Linear Independence and Bases

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

Finding an Efficient Set of Vectors that Span V

Consider the vector space $V = \mathbb{R}^3$ with the usual operations, and consider the following set of vectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} \right\}.$$

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1. Do the set of vectors above span all of $\mathbb{R}^3$? YES
2. Do we need all five vectors in the set to span all of $\mathbb{R}^3$?

NO. We only need the three standard column vectors to span $\mathbb{R}^3$. Notice

$$\begin{bmatrix}0\\-2\\5\end{bmatrix} = -2\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

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- Removing these two vectors from the set above does not change the span.
- Then also removing one of the standard column vectors would affect the span of the set.
- We need a minimum of three vectors to span all of $\mathbb{R}^3$.

Revisiting Linear Independence

Let V denote a vector space, and consider the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

- The set S of vectors is linearly independent if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$ has only the trivial solution.
- The set S of vectors is linearly dependent if the vector equation above has a non-trivial solution.

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Theorem

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of vectors is linearly dependent if and only if some $\mathbf{v}_j$ is a linear combination of the other vectors.

Basis of a Vector Space

Let H denote a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is basis for H if:

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ equals H. Namely Span $\mathcal{B} = H$.

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Note that there is not a unique basis.

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Spanning Set Theorem

Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ (which is a subspace of V).

- If one of the vectors in S is a linear combination of the remaining vectors in S, then after removing that vector from the set, the remaining vectors will still span H.
- If $H \neq \{\mathbf{0}\}$, some nonempty subset of S is a basis for H.

Example

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} \right\}$$

Standard Basis for $\mathbb{R}^n$

We saw that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$, and probably this is the most natural basis. We extend this basis to higher dimensions.

The set of standard column vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ that forms a basis for $\mathbb{R}^n$ is called the standard basis for $\mathbb{R}^n$.

Examples in $\mathbb{R}^3$

Determine whether the given set of vectors is a basis for $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}.$$

Setting $x_3 = -1$ and rearranging, we see that

$$\begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Bases for More Abstract Vector Spaces

1. Find a basis for $\mathrm{Mat}_{2\times 2}$ the vector space of all 2×2 matrices with the usual operations.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus we have $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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2. Find a basis for $\mathbb{P}_4$, the vector space of all polynomials of degree at most 4 with the usual operations.

An arbitrary vector in $\mathbb{P}_4$ is given by $p(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$. Thus we have $\mathcal{B} = \{1, t, t^2, t^3, t^4\}$.

Polynomial Vector Space Example 1

Is $\{2t^2+1, t^2-1, t+2\}$ a basis for $\mathbb{P}_2$? ($\mathbb{P}_2$ =polynomials of degree at most 2)

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We solve $c_1(2t^2+1)+c_2(t^2-1)+c_3(t+2)=0$. This gives

$$(2c_1+c_2)t^2+c_3t+(c_1-c_2)=0$$

From the linear and constant terms respectively, we see that $c_3=0$ and $c_1=c_2$. Then looking at the coefficient in front of the quadratic term, we have $2c_1+c_2=2c_1+c_1=3c_1=0$. Thus the only solution is the trivial solution, $c_1=c_2=c_3=0$. The set is linearly independent.

Note the standard basis for $\mathbb{P}_2$ is $\{t^2, t, 1\}$, which contains three vectors. We have a linearly independent set of the same number of vectors, thus this set is a basis for $\mathbb{P}_2$.

Polynomial Vector Space Example 2

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Notice that we have $2t^2 - 1 = (2t^2 + t) - (t + 1)$, or $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$. Since the set is linearly dependent, it cannot be a basis for $\mathbb{P}_2$.

Matrix Vector Space Example

Is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ 4 & 0 \end{bmatrix} \right\}$ a basis for $\text{Mat}_{2\times 2}$?

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Notice that we have

$$\begin{bmatrix} 2 & -4 \\ 4 & 0 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 4 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since one matrix can be written as a linear combination of other matrices in the set, this is a linearly dependent set, and it cannot be a basis for $\mathrm{Mat}_{2\times 2}$.

Bases for the Null Space of a Matrix

The null space of an $m \times n$ matrix A, denoted Null A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Let $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$.

Find a basis for Null A.

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Find a basis for Null A.

$$\mathbf{x} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Basis for Column Space of a Matrix

The column space of an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$, denoted Col A, is the set of all linear combinations of the columns of A.

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So every linear dep of the columns of R is a linear dep of the columns of A.

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Pivot cols of R are $\mathbf{e}_i$s; non-pivot cols are linear combs.

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$\mathbf{b} = (1, 12, 3, 17)$.

RREF of $[A|\mathbf{b}] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 - 4x_2 - 2x_4 \\ x_2 \\ -7 + x_4 \\ x_4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -7 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Set $x_2 = x_4 = 0$. Then $\mathbf{b} = \begin{bmatrix} 1 \\ 12 \\ 3 \\ 17 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix} - 7 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix}$.

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The row space of an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$, denoted Row A, is the set of all linear combinations of the rows of A. Row $A = \text{Span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\} = \text{Col } A^T$

Find a basis for Row A.

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$R = \text{RREF}(A) = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The rows of R are linear combinations of rows of A.

Thus, span rows $R \subseteq$ span rows A.

Basis for Row Space of a Matrix

The row space of an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$, denoted Row A, is the set of all linear combinations of the rows of A. Row $A = \text{Span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\} = \text{Col } A^T$

Find a basis for Row A.

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So a basis for Row R is a basis for Row A.

A basis for Row A is
$\{(1,4,0,2,0),(0,0,1,-1,0),(0,0,0,0,1)\}$.

Basis for Row Space of a Matrix

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same.

If B is in (reduced) row echelon form, the nonzero rows of B form a basis for the row space of A as well as for B.

Summary

Given a matrix A:

- Solve the equation $A\mathbf{x} = \mathbf{0}$ to find a basis for Null A.

Summary

Given a matrix A:

- Solve the equation $A\mathbf{x} = \mathbf{0}$ to find a basis for Null A.
- Find the reduced row echelon form of A.
- The pivot columns of the original matrix A form a basis for Col A.
- The non-zero rows of the RREF matrix form a basis for Row A.