## Orthogonal Sets

Linear Algebra

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### **Definition**

A set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is said to be an orthogonal set if  $\mathbf{v}_i \cdot \mathbf{v}_j = 0$  when  $i \neq j$  (ie, the vectors are pairwise orthogonal).

## Example

Is 
$$\left\{ \begin{bmatrix} 1\\-1\\0 \end{bmatrix}, \begin{bmatrix} 1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1 \end{bmatrix} \right\}$$
 an orthogonal set?

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 an orthogonal set?

Since

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$
,  $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$ , and  $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$ ,

the set is indeed an orthogonal set.

### Theorem

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is an orthogonal set of nonzero vectors in  $\mathbb{R}^n$ , then

S is a linearly independent set, and is therefore a basis for the subspace spanned by S.

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### Proof.

Suppose S is an orthogonal set and that the vectors in S are linearly dependent. This implies there exist scalars  $c_1, c_2, \ldots, c_p$ , with at least one  $c_i \neq 0$ , such that

$$\mathbf{0}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_p\mathbf{v}_p.$$

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. Then we have

$$0 = \mathbf{0} \cdot \mathbf{v}_1 = (c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_p \mathbf{v}_p) \cdot \mathbf{v}_1$$
  
=  $c_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + c_2(\mathbf{v}_2 \cdot \mathbf{v}_1) + \ldots c_p(\mathbf{v}_p \cdot \mathbf{v}_1)$   
=  $c_1(\mathbf{v}_1 \cdot \mathbf{v}_1)$ .

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Since  $\mathbf{v}_1$  is nonzero, we have  $c_1=0$ . Similarly, we can show that  $c_1=c_2=\ldots=c_p=0$ , which contradicts our original assumption. Thus S is linearly independent.

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# Orthogonal Bases

### **Definition**

An orthogonal basis for a subspace W of  $\mathbb{R}^n$  is a basis for W that is also an orthogonal set.

### Example

- ▶ The set  $S_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is an orthogonal basis for  $\mathbb{R}^3$ .
- ▶ The set  $S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^3$  that is **not** orthogonal.

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p$$

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 $\mathbf{y} \cdot \mathbf{u}_1 = (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p) \cdot \mathbf{u}_1$ 

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 $c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$  for each  $j$ .

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p$$
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 $c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$  for each  $j$ .

Suppose that  $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$  is an orthogonal basis for a subspace W of  $\mathbb{R}^n$ . Find  $\mathcal{B}$ -coordinates of  $\mathbf{y}$  in W.

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p$$

$$\mathbf{y} \cdot \mathbf{u}_1 = (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p) \cdot \mathbf{u}_1$$

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$$\mathbf{y} \cdot \mathbf{u}_1 = c_1 (\mathbf{u}_1 \cdot \mathbf{u}_1)$$

$$c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}$$

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \text{ for each } j.$$

### Theorem

Let  $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$  be an orthogonal basis for a subspace W of  $\mathbb{R}^n$ . For each  $\mathbf{y}$  in W, the weights in the linear combination

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_p \mathbf{u}_p$$
 are given by

$$c_j = rac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_i \cdot \mathbf{u}_j}$$
 for  $j = 1, 2, \dots, p$ .

### Example

Express the vector 
$$\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$
 as a linear combination of the orthogonal basis  $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ .

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$$c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{3-7}{2} = -2$$

$$c_2 = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{3+7}{2} = 5$$

$$c_3 = \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} = \frac{4}{1} = 4$$

Thus we have

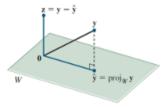
$$\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

## Orthogonal Projections

- In physics, forces are generally expressed as vectors, **F**.
- ► An object is moving with velocity **v**.
- lacktriangle We often want to decompose the force f F into two components  $f F = f F_{
  m perp} + f F_{
  m parallel}$

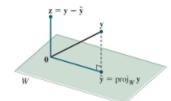
Given a nonzero vector  $\mathbf{u}$  in  $\mathbb{R}^n$ , we wish to decompose a vector  $\mathbf{y}$  in  $\mathbb{R}^n$  into the sum of two vectors, one that is parallel to  $\mathbf{u}$  and the other is orthogonal to  $\mathbf{u}$ :

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = c\mathbf{u} + \mathbf{z} = \text{proj}_{\mathbf{u}}\mathbf{y} + \mathbf{y}_{\text{perp}} = \text{proj}_{\mathbf{u}}\mathbf{y} + (\mathbf{y} - \hat{\mathbf{y}}).$$

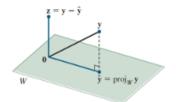


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We want to find  $\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}} \mathbf{y} = c \mathbf{u}$ .

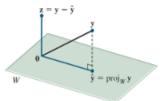


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We want to find 
$$\hat{\mathbf{y}} = \operatorname{proj}_{\mathbf{u}} \mathbf{y} = c\mathbf{u}$$
. The other component  $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$  is orthogonal to  $\mathbf{u}$ , so  $0 = (\mathbf{y} - c\mathbf{u}) \cdot \mathbf{u} = (\mathbf{y} \cdot \mathbf{u}) - c(\mathbf{u} \cdot \mathbf{u})$ .

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = c\mathbf{u} + \mathbf{z} = \mathsf{proj}_{\mathbf{u}}\mathbf{y} + \mathbf{y}_{\mathrm{perp}} = \mathsf{proj}_{\mathbf{u}}\mathbf{y} + (\mathbf{y} - \hat{\mathbf{y}}).$$



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Thus we see that

$$c = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}.$$

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Thus we see that

$$c = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}.$$

### Definition

The vector  $\hat{\mathbf{y}}$  is called the orthogonal projection of  $\mathbf{y}$  onto  $\mathbf{u}$ , and the vector  $\mathbf{z}$  is called the component of  $\mathbf{y}$  orthogonal to  $\mathbf{u}$ . We have

$$\hat{\mathbf{y}} = \mathsf{proj}_{\mathbf{u}} \mathbf{y} = \left( \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}, \qquad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

# Application of Projections

### Example

Suppose a sailboat has constant velocity given by  $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ . The wind has a constant velocity

of  $\mathbf{w} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$ . Find the component of the force that in the direction of the sailboat's motion.

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We have

$$\hat{\mathbf{w}} = \mathrm{proj}_{\mathbf{v}} \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = \frac{-10}{25} \mathbf{v} = \begin{bmatrix} -6/5 \\ -8/5 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{w} - \hat{\mathbf{w}} = \begin{bmatrix} 56/5 \\ -42/5 \end{bmatrix}$$

$$\hat{\mathbf{w}} \cdot \mathbf{z} = \begin{bmatrix} -6/5 \\ -8/5 \end{bmatrix} \cdot \begin{bmatrix} 56/5 \\ -42/5 \end{bmatrix} = 0.$$

## Orthonormal Sets

### Definition

A set  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$  is an orthonormal set if it is an orthogonal set of unit vectors.

### Example

The set  $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  is an orthonormal set that forms a basis for  $\mathbb{R}^n$ .

- Orthonormal sets (bases) are particularly nice to work with as many formulas simplify considerably.
- For example,  $\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}} \mathbf{y} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} = (\mathbf{y} \cdot \mathbf{u}) \mathbf{u}$ .

Let  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$  be an orthonormal set in  $\mathbb{R}^n$  and let  $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_p \end{bmatrix}_{n \times p}$ . Then

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_p^T \end{bmatrix}_{p \times n} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_p \end{bmatrix}_{n \times p}$$

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#### Theorem

The columns of an  $m \times n$  matrix U form an orthonormal set if and only if  $U^T U = I_{n \times n}$ .

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## Linear Mapping from Matrix whose Columns form an Orthonormal Set

#### Theorem

Let U be an  $m \times n$  matrix with orthonormal columns, and let **x** and **y** be in  $\mathbb{R}^n$ . Then

- (a)  $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ .
- (b)  $||U\mathbf{x}|| = ||\mathbf{x}||$ .
- (c)  $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$  if and only if  $\mathbf{x} \cdot \mathbf{y} = 0$ .

In other words, the linear mapping  $\mathbf{x} \mapsto U\mathbf{x}$  preserves length and orthogonality.

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#### Proof.

Statements (b) and (c) follow from statement (a). Note that

$$(U\mathbf{x})\cdot(U\mathbf{y})=(U\mathbf{x})^T(U\mathbf{y})=\mathbf{x}^T(U^TU)\mathbf{y}=\mathbf{x}^T\mathbf{y}=\mathbf{x}\cdot\mathbf{y}.$$

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