The Gram–Schmidt Orthogonalization Process

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Introduction

- We have seen that an orthogonal basis for a subspace W of $\mathbb{R}^n$ is particularly nice.
  - For $\mathbf{y}$ in $\mathbb{R}^n$, we compute the orthogonal projection of $\mathbf{y}$ onto W by

$$\hat{\mathbf{y}} = \text{proj}_W \mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_p\mathbf{u}_p \quad \text{with weights } c_i = \frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}.$$

- When we choose an orthonormal basis, the calculations are even simpler.

$$\hat{\mathbf{y}} = \text{proj}_W \mathbf{y} = (UU^T)\mathbf{y}.$$

- Recall $\hat{\mathbf{y}}$ is a useful vector to find as it is the best approximation to $\mathbf{y}$ in the subspace W.

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Example

Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ denote the subspace of $\mathbb{R}^3$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

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- Let $\mathbf{v}_1 = \mathbf{x}_1$.
- Let $\mathbf{v}_2$ be the orth proj of $\mathbf{x}_2$ onto the orth compl of $Y = \text{Span}\{\mathbf{v}_1\}$.

  By the Orth Decomp Thm, $\mathbf{x}_2 = \text{proj}_Y \mathbf{x}_2 + \mathbf{z}$, where $\mathbf{z}$ is in $Y^\perp$.

$$\text{proj}_Y \mathbf{x}_2 = \text{proj}_{\mathbf{v}_1} \mathbf{x}_2 = \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 = \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

- Hence $\mathbf{v}_2 = \mathbf{z} = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

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- Since $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{x}_2 = \mathbf{x}_2 + c\mathbf{x}_1$, $\mathbf{v}_2$ is in the subspace W.

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- Thus $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis for W.

Given a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$\mathbf{v}_1 = \mathbf{x}_1$ $\qquad$ $Y_1 = \text{Span}\{\mathbf{v}_1\}$

The Gram–Schmidt Orthogonalization Process

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Theorem

Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W.

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Theorem

Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition,

$$\operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = Y_k = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \quad \text{for } 1 \le k \le p.$$

Example

Find an orthogonal basis for the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$.

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Then we find $\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \left(\frac{36}{18}\right) \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$.

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \right\}$.

Example

Consider the matrix $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$. Find an orthogonal basis for the column space of A.

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First we find a basis for the column space by checking the reduced row echelon form of A:

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so a basis for Col } A \text{ is } \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right\}.$$

Next we use the Gram–Schmidt Process to convert the basis into an orthogonal basis.

Example, continued

$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} \right\}$. We first define $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

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Then we continue with Gram–Schmidt process:

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\frac{-40}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

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Normalizing the Basis

For the 4×3 matrix $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$, we therefore have

- An orthogonal basis for Col A is $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

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- An orthonormal basis for Col A is $\left\{ \begin{bmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{20} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} -3/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix} \right\}$.