Q1 a>

$$A=\begin{bmatrix}2 & -1 & -3 & -1\\ 1 & 2 & 3 & -1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix} \qquad A : \mathbb{R}^4 \to \mathbb{R}^4$$

$$\to \begin{bmatrix}\textcircled{2} & -1 & -3 & -1\\ 0 & -5 & -9 & 1\\ 0 & -1 & -5 & -3\\ 0 & 1 & 1 & -1\end{bmatrix}$$

$$\to \begin{bmatrix}\textcircled{2} & -1 & -3 & -1\\ 0 & \textcircled{1} & 1 & -1\\ 0 & 0 & -4 & -4\\ 0 & -5 & -9 & 1\end{bmatrix}$$

Rank = 3 , Nullity = 1

$+$ $= \dim \mathbb{R}^4$
$= 4$

$$\to \begin{bmatrix}\textcircled{2} & -1 & -3 & -1\\ 0 & \textcircled{1} & 1 & -1\\ 0 & 0 & \textcircled{-4} & -4\\ 0 & 0 & -4 & -4\end{bmatrix} \to \begin{bmatrix}\textcircled{2} & -1 & -3 & -1\\ 0 & \textcircled{1} & 1 & -1\\ 0 & 0 & \textcircled{1} & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

b> $u = (5,-1,-1)$, $v = (-1,5,8)$

$u - v = (6,-6,-9)$, $d(u,v) = \|u-v\|$

$= \sqrt{36+36+81}$

$= \sqrt{153}$

$= 3\sqrt{17}$

$\|u\| = \sqrt{27}$, $\|v\| = \sqrt{90}$

$= 3\sqrt{3}$ $\qquad = 3\sqrt{10}$

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{-18}{9\sqrt{30}} = 0.619\pi$$

or 111.4167°

c> $A = \begin{bmatrix}2 & 3 & -2\\ -1 & 4 & -1\\ -1 & 5 & -1\end{bmatrix}$ $\quad P_A(t) = t^3 - 5t^2 + 8t - 4$

$t = 1, 2$

$X = (1,1,2), (1,2,3)$

λ	1	2
AM	1	2
GM	1	1

∴ A is not diagonalizable.

Q 2 a) $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$ $P_A(t) = t^3 - 2t^2 - 38t - 1$

$A^{-1} = A^2 - 2A - 38I$ -- by Cayley Hamilton.

$= \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix}$

b) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ $P_A(t) = t^3 - 6t^2 - 15t - 8$

$t = 8, -1, -1$

$(A - 8I)(A + I) = 0$

$m_A(t) = (t-8)(t+1) \neq P_A(t)$

$\therefore$ A is derogatory.

c) $t^{10} - 5t^6 + 2t^3$

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $P_A(t) = t^3 - 4t^2 + 5t - 2$

$t^7 + 4t^6 + 11t^5 + 26t^4 + 52t^3 + 100t^2 + 192t + 374$

$t^3 - 4t^2 + 5t - 2 \overline{)\, t^{10} - 5t^6 + 2t^3}$

$t^{10} - 4t^9 + 5t^8 - 2t^7$

$- \quad + \quad - \quad +$

$4t^9 - 5t^8 + 2t^7 - 5t^6 + 2t^3$

$4t^9 - 16t^8 + 20t^7 - 8t^6$

$- \quad + \quad - \quad +$

$11t^8 - 18t^7 + 3t^6 + 2t^3$

$11t^8 - 44t^7 + 55t^6 - 22t^5$

$- \quad + \quad - \quad +$

$26t^7 - 52t^6 + 22t^5 + 2t^3$

$26t^7 - 104t^6 + 130t^5 - 52t^4$

$- \quad + \quad - \quad +$

$-52t^6 - 108t^5 + 52t^4 + 2t^3$

$52t^6 - 208t^5 + 260t^4 - 104t^3$

$- \quad + \quad - \quad +$

$100t^5 - 208t^4 + 106t^3$

$100t^5 - 208t^4 + 106t^3$

$-100t^5 \mp 400t^4 \pm 500t^3 \mp 200t^2$

$192t^4 - 394t^3 + 200t^2$

$- \; 192t^4 \mp 768t^3 \pm 960t^2 \mp 384t$

$374t^3 - 760t^2 + 384t$

$-374t^3 \mp 1496t^2$

$\pm 1870t \mp 748$

$736t^2 - 1486t + 748$

$736A^2 - 1486A + 748I$

$= \begin{bmatrix} -2 & 0 & 722 \\ 0 & -2 & 0 \\ 0 & 0 & 720 \end{bmatrix}$

MID SEMESTER TEST (MST)

Q3 $6x^2 + 3y^2 + 3z^2 - 4xy + 4xz - 2yz = 9$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\lambda = 2, 2, 8$$

$$X = \underset{X_t'}{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}, \underset{X_2}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}, \underset{X_3}{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}$$

Remove X_t' and add $X_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ so that

$X_1 \cdot X_2 = 0 = X_1 \cdot X_3 \rightarrow a + 2b = 0$ & $2a - b + c = 0$

$\rightarrow c = b - 2a = b - 2(-2b) = 5b$

$\rightarrow X_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2b \\ b \\ 5b \end{pmatrix} = b \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$

$\rightarrow X_1 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$

$\therefore A = PDP^t$ where $P = \begin{bmatrix} -2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \\ 5/\sqrt{30} & 0 & 1/\sqrt{6} \end{bmatrix}$

$\therefore 2X_1'^2 + 2X_2'^2 + 8X_3'^2 = 9$

→ Ellipsoid.

Rank = 3, Index = 3

Signature = 3, Nature = positive definite

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $\det P = -1$

Consider, $PX = X \rightarrow$

$$\left.\begin{aligned} \left(-\frac{2}{\sqrt{30}} - 1\right)x + \frac{1}{\sqrt{5}}y + \frac{2}{\sqrt{6}}z &= 0 \\ \left(\frac{1}{\sqrt{30}}x\right) + \left(\frac{2}{\sqrt{5}} - 1\right)y - \frac{1}{\sqrt{6}}z &= 0 \\ \frac{5}{\sqrt{30}}x + \left(\frac{1}{\sqrt{6}} - 1\right)z &= 0 \end{aligned}\right\}$$

$\det[P - I] = -0.12$

$\neq 0$

(unique sol.)

→ Sol Set = {(0, 0, 0)}

→ P is a combination of rotation & reflection.

Q4 a) Take $S = \{ \underset{u_2}{(1,0,-3)}, \underset{u_3}{(2,-1,0)} \}$

Add vector $v_1 = (1,0,0)$ in S

($\because$ $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & -3 \\ 2 & -1 & 0 \end{vmatrix} = 1(0-3) = -3 \neq 0$

$\rightarrow S \cup \{v_1\}$ is independent set)

$\therefore S' = S \cup \{v_1\}$ is a basis of $\mathbb{R}^3$.

$v_1 = u_1 = (1,0,0)$

$v_2 = u_2 - \text{Proj}_{v_1} u_2 = u_2 - \frac{(u_2 \cdot v_1)}{\|v_1\|^2} \cdot v_1$

$= (1,0,-3) - (1,0,0) = (0,0,-3)$

$v_3 = u_3 - \text{Proj}_{v_1} u_3 - \text{Proj}_{v_2} u_3 = u_3 - \frac{(u_3 \cdot v_1)}{\|v_1\|^2} \cdot v_1 - \frac{(u_3 \cdot v_2)}{\|v_2\|^2} \cdot v_2$

$= (2,-1,0) - 2(1,0,0) - 0$

$= (0,-1,0)$

$\therefore$ Orthonormal basis by Gram Schmidt process

$= \{ v_1' = (1,0,0), v_2' = (0,0,-1), v_3' = (0,-1,0) \}$

b) $m(T) = A = \begin{bmatrix} 2 & 3 & 4 \\ -3 & -3 & -2 \\ -2 & 1 & -1 \end{bmatrix}$ $\rightarrow |A| = -23 \neq 0$

$\rightarrow$ Rank $A = 3$ -- (surjective)

-- (all rows are independent)

$\rightarrow$ Nullity $A = 0$ -- (injective)

$\therefore$ T is bijective and $m(T^{-1}) = A^{-1} = \frac{-1}{23} \begin{bmatrix} 5 & 7 & 6 \\ 1 & 6 & -8 \\ -9 & -8 & 3 \end{bmatrix}$