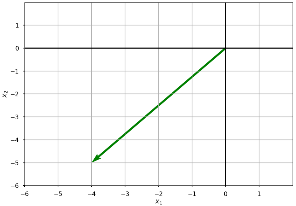
Dot Product, Length, and Orthogonality

Linear Algebra

This work was initially funded by an Institutional OER Grant from the Colorado Department of Higher Education (CDHE). For similar OER materials in other courses funded by this project in the Department of Mathematical and Statistical Sciences at the University of Colorado Denver, visit
https://github.com/CU-Denver-MathStats-OER

Calculating the Length of a Vector



$$\mathbf{v} = (-4, -5)$$

Use the Pythagorean theorem:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{(-4)^2 + (-5)^2} = \sqrt{41}$$

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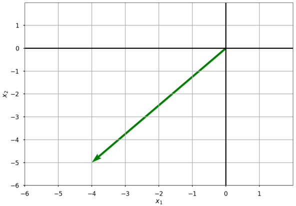

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Let $\mathbf{w} = (1, 2, 3)$ be a vector in $\mathbb{R}^3$.

$$\|\mathbf{w}\| = \sqrt{w_1^2 + w_2^2 + w_3^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

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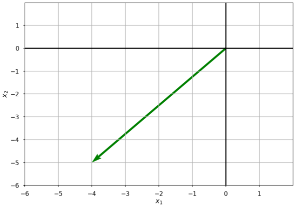

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The length (or norm) of $\mathbf{v}$ is the nonnegative scalar $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$.

The Dot Product

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ denote two $n \times 1$ column vectors in $\mathbb{R}^n$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

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Example.

If $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$ in $\mathbb{R}^3$, then $\mathbf{u} \cdot \mathbf{v} = (2)(6) + (-5)(2) + (1)(-7) = -5.$

Properties of the Dot Product

Theorem

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$.

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.

(d) $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

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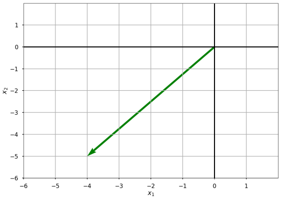
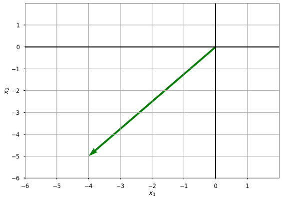
Scaling Vectors



1. Find a vector $\mathbf{w}$ that has the same direction as $\mathbf{v} = (-4, -5)$ but is twice as long.
2. Find a vector $\mathbf{u}$ that has the same direction as $\mathbf{v}$ that has length 1.



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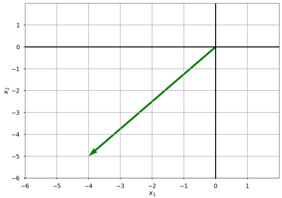

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1. Let $\mathbf{w} = 2(\mathbf{v}) = \begin{bmatrix} -8 \\ -10 \end{bmatrix}$ has the same direction as $\mathbf{v}$ and is twice as long.
2. Note that $\|\mathbf{v}\| = \sqrt{41}$. If we scale $\mathbf{v}$ by a factor of $1/\|\mathbf{v}\|$, then we will obtain a vector in the same direction as $\mathbf{v}$ with magnitude equal to 1:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{41}}\mathbf{v} = \begin{bmatrix} -4/\sqrt{41} \\ -5/\sqrt{41} \end{bmatrix}; \qquad \|\mathbf{u}\| = 1$$

Normalizing Vectors

Definition

A vector $\mathbf{u}$ whose length is 1 is called a unit vector. The process of creating a unit vector $\mathbf{u}$ in the same direction as a vector $\mathbf{v}$ is called normalizing $\mathbf{v}$. We have

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

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Compute the unit vector in the direction of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

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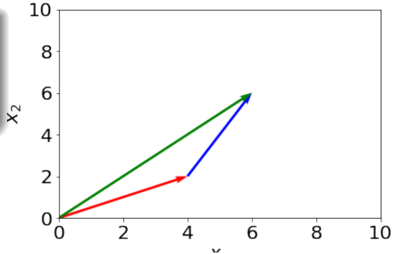
Compute the unit vector in the direction of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$.

Example

Compute the distance between $\mathbf{u} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.



Distance between two vectors

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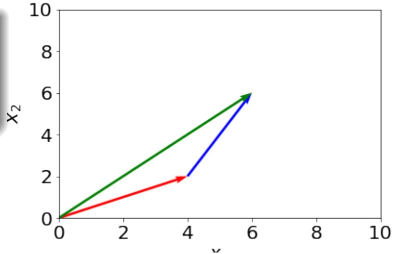


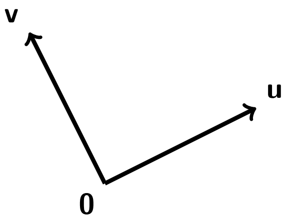
We want to find the length of vector $\mathbf{w}$ where $\mathbf{v} + \mathbf{w} = \mathbf{u}$. Thus,

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

Orthogonality

Consider two perpendicular vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.



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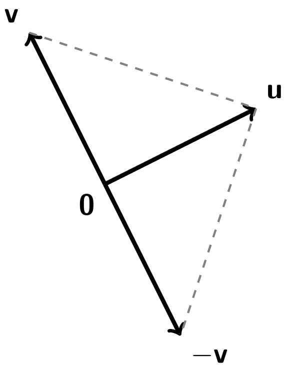
Consider two perpendicular vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

- We see that $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}-(-\mathbf{v})\|$.
- Thus $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}-(-\mathbf{v})\|^2$.

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2$$



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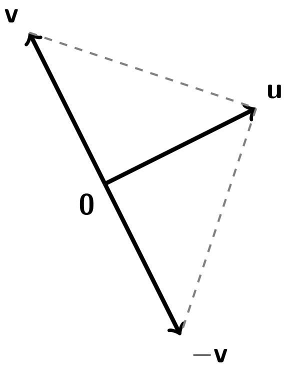

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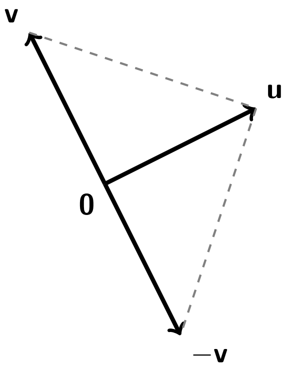

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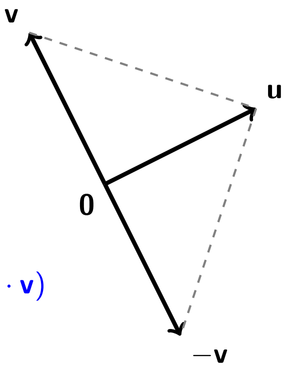

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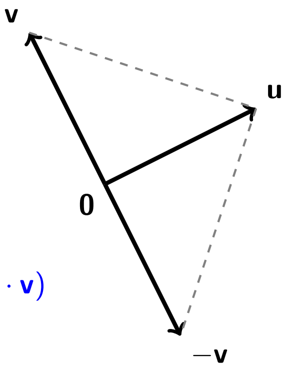
\end{aligned}$$



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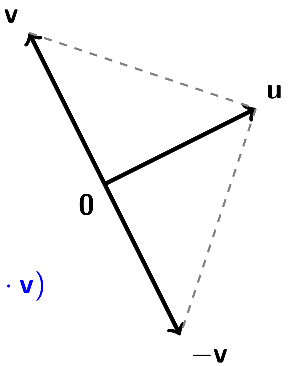

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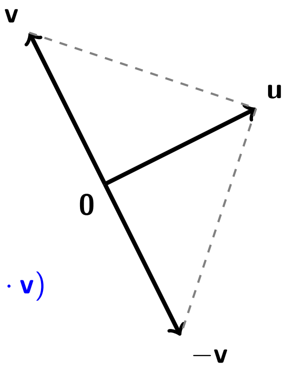

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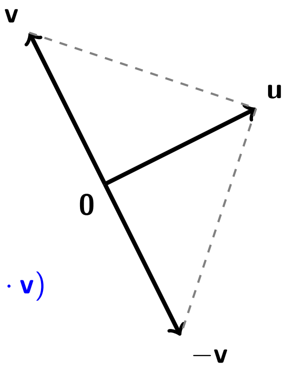

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Are $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ orthogonal?

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$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 4(-5) + (-7)(-2) + 2(3) \\ &= -20 + 14 + 6 = 0. \end{aligned}$$

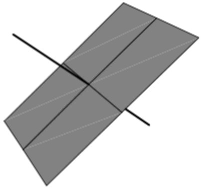
$\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Orthogonal Complements 1

Definition

If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be orthogonal to W. The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the orthogonal complement of W and is denoted $W^\perp$.

Consider the plane $W = \text{Span}\{\mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ plotted in the figure to the right.



- The vector $\mathbf{z}$ is orthogonal to W.
- The orthogonal complement $W^\perp = \{c\mathbf{z} \ : \ \text{for any scalar } c\}$.

Orthogonal Complements 2

Theorem

Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace of $\mathbb{R}^n$.

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Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace of $\mathbb{R}^n$.

Proof. We verify the following three properties.

1. $W^\perp$ is nonempty. Note that $\mathbf{0}$ is orthogonal to every vector in $\mathbb{R}^n$, and so $\mathbf{0}$ is orthogonal to every vector in W. Thus, $\mathbf{0}$ is in $W^\perp$.

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1. $W^\perp$ is nonempty. Note that $\mathbf{0}$ is orthogonal to every vector in $\mathbb{R}^n$, and so $\mathbf{0}$ is orthogonal to every vector in W. Thus, $\mathbf{0}$ is in $W^\perp$.
2. $W^\perp$ is closed under addition. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $W^\perp$. Let $\mathbf{w}$ be any vector in W. Then $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{w} \cdot \mathbf{u}) + (\mathbf{w} \cdot \mathbf{v}) = 0 + 0 = 0$. Thus, $\mathbf{u} + \mathbf{v}$ is in $W^\perp$.

Orthogonal Complements 2

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3. $W^\perp$ is closed under scalar multiplication. Let $\mathbf{u}$ be a vector in $W^\perp$, and let c be a scalar. Let $\mathbf{w}$ be any vector in W. Then $\mathbf{w} \cdot (c\mathbf{u}) = c(\mathbf{w} \cdot \mathbf{u}) = c0 = 0$. Thus, $c\mathbf{u}$ is in $W^\perp$.

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Thus, $W^\perp$ is a subspace of $\mathbb{R}^n$. $\square$

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Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{z}$ is in $W^\perp$ if and only if $\mathbf{z}$ is orthogonal to every vector in a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans W.

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Let $\mathbf{w}$ be any vector in W. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans W, there exist scalars $c_1, \ldots, c_p$ such that $\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$. We compute

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Thus, $\mathbf{z}$ is orthogonal to every vector in W, and hence $\mathbf{z}$ is in $W^\perp$. $\square$

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Thus we have bases

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We have $A^T = \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus we have basis

$$\text{Null } A^T = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

Example

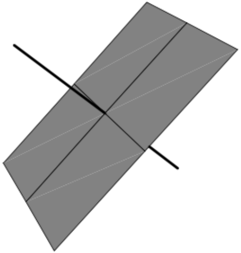
Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \end{bmatrix}$.

1. Find a basis for Null A, Col A and Row A
2. Find a basis for Null A^T.

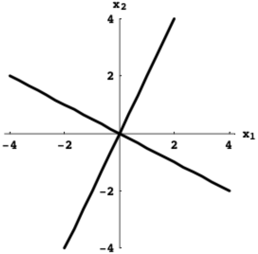
We have bases

$$\text{Null } A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \text{Row } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$\text{Col } A = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \qquad \text{Null } A^T = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$



Subspaces Nul A and Row A



Subspaces Nul A^T and Col A

Null, Col, and Row Spaces Revisited

Theorem

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A. The orthogonal complement of the column space of A is the null space of A^T.

$$(\text{Row } A)^\perp = \text{Null } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Null } A^T$$

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Proof.

Let $\mathbf{x}$ be in $(\text{Row } A)^\perp$. $A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

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Thus $\mathbf{x}$ is in Null A.

Similarly, if $\mathbf{x}$ is in Null A, then it satisfies the equation $A\mathbf{x} = \mathbf{0}$, which implies $\mathbf{x}$ is orthogonal to each row of A. Since the rows of A span Row A and by a previous theorem, $\mathbf{x}$ is orthogonal to each vector in Row A. Thus, $\mathbf{x}$ is in $(\text{Row } A)^{\perp}$.

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$$(\text{Row } A)^{\perp} = \text{Null } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Null } A^T$$

Proof.

The second statement follows by noting that Col $A =$ Row A^T:

$$(\text{Col } A)^{\perp} = (\text{Row } A^T)^{\perp} = \text{Null } A^T$$

$\square$