Applications of Determinants

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Using Determinants to Solve Systems of Equations

$$\begin{array}{rcrcl} 3x_1 & - & 2x_2 & = & 6 \\ -5x_1 & + & 4x_2 & = & 8 \end{array}$$

For a square matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$ and column vector $\mathbf{b}$, if we replace the j^{th} column of A with $\mathbf{b}$, we denote the new matrix

$$A_j(\mathbf{b}) = A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{b} & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}.$$

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Let $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$. Then

$$A_1(\mathbf{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \quad \text{and} \quad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}$, where the j^{th} entry of $\mathbf{x}$ is given by

$$x_j = \frac{\det A_j(\mathbf{b})}{\det A} \quad \text{for } j = 1, 2, \ldots, n.$$

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$$\det A = 2 \qquad \det A_1(\mathbf{b}) = 40 \qquad \det A_2(\mathbf{b}) = 54$$

$$\mathbf{x} = \begin{bmatrix} \frac{40}{2} \\ \\ \frac{54}{2} \end{bmatrix} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}$$

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$$\begin{aligned} AI_j(\mathbf{x}) &= A \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_{j-1} & \mathbf{x} & \mathbf{e}_{j+1} & \dots & \mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} A\mathbf{e}_1 & \dots & A\mathbf{e}_{j-1} & A\mathbf{x} & A\mathbf{e}_{j+1} & \dots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{b} & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} = A_j(\mathbf{b}). \end{aligned}$$

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Using the multiplicative property of determinants, we have

$$\begin{aligned} \det A_j(\mathbf{b}) &= \det\left(AI_j(\mathbf{x})\right) \\ &= (\det A)\left(\det I_j(\mathbf{x})\right) = (\det A)\, x_j. \end{aligned}$$

Therefore, we have $x_j = \dfrac{\det A_j(\mathbf{b})}{\det A}$.

□

Formula for the Inverse of a Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, when $\det A = ad - bc \neq 0$.

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Let A be an $n \times n$ matrix. The adjugate $\operatorname{adj} A$ of A (also called *classical adjoint*) is defined by

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{i1} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{i2} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ C_{1i} & C_{2i} & \dots & C_{ji} & \dots & C_{ni} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{jn} & \dots & C_{nn} \end{bmatrix},$$

where the cofactor $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Note the reversed indices!!!

The adjugate is the transpose of the matrix of cofactors.

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Proof 1.

Let $D = A^{-1}$. Then $AD = I$. To find the jth column $\mathbf{d}_j$ of D, we solve $A\mathbf{d}_j = \mathbf{e}_j$.
To find the ith entry of $\mathbf{d}_j$, we apply Cramer's Rule:

$$
\begin{aligned}
i\text{th entry of } \mathbf{d}_j = \frac{\det A_i(\mathbf{e}_j)}{\det A} &= \tfrac{1}{\det A} \det \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{e}_j & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \end{bmatrix} \\
&= \tfrac{1}{\det A} (-1)^{j+i} \det A_{ji} \quad \text{by expanding about column } i \\
&= \tfrac{1}{\det A} C_{ji}.
\end{aligned}
$$

Hence, the ijth entry of A^{-1} is $\tfrac{1}{\det A} C_{ji}$.

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Proof 2.

We show that $A\,(\operatorname{adj} A) = (\det A) I_n$.

The i, j entry of $A\,(\operatorname{adj} A)$ is $\displaystyle\sum_{k=1}^{n} a_{ik} (\operatorname{adj} A)_{kj} = \sum_{k=1}^{n} a_{ik}\, C_{jk}. = \sum_{k=1}^{n} a_{ik} (-1)^{j+k} \det A_{jk}.$

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This is the determinant of A after replacing the jth row with the ith row, which is:

$\det A$, if $i = j$; $\qquad 0$, if $i \neq j$ (since there is a repeated row).

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Since A is invertible, $A^{-1} A\,(\operatorname{adj} A) = A^{-1} (\det A) I_n$. Thus, $A^{-1} = \dfrac{1}{\det A}\,\operatorname{adj} A$.

Example

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$$\operatorname{adj} A = \begin{bmatrix} -8 & -9 & 6 \\ -2 & -2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

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$$\begin{aligned} (\operatorname{adj} A)_{21} = C_{12} &= (-1)^{1+2} \det A_{12} \\ &= (-1)^3[0 \cdot 1 - 1 \cdot (-2)] = -2. \end{aligned}$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{-2} \begin{bmatrix} -8 & -9 & 6 \\ -2 & -2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 9/2 & -3 \\ 1 & 1 & -1 \\ -1 & -3/2 & 1 \end{bmatrix}.$$

Volumes

Volume of 2d Parallelogram

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$.

The origin, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ form a parallelogram $\mathcal{P}$.

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Theorem

The area of the parallelogram $\mathcal{P}$ in $\mathbb{R}^2$ is $|\det(A)|$.

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Proof sketch:

1. Rotate the parallelogram so that $\mathbf{u}$ is on the positive x-axis.
2. Flip the parallelogram (if necessary) so that $\mathbf{v}$ is above the x-axis.
3. Horizontally scale the parallelogram so that $\mathbf{u} = (1, 0)$.
4. Horizontally shear the parallelogram so that $\mathbf{v}$ is on the y-axis.
5. Vertically scale the parallelogram so that $\mathbf{v} = (0, 1)$.

Volume of 3d Parallelopiped

Three vectors **u**, **v**, and **w** in $\mathbb{R}^3$ form a parallelopiped with vertices

$$\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{v}+\mathbf{w}.$$

`https://www.geogebra.org/m/VvR6MeR5`

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Theorem

The volume of the parallelopiped $\mathcal{P}$ in $\mathbb{R}^3$ is $|\det(A)|$, where A has rows **u**, **v**, and **w**.

This works in higher dimensions: for an $n \times n$ matrix A, the volume of the parallelotope in $\mathbb{R}^n$ corresponding to the rows of A is given by $|\det(A)|$.

Linear Transformations

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation where $\mathbf{x} \mapsto A\mathbf{x}$.
Consider the unit parallelotope $\mathcal{U}$ with volume 1 determined by the origin and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

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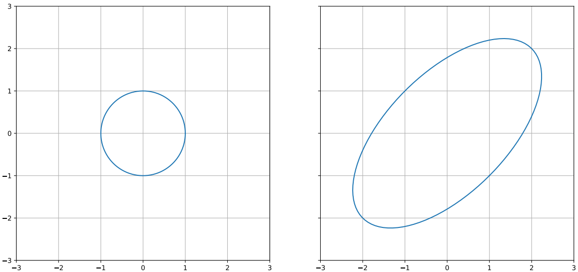
$T(\mathcal{U})$ is a parallelotope determined by the columns of $AI = A$. So its volume is $|\det A|$.

Since T is a linear transformation, the volume of any shape is scaled by $|\det A|$,
ie, if $\mathcal{R}$ is a shape in $\mathbb{R}^n$ with volume r, then $T(\mathcal{R})$ has volume $|\det A|r$.

Example: Area of an Ellipse

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, with associated matrix A.
The image of the unit circle centered at the origin is an ellipse. What is the area of the ellipse?



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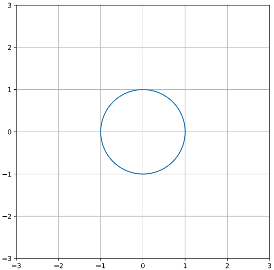

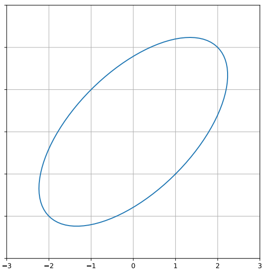

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The area of the unit circle is $\pi r^2 = \pi$.

The area of the ellipse is $|\det A|\pi = 4\pi$.

Change of Variable in Integration

Consider the integral $\int f(x)\ dx$, and a change of variables $x = g(u)$.

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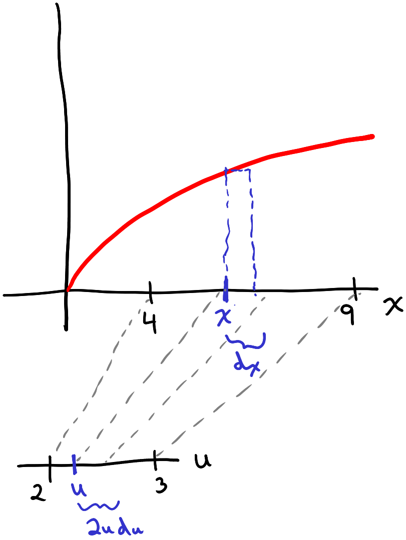
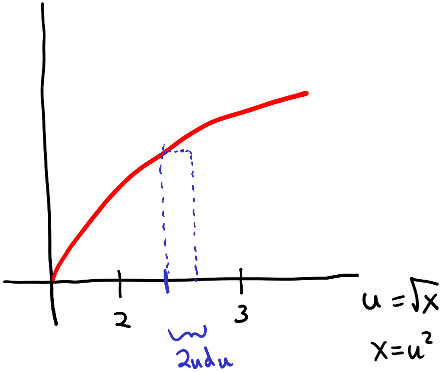
Consider $\int_a^b f(x)\ dx = \int_4^9 \sqrt{x}\ dx$. Let $x = u^2$.

Then $= \int_{\sqrt{4}}^{\sqrt{9}} \sqrt{u^2}\ 2u\ du = \int_2^3 2u^2\ du = \frac{2}{3}u^3\Big|_2^3 = \frac{2}{3}\left(3^3 - 2^3\right) = \frac{38}{3}$.


4
x
9
x
dx
2
u
3
u
2udu


2
3
2udu
u = √x
x=u²

Polar Coordinates in 2 dimensions

Consider the integral $\iint f(x,y)\,dx\,dy$, and a change of variables $\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$.

https://www.desmos.com/calculator/bfn2hgtvqp

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The correction needed describes how the area of a little rectangle $dx \times dy$ changes with respect to r and θ, and is given by the determinant of the Jacobian matrix:

$$\mathbf{J}(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

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$$\begin{aligned} \text{Then } \iint f(x,y)\; dx\, dy &= \iint f(r\cos\theta, r\sin\theta)\; \det \mathbf{J}\; dr\, d\theta \\ &= \iint f(r\cos\theta, r\sin\theta)\; r\; dr\, d\theta. \end{aligned}$$

Spherical Coordinates in 3 dimensions

Consider the integral $\iiint f(x, y, z)\ dx\, dy\, dz$, and a change of variables $\substack{x=\rho \sin\varphi \cos\theta \\ y=\rho \sin\varphi \sin\theta \\ z=\rho \cos\varphi}$.

https://www.geogebra.org/m/h9xS5ZZs

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$$\text{Then } \iiint f(x,y,z)\ dx\,dy\,dz = \iiint f(x,y,z)\ \det \mathbf{J}\ d\rho\,d\varphi\,d\theta$$

$$= \iiint f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ \rho^2\sin\varphi\ d\rho\,d\varphi\,d\theta.$$