Inner Product Spaces

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

The Dot Product

Let $\mathbf{u}$ and $\mathbf{v}$ denote two vectors in $\mathbb{R}^n$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i.$$

- The dot product is used to find lengths of vectors and distances between vectors in $\mathbb{R}^n$.
- The dot product is used to determine if vectors in $\mathbb{R}^n$ are orthogonal.
- Let's extend this concept more generally to any vector space V.

Inner Product

Definition

An inner product on a vector space V is a function that associates to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number that we denote $\langle \mathbf{u}, \mathbf{v} \rangle$. An inner product satisfies the following properties for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and scalar c:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
2. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$.
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

A vector space with an inner product is called an inner product space.

Example. The vector space $\mathbb{R}^n$ with the usual dot product is an inner product space.

Verify Properties Example 1

Let $V = \mathbb{R}^2$ and consider the function given by $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$. Show V with this function is an inner product space.

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We verify the properties of an inner product:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 = 2v_1u_1 + 3v_2u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$.
2. $\langle \mathbf{u}+\mathbf{w}, \mathbf{v} \rangle = 2(u_1+w_1)v_1+3(u_2+w_2)v_2 = 2u_1v_1+2w_1v_1+3u_2v_2+3w_2v_2 = \langle \mathbf{u}, \mathbf{v} \rangle+\langle \mathbf{w}, \mathbf{v} \rangle$.
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = 2(cu_1)v_1 + 3(cu_2)v_2 = c(2u_1v_1 + 3u_2v_2) = c\langle \mathbf{u}, \mathbf{v} \rangle$.
4. $\langle \mathbf{u}, \mathbf{u} \rangle = 2u_1^2 + 3u_2^2 \geq 0$, and equality if and only if $u_1 = u_2 = 0$.

Verify Properties Example 2

Let $V = \mathbb{R}^2$ and consider the function given by $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_2 + u_2 v_1$. Determine whether V with this function is an inner product space.

Verify Properties Example 2

Let $V = \mathbb{R}^2$ and consider the function given by $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_2 + u_2 v_1$. Determine whether V with this function is an inner product space.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_2 + u_2 v_1 = v_1 u_2 + v_2 u_1 = \langle \mathbf{v}, \mathbf{u} \rangle$.
2. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = (u_1 + w_1) v_2 + (u_2 + w_2) v_1 = (u_1 v_2 + u_2 v_1) + (w_1 v_2 + w_2 v_1) = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$.
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = (cu_1) v_2 + (cu_2) v_1 = c(u_1 v_2 + u_2 v_1) = c\langle \mathbf{u}, \mathbf{v} \rangle$.

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4. Consider the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have

$$\langle \mathbf{u}, \mathbf{u} \rangle = (1)(0) + (0)(1) = 0 \quad \text{but} \quad \mathbf{u} \neq \mathbf{0}.$$

   The last property is not satisfied.

$\mathbb{R}^2$ with the operation defined above is NOT an inner product space.

Inner Products on Polynomial Vector Spaces

Recall $\mathbb{P}_2$ denotes the vector space of all polynomials of degree at most 2. For any two vectors $p(t)$ and $q(t)$ in $\mathbb{P}_2$, let

$$\langle p(t), q(t)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Determine whether $\mathbb{P}_2$ with this function is an inner product space.

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Determine whether $\mathbb{P}_2$ with this function is an inner product space.

1. $\langle p(t), q(t) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) = q(0)p(0) + q(1)p(1) + q(2)p(2) = \langle q(t), p(t) \rangle.$
2. $$\begin{aligned} \langle p(t) + r(t), q(t) \rangle &= (p(0) + r(0))q(0) + (p(1) + r(1))q(1) + (p(2) + r(2))q(2) \\ &= \big(p(0)q(0) + p(1)q(1) + p(2)q(2)\big) + \big(r(0)q(0) + r(1)q(1) + r(2)q(2)\big) \\ &= \langle p(t), q(t) \rangle + \langle r(t), q(t) \rangle. \end{aligned}$$

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Determine whether $\mathbb{P}_2$ with this function is an inner product space.

3. $$\begin{aligned} \langle cp(t), q(t) \rangle &= (cp(0))q(0) + (cp(1))q(1) + (cp(2))q(2) \\ &= c\big(p(0)q(0) + p(1)q(1) + p(2)q(2)\big) \\ &= c\langle p(t), q(t) \rangle \end{aligned}$$

4. $\langle p(t), p(t) \rangle = p(0)p(0) + p(1)p(1) + p(2)p(2) = (p(0))^2 + (p(1))^2 + (p(2))^2 \geq 0.$

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4. $\langle p(t), p(t) \rangle = p(0)p(0) + p(1)p(1) + p(2)p(2) = (p(0))^2 + (p(1))^2 + (p(2))^2 \geq 0.$

   We have equality when $p(0) = p(1) = p(2) = 0$. If a polynomial with at most degree two has three distinct zeros, then the polynomial must be equal to zero for all t.
   This means $\langle p(t), p(t) \rangle = 0$ if and only if $p(t) = 0 + 0t + 0t^2 = \mathbf{0}$.

Lengths, Distances, and Orthogonality

Definition

Let V be an inner product space.

- We define the length (or norm) of a vector $\mathbf{v}$ to be $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
- A unit vector is a vector whose length is 1.
- The distance between vectors $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.
- Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Our vector space has been equipped with a geometric structure (we have defined distances and angles in our space).

Nearly everything discussed about dot products and orthogonality for $\mathbb{R}^n$ carries over to inner product spaces more generally (including orthonormal bases and the Gram–Schmidt process).

Polynomial Vector Space Example

Let $V = \mathbb{P}_4$ with the inner product

$$\langle p(t), q(t)\rangle = p(-2)q(-2) + p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Compute lengths and orthogonality of vectors $p_1(t) = 1$, $p_2(t) = t$, and $p_3(t) = t^2$ in $\mathbb{P}_4$.

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Compute lengths and orthogonality of vectors $p_1(t) = 1$, $p_2(t) = t$, and $p_3(t) = t^2$ in $\mathbb{P}_4$.

We create column vectors by evaluating the polynomials at the values $t = -2, -1, 0, 1, 2$:

$$p_1(t) = 1 \;\text{ has } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad p_2(t) = t \;\text{ has } \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad p_3(t) = t^2 \;\text{ has } \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

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   $$\langle 1, t^2\rangle = (1)(4) + (1)(1) + (1)(0) + (1)(1) + (1)(4) = 10 \quad \Rightarrow \text{not orthogonal}$$

Producing an Orthogonal Basis for a Polynomial Vector Space

Apply the Gram–Schmidt process to produce an orthogonal basis for the subspace V of $\mathbb{P}_4$ spanned by the vectors $p_1(t) = 1, p_2(t) = t$, and $p_3(t) = t^2$.

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$$\langle 1, 1\rangle = 5, \quad \langle t, t\rangle = 10, \quad \langle t^2, t^2\rangle = 34, \quad \langle 1, t\rangle = 0, \quad \langle 1, t^2\rangle = 10, \quad \langle t, t^2\rangle = 0.$$

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- Next $\mathbf{v}_2 = p_2(t) - \text{proj}_{\mathbf{v}_1} p_2(t) = t - 0 = t$.

$$\text{proj}_{\mathbf{v}_1} p_2(t) = c_1 \cdot 1 = 0 \quad \text{where} \quad c_1 = \frac{\langle t, 1\rangle}{\langle 1, 1\rangle} = 0.$$

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- The third (and final) vector $\mathbf{v}_3 = p_3(t) - \text{proj}_{\{\mathbf{v}_1, \mathbf{v}_2\}} p_3(t) = t^2 - 2$.

$$\text{proj}_{\{1,t\}} p_3(t) = c_1 \cdot 1 + c_2 \cdot t = (2)(1) - (0)(t) = 2 \text{ where } c_1 = \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} = \frac{10}{5}, \quad c_2 = \frac{\langle t^2, t\rangle}{\langle t, t\rangle} = 0.$$

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Our orthogonal basis is $\{1, t, t^2 - 2\}$. Note this depends on the inner product of $\mathbb{P}_4$.

An Inner Product on Functional Vector Spaces

Let $\mathcal{C}[a, b]$ denote the set of all continuous functions on the closed interval $a \leq t \leq b$.

- Using the typical addition of functions and scalar multiplication, we can verify that $\mathcal{C}[a, b]$ is a vector space.
- For two functions f and g in $\mathcal{C}[a, b]$, we define the following inner product:

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Show $\mathcal{C}[a, b]$ with the operation defined above is an inner product space.

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1. $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt = \int_a^b g(t)f(t)\,dt = \langle g, f \rangle$.
2. $\langle f + h, g \rangle = \int_a^b ((f(t) + h(t))g(t))\,\,dt = \int_a^b f(t)g(t)\,dt + \int_a^b h(t)g(t)\,dt = \langle f, g \rangle + \langle h, g \rangle$.
3. $\langle cf, g \rangle = \int_a^b (cf(t))g(t)\,dt = c\int_a^b f(t)g(t)\,dt = c\langle f, g \rangle$.
4. $\langle f, f \rangle = \int_a^b (f(t))^2\,dt \geq 0$ and equality if and only if $f(t) = 0$.

Computing Inner Products of Continuous Functions

Consider the inner product space $\mathcal{C}[0,1]$ with the usual inner product $\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$. Compute $\langle 1+t^2, 4t\rangle$.

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Consider the inner product space $\mathcal{C}[0,1]$ with the usual inner product $\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$. Compute $\langle 1+t^2, 4t\rangle$.

$$\begin{aligned}\langle 1+t^2, 4t\rangle &= \int_0^1 \left((1+t^2)4t\right) dt \\ &= \int_0^1 (4t+4t^3)\,dt \\ &= (2t^2+t^4)\Big|_0^1 \\ &= 3.\end{aligned}$$

Thus, $1+t^2$ and $4t$ are not orthogonal in $\mathcal{C}[0,1]$.

Fourier Analysis

A basis for $\mathcal{C}[0, 2\pi]$ is $\{1\} \cup \{\cos(mt) : m = 1, 2, \ldots\} \cup \{\sin(mt) : m = 1, 2, \ldots\}$.

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In fact, this is an orthogonal basis. For example, where $m \neq n$ are positive integers,

$$\begin{aligned}
\langle \cos(mt), \cos(nt) \rangle &= \int_0^{2\pi} \cos(mt)\cos(nt)\ dt \\
&= \frac{1}{2}\int_0^{2\pi} \left[\cos(mt + nt) + \cos(mt - nt)\right]\ dt \\
&= \frac{1}{2}\left[\frac{\sin(mt + nt)}{m + n} + \frac{\sin(mt - nt)}{m - n}\right]\Bigg|_0^{2\pi} = 0.
\end{aligned}$$

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In fact, this is an orthogonal basis. Any $f \in \mathcal{C}[0, 2\pi]$ can be represented by a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mt) + \sum_{m=1}^{\infty} b_m \sin(mt),$$

where for $m \geq 1$,

$$a_m = \frac{\langle f, \cos(mt)\rangle}{\langle \cos(mt), \cos(mt)\rangle} = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(mt)\ dt, \quad \text{etc.}$$

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Let W be spanned by $\{1\} \cup \{\cos(mt) : m = 1, \ldots, n\} \cup \{\sin(mt) : m = 1, \ldots, n\}$.

What is the best approximation of f by elements of W?

Python examples in Jupyter Notebook.