Complex Eigenvalues

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Example

Compute the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$.

Roots of the Characteristic Equation

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We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 5 \\ -1 & 2-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda) - (5)(-1) = \lambda^2 - 6\lambda + 13 = 0.$$

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Not easy to factor, so we use the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{-16}}{2}.$$

There are no real roots, but there are complex roots!

Complex Numbers

Definition

The imaginary number i is defined to be the number that satisfies the relation $i^2 = -1$, or equivalently $\sqrt{-1} = i$.

Example. $\sqrt{-16} = \sqrt{(16)(-1)} = (\sqrt{16})(\sqrt{-1}) = 4i$.

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Definition

A complex number is a number that can be written in the form $z = a + bi$, where a and b are real numbers and i denotes the imaginary number. The set of complex numbers is denoted $\mathbb{C}$.

- The real number a is called the real part of z and is denoted $\operatorname{Re} z$.
- The real number b is called the imaginary part of z and is denoted $\operatorname{Im} z$.

Solving Quadratic Equations

Solutions to a quadratic equation $ax^2 + bx + c = 0$ can have the following form:

- Two distinct real roots. For example, $x^2 - x - 6 = (x-3)(x+2) = 0$ has roots $x = 3$ and $x = -2$.
- One repeated real root. For example, $x^2 - 6x + 9 = (x-3)^2 = 0$ has root $x = 3$ with multiplicity 2.
- Two complex (conjugate) roots.

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Example

Solve the quadratic equation $x^2 - 6x + 13 = 0$.

Using the quadratic formula, we have

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 + 2i \text{ and } 3 - 2i.$$

Arithmetic with Complex Numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ be complex numbers. We define the following operations.

- $z_1 + z_2 = (a + c) + (b + d)i$.
- If $e \in \mathbb{R}$, $ez_1 = (ea) + (eb)i$.

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- If $z_2 \neq 0$, then we can divide by z_2: z_1/z_2.
- The conjugate $\overline{z_1}$ of z_1 is $\overline{z}_1 = a - bi$ (ie, $\mathrm{Re}\,\overline{z} = \mathrm{Re}\,z$ and $\mathrm{Im}\,\overline{z} = -\mathrm{Im}\,z$).
  - $\overline{\overline{z}} = z$ and $(\overline{z_1})(\overline{z_2}) = \overline{z_1 z_2}$.
  - $z_1\overline{z_1} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$, which is real.

Dividing Complex Numbers

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Calculations with complex numbers are best done by computer!

Finding a Basis for the Eigenspace of Complex Eigenvalues

Example

Diagonalize $A = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$.

Matrix A has two eigenvalues that we have already identified, namely $\lambda = 3 \pm 2i$.
We solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find a basis for each eigenspace. For $\lambda_1 = 3 - 2i$, we solve:

$$(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 4 - (3 - 2i) & 5 \\ -1 & 2 - (3 - 2i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + 2i & 5 \\ -1 & -1 + 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1+2i & 5 \\ -1 & -1+2i \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}$$

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A basis for the eigenspace is $\left\{ \begin{bmatrix} -1+2i \\ 1 \end{bmatrix} \right\}$.

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Putting it all together, we have $P^{-1}AP = D$, where

$$D = \begin{bmatrix} 3-2i & 0 \\ 0 & 3+2i \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -1+2i & -1-2i \\ 1 & 1 \end{bmatrix}.$$

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Check:

$$\underbrace{\begin{bmatrix} -\frac{i}{4} & \frac{1}{2}-\frac{i}{4} \\ \frac{i}{4} & \frac{1}{2}+\frac{i}{4} \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1+2i & -1-2i \\ 1 & 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 3-2i & 0 \\ 0 & 3+2i \end{bmatrix}}_{D}.$$

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What are the eigenvalues of A?

$$\begin{aligned} \det(A - \lambda I) = \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} &= (\cos\theta - \lambda)^2 + (\sin\theta)^2 \\ &= (\cos\theta)^2 + (\sin\theta)^2 - 2\lambda\cos\theta + \lambda^2 \\ &= 1 - 2\lambda\cos\theta + \lambda^2. \end{aligned}$$

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Solving the characteristic equation $\lambda^2 - (2\cos\theta)\lambda + 1 = 0$,

$$\lambda = \frac{2\cos\theta \pm \sqrt{4(\cos\theta)^2 - 4}}{2} = \cos\theta \pm \sqrt{(\cos\theta)^2 - 1}$$

Note that $(\cos\theta)^2 \leq 1$. The eigenvalues are complex when $(\cos\theta)^2 < 1$.