

PG No. 31 T_d $\bar{4}3m$ [cubic] (polar, internal polar dipole)

* Harmonics for rank 0

$$\vec{Q}_0^{(1,1)}[q](A_1)$$

** symmetry

$$1$$

** expression

$$\frac{\sqrt{3}Q_x x}{3} + \frac{\sqrt{3}Q_y y}{3} + \frac{\sqrt{3}Q_z z}{3}$$

* Harmonics for rank 1

$$\vec{Q}_{1,1}^{(1,-1)}[q](T_2), \vec{Q}_{1,2}^{(1,-1)}[q](T_2), \vec{Q}_{1,3}^{(1,-1)}[q](T_2)$$

** symmetry

$$x$$

$$y$$

$$z$$

** expression

$$Q_x$$

$$Q_y$$

$$Q_z$$

$$\vec{Q}_{1,1}^{(1,1)}[q](T_2), \vec{Q}_{1,2}^{(1,1)}[q](T_2), \vec{Q}_{1,3}^{(1,1)}[q](T_2)$$

** symmetry

$$x$$

$$y$$

$$z$$

** expression

$$\frac{\sqrt{10}Q_x(2x^2 - y^2 - z^2)}{10} + \frac{3\sqrt{10}Q_y xy}{10} + \frac{3\sqrt{10}Q_z xz}{10}$$

$$\frac{3\sqrt{10}Q_x xy}{10} - \frac{\sqrt{10}Q_y(x^2 - 2y^2 + z^2)}{10} + \frac{3\sqrt{10}Q_z yz}{10}$$

$$\frac{3\sqrt{10}Q_x xz}{10} + \frac{3\sqrt{10}Q_y yz}{10} - \frac{\sqrt{10}Q_z(x^2 + y^2 - 2z^2)}{10}$$

* Harmonics for rank 2

$$\vec{Q}_{2,1}^{(1,-1)}[q](E), \vec{Q}_{2,2}^{(1,-1)}[q](E)$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

** expression

$$-\frac{\sqrt{6}Q_x x}{6} - \frac{\sqrt{6}Q_y y}{6} + \frac{\sqrt{6}Q_z z}{3}$$

$$\frac{\sqrt{2}Q_x x}{2} - \frac{\sqrt{2}Q_y y}{2}$$

$$\vec{Q}_{2,1}^{(1,1)}[q](E), \vec{Q}_{2,2}^{(1,1)}[q](E)$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

** expression

$$-\frac{\sqrt{21}Q_x x (x^2 + y^2 - 4z^2)}{14} - \frac{\sqrt{21}Q_y y (x^2 + y^2 - 4z^2)}{14} - \frac{\sqrt{21}Q_z z (3x^2 + 3y^2 - 2z^2)}{14}$$

$$+\frac{\sqrt{7}Q_x x (3x^2 - 7y^2 - 2z^2)}{14} + \frac{\sqrt{7}Q_y y (7x^2 - 3y^2 + 2z^2)}{14} + \frac{5\sqrt{7}Q_z z (x - y) (x + y)}{14}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,-1)}[q](T_2), \tilde{\mathbb{Q}}_{2,2}^{(1,-1)}[q](T_2), \tilde{\mathbb{Q}}_{2,3}^{(1,-1)}[q](T_2)$$

** symmetry

$$\sqrt{3}yz$$

$$\sqrt{3}xz$$

$$\sqrt{3}xy$$

** expression

$$\frac{\sqrt{2}Q_y z}{2} + \frac{\sqrt{2}Q_z y}{2}$$

$$\frac{\sqrt{2}Q_x z}{2} + \frac{\sqrt{2}Q_z x}{2}$$

$$\frac{\sqrt{2}Q_x y}{2} + \frac{\sqrt{2}Q_y x}{2}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,1)}[q](T_2), \tilde{\mathbb{Q}}_{2,2}^{(1,1)}[q](T_2), \tilde{\mathbb{Q}}_{2,3}^{(1,1)}[q](T_2)$$

** symmetry

$$\sqrt{3}yz$$

$$\sqrt{3}xz$$

$$\sqrt{3}xy$$

** expression

$$\frac{5\sqrt{7}Q_x x y z}{7} - \frac{\sqrt{7}Q_y z (x^2 - 4y^2 + z^2)}{7} - \frac{\sqrt{7}Q_z y (x^2 + y^2 - 4z^2)}{7}$$

$$\frac{\sqrt{7}Q_x z (4x^2 - y^2 - z^2)}{7} + \frac{5\sqrt{7}Q_y x y z}{7} - \frac{\sqrt{7}Q_z x (x^2 + y^2 - 4z^2)}{7}$$

$$\frac{\sqrt{7}Q_x y (4x^2 - y^2 - z^2)}{7} - \frac{\sqrt{7}Q_y x (x^2 - 4y^2 + z^2)}{7} + \frac{5\sqrt{7}Q_z x y z}{7}$$

* Harmonics for rank 3

$$\tilde{\mathbb{Q}}_3^{(1,-1)}[q](A_1)$$

** symmetry

$$\sqrt{15}xyz$$

** expression

$$Q_x y z + Q_y x z + Q_z x y$$

$$\tilde{\mathbb{Q}}_3^{(1,1)}[q](A_1)$$

** symmetry

$$\sqrt{15}xyz$$

** expression

$$\frac{\sqrt{15}Q_x y z (6x^2 - y^2 - z^2)}{6} - \frac{\sqrt{15}Q_y x z (x^2 - 6y^2 + z^2)}{6} - \frac{\sqrt{15}Q_z x y (x^2 + y^2 - 6z^2)}{6}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,-1)}[q](T_1), \tilde{\mathbb{Q}}_{3,2}^{(1,-1)}[q](T_1), \tilde{\mathbb{Q}}_{3,3}^{(1,-1)}[q](T_1)$$

** symmetry

$$\frac{\sqrt{15}x (y - z) (y + z)}{2}$$

$$-\frac{\sqrt{15}y(x-z)(x+z)}{2}$$

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$\frac{Q_x(y-z)(y+z)}{2} + Q_yxy - Q_zxz$$

$$-Q_xxy - \frac{Q_y(x-z)(x+z)}{2} + Q_zyz$$

$$Q_xxz - Q_yyz + \frac{Q_z(x-y)(x+y)}{2}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,1)}[q](T_1), \tilde{\mathbb{Q}}_{3,2}^{(1,1)}[q](T_1), \tilde{\mathbb{Q}}_{3,3}^{(1,1)}[q](T_1)$$

** symmetry

$$\frac{\sqrt{15}x(y-z)(y+z)}{2}$$

$$-\frac{\sqrt{15}y(x-z)(x+z)}{2}$$

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{15}Q_x(y-z)(y+z)(6x^2-y^2-z^2)}{12} - \frac{\sqrt{15}Q_yxy(2x^2-5y^2+9z^2)}{12} + \frac{\sqrt{15}Q_zxz(2x^2+9y^2-5z^2)}{12}$$

$$-\frac{\sqrt{15}Q_xxy(5x^2-2y^2-9z^2)}{12} + \frac{\sqrt{15}Q_y(x-z)(x+z)(x^2-6y^2+z^2)}{12} - \frac{\sqrt{15}Q_zyz(9x^2+2y^2-5z^2)}{12}$$

$$\frac{\sqrt{15}Q_xxz(5x^2-9y^2-2z^2)}{12} + \frac{\sqrt{15}Q_yyz(9x^2-5y^2+2z^2)}{12} - \frac{\sqrt{15}Q_z(x-y)(x+y)(x^2+y^2-6z^2)}{12}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,-1)}[q](T_2), \tilde{\mathbb{Q}}_{3,2}^{(1,-1)}[q](T_2), \tilde{\mathbb{Q}}_{3,3}^{(1,-1)}[q](T_2)$$

** symmetry

$$\frac{x(2x^2-3y^2-3z^2)}{2}$$

$$-\frac{y(3x^2-2y^2+3z^2)}{2}$$

$$-\frac{z(3x^2+3y^2-2z^2)}{2}$$

** expression

$$\frac{\sqrt{15}Q_x(2x^2-y^2-z^2)}{10} - \frac{\sqrt{15}Q_yxy}{5} - \frac{\sqrt{15}Q_zxz}{5}$$

$$-\frac{\sqrt{15}Q_xxy}{5} - \frac{\sqrt{15}Q_y(x^2-2y^2+z^2)}{10} - \frac{\sqrt{15}Q_zyz}{5}$$

$$-\frac{\sqrt{15}Q_xxz}{5} - \frac{\sqrt{15}Q_yyz}{5} - \frac{\sqrt{15}Q_z(x^2+y^2-2z^2)}{10}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,1)}[q](T_2), \tilde{\mathbb{Q}}_{3,2}^{(1,1)}[q](T_2), \tilde{\mathbb{Q}}_{3,3}^{(1,1)}[q](T_2)$$

** symmetry

$$\frac{x(2x^2-3y^2-3z^2)}{2}$$

$$-\frac{y(3x^2-2y^2+3z^2)}{2}$$

$$-\frac{z(3x^2+3y^2-2z^2)}{2}$$

** expression

$$\begin{aligned} & \frac{Q_x (8x^4 - 24x^2y^2 - 24x^2z^2 + 3y^4 + 6y^2z^2 + 3z^4)}{12} + \frac{5Q_yxy (4x^2 - 3y^2 - 3z^2)}{12} + \frac{5Q_zxz (4x^2 - 3y^2 - 3z^2)}{12} \\ & - \frac{5Q_xxy (3x^2 - 4y^2 + 3z^2)}{12} + \frac{Q_y (3x^4 - 24x^2y^2 + 6x^2z^2 + 8y^4 - 24y^2z^2 + 3z^4)}{12} - \frac{5Q_zyz (3x^2 - 4y^2 + 3z^2)}{12} \\ & - \frac{5Q_xxz (3x^2 + 3y^2 - 4z^2)}{12} - \frac{5Q_yyz (3x^2 + 3y^2 - 4z^2)}{12} + \frac{Q_z (3x^4 + 6x^2y^2 - 24x^2z^2 + 3y^4 - 24y^2z^2 + 8z^4)}{12} \end{aligned}$$

* Harmonics for rank 4

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](A_1)$$

** symmetry

$$\frac{\sqrt{21} (x^4 - 3x^2y^2 - 3x^2z^2 + y^4 - 3y^2z^2 + z^4)}{6}$$

** expression

$$\frac{\sqrt{3}Q_xx (2x^2 - 3y^2 - 3z^2)}{6} - \frac{\sqrt{3}Q_yy (3x^2 - 2y^2 + 3z^2)}{6} - \frac{\sqrt{3}Q_zz (3x^2 + 3y^2 - 2z^2)}{6}$$

$$\tilde{\mathbb{Q}}_4^{(1,1)}[q](A_1)$$

** symmetry

$$\frac{\sqrt{21} (x^4 - 3x^2y^2 - 3x^2z^2 + y^4 - 3y^2z^2 + z^4)}{6}$$

** expression

$$\begin{aligned} & \frac{\sqrt{1155}Q_xx (x^4 - 5x^2y^2 - 5x^2z^2 + 3y^4 - 3y^2z^2 + 3z^4)}{66} + \frac{\sqrt{1155}Q_yy (3x^4 - 5x^2y^2 - 3x^2z^2 + y^4 - 5y^2z^2 + 3z^4)}{66} \\ & + \frac{\sqrt{1155}Q_zz (3x^4 - 3x^2y^2 - 5x^2z^2 + 3y^4 - 5y^2z^2 + z^4)}{66} \end{aligned}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E), \tilde{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E)$$

** symmetry

$$- \frac{\sqrt{15} (x^4 - 12x^2y^2 + 6x^2z^2 + y^4 + 6y^2z^2 - 2z^4)}{12}$$

$$\frac{\sqrt{5} (x - y) (x + y) (x^2 + y^2 - 6z^2)}{4}$$

** expression

$$\begin{aligned} & - \frac{\sqrt{105}Q_xx (x^2 - 6y^2 + 3z^2)}{42} + \frac{\sqrt{105}Q_yy (6x^2 - y^2 - 3z^2)}{42} - \frac{\sqrt{105}Q_zz (3x^2 + 3y^2 - 2z^2)}{42} \\ & \frac{\sqrt{35}Q_xx (x^2 - 3z^2)}{14} - \frac{\sqrt{35}Q_yy (y^2 - 3z^2)}{14} - \frac{3\sqrt{35}Q_zz (x - y) (x + y)}{14} \end{aligned}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,1)}[q](E), \tilde{\mathbb{Q}}_{4,2}^{(1,1)}[q](E)$$

** symmetry

$$- \frac{\sqrt{15} (x^4 - 12x^2y^2 + 6x^2z^2 + y^4 + 6y^2z^2 - 2z^4)}{12}$$

$$\frac{\sqrt{5} (x - y) (x + y) (x^2 + y^2 - 6z^2)}{4}$$

** expression

$$\begin{aligned} & - \frac{\sqrt{33}Q_xx (5x^4 - 88x^2y^2 + 38x^2z^2 + 33y^4 + 66y^2z^2 - 30z^4)}{132} - \frac{\sqrt{33}Q_yy (33x^4 - 88x^2y^2 + 66x^2z^2 + 5y^4 + 38y^2z^2 - 30z^4)}{132} \\ & + \frac{\sqrt{33}Q_zz (3x^4 + 132x^2y^2 - 50x^2z^2 + 3y^4 - 50y^2z^2 + 10z^4)}{132} \\ & \frac{\sqrt{11}Q_xx (5x^4 - 4x^2y^2 - 46x^2z^2 - 9y^4 + 66y^2z^2 + 12z^4)}{44} \\ & + \frac{\sqrt{11}Q_yy (9x^4 + 4x^2y^2 - 66x^2z^2 - 5y^4 + 46y^2z^2 - 12z^4)}{44} + \frac{21\sqrt{11}Q_zz (x - y) (x + y) (x^2 + y^2 - 2z^2)}{44} \end{aligned}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,-1)}[q](T_1), \tilde{\mathbb{Q}}_{4,2}^{(1,-1)}[q](T_1), \tilde{\mathbb{Q}}_{4,3}^{(1,-1)}[q](T_1)$$

** symmetry

$$\frac{\sqrt{35}yz(y-z)(y+z)}{2}$$

$$-\frac{\sqrt{35}xz(x-z)(x+z)}{2}$$

$$\frac{\sqrt{35}xy(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{5}Q_yz(3y^2-z^2)}{4} + \frac{\sqrt{5}Q_z y(y^2-3z^2)}{4}$$

$$-\frac{\sqrt{5}Q_xz(3x^2-z^2)}{4} - \frac{\sqrt{5}Q_zx(x^2-3z^2)}{4}$$

$$\frac{\sqrt{5}Q_xy(3x^2-y^2)}{4} + \frac{\sqrt{5}Q_yx(x^2-3y^2)}{4}$$

$$\vec{Q}_{4,1}^{(1,1)}[q](T_1), \vec{Q}_{4,2}^{(1,1)}[q](T_1), \vec{Q}_{4,3}^{(1,1)}[q](T_1)$$

** symmetry

$$\frac{\sqrt{35}yz(y-z)(y+z)}{2}$$

$$-\frac{\sqrt{35}xz(x-z)(x+z)}{2}$$

$$\frac{\sqrt{35}xy(x-y)(x+y)}{2}$$

** expression

$$\frac{9\sqrt{77}Q_xxyz(y-z)(y+z)}{22} - \frac{\sqrt{77}Q_yz(3x^2y^2-x^2z^2-6y^4+11y^2z^2-z^4)}{22} - \frac{\sqrt{77}Q_z y(x^2y^2-3x^2z^2+y^4-11y^2z^2+6z^4)}{22}$$

$$-\frac{\sqrt{77}Q_xz(6x^4-3x^2y^2-11x^2z^2+y^2z^2+z^4)}{22} - \frac{9\sqrt{77}Q_yxyz(x-z)(x+z)}{22} + \frac{\sqrt{77}Q_zx(x^4+x^2y^2-11x^2z^2-3y^2z^2+6z^4)}{22}$$

$$\frac{\sqrt{77}Q_xy(6x^4-11x^2y^2-3x^2z^2+y^4+y^2z^2)}{22} - \frac{\sqrt{77}Q_yx(x^4-11x^2y^2+x^2z^2+6y^4-3y^2z^2)}{22} + \frac{9\sqrt{77}Q_zxyz(x-y)(x+y)}{22}$$

$$\vec{Q}_{4,1}^{(1,-1)}[q](T_2), \vec{Q}_{4,2}^{(1,-1)}[q](T_2), \vec{Q}_{4,3}^{(1,-1)}[q](T_2)$$

** symmetry

$$\frac{\sqrt{5}yz(6x^2-y^2-z^2)}{2}$$

$$-\frac{\sqrt{5}xz(x^2-6y^2+z^2)}{2}$$

$$-\frac{\sqrt{5}xy(x^2+y^2-6z^2)}{2}$$

** expression

$$\frac{3\sqrt{35}Q_xxyz}{7} + \frac{\sqrt{35}Q_yz(6x^2-3y^2-z^2)}{28} + \frac{\sqrt{35}Q_z y(6x^2-y^2-3z^2)}{28}$$

$$-\frac{\sqrt{35}Q_xz(3x^2-6y^2+z^2)}{28} + \frac{3\sqrt{35}Q_yxyz}{7} - \frac{\sqrt{35}Q_zx(x^2-6y^2+3z^2)}{28}$$

$$-\frac{\sqrt{35}Q_xy(3x^2+y^2-6z^2)}{28} - \frac{\sqrt{35}Q_yx(x^2+3y^2-6z^2)}{28} + \frac{3\sqrt{35}Q_zxyz}{7}$$

$$\vec{Q}_{4,1}^{(1,1)}[q](T_2), \vec{Q}_{4,2}^{(1,1)}[q](T_2), \vec{Q}_{4,3}^{(1,1)}[q](T_2)$$

** symmetry

$$\frac{\sqrt{5}yz(6x^2-y^2-z^2)}{2}$$

$$-\frac{\sqrt{5}xz(x^2-6y^2+z^2)}{2}$$

$$-\frac{\sqrt{5}xy(x^2+y^2-6z^2)}{2}$$

** expression

$$\frac{21\sqrt{11}Q_xxyz(2x^2-y^2-z^2)}{22} - \frac{\sqrt{11}Q_yz(6x^4-51x^2y^2+5x^2z^2+6y^4+5y^2z^2-z^4)}{22} - \frac{\sqrt{11}Q_z y(6x^4+5x^2y^2-51x^2z^2-y^4+5y^2z^2+6z^4)}{22}$$

$$-\frac{\sqrt{11}Q_xz(6x^4-51x^2y^2+5x^2z^2+6y^4+5y^2z^2-z^4)}{22} - \frac{21\sqrt{11}Q_yxyz(x^2-2y^2+z^2)}{22} + \frac{\sqrt{11}Q_zx(x^4-5x^2y^2-5x^2z^2-6y^4+51y^2z^2-6z^4)}{22}$$

$$-\frac{\sqrt{11}Q_xy(6x^4+5x^2y^2-51x^2z^2-y^4+5y^2z^2+6z^4)}{22} + \frac{\sqrt{11}Q_yx(x^4-5x^2y^2-5x^2z^2-6y^4+51y^2z^2-6z^4)}{22} - \frac{21\sqrt{11}Q_zxyz(x^2+y^2-2z^2)}{22}$$