Orthogonal Sets

Linear Algebra

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`https://github.com/CU-Denver-MathStats-OER`

Orthogonal Sets

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ when $i \neq j$ (ie, the vectors are pairwise orthogonal).

Example

Is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal set?

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Since

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \quad \mathbf{v}_1 \cdot \mathbf{v}_3 = 0, \quad \text{and } \mathbf{v}_2 \cdot \mathbf{v}_3 = 0,$$

the set is indeed an orthogonal set.

Linear Independence of Orthogonal Sets

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is a linearly independent set, and is therefore a basis for the subspace spanned by S.

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Proof.

Suppose S is an orthogonal set and that the vectors in S are linearly dependent. This implies there exist scalars $c_1, c_2, \ldots, c_p$, with at least one $c_i \neq 0$, such that
$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$.

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$$\begin{aligned} 0 = \mathbf{0} \cdot \mathbf{v}_1 &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p) \cdot \mathbf{v}_1 \\ &= c_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + c_2(\mathbf{v}_2 \cdot \mathbf{v}_1) + \ldots c_p(\mathbf{v}_p \cdot \mathbf{v}_1) \\ &= c_1(\mathbf{v}_1 \cdot \mathbf{v}_1). \end{aligned}$$

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Since $\mathbf{v}_1$ is nonzero, we have $c_1 = 0$. Similarly, we can show that $c_1 = c_2 = \ldots = c_p = 0$, which contradicts our original assumption. Thus S is linearly independent. $\square$

Orthogonal Bases

Definition

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Example

- The set $S_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$.
- The set $S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ that is not orthogonal.

Finding Coordinates With an Orthogonal Basis

Suppose that $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$.

Find $\mathcal{B}$-coordinates of $\mathbf{y}$ in W.

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$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_p\mathbf{u}_p$$

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\mathbf{y}\cdot\mathbf{u}_1 &= c_1(\mathbf{u}_1\cdot\mathbf{u}_1) \\
c_1 &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}
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c_j &= \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \text{ for each } j.
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\end{aligned}$$

Theorem

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination
$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_p\mathbf{u}_p$ are given by

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \quad \text{for } j = 1, 2, \ldots, p.$$

Finding Coordinates with an Orthogonal Basis

Example

Express the vector $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of the orthogonal basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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$$c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{3 - 7}{2} = -2$$

$$c_2 = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{3 + 7}{2} = 5$$

$$c_3 = \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} = \frac{4}{1} = 4$$

Thus we have

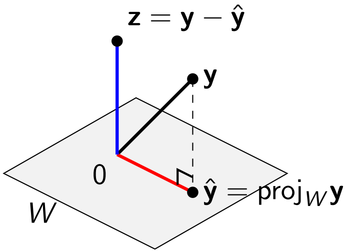
$$\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Orthogonal Projections

- In physics, forces are generally expressed as vectors, $\mathbf{F}$.
- An object is moving with velocity $\mathbf{v}$.
- We often want to decompose the force $\mathbf{F}$ into two components $\mathbf{F} = \mathbf{F}_{\text{perp}} + \mathbf{F}_{\text{parallel}}$

Given a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$, we wish to decompose a vector $\mathbf{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one that is parallel to $\mathbf{u}$ and the other is orthogonal to $\mathbf{u}$:

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = c\mathbf{u} + \mathbf{z} = \text{proj}_{\mathbf{u}}\mathbf{y} + \mathbf{y}_{\text{perp}} = \text{proj}_{\mathbf{u}}\mathbf{y} + (\mathbf{y} - \hat{\mathbf{y}}).$$



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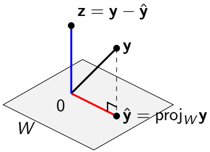

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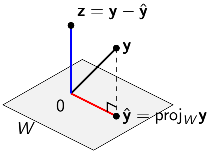

We want to find $\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}}\mathbf{y} = c\mathbf{u}$. The other component $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\mathbf{u}$, so

$$0 = (\mathbf{y} - c\mathbf{u}) \cdot \mathbf{u} = (\mathbf{y} \cdot \mathbf{u}) - c(\mathbf{u} \cdot \mathbf{u}).$$

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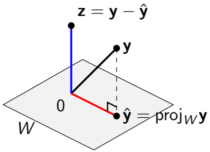

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Thus we see that

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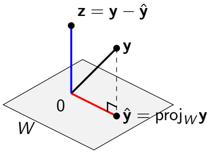

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Definition

The vector $\hat{\mathbf{y}}$ is called the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$, and the vector $\mathbf{z}$ is called the component of $\mathbf{y}$ orthogonal to $\mathbf{u}$. We have

$$\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}}\mathbf{y} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}, \qquad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

Application of Projections

Example

Suppose a sailboat has constant velocity given by $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The wind has a constant velocity of $\mathbf{w} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$. Find the component of the force that in the direction of the sailboat's motion.

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We have

$$\hat{\mathbf{w}} = \text{proj}_{\mathbf{v}}\mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \frac{-10}{25}\mathbf{v} = \begin{bmatrix} -6/5 \\ -8/5 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{w} - \hat{\mathbf{w}} = \begin{bmatrix} 56/5 \\ -42/5 \end{bmatrix}$$

And we can verify that

$$\hat{\mathbf{w}} \cdot \mathbf{z} = \begin{bmatrix} -6/5 \\ -8/5 \end{bmatrix} \cdot \begin{bmatrix} 56/5 \\ -42/5 \end{bmatrix} = 0.$$

Orthonormal Sets

Definition

A set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors.

Example

The set $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal set that forms a basis for $\mathbb{R}^n$.

- Orthonormal sets (bases) are particularly nice to work with as many formulas simplify considerably.
- For example, $\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}}\mathbf{y} = \left(\dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = (\mathbf{y}\cdot\mathbf{u})\mathbf{u}$.

Matrices Constructed from Orthonormal Sets

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthonormal set in $\mathbb{R}^n$ and let $U = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p\end{bmatrix}_{n\times p}$. Then

$$U^T U = \begin{bmatrix}\mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_p^T\end{bmatrix}_{p\times n} \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p\end{bmatrix}_{n\times p}$$

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$$U^TU = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_p^T \end{bmatrix}_{p\times n} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p \end{bmatrix}_{n\times p} = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \ldots & \mathbf{u}_1^T\mathbf{u}_p \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \ldots & \mathbf{u}_2^T\mathbf{u}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{u}_p^T\mathbf{u}_1 & \mathbf{u}_p^T\mathbf{u}_2 & \ldots & \mathbf{u}_p^T\mathbf{u}_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}_{p\times p}$$

Theorem

The columns of an $m \times n$ matrix U form an orthonormal set if and only if $U^TU = I_{n\times n}$.

Linear Mapping from Matrix whose Columns form an Orthonormal Set

Theorem

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

(a) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

(b) $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

In other words, the linear mapping $\mathbf{x} \mapsto U\mathbf{x}$ preserves length and orthogonality.

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Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

(a) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

(b) $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

In other words, the linear mapping $\mathbf{x} \mapsto U\mathbf{x}$ preserves length and orthogonality.

Proof.

Statements (b) and (c) follow from statement (a). Note that

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T(U\mathbf{y}) = \mathbf{x}^T(U^TU)\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

□