Linear Transformations

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Interpretations of the Matrix Equation $A\mathbf{x} = \mathbf{b}$

- Our focus has been on using matrices and vectors to solve systems of linear equations.
- We have applied these techniques to determine whether a vector is in the span of a set of vectors.
- Now we will interpret the matrix-vector product $A\mathbf{x}$ as a function mapping from one space into another.

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Example

- $f : x \mapsto x^2$ maps from $\mathbb{R} \to \mathbb{R}$.
- $f : x \mapsto 7x$ maps from $\mathbb{R} \to \mathbb{R}$.
- $f : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix}$ maps from $\mathbb{R}^2 \to \mathbb{R}^2$.
- $f : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$ maps from $\mathbb{R}^2 \to \mathbb{R}^3$.

Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

A transformation (also called a function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\mathbf{x}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

- The set of inputs, $\mathbb{R}^n$, is called the domain of T.
- The space where the inputs are mapped to, $\mathbb{R}^m$, is call the codomain of T.
- For each $\mathbf{x}$ in the domain, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$.
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$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ 0 \end{bmatrix}$$

Matrix Transformations

Let A be an $m \times n$ matrix. Then we can define a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ where

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1. $A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$. $\quad T(\mathbf{x}) = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix}$ maps from $\mathbb{R}^2 \to \mathbb{R}^2$.

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2. $T : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$ maps from $\mathbb{R}^2 \to \mathbb{R}^3$.

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 1 & 0 \\ 5 & 3 & -9 \end{bmatrix}$, and let T be the matrix transformation $T : \mathbf{x} \mapsto A\mathbf{x}$.

1. Find the image of the vectors $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

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2. Is $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the range of T? If so, what is the preimage?

Linear Transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it satisfies:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.
2. $T(c\mathbf{v}) = c\big(T(\mathbf{v})\big)$ for all scalars c and vectors $\mathbf{v}$ in $\mathbb{R}^n$.

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$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ 0 \end{bmatrix}$ is a linear transformation.

Matrix Transformations are Linear Transformations

Theorem

Every matrix transformation is a linear transformation. Namely, if $T : \mathbb{R}^n \to \mathbb{R}^m$ can be expressed as $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A, then T is a linear transformation.

Proof.

Image of the Zero Vector

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.

Proof.

Example

Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4 : \mathbf{x} \mapsto A\mathbf{x}$ where $A = \begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$.

1. Find all $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

2. Find all $\mathbf{x}$ (if any) such that $T(\mathbf{x}) = \begin{bmatrix} 7 \\ 5 \\ 9 \\ 7 \end{bmatrix}$.