Linear Independence

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Redundancy when taking Span

We've seen that the Span of a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors does not change if we add a vector $\mathbf{w}$ that is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

So there is redundancy in S for the Span if a vector $\mathbf{v}_j \in S$ can be written as a linear combination of the other vectors in S.

How can we tell if a vector in S can be written as a linear combination of the other vectors?

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This might require solving n systems of linear equations!

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Suppose there is a nontrivial solution $c_1, c_2, \ldots, c_n$ (ie, not all c_js are zero).
In particular, suppose that $c_j \neq 0$. Then we solve for $\mathbf{v}_j$:

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_j\mathbf{v}_j + \cdots + c_n\mathbf{v}_n = \mathbf{0} \\ c_j\mathbf{v}_j &= -c_1\mathbf{v}_1 - c_2\mathbf{v}_2 + \cdots - c_{j-1}\mathbf{v}_{j-1} - c_{j+1}\mathbf{v}_{j+1} - \cdots - c_n\mathbf{v}_n \\ \mathbf{v}_j &= -\frac{c_1}{c_j}\mathbf{v}_1 - \frac{c_2}{c_j}\mathbf{v}_2 - \cdots - \frac{c_{j-1}}{c_j}\mathbf{v}_{j-1} - \frac{c_{j+1}}{c_j}\mathbf{v}_{j+1} - \cdots - \frac{c_n}{c_j}\mathbf{v}_n. \end{aligned}$$

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$$c_j\mathbf{v}_j = -c_1\mathbf{v}_1 - c_2\mathbf{v}_2 + \cdots - c_{j-1}\mathbf{v}_{j-1} - c_{j+1}\mathbf{v}_{j+1} - \cdots - c_n\mathbf{v}_n$$

$$\mathbf{v}_j = -\frac{c_1}{c_j}\mathbf{v}_1 - \frac{c_2}{c_j}\mathbf{v}_2 - \cdots - \frac{c_{j-1}}{c_j}\mathbf{v}_{j-1} - \frac{c_{j+1}}{c_j}\mathbf{v}_{j+1} - \cdots - \frac{c_n}{c_j}\mathbf{v}_n.$$

So $\mathbf{v}_j$ can be written as a linear combination of the other vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$.

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So $\mathbf{v}_j$ can be written as a linear combination of the other vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$.

This is if and only if: if $\mathbf{v_j}$ can be written as a linear combination of the other vectors, then there exists a nontrivial solution to the homogeneous system.

Revisiting Vector Equations

If A is an $m \times n$ matrix with columns vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ (each $\mathbf{a}_i$ is in $\mathbb{R}^m$), the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0},$$

which has corresponding augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{0} \end{bmatrix}.$$

- Homogeneous linear systems always have a trivial solution $x_1 = x_2 = \ldots = x_n = 0$.
- If a non-trivial solution exists, then:
  - At least one $x_j \neq 0$.
  - At least one $\mathbf{a}_j$ can be written as a linear combination of the other column vectors.
- If the trivial solution is the only solution, then it is not possible to write any column vector as a linear combination of the other column vectors.

Example

Determine whether the equation

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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has a nontrivial solution.

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 1 & 6 & -10 & 0 \\ 0 & 3 & -6 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Since x_3 is a free variable, nontrivial solutions exist!

Interpreting the Solution

Determine whether the equation

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has a non-trivial solution.

Solution set is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

So we have

$$-2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix}$$

Linear Independence

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ denote a set of n vectors each in $\mathbb{R}^m$. We say the set of vectors is linearly independent if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$ has only a trivial solution.

Otherwise, a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be linearly dependent if there exists weights $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}.$$

Example

The set $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} \right\}$ is linearly dependent.

Example

Determine whether the set of vectors $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Example

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$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

There is a unique solution, the trivial solution $\mathbf{x} = \mathbf{0}$, so the vectors are linearly independent.

> The columns of matrix A are linearly independent if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

The Invertible Matrix Theorem redux 2

Let A be a square $n \times n$ matrix. Then all of the following statements are equivalent.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix I_n.

(c) A has n pivots.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I_n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

(l) A^T is an invertible matrix.

(m) $\det A \neq 0$.

What if S contains $\mathbf{0}$?

Theorem

If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors in $\mathbb{R}^m$ contains the zero vector $\mathbf{0}$, then the set is linearly dependent.

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Proof.

Without any loss of generality, suppose $\mathbf{v}_1 = \mathbf{0}$. Then we see the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has a non-trivial solution. Namely, we set $x_1 = 1$ (or any non-zero value) and set $x_i = 0$ for each $i > 1$. This gives

$$1\mathbf{0} + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_n = \mathbf{0} + \mathbf{0} + \ldots + \mathbf{0} = \mathbf{0}.$$

$\square$

Small Sets Vectors

1. If possible, give an example of a set of 1 vector in $\mathbb{R}^3$ so the set is linearly independent.

2. If possible, give an example of a set of 2 vectors in $\mathbb{R}^3$ so the set is linearly independent.

3. If possible, give an example of a set of 3 vectors in $\mathbb{R}^3$ so the set is linearly independent.

4. If possible, give an example of a set of 4 vectors in $\mathbb{R}^3$ so the set is linearly independent.

Number of Vectors in Set S and the Dimension of the Vectors in S

Theorem

Any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors in $\mathbb{R}^m$ is linearly dependent if $n > m$. That is, if a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

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The columns of matrix A are linearly independent if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Proof.

Let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ denote the $m \times n$ matrix. Then the equation $A\mathbf{x} = \mathbf{0}$ has m equations and n variables.

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Proof.

Let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ denote the $m \times n$ matrix. Then the equation $A\mathbf{x} = \mathbf{0}$ has m equations and n variables. Since $n > m$, we have more variables than we have equations, so there must be at least one free variable. Thus $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution, and the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent. □

Characterization of Linearly Dependent Sets

Theorem

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. If fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

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Proof.

Extra statement: If S is linearly dependent, then by definition there exists weights $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$. Suppose j is the largest index where $c_j \neq 0$.

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Proof.

Extra statement: If S is linearly dependent, then by definition there exists weights $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$. Suppose j is the largest index where $c_j \neq 0$. Then we can subtract all terms except for $c_j\mathbf{v}_j$ from both sides, and then divide both sides by c_j, giving

$$\mathbf{v}_j = -\frac{c_1}{c_j}\mathbf{v}_1 - \frac{c_2}{c_j}\mathbf{v}_2 - \ldots - \frac{c_{j-1}}{c_j}\mathbf{v}_{j-1} - \frac{0}{c_j}\mathbf{v}_{j+1} - \ldots - \frac{0}{c_j}\mathbf{v}_n.$$

Thus we see $\mathbf{v}_j$ is a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$. $\square$