**21-241 Matrices and Linear Transformations** **Fall 2021**
**Homework 8** **Due Friday, October 29**

## Required Problems

1. Let $A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of $A$ by hand.

   (b) Find $A^{-1}$ and find the eigenvalues and eigenvectors of $A^{-1}$ by hand.

   (c) What do you notice about the eigenvectors of $A$ and $A^{-1}$?

   (d) What do you notice about the eigenvalues of $A$ and $A^{-1}$?

   (e) Let $A$ be an invertible square matrix. Show that if $\mathbf{x}$ is an eigenvector for $A$ with eigenvalue $\lambda$, then it is also an eigenvector for $A^{-1}$. [Hint: Start with the equation $A\mathbf{x} = \lambda\mathbf{x}$, multiply both sides by $A^{-1}$, and find an expression for $A^{-1}\mathbf{x}$.] What is the corresponding eigenvalue?

2. Suppose an $n \times n$ matrix $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. We will prove that the determinant of $A$ equals the product $\lambda_1\lambda_2 \cdots \lambda_n$ in two ways.

   (a) Prove that the determinant of $A$ equals the product $\lambda_1\lambda_2 \cdots \lambda_n$ by starting with the polynomial $\det(A - \lambda I)$ separated into its $n$ factors, i.e.

   $$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

   [If $p$ is a polynomial of degree $n \geq 1$ in the variable $\lambda$, and $a$ is a root of $p$ (i.e. $p(a) = 0$) then $p = (a - \lambda)p'$ where $p'$ is a polynomial of degree $n - 1$. The equality above holds since, using the cofactor formula, we can argue that the leading term in $\det(A - \lambda I)$ is always $(-\lambda)^n$.]
   Then set $\lambda = 0$.

   (b) Prove that if $A$ is diagonalizable, the determinant of $A$ equals the product $\lambda_1\lambda_2 \cdots \lambda_n$ using the fact that determinant is multiplicative.

   (c) Verify that determinant of $A$ equals the product $\lambda_1\lambda_2$ for the matrix $A$ in Problem 1.

3. (a) Prove that if $\Lambda_1$ and $\Lambda_2$ are $n \times n$ diagonal matrices, then $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$

   (b) Suppose $A$ and $B$ are $n \times n$ matrices, and have the same set of $n$ independent eigenvectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Use diagonalization to prove that $AB = BA$.

4. Let $A = \begin{pmatrix} .6 & .9 \\ .4 & .1 \end{pmatrix}$.

   (a) Find $\Lambda$ and $X$ to diagonalize $A$.

   (b) What is the limit of $\Lambda^k$ as $k \to \infty$? I.e. for large values of $k$, what matrix does $\Lambda^k$ approach?

(c) What is the limit of $X\Lambda^k X^{-1}$?

(d) What vector is in the columns of the limiting matrix for $X\Lambda^k X^{-1}$?

5. This problem gives an example of a Markov matrix which is not a positive Markov matrix, and how such matrices may not have attracting steady states if there are multiple eigenvalues with magnitude 1. Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find the characteristic polynomial for $P$. [Hint: You can do this using the permutation formula for determinant by finding all ways to choose a nonzero entry from each row and column of $P - \lambda I$. Alternatively you could use the Row 4 cofactor formula on $P - \lambda I$ and then use the fact that the determinant of a triangular matrix is the product of the entries on the diagonal.]

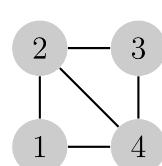
(b) Find the steady state vector $\mathbf{v}$ corresponding to the eigenvalue $\lambda = 1$. I.e. find the vector $\mathbf{v}$ where $P\mathbf{v} = \mathbf{v}$.

(c) Let $\mathbf{u}_0 = (1, 0, 0, 0)$. If $\mathbf{u}_{k+1} = P\mathbf{u}_k$, what are $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$? Does the sequence $\{\mathbf{u}_k\}$ converge to $\mathbf{v}$? Why or why not?

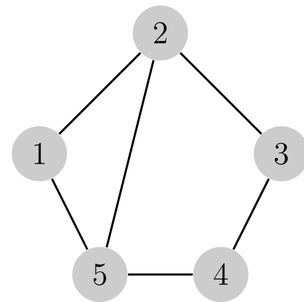
(d) What are the four eigenvalues for $P$?

6. This problem illustrates how to use a Markov matrix to analyze a random walk on a graph. Suppose a person walks from node to node on a graph, at each time step choosing randomly to move to a neighboring node, each with equal probability. For each of the graphs below, what Markov matrix describes the transition probabilities for the random walk? [For grading purposes, please let row $i$ and column $i$ correspond to node $i$, as labeled.] What proportion of the time would we expect to find the random walker at each node?

(a)




(b)




Optional: For each vertex, look at how many neighbors it has, and the long run probability of a walker being there. Can you prove there is a relationship? What is it?

## Optional Problems

7. Let $A$ be a matrix, and $\mathbf{x}$ and $\mathbf{y}$ be eigenvectors for $A$. Prove or disprove each of the following statements.
   (a) For all scalars $c \neq 0$, the vector $c\mathbf{x}$ is an eigenvector for $A$.
   (b) For all integers $k \geq 1$, $\mathbf{x}$ is an eigenvector for $A^k$.
   (c) The vector $\mathbf{x} + \mathbf{y}$ is always an eigenvector for $A$.
8. (Strang 6.1.12) Find three eigenvectors for this matrix $P$ (Projection matrices have $\lambda = 1$ and 0.):

$$\textbf{Projection matrix} \qquad P = \begin{pmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   If two eigenvectors share the same $\lambda$, so do all of their linear combinations. Find an eigenvector of $P$ with no zero components.
9. (Strang 6.1.25) Suppose $A$ and $B$ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then $A = B$. Reason: Any vector $\mathbf{x}$ is a linear combination $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$. What is $A\mathbf{x}$? What is $B\mathbf{x}$?
10. Prove that if $A_1$ is similar to $A_2$ and $A_2$ is similar to $A_3$, then $A_1$ is similar to $A_3$.
11. Prove or disprove:
    (a) If $\mathbf{x}$ is an eigenvector for $A$ and $B$, then $\mathbf{x}$ is an eigenvector for $AB$ and $BA$.
    (b) If $\lambda$ is an eigenvalue for $A$ and $B$, then $\lambda^2$ is an eigenvalue for $AB$ and $BA$.
12. List all matrices that are similar to the identity matrix.
13. Prove that the eigenvalues of a triangular matrix are the entries on the diagonal.

14. The trace of a matrix is the sum of the diagonal entries. Prove that the sum of the eigenvalues is equal to the trace.

15. Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors for $A$ with eigenvalues $\lambda_1$ and $\lambda_2$. Under what conditions on $\lambda_1$ and $\lambda_2$ is $\mathbf{x}_1 + \mathbf{x}_2$ an eigenvector for $A$?

16. We have seen how it is possible to find eigenvalues and eigenvectors of a matrix by finding roots of its characteristic polynomial. In this problem you will show how to do the reverse: You can find the roots of a polynomial by finding the eigenvectors of its "companion matrix." Let $p$ be the degree $n$ polynomial $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + z^n$.

    Note that the coefficient of $z^n$ is 1.

    Define the companion matrix for $p$ to be the $n \times n$ matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 1 \\ -c_0 & -c_1 & \dots & -c_{n-2} & -c_{n-1} \end{pmatrix}.$$

    (a) Show that $\det(C - \lambda I) = p(\lambda)$.
    (b) Prove that $z$ is a root of $p$ if and only if it is an eigenvalue of $C$ with eigenvector $(1, z, z^2, \dots, z^{n-1})$.
    (c) Explain how to determine the roots of any degree $n$ polynomial (even if its leading coefficient is not 1) if you know how to find eigenvectors for a matrix. [This is actually how some polynomial solvers proceed: Rather than solving the polynomial they instead find the eigenvectors of its companion matrix. ]