Change of Basis

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Change of Coordinate Mappings

Let V be a vector space with two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$.
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We have that $\mathbf{x} = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + \ldots + d_n\mathbf{b}_n$. We want to write $\mathbf{x}$ as a linear combin of $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$.

Since $\mathcal{C}$ is a basis, each $\mathbf{b}_j$ is a linear combin of the elements of $\mathcal{C}$: $\mathbf{b}_j = a_{1j}\mathbf{c}_1 + a_{2j}\mathbf{c}_2 + \ldots + a_{nj}\mathbf{c}_n$.

Substituting and grouping the $\mathbf{c}_i$, we have

$$\mathbf{x} = \left(\sum_{j=1}^{n} a_{1j} d_j\right) \mathbf{c}_1 + \left(\sum_{j=1}^{n} a_{2j} d_j\right) \mathbf{c}_2 + \ldots + \left(\sum_{j=1}^{n} a_{ij} d_j\right) \mathbf{c}_i + \ldots + \left(\sum_{j=1}^{n} a_{nj} d_j\right) \mathbf{c}_n.$$

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Thus, $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$

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$$\text{Thus, } [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \dots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} [\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}}.$$

Change of Basis Matrix

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be bases for a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}.$$

The map $\mathbf{z} \mapsto \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\mathbf{z}$ is a linear transformation from $\mathbb{R}^n$ ($\mathcal{B}$-coordinates) to $\mathbb{R}^n$ ($\mathcal{C}$-coordinates).

The columns of the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, denoted $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$, are the $\mathcal{C}$-coordinate vectors in the basis $\mathcal{B}$.

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \ldots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}.$$

Example

Let $V = \mathbb{P}^2$ and consider two bases $\mathcal{B} = \left\{1, t, t^2\right\}$ and $\mathcal{C} = \left\{1, 1+t, 1+t+t^2\right\}$.
Write $\mathbf{x} = 3 - 2t + 7t^2$ in terms of the basis $\mathcal{C}$.

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We compute the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \dots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

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We can then compute

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \\ 7 \end{bmatrix}.$$

So $3 - 2t + 7t^2 = 5(1) - 9(1 + t) + 7(1 + t + t^2)$.

Example

Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V. Suppose we know that

$$\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \quad \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3, \quad \text{and} \quad \mathbf{a}_3 = -\mathbf{b}_2 + 2\mathbf{b}_3$$

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1. Find the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$.
2. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

We know that

$$[\mathbf{a}_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}_{\mathcal{B}}, [\mathbf{a}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}}, [\mathbf{a}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}_{\mathcal{B}}$$

Thus we have

$$\underset{\mathcal{B}\leftarrow\mathcal{A}}{P} = \begin{bmatrix} [\mathbf{a}_1]_{\mathcal{B}} & [\mathbf{a}_2]_{\mathcal{B}} & [\mathbf{a}_3]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

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2. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

We have $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}_{\mathcal{A}}$. Using our change of basis matrix we have

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}_{\mathcal{A}} = \begin{bmatrix} 8 \\ 0 \\ 6 \end{bmatrix}_{\mathcal{B}}.$$

Example in $\mathbb{R}^n$

Let $V = \mathbb{R}^2$ and consider two bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$.

Write $\begin{bmatrix} 1 \\ 6 \end{bmatrix}_{\mathcal{B}}$ in terms of the basis $\mathcal{C}$.

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We find $[\mathbf{b}_1]_{\mathcal{C}}$ and $[\mathbf{b}_2]_{\mathcal{C}}$ by solving

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More efficient to solve both by row reducing

$$\left[\mathbf{c}_1 \quad \mathbf{c}_2 \;\middle|\; \mathbf{b}_1 \quad \mathbf{b}_2\right] = \left[\begin{array}{cc|cc} 5 & 2 & 2 & -6 \\ 3 & -2 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{2}{5} & \frac{2}{5} & -\frac{6}{5} \\ 0 & -\frac{16}{5} & -\frac{1}{5} & \frac{23}{5} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{16} & \frac{-10}{16} \\ 0 & 1 & -\frac{1}{16} & \frac{-23}{16} \end{array}\right]$$

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$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{3}{16} & \frac{-10}{16} \\ -\frac{1}{16} & \frac{-23}{16} \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -27/8 \\ -137/16 \end{bmatrix}_{\mathcal{C}}.$$

Finding the Change of Basis Matrix for Two Bases of $\mathbb{R}^n$

Let $V = \mathbb{R}^n$ with two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$.
(*Note that $\mathbf{b}_i$ and $\mathbf{c}_j$ are written in terms of the standard basis for $\mathbb{R}^n$.*)

Then the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can be found by row reducing

$$\left[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \ldots \quad \mathbf{c}_n \quad \middle| \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \ldots \quad \mathbf{b}_n\right] \quad \rightarrow \quad \left[I_n \quad \middle| \quad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right].$$

Note this is reminiscent of computing A^{-1} by row reducing $[A \mid I] \rightarrow [I \mid A^{-1}]$.

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Let V be a vector space with two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$.

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Recall that the $\mathcal{C}$-coordinate mapping $V \to \mathbb{R}^n$ where $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{C}}$ is an isomorphism.

Since $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a linearly independent set, so is $\{[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, \ldots, [\mathbf{b}_n]_{\mathcal{C}}\}$.

Thus the cols of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly indep, and by the Invertible Matrix Theorem $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is invertible.

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What is $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}$? Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$, $\quad [\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}[\mathbf{x}]_{\mathcal{C}}$. Thus, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

Example

Let $V = \mathbb{P}^2$ and consider two bases $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{1, 1+t, 1+t+t^2\}$.

Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

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Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

We compute the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$:

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} [1]_{\mathcal{B}} & [1+t]_{\mathcal{B}} & [1+t+t^2]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we can compute

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix Vector Space

Let V denote the vector space of symmetric 2×2 matrices. We have a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Consider the set $\mathcal{C} = \left\{ \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

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We know that

$$[\mathbf{c}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}}$$

The change of basis matrix below is invertible,
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The change of basis matrix below is invertible, so $\mathcal{C}$ is also a basis.

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & -1 \\ 2 & 4 & 1 \end{bmatrix}$$

2. Write the matrix corresponding to $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}_{\mathcal{C}}$ in $\mathcal{B}$ coordinates.

Matrix Vector Space

Let V denote the vector space of symmetric 2×2 matrices. We have a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Consider the set $\mathcal{C} = \left\{ \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

1. Show that $\mathcal{C}$ is a basis for V.

We know that

$$[\mathbf{c}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}}$$

The change of basis matrix below is invertible, so $\mathcal{C}$ is also a basis.

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & -1 \\ 2 & 4 & 1 \end{bmatrix}$$

2. Write the matrix corresponding to $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}_{\mathcal{C}}$ in $\mathcal{B}$ coordinates.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & -1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} -13 \\ 3 \\ 3 \end{bmatrix}_{\mathcal{B}}$$

which is the matrix $\begin{bmatrix} -13 & 3 \\ 3 & 3 \end{bmatrix}$.