Eigenvalues and Eigenvectors

Linear Algebra

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https://github.com/CU-Denver-MathStats-OER

Introduction

The concepts of eigenvalues and eigenvectors of a matrix are useful throughout pure and applied mathematics.

- Eigenvalues are used to study difference equations and continuous dynamical systems.
- They provide critical information in engineering design, and they arise naturally in such fields as physics and chemistry.

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How do $\mathbf{x}$ and $A\mathbf{x}$ relate?

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a) Compute $A\mathbf{u}$, where $\mathbf{u} = \begin{bmatrix} 1+\sqrt{3} \\ 1 \end{bmatrix}$.

b) Compute $A\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1+\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1-\sqrt{3} \end{bmatrix} = (1-\sqrt{3})\mathbf{u}.$$

$$\begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \neq c\mathbf{v}.$$

Eigenvalues and Eigenvectors

Consider the matrix $A = \begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix}$ and vectors $\mathbf{u} = \begin{bmatrix} 1+\sqrt{3} \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1+\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1-\sqrt{3} \end{bmatrix} = (1-\sqrt{3})\mathbf{u}.$$

$$\begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \neq c\mathbf{v}.$$

Definition

An eigenvalue of an $n \times n$ matrix A is a scalar λ such that there exists a nonzero vector $\mathbf{x}$ where $A\mathbf{x} = \lambda\mathbf{x}$. The nonzero vector $\mathbf{x}$ is called an eigenvector corresponding to λ.

Example

Show that $\lambda = 4$ is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

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- This gives the equivalent equation $A\mathbf{x} - 4\mathbf{x} = \mathbf{0}$.
- Note that we can represent the scalar product $4\mathbf{x} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{x} = 4I_2\mathbf{x}$.
- We solve the equation $A\mathbf{x} - 4\mathbf{x} = A\mathbf{x} - 4I_2\mathbf{x} = (A - 4I)\mathbf{x} = \mathbf{0}$.

$$(A - 4I)\mathbf{x} = \left(\begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \mathbf{x} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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We have $\begin{bmatrix} -4 & -2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which gives the solution set $\mathbf{x} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$.

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Since the equation $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution, $\lambda = 4$ is an eigenvalue.

We see that $\mathbf{v} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector. In fact, any nonzero scalar multiple of $\mathbf{v}$, such as $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, is also an eigenvector corresponding to $\lambda = 4$.

Setting Up A Matrix Equation

A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$.

In general, we can rewrite $A\mathbf{x} = \lambda\mathbf{x}$ as a homogeneous matrix equation:

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

The set of all solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is called the eigenspace of A corresponding to the eigenvalue λ.

The eigenspace of $\begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ corresponding to $\lambda = 4$ is Span $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

Finding a Basis for the Eigenspace

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 2$.

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$$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

The augmented matrix for the equation $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have the solution set

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix}.$$

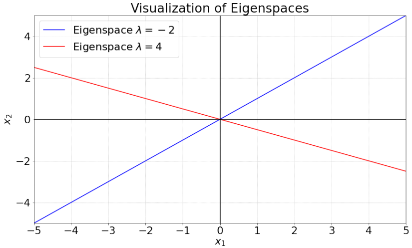
A basis for the eigenspace corresponding to $\lambda = 2$ is

$$\mathcal{B}_{\lambda=2} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Geometric Picture

Let $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$. Find bases for the eigenspaces corresponding to the eigenvalues $\lambda = 4$ and $\lambda = -2$.

$$\mathcal{B}_{\lambda=4} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B}_{\lambda=-2} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$



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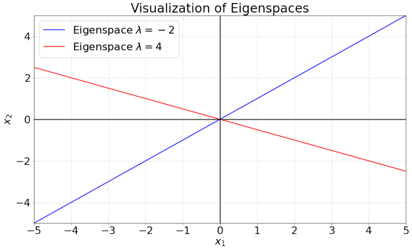

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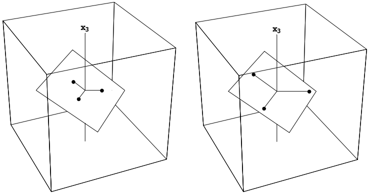

Finding Eigenvalues of A^2, A^3, A^k

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.

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   We see λ^2 is an eigenvalue of the matrix $A^2 = AA$.

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In general, if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k, where k is a positive integer.

Example 2

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$$(A - 5I)\mathbf{x} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

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Since the equation $A\mathbf{x} = 5\mathbf{x}$ has only the trivial solution, $\lambda = 5$ is NOT an eigenvalue of A.

Eigenvalues of a Triangular Matrix

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Proof. Let's consider the 3×3 case. Let A be a 3×3 upper triangular matrix. Then

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

The scalar λ is an eigenvalue of A if and only if there is a nontrivial solution to the homogeneous equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. We have a nontrivial solution when $(A - \lambda I)$ has at least one free variable.

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- If we set $\lambda = a_{11}$, then x_1 is free since there is no pivot in column 1.
- If we set $\lambda = a_{22}$, then x_2 is free since there is no pivot in column 2 (assuming that $\lambda \neq a_{11}$).
- If we set $\lambda = a_{33}$, then x_3 is free since there is no pivot in column 3 (assuming that $\lambda \neq a_{11}, a_{22}$).

Thus the eigenvalues of A are $\lambda = a_{11}$, a_{22}, or a_{33}. The argument follows similarly for larger matrices and for lower triangular matrices as well. $\square$

Linear Independence of Eigenvectors

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If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

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Proof. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. Thus there are scalars c_i, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}. \tag{1}$$

Among all such c_is that are not all zero, choose the scalars such that the fewest c_is are nonzero. Since the v_is are eigenvectors and not the zero vector, at least two of the c_is must be nonzero. Assume that c_j and c_k are nonzero.

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Multiplying both sides of equation (1) by A gives

$$A(c_1\mathbf{v}_1 + \ldots + c_j\mathbf{v}_j + \ldots + c_p\mathbf{v}_p) = c_1\lambda_1\mathbf{v}_1 + \ldots + c_j\lambda_j\mathbf{v}_j + \ldots + c_p\lambda_p\mathbf{v}_p = \mathbf{0}. \tag{2}$$

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Multiplying both sides of equation (1) by λ_j and subtracting from equation (2) gives

$$c_1(\lambda_1 - \lambda_j)\mathbf{v}_1 + c_2(\lambda_2 - \lambda_j)\mathbf{v}_2 + \ldots + 0\mathbf{v}_j + \ldots + c_p(\lambda_p - \lambda_j)\mathbf{v}_p = \mathbf{0}. \tag{3}$$

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$$c_1(\lambda_1 - \lambda_j)\mathbf{v}_1 + c_2(\lambda_2 - \lambda_j)\mathbf{v}_2 + \ldots + 0\mathbf{v}_j + \ldots + c_p(\lambda_p - \lambda_j)\mathbf{v}_p = \mathbf{0}. \tag{3}$$

Note that $c_k(\lambda_k - \lambda_j) \neq 0$ since $\lambda_k \neq \lambda_j$. Thus, we have expressed $\mathbf{0}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with not all scalars 0 that has fewer zero scalars than what we assumed was the fewest. Contradiction! $\square$