The Gram–Schmidt Orthogonalization Process

Linear Algebra

This work was initially funded by an Institutional OER Grant from the Colorado Department of Higher Education (CDHE). For similar OER materials in other courses funded by this project in the Department of Mathematical and Statistical Sciences at the University of Colorado Denver, visit
https://github.com/CU-Denver-MathStats-OER

Introduction

- We have seen that an orthogonal basis for a subspace W of $\mathbb{R}^n$ is particularly nice.
  - For $\mathbf{y}$ in $\mathbb{R}^n$, we compute the orthogonal projection of $\mathbf{y}$ onto W by

$$\hat{\mathbf{y}} = \mathrm{proj}_W \mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_p\mathbf{u}_p \quad \text{with weights } c_i = \frac{\mathbf{y}\cdot\mathbf{u}_i}{\mathbf{u}_i\cdot\mathbf{u}_i}.$$

- When we choose an orthonormal basis, the calculations are even simpler.

$$\hat{\mathbf{y}} = \mathrm{proj}_W \mathbf{y} = (UU^T)\mathbf{y}.$$

- Recall $\hat{\mathbf{y}}$ is a useful vector to find as it is the best approximation to $\mathbf{y}$ in the subspace W.

Introduction

- We have seen that an orthogonal basis for a subspace W of $\mathbb{R}^n$ is particularly nice.
  - For $\mathbf{y}$ in $\mathbb{R}^n$, we compute the orthogonal projection of $\mathbf{y}$ onto W by

$$\hat{\mathbf{y}} = \text{proj}_W \mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_p\mathbf{u}_p \quad \text{with weights } c_i = \frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}.$$

- When we choose an orthonormal basis, the calculations are even simpler.

$$\hat{\mathbf{y}} = \text{proj}_W \mathbf{y} = (UU^T)\mathbf{y}.$$

- Recall $\hat{\mathbf{y}}$ is a useful vector to find as it is the best approximation to $\mathbf{y}$ in the subspace W.

Example

Let $W = \text{Span}\,\{\mathbf{x}_1, \mathbf{x}_2\}$ denote the subspace of $\mathbb{R}^3$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

How can we find an orthogonal basis for W?

Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ denote a subspace of $\mathbb{R}^3$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

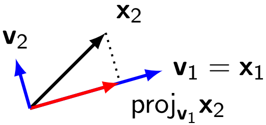
How can we find an orthogonal basis for W?

Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ denote a subspace of $\mathbb{R}^3$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

How can we find an orthogonal basis for W?

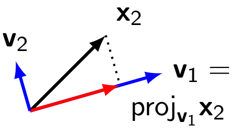
We want orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$ such that $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

- Let $\mathbf{v}_1 = \mathbf{x}_1$.



Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ denote a subspace of $\mathbb{R}^3$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

How can we find an orthogonal basis for W?

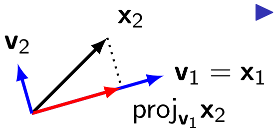


We want orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$ such that $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

- Let $\mathbf{v}_1 = \mathbf{x}_1$.
- Let $\mathbf{v}_2$ be the orth proj of $\mathbf{x}_2$ onto the orth compl of $Y = \text{Span}\{\mathbf{v}_1\}$.
  By the Orth Decomp Thm, $\mathbf{x}_2 = \text{proj}_Y \mathbf{x}_2 + \mathbf{z}$, where $\mathbf{z}$ is in $Y^\perp$.
  $$\text{proj}_Y \mathbf{x}_2 = \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$
- Hence $\mathbf{v}_2 = \mathbf{z} = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ denote a subspace of $\mathbb{R}^3$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

How can we find an orthogonal basis for W?



We want orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$ such that $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

- Let $\mathbf{v}_1 = \mathbf{x}_1$.
- Let $\mathbf{v}_2$ be the orth proj of $\mathbf{x}_2$ onto the orth compl of $Y = \text{Span}\{\mathbf{v}_1\}$.

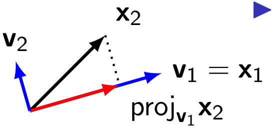
  By the Orth Decomp Thm, $\mathbf{x}_2 = \text{proj}_Y \mathbf{x}_2 + \mathbf{z}$, where $\mathbf{z}$ is in $Y^\perp$.

$$\text{proj}_Y \mathbf{x}_2 = \text{proj}_{\mathbf{v}_1} \mathbf{x}_2 = \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 = \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

- Hence $\mathbf{v}_2 = \mathbf{z} = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.
- Since $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \mathbf{x}_2 + c\mathbf{x}_1$, $\mathbf{v}_2$ is in the subspace W.

Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ denote a subspace of $\mathbb{R}^3$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

How can we find an orthogonal basis for W?



We want orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$ such that $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

- Let $\mathbf{v}_1 = \mathbf{x}_1$.
- Let $\mathbf{v}_2$ be the orth proj of $\mathbf{x}_2$ onto the orth compl of $Y = \text{Span}\{\mathbf{v}_1\}$.

  By the Orth Decomp Thm, $\mathbf{x}_2 = \text{proj}_Y \mathbf{x}_2 + \mathbf{z}$, where $\mathbf{z}$ is in $Y^\perp$.

$$\text{proj}_Y \mathbf{x}_2 = \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \frac{15}{45}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

- Hence $\mathbf{v}_2 = \mathbf{z} = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.
- Since $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \mathbf{x}_2 + c\mathbf{x}_1$, $\mathbf{v}_2$ is in the subspace W.
- Thus $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis for W.

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$\mathbf{v}_1 = \mathbf{x}_1$ $\qquad$ $Y_1 = \text{Span}\{\mathbf{v}_1\}$

The Gram–Schmidt Orthogonalization Process

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1 \qquad Y_1 = \mathrm{Span}\{\mathbf{v}_1\}$$

$$\mathbf{v}_2 = \mathrm{proj}_{Y_1^\perp}\mathbf{x}_2 = \mathbf{x}_2 - \mathrm{proj}_{Y_1}\mathbf{x}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 \qquad Y_2 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

The Gram–Schmidt Orthogonalization Process

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1 \qquad Y_1 = \text{Span}\{\mathbf{v}_1\}$$

$$\mathbf{v}_2 = \text{proj}_{Y_1^\perp}\mathbf{x}_2 = \mathbf{x}_2 - \text{proj}_{Y_1}\mathbf{x}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 \qquad Y_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

$$\mathbf{v}_3 = \text{proj}_{Y_2^\perp}\mathbf{x}_3 = \mathbf{x}_3 - \text{proj}_{Y_2}\mathbf{x}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 \qquad Y_3 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

The Gram–Schmidt Orthogonalization Process

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1 \qquad Y_1 = \mathrm{Span}\{\mathbf{v}_1\}$$

$$\mathbf{v}_2 = \mathrm{proj}_{Y_1^\perp}\mathbf{x}_2 = \mathbf{x}_2 - \mathrm{proj}_{Y_1}\mathbf{x}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 \qquad Y_2 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

$$\mathbf{v}_3 = \mathrm{proj}_{Y_2^\perp}\mathbf{x}_3 = \mathbf{x}_3 - \mathrm{proj}_{Y_2}\mathbf{x}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 \qquad Y_3 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

$$\vdots$$

$$\begin{aligned}\mathbf{v}_p &= \mathrm{proj}_{Y_{p-1}^\perp}\mathbf{x}_p = \mathbf{x}_p - \mathrm{proj}_{Y_{p-1}}\mathbf{x}_p \\ &= \mathbf{x}_p - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 - \ldots - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_{p-1}}{\mathbf{v}_{p-1}\cdot\mathbf{v}_{p-1}}\right)\mathbf{v}_{p-1}. \qquad Y_p = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}\end{aligned}$$

The Gram–Schmidt Orthogonalization Process

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1 \qquad Y_1 = \mathrm{Span}\{\mathbf{v}_1\}$$

$$\mathbf{v}_2 = \mathrm{proj}_{Y_1^\perp}\mathbf{x}_2 = \mathbf{x}_2 - \mathrm{proj}_{Y_1}\mathbf{x}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 \qquad Y_2 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

$$\mathbf{v}_3 = \mathrm{proj}_{Y_2^\perp}\mathbf{x}_3 = \mathbf{x}_3 - \mathrm{proj}_{Y_2}\mathbf{x}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 \qquad Y_3 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

$$\vdots$$

$$\begin{aligned}\mathbf{v}_p &= \mathrm{proj}_{Y_{p-1}^\perp}\mathbf{x}_p = \mathbf{x}_p - \mathrm{proj}_{Y_{p-1}}\mathbf{x}_p \\ &= \mathbf{x}_p - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 - \ldots - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_{p-1}}{\mathbf{v}_{p-1}\cdot\mathbf{v}_{p-1}}\right)\mathbf{v}_{p-1}. \qquad Y_p = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}\end{aligned}$$

Theorem

Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W.

The Gram–Schmidt Orthogonalization Process

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1 \qquad Y_1 = \operatorname{Span}\{\mathbf{v}_1\}$$

$$\mathbf{v}_2 = \operatorname{proj}_{Y_1^\perp}\mathbf{x}_2 = \mathbf{x}_2 - \operatorname{proj}_{Y_1}\mathbf{x}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 \qquad Y_2 = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

$$\mathbf{v}_3 = \operatorname{proj}_{Y_2^\perp}\mathbf{x}_3 = \mathbf{x}_3 - \operatorname{proj}_{Y_2}\mathbf{x}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 \qquad Y_3 = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

$$\vdots$$

$$\begin{aligned}\mathbf{v}_p &= \operatorname{proj}_{Y_{p-1}^\perp}\mathbf{x}_p = \mathbf{x}_p - \operatorname{proj}_{Y_{p-1}}\mathbf{x}_p \\ &= \mathbf{x}_p - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2 - \ldots - \left(\frac{\mathbf{x}_p\cdot\mathbf{v}_{p-1}}{\mathbf{v}_{p-1}\cdot\mathbf{v}_{p-1}}\right)\mathbf{v}_{p-1}. \qquad Y_p = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}\end{aligned}$$

Theorem

Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition,

$$\operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = Y_k = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \quad \text{for } 1 \le k \le p.$$

Example

Find an orthogonal basis for the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$.

Example

Find an orthogonal basis for the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$.

We have $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$.

Example

Find an orthogonal basis for the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$.

We have $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$.

Then we find $\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \left(\frac{36}{18}\right) \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$.

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \right\}$.

Example

Consider the matrix $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$. Find an orthogonal basis for the column space of A.

Example

Consider the matrix $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$. Find an orthogonal basis for the column space of A.

First we find a basis for the column space by checking the reduced row echelon form of A:

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so a basis for Col } A \text{ is } \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right\}.$$

Next we use the Gram–Schmidt Process to convert the basis into an orthogonal basis.

Example, continued

$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right\}$. We first define $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

Example, continued

$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right\}$. We first define $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

Then we continue with Gram–Schmidt process:

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\frac{-40}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right\}$. We first define $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

Then we continue with Gram–Schmidt process:

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\frac{-40}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \left(\frac{30}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{-10}{20} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Normalizing the Basis

For the 4×3 matrix $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$, we therefore have

- An orthogonal basis for Col A is $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

Normalizing the Basis

For the 4×3 matrix $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$, we therefore have

- An orthogonal basis for Col A is $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

- An orthonormal basis for Col A is $\left\{ \begin{bmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{20} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} -3/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix} \right\}$.