The Determinant of a Matrix

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

The Invertible Matrix Theorem redux

Let A be a square $n \times n$ matrix. Then all of the following statements are equivalent.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix I_n.

(c) A has n pivots.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(h) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(i) The columns of A span $\mathbb{R}^n$.

(j) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(l) There is an $n \times n$ matrix C such that $CA = I_n$.

(m) There is an $n \times n$ matrix D such that $AD = I_n$.

(n) A^T is an invertible matrix.

Inverse of a 2×2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

when $ad - bc \neq 0$.
If $ad - bc = 0$, then A is not invertible.

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- If both a and c are equal to 0, A is not invertible. Assume $a \neq 0$.
- Then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

- As long as $ad - bc \neq 0$, A has a pivot in every column and is invertible.

The determinant $\det A$ of a 2×2 matrix A is $\det A = ad - bc = a_{11}a_{22} - a_{12}a_{21}$.

Examples of Determinants of 2×2 Matrices

Compute the determinant of each matrix and determine whether the matrix has an inverse or not? If the matrix is invertible, give its inverse.

a) $\begin{bmatrix} 2 & -4 \\ 3 & 8 \end{bmatrix}$

b) $\begin{bmatrix} -5 & 2 \\ -10 & 4 \end{bmatrix}$

Determinant of larger matrices

We want a function called the determinant defined on square $n \times n$ matrices that produces a scalar with the following properties:

1. $\det A \neq 0$ if and only if A is invertible.

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We want a function called the determinant defined on square $n \times n$ matrices that produces a scalar with the following properties:

1. $\det A \neq 0$ if and only if A is invertible.
2. $\det(AB) = (\det A)(\det B)$.
3. $\det(I_n) = 1$.
4. det is multilinear.

This ends up making a complicated (but useful!) function.

Determinant of a 1×1 Matrix

Let $A = \begin{bmatrix} a_{11} \end{bmatrix}$. What is $\det A$?

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Let $A = \begin{bmatrix} a_{11} \end{bmatrix}$. What is $\det A$?

A is invertible if and only if $a_{11} \neq 0 \quad \Rightarrow \quad \det A = a_{11}$.

Determinant of a 3×3 Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

- The submatrix obtained by deleting entries in the first row and first column we denote $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$.
- The submatrix obtained by deleting entries in the first row and second column we denote $A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$.
- In general, the submatrix obtained by deleting entries in the i^{th} row and j^{th} column we denote A_{ij}.

Determinant of a 3×3 Matrix

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

We can compute the determinant of matrix A as follows:

$$\begin{aligned} \det A &= a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) \\ &= (3)\left(\det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) - (-1)\left(\det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) + (2)\left(\det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\right) \\ &= (3)(2) - (-1)(1) + (2)(-4) \\ &= -1 \end{aligned}$$

Determinant of a 3×3 Matrix

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

$$\begin{aligned} \det A &= a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) \\ &= (3)(2) - (-1)(1) + (2)(-4) = -1 \end{aligned}$$

Or we have:

$$\begin{aligned} \det A &= -a_{21}(\det A_{21}) + a_{22}(\det A_{22}) - a_{23}(\det A_{23}) = -(1)\left(\det \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\right) + (2)\left(\det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\right) - 0 \\ &= -(1)(-1) + (2)(-1) - 0 = -1 \end{aligned}$$

Or we have

$$\begin{aligned} \det A &= a_{31}(\det A_{31}) - a_{32}(\det A_{32}) + a_{33}(\det A_{33}) = (2)\left(\det \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}\right) - 0 + (1)\left(\det \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\right) \\ &= (2)(-4) - 0 + (1)(7) = -1 \end{aligned}$$

Determinant of a 3×3 Matrix

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

We have

- $\det A = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13})$, or
- $\det A = -a_{21}(\det A_{21}) + a_{22}(\det A_{22}) - a_{23}(\det A_{23})$, or
- $\det A = a_{31}(\det A_{31}) - a_{32}(\det A_{32}) + a_{33}(\det A_{33})$.

Choosing to expand across a row with 0's requires less calculations!

The sign associated with the coefficient a_{ij} in front of each determinant is positive if $i + j$ is even and negative if $i + j$ is odd.

Examples of Determinants of 3×3 Matrices

Compute the determinant of each matrix.

a) $\begin{bmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 7 & -5 \\ 0 & -2 & 16 \\ 0 & 0 & 5 \end{bmatrix}$

Mnemonic Device for 2×2 and 3×3 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

Mnemonic Device for 2×2 and 3×3 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad$$

$$+$$

Mnemonic Device for 2 × 2 and 3 × 3 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad \; - bc$$

Mnemonic Device for 2×2 and 3×3 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad \; -bc$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Mnemonic Device for 2×2 and 3×3 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad \; -bc$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Mnemonic Device for 2×2 and 3×3 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = +ad - bc$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

Mnemonic Device for 2×2 and 3×3 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = {\color{red}+ad} \; {\color{blue}-bc}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} = \begin{matrix} {\color{red}+a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}} \\ {\color{blue}-a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}} \end{matrix}$$

Determinants of 4×4 Matrices

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 10 & 2 & -3 \\ 2 & 5 & -1 & -6 \\ 0 & 7 & -5 & 1 \end{bmatrix}.$$

Then we have

$$\det A = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) - a_{14}(\det A_{14})$$

$$(1)\left(\det \begin{bmatrix} 10 & 2 & -3 \\ 5 & -1 & -6 \\ 7 & -5 & 1 \end{bmatrix}\right) - (2)\left(\det \begin{bmatrix} 4 & 2 & -3 \\ 2 & -1 & -6 \\ 0 & -5 & 1 \end{bmatrix}\right) + (3)\left(\det \begin{bmatrix} 4 & 10 & -3 \\ 2 & 5 & -6 \\ 0 & 7 & 1 \end{bmatrix}\right) - (-2)\left(\det \begin{bmatrix} 4 & 10 & 2 \\ 2 & 5 & -1 \\ 0 & 7 & -5 \end{bmatrix}\right)$$

$$\det \begin{bmatrix} 10 & 2 & -3 \\ 5 & -1 & -6 \\ 7 & -5 & 1 \end{bmatrix} = (10)\left(\det \begin{bmatrix} -1 & -6 \\ -5 & 1 \end{bmatrix}\right) - (2)\left(\det \begin{bmatrix} 5 & -6 \\ 7 & 1 \end{bmatrix}\right) + (-3)\left(\det \begin{bmatrix} 5 & -1 \\ 7 & -5 \end{bmatrix}\right)$$

and so on!!!!

Example of Determinant of a 4×4 Matrix

Compute the determinant of $A = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 2 & 3 & -4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

Example of Determinant of a 4×4 Matrix

Compute the determinant of $A = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 2 & 3 & -4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Determinants of $n \times n$ Matrices

Let $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 10 & 2 & -3 \\ 2 & 5 & -1 & -6 \\ 0 & 7 & -5 & 1 \end{bmatrix}$.

Then we have

$\det A = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) - a_{14}(\det A_{14})$

In general, if A is an $n \times n$ matrix, then we define:

$$\begin{aligned} \det A &= a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + \ldots + (-1)^{1+n} a_{1n}(\det A_{1n}) \\ &= \sum_{j=1}^{n} (-1)^{1+j}\ a_{1j}\ \det\ A_{1j} \\ &= \sum_{j=1}^{n} (-1)^{2+j}\ a_{2j}\ \det\ A_{2j} = \sum_{j=1}^{n} (-1)^{i+j}\ a_{ij}\ \det\ A_{ij} \end{aligned}$$

Computing Determinants with Python

Compute the determinant of $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

We can use `A.det()` in SymPy.

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Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 4 & 10 & 2 & -3 \\ 2 & 5 & -1 & -6 \\ 0 & 7 & -5 & 1 \end{bmatrix}.$$

Cofactor Expansion

In general, if A is an $n \times n$ matrix, then we define:

$$\det A = \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j}\ \det\ A_{1j} = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + \ldots + (-1)^{1+n} a_{1n}(\det A_{1n})$$

- We call the expression $(-1)^{i+j} \det\ A_{ij}$ the (i, j) cofactor.
- We can rewrite the formulas in the definition above in terms of cofactors:

$$\det\ A = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \ldots + a_{1n} C_{1n}.$$

- The formula above is the cofactor expansion across the 1st row.
- In general, we can use the cofactor expansion across any row:

$$\det\ A = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in}.$$

Cofactor Expansion Across Rows or Down Columns

Theorem

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

- The cofactor expansion across the i^{th} row is given by:

$$\det\ A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

- The cofactor expansion down the j^{th} column is given by:

$$\det\ A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Comparing Methods

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

We can compute the determinant of matrix A using the cofactor expansion across the first row:

$$\begin{aligned} \det A &= a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13}) \\ &= (3)\left(\det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) - (-1)\left(\det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) + (2)\left(\det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\right) \\ &= (3)(2) - (-1)(1) + (2)(-4) = -1 \end{aligned}$$

We can compute the determinant of matrix A using the cofactor expansion down column 2:

$$\begin{aligned} \det A &= -a_{12}(\det A_{12}) + a_{22}(\det A_{22}) - a_{32}(\det A_{32}) \\ &= -(-1)\left(\det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) + (2)\left(\det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\right) - (0)\left(\det \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\right) \\ &= (1)(1) + (2)(-1) - 0 = -1 \end{aligned}$$