The Dimension of a Vector Space

Linear Algebra

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Considering Bases for $\mathbb{P}_2$

Given a vector space V, then $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V if

1. $\mathcal{B}$ is a linearly independent set, and
2. $\text{Span}\,\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} = V$.

Example

Let $V = \mathbb{P}_2$, the vector space of degree two (or less) polynomials with usual polynomial addition and scalar multiplication.

- $\mathcal{B} = \{1, t, t^2\}$ is the standard basis.
- $\mathcal{B}_1 = \{1 - 2t, 5 + t, -8 + 3t^2\}$ is also a basis.
- Is $\mathcal{B}_2 = \{1 - 2t + t^2, 5 + t - t^2\}$ a basis?
- Is $\mathcal{B}_3 = \{1 - 2t + t^2, 5 + t - t^2, 2 + 7t, 6t + 2t^2\}$ a basis?

How Many Vectors Are Needed to Form a Basis for V?

Theorem

If a vector space V has a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then any set containing more than n vectors of V must be linearly dependent.

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Proof. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be a set with p vectors from V, where $p > n$.
Suppose that $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p = \mathbf{0}$. Since $\mathcal{B}$ is a basis for V, each $\mathbf{u}_j$ can be written as a linear combination of the elements of $\mathcal{B}$: $\mathbf{u}_j = a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \ldots + a_{nj}\mathbf{v}_n$.
Substituting and grouping the $\mathbf{v}_i$, we have

$$\left(\sum_{j=1}^{p} a_{1j}c_j\right)\mathbf{v}_1 + \left(\sum_{j=1}^{p} a_{2j}c_j\right)\mathbf{v}_2 + \ldots + \left(\sum_{j=1}^{p} a_{ij}c_j\right)\mathbf{v}_i + \ldots + \left(\sum_{j=1}^{p} a_{nj}c_j\right)\mathbf{v}_n = \mathbf{0}.$$

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Since the $\mathcal{B}$ is a basis and thus linearly indep, the only linear combination of the $\mathbf{v}_i$s equal to $\mathbf{0}$ is where all the scalar weights are 0.

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Thus we wish to solve the homogeneous linear system

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{np} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

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Since $p > n$, there are more columns than rows and so there is a nontrivial solution for the c_js.
Thus $\mathcal{U}$ is a linearly dependent set. $\square$

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Proof. Let $\mathcal{B}_1$ be a basis for V consisting of n vectors and let $\mathcal{B}_2$ denote another basis for V that consists of m vectors. By the previous theorem, we know $\mathcal{B}_2$ cannot have more than n vectors, so $m \leq n$.

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Similarly, since $\mathcal{B}_2$ is a basis with m vectors, then we know (also by the previous theorem) that $\mathcal{B}_1$ cannot contain more than m vectors, so we have $n \leq m$. Since $m \leq n$ and $n \leq m$, we have $m = n$.

$\square$

The Dimension of a Vector Space

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Example

- $\mathbb{P}_2$ has dimension 3.
- $\mathrm{Mat}_{2\times 2}$ has dimension 4.
- The vector space of polynomials (of any degree) is infinite dimensional.

Example

Find a basis and state the dimension of the subspace:

$$H = \left\{ \begin{bmatrix} 3a - b \\ 6a \\ 2a + b \end{bmatrix} \;:\; a, b \text{ in } \mathbb{R} \right\}$$

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$$H = \left\{ \begin{bmatrix} 3a - b \\ 6a \\ 2a + b \end{bmatrix} \;:\; a, b \text{ in } \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \;:\; a, b \text{ in } \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Subspaces of a Vector Space

All subspaces of $\mathbb{R}^3$ can be classified by their dimension:

- 0-dimensional subspaces: There is only one, the zero subspace $H_0 = \{\mathbf{0}\}$.
- 1-dimensional subspaces: Lines through the origin. Any subspace spanned by a single, nonzero-vector $H_1 = \text{Span}\{\mathbf{v}_1\}$.
- 2-dimensional subspaces: Planes that contain the origin. Any subspace spanned by two linearly independent vectors $H_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
- 3-dimensional subspaces: The entire vector space $\mathbb{R}^3$ is the only subspace with three dimensions. Any subspace spanned by three linearly independent vectors $H_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The Basis Theorem

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Determine whether the following set $S = \{1 - t, 1 + t, 2t^2\}$ is a basis for $\mathbb{P}_2$.

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Theorem (The Basis Theorem)

Let V be a p-dimensional vector space. Any linearly independent set of p vectors in V is automatically a basis for V. Any set of exactly p vectors that span V is automatically a basis for V.

Dimension of Matrix Subspaces

Find the dimensions of Null A and Col A for $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$

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Find RREF of A.

- We see Null A has dimension 2 since we have a basis $\mathcal{B}_{\text{null}} = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.
- We see Col A has dimension 3 since we have a basis $\mathcal{B}_{\text{col}} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$.

Rank and Nullity of a Matrix

- The rank of an $m \times n$ matrix A is the dimension of the column space.
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1. If A is a 9×12 matrix with rank 3, what is the nullity of A?
2. If A is a 10×6 matrix with nullity 4, what is the rank of A?

Rank and The Invertible Matrix Theorem

Theorem (The Invertible Matrix Theorem (continued))

Let A be an $n \times n$ matrix. Then the following are equivalent statements:

(a) A is an invertible matrix.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(h) The columns of A span $\mathbb{R}^n$.

(n) The columns of A form a basis for $\mathbb{R}^n$.

(o) Col $A = \mathbb{R}^n$.

(p) rank $A = n$. (*A has "full rank"*)

(q) nullity $A = 0$.

(r) Null $A = \{\mathbf{0}\}$.