

PG No. 18 D_3 32 (321 setting) [trigonal] (polar, internal polar dipole)

* Harmonics for rank 0

$$\bar{Q}_0^{(1,1)}[q](A_1)$$

** symmetry

$$1$$

** expression

$$\frac{\sqrt{3}Q_x x}{3} + \frac{\sqrt{3}Q_y y}{3} + \frac{\sqrt{3}Q_z z}{3}$$

* Harmonics for rank 1

$$\bar{Q}_1^{(1,-1)}[q](A_2)$$

** symmetry

$$z$$

** expression

$$Q_z$$

$$\bar{Q}_1^{(1,1)}[q](A_2)$$

** symmetry

$$z$$

** expression

$$\frac{3\sqrt{10}Q_x x z}{10} + \frac{3\sqrt{10}Q_y y z}{10} - \frac{\sqrt{10}Q_z (x^2 + y^2 - 2z^2)}{10}$$

$$\bar{Q}_{1,1}^{(1,-1)}[q](E), \bar{Q}_{1,2}^{(1,-1)}[q](E)$$

** symmetry

$$x$$

$$y$$

** expression

$$Q_x$$

$$Q_y$$

$$\bar{Q}_{1,1}^{(1,1)}[q](E), \bar{Q}_{1,2}^{(1,1)}[q](E)$$

** symmetry

$$x$$

$$y$$

** expression

$$\frac{\sqrt{10}Q_x (2x^2 - y^2 - z^2)}{10} + \frac{3\sqrt{10}Q_y x y}{10} + \frac{3\sqrt{10}Q_z x z}{10}$$

$$\frac{3\sqrt{10}Q_x x y}{10} - \frac{\sqrt{10}Q_y (x^2 - 2y^2 + z^2)}{10} + \frac{3\sqrt{10}Q_z y z}{10}$$

* Harmonics for rank 2

$$\bar{Q}_2^{(1,-1)}[q](A_1)$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{6}Q_x x}{6} - \frac{\sqrt{6}Q_y y}{6} + \frac{\sqrt{6}Q_z z}{3}$$

$$\bar{Q}_2^{(1,1)}[q](A_1)$$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{21}Q_x x (x^2 + y^2 - 4z^2)}{14} - \frac{\sqrt{21}Q_y y (x^2 + y^2 - 4z^2)}{14} - \frac{\sqrt{21}Q_z z (3x^2 + 3y^2 - 2z^2)}{14}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,-1)}[q](E, 1), \tilde{\mathbb{Q}}_{2,2}^{(1,-1)}[q](E, 1)$$

** symmetry

$$\sqrt{3}yz$$

$$-\sqrt{3}xz$$

** expression

$$\frac{\sqrt{2}Q_y z}{2} + \frac{\sqrt{2}Q_z y}{2}$$

$$-\frac{\sqrt{2}Q_x z}{2} - \frac{\sqrt{2}Q_z x}{2}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,-1)}[q](E, 2), \tilde{\mathbb{Q}}_{2,2}^{(1,-1)}[q](E, 2)$$

** symmetry

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

$$-\sqrt{3}xy$$

** expression

$$\frac{\sqrt{2}Q_x x}{2} - \frac{\sqrt{2}Q_y y}{2}$$

$$-\frac{\sqrt{2}Q_x y}{2} - \frac{\sqrt{2}Q_y x}{2}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,1)}[q](E, 1), \tilde{\mathbb{Q}}_{2,2}^{(1,1)}[q](E, 1)$$

** symmetry

$$\sqrt{3}yz$$

$$-\sqrt{3}xz$$

** expression

$$\frac{5\sqrt{7}Q_x x y z}{7} - \frac{\sqrt{7}Q_y z (x^2 - 4y^2 + z^2)}{7} - \frac{\sqrt{7}Q_z y (x^2 + y^2 - 4z^2)}{7}$$

$$-\frac{\sqrt{7}Q_x z (4x^2 - y^2 - z^2)}{7} - \frac{5\sqrt{7}Q_y x y z}{7} + \frac{\sqrt{7}Q_z x (x^2 + y^2 - 4z^2)}{7}$$

$$\tilde{\mathbb{Q}}_{2,1}^{(1,1)}[q](E, 2), \tilde{\mathbb{Q}}_{2,2}^{(1,1)}[q](E, 2)$$

** symmetry

$$\frac{\sqrt{3}(x-y)(x+y)}{2}$$

$$-\sqrt{3}xy$$

** expression

$$\frac{\sqrt{7}Q_x x (3x^2 - 7y^2 - 2z^2)}{14} + \frac{\sqrt{7}Q_y y (7x^2 - 3y^2 + 2z^2)}{14} + \frac{5\sqrt{7}Q_z z (x-y)(x+y)}{14}$$

$$-\frac{\sqrt{7}Q_x y (4x^2 - y^2 - z^2)}{7} + \frac{\sqrt{7}Q_y x (x^2 - 4y^2 + z^2)}{7} - \frac{5\sqrt{7}Q_z x y z}{7}$$

* Harmonics for rank 3

$$\tilde{\mathbb{Q}}_3^{(1,-1)}[q](A_1)$$

** symmetry

$$\frac{\sqrt{10}x (x^2 - 3y^2)}{4}$$

** expression

$$\frac{\sqrt{6}Q_x(x-y)(x+y)}{4} - \frac{\sqrt{6}Q_yxy}{2}$$

$$\bar{Q}_3^{(1,1)}[q](A_1)$$

** symmetry

$$\frac{\sqrt{10}x(x^2-3y^2)}{4}$$

** expression

$$\frac{\sqrt{10}Q_x(4x^4-21x^2y^2-3x^2z^2+3y^4+3y^2z^2)}{24} + \frac{\sqrt{10}Q_yxy(13x^2-15y^2+6z^2)}{24} + \frac{7\sqrt{10}Q_zxz(x^2-3y^2)}{24}$$

$$\bar{Q}_3^{(1,-1)}[q](A_2, 1)$$

** symmetry

$$-\frac{z(3x^2+3y^2-2z^2)}{2}$$

** expression

$$-\frac{\sqrt{15}Q_xxz}{5} - \frac{\sqrt{15}Q_yyz}{5} - \frac{\sqrt{15}Q_z(x^2+y^2-2z^2)}{10}$$

$$\bar{Q}_3^{(1,-1)}[q](A_2, 2)$$

** symmetry

$$\frac{\sqrt{10}y(3x^2-y^2)}{4}$$

** expression

$$\frac{\sqrt{6}Q_xxy}{2} + \frac{\sqrt{6}Q_y(x-y)(x+y)}{4}$$

$$\bar{Q}_3^{(1,1)}[q](A_2, 1)$$

** symmetry

$$-\frac{z(3x^2+3y^2-2z^2)}{2}$$

** expression

$$-\frac{5Q_xxz(3x^2+3y^2-4z^2)}{12} - \frac{5Q_yyz(3x^2+3y^2-4z^2)}{12} + \frac{Q_z(3x^4+6x^2y^2-24x^2z^2+3y^4-24y^2z^2+8z^4)}{12}$$

$$\bar{Q}_3^{(1,1)}[q](A_2, 2)$$

** symmetry

$$\frac{\sqrt{10}y(3x^2-y^2)}{4}$$

** expression

$$\frac{\sqrt{10}Q_xxy(15x^2-13y^2-6z^2)}{24} - \frac{\sqrt{10}Q_y(3x^4-21x^2y^2+3x^2z^2+4y^4-3y^2z^2)}{24} + \frac{7\sqrt{10}Q_zyz(3x^2-y^2)}{24}$$

$$\bar{Q}_{3,1}^{(1,-1)}[q](E, 1), \bar{Q}_{3,2}^{(1,-1)}[q](E, 1)$$

** symmetry

$$-\frac{\sqrt{6}x(x^2+y^2-4z^2)}{4}$$

$$-\frac{\sqrt{6}y(x^2+y^2-4z^2)}{4}$$

** expression

$$-\frac{\sqrt{10}Q_x(3x^2+y^2-4z^2)}{20} - \frac{\sqrt{10}Q_yxy}{10} + \frac{2\sqrt{10}Q_zxz}{5}$$

$$-\frac{\sqrt{10}Q_xxy}{10} - \frac{\sqrt{10}Q_y(x^2+3y^2-4z^2)}{20} + \frac{2\sqrt{10}Q_zyz}{5}$$

$$\bar{Q}_{3,1}^{(1,-1)}[q](E, 2), \bar{Q}_{3,2}^{(1,-1)}[q](E, 2)$$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$Q_x y z + Q_y x z + Q_z x y$$

$$Q_x x z - Q_y y z + \frac{Q_z(x-y)(x+y)}{2}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,1)}[q](E, 1), \tilde{\mathbb{Q}}_{3,2}^{(1,1)}[q](E, 1)$$

** symmetry

$$-\frac{\sqrt{6}x(x^2+y^2-4z^2)}{4}$$

$$-\frac{\sqrt{6}y(x^2+y^2-4z^2)}{4}$$

** expression

$$-\frac{\sqrt{6}Q_x(4x^4+3x^2y^2-27x^2z^2-y^4+3y^2z^2+4z^4)}{24} - \frac{5\sqrt{6}Q_yxy(x^2+y^2-6z^2)}{24} - \frac{5\sqrt{6}Q_zxz(3x^2+3y^2-4z^2)}{24}$$

$$-\frac{5\sqrt{6}Q_xxy(x^2+y^2-6z^2)}{24} + \frac{\sqrt{6}Q_y(x^4-3x^2y^2-3x^2z^2-4y^4+27y^2z^2-4z^4)}{24} - \frac{5\sqrt{6}Q_zyz(3x^2+3y^2-4z^2)}{24}$$

$$\tilde{\mathbb{Q}}_{3,1}^{(1,1)}[q](E, 2), \tilde{\mathbb{Q}}_{3,2}^{(1,1)}[q](E, 2)$$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z(x-y)(x+y)}{2}$$

** expression

$$\frac{\sqrt{15}Q_xyz(6x^2-y^2-z^2)}{6} - \frac{\sqrt{15}Q_yxz(x^2-6y^2+z^2)}{6} - \frac{\sqrt{15}Q_zxy(x^2+y^2-6z^2)}{6}$$

$$\frac{\sqrt{15}Q_xxz(5x^2-9y^2-2z^2)}{12} + \frac{\sqrt{15}Q_yyz(9x^2-5y^2+2z^2)}{12} - \frac{\sqrt{15}Q_z(x-y)(x+y)(x^2+y^2-6z^2)}{12}$$

* Harmonics for rank 4

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](A_1, 1)$$

** symmetry

$$\frac{3x^4}{8} + \frac{3x^2y^2}{4} - 3x^2z^2 + \frac{3y^4}{8} - 3y^2z^2 + z^4$$

** expression

$$\frac{3\sqrt{7}Q_xx(x^2+y^2-4z^2)}{28} + \frac{3\sqrt{7}Q_yy(x^2+y^2-4z^2)}{28} - \frac{\sqrt{7}Q_zz(3x^2+3y^2-2z^2)}{7}$$

$$\tilde{\mathbb{Q}}_4^{(1,-1)}[q](A_1, 2)$$

** symmetry

$$\frac{\sqrt{70}yz(3x^2-y^2)}{4}$$

** expression

$$\frac{3\sqrt{10}Q_xxyz}{4} + \frac{3\sqrt{10}Q_yz(x-y)(x+y)}{8} + \frac{\sqrt{10}Q_zy(3x^2-y^2)}{8}$$

$$\tilde{\mathbb{Q}}_4^{(1,1)}[q](A_1, 1)$$

** symmetry

$$\frac{3x^4}{8} + \frac{3x^2y^2}{4} - 3x^2z^2 + \frac{3y^4}{8} - 3y^2z^2 + z^4$$

** expression

$$\frac{3\sqrt{55}Q_x x (x^4 + 2x^2y^2 - 12x^2z^2 + y^4 - 12y^2z^2 + 8z^4)}{88} + \frac{3\sqrt{55}Q_y y (x^4 + 2x^2y^2 - 12x^2z^2 + y^4 - 12y^2z^2 + 8z^4)}{88} + \frac{\sqrt{55}Q_z z (15x^4 + 30x^2y^2 - 40x^2z^2 + 15y^4 - 40y^2z^2 + 8z^4)}{88}$$

$$\bar{Q}_4^{(1,1)}[q](A_1, 2)$$

** symmetry

$$\frac{\sqrt{70}yz (3x^2 - y^2)}{4}$$

** expression

$$\frac{3\sqrt{154}Q_x xy z (7x^2 - 5y^2 - 2z^2)}{44} - \frac{3\sqrt{154}Q_y z (x^4 - 9x^2y^2 + x^2z^2 + 2y^4 - y^2z^2)}{44} - \frac{\sqrt{154}Q_z y (3x^2 - y^2) (x^2 + y^2 - 8z^2)}{44}$$

$$\bar{Q}_4^{(1,-1)}[q](A_2)$$

** symmetry

$$\frac{\sqrt{70}xz (x^2 - 3y^2)}{4}$$

** expression

$$\frac{3\sqrt{10}Q_x z (x - y) (x + y)}{8} - \frac{3\sqrt{10}Q_y xyz}{4} + \frac{\sqrt{10}Q_z x (x^2 - 3y^2)}{8}$$

$$\bar{Q}_4^{(1,1)}[q](A_2)$$

** symmetry

$$\frac{\sqrt{70}xz (x^2 - 3y^2)}{4}$$

** expression

$$\frac{3\sqrt{154}Q_x z (2x^4 - 9x^2y^2 - x^2z^2 + y^4 + y^2z^2)}{44} + \frac{3\sqrt{154}Q_y xyz (5x^2 - 7y^2 + 2z^2)}{44} - \frac{\sqrt{154}Q_z x (x^2 - 3y^2) (x^2 + y^2 - 8z^2)}{44}$$

$$\bar{Q}_{4,1}^{(1,-1)}[q](E, 1), \bar{Q}_{4,2}^{(1,-1)}[q](E, 1)$$

** symmetry

$$- \frac{\sqrt{10}yz (3x^2 + 3y^2 - 4z^2)}{4}$$

$$\frac{\sqrt{10}xz (3x^2 + 3y^2 - 4z^2)}{4}$$

** expression

$$- \frac{3\sqrt{70}Q_x xyz}{28} - \frac{\sqrt{70}Q_y z (3x^2 + 9y^2 - 4z^2)}{56} - \frac{3\sqrt{70}Q_z y (x^2 + y^2 - 4z^2)}{56}$$

$$\frac{\sqrt{70}Q_x z (9x^2 + 3y^2 - 4z^2)}{56} + \frac{3\sqrt{70}Q_y xyz}{28} + \frac{3\sqrt{70}Q_z x (x^2 + y^2 - 4z^2)}{56}$$

$$\bar{Q}_{4,1}^{(1,-1)}[q](E, 2), \bar{Q}_{4,2}^{(1,-1)}[q](E, 2)$$

** symmetry

$$\frac{\sqrt{35} (x^2 - 2xy - y^2) (x^2 + 2xy - y^2)}{8}$$

$$\frac{\sqrt{35}xy (x - y) (x + y)}{2}$$

** expression

$$\frac{\sqrt{5}Q_x x (x^2 - 3y^2)}{4} - \frac{\sqrt{5}Q_y y (3x^2 - y^2)}{4}$$

$$\frac{\sqrt{5}Q_x y (3x^2 - y^2)}{4} + \frac{\sqrt{5}Q_y x (x^2 - 3y^2)}{4}$$

$$\bar{Q}_{4,1}^{(1,-1)}[q](E, 3), \bar{Q}_{4,2}^{(1,-1)}[q](E, 3)$$

** symmetry

$$- \frac{\sqrt{5} (x - y) (x + y) (x^2 + y^2 - 6z^2)}{4}$$

$$\frac{\sqrt{5}xy(x^2 + y^2 - 6z^2)}{2}$$

** expression

$$-\frac{\sqrt{35}Q_x x(x^2 - 3z^2)}{14} + \frac{\sqrt{35}Q_y y(y^2 - 3z^2)}{14} + \frac{3\sqrt{35}Q_z z(x - y)(x + y)}{14}$$

$$\frac{\sqrt{35}Q_x y(3x^2 + y^2 - 6z^2)}{28} + \frac{\sqrt{35}Q_y x(x^2 + 3y^2 - 6z^2)}{28} - \frac{3\sqrt{35}Q_z xyz}{7}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,1)}[q](E, 1), \tilde{\mathbb{Q}}_{4,2}^{(1,1)}[q](E, 1)$$

** symmetry

$$-\frac{\sqrt{10}yz(3x^2 + 3y^2 - 4z^2)}{4}$$

$$\frac{\sqrt{10}xz(3x^2 + 3y^2 - 4z^2)}{4}$$

** expression

$$-\frac{21\sqrt{22}Q_x xyz(x^2 + y^2 - 2z^2)}{44} + \frac{\sqrt{22}Q_y z(3x^4 - 15x^2y^2 - x^2z^2 - 18y^4 + 41y^2z^2 - 4z^4)}{44} + \frac{3\sqrt{22}Q_z y(x^4 + 2x^2y^2 - 12x^2z^2 + y^4 - 12y^2z^2 + 8z^4)}{44}$$

$$\frac{\sqrt{22}Q_x z(18x^4 + 15x^2y^2 - 41x^2z^2 - 3y^4 + y^2z^2 + 4z^4)}{44} + \frac{21\sqrt{22}Q_y xyz(x^2 + y^2 - 2z^2)}{44} - \frac{3\sqrt{22}Q_z x(x^4 + 2x^2y^2 - 12x^2z^2 + y^4 - 12y^2z^2 + 8z^4)}{44}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,1)}[q](E, 2), \tilde{\mathbb{Q}}_{4,2}^{(1,1)}[q](E, 2)$$

** symmetry

$$\frac{\sqrt{35}(x^2 - 2xy - y^2)(x^2 + 2xy - y^2)}{8}$$

$$\frac{\sqrt{35}xy(x - y)(x + y)}{2}$$

** expression

$$\frac{\sqrt{77}Q_x x(5x^4 - 46x^2y^2 - 4x^2z^2 + 21y^4 + 12y^2z^2)}{88} + \frac{\sqrt{77}Q_y y(21x^4 - 46x^2y^2 + 12x^2z^2 + 5y^4 - 4y^2z^2)}{88} + \frac{9\sqrt{77}Q_z z(x^2 - 2xy - y^2)(x^2 + 2xy - y^2)}{88}$$

$$\frac{\sqrt{77}Q_x y(6x^4 - 11x^2y^2 - 3x^2z^2 + y^4 + y^2z^2)}{22} - \frac{\sqrt{77}Q_y x(x^4 - 11x^2y^2 + x^2z^2 + 6y^4 - 3y^2z^2)}{22} + \frac{9\sqrt{77}Q_z xyz(x - y)(x + y)}{22}$$

$$\tilde{\mathbb{Q}}_{4,1}^{(1,1)}[q](E, 3), \tilde{\mathbb{Q}}_{4,2}^{(1,1)}[q](E, 3)$$

** symmetry

$$-\frac{\sqrt{5}(x - y)(x + y)(x^2 + y^2 - 6z^2)}{4}$$

$$\frac{\sqrt{5}xy(x^2 + y^2 - 6z^2)}{2}$$

** expression

$$-\frac{\sqrt{11}Q_x x(5x^4 - 4x^2y^2 - 46x^2z^2 - 9y^4 + 66y^2z^2 + 12z^4)}{44} - \frac{\sqrt{11}Q_y y(9x^4 + 4x^2y^2 - 66x^2z^2 - 5y^4 + 46y^2z^2 - 12z^4)}{44} - \frac{21\sqrt{11}Q_z z(x - y)(x + y)(x^2 + y^2 - 2z^2)}{44}$$

$$\frac{\sqrt{11}Q_x y(6x^4 + 5x^2y^2 - 51x^2z^2 - y^4 + 5y^2z^2 + 6z^4)}{22} - \frac{\sqrt{11}Q_y x(x^4 - 5x^2y^2 - 5x^2z^2 - 6y^4 + 51y^2z^2 - 6z^4)}{22} + \frac{21\sqrt{11}Q_z xyz(x^2 + y^2 - 2z^2)}{22}$$