The Characteristic Equation

Linear Algebra

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$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \det\begin{bmatrix} 0-\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix} \\
&= (0-\lambda)(5-\lambda) - (1)(-6) \\
&= \lambda^2 - 5\lambda + 6
\end{aligned}$$

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- Does $A\mathbf{x} = \lambda\mathbf{x}$ have a nontrivial solution?
- Does $(A - \lambda I)\mathbf{x} = \mathbf{0}$ have a nontrivial solution?
- Is the matrix $(A - \lambda I)$ NOT invertible?
- Is $\det(A - \lambda I) = 0$?

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To find all eigenvalues, we solve the following equation:

$$\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 = 0.$$

- The scalar equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A.
- The polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of A.
- A scalar λ is an eigenvalue of A if and only if λ satisfies $\det(A - \lambda I) = 0$.

Example

Find and solve the characteristic equation

$$\text{of } A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}.$$

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$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & -5-\lambda & 0 \\ 1 & 8 & 1-\lambda \end{bmatrix} \\ &= (-5-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \\ &= -(5+\lambda)\left[(1-\lambda)^2 - 1\right] \\ &= -(5+\lambda)(\lambda^2 - 2\lambda) \\ &= -\lambda(5+\lambda)(\lambda - 2) \end{aligned}$$

The characteristic equation $-\lambda(5+\lambda)(\lambda-2) = 0$ has solutions $\lambda = 0, -5$, and 2.

So the eigenvalues of A are $\lambda = 0, -5$, and 2.

Invertible Matrix Theorem Continued, Even More!

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- Zero is an eigenvalue of A if and only if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$, which happens if and only if $0 = \det(A - 0I) = \det A$.

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- $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if A is NOT invertible.
- Hence A is invertible if and only if 0 is NOT an eigenvalue.

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- $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if A is NOT invertible.
- Hence A is invertible if and only if 0 is NOT an eigenvalue.

Theorem (The Invertible Matrix Theorem (continued))

Let A be an $n \times n$ matrix. Then the following are equivalent statements:

(a) A is an invertible matrix.

(r) The number 0 is not an eigenvalue of A.

Eigenvalues of Triangular Matrices

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 & 0 \\ 6 & 7 & 8 & 9 & 0 & 0 \\ -1 & -2 & -3 & -4 & -5 & 0 \\ -6 & -7 & -8 & -9 & -10 & 1 \end{bmatrix}.$$

- If A is a triangular matrix, then the entries on the main diagonal are the eigenvalues of A.
- The algebraic multiplicity of an eigenvalue is the multiplicity of the corresponding root of the characteristic equation of A.

Similar Matrices

Definition

If A and B are two $n \times n$ matrices, we say A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or equivalently $A = PBP^{-1}$.

- If A is similar to B, then B is similar to A.

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$$A = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \text{ is similar to } B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ since}$$

$$P^{-1}AP = \begin{bmatrix} -6 & -4 & -1 \\ -3 & -2 & -1 \\ 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = B.$$

Eigenvalues of Similar Matrices

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\begin{aligned}
B - \lambda I &= P^{-1}AP - \lambda P^{-1}P \\
&= P^{-1}AP - P^{-1}(\lambda I)P \\
&= P^{-1}(A - \lambda I)P.
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This gives

$$\begin{aligned} \det(B - \lambda I) &= \det\left(P^{-1}(A - \lambda I)P\right) \\ &= (\det P^{-1})(\det(A - \lambda I))(\det P) \\ &= (\det P^{-1})(\det P)(\det(A - \lambda I)) \\ &= \det(A - \lambda I). \end{aligned}$$

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Since $\det(B - \lambda I) = \det(A - \lambda I)$, the two similar matrices have the same characteristic polynomial, and thus the same eigenvalues. □

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Recall A is similar to the diagonal matrix B below.

$$\begin{bmatrix} -6 & -4 & -1 \\ -3 & -2 & -1 \\ 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Since B is a triangular (diagonal) matrix, the eigenvalues of B are $\lambda = -1, 3$, where $\lambda = -1$ has multiplicity 2. Thus A has eigenvalues $\lambda = -1, 3$, where $\lambda = -1$ has multiplicity 2.

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Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix}$$

Both matrices have eigenvalues $\lambda = 1, 5$, however they are not similar matrices.

Similarity is not the same as row equivalence. The two matrices above are row equivalent, but they are not similar since there is no matrix P such that $P^{-1}AP = B$.