HW Set 3 - Solutions

[1.] When you solve a DE your answer/ solution is a function or family of functions.

[2.] (a) $\frac{dy}{dt} = t^4 y$ (separable)

$\int \frac{1}{y}\, dy = \int t^4\, dt$

$\ln|y| = \frac{1}{5}t^5 + c$

$|y| = e^{\frac{1}{5}t^5 + c} = \underbrace{e^c}_{(+)\text{ constant}} e^{\frac{1}{5}t^5} = \underbrace{c}_{\text{"new" } c} e^{\frac{1}{5}t^5}$ where $c \geq 0$ ($c=0$ ok b/c $y=0$ is a soln)

$y = \pm c e^{\frac{1}{5}t^5}$ or $\boxed{y = ce^{\frac{1}{5}t^5}}$ where c = any constant

(b) $\frac{dy}{dt} = 2y+1$ (separable)

$\int \frac{1}{2y+1}\, dy = \int dt$

$\frac{1}{2}\ln|2y+1| = t + c$

$\ln|2y+1| = 2t + c$ (new c)

$|2y+1| = e^{2t+c}$

$|2y+1| = ce^{2t}$, $c \geq 0$ (if $c=0$, $y=-\frac{1}{2}$, which is a soln)

$2y+1 = ce^{2t}$, c = any constant

$\boxed{y = \frac{1}{2}(ce^{2t} - 1)}$

(c) $\frac{dy}{dt} = ty^{1/3}$ (separable)

$\int y^{-1/3}\, dy = \int t\, dt$

$\frac{3}{2}y^{2/3} = \frac{1}{2}t^2 + c$

$y^{2/3} = 3t^2 + c$

$\boxed{y = (3t^2 + c)^{3/2}}$, c = any constant

(d)

$\frac{dy}{dt} = \frac{t}{y+1}$

$\int (y+1)\, dy = \int t\, dt$

$\frac{1}{2}y^2 + y = \frac{1}{2}t^2 + c$ (c=any)

* $y^2 + 2y = t^2 + c$ (new c)

$(y+1)^2 - 1 = t^2 + c$

$(y+1)^2 = t^2 + 1 + c$ (or make new $\tilde{c}$, $\tilde{c} = 1 + c$)

$y + 1 = \pm(t^2 + 1 + c)^{1/2}$

complete the square on LHS:

* $(y^2 + 2y + 1) - 1 = (y+1)^2 - 1$

$\boxed{y = -1 \pm (t^2 + 1 + c)^{1/2}}$

for grading purposes, it's ok if they left y implicitly defined in Step *

(e) $2\frac{dy}{dx} = xy(x+1)$

$2\frac{dy}{dx} = y(x^2+x)$

$\int \frac{2}{y}\,dy = \int (x^2+x)\,dx$

$2\ln|y| = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$

$\ln|y| = \frac{1}{6}x^3 + \frac{1}{4}x^2 + C$ (new c)

$|y| = Ce^{\frac{1}{6}x^3 + \frac{1}{4}x^2}$, $C = \geq 0$

$\boxed{y = Ce^{\frac{1}{6}x^3+\frac{1}{4}x^2}}$, C = any constant

(f) $\frac{dP}{dt} = 4(1-3P)$

$\int \frac{1}{1-3P}\,dP = \int 4\,dt$

$-\frac{1}{3}\ln|1-3P| = 4t + C$

$\ln|1-3P| = -12t + C$

$|1-3P| = e^{-12t+C}$

$1-3P = Ce^{-12t}$

$\boxed{P = \frac{1}{3}(1-Ce^{-12t})}$, C = any constant

3. Solve the following IVPs

(a) $\frac{dy}{dt} = \frac{-t}{y}$; $y(0)=4$

$\int y\,dy = \int -t\,dt$

$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C$

$y^2 = -t^2 + C$

$y = \pm\sqrt{C-t^2}$

Use initial condition to find C:

$4 = \pm\sqrt{C-0}$, $C = 16$

$\boxed{\therefore\ y(t) = \sqrt{16-t^2}}$

only want $+\sqrt{\ }$, not $-\sqrt{\ }$

(b) $\frac{dy}{dt} = -(y)^{1/3}$; $y(0)=27$

$\int -y^{-1/3}\,dy = \int dt$

$-\frac{3}{2}y^{2/3} = t + C$

$y^{2/3} = -\frac{2}{3}t + C$

$y = \left(-\frac{2}{3}t + C\right)^{3/2}$

$27 = \left(-\frac{2}{3}\cdot 0 + C\right)^{3/2}$

$(\sqrt[3]{27})^2 = C = 9$

$\therefore\ \boxed{y(t) = \left(-\frac{2}{3}t + 9\right)^{3/2}}$

(c) $\frac{dy}{dx} = \frac{x(y-2)}{x^2+4}$; $y(1)=5$

$\int \frac{1}{y-2}\,dy = \int \frac{x}{x^2+4}\,dx$ $\quad u = x^2+4$, $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$

$\ln|y-2| = \int \frac{\frac{1}{2}}{u}\,du$

$\ln|y-2| = \frac{1}{2}\ln|x^2+4| + c$

$|y-2| = e^{\frac{1}{2}\ln|x^2+4|+c}$

$|y-2| = e^{\ln|x^2+4|^{1/2}+c} = e^c\,|x^2+4|^{1/2}$

$|y-2| = (\sqrt{x^2+4})(c)$ (new c, $c>0$)

$y-2 = c\sqrt{x^2+4}$, c = any constant

$y = c\sqrt{x^2+4}+2$

IV: $5 = c\sqrt{1^2+4}+2$

$5 = c\sqrt{5}+2$

$c = \frac{3}{\sqrt{5}}$

$\boxed{y(t) = \frac{3}{\sqrt{5}}\sqrt{x^2+4}+2}$

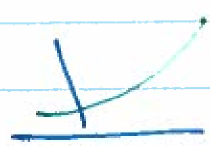
4. Want $\frac{dy}{dt} = \ldots$, for which $y(t)=6$ is a solution but $y(t)=8$ is not a soln.

Let $\frac{dy}{dt} = (y-6)(y+1)$.

$y(t)=6$ is a solution b/c it satisfies the DE $(0=0)$. If, however, $y(t)=8$, then $\frac{dy}{dt}=0$ but $(y-6)(y+1)$ becomes 18 ($\neq 0$!).

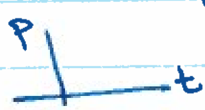
Thus $y(t)=8$ is not a solution.

5. $\frac{dP}{dt} = 0.2P$ $\rightsquigarrow$



This looks like the graph of an exponential function, which also happens to be the form of the gen'l soln to this eqt ($P(t) = ke^{.2t}$)

Thus, we're looking at



, b/c these would be the appropriate axes for displaying solution curves.

$\boxed{6.}$ $\frac{dy}{dt} = y - t$. Want to know if any of these are solns:

(i) $y = t+2$, (ii) $y = e^t - 1$, (iii) $y = e^t + t + 1$, (iv) $y = t$

(a) Read with meaning (answers will vary!): the rate of change of the function y at a given time, t, is equal to the function y itself at that time minus the time. (Probably a student will come up with something more interesting!)

(b) Cornelia could plot slope fields / tangent vector fields for the DE to see if any of the functions (i)-(iv) appear to satisfy the slopes of $\frac{dy}{dt}$. At a glance it should be relatively easy to see if there are any straight line solns (like (i) or (iv)) or if the solns contain exponential behavior (like (ii), (iii)). (Then again, this would require knowing what e^t+t+1 looks like...!)

→ At a glance, $y = t+2$ looks like a contender from slope field...

(c) Test each function...

(i) $y = t+2$

$\frac{dy}{dt}$	$y-t$
1	$t+2-t$
$\neq$	$=2$

Not a soln.

(ii) $y = e^t - 1$

$\frac{dy}{dt}$	$y-t$
e^t	e^t-1-t
$\neq$	

Not a soln

(iii) $y = e^t + t + 1$

$\frac{dy}{dt}$	$y-t$
e^t+1	$e^t+t+1-t$
	$= e^t+1$

Yes! A soln

(iv) $y = t$

$\frac{dy}{dt}$	$y-t$
1	0

Not a soln.

Only $y(t) = e^t + t + 1$ is a soln.