CMIMC 2018

Team Round

Set One

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 problems to be solved in 30 minutes. The problems are not ordered.
3. Write your team name and team ID on both answer sheets. Choose one to be the ten-minute submission and the other to be the thirty-minute submission, and circle the appropriate choice.
4. Each problem is worth up to 8 points. Points earned per problem are halved for a thirty-minute submission and quartered if only one leg is correct.
5. Write your answers in the corresponding boxes on the answer sheets.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to the registration desk on the first floor of the University Center by 1:30 PM.

CMIMC 2018

Team (Set One)

1-1. Let ABC be a triangle with $BC = 30$, $AC = 50$, and $AB = 60$. Circle ω_B is the circle passing through A and B tangent to BC at B; ω_C is defined similarly. Suppose the tangent to $\odot(ABC)$ at A intersects ω_B and ω_C for the second time at X and Y respectively. Compute XY.

2-1. Suppose that a and b are non-negative integers satisfying $a + b + ab + a^b = 42$. Find the sum of all possible values of $a + b$.

3-1. Let Ω be a semicircle with endpoints A and B and diameter 3. Points X and Y are located on the boundary of Ω such that the distance from X to AB is $\frac{5}{4}$ and the distance from Y to AB is $\frac{1}{4}$. Compute

$$(AX + BX)^2 - (AY + BY)^2.$$

4-1. Define an integer $n \geq 0$ to be *two-far* if there exist integers a and b such that a, b, and $n+a+b$ are all powers of two. If N is the number of two-far integers less than 2048, find the remainder when N is divided by 100.

5-1. How many ordered triples (a, b, c) of integers satisfy the inequality

$$a^2 + b^2 + c^2 \leq a + b + c + 2?$$

6-2. Let $T = TNYWR$. Compute the number of ordered triples (a, b, c) such that a, b, and c are distinct positive integers and $a + b + c = T$.

7-2. Let $T = TNYWR$. A total of $2T$ students go on a road trip. They take two cars, each of which seats T people. Call two students *friendly* if they sat together in the same car going to the trip and in the same car going back home. What is the smallest possible number of friendly pairs of students on the trip?

8-2. Let $T = TNYWR$, and let $T = 10X + Y$ for an integer X and a digit Y. Suppose that a and b are real numbers satisfying $a + \frac{1}{b} = Y$ and $\frac{b}{a} = X$. Compute $(ab)^4 + \frac{1}{(ab)^4}$.

9-2. Let $T = TNYWR$. The solutions in z to the equation

$$\left(z + \frac{T}{z}\right)^2 = 1$$

form the vertices of a quadrilateral in the complex plane. Compute the area of this quadrilateral.

10-2. Let $T = TNYWR$. Circles ω_1 and ω_2 intersect at P and Q. The common external tangent ℓ to the two circles closer to Q touches ω_1 and ω_2 at A and B respectively. Line AQ intersects ω_2 at X while BQ intersects ω_1 again at Y. Let M and N denote the midpoints of $\overline{AY}$ and $\overline{BX}$, also respectively. If $AQ = \sqrt{T}$, $BQ = 7$, and $AB = 8$, then find the length of MN.