Matrix Arithmetic

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

Representations of Matrices

- If A is an $m \times n$ matrix, it has m rows and n columns.
- The j^{th} column vector is denoted $\mathbf{a}_j$.
- We have n column vectors, and each $\mathbf{a}_j$ is in $\mathbb{R}^m$.
- The entry in the i^{th} row and j^{th} column is denoted a_{ij}.
- Rows First! First we give the row index, then the column index.
- Recall that in Python, indexing starts at 0.

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_j & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

Types of Matrices

- An $m \times n$ whose entries are all 0 is called a zero matrix, $M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
- A matrix that has the same number of rows and columns is called a square matrix.
- We call the entries along the diagonal of a square matrix the diagonal entries.
- A square matrix whose off-diagonal entries are all 0 is called a diagonal matrix.
- An $n \times n$ matrix with each $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$ is called the identity matrix and is denoted I_n.
- An $n \times n$ matrix for which $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$ is called a symmetric matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}, I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 0 & 8 \\ 3 & 0 & 11 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

Arithmetic with Matrices

- Let A and B be $m \times n$ matrices of the same size. The sum $C = A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries of A and B. In other words, $c_{ij} = a_{ij} + b_{ij}$.
- Let A denote an $m \times n$ matrix and r a scalar. To compute scalar multiple rA we multiply every entry in A by the scalar r. Thus, if $C = rA$, then $c_{ij} = ra_{ij}$.
- Two matrices A and B are equal if they have the same size and $a_{ij} = b_{ij}$ for all entries.

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Example

Let $A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 0 & 5 \end{bmatrix}$. Compute $2A - B$.

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices of the same size, and let r and s denote scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Matrix Multiplication

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p\end{bmatrix}.$$

Let $A = \begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}$ and $B = \begin{bmatrix}2 & 1 & 0 & 5\\ 6 & 0 & -1 & 2\\ 1 & 2 & 1 & 1\end{bmatrix}$. We define the product AB as follows:

$$\begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}2\\ 6\\ 1\end{bmatrix} = \begin{bmatrix}1\\ -7\end{bmatrix} \qquad \begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ 2\end{bmatrix} = \begin{bmatrix}5\\ 4\end{bmatrix} \qquad \begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}0\\ -1\\ 1\end{bmatrix} = \begin{bmatrix}3\\ 3\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0 & 3\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}5\\ -2\\ 1\end{bmatrix} = \begin{bmatrix}-2\\ 15\end{bmatrix} \qquad AB = \begin{bmatrix}1 & 5 & 3 & -2\\ -7 & 4 & 3 & 15\end{bmatrix}$$

Checking Matrix Sizes

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. The matrix product AB:

- Is only defined when A has the same number of columns as B has rows.
- Is undefined when A has a different number of columns as B has rows.
- Results in an $m \times p$ matrix when it is defined.

inner sizes match, outer sizes give result: $A_{m\times n}B_{n\times p} = (AB)_{m\times p}$.

$$AB = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 1 & 0 & 5 \\ 6 & 0 & -1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}_{3\times 4}$$

$$= \begin{bmatrix} 1 & 5 & 3 & -2 \\ -7 & 4 & 3 & 15 \end{bmatrix}_{2\times 4}$$

Is the Operation Defined?

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E = \begin{bmatrix} 4 & 6 \\ 0 & 3 \\ 1 & 0 \end{bmatrix},\ F = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1. For which matrices is addition with A defined? [In what order?]
2. For which matrices is addition with B defined?
3. For which matrices is addition with C defined?
4. For which matrices is multiplication with A defined? [In what order?]
5. For which matrices is multiplication with B defined?
6. For which matrices is multiplication with C defined?

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 4 & 6 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}, \ F = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Evaluate each of the products below:

a) AD

b) DA

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E = \begin{bmatrix} 4 & 6 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Evaluate each of the products below:

a) BF

b) FB

c) B^2

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 4 & 6 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Evaluate each of the products below:

a) $A(BC)$

b) $(AB)C$

Finding the ij^{th} Entry

$$AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \left[\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right] = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{i1} & b_{i2} & \dots & b_{ij} & \dots & b_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix}$$

$$\Big(AB\Big)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Properties of Matrix Multiplication

Let A, B, and C be matrices for which the indicated sums and products are defined. Let r denote a scalar.

a. $A(BC) = (AB)C$ (associative law)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ (scalar multiplication)

e. $I_m A = A = A I_m$ (identity for a square $m \times m$ matrix A)

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Partial proof for identity.

Note that $I_m = [\mathbf{e}_1\ \mathbf{e}_2\ \ldots\ \mathbf{e}_m]$. Then

$$AI_m = A[\mathbf{e}_1\ \mathbf{e}_2\ \ldots\ \mathbf{e}_m] = [A\mathbf{e}_1\ A\mathbf{e}_2\ \ldots\ A\mathbf{e}_m] = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_m] = A.$$

$\square$

The Transpose of a Matrix

Let A denote an $m \times n$ matrix. The transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Example

Give the transpose of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -5 & -6 & -7 & -8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Properties of Transpose

Let A and B be matrices for which the indicated sums and products are defined. Let r denote a scalar.

a. $\left(A^T\right)^T = A$

b. $(A+B)^T = A^T + B^T$

c. $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

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Other proof for identity.

We previously showed that $A_{m\times m} I_m = A$. We wish to show that $I_m A = A$.

$$I_m A = \left((I_m A)^T\right)^T = \left(A^T I_m^T\right)^T = \left(A^T I_m\right)^T = \left(A^T\right)^T = A. \qquad \square$$