Guess and Test for Nonhomogeneous Cases

So far we have been using patterns recognized in wisely guessing the form of solutions to homogeneous second order differential equations of the form $ay'' + by' + cy = 0$. Can we adjust our guesses to handle nonhomogeneous differential equations as well?

1. Find a solution to the following **nonhomogeneous** differential equation:

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18.$$

forcing function is constant

Guess $x_p = 2$ is a constant

What is your best guess for a function whose second derivative plus 10 times its first derivative plus 9 times the function itself sum to 18? Test out your guess to see if it works. If it doesn't work keep trying.

Test $x = 18$:

$0 + 10(0) + 9(18) = 36 \neq 18$ $x_p = 18$ not a solution

$0 + 0 + 9(2) = 18 = 18$ ✓

Homogeneous: $x'' + 10x' + 9x = 0$

$r^2 + 10r + 9 = 0$

$(r+9)(r+1) = 0$

$x_h = C_1 e^{-9t} + C_2 e^{-t} + 18$

t t

a particular solution

$x'' + 10x' + 9x = 18$

$x_p = 18$ $x_p = 2$

$x_p' = 0$ $x_p' = 0$

$x_p'' = 0$ $x_p'' = 0$

The solution you found in the previous problem is called the **particular solution** to the nonhomogeneous differential equation it is not the general solution. For now we will focus on how we can find the particular solution by wisely guessing their general form based on the nonhomogeneous part of the differential equation. We will soon combine what we know about the homogeneous case and particular solutions to find general solutions.

2. Find a solution to the following nonhomogeneous differential equation:

$$\boxed{\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18t.}$$

Linear

Guess $\boxed{x = At + B}$

What is your best guess? Test out your guess to see if it works. If it doesn't work keep trying.

$\boxed{x = 2t}$ $\quad x' = 2 \quad x'' = 0$

$0 + 10(2) + 9(2t) = 18t$

$18t + \underbrace{20} = 18t$

$x = 2t + B$

$x = 2t - 20$

$\boxed{x = 2t + B}$ $\rightarrow$ Plug and find value of B that works

$x' = 2$

$x'' = 0$

$0 + 10(2) + 9(2t + B) = 18t$

$20 + 18t + 9B = 18t$

$20 + 9B = 0$

$B = -20/9$

$$\boxed{x_p = 2t - 20/9}$$

3. Based on the previous examples, what would be a good guess for the general form of the particular solution to

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18t^3.$$

New

Your guess should have depend on constants whose values you do not need to determine for this example.

$$\boxed{x_p = At^3 + Bt^2 + Ct + D}$$

① Reasonable Guess

② Plug it in

③ Solve for

undetermined coefficients

Method of undetermined coefficients

coefficients

4. Sean and Phil are trying to find the particular solution to $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = \boxed{85\sin(2t)}$. Sean guesses $x(t) = A\sin(2t)$ for the particular solution and Phil guesses $x(t) = B\cos(2t)$.

   (a) Do you think these are reasonable guesses? Explain why or why not.

x_p = polynomial, not reasonable

$x_p = A\sin(2t)$ seems reasonable since taking derivative of $\sin(2t)$ gives $\cos(2t)$ and $\sin(2t)$ back

$x_p = B\cos(2t)$

   (b) For each of their guesses, can you find a value of A or B such that their guess is a solution? If yes, write down the general solution. If no, come up with a different guess for the particular solution and show that your guess is correct.

Try $A\sin(2t) = x_p$ $\rightarrow$ $x_p' = 2A\cos(2t)$

$x_p'' = -4A\sin(2t)$

can't work

$x_p = B\cos(2t)$ doesn't work either

$\boxed{x_p = A\sin(2t) + B\cos(2t)}$

$$x'' + 10x' + 9x = 85\sin(2t)$$

⓪ Find homogeneous solution

① Guess that x_p has same form as forcing function

② Plug it in check.

$$9 \quad x_p = A\sin(2t) + B\cos(2t)$$

$$10 \quad x_p' = 2A\cos(2t) - 2B\sin(2t)$$

$$1 \quad x_p'' = -4A\sin(2t) - 4B\cos(2t)$$

$$\overbrace{\left(-4A\sin(2t) - 4B\cos(2t)\right)}^{x_p''} + \overbrace{\left(20A\cos(2t) - 20B\sin(2t)\right)}^{10x_p'}$$

$$+ \overbrace{\left(9A\sin(2t) + 9B\cos(2t)\right)}^{9x_p} = 85\sin(2t) + 0\cos(2t)$$

③ Solve for undetermined coefficients

$$\left(\boxed{-4A - 20B + 9A}\right)\sin(2t) = 85\sin(2t)$$

$$\left(-4B + 20A + 9B\right)\cos(2t) = 0\cos(2t)$$

$$5A - 20B = 85 \qquad A = 1$$

$$20A + 5B = 0 \qquad B = -4$$

$$\boxed{x_p = \sin(2t) - 4\cos(2t)}$$

is one solution

General solution: $x = \boxed{x_{homogeneous}} + x_p$

$f(t)$	Guess
Polynomial degree n	$x_p = A_n t^n + \cdots + A_1 t + A_0$
$\sin(\beta t)$ or $\cos(\beta t)$	$x_p = A\sin(\beta t) + B\cos(\beta t)$ *
e^{kt}	$x_p = Ae^{kt}$

$$ax'' + bx' + cx = e^{kt}$$

* If forcing function's initial guess is a homogeneous solution there is resonance, so we need to multiply our first guess by t.

5. Consider the nonhomogeneous differential equation

$\boxed{x'' + 25x = 0}$

$$\frac{d^2x}{dt^2} + 25x = 10\cos(5t)$$

$0 = 10\cos(5t)$

(a) Suppose you wish to find the particular solution to this differential equation. Explain why a guess of the form $x(t) = A\cos(5t) + B\sin(5t)$ is doomed to fail.

$r^2 + 25 = 0$

$r = \pm 5i$

$x_h = C_1\sin(5t) + C_2\cos(5t)$

solution to homogeneous

(b) Nevertheless, explain why your particular solution must have terms that *look like* $\cos(5t)$ and $\sin(5t)$.

when our initial guess is a homogeneous solution, then we have $\boxed{\text{resonance}}$

not worked on in class

(c) For an unknown differentiable function $f(t)$, write down the first and second derivatives of $tf(t)$, what do you notice?

$(tf(t))' = f(t) + tf'(t)$

$(tf(t))'' = f'(t) + f'(t) + tf''(t)$

Derivative of $tf(t)$ contains terms with $f(t)$ and $f'(t)$

(d) Explain why a guess of $At\cos(5t)$ is insufficient to find the particular solution.

The first and second derivatives of $At\cos(5t)$ will also have $\sin(5t)$ terms that need to match right-side

(e) Use the guess $x(t) = t(A\cos(5t) + B\sin(5t))$ to find a particular solution to the above equation.

$x_p = At\cos(5t) + Bt\sin(5t)$

$x_p' =$ Product & Chain Rules

Exercise: Solve for x_p.

solution next page

Solution part e :

$$x_p' = A\cos(5t) - 5At\sin(5t) + B\sin(5t) + 5Bt\cos(5t)$$

$$x_p'' = -5A\sin(5t) - 5A\sin(5t) - 25At\cos(5t) + 5B\cos(5t) + 5B\cos(5t) - 25Bt\sin(5t)$$

Substituting into $x'' + 25x = 10\cos(5t)$ gives

$$(-5A\sin(5t) - 5A\sin(5t) - 25At\cos(5t) + 5B\cos(5t) + 5B\cos(5t) - 25Bt\sin(5t)) + 25(At\cos(5t) + Bt\sin(5t)) = 10\cos(5t)$$

Canceling terms and grouping remaining terms gives

$$-10A\sin(5t) + 10B\cos(5t) = 10\cos(5t)$$

which gives $B = 1$ and $A = 0$.

$$\boxed{x_p = t\sin(5t)}$$

The previous exercise is an example of **resonance**, which occurs when an external force has the same properties (such as frequency) as the general homogeneous solution. Practically speaking, when the homogeneous and nonhomogeneous parts of the differential equation have resonance this creates a huge increase in energy (that may even cause a bridge to collapse).

6. For each of the examples circle the word(s) that correctly describe whether the nonhomogeneous differential equation has resonance or not.

   $x_p = A\cos(5t) + B\sin(5t)$

   (a) $x'' + 25x = \boxed{10\cos(5t)}$ (does or does not) have resonance since $x_H = C_1\cos(5t) + C_2\sin(5t)$.

   $x_p = Ae^{5t}$

   (b) $x'' + 25x = \boxed{10e^{5t}}$ (does or does not) resonance since $x_H = C_1\cos(5t) + C_2\sin(5t)$.

   (c) $x'' - 3x' - 10x = 10\cos(5t)$ (does or does not) resonance since $x_H = C_1e^{5t} + C_2e^{-2t}$.

   $x_p = A\cos(5t) + B\sin(5t)$

   (d) $x'' - 3x' - 10x = 10e^{5t}$ (does or does not) resonance since $x_H = C_1e^{5t} + C_2e^{-2t}$.

   (e) $x'' + 2x' + 17x = 6e^{-t}\sin(4t)$ (does or does not) resonance since $x_H = C_1e^{-t}\cos(4t) + C_2e^{-t}\sin(4t)$.

   (f) $x'' + 2x' + 17x = 6\sin(4t)$ (does or does not) resonance since $x_H = C_1e^{-t}\cos(4t) + C_2e^{-t}\sin(4t)$.

   $x_p = A\sin(4t) + B\cos(4t)$

   Whenever resonance is present between the homogeneous solution and nonhomogeneous forcing function, we can adjust our initial guess by multiplying by a factor of t. For example, since $x'' + 25x = 10\cos(5t)$ has resonance our guess for the particular solution is

$$x_p = t\big(A\cos(5t) + B\sin(5t)\big).$$

7. For each of the examples in the previous question where there was resonance, give the initial guess for the particular solution. Do not solve for the values of the undetermined coefficients.

Resonance

a) $x_p = t\big(A\cos(5t) + B\sin(5t)\big)$

b) $x_p = Ae^{5t}$ → no resonance

c) $x_p = A\cos(5t) + B\sin(5t)$

e) $x_p = t\big(Ae^{-t}\cos(4t) + Be^{-t}\sin(4t)\big)$

Welcome! Today we will finish worksheet 11 and work on worksheet 12

Last time: Solve $ay'' + by' + cy = f(t)$

$f(t)$ ↑ forcing function.

① Find homogeneous solution y_H to $ay'' + by' + cy = 0$ to make sure there is no resonance with $f(t)$

② Find a good guess and adjust if y_p is a homogeneous solution and there is resonance

$f(t)$	y_p
Ct^n	$A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$
$C\cos(\beta t)$ or $\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
Ce^{rt}	Ae^{rt}

③ Plug in guess for y_p and solve for all undetermined coefficients.

8. What would be a good guess for the general form of the particular solution to

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 5e^{3t}?$$

If $r=3$ is a root resonance

Do not find the values of the undetermined coefficients.

① Find homogeneous solution: $y'' - 6y' + 9y = 0 \quad (r-3)^2 = 0$

$$\boxed{y_h = c_1 e^{3t} + c_2 t e^{3t}}$$

② $f(t) = 5e^{3t} \to$ Guess $y_p = Ae^{3t}$ $\quad y_p = Ate^{3t}$

$$\boxed{y_p = At^2 e^{3t}}$$

9. What would be a good guess for the general form of the particular solution to

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 5e^{3t}\cos(2t)?$$

$y_h = c_1 e^{3t} + c_2 t e^{3t}$ same $\quad r = 3 \pm 2i$

Do not find the values of the undetermined coefficients.

$$\boxed{y_p = Ae^{3t}\cos(2t) + Be^{3t}\sin(2t)}$$

no resonance

10. What would be a good guess for the general form of the particular solution to

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 5t^2e^{3t}\cos(2t)?$$

Do not find the values of the undetermined coefficients.

?

See next worksheet 12 where this is explained.