

# The Determinant of a Matrix

## Linear Algebra

These materials were created by Adam Spiegler, Stephen Hartke, and others, and are licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

This work was initially funded by an Institutional OER Grant from the Colorado Department of Higher Education (CDHE). For similar OER materials in other courses funded by this project in the Department of Mathematical and Statistical Sciences at the University of Colorado Denver, visit <https://github.com/CU-Denver-MathStats-OER>

# The Invertible Matrix Theorem redux

Let  $A$  be a square  $n \times n$  matrix. Then all of the following statements are **equivalent**.

- (a)  $A$  is an invertible matrix.
- (b)  $A$  is row equivalent to the  $n \times n$  identity matrix  $I_n$ .
- (c)  $A$  has  $n$  pivots.
- (d) The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- (e) The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one.
- (h) The equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- (i) The columns of  $A$  span  $\mathbb{R}^n$ .
- (j) The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- (l) There is an  $n \times n$  matrix  $C$  such that  $CA = I_n$ .
- (m) There is an  $n \times n$  matrix  $D$  such that  $AD = I_n$ .
- (n)  $A^T$  is an invertible matrix.

# Inverse of a $2 \times 2$ Matrix

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

when  $ad - bc \neq 0$ .

If  $ad - bc = 0$ , then  $A$  is not invertible.

Consider  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

► If both  $a$  and  $c$  are equal to 0,  $A$  is not invertible. Assume  $a \neq 0$ .

► Then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

► As long as  $ad - bc \neq 0$ ,  $A$  has a pivot in every column and is invertible.

The **determinant**  $\det A$  of a  $2 \times 2$  matrix  $A$  is  $\det A = ad - bc = a_{11}a_{22} - a_{12}a_{21}$ .

## Examples of Determinants of $2 \times 2$ Matrices

Compute the determinant of each matrix and determine whether the matrix has an inverse or not? If the matrix is invertible, give its inverse.

a)  $\begin{bmatrix} 2 & -4 \\ 3 & 8 \end{bmatrix}$

b)  $\begin{bmatrix} -5 & 2 \\ -10 & 4 \end{bmatrix}$

# Determinant of larger matrices

We want a function called the **determinant** defined on square  $n \times n$  matrices that produces a scalar with the following properties:

1.  $\det A \neq 0$  if and only if  $A$  is **invertible**.

# Determinant of larger matrices

We want a function called the **determinant** defined on square  $n \times n$  matrices that produces a scalar with the following properties:

1.  $\det A \neq 0$  if and only if  $A$  is **invertible**.
2.  $\det(AB) = (\det A)(\det B)$ .
3.  $\det(I_n) = 1$ .
4.  $\det$  is multilinear.

This ends up making a complicated (**but useful!**) function.

# Determinant of a $1 \times 1$ Matrix

Let  $A = [a_{11}]$ . What is  $\det A$ ?

# Determinant of a $1 \times 1$ Matrix

Let  $A = [a_{11}]$ . What is  $\det A$ ?

$A$  is invertible if and only if  $a_{11} \neq 0 \Rightarrow \det A = a_{11}$ .



# Determinant of a $3 \times 3$ Matrix

Let  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ .

- ▶ The **submatrix** obtained by deleting entries in the first row and first column we denote  $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ .
- ▶ The submatrix obtained by deleting entries in the first row and second column we denote  $A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ .
- ▶ In general, the submatrix obtained by deleting entries in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column we denote  $A_{ij}$ .

# Determinant of a $3 \times 3$ Matrix

$$\text{Let } A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

We can compute the determinant of matrix  $A$  as follows:

$$\begin{aligned} \det A &= a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) \\ &= (3) \left( \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) - (-1) \left( \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right) + (2) \left( \det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right) \\ &= (3)(2) - (-1)(1) + (2)(-4) \\ &= -1 \end{aligned}$$

# Determinant of a $3 \times 3$ Matrix

$$\text{Let } A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

$$\begin{aligned} \det A &= a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) \\ &= (3)(2) - (-1)(1) + (2)(-4) = -1 \end{aligned}$$

Or we have:

$$\begin{aligned} \det A &= -a_{21}(\det A_{21}) + a_{22}(\det A_{22}) - a_{23}(\det A_{23}) = -(1) \left( \det \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \right) + (2) \left( \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \right) - 0 \\ &= -(1)(-1) + (2)(-1) - 0 = -1 \end{aligned}$$

Or we have

$$\begin{aligned} \det A &= a_{31}(\det A_{31}) - a_{32}(\det A_{32}) + a_{33}(\det A_{33}) = (2) \left( \det \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix} \right) - 0 + (1) \left( \det \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \right) \\ &= (2)(-4) - 0 + (1)(7) = -1 \end{aligned}$$

# Determinant of a $3 \times 3$ Matrix

Let  $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ .

We have

- ▶  $\det A = a_{11} (\det A_{11}) - a_{12} (\det A_{12}) + a_{13} (\det A_{13})$ , or
- ▶  $\det A = -a_{21} (\det A_{21}) + a_{22} (\det A_{22}) - a_{23} (\det A_{23})$ , or
- ▶  $\det A = a_{31} (\det A_{31}) - a_{32} (\det A_{32}) + a_{33} (\det A_{33})$ .

Choosing to expand across a row with 0's requires less calculations!

The sign associated with the coefficient  $a_{ij}$  in front of each determinant is positive if  $i + j$  is even and negative if  $i + j$  is odd.

# Examples of Determinants of $3 \times 3$ Matrices

Compute the determinant of each matrix.

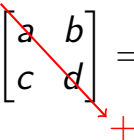
a) 
$$\begin{bmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{bmatrix}$$

b) 
$$\begin{bmatrix} 3 & 7 & -5 \\ 0 & -2 & 16 \\ 0 & 0 & 5 \end{bmatrix}$$

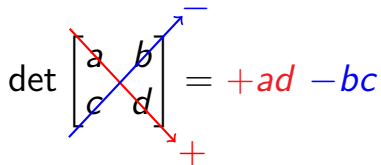
## Mnemonic Device for $2 \times 2$ and $3 \times 3$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

# Mnemonic Device for $2 \times 2$ and $3 \times 3$ Matrices

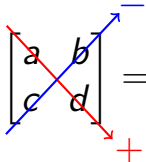
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad$$


# Mnemonic Device for $2 \times 2$ and $3 \times 3$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad - bc$$


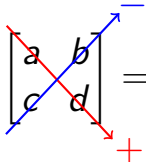


# Mnemonic Device for $2 \times 2$ and $3 \times 3$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad - bc$$


$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

# Mnemonic Device for $2 \times 2$ and $3 \times 3$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad - bc$$


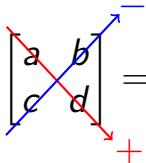
$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

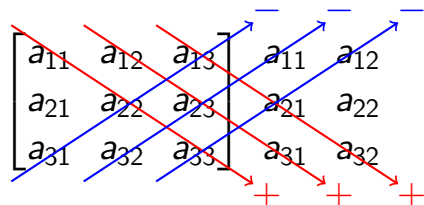
# Mnemonic Device for $2 \times 2$ and $3 \times 3$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad - bc$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

# Mnemonic Device for $2 \times 2$ and $3 \times 3$ Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad - bc$$


$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$


# Determinants of $4 \times 4$ Matrices

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 10 & 2 & -3 \\ 2 & 5 & -1 & -6 \\ 0 & 7 & -5 & 1 \end{bmatrix}.$$

Then we have

$$\det A = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) - a_{14}(\det A_{14})$$

$$(1) \left( \det \begin{bmatrix} 10 & 2 & -3 \\ 5 & -1 & -6 \\ 7 & -5 & 1 \end{bmatrix} \right) - (2) \left( \det \begin{bmatrix} 4 & 2 & -3 \\ 2 & -1 & -6 \\ 0 & -5 & 1 \end{bmatrix} \right) + (3) \left( \det \begin{bmatrix} 4 & 10 & -3 \\ 2 & 5 & -6 \\ 0 & 7 & 1 \end{bmatrix} \right) - (-2) \left( \det \begin{bmatrix} 4 & 10 & 2 \\ 2 & 5 & -1 \\ 0 & 7 & -5 \end{bmatrix} \right)$$

$$\det \begin{bmatrix} 10 & 2 & -3 \\ 5 & -1 & -6 \\ 7 & -5 & 1 \end{bmatrix} = (10) \left( \det \begin{bmatrix} -1 & -6 \\ -5 & 1 \end{bmatrix} \right) - (2) \left( \det \begin{bmatrix} 5 & -6 \\ 7 & 1 \end{bmatrix} \right) + (-3) \left( \det \begin{bmatrix} 5 & -1 \\ 7 & -5 \end{bmatrix} \right)$$

and so on!!!!

## Example of Determinant of a $4 \times 4$ Matrix

Compute the determinant of  $A = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 2 & 3 & -4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ .

## Example of Determinant of a $4 \times 4$ Matrix

Compute the determinant of  $A = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 2 & 3 & -4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ .

### Theorem

If  $A$  is a **triangular matrix**, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .

# Determinants of $n \times n$ Matrices

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 10 & 2 & -3 \\ 2 & 5 & -1 & -6 \\ 0 & 7 & -5 & 1 \end{bmatrix}.$$

Then we have

$$\det A = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) - a_{14}(\det A_{14})$$

In general, if  $A$  is an  $n \times n$  matrix, then we define:

$$\det A = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + \dots + (-1)^{1+n} a_{1n}(\det A_{1n})$$

$$= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= \sum_{j=1}^n (-1)^{2+j} a_{2j} \det A_{2j} = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{ij}$$



# Computing Determinants with Python

Compute the determinant of  $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ .

We can use `A.det()` in SymPy.

# Computing Determinants with Python

Compute the determinant of  $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ .

We can use `A.det()` in SymPy.

Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 10 & 2 & -3 \\ 2 & 5 & -1 & -6 \\ 0 & 7 & -5 & 1 \end{bmatrix}.$$

# Cofactor Expansion

In general, if  $A$  is an  $n \times n$  matrix, then we define:

$$\det A = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j} = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + \dots + (-1)^{1+n} a_{1n}(\det A_{1n})$$

- ▶ We call the expression  $(-1)^{i+j} \det A_{ij}$  the  $(i, j)$  **cofactor**.
- ▶ We can rewrite the formulas in the definition above in terms of cofactors:

$$\det A = \sum_{j=1}^n a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}.$$

- ▶ The formula above is the **cofactor expansion across the 1st row**.
- ▶ In general, we can use the cofactor expansion **across any row**:

$$\det A = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}.$$

# Cofactor Expansion Across Rows or Down Columns

## Theorem

The determinant of an  $n \times n$  matrix  $A$  can be computed by a cofactor expansion **across any row** or **down any column**.

- ▶ The cofactor expansion across the  $i^{\text{th}}$  row is given by:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}.$$

- ▶ The cofactor expansion down the  $j^{\text{th}}$  column is given by:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}.$$

# Comparing Methods

$$\text{Let } A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

We can compute the determinant of matrix  $A$  using the **cofactor expansion across the first row**:

$$\begin{aligned} \det A &= a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) \\ &= (3) \left( \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) - (-1) \left( \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right) + (2) \left( \det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right) \\ &= (3)(2) - (-1)(1) + (2)(-4) = -1 \end{aligned}$$

We can compute the determinant of matrix  $A$  using the **cofactor expansion down column 2**:

$$\begin{aligned} \det A &= -a_{12}(\det A_{12}) + a_{22}(\det A_{22}) - a_{32}(\det A_{32}) \\ &= -(-1) \left( \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right) + (2) \left( \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \right) - (0) \left( \det \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \right) \\ &= (1)(1) + (2)(-1) - 0 = -1 \end{aligned}$$