Dimension-reduced Interior Point Method

Discussion 5

September 2, 2022

Negative Curvature of Hessian of Potential

Accelerate computation of negative curvature

$$\nabla_{\mathbf{x}\mathbf{x}}\varphi(\mathbf{x}) = -\frac{\rho\nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top}{f(\mathbf{x})^2} + \frac{\rho\mathbf{A}^\top\mathbf{A}}{f(\mathbf{x})} + \mathbf{X}^{-2}.$$

The most expensive step in potential reduction

$$\mathbf{A} = \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{A} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times 1} & -\mathbf{b} \\ -\mathbf{A}^\top & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{c} \\ \mathbf{b}^\top & -\mathbf{c}^\top & \mathbf{0}_{1\times n} & -1 & 0 \end{pmatrix}, \quad \nabla f(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

$$\mathbf{A}^\top\mathbf{A} = \begin{pmatrix} \mathbf{A}_l\mathbf{A}_l^\top + \mathbf{b}\mathbf{b}^\top & -\mathbf{b}\mathbf{c}^\top & \mathbf{A}_l & -\mathbf{b} & -\mathbf{A}_l\mathbf{c} \\ -\mathbf{c}\mathbf{b}^\top & \mathbf{A}_l^\top\mathbf{A}_l & \mathbf{0}_{n\times n} & \mathbf{c} & -\mathbf{A}_l^\top\mathbf{b} \\ \mathbf{A}^\top & \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} & -\mathbf{c} \\ -\mathbf{b}^\top & \mathbf{c}^\top & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times 1} \\ -\mathbf{c}^\top\mathbf{A}_l^\top & -\mathbf{b}^\top\mathbf{A}_l & -\mathbf{c}^\top & \mathbf{0}_{1\times 1} & \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \end{pmatrix}$$

Some properties

- Known lower bound of negative eigen-value

$$
\begin{aligned}
\nabla_{\mathbf{xx}}\varphi(\mathbf{x}) &= -\frac{\rho\nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top}{f(\mathbf{x})^2} + \frac{\rho\mathbf{A}^\top\mathbf{A}}{f(\mathbf{x})} + \mathbf{X}^{-2} \\
&\succeq -\frac{\rho\nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top}{f(\mathbf{x})^2} \\
&\succeq -\frac{\rho\|\mathbf{A}^\top\mathbf{A}\|}{f(\mathbf{x})}
\end{aligned}
$$

- One negative eigen-value in $\left[-\frac{\rho\|\mathbf{A}^\top\mathbf{A}\|}{f(\mathbf{x})}, 0\right)$

Therefore, we

- seek a unique eigenvalue in a known range
- usable algorithm: Lanczos, FEAST method

Computationally $\nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top$ is dense and needs special treatment.

More Agressive use of Direction

Currently the trust region radius is restricted by $\mathbf{x}+\mathbf{d} \geq \mathbf{0}$ and

$$\|\mathbf{X}^{-1}\mathbf{d}\| \leq \beta \leq 1.$$

- $\|\mathbf{X}^{-1}\mathbf{d}\| \geq \|\mathbf{X}^{-1}\mathbf{d}\|_\infty$ guarantees feasibility and trackability
- Too conservative in general and generally more aggressive steps can be taken

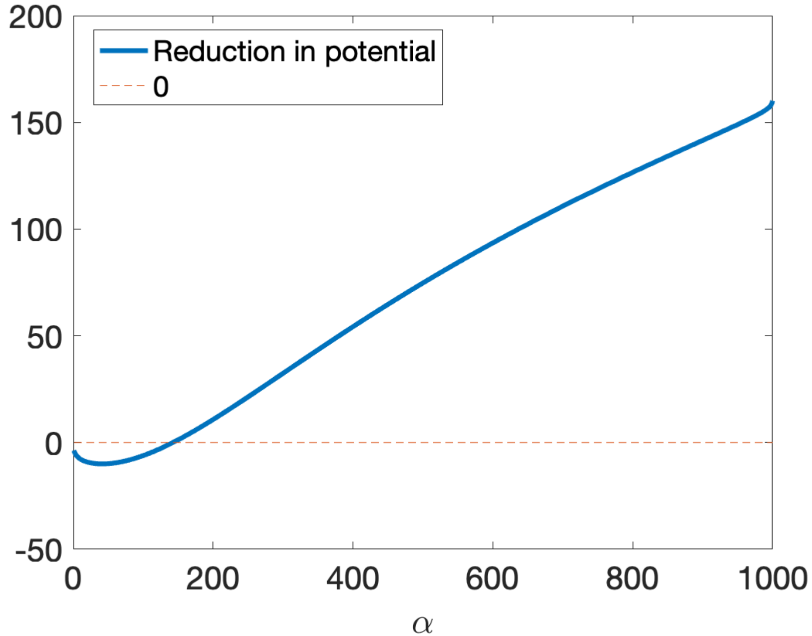

Figure 1. Better if larger step is taken

A Line-search Strategy

A linesearch strategy is employed

- A direction is assembled $\mathbf{d} = \alpha_g \nabla\varphi + \alpha_m \mathbf{m}^k$
- A linesearch is done to identify $\sigma \geq 1$ and

$$\varphi(\mathbf{x} + \sigma\mathbf{d}) \leq \varphi(\mathbf{x} + \mathbf{d})$$

- Reduce 30% $\sim$ 40% of iterations

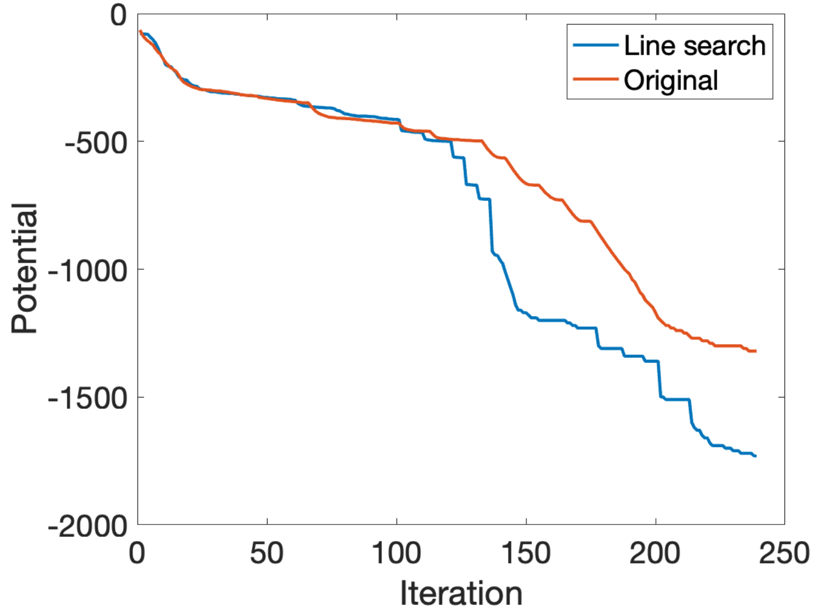

Figure 2. Reduction by line-search

Current work

Summary of the current results

- An algorithm solving LPs to 10^{-6} accuracy
- The most expensive step is to approximately find

$$\mathbf{v}_{\lambda_{\min}}\left\{-\frac{\rho\nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top}{f(\mathbf{x})^2}+\frac{\rho\mathbf{A}^\top\mathbf{A}}{f(\mathbf{x})}+\mathbf{X}^{-2}\right\}.$$

  $\mathcal{O}(n^2)$ complexity and matrix-free. Still looking for ways and structures to further improve.
- Now ready to transform Matlab into C implementation.

Adaptive ρ works but less significant compared to line-search.