

Homogeneous Systems of Linear Equations

Linear Algebra

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Homogeneous Solutions

A system of equations is **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$.

$$x_1 + x_2 + x_3 = 7$$

$$x_1 - x_2 + 2x_3 = 7$$

$$5x_1 + x_2 + x_3 = 11$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 11 \end{bmatrix}$$

NOT a homogeneous system.

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YES, this is a homogeneous system.

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Every homogeneous system has a **trivial solution** $\mathbf{x} = \mathbf{0}$, and so is **consistent**.

Does the homogeneous system have a **nontrivial solution**? ie, infinite number of solutions?

Existence of Nontrivial Solutions

Recall that if the system $A\mathbf{x} = \mathbf{b}$ is **consistent**, then it has:

- ▶ A **unique solution** if, when forming the RREF of A , there is a pivot position in every column (ie, all variables are **basic**).
- ▶ An **infinite number** of solutions if there is a column of A without a pivot position (ie, there is at least one **free variable**).

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the solution set has at least one free variable.

Example

Find the solution set to the **homogeneous** system

$$\begin{array}{rrcr} x_1 & + & 3x_2 & + & 7x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \\ -2x_1 & - & 4x_2 & - & 8x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ -2 & -4 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcolor{red}{1} & 0 & \textcolor{blue}{-2} & 0 \\ 0 & \textcolor{red}{1} & \textcolor{blue}{3} & 0 \\ 0 & 0 & \textcolor{blue}{0} & 0 \end{bmatrix}$$

Notice x_3 is a free variable, and the solution set can therefore be given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = x_3 \mathbf{v}, \text{ where } \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Example

Find the solution set to the homogeneous system

$$\begin{array}{rcrcrcrcrcl} 3x_1 & - & 2x_2 & + & x_3 & = & 0 \\ -6x_1 & + & 4x_2 & - & 2x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 3 & -2 & 1 & 0 \\ -6 & 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Both x_2 and x_3 are free variables, and the solution set can therefore be given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 - \frac{1}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} = x_2 \mathbf{u} + x_3 \mathbf{v}.$$

Nonhomogeneous Linear Systems

Find the solution set to the **nonhomogeneous** system

$$\begin{array}{rcccccccl} x_1 & + & 3x_2 & + & 7x_3 & = & 4 \\ & & x_2 & + & 3x_3 & = & 5 \\ -2x_1 & - & 4x_2 & - & 8x_3 & = & 2 \end{array}$$

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$$\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ -2 & -4 & -8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -11 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice x_3 is a free variable, and the solution set can therefore be given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 + 2x_3 \\ 5 - 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}).$$

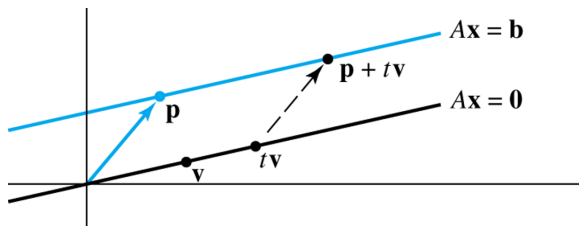
Comparing Homogeneous and Nonhomogeneous Solutions

$$\begin{aligned}x_1 + 3x_2 + 7x_3 &= 0 \\x_2 + 3x_3 &= 0 \\-2x_1 - 4x_2 - 8x_3 &= 0\end{aligned}$$

$$\begin{aligned}x_1 + 3x_2 + 7x_3 &= 4 \\x_2 + 3x_3 &= 5 \\-2x_1 - 4x_2 - 8x_3 &= 2\end{aligned}$$

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$$\mathbf{x} = \begin{bmatrix} -11 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \mathbf{p} + t\mathbf{v} \quad (t \in \mathbb{R}).$$



Theorem

Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and has a solution $\mathbf{x} = \mathbf{p}$. If \mathbf{v} is **any** solution to the **homogeneous** equation $A\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = \mathbf{p} + \mathbf{v}$$

is a solution to the **nonhomogeneous** equation $A\mathbf{x} = \mathbf{b}$.

Proof.

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Proof.

We have $A\mathbf{x} = A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. □

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Theorem

Every solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{p} + \mathbf{v}$ for **some** solution \mathbf{v} of $A\mathbf{x} = \mathbf{0}$.

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Theorem

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Theorem

Every solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{p} + \mathbf{v}$ for **some** solution \mathbf{v} of $A\mathbf{x} = \mathbf{0}$.

Proof.

Let \mathbf{y} be a solution to $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{v} = \mathbf{y} - \mathbf{p}$. Then $A\mathbf{v} = A(\mathbf{y} - \mathbf{p}) = A\mathbf{y} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Hence, \mathbf{v} is a solution to $A\mathbf{x} = \mathbf{0}$. □