Vector Spaces and Subspaces

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

Introduction

We have seen that vector addition and scalar multiplication are key concepts for linear algebra:

- linear combinations,
- vector equations (linear systems and matrix equations),
- linear transformations.

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So far our vectors have been vectors in $\mathbb{R}^n$.

We will now generalize "vectors" to be any objects that we can add and do scalar mult.

Vector Spaces

A vector space is a nonempty set V of objects called **vectors** on which we define two operations, called addition and scalar multiplication, for which the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c and d:

1. Closed under addition:
   $\mathbf{u} + \mathbf{v}$ is in V.
2. Addition is commutative:
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. Addition is associative:
   $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. Additive identity (zero vector):
   There exists $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. Additive inverse: For each $\mathbf{u}$ in V, there exists a $-\mathbf{u}$ in V where $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. Closed under scalar multiplication:
   $c\mathbf{u}$ is in V.
7. Distributive property 2:
   $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
8. Distributive property 1:
   $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
9. Scalar multiplication is associative:
   $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. Multiplicative identity:
    $1\mathbf{u} = \mathbf{u}$

Examples

$V = \mathbb{R}^2$ with usual vector addition and scalar multiplication.

1. $\mathbf{u} + \mathbf{v}$ is in $\mathbb{R}^2$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in $\mathbb{R}^2$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
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$V = \mathbb{P}_2$ denote the set of all polynomials of degree at most two, with the usual operations.

$$V = \{a_2x^2 + a_1x + a_0 : a_2, a_1, a_0 \in \mathbb{R}\}$$

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$$V = \{a_2x^2 + a_1x + a_0 : a_2, a_1, a_0 \in \mathbb{R}\}$$

1. $(a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$
2. $f(x) + g(x) = g(x) + f(x)$
3. $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$
4. $\mathbf{0} = 0x^2 + 0x + 0 = 0$
5. $-f(x) = -a_2x^2 - a_1x - a_0$.
6. $cf(x) = ca_2x^2 + ca_1x + ca_0$ in $\mathbb{P}_2$
7. $c(f(x) + g(x)) = cf(x) + cg(x)$
8. $(c + d)f(x) = cf(x) + df(x)$
9. $c(df(x)) = (cd)f(x)$
10. $1f(x) = f(x)$

Is it a Vector Space?

$V = \mathbb{R}^3$ with usual scalar multiplication and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_2 \\ u_2 + v_2 \\ u_3 + v_2 \end{bmatrix}$.

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3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \begin{bmatrix} (u_1 + v_2) + w_2 & (u_2 + v_2) + w_2 & (u_3 + v_2) + w_2 \end{bmatrix}$
   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} u_1 + (v_2 + w_2) & u_2 + (v_2 + w_2) & u_3 + (v_2 + w_2) \end{bmatrix}$
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$, then $\mathbf{0} = \begin{bmatrix} a & 0 & b \end{bmatrix}$.
5. $\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ has no additive inverse.
6. $c\mathbf{u}$ is in $\mathbb{R}^3$.
7. $c(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} c(u_1 + v_2) & c(u_2 + v_2) & c(u_3 + v_2) \end{bmatrix}$
   $c\mathbf{u} + c\mathbf{v} = \begin{bmatrix} cu_1 + cv_2 & cu_2 + cv_2 & cu_3 + cv_2 \end{bmatrix}$
8. $(c + d)\mathbf{u} = \begin{bmatrix} (c + d)u_1 & (c + d)u_2 & (c + d)u_3 \end{bmatrix}$
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9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Uniqueness of Zero Vector and Additive Inverse

If V is a vector space, then there exists a unique zero vector $\mathbf{0}$.

Proof. Suppose $\mathbf{w}$ in V with $\mathbf{u} + \mathbf{w} = \mathbf{u} = \mathbf{w} + \mathbf{u}$ for all $\mathbf{u}$ in V.
Thus, if $\mathbf{u} = \mathbf{0}$, then we have $\mathbf{0} + \mathbf{w} = \mathbf{0}$, and so $\mathbf{w} = \mathbf{0}$.

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If V is a vector space, then for each $\mathbf{u}$ in V there exists a unique additive inverse $-\mathbf{u}$.

Proof. Suppose for $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{w} = \mathbf{0} = \mathbf{u} + \mathbf{z}$.
Add $\mathbf{z}$ to both sides of $\mathbf{u} + \mathbf{w} = \mathbf{0}$. $\implies$ $\mathbf{w} = \mathbf{z}$.

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4. $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
5. $-\mathbf{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ -u_3 \end{bmatrix}$
6. $c\mathbf{u}$ is in $\mathbb{R}^3$.
7. $c(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} -c(u_1 + v_1) & -c(u_2 + v_2) & -c(u_3 + v_3) \end{bmatrix}$
   $c\mathbf{u} + c\mathbf{v} = \begin{bmatrix} -cu_1 - cv_1 & -cu_2 - cv_2 & -cu_3 - cv_3 \end{bmatrix}$
8. $(c + d)\mathbf{u} = \begin{bmatrix} -(c + d)u_1 & -(c + d)u_2 & -(c + d)u_3 \end{bmatrix}$
   $c\mathbf{u} + d\mathbf{u} = \begin{bmatrix} -cu_1 - du_1 & -cu_2 - du_2 & -cu_3 - du_3 \end{bmatrix}$
9. $c(d\mathbf{u}) \neq (cd)\mathbf{u}$
10. Multiplicative identity fails since $1\mathbf{u} \neq \mathbf{u}$.

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Let A, B, and C denote 3×3 matrices and c and d scalars.

1. $A + B$ is a 3×3 matrix.
2. $A + B = B + A$
3. $(A + B) + C = A + (B + C)$
4. $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
5. $-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}$
6. cA is a 3×3 matrix.
7. $c(A + B) = cA + cB$
8. $(c + d)A = cA + dA$
9. $(cd)A = c(dA)$
10. $1A = A$

Is it a Vector Space?

V is all 3×3 matrices with determinant 1.

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V is all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with usual operations.

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7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
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No!
Addition is not commutative: $\sin(x^2) \neq (\sin x)^2$.

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$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} \\ b_{21} & a_{22} + b_{22} \end{bmatrix}$$

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Subspaces

Definition

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A subspace of a vector space V is a subset H of V such that
H along with the addition and scalar multiplication of V forms a vector space.

Some of the vector space properties are satisfied for H since V is a vector space.

If a subset H of a vector space V has the following three properties, it is a subspace.

1. Nonempty. There exists some vector $\mathbf{u}$ in H.
2. Closed under vector addition. If $\mathbf{u}$ and $\mathbf{v}$ are both in H, then their sum $\mathbf{u}+\mathbf{v}$ is in H.
3. Closed under scalar multiplication. If $\mathbf{u}$ is in H, then for all scalars c, $c\mathbf{u}$ is in H.

Let $V = \mathbb{R}^3$ with usual operations. Consider the subset $H = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$.

The Zero Vector and Additive Inverses

Theorem

Let V be a vector space. For any vector $\mathbf{u}$ in V, $0\mathbf{u} = \mathbf{0}$.

Thus, nonempty plus closure of scalar mult implies that the zero vector is in a subspace.

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Proof. Note that $\mathbf{0} = 0\mathbf{u} = (1-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u} = \mathbf{u} + (-1)\mathbf{u}$.

Since the additive inverse is unique, $(-1)\mathbf{u} = -\mathbf{u}$.

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{R}^4$ with usual operations.

$$H = \left\{ \begin{bmatrix} 2a+3b \\ -d \\ 0 \\ 6c+2a-b \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}.$$

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1. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in H.
2. $\begin{bmatrix} 2a_1+3b_1 \\ -d_1 \\ 0 \\ 6c_1+2a_1-b_1 \end{bmatrix} + \begin{bmatrix} 2a_2+3b_2 \\ -d_2 \\ 0 \\ 6c_2+2a_2-b_2 \end{bmatrix} = \begin{bmatrix} 2(a_1+a_2)+3(b_1+b_2) \\ -(d_1+d_2) \\ 0 \\ 6(c_1+c_2)+2(a_1+a_2)-(b_1+b_2) \end{bmatrix}.$
3. $k\begin{bmatrix} 2a+3b \\ -d \\ 0 \\ 6c+2a-b \end{bmatrix} = \begin{bmatrix} 2(ka)+3(kb) \\ -(kd) \\ 0 \\ 6(kc)+2(ka)-(kb) \end{bmatrix}.$

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So H is a subspace of $\mathbb{R}^4$.

Is it a Subspace?

1. Nonempty.
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Let $V = \mathbb{R}^3$ with usual operations.

$$H = \left\{ \begin{bmatrix} 3a+b \\ a+5 \\ 2a-5b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}.$$

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$$H = \left\{ \begin{bmatrix} 3a+b \\ a+5 \\ 2a-5b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}.$$

1. Nonempty: $\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ is in H.
2. Consider $\begin{bmatrix} 3(3)+0 \\ 3+5 \\ 2(3)-0 \end{bmatrix} + \begin{bmatrix} 0+1 \\ 0+5 \\ 0-5(1) \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(3)+1 \\ 3+10 \\ 2(3)-5(1) \end{bmatrix}$ is NOT in H.
3. If $k = 0$, then $k\mathbf{h} = \mathbf{0}$ is NOT in H. We need $a = -5$, which means $b = 15$. This gives $2a - 5b = -10 - 75 = -85$. We cannot find values for a, b, and c in $\mathbb{R}$ such that $\mathbf{0}$ is in H.

The Span of a Set of Vectors

Theorem

Let $\mathbf{v}_1, \ldots \mathbf{v}_p$ be vectors from a vector space V. Then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

- We call Span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.
- Given any subspace of H of V, a spanning set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H =$ Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

The Span of a Set of Vectors

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Let $\mathbf{v}_1, \ldots \mathbf{v}_p$ be vectors from a vector space V. Then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

- We call $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.
- Given any subspace of H of V, a spanning set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Proof.

1. $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_p$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. So $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is nonempty.

Theorem

Let $\mathbf{v}_1, \ldots \mathbf{v}_p$ be vectors from a vector space V. Then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

- We call $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.
- Given any subspace of H of V, a spanning set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Proof.

1. $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_p$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. So $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is nonempty.
2. Let $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_p\mathbf{v}_p$ and $\mathbf{y} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_p\mathbf{v}_p$. Then we have

   $\mathbf{x} + \mathbf{y} = (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \ldots + (a_p + b_p)\mathbf{v}_p$. So $\mathbf{x} + \mathbf{y}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

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$$\mathbf{x} + \mathbf{y} = (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \ldots + (a_p + b_p)\mathbf{v}_p. \text{ So } \mathbf{x} + \mathbf{y} \text{ is in Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}.$$
3. Let $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_p\mathbf{v}_p$. Then for a scalar c, we have
$$c\mathbf{x} = (ca_1)\mathbf{v}_1 + (ca_2)\mathbf{v}_2 + \ldots + (ca_p)\mathbf{v}_p. \text{ So } c\mathbf{x} \text{ is in Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}.$$

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{R}^4$ with usual operations.

$$H = \left\{ \begin{bmatrix} 2a + 3b \\ -d \\ 0 \\ 6c + 2a - b \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}.$$

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$$H = \left\{ \begin{bmatrix} 2a+3b \\ -d \\ 0 \\ 6c+2a-b \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}.$$

Notice that $\begin{bmatrix} 2a+3b \\ -d \\ 0 \\ 6c+2a-b \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix} + d \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. Thus we have:

$$H = \text{Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus, H is a subspace of $\mathbb{R}^4$.

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathrm{Mat}_{2\times 2}$ with the usual operations.

$$H = \left\{ \begin{bmatrix} a & c \\ c & b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathrm{Mat}_{2\times 2}$ with the usual operations.

$$H = \left\{ \begin{bmatrix} a & c \\ c & b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

YES! $H = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathrm{Mat}_{2\times 2}$ with the usual operations.

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Let $V = \mathrm{Mat}_{3\times 3}$ with the usual operations.

$$H = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \text{ with } a + b + c = 0 \right\}$$

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \text{Mat}_{2\times 2}$ with the usual operations.

$$H = \left\{ \begin{bmatrix} a & c \\ c & b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

YES! $H = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Let $V = \text{Mat}_{3\times 3}$ with the usual operations.

$$H = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \text{ with } a + b + c = 0 \right\}$$

YES! $H = \text{Span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{P}$ (all polynomials) with usual operations.
H is the set of polynomials of degree at most 2.

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{P}$ (all polynomials) with usual operations.
H is the set of polynomials of degree at most 2.
YES! $H = \text{Span}\{x^2, x, 1\}$.

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{P}$ (all polynomials) with usual operations.
H is the set of polynomials of degree at most 2.
YES! $H = \text{Span}\{x^2, x, 1\}$.

H is the set of polynomials with integer coefficients.

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{P}$ (all polynomials) with usual operations.
H is the set of polynomials of degree at most 2.
YES! $H = \text{Span}\{x^2, x, 1\}$.

H is the set of polynomials with integer coefficients.
No! Not closed under scalar mult (scalars are reals).

Is it a Subspace?

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2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{P}$ (all polynomials) with usual operations.
H is the set of polynomials of degree at most 2.
YES! $H = \text{Span}\{x^2, x, 1\}$.

H is the set of polynomials with integer coefficients.
No! Not closed under scalar mult (scalars are reals).

Let V be all continuous functions with the usual operations.
H is the set of all differentiable functions.

Is it a Subspace?

1. Nonempty.
2. Closed under addition.
3. Closed under scalar mult.

Let $V = \mathbb{P}$ (all polynomials) with usual operations.
H is the set of polynomials of degree at most 2.
YES! $H = \text{Span}\{x^2, x, 1\}$.

H is the set of polynomials with integer coefficients.
No! Not closed under scalar mult (scalars are reals).

Let V be all continuous functions with the usual operations.
H is the set of all differentiable functions.
YES! But cannot be described as a finite span.