Matrix Equations

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

Revisiting Linear Combinations

As we have previously seen, a very fundamental question in linear algebra is determining whether a vector $\mathbf{b}$ in $\mathbb{R}^n$ can be written as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ all in in $\mathbb{R}^n$.

Example

Determine whether $\mathbf{b} = \begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

Do there exist real numbers (called weights) x_1 and x_2 such that

$$\begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} ?$$

Multiplying a Matrix and a Vector

Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ (each column in the matrix is in $\mathbb{R}^m$), and let $\mathbf{x}$ denote a column vector in $\mathbb{R}^n$. Then we define the product of matrix A and column vector $\mathbf{x}$ as

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Example

Do there exist real numbers (called weights) x_1 and x_2 such that

$$\begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}?$$

Example

If possible, compute the product $A\mathbf{x}$.

1. $A = \begin{bmatrix} 2 & -4 & 3 & 1 \\ 6 & 2 & 1 & 9 \\ 1 & 0 & 2 & -1 \end{bmatrix}_{3\times 4}$ and $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 5 \\ 2 \end{bmatrix}_{4\times 1}$

2. $A = \begin{bmatrix} 2 & -4 & 3 & 1 \\ 6 & 2 & 1 & 9 \\ 1 & 0 & 2 & -1 \end{bmatrix}_{3\times 4}$ and $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}_{3\times 1}$

(!!) $A\mathbf{x}$ is only defined if the number of columns of A equals the number of rows in $\mathbf{x}$.

Determine whether $\mathbf{b} = \begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix}$ is in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

We can set up the following matrix equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix}$$

Which is equivalent to solving the system of linear equations:

$$\begin{array}{rcrcr} x_1 & & & = & 6 \\ & & x_2 & = & -2 \\ -2x_1 & + & -x_2 & = & -10 \end{array}.$$

If A is an $m \times n$ matrix with columns columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$$

which has corresponding augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

If A is an $m \times n$ matrix with columns vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b},$$

which has corresponding augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ has a solution (is consistent) if and only if $\mathbf{b}$ is a linear combination of the columns of matrix A.

- We have previously considered whether a specified vector is in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.
- A more abstract question is whether all vectors are in the Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Are all vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$ in Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix} \right\}$?

$$\begin{bmatrix} 1 & 1 & 6 & b_1 \\ 1 & 0 & 8 & b_2 \\ -3 & -3 & 6 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 6 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ -3 & -3 & 6 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 6 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ 0 & 0 & 24 & b_3 + 3b_1 \end{bmatrix}$$

No matter the values of b_1, b_2, and b_3, we see that x_1, x_2, and x_3 are all basic variables. The system is consistent, which means all vectors are in the span.

Let A be an $m \times n$ matrix. Then the following statements are all equivalent.

1. For any vector $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. All vectors $\mathbf{b}$ in $\mathbb{R}^m$ can be written as a linear combination of the columns of A.
3. The columns of A span all of $\mathbb{R}^m$.
4. When forming the RREF of A, there is a pivot position in every row.

Example

Do the vectors $\begin{bmatrix} 1 \\ 3 \\ 6 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ 9 \\ -4 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ span all of $\mathbb{R}^4$?

Example

Do the vectors $\begin{bmatrix} 1 \\ 3 \\ 6 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ 9 \\ -4 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ span all of $\mathbb{R}^4$?

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 1 & 4 & 2 \\ 6 & 2 & 9 & 2 \\ -2 & 0 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 2 & 3 & -10 \\ 0 & 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the matrix A does NOT have a pivot position in every row, the vectors do NOT span all of $\mathbb{R}^4$.

Properties of Matrix-Vector Products

Let A be an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and c be a scalar. We have

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- $A(c\mathbf{v}) = c\,(A\mathbf{v})$

Proof.

Properties of Matrix-Vector Products

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- $A(c\mathbf{v}) = c\,(A\mathbf{v})$

Proof.

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ denote the column vectors of A. We have

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} &= (u_1+v_1)\mathbf{a}_1 + (u_2+v_2)\mathbf{a}_2 + \ldots + (u_n+v_n)\mathbf{a}_n \\ &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \ldots + u_n\mathbf{a}_n) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots + v_n\mathbf{a}_n) \\ &= A\mathbf{u} + A\mathbf{v}. \end{aligned}$$

□