Class 08: Confidence Intervals

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Agenda

- Confidence Intervals
  - Analytic
  - Bootstraps
- Working Groups
- Looking Ahead

A survey

Define. . .

- confidence interval
- estimand
- estimator

Confidence Intervals

Some definitions:

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  - A function from real numbers to probabilities.
  - Usually denoted by capital latin letters.
  - Random variables have population properties, $P(X = 4)$, $E[X]$, $Var(X)$. These are normally denoted by greek letters (μ, σ).

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- Estimator
  - The formula for constructing an estimate.

Estimators

estimand

Ingredients

- 150g unsalted butter, plus extra for greasing
- 150g plain chocolate, broken into pieces
- 150g plain flour
- ½ tsp baking powder
- ½ tsp bicarbonate of soda
- 200g light muscovado sugar
- 2 large eggs

Method

1. Heat the oven to 160C/140C fan/gas 3. Grease and base line a 1 litre heatproof glass pudding basin and a 450g loaf tin with baking parchment.

2. Put the butter and chocolate into a saucepan and melt over a low heat, stirring. When the chocolate has all melted remove from the heat.

estimator

estimate

An example

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What can we say about the estimand given this estimator?

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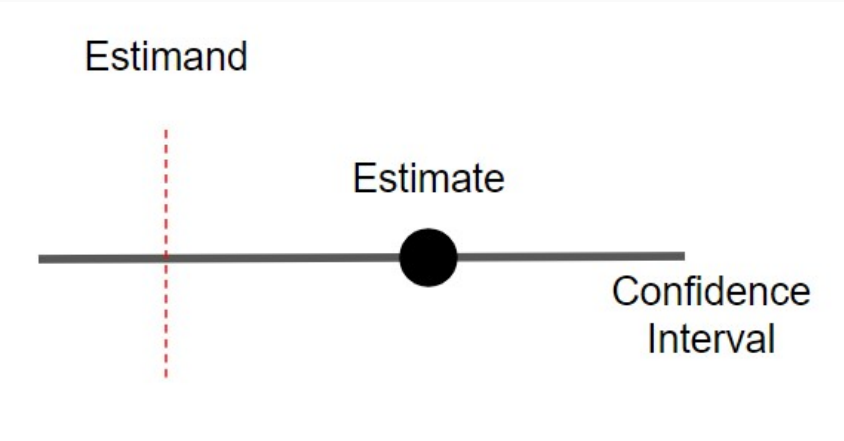
- We make assumptions about the population distribution.
- We can use math ("analytic") or computing ("bootstrapping") to say how likely it would be to see the data that we saw, given values of the population estimand.

Confidence Intervals

Confidence Interval: An interval calculated on the sample such that, if you were to resample multiple times, they would cover the true estimand α fraction of the time.

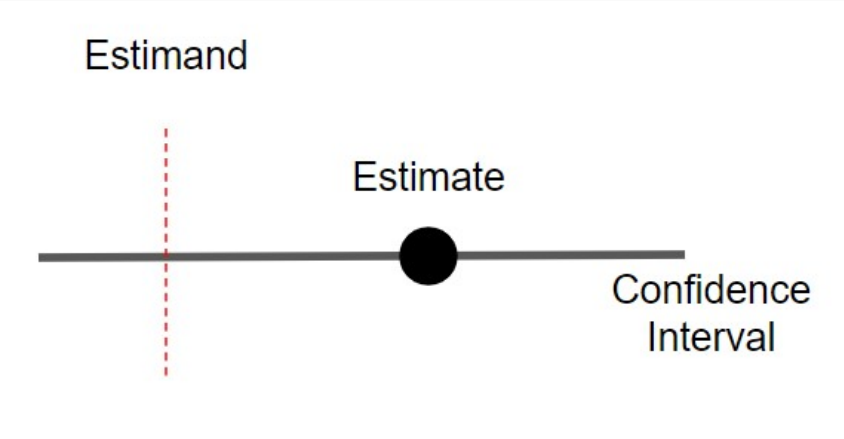
For example, a 95% confidence interval should cover the true value 95% of the time across samples. (of course, we only ever see one.)



Confidence Intervals

Confidence intervals. . .

- are Random Variables themselves.
- *either cover or don't for one sample*
  - the probabilistic interpretation is over repeated samples.
- Can have better or worse precision.
- May rely on assumptions that are good or bad.



When you're estimating a model.

If you're having trouble thinking about your confidence intervals, ask yourself some of the following:

- What would it mean to draw another random sample of your data?
- What is your population that your sample is a subset of?

A third example

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Two methods to calculate confidence intervals

- Analytic
- Bootstrapping (computational)

Analytic Confidence Intervals

Use math and assumptions to estimate what the variance of your estimator is.

For example:

- Suppose we want to know the mean $E[X]$.

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- If we assume the Central Limit Theorem, then $\bar{X} \sim N(E[X], Var(X)/N)$
- Therefore, there is a 95% probability that $\bar{X}$ will be between $\mu_X - 1.96\sqrt{Var(X)/N}$ and $\mu_X + 1.96\sqrt{Var(X)/N}$.

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- We can plug in the estimated value of $\widehat{Var(X)}$ (call it $\hat{\sigma}^2_X$), to get $(\bar{x} - 1.96\hat{\sigma}_X/\sqrt{N}, \bar{x} + 1.96\hat{\sigma}_X/\sqrt{N})$

Analytic Confidence Intervals

Benefits

- Quick to compute once you've done the math.
- Well studied, well-known assumptions.

Drawbacks

- Need to do hard math if they aren't already solved.
- Assumptions may not hold.

Bootstrapped Confidence Intervals

Efron (1979): - Instead of focusing $\bar{X}$ as our estimate, consider the whole sample x as an estimate of the population distribution. ($\hat{F}(X)$) - We can now actually sample infinitely many of our estimates, and look at their distribution.

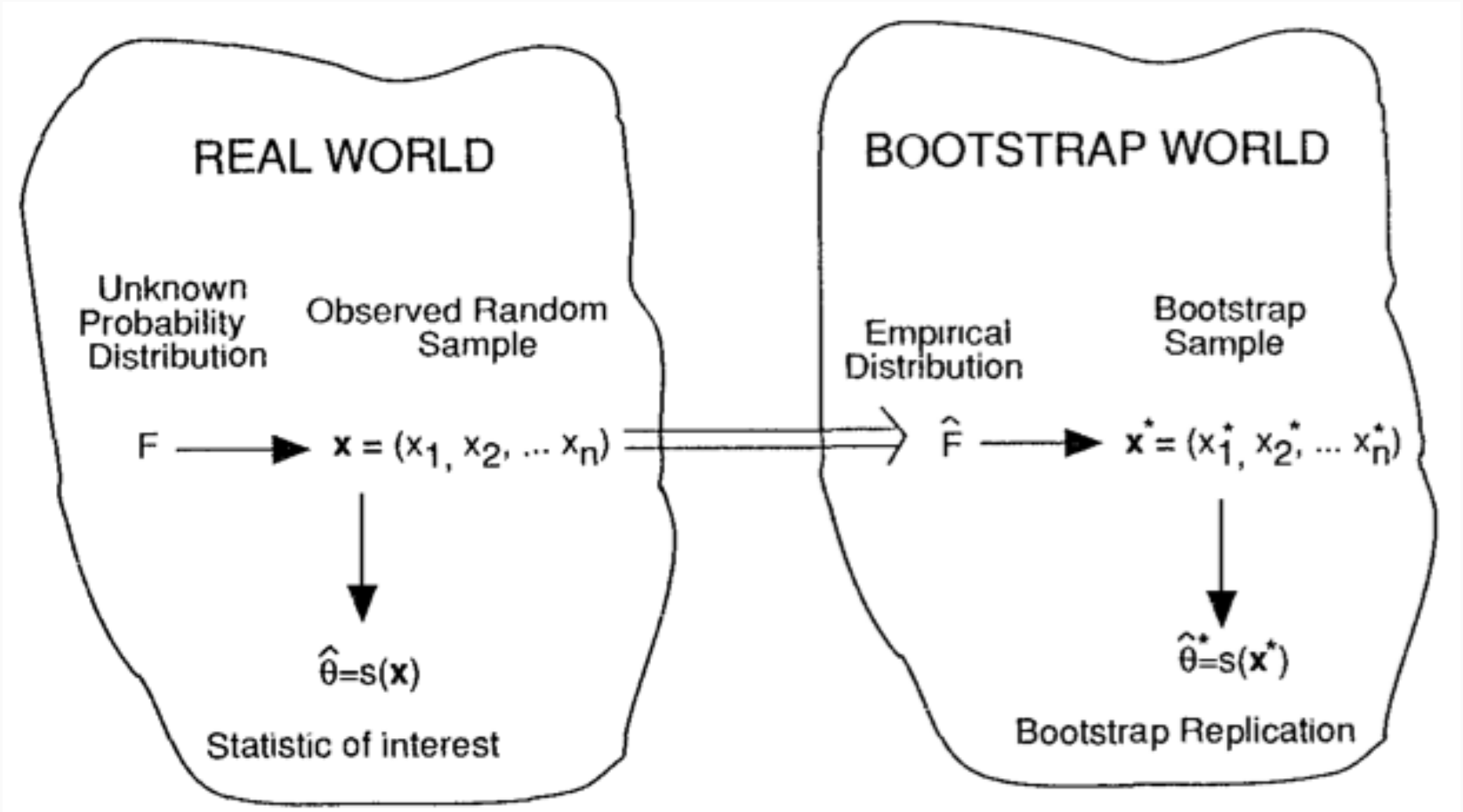
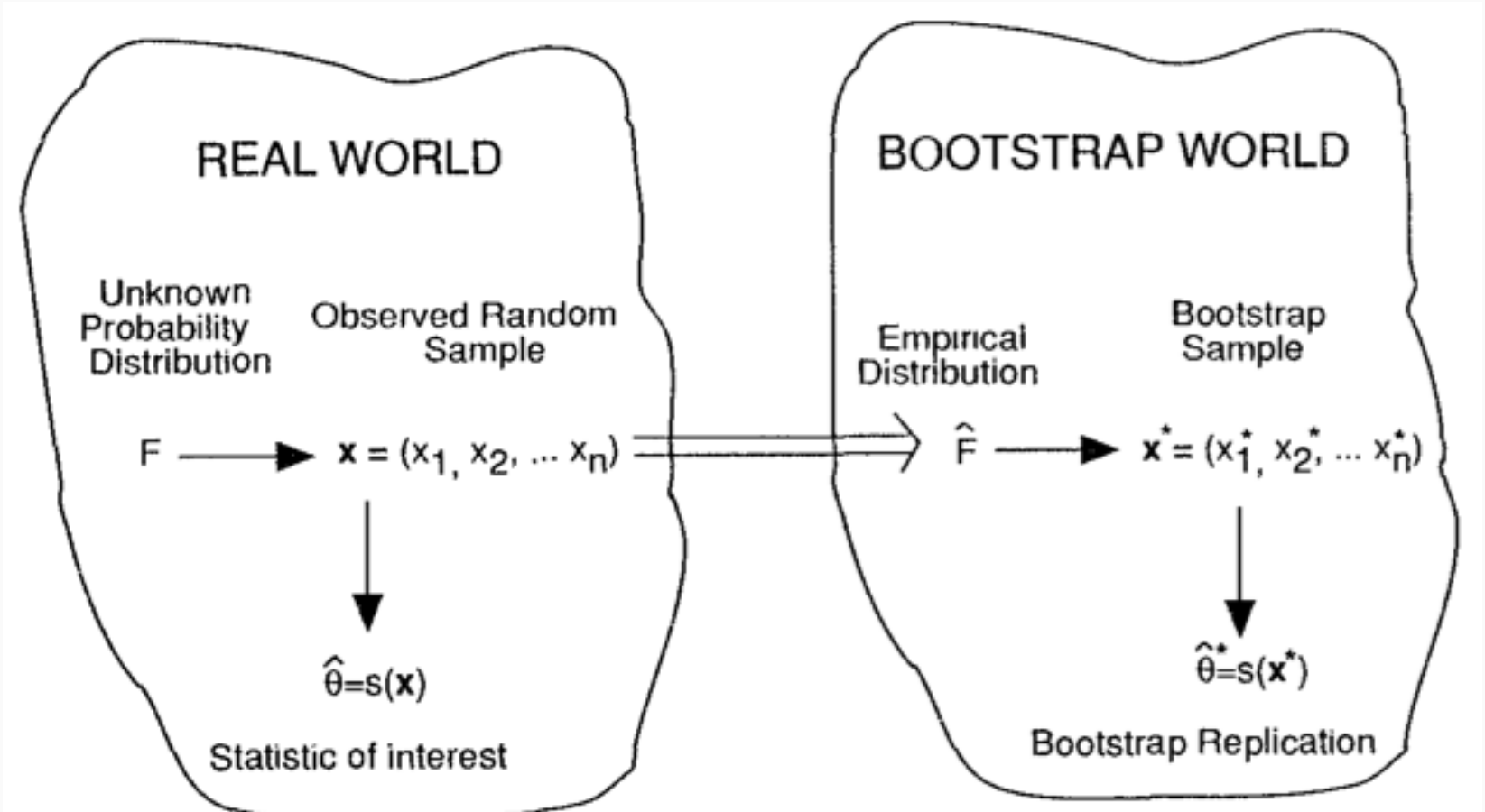


Figure 1: Efron & Tibishirani 1993

Bootstrapped Confidence Intervals

Percentile Bootstrap:

- Take your (single) sample of N.
- Construct your estimate $\hat{theta}$.
- For N_{boot} times...
  - Resample N items from your sample (with replacement).
  - Calculate $\hat{theta}_b$ on that sample.
- A 95% confidence interval is the 2.5th and 97.5th percentile of those bootstraps.



Percentile Bootstrap Confidence Intervals

Benefits

- Easy to calculate without math.
- Quite robust to skew in the data, complicated metrics.

Drawbacks

- Computationally intensive.
- Edge cases where they fail.

The usefulness of Simulations

- You can “play god” by drawing many samples.
- Understand distributions and performance.
- Can test edge cases.

Demo!

Looking Ahead

- Sign-Ups: Mid-Point Presentation dates.
  - Will be assigned two peers to provide feedback.
  - Submit to Canvas. I will share with author.
  - I will share a prompt shortly.
- 03-18 Mid-Point Presentations A, Intro to Code Review
- 03-25 Review of two peer projects, Mid-Point Presentations B

Working Groups

- In groups of 3. . .
  - Share the current state of your draft.
  - What is the state of your analysis?
  - What is your current blocker?
  - What is the current weakest link in your project?

Each person gets 10 minutes to discuss and actively debug.