Applications to Computer Graphics

Linear Algebra

This work was initially funded by an Institutional OER Grant from the Colorado Department of Higher Education (CDHE). For similar OER materials in other courses funded by this project in the Department of Mathematical and Statistical Sciences at the University of Colorado Denver, visit
https://github.com/CU-Denver-MathStats-OER

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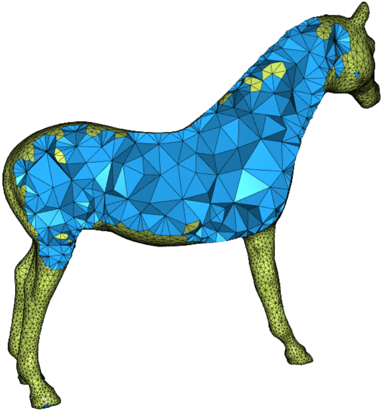


https://www.cg.tu-berlin.de/research/projects/harmonic-triangulations/

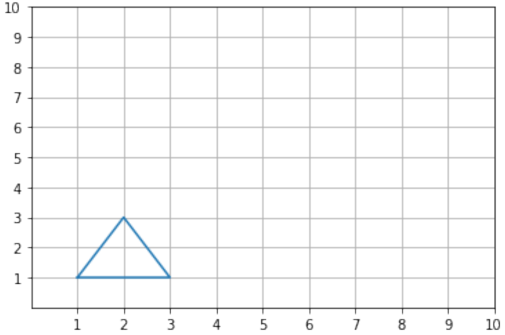
Image Credit: Star Wars low poly portraits, designed by Vladan Filipovic

Let's examine some ways to manipulate and display graphical images using matrices.

Scaling One Triangle

Consider one triangular piece. If we want to scale the triangle by a factor of three, then we apply the transformation

$$S : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}.$$



We have:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

Vertical Scaling

If we would like to vertically scale the triangle by a factor of 2, we have

$$V : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

We have:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

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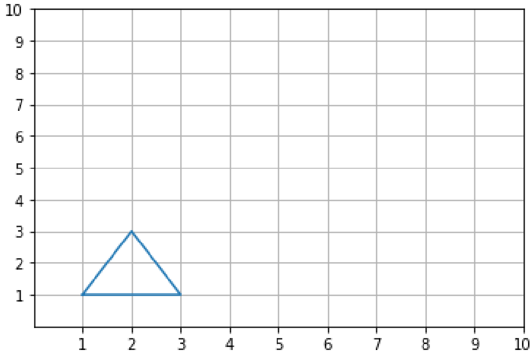


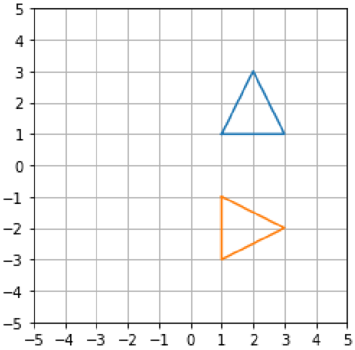
Rotation

If we would like to rotate the triangle counter-clockwise by an angle of θ around the origin, we have

$$R : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}.$$

For example a counterclockwise rotation by $\theta = \frac{3\pi}{2}$ would be given by

$$R : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$



We have:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Composing Maps

Imagine we want to apply three transformations to the triangle in the following order:

1. Scale the triangle by a factor of three.
2. Then vertically scale by a factor of 2 (in vertical direction only).
3. Finally rotate counter-clockwise around the origin by $\theta = 270^\circ = \frac{3\pi}{2}$ radians.

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First apply the scaling matrix using matrix S:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Then apply the vertical scaling using V:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Finally, we rotate this result counter-clockwise by $\theta = \frac{3\pi}{2}$ using R:

$$\begin{bmatrix} \cos\left(\frac{3\pi}{2}\right) & -\sin\left(\frac{3\pi}{2}\right) \\ \sin\left(\frac{3\pi}{2}\right) & \cos\left(\frac{3\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

The image under the composition of these three transformations is $R\Big(V\big(S(\mathbf{x})\big)\Big) = (RVS)(\mathbf{x})$

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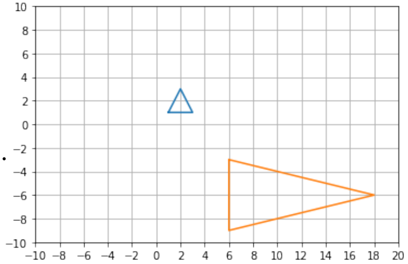
$$M : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto RVS\mathbf{x}.$$

where

$$RVS = \begin{bmatrix} \cos\left(\frac{3\pi}{2}\right) & -\sin\left(\frac{3\pi}{2}\right) \\ \sin\left(\frac{3\pi}{2}\right) & \cos\left(\frac{3\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -3 & 0 \end{bmatrix}.$$



Thus we have

$$\begin{bmatrix} 0 & 6 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Composing Linear Transformations

The composition of linear transformations with associated matrices A, B, and C in the following order:

1. first apply transformation given by matrix A,
2. then apply transformation given by matrix B, and
3. finally apply transformation given by matrix C

is equivalent to the linear transformation with associated matrix given by CBA.

Note we can extend this idea to compose any number of linear transformations.

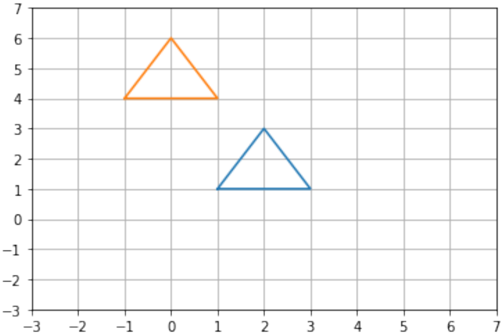
Translations

To translate the vertex of the triangle at $(1, 1)$ to the left by 2 units and up by 3 units, we add vectors:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Doing the same for the other two vertices, we have

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$



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- How can we compose translations with the other linear transformations if translation cannot be represented as matrix multiplication?

One common way is to introduce homogeneous coordinates:

- Each point (x, y) in $\mathbb{R}^2$ can be identified with the point $(x, y, 1)$ in $\mathbb{R}^3$.
- We say (x, y) has homogeneous coordinates $(x, y, 1)$.

Homogeneous Coordinates

Now we can perform translation by matrix multiplication. For example, if we want to shift the point $(3, 1)$ to the left by 2 units and up by the 3 units, then we have the product

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 1+3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

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In general, if we want to translate by h units in the horizontal direction and k units in the vertical direction, using homogeneous coordinates we have:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}.$$

Let's perform a composition of three different transformations to the triangle in the following order:

1. Scale the triangle by a factor of 3.
2. Rotate the triangle counter-clockwise around the origin by $\frac{\pi}{2}$.
3. Translate to the right by 5 units and down by 2 units.

We have the following three matrices for the scaling S, rotation R, and translation B, respectively,

$$S = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can now compose all three maps by multiplying matrices:

$$BRS = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Scale the triangle by a factor of 3.
2. Rotate the triangle counter-clockwise around the origin by $\frac{\pi}{2}$.
3. Translate to the right by 5 units and down by 2 units.

Then we have each vertex (given in homogeneous coordinates) mapped as follows

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}.$$