Systems of Linear Equations

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

Linear Equations

A linear equation of $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where

- b and the coefficients $a_1, a_2, \ldots, a_n$ are constants (real numbers), and
- we have n variables denoted $x_1, x_2, \ldots, x_n$.

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Example

Determine whether the equation is linear in x_1, x_2, and x_3.

a) $\cos\left(\frac{\pi}{3}\right)x_1 + e^4x_2 - \frac{x_3}{\pi} = -5$

b) $3x_1 + 2x_1x_2 - 3x_3 = 8$

c) $2x_1 + 4\sqrt{x_2} - 3x_3 = 8$

Systems of Linear Equations

A system of linear equations is a collection of one or more linear equations involving the same variables $x_1, \ldots x_n$.

A system of two linear equations with two variables, x_1 and x_2.

$$2x_1 - 9x_2 = 12$$
$$4x_1 + 2x_2 = -16$$

A system of three linear equations with four variables, x_1, x_2, x_3, and x_4.

$$8x_1 + 7x_2 - x_3 - 5x_4 = 10$$
$$2x_1 - 6x_2 - 12x_4 = 0$$
$$0.5x_1 - 0.01x_2 + 2.1x_3 - 1.5x_4 = -2$$

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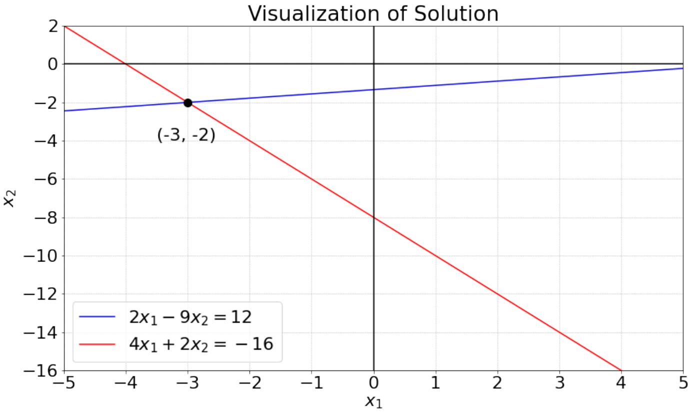
We frequently refer to a system of linear equations as a linear system.

Visualizing the Solution

A solution to a linear system is an ordered list of numbers $(s_1, s_2, \ldots s_n)$ that make all of the equations in the system true when we substitute $x_1 = s_1, \ldots, x_n = s_n$.

For example, $(x_1, x_2) = (-3, -2)$ is a solution to the system

$$2x_1 - 9x_2 = 12$$
$$4x_1 + 2x_2 = -16$$



Review: Solving a Linear System

You probably already have some strategies for solving some systems of equations. For example:

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1. Multiply top equation by -2 and add to bottom equation.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 0x_1 + 20x_2 &= -40 \end{aligned}$$

2. Divide bottom equation by 20.

$$\begin{aligned} 2x_1 - 9x_2 &= 12 \\ 0x_1 + x_2 &= -2 \end{aligned}$$

3. Add 9 times bottom equation to top equation:

$$\begin{aligned} 2x_1 + 0x_2 &= -6 \\ 0x_1 + x_2 &= -2 \end{aligned}$$

4. Divide top equation by 2:

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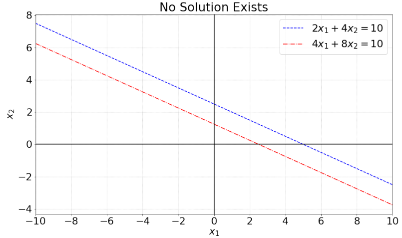

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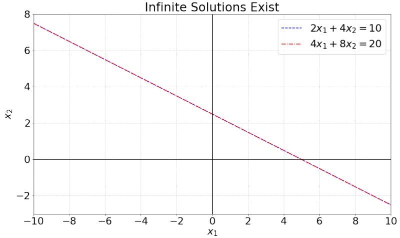

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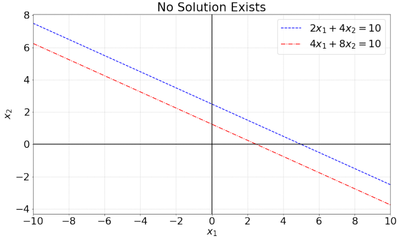

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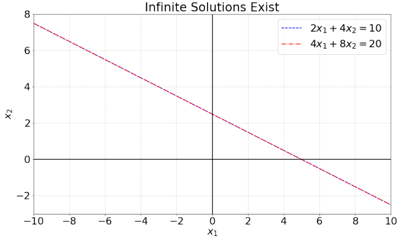

A system of linear equations may have:

- No solution. Such systems are called inconsistent, or
- Solutions (called consistent). How many solutions?
  - Exactly one solution.
  - Infinitely many solutions.

Matrix Notation

- Good news: There is a systemic way to solve a linear system (if a solution exists).
- Bad news: It can involve many algebraic steps.
- We can use a rectangular array called a matrix to help organize the work.

System of linear equations:

$$
\begin{aligned}
2x_1 - 9x_2 &= 12 \\
4x_1 + 2x_2 &= -16
\end{aligned}
$$

The coefficient matrix:

$$
\begin{bmatrix} 2 & -9 \\ 4 & 2 \end{bmatrix}
$$

The augmented matrix:

$$
\begin{bmatrix} 2 & -9 & 12 \\ 4 & 2 & -16 \end{bmatrix}
$$

Example

Give the augmented matrix for the system of linear equations.

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\begin{aligned}
&8x_1 + 7x_2 - x_3 - 5x_4 = 10 \\
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Solution

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\begin{bmatrix}
8 & 7 & -1 & -5 & 10 \\
2 & -6 & 0 & -12 & 0 \\
0.5 & -0.01 & 2.1 & -1.5 & -2
\end{bmatrix}
$$

Describing the Size of a Matrix

How many rows does the augmented matrix have? How many columns?

$$\begin{bmatrix} 8 & 7 & -1 & -5 & 10 \\ 2 & -6 & 0 & -12 & 0 \\ 0.5 & -0.01 & 2.1 & -1.5 & -2 \end{bmatrix}$$

- The augmented matrix above has 3 rows. This means the system consists of 3 equations.
- The augmented matrix above has 5 columns. This means there are 4 variables.

Thus the augmented matrix above is a 3×5 (read as "3 by 5") matrix.

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Caution! The order matters when describing the size of a matrix.
Always rows first, then columns second!

Solving a Linear System Using Matrices

Solve the following system.

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Thus, the solution is $(x_1, x_2) = (-3, -2)$.

Going One Dimension Up

Solve the following system.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned}$$

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Solve the following system.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned} \sim \begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 7 \\ 5 & 1 & 1 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 5 & 1 & 1 & 11 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & -4 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -6 & -24 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Thus $(x_1, x_2, x_3) = (1, 2, 4)$ is a solution of the linear system.

Elementary Row Operations

There are three operations we can apply to the rows of an augmented matrix:

1. Change one row by adding a multiple of another row to it.
2. Swapping two rows.
3. Scaling one row by multiplying all entries in the row by a nonzero constant.

$$\begin{bmatrix} 1 & -8 & 0 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

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Two matrices are row equivalent if one matrix can be transformed into the other using elementary row operations.

Equivalent Systems

- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solutions.

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 7 \\ 5 & 1 & 1 & 11 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{ are row equivalent matrices!}$$

- If two linear systems have the same solutions, we say the two linear systems are equivalent.

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 4 \end{aligned} \quad \text{are equivalent systems!}$$

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Proof. Suppose L_1 is a linear system on variables $x_1, \ldots, x_n$, and L_2 is the resulting linear system after applying one elementary row operation. Suppose $(s_1, \ldots, s_n)$ is a solution to L_1. Then $(s_1, \ldots, s_n)$ is also a solution to L_2.

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1. Swapping two rows.
2. Scaling a row: $a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \quad \Rightarrow \quad ca_{11}x_1 + \ldots + ca_{1n}x_n = cb_1$.

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3. Adding a multiple of a row to another row:

$$\begin{array}{rrrcl} a_{11}x_1 & +\ldots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 & +\ldots & +a_{2n}x_n & = & b_2 \end{array} \Rightarrow \begin{array}{rrrcr} (a_{11}+ca_{21})x_1 & +\ldots & +(a_{1n}+ca_{2n})x_n & = & b_1+cb_2 \\ a_{21}x_1 & +\ldots & +a_{2n}x_n & = & b_2 \end{array}$$

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Is every solution to L_2 also a solution to L_1?

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Is every solution to L_2 also a solution to L_1?

Note that every elementary row operation is reversible (why we scale by nonzero constants!).
Hence, L_2 can be transformed into L_1 by applying one elementary row operation.
From the same argument as above, every solution to L_2 is also a solution to L_1.

□