Dimension-reduced Interior Point Method

Discussion 3

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The original direction composes of **projected gradient** and **momentum**

$$\begin{aligned}\mathbf{d}^k &\leftarrow \alpha^g \mathbf{P}_\Delta[\nabla\varphi(\mathbf{x}^k)] + \alpha^m(\mathbf{x}^k - \mathbf{x}^{k-1})\\ \mathbf{x}^k &\leftarrow \mathbf{x}^k + \mathbf{d}^k\end{aligned}$$

We can also consider **scaled projected gradient**

$$\mathbf{p}(\mathbf{x}^k) := \frac{\mathbf{X}^k\Big(\mathbf{I} - \frac{\mathbf{X}^k\mathbf{e}\mathbf{e}^\top\mathbf{X}^k}{\|\mathbf{x}^k\|^2}\Big)\mathbf{X}^k\nabla\varphi(\mathbf{x}^k)}{\Big\|\Big(\mathbf{I} - \frac{\mathbf{X}^k\mathbf{e}\mathbf{e}^\top\mathbf{X}^k}{\|\mathbf{x}^k\|^2}\Big)\mathbf{X}^k\nabla\varphi(\mathbf{x}^k)\Big\|}$$

and build up $\mathbf{d}^k$.

- Empirically better than projected gradient

(m, n)/Iteration	Projected gradient	Scaled Projected Gradient
$(50, 100)$	37	33
$(200, 1000)$	45	39
$(500, 2000)$	46	34

- Need more careful tuning for higher accuracy

A preliminary attempt by simplifying the dual problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \max_{\mathbf{y}} \quad & \mathbf{b}^\top \mathbf{y} & \qquad & \mathbf{b}^\top \mathbf{y}^+ - \mathbf{b}^\top \mathbf{y}^- \\ \text{subject to} \quad & \mathbf{A}^\top \mathbf{y} + \mathbf{s} = \mathbf{c} & & \mathbf{A}^\top \mathbf{y}^+ - \mathbf{A}^\top \mathbf{y}^- + \mathbf{s} = \mathbf{c} \\ & \mathbf{s} \geq \mathbf{0} & & \mathbf{y}^+, \mathbf{y}^-, \mathbf{s} \geq \mathbf{0} \end{aligned}$$

and

$$\begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{A} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times 1} & -\mathbf{b} \\ -\mathbf{A}^\top & \mathbf{A}^\top & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{c} \\ \mathbf{b}^\top & -\mathbf{b}^\top & -\mathbf{c}^\top & \mathbf{0}_{1\times n} & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y}^+ \\ \mathbf{y}^- \\ \mathbf{x} \\ \mathbf{s} \\ \kappa \\ \tau \end{pmatrix} = \mathbf{0}_{m+n+1},$$

Solving LPs

Simplex constrained QP formulation

$$\begin{aligned} \min_{\mathbf{u}=(\mathbf{y},\mathbf{x},\mathbf{s},\kappa,\tau)} \quad & f(\mathbf{u}) := \frac{1}{2}\|\hat{\mathbf{A}}\mathbf{u}\|^2 \\ \text{subject to} \quad & \mathbf{e}^\top \mathbf{u} = 1 \\ & \mathbf{u} \geq \mathbf{0}. \end{aligned}$$

Using the potential function

$$\varphi(\mathbf{u}) := \rho \log\left(f(\mathbf{u})\right) - B(\mathbf{y}^+) - B(\mathbf{y}^-) - B(\mathbf{x}) - B(\mathbf{s}) - \log\kappa - \log\tau$$

and apply the dimension-reduced method.

Preliminary results

Solving synthetic LPs, 20000 iterations

m	n	$\|\mathbf{Ax}-\mathbf{b}\|$	$\|\mathbf{A}^\top\mathbf{y}+\mathbf{s}-\mathbf{c}\|$	$\mathbf{c}^\top\mathbf{x}-\mathbf{b}^\top\mathbf{y}$
10	100	3e-06	4e-09	2e-05
50	200	6e-04	9e-07	2e-03
100	500	3e-05	3e-07	4e-04
500	1000	2e-03	5e-06	1e-03

Table 1. Synthetic tests

- Possible to solve LPs to low accuracy
- Needs much tuning for real-life LPs (e.g., `Netlib`)
- Or just using HSD without $\mathbf{y}=\mathbf{y}^+-\mathbf{y}^-$ / primal-dual potential function