The Matrix of a Linear Transformation

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

The Matrix of a Linear Transformation

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ denote a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = T(\mathbf{e}_1) = \begin{bmatrix}3\\-1\\1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T(\mathbf{e}_2) = \begin{bmatrix}0\\2\\1\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T(\mathbf{e}_3) = \begin{bmatrix}1\\0\\1\\-1\end{bmatrix}.$$

Find the image of an arbitrary vector $\mathbf{x}$ in $\mathbb{R}^3$.

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Find the image of an arbitrary vector $\mathbf{x}$ in $\mathbb{R}^3$.

$$\begin{aligned} T(\mathbf{x}) &= T\left(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3\right) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + x_3 T(\mathbf{e}_3) \\ &= x_1 \begin{bmatrix}3\\-1\\1\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\2\\1\\3\end{bmatrix} + x_3 \begin{bmatrix}1\\0\\1\\-1\end{bmatrix} = \begin{bmatrix}3 & 0 & 1\\-1 & 2 & 0\\1 & 1 & 1\\0 & 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = A\mathbf{x}. \end{aligned}$$

Every Linear Transformation is a Matrix Transformation

Theorem

For any linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, there is a unique matrix A (called the associated matrix for the linear transformation) such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. The matrix A can be found by

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}.$$

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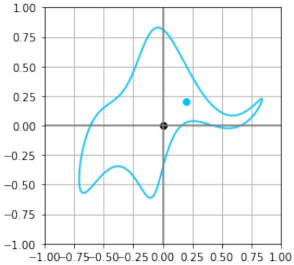
Proof.

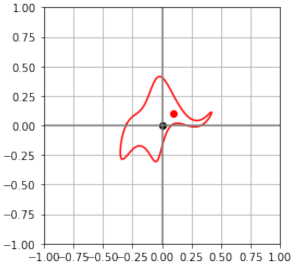
$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \ldots + x_n T(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}. \end{aligned}$$

□

Consider the linear transformation given by $T : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto 0.5\mathbf{x}$. Find the associated matrix A for this linear transformation.

We have $T(\mathbf{e}_1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$ giving the matrix $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$.

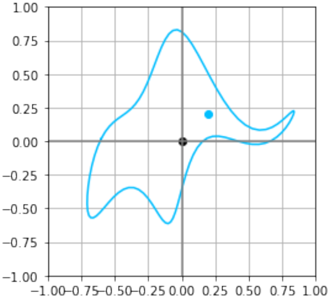


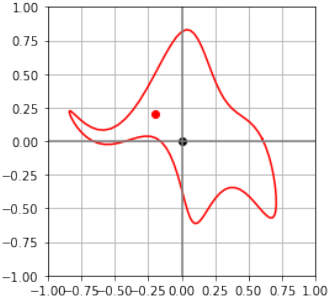


Geometric Interpretation in $\mathbb{R}^2$

Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

We have $T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ giving the geometric interpretation seen below.





Contractions and Expansions

Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$.

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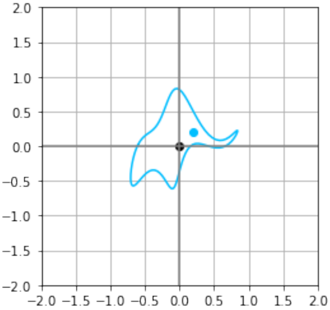
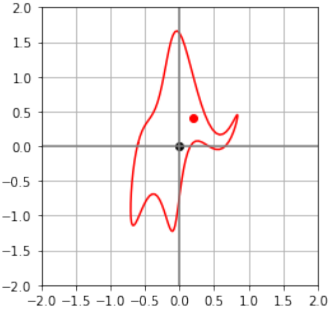
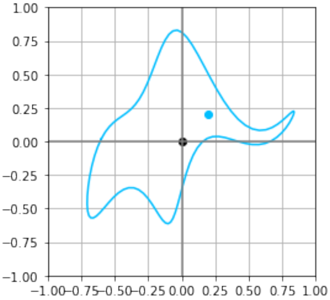
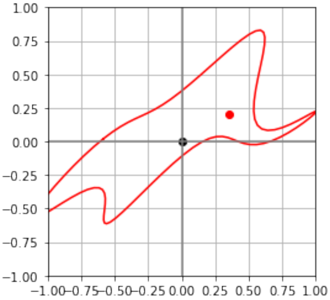




Shear Transformations

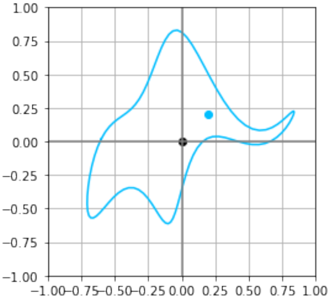
Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0.75 \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

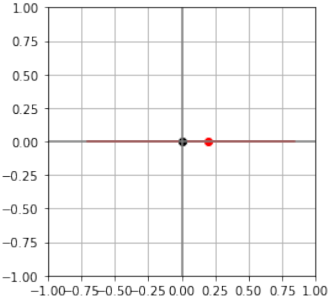




Projections

Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x}$.





Rotations

Consider the linear transformation T of $\mathbb{R}^2$ that rotates by the angle θ. What is the associated matrix of T?

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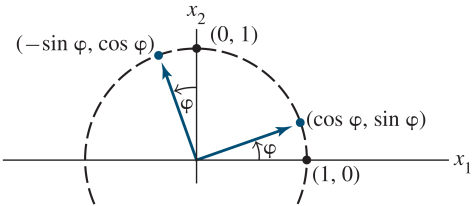

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}$$

Properties of Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The mapping T is said to be:

- One-to-one if each $\mathbf{b}$ in $\mathbb{R}^m$ has **at most one** $\mathbf{x}$ in $\mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.
- Onto if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of **at least one** $\mathbf{x}$ in $\mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.

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Example

- The dilation transformation $T(\mathbf{x}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is both one-to-one and onto.
- The projection transformation $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is neither one-to-one nor onto.

Examples

Determine whether each of the transformations is one-to-one and/or onto.

$$T : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad R : \mathbb{R}^3 \to \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} -x_2 \\ -x_3 \\ -x_1 \end{bmatrix} \qquad S : \mathbb{R}^2 \to \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ 0 \\ x_1 \end{bmatrix}$$

Theorem

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

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The forward direction we prove using proof by contradiction. Let's assume that T is a one-to-one mapping, and that $T(\mathbf{x}) = \mathbf{0}$ has a non-trivial solution $\mathbf{v}$ in $\mathbb{R}^n$. Then T is not one-to-one since $T(\mathbf{0}) = T(\mathbf{v}) = \mathbf{0}$. Thus we have a contradiction.

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The reverse direction we prove using proof by contradiction as well. Suppose $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, and T is not one-to-one. This means there exist $\mathbf{u} \neq \mathbf{v}$ both in $\mathbb{R}^n$ with $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{b}$ in $\mathbb{R}^m$.

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$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Since $\mathbf{u} \neq \mathbf{v}$, we have found a non-trivial solution such that $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. Thus we have a contradiction. $\square$

Summary

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix A is one-to-one if and only if

- $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.
- The solution set to $A\mathbf{x} = \mathbf{0}$ has no free variables.
- The matrix A has a pivot in every column.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix A is onto if and only if

- For any $\mathbf{b}$ in $\mathbb{R}^m$ there exists at least one $\mathbf{x}$ in $\mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$.
- All vectors $\mathbf{b}$ in $\mathbb{R}^m$ can be written as a linear combination of the columns of A.
- The matrix A has a pivot in every row.
- The columns of A span all of $\mathbb{R}^m$.