Diagonalization

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Diagonal Matrices

Diagonal matrices are particularly nice to work with in many contexts, and can simplify calculations considerably.

Consider the matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

- We know $\det D = (2)(3)(5) = 30$, and thus D is invertible.
- We know that D has eigenvalues $\lambda = 2, 3, 5$.
- We can easily raise D to powers:

$$D^k = \underbrace{DD \dots D}_{k \text{ times}} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 5^k \end{bmatrix}.$$

Example

Recall this example where A is similar to the diagonal matrix B.

$$\underbrace{\begin{bmatrix} -6 & -4 & -1 \\ -3 & -2 & -1 \\ 5 & 3 & 1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{B}.$$

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Definition

An $n \times n$ matrix A is said to be diagonalizable if it is similar to a diagonal matrix D.

- Not all matrices are diagonalizable.
- How do we determine if a matrix is diagonalizable?
  If so, how can we find an invertible matrix P such that $P^{-1}AP = D$?

Diagonalizable Matrices

Suppose an $n \times n$ matrix A is diagonalizable. Then A is is similar to a diagonal matrix D, and there is an invertible matrix P such that $P^{-1}AP = D$. Equivalently, $AP = PD$.

Let $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix}$ and $D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$.

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Since $A\mathbf{v}_i = d_i\mathbf{v}_i$, d_i is an eigenvalue of A with corresponding eigenvector $\mathbf{v}_i$.

Since P is invertible, the columns of P are linearly independent.

Thus, the $\mathbf{v}_i$s are n linearly independent eigenvectors of A.

Diagonalization Theorem

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

If A is diagonalizable, then $P^{-1}AP = D$ where:

- P is an $n \times n$ matrix whose columns are the n linearly independent eigenvectors.
- D is an $n \times n$ diagonal matrix whose diagonal entries are the eigenvalues of A that correspond, respectively, to the eigenvectors in P.

$$P^{-1}A\underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}}_{D}$$

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Thus $\lambda_1 = 5$ with eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 4$ with eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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Since the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we have

$$P^{-1}AP = \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} = D.$$

Linear Independence of Eigenvectors

Theorem

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

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Proof. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. Thus there are scalars c_i, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}. \qquad (1)$$

Among all such c_is that are not all zero, choose the scalars such that the fewest c_is are nonzero. Since the v_is are eigenvectors and not the zero vector, at least two of the c_is must be nonzero. Assume that c_j and c_k are nonzero.

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Multiplying both sides of equation (1) by A gives

$$A(c_1\mathbf{v}_1 + \ldots + c_j\mathbf{v}_j + \ldots + c_p\mathbf{v}_p) = c_1\lambda_1\mathbf{v}_1 + \ldots + c_j\lambda_j\mathbf{v}_j + \ldots + c_p\lambda_p\mathbf{v}_p = \mathbf{0}. \tag{2}$$

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Multiplying both sides of equation (1) by λ_j and subtracting from equation (2) gives

$$c_1(\lambda_1 - \lambda_j)\mathbf{v}_1 + c_2(\lambda_2 - \lambda_j)\mathbf{v}_2 + \ldots + 0\mathbf{v}_j + \ldots + c_p(\lambda_p - \lambda_j)\mathbf{v}_p = \mathbf{0}. \tag{3}$$

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Note that $c_k(\lambda_k - \lambda_j) \neq 0$ since $\lambda_k \neq \lambda_j$. Thus, we have expressed $\mathbf{0}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with not all scalars 0 that has fewer nonzero scalars than what we assumed was the fewest.

Contradiction! □

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If possible, diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

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1. Find the eigenvalues by solving the characteristic equation $\det(A - \lambda I) = \mathbf{0}$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)^2(1-\lambda) = 0$$

A has eigenvalues $\lambda = 2$ (with multiplicity 2) and $\lambda = 1$.

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if $\mathbf{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Thus the set $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

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3. Construct matrix P from the vectors above.

$$P = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

4. Construct matrix D from the corresponding eigenvalues.

Checking our Work

1. Find the eigenvalues by solving the characteristic equation $\det(A - \lambda I) = \mathbf{0}$.
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3. Construct matrix P from the vectors above.
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5. Verify your results.

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$$P^{-1}AP = \left(\begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D.$$

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Or you can check that

$$AP = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad PD = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

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Solving for A, we have that $A = PDP^{-1}$. Thus,

$$A^7 = \underbrace{(PDP^{-1})(PDP^{-1})\dots(PDP^{-1})}_{7 \text{ times}} = P\underbrace{D(P^{-1}P)D(P^{-1}P)\dots(P^{-1}P)D}_{7 \text{ times}}P^{-1} = PD^7P^{-1}$$

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$$= \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^7 & 0 & 0 \\ 0 & 2^7 & 0 \\ 0 & 0 & 2^7 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 128 & 0 & 0 \\ 127 & 128 & 127 \\ -127 & 0 & 1 \end{bmatrix}$$

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In general, $A^k = (PDP^{-1})^k = PD^kP^{-1}$ for a positive integer k.

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For $\lambda = 4$ we have

$$(A - 4I)\mathbf{x} = \begin{bmatrix} -2 & 4 & 6 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \text{ when } \mathbf{x} = x_3 \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \text{ thus a basis is } \left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

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Since we do **NOT** have three linearly independent eigenvectors, A is **NOT diagonalizable.**

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Theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Matrices with Eigenvalues of Higher Multiplicity

Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

(a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k (called the *geometric multiplicity*) is at most the algebraic multiplicity of the eigenvalue λ_k.

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(c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to eigenvalue λ_k, then all the vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ form an eigenvector basis for $\mathbb{R}^n$.

Reviewing Previous Examples

The matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

has eigenvalues:

- $\lambda = 2$ with eigenspace dimension equal to 2, and
- $\lambda = 1$ with eigenspace dimension equal to 1.
- $2 + 1 = 3$, so A is diagonalizable.

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- Since A has three distinct eigenvalues, it is diagonalizable.

Interpretation of Similarity

Suppose that A is an $n \times n$ diagonalizable matrix: $A = PDP^{-1}$.

The columns of P are bases for the eigenspaces of A, and together form a basis $\mathcal{B}$ for $\mathbb{R}^n$.

Consider the linear transform $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x} = PDP^{-1}\mathbf{x}$.

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P is the change of basis matrix from $\mathcal{B}$ to the standard basis.

P^{-1} is the change of basis matrix from the standard basis to $\mathcal{B}$.

So to perform T on a vector x with respect to the standard basis, we can

1. change to the basis $\mathcal{B}$ by multiplying by P^{-1},
2. then scale each coordinate by multiplying by D,
3. then change back to the standard basis by multiplying by P.

Example

Consider the linear transform $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1.5 & 0.5 \\ 1 & 1 \end{bmatrix} \mathbf{x}$.

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \right)^{-1}. \end{aligned}$$

Desmos:
`https://www.desmos.com/calculator/n7l5cf9xjo`

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