Coordinate Systems

Linear Algebra

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Basis of a Vector Space

Recall:

Let H be a subspace of a vector space V. A set $\mathcal{B}$ of vectors in V is a basis for H if:

- $\mathcal{B}$ is a linearly independent set, and
- the subspace spanned by $\mathcal{B}$ equals H (ie, $\text{Span}\ \mathcal{B} = H$).

The Uniqueness Representation Theorem

One nice result of choosing a basis $\mathcal{B}$ for a vector space V is to impose a coordinate system on V.

Example

- If $V = \mathbb{R}^4$ with standard basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, then

$$\begin{bmatrix} 2 \\ -5 \\ 17 \\ -8 \end{bmatrix} = 2\mathbf{e}_1 - 5\mathbf{e}_2 + 17\mathbf{e}_3 - 8\mathbf{e}_4.$$

- Consider the vector space $V = \mathrm{Mat}_{2\times 2}$ and the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

  Then an element in $\mathrm{Mat}_{2\times 2}$ such as $\begin{bmatrix} 2 & -5 \\ 17 & -8 \end{bmatrix}$ can be written as the linear combination:

$$\begin{bmatrix} 2 & -5 \\ 17 & -8 \end{bmatrix} = 2\mathbf{v}_1 - 5\mathbf{v}_2 + 17\mathbf{v}_3 - 8\mathbf{v}_4.$$

The Uniqueness Representation Theorem

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ (our coordinates) such that

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n. \tag{1}$$

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$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n. \tag{1}$$

Proof. Since $\mathcal{B}$ spans V, we know there exists scalars such that equation (1) above holds. Suppose the representation is not unique, and there exist scalars $d_1, d_2, \ldots, d_n$ such that

$$\mathbf{x} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + \ldots + d_n\mathbf{b}_n.$$

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Proof. Since $\mathcal{B}$ spans V, we know there exists scalars such that equation (1) above holds. Suppose the representation is not unique, and there exist scalars $d_1, d_2, \ldots, d_n$ such that

$$\mathbf{x} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + \ldots + d_n\mathbf{b}_n.$$

Subtracting the two gives

$$\mathbf{0} = (c_1 - d_1)\mathbf{b}_1 + (c_2 - d_2)\mathbf{b}_2 + \ldots + (c_n - d_n)\mathbf{b}_n. \tag{2}$$

Since we assumed $\mathcal{B}$ is a basis for V, equation (2) has only the trivial solution, so $c_j = d_j$ for all $1 \leq j \leq n$. Thus we see the representation is unique. $\square$

Coordinates Relative to a Basis

Definition

Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V, and $\mathbf{x}$ is in V. The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the unique scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n$.

Definition

If $c_1, c_2, \ldots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

Mapping V to $\mathbb{R}^n$

Definition

The mapping $V \to \mathbb{R}^n : \mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping (determined by $\mathcal{B}$).

$$\mathrm{Mat}_{2\times 2} \to \mathbb{R}^4 : \begin{bmatrix} 2 & -5 \\ 17 & -8 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ -5 \\ 17 \\ -8 \end{bmatrix}$$

Example

Let $V = \mathbb{R}^2$ and consider the vector $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

Consider two different bases for V:

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Example

Let $V = \mathbb{R}^2$ and consider the vector $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

We have $\mathbf{x} = 3\mathbf{e}_1 - 5\mathbf{e}_2$, so

$$[\mathbf{x}]_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

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We solve the vector equation $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

This system of linear equations has augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

Thus we have $[\mathbf{x}]_{\mathcal{B}_2} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ since $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Geometric Interpretation

Let $V = \mathbb{R}^2$ and consider the vector $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

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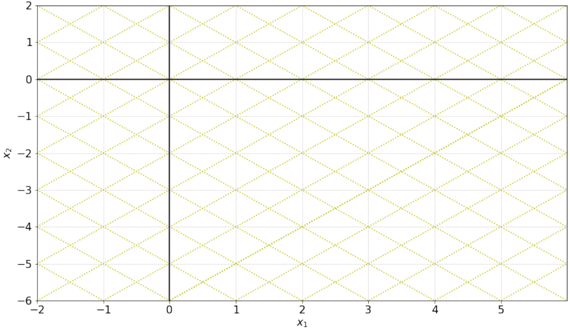

We have $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$[\mathbf{x}]_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

We have $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$[\mathbf{x}]_{\mathcal{B}_2} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

Let $V = \mathbb{R}^2$ and consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$.

Suppose vector $\mathbf{x}$ in $\mathbb{R}^2$ has $\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

How can we determine what the coordinates of this vector are in the standard basis?

Change of Coordinates Matrix for Basis in $\mathbb{R}^n$

Let $V = \mathbb{R}^2$ and consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$.

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$$2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} - 6 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 20 \\ -28 \end{bmatrix}$$

Change of Coordinates Matrix for Basis in $\mathbb{R}^n$

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The matrix $P = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ is called the change of coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^2$.

In general, if $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, then the change of coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$ is given by the matrix $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{bmatrix}$.

Let $V = \mathbb{R}^2$ and consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$.

Let $\mathbf{x} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$. Find the $\mathcal{B}$-coordinates of $\mathbf{x}$.

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- We have the change of coordinate matrix $P = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ which maps $[\mathbf{x}]_{\mathcal{B}}$ to the standard basis.
- We want to do the inverse of this map (map standard basis to $[\mathbf{x}]_{\mathcal{B}}$).
- The inverse matrix P^{-1} of P will do the job.

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In this example, we have

$$P = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -2 & -1.5 \\ -1 & -0.5 \end{bmatrix}.$$

This gives

$$[\mathbf{x}]_{\mathcal{B}} = P^{-1}\mathbf{x} = \begin{bmatrix} -2 & -1.5 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} 8 \\ -11 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -2.5 \end{bmatrix}.$$

Coordinate Mappings from V into $\mathbb{R}^n$

Given a vector space V with basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, we can define the coordinate map

$$T : V \to \mathbb{R}^n : \mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}.$$

> The coordinate mapping above is a one-to-one linear transformation from V onto $\mathbb{R}^n$.
> A linear transformation that is both one-to-one and onto is called an isomorphism.

Thus, V and $\mathbb{R}^n$ have the "same shape" with respect to vector space properties (ie, vector addition and scalar multiplication).

Example

Let $V = \mathbb{P}_3$ with basis $\mathcal{B} = \{1, t, t^2, t^3\}$. Then the coordinate mapping onto $\mathbb{R}^4$ is given by

$$T(1) = \mathbf{e}_1,\ T(t) = \mathbf{e}_2,\ T(t^2) = \mathbf{e}_3, \text{ and } T(t^3) = \mathbf{e}_4.$$

Find the $\mathcal{B}$-coordinates of the vector (polynomial) $\mathbf{x} = 2 + 2t^2 - 5t^3$.

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Find the $\mathcal{B}$-coordinates of the vector (polynomial) $\mathbf{x} = 2 + 2t^2 - 5t^3$.

We have $T(\mathbf{x}) = T(2 + 0t + 2t^2 - 5t^3) = 2\mathbf{e}_1 + 0\mathbf{e}_2 + 2\mathbf{e}_3 - 5\mathbf{e}_4$, which gives $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -5 \end{bmatrix}$.

Let $V = \mathbb{P}_3$ with basis $\mathcal{B} = \left\{1, t, t^2, t^3\right\}$. Determine whether polynomials $\mathbf{v}_1 = 2 + 2t^2 - 5t^3$, $\mathbf{v}_2 = 5 + t^3$, $\mathbf{v}_3 = 7t - 3t^2$, and $\mathbf{v}_4 = -1 + 7t + t^2 - 11t^3$ form a linearly independent set.

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We have $[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -5 \end{bmatrix}, \ [\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ [\mathbf{v}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 7 \\ -3 \\ 0 \end{bmatrix}, \ [\mathbf{v}_4]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 7 \\ 1 \\ -11 \end{bmatrix}$.

Since $\mathbb{P}_3$ and $\mathbb{R}^4$ are isomorphic,
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is lin indep in $\mathbb{P}_3$ if and only if $\{[\mathbf{v}_1]_{\mathcal{B}}, [\mathbf{v}_2]_{\mathcal{B}}, [\mathbf{v}_3]_{\mathcal{B}}, [\mathbf{v}_4]_{\mathcal{B}}\}$ is lin indep in $\mathbb{R}^4$.

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$$\begin{bmatrix} 2 & 5 & 0 & -1 \\ 0 & 0 & 7 & 7 \\ 2 & 0 & -3 & 1 \\ -5 & 1 & 0 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Setting the free variable $x_4 = 1$, we obtain:
$-2[\mathbf{v}_1]_{\mathcal{B}} + 1[\mathbf{v}_2]_{\mathcal{B}} - 1[\mathbf{v}_3]_{\mathcal{B}} + 1[\mathbf{v}_4]_{\mathcal{B}} = \mathbf{0}$.

So we have $-2(2 + 2t^2 - 5t^3) + 1(5 + t^3) - 1(7t - 3t^2) + 1(-1 + 7t + t^2 - 11t^3) = 0$.