Orthogonal Diagonalization and Symmetric Matrices

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

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there exist n linearly independent eigenvectors.

$\Leftrightarrow$ there is a basis for $\mathbb{R}^n$ consisting of eigenvectors.

Ques. When does there exist an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors?

Definition

A symmetric matrix is a square matrix A such that $A^T = A$.

- Entries on the main diagonal can be anything.
- Entries above and below the main diagonal come in mirrored pairs, $a_{ij} = a_{ji}$.

Here are some examples of symmetric matrices:

$$\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 & 7 \\ -2 & 12 & 5 \\ 7 & 5 & -17 \end{bmatrix}.$$

Here are some examples of matrices that are not symmetric:

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Review: Diagonalizing a Matrix

Diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

1. Find the eigenvalues of A by solving $\det(A - \lambda I) = 0$. We have $\lambda = 8, 6, 3$.

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2. If possible, find n linearly independent eigenvectors.

   We have $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = 8$, $\lambda_2 = 6$ and $\lambda_3 = 3$, respectively.

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3. Then $A = PDP^{-1}$. We have $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Choosing An Orthonormal Basis

We have $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = 8$, $\lambda_2 = 6$ and $\lambda_3 = 3$, respectively.

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We can instead construct column vectors for P that are orthonormal:

We have $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ corresponding to $\lambda_1 = 8$, $\lambda_2 = 6$ and $\lambda_3 = 3$, respectively.

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Now we have $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ and $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Orthogonal Matrices

Definition

A matrix P is called orthogonal if it is a square matrix with orthonormal columns. If P is an orthogonal matrix, we have shown that $P^TP = I$ and $PP^T = I$, therefore $P^{-1} = P^T$.

$$A = PDP^{-1} = PDP^T$$

$$\begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

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$$\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 = (A\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 \quad \text{by def of eigenvector}$$

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Therefore, $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, which implies $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ since we assumed λ_1 and λ_2 are not equal. $\square$

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Since $A^T = A$, we see that A is a symmetric matrix.

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($\Leftarrow$) Converse is difficult! Hard part: dim of each eigenspace equals the algebraic multiplicity. $\square$

Example

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2. Find a basis for the eigenspace of each eigenvector.

$$\lambda_1 = -3 \;\text{ has }\; \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \;\text{ and }\; \lambda_2 = 15 \;\text{ has }\; \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

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3. Using the Gram–Schmidt process, find an orthogonal basis for the eigenspaces.

Making Things Orthogonal

We have $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$.

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- $\mathbf{w}_1$ and $\mathbf{w}_2$ are NOT orthogonal. We set $\mathbf{v}_1 = \mathbf{w}_1$.
- We find the component of $\mathbf{w}_2$ orthogonal to $\mathbf{v}_1$

$$\mathrm{proj}_{\mathbf{v}_1}\mathbf{w}_2 = \frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{-1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

Our next vector is $\mathbf{v}_2 = \mathbf{w}_2 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{w}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix}$.

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- $\mathbf{w}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, so we set $\mathbf{v}_3 = \mathbf{w}_3$.

Normalizing the Vectors

We now have orthogonal eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$.

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4. Normalize each column vector to find possible columns of P.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ \frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$

We can check that $A = PDP^T$:

$$A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ \frac{1}{\sqrt{2}} & \sqrt{2}/6 & -2/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{2}/6 & \sqrt{2}/6 & 2\sqrt{2}/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

The Spectral Theorem

The set of eigenvalues of A is sometimes called the spectrum of A.

Theorem (The Spectral Theorem)

If A is a symmetric $n \times n$ matrix, then A is orthogonally diagonalizable (with real eigenvalues).

There exist orthogonal $P = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{bmatrix}$ and diagonal $D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$ such that

$$A = PDP^T = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

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The matrices $\mathbf{u}_i \mathbf{u}_i^T$ are $n \times n$ matrices, rank 1, and are orthogonal projection matrices.

The product $\mathbf{u}\mathbf{v}^T$ of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is sometimes called the *outer product*, and has rank ≤ 1.