First-order Potential Reduction Method

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1 Dimension-reduced Method for Potential Reduction

In this section, we discuss the application of dimension-reduced method to potential reduction. For brevity, we for now only consider the primal potential reduction and focus on the simplex-constrained QP.

$$\begin{array}{rl} \min\limits_{\mathbf{x}} & \frac{1}{2}\|\mathbf{A}\mathbf{x}\|^2 \quad =: f(\mathbf{x}) \\ \text{subject to} & \mathbf{e}^\top \mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

and we adopt the potential function

$$\varphi(\mathbf{x}) := \rho \log(f(\mathbf{x})) - \sum_{i=1}^{n} \log x_i,$$

whose gradient is given by

$$\nabla \varphi(\mathbf{x}) = \frac{\rho \nabla f(\mathbf{x})}{f(\mathbf{x})} - \mathbf{X}^{-1}\mathbf{e}.$$

At each iteration, we evaluate the gradient $\nabla \varphi(\mathbf{x}^k)$, let $\Delta := \mathbf{x}^{k+1} - \mathbf{x}^k$ and solve following subproblem

$$\begin{array}{rc} \min\limits_{\Delta} & \langle \nabla \varphi(\mathbf{x}^k), \Delta \rangle \\ \text{subject to} & \mathbf{e}^\top \Delta = 0 \\ & \left\| (\mathbf{X}^k)^{-1} \Delta \right\| \leq \beta. \end{array}$$

Starting from the basic potential reduction, we extend it by incorporating momentum term for faster convergence.

1.1 Two directions

In this section, we consider two direction extension of the potential reduction framework. In a word, by keeping track of one recent history iterate, we update

$$\begin{aligned} \mathbf{d}^k &\leftarrow \alpha^g \mathbf{P}_\Delta\left[\nabla\varphi\left(\mathbf{x}^k\right)\right] + \alpha^m\left(\mathbf{x}^k - \mathbf{x}^{k-1}\right) \\ \mathbf{x}^k &\leftarrow \mathbf{x}^k + \mathbf{d}^k \end{aligned}$$

where $\mathbf{P}_\Delta[\cdot]$ is the orthogonal projection onto $\mathbf{e}^\top\mathbf{x} = 0$. Note that we compute α^g, α^d through the following model

$$\begin{aligned} \min_{\mathbf{d},\alpha^g,\alpha^m} \quad & \tfrac{1}{2}\mathbf{d}^\top\mathbf{H}\mathbf{d} + \mathbf{h}^\top\mathbf{d} \\ \text{subject to} \quad & \left\|\mathbf{X}^{-1}\mathbf{d}\right\| \le \Delta \\ & \mathbf{d} = \alpha^g\mathbf{g}^k + \alpha^m\mathbf{m}^k \end{aligned}$$

where $\mathbf{g}^k := \mathbf{P}_\Delta\left[\nabla\varphi\left(\mathbf{x}^k\right)\right]$, $\mathbf{m}^k := \mathbf{x}^k - \mathbf{x}^{k-1}$. Alternatively, we define $\mathbf{G} := \begin{pmatrix} | & | \\ \mathbf{g}^k & \mathbf{m}^k \\ | & | \end{pmatrix}, \alpha = \begin{pmatrix} \alpha^g \\ \alpha^m \end{pmatrix}$ and $\mathbf{d} = \mathbf{G}\alpha$, giving

$$\begin{aligned} \min_{\alpha} \quad & \tfrac{1}{2}\alpha^\top\mathbf{G}^\top\mathbf{H}\mathbf{G}^\top\alpha + \mathbf{h}^\top\mathbf{G}\alpha \\ \text{subject to} \quad & \left\|\mathbf{X}^{-1}\mathbf{G}\alpha\right\| \le \Delta, \end{aligned}$$

or

$$\begin{aligned} \min_{\alpha} \quad & \tfrac{1}{2}\alpha^\top\widetilde{\mathbf{H}}\alpha + \widetilde{\mathbf{h}}\alpha \quad =: m(\alpha) \\ \text{subject to} \quad & \|\mathbf{M}\alpha\| \le \Delta \end{aligned}$$

for

$$\begin{aligned} \widetilde{\mathbf{H}} &:= \begin{pmatrix} \left\langle \mathbf{g}^k, \nabla^2_{\mathbf{x},\mathbf{x}}\varphi\left(\mathbf{x}^k\right)\mathbf{g}^k\right\rangle & \left\langle \mathbf{g}^k, \nabla^2_{\mathbf{x},\mathbf{x}}\varphi\left(\mathbf{x}^k\right)\mathbf{m}^k\right\rangle \\ \left\langle \mathbf{m}^k, \nabla^2_{\mathbf{x},\mathbf{x}}\varphi\left(\mathbf{x}^k\right)\mathbf{g}^k\right\rangle & \left\langle \mathbf{m}^k, \nabla^2_{\mathbf{x},\mathbf{x}}\varphi\left(\mathbf{x}^k\right)\mathbf{m}^k\right\rangle \end{pmatrix} \\ \widetilde{\mathbf{h}} &:= \begin{pmatrix} \left\|\mathbf{g}^k\right\|^2 \\ \left\langle \mathbf{g}^k, \mathbf{m}^k\right\rangle \end{pmatrix} \\ \mathbf{M} &:= \begin{pmatrix} \left\|\left(\mathbf{X}^k\right)^{-1}\mathbf{g}^k\right\|^2 & \left\langle \mathbf{g}^k, \left(\mathbf{X}^k\right)^{-2}\mathbf{m}^k\right\rangle \\ \left\langle \mathbf{m}^k, \left(\mathbf{X}^k\right)^{-2}\mathbf{g}^k\right\rangle & \left\|\left(\mathbf{X}^k\right)^{-1}\mathbf{m}^k\right\|^2 \end{pmatrix}. \end{aligned}$$

Note that $\nabla^2_{\mathbf{x},\mathbf{x}}\varphi\left(\mathbf{x}^k\right) = -\frac{\rho\nabla f\left(\mathbf{x}^k\right)\nabla f\left(\mathbf{x}^k\right)^\top}{f\left(\mathbf{x}^k\right)^2} + \rho\frac{\mathbf{A}^\top\mathbf{A}}{f\left(\mathbf{x}^k\right)} + \left(\mathbf{X}^k\right)^{-2}$ and we evaluate the above relations via

$$
\begin{aligned}
\left\langle \mathbf{a}, \nabla^2_{\mathbf{x},\mathbf{x}} \varphi\left(\mathbf{x}^k\right) \mathbf{a} \right\rangle &= \left\langle \mathbf{a}, -\frac{\rho \nabla f\left(\mathbf{x}^k\right) \nabla f\left(\mathbf{x}^k\right)^\top \mathbf{a}}{f\left(\mathbf{x}^k\right)^2} \right\rangle + \frac{\|\mathbf{A}\mathbf{a}\|^2}{f\left(\mathbf{x}^k\right)} + \left\| \left(\mathbf{X}^k\right)^{-1} \mathbf{a} \right\|^2 \\
&= -\rho \left(\frac{\nabla f\left(\mathbf{x}^k\right)^\top \mathbf{a}}{f\left(\mathbf{x}^k\right)} \right)^2 + \frac{\|\mathbf{A}\mathbf{a}\|^2}{f\left(\mathbf{x}^k\right)} + \left\| \left(\mathbf{X}^k\right)^{-1} \mathbf{a} \right\|^2 \\
\left\langle \mathbf{a}, \nabla^2_{\mathbf{x},\mathbf{x}} \varphi\left(\mathbf{x}^k\right) \mathbf{b} \right\rangle &= \left\langle \mathbf{a}, -\frac{\rho \nabla f\left(\mathbf{x}^k\right) \nabla f\left(\mathbf{x}^k\right)^\top \mathbf{b}}{f\left(\mathbf{x}^k\right)^2} \right\rangle + \frac{\langle \mathbf{A}\mathbf{a}, \mathbf{A}\mathbf{b} \rangle}{f\left(\mathbf{x}^k\right)} + \left\langle \mathbf{a}, \left(\mathbf{X}^k\right)^{-2} \mathbf{b} \right\rangle \\
&= -\rho \left(\frac{\nabla f\left(\mathbf{x}^k\right)^\top \mathbf{a}}{f\left(\mathbf{x}^k\right)} \right) \left(\frac{\nabla f\left(\mathbf{x}^k\right)^\top \mathbf{b}}{f\left(\mathbf{x}^k\right)} \right) + \frac{\langle \mathbf{A}\mathbf{a}, \mathbf{A}\mathbf{b} \rangle}{f\left(\mathbf{x}^k\right)} + \left\langle \mathbf{a}, \left(\mathbf{X}^k\right)^{-2} \mathbf{b} \right\rangle .
\end{aligned}
$$

To ensure feasibility, we always choose $\Delta \leq 1$ and adjust it based on the trust-region rule.

2 Potential Reduction for LP

In this section, we discuss the potential reduction method on LP HSD model.

$$
\begin{array}{cc}
\min\limits_{\mathbf{x} \in \mathbb{R}^n} & \mathbf{c}^\top \mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{array}
$$

$$
\begin{array}{cc}
\max\limits_{\mathbf{y}} & \mathbf{b}^\top \mathbf{y} \\
\text{subject to} & \mathbf{A}^\top \mathbf{y} + \mathbf{s} = \mathbf{c} \\
& \mathbf{s} \geq \mathbf{0}
\end{array}
$$

and

$$
\begin{aligned}
\mathbf{A}\mathbf{x} - \mathbf{b}\tau &= \mathbf{0} \\
-\mathbf{A}^\top \mathbf{y} - \mathbf{s} + \mathbf{c}\tau &= \mathbf{0} \\
\mathbf{b}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x} - \kappa &= 0 \\
\mathbf{e}_n^\top \mathbf{x} + \mathbf{e}_n^\top \mathbf{s} + \kappa + \tau &= 1
\end{aligned}
$$

2.1 Potential Reduction for HSD

In this section we consider the original HSD formulation

$$
\begin{aligned}
\mathbf{A}\mathbf{x} - \mathbf{b}\tau &= \mathbf{0} \\
-\mathbf{A}^\top \mathbf{y} - \mathbf{s} + \mathbf{c}\tau &= \mathbf{0} \\
\mathbf{b}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x} - \kappa &= 0 \\
\mathbf{e}_n^\top \mathbf{x} + \mathbf{e}_n^\top \mathbf{s} + \kappa + \tau &= 1
\end{aligned}
$$

and we have

$$
\begin{aligned}
\begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{A} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times 1} & -\mathbf{b} \\ -\mathbf{A}^\top & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{c} \\ \mathbf{b}^\top & -\mathbf{c}^\top & \mathbf{0}_{1\times n} & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \\ \mathbf{s} \\ \kappa \\ \tau \end{pmatrix} &= \mathbf{0} \\
\mathbf{e}_n^\top \mathbf{x} + \mathbf{e}_n^\top \mathbf{s} + \kappa + \tau &= 1.
\end{aligned}
$$

In this method, the dual variable $\mathbf{y}$ is free and needs special treatment. First we consider the potential function

$$
\begin{aligned}
f(\mathbf{x}, \mathbf{y}, \mathbf{s}, \kappa, \tau) &:= \frac{1}{2} \left\| \widetilde{\mathbf{A}} \mathbf{u} \right\|^2 \\
\varphi(\mathbf{x}, \mathbf{y}, \mathbf{s}, \kappa, \tau) &:= \rho \log(f(\mathbf{x})) - B(\mathbf{x}) - B(\mathbf{s}) - \log \kappa - \log \tau
\end{aligned}
$$

and

$$
\begin{aligned}
& \nabla f(\mathbf{x}, \mathbf{y}, \mathbf{s}, \kappa, \tau) \\
={}& \widetilde{\mathbf{A}}^\top \widetilde{\mathbf{A}} \mathbf{u} \\
={}& \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{A} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times 1} & -\mathbf{b} \\ -\mathbf{A}^\top & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{c} \\ \mathbf{b}^\top & -\mathbf{c}^\top & \mathbf{0}_{1\times n} & -1 & 0 \end{pmatrix}^\top \begin{pmatrix} \mathbf{A}\mathbf{x} - \mathbf{b}\tau & =: \mathbf{r}_1 \\ -\mathbf{A}^\top \mathbf{y} - \mathbf{s} + \mathbf{c}\tau & =: \mathbf{r}_2 \\ \mathbf{b}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x} - \kappa & =: r_3 \end{pmatrix} \\
={}& \begin{pmatrix} \mathbf{0}_{m\times m} & -\mathbf{A} & \mathbf{b} \\ \mathbf{A}^\top & \mathbf{0}_{n\times n} & -\mathbf{c} \\ \mathbf{0}_{n\times m} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times m} & \mathbf{0}_{1\times n} & -1 \\ -\mathbf{b}^\top & \mathbf{c}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -\mathbf{A}\mathbf{r}_2 + \mathbf{b} r_3 \\ \mathbf{A}^\top \mathbf{r}_1 - \mathbf{c} r_3 \\ -\mathbf{r}_2 \\ -r_3 \\ -\mathbf{b}^\top \mathbf{r}_1 + \mathbf{c}^\top \mathbf{r}_2 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla \varphi(\mathbf{u}) &= \frac{\rho \nabla f(\mathbf{u})}{f(\mathbf{u})} - \begin{pmatrix} \mathbf{X}^{-1}\mathbf{e} \\ \mathbf{0}_m \\ \mathbf{S}^{-1}\mathbf{e} \\ \kappa^{-1} \\ \tau^{-1} \end{pmatrix} \\
\nabla^2_{\mathbf{u},\mathbf{u}} \varphi(\mathbf{u}) &= -\frac{\rho \nabla f(\mathbf{u}) \nabla f(\mathbf{u})^\top}{f(\mathbf{u})^2} + \rho \frac{\widetilde{\mathbf{A}}^\top \widetilde{\mathbf{A}}}{f(\mathbf{u})} + \operatorname{diag} \begin{pmatrix} \mathbf{X}^{-2}\mathbf{e} \\ \mathbf{0}_m \\ \mathbf{S}^{-2}\mathbf{e} \\ \kappa^{-2} \\ \tau^{-2} \end{pmatrix}.
\end{aligned}
$$

In view of the subproblem update, we have

$$
\begin{aligned}
\min_{\mathbf{d}, \alpha^g, \alpha^m} \quad & \tfrac{1}{2} \mathbf{d}^\top \mathbf{H} \mathbf{d} + \mathbf{h}^\top \mathbf{d} \\
\text{subject to} \quad & \left\| \mathbf{X}^{-1} \mathbf{d}_{\mathbf{x}} \right\|^2 + \left\| \mathbf{S}^{-1} \mathbf{d}_{\mathbf{s}} \right\|^2 + \left| \kappa^{-1} \mathbf{d}_\kappa \right| + \left| \tau^{-1} \mathbf{d}_\tau \right| + \left\| \theta^{-1} \mathbf{d}_{\mathbf{y}} \right\|^2 \le \beta \le 1 \\
& \mathbf{d} = \alpha^g \mathbf{g}^k + \alpha^m \mathbf{m}^k.
\end{aligned}
$$

$$
\begin{array}{cc}
\min\limits_{\mathbf{d},\alpha^g,\alpha^m} & \frac{1}{2}\mathbf{d}^\top\mathbf{H}\mathbf{d}+\mathbf{h}^\top\mathbf{d} \\
\text{subject to} & \left\|\mathbf{X}^{-1}\mathbf{d}_\mathbf{x}\right\|^2+\left\|\mathbf{S}^{-1}\mathbf{d}_\mathbf{s}\right\|^2+\left|\kappa^{-1}\mathbf{d}_\kappa\right|^2+\left|\tau^{-1}\mathbf{d}_\tau\right|^2\le\beta_1\le 1 \\
 & \mathbf{d}=\alpha^g\mathbf{g}^k+\alpha^m\mathbf{m}^k.
\end{array}
$$

and

$$
\begin{array}{cc}
\min\limits_{\mathbf{d},\alpha^g,\alpha^m} & \frac{1}{2}\mathbf{d}^\top\mathbf{H}\mathbf{d}+\mathbf{h}^\top\mathbf{d} \\
\text{subject to} & \left\|\mathbf{U}^{-1}\mathbf{d}_\mathbf{u}\right\|\le\beta\le 1 \\
 & \mathbf{d}=\alpha^g\mathbf{g}^k+\alpha^m\mathbf{m}^k.
\end{array}
$$

$$
\left(\mathbf{b}^\top\mathbf{y}-\mathbf{c}^\top\mathbf{x}-\kappa\right)^2+\left\|-\mathbf{A}^\top\mathbf{y}-\mathbf{s}+\mathbf{c}\tau\right\|^2
$$

2.2 Further simplification

There are some basic operations to implement

Residual setup

$$
\begin{array}{rcl}
\mathbf{r}_1 & = & \mathbf{A}\mathbf{x}-\mathbf{b}\tau \\
\mathbf{r}_2 & = & -\mathbf{A}^\top\mathbf{y}-\mathbf{s}+\mathbf{c}\tau \\
r_3 & = & \mathbf{b}^\top\mathbf{y}-\mathbf{c}^\top\mathbf{x}-\kappa.
\end{array}
$$

Objective value

$$
f=\frac{1}{2}\left[\|\mathbf{r}_1\|^2+\|\mathbf{r}_2\|^2+r_3^2\right]
$$

Gradient setup

$$
\nabla f=\begin{pmatrix} -\mathbf{A}\mathbf{r}_2+\mathbf{b}r_3 \\ \mathbf{A}^\top\mathbf{r}_1-\mathbf{c}r_3 \\ -\mathbf{r}_2 \\ -r_3 \\ -\mathbf{b}^\top\mathbf{r}_1+\mathbf{c}^\top\mathbf{r}_2 \end{pmatrix}
$$

$$
\nabla\varphi=\frac{\rho\nabla f}{f}-\begin{pmatrix} \mathbf{X}^{-1}\mathbf{e} \\ \mathbf{0}_m \\ \mathbf{S}^{-1}\mathbf{e} \\ \kappa^{-1} \\ \tau^{-1} \end{pmatrix}
$$

Hessian (no actual setup)

$$\widetilde{\mathbf{A}}^\top\widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{0}_{m\times m} & -\mathbf{A} & \mathbf{b} \\ \mathbf{A}^\top & \mathbf{0}_{n\times n} & -\mathbf{c} \\ \mathbf{0}_{n\times m} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times m} & \mathbf{0}_{1\times n} & -1 \\ -\mathbf{b}^\top & \mathbf{c}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{A} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times 1} & -\mathbf{b} \\ -\mathbf{A}^\top & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{c} \\ \mathbf{b}^\top & -\mathbf{c}^\top & \mathbf{0}_{1\times n} & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{A}\mathbf{A}^\top + \mathbf{b}\mathbf{b}^\top & -\mathbf{b}\mathbf{c}^\top & \mathbf{A} & -\mathbf{b} & -\mathbf{A}\mathbf{c} \\ -\mathbf{c}\mathbf{b}^\top & \mathbf{A}^\top\mathbf{A} & \mathbf{0}_{n\times n} & \mathbf{c} & -\mathbf{A}^\top\mathbf{b} \\ \mathbf{A}^\top & \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} & -\mathbf{c} \\ -\mathbf{b}^\top & \mathbf{c}^\top & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times 1} \\ -\mathbf{c}^\top\mathbf{A}^\top & -\mathbf{b}^\top\mathbf{A} & -\mathbf{c}^\top & \mathbf{0}_{1\times 1} & \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \end{pmatrix}$$

$$\nabla^2\varphi = -\frac{\rho\nabla f\nabla f^\top}{f^2} + \frac{\rho\widetilde{\mathbf{A}}^\top\widetilde{\mathbf{A}}}{f} + \begin{pmatrix} \mathbf{X}^{-2} & & & & \\ & \mathbf{0}_m & & & \\ & & \mathbf{S}^{-2} & & \\ & & & \kappa^{-2} & \\ & & & & \tau^{-2} \end{pmatrix}.$$

Hessian-vector (with projection)

$$\mathbf{u} = \mathbf{x} - \frac{\mathbf{e}^\top\mathbf{x}}{n}\cdot\mathbf{e}$$

$$\nabla^2\varphi\mathbf{u} = -\frac{\rho\left(\nabla f^\top\mathbf{u}\right)}{f^2}\nabla f + \frac{\rho}{f}\widetilde{\mathbf{A}}^\top\left(\widetilde{\mathbf{A}}\mathbf{u}\right) + \begin{pmatrix} \mathbf{X}^{-2} & & & & \\ & \mathbf{0}_{m\times m} & & & \\ & & \mathbf{S}^{-2} & & \\ & & & \kappa^{-2} & \\ & & & & \tau^{-2} \end{pmatrix}\mathbf{u}.$$

Minimal eigenvalue

To evaluate the minimum eigen-value of $\mathbf{P}_\Delta\nabla^2\varphi\mathbf{P}_\Delta$ and the corresponding eigen-vector

$$\mathbf{X}\nabla^2\varphi\mathbf{X} = -\frac{4\rho\mathbf{X}\mathbf{A}^\top\mathbf{A}\mathbf{x}\mathbf{x}^\top\mathbf{A}^\top\mathbf{A}\mathbf{X}}{\|\mathbf{A}\mathbf{x}\|^4} + \frac{2\rho\mathbf{X}\mathbf{A}^\top\mathbf{A}\mathbf{X}}{\|\mathbf{A}\mathbf{x}\|^2} + \mathbf{I}$$

Note that

$$\begin{aligned} & \mathbf{X}\nabla^2\varphi\left(\mathbf{u}\right)\mathbf{X} \\ = \; & -\frac{\rho\mathbf{X}\nabla f\left(\mathbf{u}\right)\nabla f\left(\mathbf{u}\right)^\top\mathbf{X}}{f\left(\mathbf{u}\right)^2} + \frac{\rho\mathbf{X}\mathbf{A}^\top\mathbf{A}\mathbf{X}}{f\left(\mathbf{u}\right)} + \mathbf{D}. \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{v}} \quad & \left\langle\mathbf{X}\mathbf{v}, \nabla^2\varphi\left(\mathbf{u}\right)\mathbf{X}\mathbf{v}\right\rangle \\ \text{subject to} \quad & \mathbf{e}^\top\mathbf{X}\mathbf{v} = 0 \\ & \|\mathbf{X}\mathbf{v}\| = 1 \end{aligned}$$

$$\begin{array}{ll} \min_{\mathbf{v}} & \left\langle \mathbf{v}, \left(\mathbf{I} - \frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\right) \left(\mathbf{X}\nabla^2\varphi\left(\mathbf{u}\right)\mathbf{X}\right) \left(\mathbf{I} - \frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\right) \mathbf{v} \right\rangle \\ \text{subject to} & \|\mathbf{X}\mathbf{v}\| = 1 \end{array}$$

$$\begin{aligned} & \left(\mathbf{I} - \frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\right) \mathbf{H} \left(\mathbf{I} - \frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\right) \\ = \quad & \mathbf{H} - \frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\mathbf{H} - \mathbf{H}\frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2} + \mathbf{x}^\top\mathbf{H}\mathbf{x}\frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^4} \end{aligned}$$

3 Numerical Experiments

Problem	PInfeas	DInfeas.	Compl.	Problem	PInfeas	DInfeas.	Compl.
DLITTLE	1.347e-10	2.308e-10	2.960e-09	KB2	5.455e-11	6.417e-10	7.562e-11
AFIRO	7.641e-11	7.375e-11	3.130e-10	LOTFI	2.164e-09	4.155e-09	8.663e-08
AGG2	3.374e-08	4.859e-08	6.286e-07	MODSZK1	1.527e-06	5.415e-05	2.597e-04
AGG3	2.248e-05	1.151e-06	1.518e-05	RECIPELP	5.868e-08	6.300e-08	1.285e-07
BANDM	2.444e-09	4.886e-09	3.769e-08	SC105	7.315e-11	5.970e-11	2.435e-10
BEACONFD	5.765e-12	9.853e-12	1.022e-10	SC205	6.392e-11	5.710e-11	2.650e-10
BLEND	2.018e-10	3.729e-10	1.179e-09	SC50A	1.078e-05	6.098e-06	4.279e-05
BOEING2	1.144e-07	1.110e-08	2.307e-07	SC50B	4.647e-11	3.269e-11	1.747e-10
BORE3D	2.389e-08	5.013e-08	1.165e-07	SCAGR25	1.048e-07	5.298e-08	1.289e-06
BRANDY	2.702e-05	7.818e-06	1.849e-05	SCAGR7	1.087e-07	1.173e-08	2.601e-07
CAPRI	7.575e-05	4.488e-05	4.880e-05	SCFXM1	4.323e-06	5.244e-06	8.681e-06
E226	2.656e-06	4.742e-06	2.512e-05	SCORPION	1.674e-09	1.892e-09	1.737e-08
FINNIS	8.577e-07	8.367e-07	1.001e-05	SCTAP1	5.567e-07	8.430e-07	5.081e-06
FORPLAN	5.874e-07	2.084e-07	4.979e-06	SEBA	2.919e-11	5.729e-11	1.448e-10
GFRD-PNC	4.558e-05	1.052e-05	4.363e-05	SHARE1B	3.367e-07	1.339e-06	3.578e-06
GROW7	1.276e-04	4.906e-06	1.024e-04	SHARE2B	2.142e-04	2.014e-05	6.146e-05
ISRAEL	1.422e-06	1.336e-06	1.404e-05	STAIR	5.549e-04	8.566e-06	2.861e-05
STANDATA	5.645e-08	2.735e-07	5.130e-06	STANDGUB	2.934e-08	1.467e-07	2.753e-06
STOCFOR1	6.633e-09	9.701e-09	4.811e-08	VTP-BASE	1.349e-10	5.098e-11	2.342e-10

Table 1: Solving NETLIB LPs in **1000** iterations