The Invertible Matrix Theorem

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Equivalent Statements

Let A be a square $n \times n$ matrix.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix I_n.

(c) A has n pivots when forming the reduced row echelon form of A.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

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(d)⇒(b) If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, that means the augmented matrix $[A\mathbf{0}]$ has no free variables. Thus A has n pivots.

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(a)⇒(d) If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any b in $\mathbb{R}^n$.

(b)⇒(a) If A is row equivalent to I_n, then there are elementary matrices E_i such that $E_p E_{p-1} \cdots E_1 A = I_n$. Thus, $A^{-1} = E_p E_{p-1} \cdots E_1$, so A is invertible.

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Statements (a), (b), (c), (d) are all equivalent!

Equivalence of Statements

(a) A is an invertible matrix.

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(j) There is an $n \times n$ matrix C such that $CA = I_n$.

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(a)$\Rightarrow$(j) If A is invertible, then $C = A^{-1}$ is a matrix such that $CA = I_n$.

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Statements (a), (b), (c), (d), and (j) are all equivalent!

Adding to the Chain of Equivalence

(a) A is an invertible matrix.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

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(a)⇒(k) If A is invertible, then there is an $n \times n$ matrix $D = A^{-1}$ such that $AD = I_n$.

(k)⇒(g) If there is an $n \times n$ matrix D such that $AD = I_n$, then the equation $AD\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$. Thus, for any $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = D\mathbf{b}$.

(g)⇒(a) If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, each row in A must have a pivot. Since A is $n \times n$, A has n pivots, and A therefore is invertible.

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Statements (a), (b), (c), (d), (g), (j) and (k) are all equivalent statements!

The Invertible Matrix Theorem (so far)

Let A be a square $n \times n$ matrix. Then all of the following statements are equivalent.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix I_n.

(c) A has n pivots.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I_n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

(l) A^T is an invertible matrix.

Example

Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}.$$

Does A have an inverse? Explain why or why not.

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Does A have an inverse? Explain why or why not.

- A is not row equivalent to I_3.
- The columns of A do not span $\mathbb{R}^3$.
- A^T is not invertible.
- And so on ...