LP and its potential function

We start from the simple case of simplex-constrained QP

$$\begin{aligned} \min \quad & f(x) = \frac{1}{2}\|Ax\|_2^2 \\ \text{s.t.} \quad & e^T x = 1, \quad x \geq 0. \end{aligned}$$

Define the potential function and its gradient and Hessian

$$\begin{aligned} &\phi(x) = \rho \log(f(x)) - \sum_{i=1}^{n} \log x_i, \\ &\nabla \phi(x) = \frac{\rho \nabla f(x)}{f(x)} - X^{-1} e, \\ &\nabla^2 \phi(x) = H = -\frac{\rho}{f(x)^2} \nabla f(x) \nabla f(x)^T + \rho \frac{A^T A}{f(x)} + X^{-2}. \end{aligned}$$

Full-Dimension DRSOM

Assume we solve the full-dimension problem in each iterate:

$$\begin{aligned} \min_{d} \quad & \nabla\phi(x)^T d + \frac{1}{2} d^T H d \\ \text{s.t.} \quad & e^T d = 0, \quad \|X^{-1} d\|_2 \le \beta. \end{aligned} \tag{1}$$

Theorem 1

By choosing $\beta \le c\sqrt{f(x)}$ for some $c > 0$, there exists some $M > 0$ such that

$$\phi(x+d) - \phi(x) \le \nabla\phi(x)^T d + \frac{1}{2} d^T H d + \frac{M\beta^3}{2f(x)^{\frac{3}{2}}} + \beta^3.$$

Proof of Theorem 1

Lemma 1

For any $x > -1$, we have

$$\left|\log(1+x) - x + \frac{x^2}{2}\right| \le \frac{|x|^3}{3}.$$

According to Lemma 1, we have

$$\log(f(x+d)) - \log(f(x)) = \log\left(1 + \frac{\nabla f(x)^T d}{f(x)} + \frac{\|Ad\|_2^2}{2f(x)}\right)$$

$$\le \frac{\nabla f(x)^T d}{f(x)} + \frac{\|Ad\|_2^2}{2f(x)} - \frac{d\nabla f(x)\nabla f(x)^T d}{2f(x)^2} + \frac{M\beta^3}{2f(x)^{\frac{3}{2}}}$$

and

$$-\sum_{i=1}^{n}\log(x_i + d_i) + \sum_{i=1}^{n}\log(x_i) = -\sum_{i=1}^{n}\log(1 + \frac{d_i}{x_i})$$

$$\le -d^T X^{-1} e + d^T X^{-2} d + \sum_{i=1}^{n}\left|\frac{d_i}{x_i}\right|^3 \le -d^T X^{-1} e + d^T X^{-2} d + \beta^3.$$

Optimality Condition

The problem (1) can be reformulated as

$$\begin{aligned} \min_{d} \quad & \nabla\phi(x)^T X d' + \frac{1}{2} d'^T Q d' \\ \text{s.t.} \quad & e^T X d' = 0, \quad \|d'\|_2 \le \beta. \end{aligned} \tag{2}$$

where $Q = XHX$. Its optimality condition is

$$\begin{aligned} & (Q + \mu I)d' = -X(\nabla\phi(x) - \lambda e), \\ & e^T X d' = 0, \quad \|d'\|_2 \le \beta \\ & Q + \mu I \succeq 0, \\ & \mu \ge 0, \quad \mu(\beta - \|d'\|_2) = 0. \end{aligned}$$

Decrease in each step

Firstly,

$$\nabla \phi(x)^T X d' + \frac{1}{2} d'^T Q d' = -d'^T (Q + \mu I) d' + \frac{1}{2} d'^T Q d' \leq -\frac{\mu}{2} \|d'\|_2^2.$$

Assumption 1 (Scaled Lipschitz)

The scaled hessian $Q = XHX$ is bouned by

$$\|Q\|_2 \leq c f(x)^{-\frac{3}{4}}.$$

According to Ye(2005),

$$\|(Q + \mu I) d'\| = \|X(\nabla \phi(x) - \lambda e)\|_2 \geq 1.$$

By choosing $\beta \leq \frac{f(x)^{\frac{3}{4}}}{2c}$, then

$$1 \leq \|(Q + \mu I) d'\| \leq (\|Q\|_2 + \mu)\beta \leq \frac{1}{2} + \mu\beta$$

which implies $\mu \geq 1/2\beta$.

Decrease in each step

Thus, if $\beta \leq \frac{f(x)^{\frac{3}{4}}}{2c}$, we have

$$\nabla \phi(x)^T X d' + \frac{1}{2} d'^T Q d' \leq -\frac{\beta}{4}.$$

By choosing $\beta = O(f(x)^{\frac{3}{4}})$, theorem 1 shows

$$\phi(x+d) - \phi(x) \leq -\frac{\beta}{4} + \frac{M\beta^3}{2f(x)^{\frac{3}{2}}} + \beta^3 \leq -c_0 f(x)^{\frac{3}{4}}.$$

If $f(x^k) \geq \epsilon f(x^0)$ for $k = 1, \ldots, K$, we must have

$$\rho \log(\frac{1}{\epsilon}) \geq \phi(x^0) - \phi(x^K) \geq \sum_{k=1}^{K} c_0 f(x^k) \geq c_0 f(x_0)^{\frac{3}{4}} \epsilon^{\frac{3}{4}} K$$

that is,

$$K \leq \frac{\rho/(c_0 f(x^0)^{\frac{3}{4}})}{\epsilon^{\frac{3}{4}}} \log(\frac{1}{\epsilon}).$$