z^{(1,G)} \end{pmatrix}$$

$$\begin{aligned} C_x^{(1,G)} &= G_x \\ C_y^{(1,G)} &= G_y \\ C_z^{(1,G)} &= G_z \end{aligned}$$

$$S^{(3,G)} = \begin{pmatrix} S_{1x}^{(3,G)} & S_{1y}^{(3,G)} & S_{1z}^{(3,G)} \\ S_{2x}^{(3,G)} & S_{2y}^{(3,G)} & S_{2z}^{(3,G)} \\ S_{3x}^{(3,G)} & S_{3y}^{(3,G)} & S_{3z}^{(3,G)} \\ S_{4x}^{(3,G)} & S_{4y}^{(3,G)} & S_{4z}^{(3,G)} \\ S_{5x}^{(3,G)} & S_{5y}^{(3,G)} & S_{5z}^{(3,G)} \\ S_{6x}^{(3,G)} & S_{6y}^{(3,G)} & S_{6z}^{(3,G)} \end{pmatrix}$$

$$\begin{aligned} S_{1x}^{(3,G)} &= 2G_x[1] + G_x[2] + 2G_x^{\alpha} \\ S_{1y}^{(3,G)} &= G_y[2] - G_y^{\alpha} - G_y^{\beta} + 2Q_{zx}[1] \\ S_{1z}^{(3,G)} &= G_z[2] - G_z^{\alpha} + G_z^{\beta} - 2Q_{xy}[1] \\ S_{2x}^{(3,G)} &= G_x[2] - G_x^{\alpha} + G_x^{\beta} - 2Q_{yz}[1] \\ S_{2y}^{(3,G)} &= 2G_y[1] + G_y[2] + 2G_y^{\alpha} \\ S_{2z}^{(3,G)} &= G_z[2] - G_z^{\alpha} - G_z^{\beta} + 2Q_{xy}[1] \\ S_{3x}^{(3,G)} &= G_x[2] - G_x^{\alpha} - G_x^{\beta} + 2Q_{yz}[1] \\ S_{3y}^{(3,G)} &= G_y[2] - G_y^{\alpha} + G_y^{\beta} - 2Q_{zx}[1] \\ S_{3z}^{(3,G)} &= 2G_z[1] + G_z[2] + 2G_z^{\alpha} \\ S_{4x}^{(3,G)} &= G_{xyz} - 3Q_u[1] - Q_v[1] \\ S_{4y}^{(3,G)} &= G_z[1] - G_z^{\alpha} - G_z^{\beta} - Q_{xy}[1] \\ S_{4z}^{(3,G)} &= G_y[1] - G_y^{\alpha} + G_y^{\beta} + Q_{zx}[1] \\ S_{5x}^{(3,G)} &= G_z[1] - G_z^{\alpha} + G_z^{\beta} + Q_{xy}[1] \\ S_{5y}^{(3,G)} &= G_{xyz} + 3Q_u[1] - Q_v[1] \\ S_{5z}^{(3,G)} &= G_x[1] - G_x^{\alpha} - G_x^{\beta} - Q_{yz}[1] \\ S_{6x}^{(3,G)} &= G_y[1] - G_y^{\alpha} - G_y^{\beta} - Q_{zx}[1] \\ S_{6y}^{(3,G)} &= G_x[1] - G_x^{\alpha} + G_x^{\beta} + Q_{yz}[1] \\ S_{6z}^{(3,G)} &= G_{xyz} + 2Q_v[1] \end{aligned}$$

$$A^{(3,G)} = \begin{pmatrix} A_{4x}^{(3,G)} & A_{4y}^{(3,G)} & A_{4z}^{(3,G)} \\ A_{5x}^{(3,G)} & A_{5y}^{(3,G)} & A_{5z}^{(3,G)} \\ A_{6x}^{(3,G)} & A_{6y}^{(3,G)} & A_{6z}^{(3,G)} \end{pmatrix}$$

$$A_{4x}^{(3,G)} = Q_0 - Q_u[2] + Q_v[2]$$
$$A_{4y}^{(3,G)} = G_z[3] + Q_{xy}[2]$$
$$A_{4z}^{(3,G)} = -G_y[3] + Q_{zx}[2]$$
$$A_{5x}^{(3,G)} = -G_z[3] + Q_{xy}[2]$$
$$A_{5y}^{(3,G)} = Q_0 - Q_u[2] - Q_v[2]$$
$$A_{5z}^{(3,G)} = G_x[3] + Q_{yz}[2]$$
$$A_{6x}^{(3,G)} = G_y[3] + Q_{zx}[2]$$
$$A_{6y}^{(3,G)} = -G_x[3] + Q_{yz}[2]$$
$$A_{6z}^{(3,G)} = Q_0 + 2Q_u[2]$$