The Inverse of a Matrix

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

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Can we solve the matrix equation $A\mathbf{x} = \mathbf{b}$ in a similar way?

The Inverse of a Matrix

Let A denote a square $n \times n$ matrix. The inverse of A (if it exists) is denoted A^{-1} and it is the unique matrix such that

- $AA^{-1} = I_n$ and
- $A^{-1}A = I_n$.

For example if $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$.

Check:

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} =$$

Solving the Matrix Equation $A\mathbf{x} = \mathbf{b}$

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

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First, we show that $A^{-1}\mathbf{b}$ is a solution:

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b}.$$

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Second, we show the solution is unique. Suppose $\mathbf{y}$ is a solution. Then $A\mathbf{y} = \mathbf{b}$. Multiplying both sides on the left by A^{-1} and simplifying, we obtain:

$$A^{-1}(A\mathbf{y}) = A^{-1}\mathbf{b} \quad \Rightarrow \quad (A^{-1}A)\mathbf{y} = A^{-1}\mathbf{b} \quad \Rightarrow \quad I_n\mathbf{y} = A^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{y} = A^{-1}\mathbf{b}.$$

$\square$

Properties of Inverses

If A is an invertible matrix, then A^{-1} is invertible and its inverse is

$$\left(A^{-1}\right)^{-1} = A.$$

Why?

- A matrix that is not invertible is called a singular matrix.
- A matrix that is invertible is called a nonsingular matrix.

Properties of Inverses

If A and B are $n \times n$ invertible matrices, then AB is an $n \times n$ invertible matrix. The inverse $(AB)^{-1}$ is obtained by taking the product of inverses in the reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Why?

Properties of Inverses

If A is an invertible matrix, then so is A^T, and the inverse is

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

Why?

Elementary Matrices

An elementary matrix is one that is obtained by performing an elementary row operation on an identity matrix.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Elementary Row Operations via Elementary Matrices

Theorem

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is an elementary matrix created by performing the same row operation on I_m.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3\times 4} \qquad E_2A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -13 & -14 & -15 & -16 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3\times 4}$$

Elementary Matrices are Invertible

Every elementary matrix E is invertible, and the inverse E^{-1} is formed by applying the inverse elementary row operation to I_m.

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Write out matrices to represent E_1^{-1}, E_2^{-1}, and E_3^{-1}.

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($\Rightarrow$) Suppose that A is invertible. Then $A\mathbf{x} = \mathbf{b}$ is solvable for every $\mathbf{b}$ in $\mathbb{R}^n$. Thus, there must be a pivot in every row of A. Since A is square, the RREF of A is I_n.

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($\Leftarrow$) Suppose that the RREF of A is I_n. Then there is a sequence $R_1, R_2, \ldots, R_p$ of elementary row operations that transforms A into I_n. Let E_i be the elementary matrix corresponding to R_i. Then

$$E_p E_{p-1} \ldots E_2 E_1 A = I_n$$

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$$\underbrace{E_p E_{p-1} \ldots E_2 E_1}_{A^{-1}} A = I_n$$

□

An Algorithm for Finding A^{-1}

Row reduce the augmented matrix $\left[A \mid I_n\right]$.

- If RREF of A is I_n, then RREF of $\left[A \mid I_n\right]$ is $\left[I_n \mid A^{-1}\right]$.
- Otherwise, A does not have an inverse.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & \frac{2}{5} \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

Practice

If possible, find the inverse of the given matrix.

a) $\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

b) $\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$

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$$\begin{bmatrix} a \\ b \end{bmatrix} = k \begin{bmatrix} c \\ d \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{rcl} a & = & kc \\ b & = & kd \end{array} \quad \Rightarrow \quad \frac{a}{c} = k = \frac{b}{d} \quad \Rightarrow \quad ad = bc \quad \Leftrightarrow \quad ad - bc = 0$$

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Theorem

A is not invertible if and only if $ad - bc = 0$.

If A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Practice

If possible, find the inverse of the given matrix.

a) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ -4 & 0 & -2 \end{bmatrix}$

Solving Systems of Linear Equations

Solve the system of equations

$$\begin{aligned} x_1 + 3x_3 &= 2 \\ -2x_2 - 2x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

The equation above has matrix equation $A\mathbf{x} = \mathbf{b}$ given by

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We compute the solution by $\mathbf{x} = A^{-1}\mathbf{b}$:

$$\mathbf{x} = \begin{bmatrix} 4 & 4.5 & -3 \\ 1 & 1 & -1 \\ -1 & -1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 12.5 \\ 3 \\ -3.5 \end{bmatrix}$$

Solving Systems with Inverse Matrices

1. Write the system of linear equations as a matrix equation $A\mathbf{x} = \mathbf{b}$.
2. Find A^{-1} (if possible).
3. Multiply both sides of the equation in step (1) on the left by A^{-1}.
4. The solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Note. If the number of equations and variables are not equal, then A will not be a square matrix, and it will not be possible to find A^{-1}. Use row reduction on the augmented matrix instead.

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How does solving $A\mathbf{x} = \mathbf{b}$ with inverse matrices compare to finding the RREF of the augmented matrix?