Properties of Determinants

Linear Algebra

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Cofactor Expansion Definition of Determinant

For an $n \times n$ matrix A, the cofactor expansion about row i is $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$.

This recursive definition is too time-consuming for large matrices.

We will discover a more efficient method of calculating the determinant by studying how the elementary row operations change the determinant.

Scaling a Row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\det I_4 = 1$ $\quad$ $\det E =$

Scaling a Row

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&1&0&0\\\color{blue}{0}&\color{blue}{0}&\color{blue}{5}&\color{blue}{0}\\0&0&0&1\end{bmatrix} \qquad \begin{bmatrix}1&2&3&4\\5&6&7&8\\-1&3&7&9\\4&-2&-3&2\end{bmatrix} \to \begin{bmatrix}1&2&3&4\\5&6&7&8\\\color{blue}{-5}&\color{blue}{15}&\color{blue}{35}&\color{blue}{45}\\4&-2&-3&2\end{bmatrix}$$

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If B is obtained by scaling one row of A by k, then $\det B = k(\det A)$.

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> Let A and B be $n \times n$ matrices, and let k be a scalar.
> If B is obtained by scaling one row of A by k, then $\det B = k(\det A)$.

Proof. Let A be an $n \times n$ matrix, and let B be the matrix obtained by multiplying the row i of A by the scalar k. We calculate the determinant of B by cofactor expansion across row i.

$$\det B = \sum_{j=1}^{n} (-1)^{i+j} b_{ij} \det B_{ij} = \sum_{j=1}^{n} (-1)^{i+j} (k \cdot a_{ij}) \det A_{ij} = k \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \det A.$$

Adding a Multiple of a Row to another Row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ \color{red}{0} & \color{red}{1} & \color{red}{3} & \color{red}{0} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Let A and B be $n \times n$ matrices.
If B is obtained from A by adding a multiple of a row to another, then $\det B = \det A$.

Adding a Multiple of a Row to another Row

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Let A and B be $n \times n$ matrices.
If B is obtained from A by adding a multiple of a row to another, then $\det B = \det A$.

Proof. We'll give a proof later.

Swapping Two Rows

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\ \color{green}0&\color{green}0&\color{green}0&\color{green}1\\0&0&1&0\\ \color{green}0&\color{green}1&\color{green}0&\color{green}0\end{bmatrix}$$

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Let A and B be $n \times n$ matrices.
If B is obtained by swapping two rows of A, then $\det B = -\det A$.

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Elementary Row Operations and Determinants

Let A and B denote $n \times n$ matrices, and let k denote a scalar.

- If B is obtained from A by scaling one row of A by k, then $\det B = k(\det A)$.
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If $n = 2$, then we can directly calculate the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix}$$

$\det A = ad - bc$ $\qquad$ $\det B = a(d+kb) - b(c+ka)$
$= ad - bc + kab - kab = ad - bc$

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$\det B = bc - ad = -(ad - bc)$

Proof continued

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Assume that $n > 2$. Let i be a row other than rows r and s.
We calculate the determinant of matrix B by doing a cofactor expansion across row i.

$$\det B = \sum_{j=1}^{n} (-1)^{i+j}\, b_{ij}\, \det B_{ij}$$

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Example of Computing Determinant

Compute the determinant of $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

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Row-reduce A, tracking how the elementary row ops change the determinant.

$$\det \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{bmatrix} = -3\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & -5 \end{bmatrix}$$

$$= 15\det \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = 15 \cdot 1 = 15$$

Example

A square matrix A is invertible if and only if $\det A \neq 0$.

Compute the determinant of $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

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Since A is not invertible, we have $\det A = 0$.

Elementary Matrices

Scaling a row:

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 5$$

Add mult of row to another:

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1$$

Swapping two rows:

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = -1$$

If E is an elementary matrix, then $\det(EA) = (\det E)(\det A)$.

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Proof.

Suppose A is not invertible and AB is invertible. Since AB is invertible, there exists an inverse matrix M such that $(AB)M = I_n$. Since there exists a matrix $D = BM$ such that $A(BM) = I_n$, by the Invertible Matrix Theorem, it follows that A is invertible. Thus, we have a contradiction, so our original assumption must not be correct. Therefore, if A is not invertible, it must follow that AB is not invertible. $\square$

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If AB is invertible, then both A and B are invertible.

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Case 1: Suppose either A or B is not invertible. Then we just showed that AB is not invertible. In this case, $\det(AB) = 0$ and $(\det A)(\det B) = 0$, so the property holds.

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Case 2: Suppose both A and B are invertible. This means A is row equivalent to I_n. Thus, we have $A = E_p \dots E_2 E_1 I_n$ where each E_i denotes an elementary matrix.

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$$\begin{aligned} \det(AB) = \det(E_p \dots E_2 E_1 B) &= \det(E_p) \det(E_{p-1} \dots E_2 E_1 B) \\ &= \det(E_p) \det(E_{p-1}) \det(E_{p-2} \dots E_2 E_1 B) \\ &= \dots \\ &= \det(E_p) \det(E_{p-1}) \dots (\det E_1)(\det B) \\ &= \det(E_p E_{p-1} \dots E_2 E_1)(\det B) \\ &= (\det A)(\det B) \end{aligned}$$

□

Determinants of A and A^{-1}

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Proof.

Let A be an invertible matrix with inverse A^{-1}. This means $AA^{-1} = I_n$. Then we have $\det\left(AA^{-1}\right) = \det I_n = 1$. Using the previous result about the determinant for a product of matrices, we therefore have

$$1 = \det\left(AA^{-1}\right) = \left(\det A\right)\left(\det A^{-1}\right).$$

Therefore we have $\det A^{-1} = \dfrac{1}{\det A}$. $\square$