Markov Chains

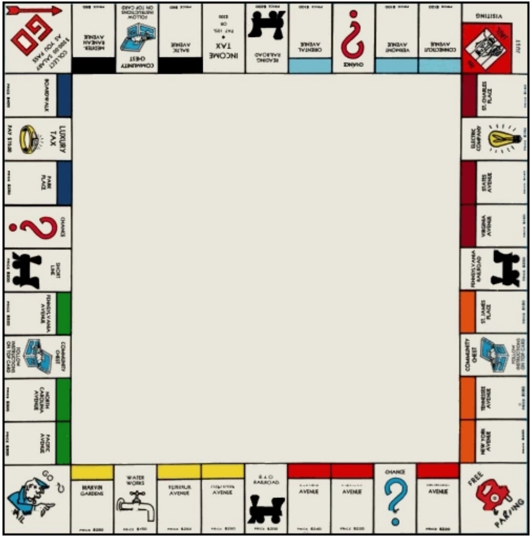
Linear Algebra

This work was initially funded by an Institutional OER Grant from the Colorado Department of Higher Education (CDHE). For similar OER materials in other courses funded by this project in the Department of Mathematical and Statistical Sciences at the University of Colorado Denver, visit
https://github.com/CU-Denver-MathStats-OER

Monopoly is a board game where tokens are moved around the board. Players start on "Go".

On each turn, a player rolls two dice, and then advances their token by the number of spaces equal to the sum of the rolled dice.

What is the most landed on spot in Monopoly?



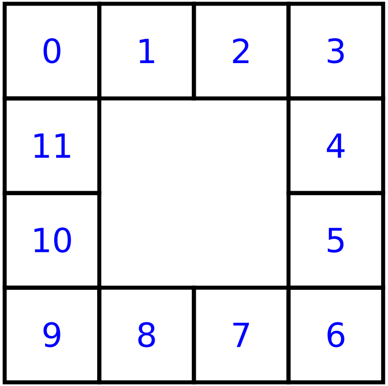
Simplified Monopoly

We'll use a board with 12 spaces (4 on each side), and roll 1 4-sided die (numbered 1,2,3,4) on each turn.

Players start on space 0 ("Go").

If a player lands on space 9 ("Go to Jail"), then when they leave on their next turn, they go to spaces $4, \ldots, 7$.

Where will a player be after their first turn?



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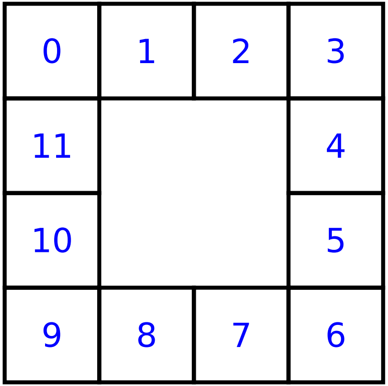
Where will a player be after their first turn?

A state is the current space a player's token is on.

A state vector gives a probability distribution on the states.

$$\mathbf{x}_1 = \begin{bmatrix} 0 & .25 & .25 & .25 & .25 & 0 & 0 & \ldots & 0 \end{bmatrix}$$



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Let $\mathbf{p}_i$ give the probability distribution on states after one turn if the player starts on space i.

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Switch to Jupyter Notebook for calculations.

Markov Chains

Definition

A Markov chain is a mathematical model used to describe a random process where:

- At each discrete time $k \geq 0$, the system is in one of n states.
- There are probabilities that describe how likely it is the system transitions from one state to another. These probabilities depend only on the current state.

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Since a Markov chain describes a random process, we let the state vector $\mathbf{x}_k$ be the probability distribution over which state the system is in at time k.

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$$\mathbf{x}_{k+1} = P\mathbf{x}_k = P(P\mathbf{x}_{k-1}) = P^2\mathbf{x}_{k-1} = \cdots = P^{k+1}\mathbf{x}_0.$$

A Simple Weather Model

- Imagine that there are two possible states for weather: no rain (sunny) or rain.
- You can always directly observe the current weather state.
- The current weather has some bearing on what the next day's weather will be.

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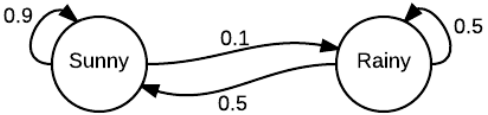

We can build a model to predict tomorrow's weather based on the weather today.

$$P = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$$

- We organize the probabilities of transitioning from one state to another in a transition matrix.
- p_{ij} is the probability of transitioning from state j to state i.
- This will be a stochastic matrix. The entries in each column vector add up to 1.

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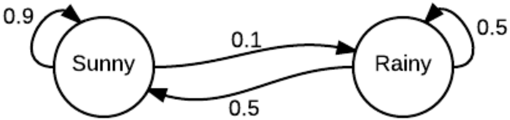

Today the weather is sunny.

state 1 = sunny, state 2 = rainy

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What is the probability that it rains three days from now?

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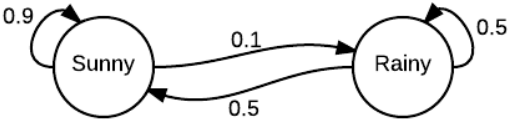

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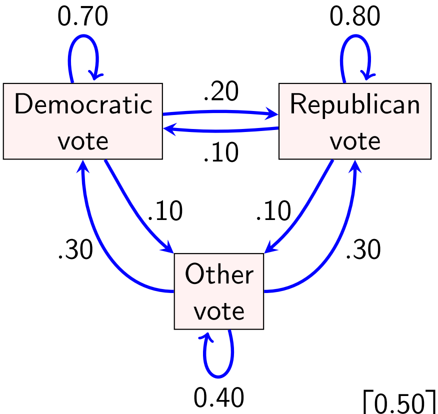
$$\begin{aligned} \mathbf{x}_3 = P\mathbf{x}_2 = PP\mathbf{x}_1 = PPP\mathbf{x}_0 &= P^3\mathbf{x}_0 \\ &= \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.844 \\ 0.156 \end{bmatrix} \end{aligned}$$

So the probability it rains in three days, given that today is sunny, is 0.156.

Modeling Election Outcomes

Consider the following model:



With initial state vector $\mathbf{x}_0 = \begin{bmatrix} 0.50 \\ 0.45 \\ 0.05 \end{bmatrix}$

1. What is the predicted outcome in the next election cycle?
2. What is the predicted outcome in two election cycles?

$$\begin{bmatrix} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.8 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.45 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.41 \\ 0.475 \\ 0.115 \end{bmatrix}$$

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Long Term Predictions

Often we are interested in making predictions far out in time.

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- Based on current infection rates, what is the long term prediction for the total number of infected people?

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The limiting distribution does not depend on the initial state vector $\mathbf{x}_0$.

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If P is a stochastic matrix, then a steady state vector (or equilibrium vector) for P is a probability vector $\mathbf{q}$ (nonnegative entries that sum to 1) such that

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(*Monopoly example in Jupyter Notebook.*)

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Consider the column vector of all 1's that we denote $\mathbf{e}$. Then we have

$$P^T\mathbf{e} = \begin{bmatrix} \sum_{j=1}^n p_{j1} \\ \sum_{j=1}^n p_{j2} \\ \vdots \\ \sum_{j=1}^n p_{jn} \end{bmatrix} = \mathbf{e}.$$

Therefore, $\mathbf{e}$ is an eigenvector of P^T corresponding to the eigenvalue $\lambda = 1$. $\square$