Orthogonal Projections

Linear Algebra

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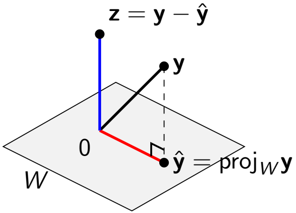
https://github.com/CU-Denver-MathStats-OER

Orthogonal Projection onto a Subspace

Given a subspace W of $\mathbb{R}^n$, in many instances it is useful to decompose a vector $\mathbf{y}$ in $\mathbb{R}^n$ into the sum of two vectors $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$:

- The orthogonal projection of $\mathbf{y}$ onto W is $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$.
- The component of $\mathbf{y}$ orthogonal to W is $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.



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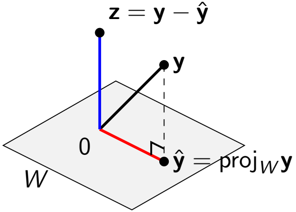

Theorem (The Orthogonal Decomposition Theorem)

Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$. If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal basis of W, then

$$\hat{\mathbf{y}} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_p\mathbf{u}_p \text{ with weights } c_i = \frac{\mathbf{y}\cdot\mathbf{u}_i}{\mathbf{u}_i\cdot\mathbf{u}_i}, \quad \text{and } \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

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Proof. Note that $\hat{\mathbf{y}}$ is in W, and that $\hat{\mathbf{y}}$ and $\mathbf{z}$ sum to $\mathbf{y}$. It remains to show that $\mathbf{z}$ is in $W^\perp$.

$$\begin{aligned}\mathbf{z}\cdot\mathbf{u}_i = (\mathbf{y}-\hat{\mathbf{y}})\cdot\mathbf{u}_i &= (\mathbf{y}\cdot\mathbf{u}_i) - \hat{\mathbf{y}}\cdot\mathbf{u}_i \\ &= (\mathbf{y}\cdot\mathbf{u}_i) - [c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_p\mathbf{u}_p]\cdot\mathbf{u}_i\end{aligned}$$

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&= (\mathbf{y}\cdot\mathbf{u}_i) - c_1(\mathbf{u}_1\cdot\mathbf{u}_i) - \ldots - c_i(\mathbf{u}_i\cdot\mathbf{u}_i) - \ldots - c_p(\mathbf{u}_p\cdot\mathbf{u}_i)
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&= (\mathbf{y}\cdot\mathbf{u}_i) - c_1(\mathbf{u}_1\cdot\mathbf{u}_i) - \ldots - c_i(\mathbf{u}_i\cdot\mathbf{u}_i) - \ldots - c_p(\mathbf{u}_p\cdot\mathbf{u}_i) \\
&= (\mathbf{y}\cdot\mathbf{u}_i) - c_i(\mathbf{u}_i\cdot\mathbf{u}_i) = (\mathbf{y}\cdot\mathbf{u}_i) - \left(\frac{\mathbf{y}\cdot\mathbf{u}_i}{\mathbf{u}_i\cdot\mathbf{u}_i}\right)(\mathbf{u}_i\cdot\mathbf{u}_i) = 0.
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Since $\mathbf{z}$ is orthogonal to each $\mathbf{u}_i$ in the basis, $\mathbf{z}$ is in $W^\perp$ by a previous theorem.

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Proof.

For uniqueness, suppose that $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \hat{\mathbf{y}}_1 + \mathbf{z}_1$. Then $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1 = \mathbf{z}_1 - \mathbf{z}$.

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But $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1 \in W$ and $\mathbf{z}_1 - \mathbf{z} \in W^{\perp}$, which implies that $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1 = \mathbf{z}_1 - \mathbf{z} = \mathbf{0}$.

Thus, $\hat{\mathbf{y}}_1 = \hat{\mathbf{y}}$ and $\mathbf{z}_1 = \mathbf{z}$. Therefore, $\hat{\mathbf{y}}$ and $\mathbf{z}$ are the only vectors satisfying the conditions. □

Example

Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ where $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$. Write $\mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$ as a sum of a vector in W and a vector orthogonal to W.

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The orthogonal projection of $\mathbf{y}$ onto W is $\hat{\mathbf{y}} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ where we have weights

$$c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{9}{9} = 1 \text{ and } c_2 = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{26}{20} = \frac{13}{10}.$$

This gives

$$\hat{\mathbf{y}} = (1)\mathbf{u}_1 + \left(\frac{13}{10}\right)\mathbf{u}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \frac{13}{10}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -13/5 \\ 3 \\ 26/5 \end{bmatrix},$$

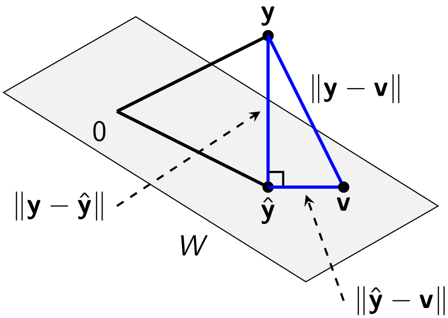
and thus

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} -13/5 \\ 3 \\ 26/5 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 0 \\ 4/5 \end{bmatrix}.$$

Theorem (The Best Approximation Theorem)

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \quad \text{for all } \mathbf{v} \neq \mathbf{y} \text{ in } W.$$



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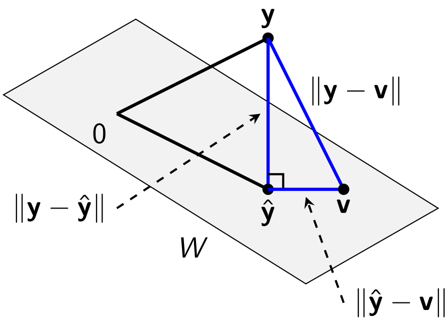
Proof.

For any $\mathbf{v} \in W$, $\hat{\mathbf{y}} - \mathbf{v}$ is orthogonal to $\mathbf{y} - \hat{\mathbf{y}}$.



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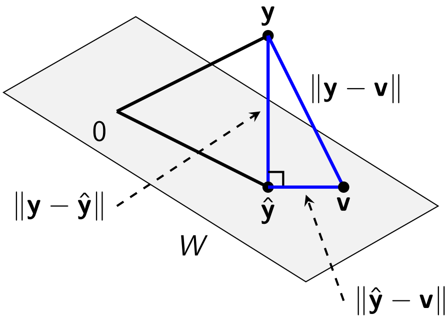
For any $\mathbf{v} \in W$, $\hat{\mathbf{y}}-\mathbf{v}$ is orthogonal to $\mathbf{y}-\hat{\mathbf{y}}$.
By Pyth., $\|\mathbf{y}-\mathbf{v}\|^2 = \|\mathbf{y}-\hat{\mathbf{y}}\|^2 + \|\hat{\mathbf{y}}-\mathbf{v}\|^2$.
So $\|\mathbf{y}-\mathbf{v}\| > \|\mathbf{y}-\hat{\mathbf{y}}\|$ for $\mathbf{v} \neq \hat{\mathbf{y}}$. □



The Best Approximation Theorem

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For any $\mathbf{v} \in W$, $\hat{\mathbf{y}} - \mathbf{v}$ is orthogonal to $\mathbf{y} - \hat{\mathbf{y}}$.
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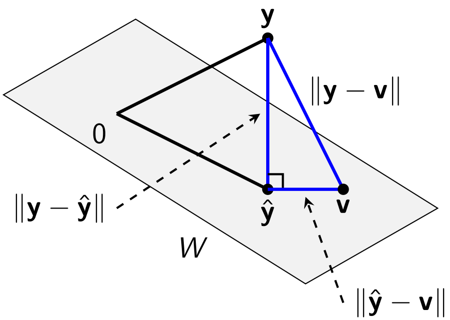

- The vector $\hat{\mathbf{y}}$ is called the best approximation to $\mathbf{y}$ by elements of W.
- The distance $\|\mathbf{y} - \hat{\mathbf{y}}\| = \|\mathbf{z}\|$ is the error of the approximation.

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$. Find the best approximation of $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ in the subspace $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

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Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$. Find the best approximation of $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ in the subspace $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

From the Best Approximation Theorem, we know that the orthogonal projection of $\mathbf{y}$ onto W is the closest point in W to $\mathbf{y}$. The orthogonal projection of $\mathbf{y}$ onto W is $\hat{\mathbf{y}} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ where we have weights $c_1 = \dfrac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1} = \dfrac{0}{14}$ and $c_2 = \dfrac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2} = \dfrac{28}{42} = \dfrac{2}{3}$.

Therefore, the best approximation for $\mathbf{y}$ in W is

$$\hat{\mathbf{y}} = (0)\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} \quad \text{with} \quad \|\mathbf{z}\| = \left\| \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix} \right\| \approx 4.0415.$$

Example

Let $\mathbf{u}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and define the subspace $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ of $\mathbb{R}^3$.

Let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$. Compute $U^T U$ and UU^T.

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We have

$$U^T U = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So U has orthonormal columns!

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$$UU^T = \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 5/9 & 2/9 & 4/9 \\ 2/9 & 8/9 & -2/9 \\ 4/9 & -2/9 & 5/9 \end{bmatrix}$$

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Let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$. Compute $(UU^T)\mathbf{y}$ and $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$ for $\mathbf{y} = (1, -2, 4)$.

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We have

$$(UU^T)\mathbf{y} = \begin{bmatrix} 5/9 & 2/9 & 4/9 \\ 2/9 & 8/9 & -2/9 \\ 4/9 & -2/9 & 5/9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 17/9 \\ -22/9 \\ 28/9 \end{bmatrix},$$

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Let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$. Compute $(UU^T)\mathbf{y}$ and $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$ for $\mathbf{y} = (1, -2, 4)$.

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and

$$\hat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 = \frac{13}{3}\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 = \frac{13}{3}\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 17/9 \\ -22/9 \\ 28/9 \end{bmatrix}.$$

Orthogonal projection onto a Subspace using an Orthonormal Basis

The formula for $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$ is simplified when the basis for W is an orthonormal basis.

Theorem

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \ldots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p = UU^T\mathbf{y} \quad \text{for all } \mathbf{y} \text{ in } \mathbb{R}^n,$$

where the matrix $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p \end{bmatrix}$.

The matrix UU^T is called the orthogonal projection matrix onto W.

Orthogonal projection onto a Subspace using an Orthonormal Basis

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Proof.

Note that $U^T\mathbf{y} = \begin{bmatrix} \mathbf{u}_1^T\mathbf{y} \\ \mathbf{u}_2^T\mathbf{y} \\ \vdots \\ \mathbf{u}_p^T\mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{y} \\ \mathbf{u}_2 \cdot \mathbf{y} \\ \vdots \\ \mathbf{u}_p \cdot \mathbf{y} \end{bmatrix}$. Thus, $UU^T\mathbf{y} = (\mathbf{u}_1 \cdot \mathbf{y})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{y})\mathbf{u}_2 + \ldots + (\mathbf{u}_p \cdot \mathbf{y})\mathbf{u}_p$.