Systems of Linear Equations

Linear Algebra

This work was initially funded by an Institutional OER Grant from the Colorado Department of Higher Education (CDHE). For similar OER materials in other courses funded by this project in the Department of Mathematical and Statistical Sciences at the University of Colorado Denver, visit
https://github.com/CU-Denver-MathStats-OER

Linear Equations

A linear equation of $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where

- b and the coefficients $a_1, a_2, \ldots, a_n$ are constants (real numbers), and
- we have n variables denoted $x_1, x_2, \ldots, x_n$.

Linear Equations

A linear equation of $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where

- b and the coefficients $a_1, a_2, \ldots, a_n$ are constants (real numbers), and
- we have n variables denoted $x_1, x_2, \ldots, x_n$.

Example

Determine whether the equation is linear in x_1, x_2, and x_3.

a) $\cos\left(\frac{\pi}{3}\right)x_1 + e^4 x_2 - \frac{x_3}{\pi} = -5$

b) $3x_1 + 2x_1x_2 - 3x_3 = 8$

c) $2x_1 + 4\sqrt{x_2} - 3x_3 = 8$

Systems of Linear Equations

A system of linear equations is a collection of one or more linear equations involving the same variables $x_1, \ldots x_n$.

A system of two linear equations with two variables, x_1 and x_2.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned}$$

A system of three linear equations with four variables, x_1, x_2, x_3, and x_4.

$$\begin{aligned} 8x_1 + 7x_2 - x_3 - 5x_4 &= 10 \\ 2x_1 - 6x_2 - 12x_4 &= 0 \\ 0.5x_1 - 0.01x_2 + 2.1x_3 - 1.5x_4 &= -2 \end{aligned}$$

Systems of Linear Equations

A system of linear equations is a collection of one or more linear equations involving the same variables $x_1, \ldots x_n$.

A system of two linear equations with two variables, x_1 and x_2.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned}$$

A system of three linear equations with four variables, $x_1, x_2, x_3,$ and x_4.

$$\begin{array}{rrrrcr} 8x_1 & +7x_2 & -x_3 & -5x_4 & = & 10 \\ 2x_1 & -6x_2 & & -12x_4 & = & 0 \\ 0.5x_1 & -0.01x_2 & +2.1x_3 & -1.5x_4 & = & -2 \end{array}$$

Systems of Linear Equations

A system of linear equations is a collection of one or more linear equations involving the same variables $x_1, \ldots x_n$.

A system of two linear equations with two variables, x_1 and x_2.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned}$$

A system of three linear equations with four variables, x_1, x_2, x_3, and x_4.

$$\begin{array}{rrrrcr} 8x_1 & +7x_2 & -x_3 & -5x_4 & = & 10 \\ 2x_1 & -6x_2 & & -12x_4 & = & 0 \\ 0.5x_1 & -0.01x_2 & +2.1x_3 & -1.5x_4 & = & -2 \end{array}$$

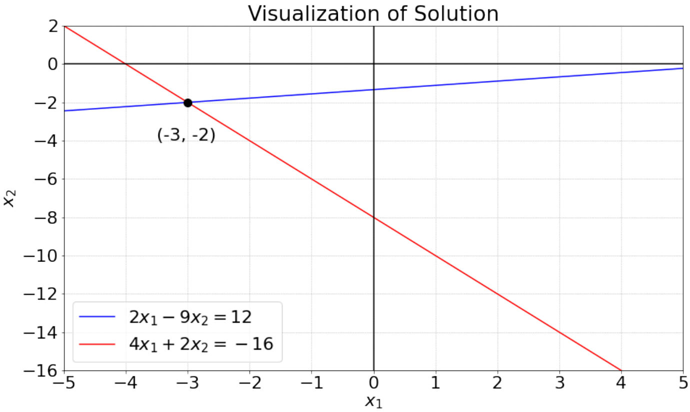
We frequently refer to a system of linear equations as a linear system.

Visualizing the Solution

A solution to a linear system is an ordered list of numbers $(s_1, s_2, \ldots s_n)$ that make all of the equations in the system true when we substitute $x_1 = s_1, \ldots, x_n = s_n$.

For example, $(x_1, x_2) = (-3, -2)$ is a solution to the system

$$2x_1 - 9x_2 = 12$$
$$4x_1 + 2x_2 = -16$$



Review: Solving a Linear System

You probably already have some strategies for solving some systems of equations. For example:

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned}$$

Review: Solving a Linear System

You probably already have some strategies for solving some systems of equations. For example:

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned}$$

1. Multiply top equation by -2 and add to bottom equation.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 0x_1 + 20x_2 &= -40 \end{aligned}$$

2. Divide bottom equation by 20.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 0x_1 + x_2 &= -2 \end{aligned}$$

3. Add 9 times bottom equation to top equation:

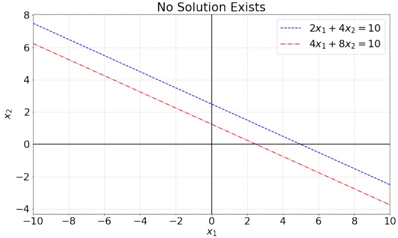
$$\begin{aligned} 2x_1 + 0x_2 &= -6 \\ 0x_1 + x_2 &= -2 \end{aligned}$$

4. Divide top equation by 2:

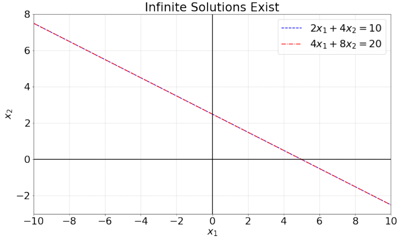
$$\begin{aligned} x_1 &= -3 \\ x_2 &= -2 \end{aligned}$$

How Many Solutions Exist?

$$2x_1 + 4x_2 = 10$$
$$4x_1 + 8x_2 = 10$$

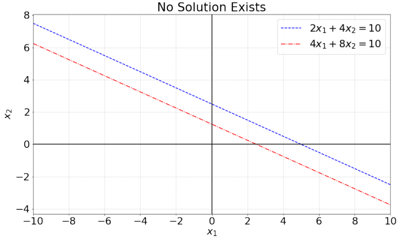


$$2x_1 + 4x_2 = 10$$
$$4x_1 + 8x_2 = 20$$

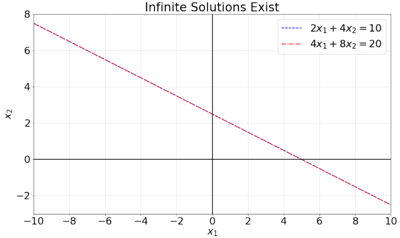


How Many Solutions Exist?

$$2x_1 + 4x_2 = 10$$
$$4x_1 + 8x_2 = 10$$



$$2x_1 + 4x_2 = 10$$
$$4x_1 + 8x_2 = 20$$



A system of linear equations may have:

- No solution. Such systems are called inconsistent, or
- Solutions (called consistent). How many solutions?
  - Exactly one solution.
  - Infinitely many solutions.

Matrix Notation

- Good news: There is a systemic way to solve a linear system (if a solution exists).
- Bad news: It can involve many algebraic steps.
- We can use a rectangular array called a matrix to help organize the work.

System of linear equations:

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned}$$

The coefficient matrix:

$$\begin{bmatrix} 2 & -9 \\ 4 & 2 \end{bmatrix}$$

The augmented matrix:

$$\begin{bmatrix} 2 & -9 & 12 \\ 4 & 2 & -16 \end{bmatrix}$$

Example

Example

Give the augmented matrix for the system of linear equations.

$$
\begin{aligned}
&8x_1 + 7x_2 - x_3 - 5x_4 = 10 \\
&2x_1 - 6x_2 - 12x_4 = 0 \\
&0.5x_1 - 0.01x_2 + 2.1x_3 - 1.5x_4 = -2
\end{aligned}
$$

Example

Example

Give the augmented matrix for the system of linear equations.

$$
\begin{aligned}
&8x_1 + 7x_2 - x_3 - 5x_4 = 10\\
&2x_1 - 6x_2 - 12x_4 = 0\\
&0.5x_1 - 0.01x_2 + 2.1x_3 - 1.5x_4 = -2
\end{aligned}
$$

Solution

$$
\begin{bmatrix}
8 & 7 & -1 & -5 & 10\\
2 & -6 & 0 & -12 & 0\\
0.5 & -0.01 & 2.1 & -1.5 & -2
\end{bmatrix}
$$

Describing the Size of a Matrix

How many rows does the augmented matrix have? How many columns?

$$\begin{bmatrix} 8 & 7 & -1 & -5 & 10 \\ 2 & -6 & 0 & -12 & 0 \\ 0.5 & -0.01 & 2.1 & -1.5 & -2 \end{bmatrix}$$

- The augmented matrix above has 3 rows. This means the system consists of 3 equations.
- The augmented matrix above has 5 columns. This means there are 4 variables.

Thus the augmented matrix above is a 3×5 (read as "3 by 5") matrix.

Describing the Size of a Matrix

How many rows does the augmented matrix have? How many columns?

$$\begin{bmatrix} 8 & 7 & -1 & -5 & 10 \\ 2 & -6 & 0 & -12 & 0 \\ 0.5 & -0.01 & 2.1 & -1.5 & -2 \end{bmatrix}$$

- The augmented matrix above has 3 rows. This means the system consists of 3 equations.
- The augmented matrix above has 5 columns. This means there are 4 variables.

Thus the augmented matrix above is a 3×5 (read as "3 by 5") matrix.

Caution! The order matters when describing the size of a matrix.
Always rows first, then columns second!

Solving a Linear System Using Matrices

Solve the following system.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned} \rightarrow \begin{bmatrix} 2 & -9 & 12 \\ 4 & 2 & -16 \end{bmatrix}$$

Solving a Linear System Using Matrices

Solve the following system.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 4x_1 + 2x_2 &= -16 \end{aligned} \rightarrow \begin{bmatrix} 2 & -9 & 12 \\ 4 & 2 & -16 \end{bmatrix}$$

1. Multiply top equation by -2 and add to bottom equation.

$$\begin{bmatrix} 2 & -9 & 12 \\ 0 & 20 & -40 \end{bmatrix}$$

2. Divide bottom equation by 20.

$$\begin{bmatrix} 2 & -9 & 12 \\ 0 & 1 & -2 \end{bmatrix}$$

3. Add 9 times bottom equation to top equation:

$$\begin{bmatrix} 2 & 0 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

4. Divide top equation by 2:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

Thus, the solution is $(x_1, x_2) = (-3, -2)$.

Going One Dimension Up

Solve the following system.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned}$$

Going One Dimension Up

Solve the following system.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned} \sim \begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 7 \\ 5 & 1 & 1 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 5 & 1 & 1 & 11 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & -4 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -6 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Thus $(x_1, x_2, x_3) = (1, 2, 4)$ is a solution of the linear system.

Elementary Row Operations

There are three operations we can apply to the rows of an augmented matrix:

1. Change one row by adding a multiple of another row to it.
2. Swapping two rows.
3. Scaling one row by multiplying all entries in the row by a nonzero constant.

$$\begin{bmatrix} 1 & -8 & 0 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Elementary Row Operations

There are three operations we can apply to the rows of an augmented matrix:

1. Change one row by adding a multiple of another row to it.
2. Swapping two rows.
3. Scaling one row by multiplying all entries in the row by a nonzero constant.

$$\begin{bmatrix} 1 & -8 & 0 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Two matrices are row equivalent if one matrix can be transformed into the other using elementary row operations.

Equivalent Systems

- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 7 \\ 5 & 1 & 1 & 11 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{ are row equivalent matrices!}$$

- If two linear systems have the same solutions, we say the two linear systems are equivalent.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 4 \end{aligned} \quad \text{are equivalent systems!}$$

Equivalent Systems

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

Equivalent Systems

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

Proof. Suppose L_1 is a linear system on variables $x_1, \ldots, x_n$, and L_2 is the resulting linear system after applying one elementary row operation. Suppose $(s_1, \ldots, s_n)$ is a solution to L_1. Then $(s_1, \ldots, s_n)$ is also a solution to L_2.

1. Swapping two rows.

Equivalent Systems

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

Proof. Suppose L_1 is a linear system on variables $x_1, \ldots, x_n$, and L_2 is the resulting linear system after applying one elementary row operation. Suppose $(s_1, \ldots, s_n)$ is a solution to L_1. Then $(s_1, \ldots, s_n)$ is also a solution to L_2.

1. Swapping two rows.
2. Scaling a row: $a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \quad \Rightarrow \quad ca_{11}x_1 + \ldots + ca_{1n}x_n = cb_1$.

Equivalent Systems

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

Proof. Suppose L_1 is a linear system on variables $x_1, \dots, x_n$, and L_2 is the resulting linear system after applying one elementary row operation. Suppose $(s_1, \dots, s_n)$ is a solution to L_1. Then $(s_1, \dots, s_n)$ is also a solution to L_2.

1. Swapping two rows.
2. Scaling a row: $a_{11}x_1 + \dots + a_{1n}x_n = b_1 \quad \Rightarrow \quad ca_{11}x_1 + \dots + ca_{1n}x_n = cb_1$.
3. Adding a multiple of a row to another row:

$$\begin{array}{ccccc} a_{11}x_1 & +\dots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 & +\dots & +a_{2n}x_n & = & b_2 \end{array} \Rightarrow \begin{array}{ccccc} (a_{11}+ca_{21})x_1 & +\dots & +(a_{1n}+ca_{2n})x_n & = & b_1+cb_2 \\ a_{21}x_1 & +\dots & +a_{2n}x_n & = & b_2 \end{array}$$

Equivalent Systems

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

Proof. Suppose L_1 is a linear system on variables $x_1, \ldots, x_n$, and L_2 is the resulting linear system after applying one elementary row operation. Suppose $(s_1, \ldots, s_n)$ is a solution to L_1. Then $(s_1, \ldots, s_n)$ is also a solution to L_2.

Is every solution to L_2 also a solution to L_1?

Equivalent Systems

Theorem

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

Proof. Suppose L_1 is a linear system on variables $x_1, \ldots, x_n$, and L_2 is the resulting linear system after applying one elementary row operation. Suppose $(s_1, \ldots, s_n)$ is a solution to L_1. Then $(s_1, \ldots, s_n)$ is also a solution to L_2.

Is every solution to L_2 also a solution to L_1?

Note that every elementary row operation is reversible (why we scale by nonzero constants!).
Hence, L_2 can be transformed into L_1 by applying one elementary row operation.
From the same argument as above, every solution to L_2 is also a solution to L_1.

$\square$