Homogeneous Systems of Linear Equations

Linear Algebra

This work was initially funded by an Institutional OER Grant from the Colorado Department of Higher Education (CDHE). For similar OER materials in other courses funded by this project in the Department of Mathematical and Statistical Sciences at the University of Colorado Denver, visit
https://github.com/CU-Denver-MathStats-OER

Homogeneous Solutions

A system of equations is homogeneous if it can be written in the form $A\mathbf{x} = \mathbf{0}$.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 11 \end{bmatrix}$$

NOT a homogeneous system.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \\ 5x_1 + x_2 + x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

YES, this is a homogeneous system.

Homogeneous Solutions

A system of equations is homogeneous if it can be written in the form $A\mathbf{x} = \mathbf{0}$.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 11 \end{bmatrix}$$

NOT a homogeneous system.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \\ 5x_1 + x_2 + x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

YES, this is a homogeneous system.

Every homogeneous system has a trivial solution $\mathbf{x} = \mathbf{0}$, and so is consistent.

Does the homogeneous system have a nontrivial solution? ie, infinite number of solutions?

Existence of Nontrivial Solutions

Recall that if the system $A\mathbf{x} = \mathbf{b}$ is consistent, then it has:

- A unique solution if, when forming the RREF of A, there is a pivot position in every column (ie, all variables are basic).
- An infinite number of solutions if there is a column of A without a pivot position (ie, there is at least one free variable).

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the solution set has at least one free variable.

Example

Find the solution set to the homogeneous system

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & 7x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \\ -2x_1 & - & 4x_2 & - & 8x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ -2 & -4 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice x_3 is a free variable, and the solution set can therefore be given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = x_3\mathbf{v}\ , \text{ where } \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Example

Find the solution set to the homogeneous system

$$\begin{array}{rcrcrcl} 3x_1 & - & 2x_2 & + & x_3 & = & 0 \\ -6x_1 & + & 4x_2 & - & 2x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 3 & -2 & 1 & 0 \\ -6 & 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Both x_2 and x_3 are free variables, and the solution set can therefore be given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 - \frac{1}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}.$$

Nonhomogeneous Linear Systems

Find the solution set to the nonhomogeneous system

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 7x_3 & = & 4 \\ & & x_2 & + & 3x_3 & = & 5 \\ -2x_1 & - & 4x_2 & - & 8x_3 & = & 2 \end{array}$$

Nonhomogeneous Linear Systems

Find the solution set to the nonhomogeneous system

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 7x_3 & = & 4 \\ & & x_2 & + & 3x_3 & = & 5 \\ -2x_1 & - & 4x_2 & - & 8x_3 & = & 2 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ -2 & -4 & -8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -11 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice x_3 is a free variable, and the solution set can therefore be given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 + 2x_3 \\ 5 - 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}).$$

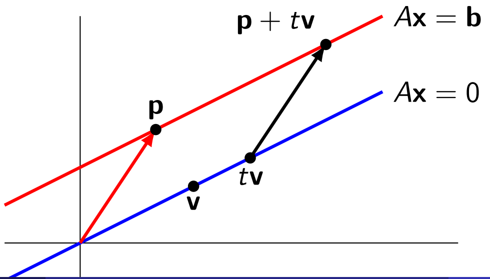
Comparing Homogeneous and Nonhomogeneous Solutions

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & 7x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \\ -2x_1 & - & 4x_2 & - & 8x_3 & = & 0 \end{array}$$

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = t\mathbf{v} \quad (t \text{ in } \mathbb{R}).$$

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & 7x_3 & = & 4 \\ & & x_2 & + & 3x_3 & = & 5 \\ -2x_1 & - & 4x_2 & - & 8x_3 & = & 2 \end{array}$$

$$\mathbf{x} = \begin{bmatrix} -11 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}).$$



Theorem

Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and has a solution $\mathbf{x} = \mathbf{p}$. If $\mathbf{v}$ is **any** solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = \mathbf{p} + \mathbf{v}$$

is a solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$.

Proof.

Theorem

Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and has a solution $\mathbf{x} = \mathbf{p}$. If $\mathbf{v}$ is **any** solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = \mathbf{p} + \mathbf{v}$$

is a solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$.

Proof.

We have $A\mathbf{x} = A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. $\square$

Theorem

Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and has a solution $\mathbf{x} = \mathbf{p}$. If $\mathbf{v}$ is **any** solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = \mathbf{p} + \mathbf{v}$$

is a solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$.

Proof.

We have $A\mathbf{x} = A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. $\square$

Theorem

Every solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{p} + \mathbf{v}$ for **some** solution $\mathbf{v}$ of $A\mathbf{x} = \mathbf{0}$.

Proof.

Theorem

Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and has a solution $\mathbf{x} = \mathbf{p}$. If $\mathbf{v}$ is **any** solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = \mathbf{p} + \mathbf{v}$$

is a solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$.

Proof.

We have $A\mathbf{x} = A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. □

Theorem

Every solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{p} + \mathbf{v}$ for **some** solution $\mathbf{v}$ of $A\mathbf{x} = \mathbf{0}$.

Proof.

Let $\mathbf{y}$ be a solution to $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{v} = \mathbf{y} - \mathbf{p}$. Then $A\mathbf{v} = A(\mathbf{y} - \mathbf{p}) = A\mathbf{y} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Hence, $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$. □