Homework Set 7 Solutions

1. For this problem, use the coffee cooling rate of change equation

$$\frac{dC}{dt} = -0.4C + 28.$$

(a) Is there ever a time when two cups of coffee, one at initially 160°F and one at 180°F, are the exact same temperature? Answer this question according to the uniqueness theorem. Comment on whether your answer matches what you expect to happen in real life?

(b) How long will it take a cup of hot coffee that is initially 180°F to cool down to 100°F? Use the reverse product rule to figure this out and then check the reasonableness of your answer with Euler's method.

(a) No, the the two cups of coffee will never be exactly the same temperature. In order for the cups to have the same temperature, the solution functions corresponding to the cups must intersect at some point. Let $f(t, C) = -0.4C + 28$. Then

$$\frac{\partial f}{\partial C} = -0.4$$

Observe that f and $\frac{\partial f}{\partial C}$ are continuous for all points. By the Uniqueness Theorem, we can conclude that there is a unique solution passing through any point. Therefore, the two cups will never be the same temperature. In real life, we will expect both cups in the long term to get essentially to room temperature (within the margin of error of any thermometer). Thus, while theoretically the cups will never be the same temperature, we might eventually measure them to be the same.

(b) First, we solve this initial value problem using the reverse-product rule technique.

$$\begin{aligned} C' + 0.4C &= 28 \\ C'u + 0.4uC &= 28u \end{aligned}$$

Let $u' = 0.4u$. Then $u = e^{0.4t}$.

$$\begin{aligned} C'u + u'C &= 28u \\ (Cu)' &= 28u \\ \int (Ce^{0.4t})'\,dt &= \int 28e^{0.4t}\,dt \\ Ce^{0.4t} &= \frac{28}{0.4}e^{0.4t} + K \\ C &= 70 + Ke^{-0.4t} \end{aligned}$$

Using the initial condition $C(0) = 180$, we obtain

$$\begin{aligned} 180 &= 70 + Ke^0 \\ K &= 110 \end{aligned}$$

Thus $C(t) = 70 + 110e^{-0.4t}$. The cup has cooled down to 100 degrees when $C(t) = 100$, so when

$$\begin{aligned} 70 + 110e^{-0.4t} &= 100 \\ 110e^{0.4t} &= 30 \\ e^{0.4t} &= \frac{3}{11} \\ 0.4t &= \ln\left(\frac{3}{11}\right) \\ t &= \frac{1}{0.4}\left(\frac{3}{11}\right) \approx 3.25 \end{aligned}$$

Thus, it will take approximately 3.25 minutes to cool down to 100 degrees. To check to see whether this is reasonable, we can use Euler's method to approximate $C(3)$ using a step size of 1:

t	C	$\frac{dC}{dt}$
0	180	-44
1	136	-26.4
2	109.6	-15.84
3	93.76	

Thus, the Euler approximation shows the cup reaching 100 degrees at around 2.5 minutes, which is faster than our exact solution, but it's in the same ballpark.

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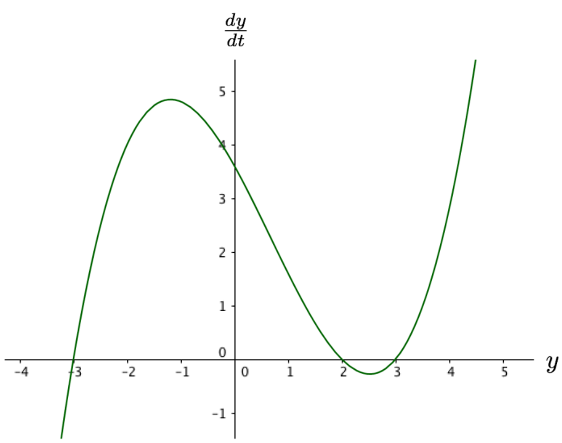
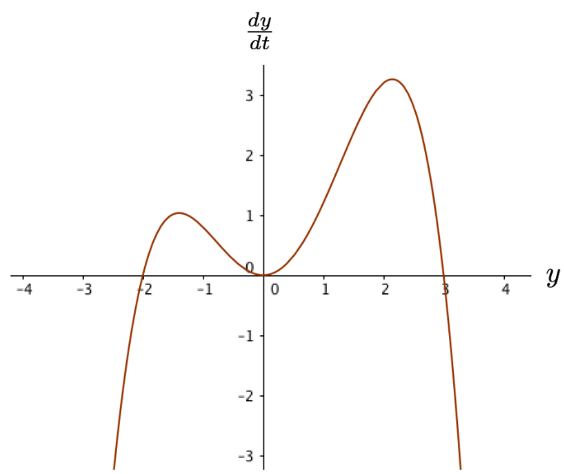
2. For each part below you are provided with an autonomous derivative graph. Figure out the long-term behavior of every possible solution function. Illustrate your conclusions with representative $y(t)$ solution graphs and summarize your findings about the long-term behavior of different solutions in paragraph form.

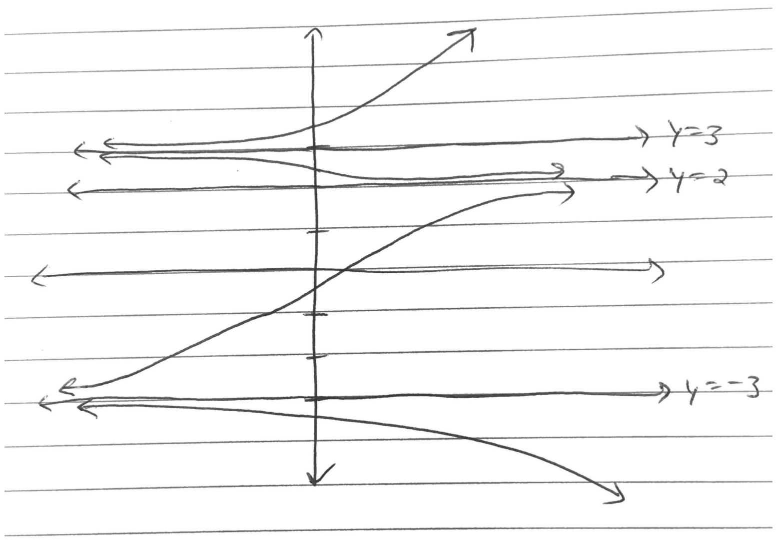


(a)

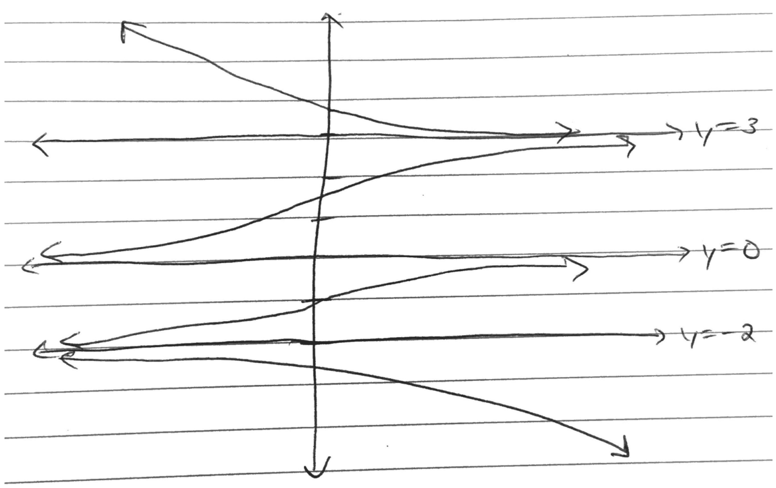


(b)

(a) Based on the dy/dt vs y graph, we have three equilibrium solutions: $y = -3, y = 2, y = 3$. If $y < -3$, it decreases toward infinity. If $-3 < y < 2$, it increases toward 2. If $2 < y < 3$, it decreases toward 2. If $3 < y$, it increases toward infinity.



(b) Based on the dy/dt vs y graph, we have three equilibrium solutions: $y = -2, y = 0, y = 3$. If $y < -2$, it decreases toward infinity. If $-2 < y < 0$, it increases toward 0. If $0 < y < 3$, it increases toward 3. If $3 < y$, it decreases toward 3.



3. For each part in problem 2, create a phase line and classify each equilibrium solution as either an attractor, repeller, or node.

(a) The solutions $y = -3$ and $y = 3$ are unstable and the solution $y = 2$ is stable.

(b) The solution $y = -2$ is unstable, the solution $y = 3$ is stable, and the solution $y = 0$ is neither (converges below, diverges above).

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4. For problem 2b, use the uniqueness theorem to determine if any of the non-constant solution functions ever reach the equilibrium solution of $y(t) = 0$ in a finite amount of time.

We can see that the partial derivative of this differential equation with respect to y is continuous as this function is differentiable for all of y. Thus, by the uniqueness theorem, no non-constant solution functions will reach the equilibrium solution of $y = 0$ in a finite amount of time.

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5. Given an autonomous differential equation $\frac{dy}{dt} = f(y)$, give a general strategy for how to use an autonomous derivative graph to determine the long term behavior of solution functions.

Answers will vary.

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6. Suppose you wish to predict future values of some quantity, y, using an autonomous differential equation (that is, dy/dt depends explicitly only on y). Experiments have been performed that give the following information:

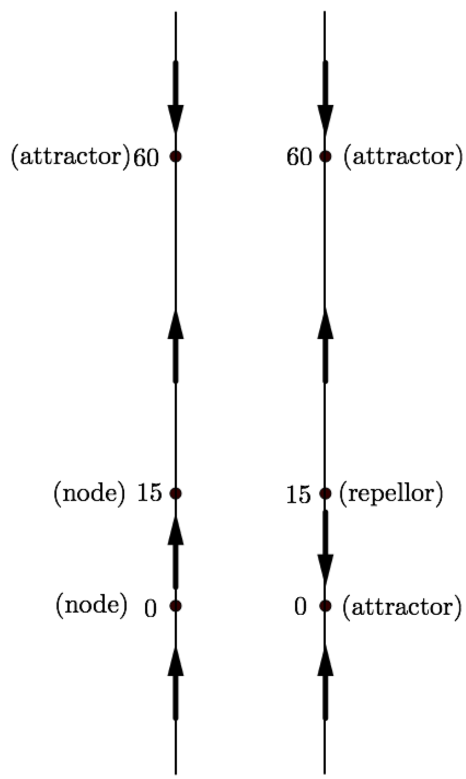
   - The only equilibrium solutions are $y(t) = 0$, $y(t) = 15$, and $y(t) = 60$
   - If the value of y is 100, the quantity decreases
   - If the value of y is 30, the quantity increases
   - If the value of y is negative, the quantity increases

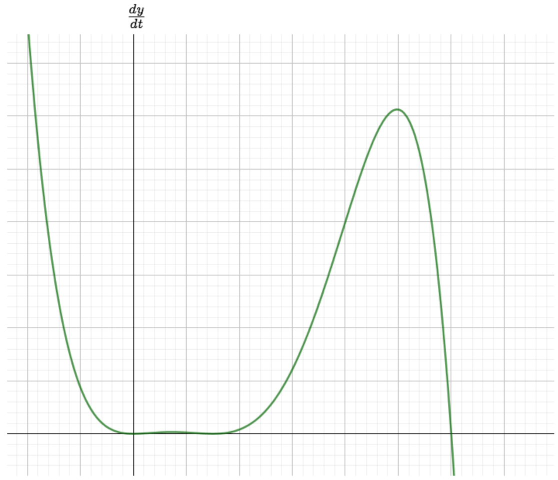
   (a) How many different phase lines match the above? Sketch all possible phase lines.

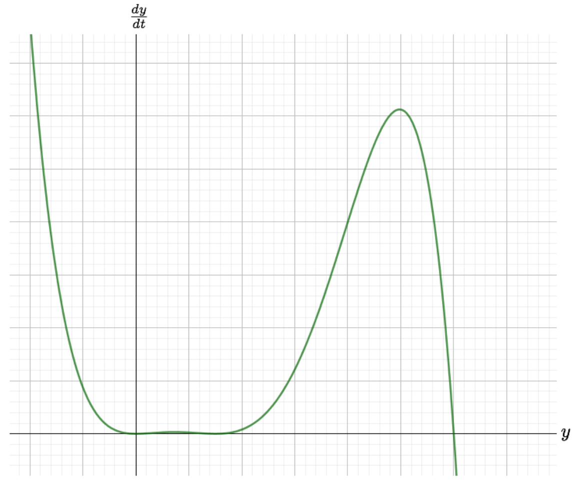
   (b) Provide a rough sketch of an autonomous derivative graph for each of your phase lines in part 6a.

   (c) For each of your different sketches in part 6b, develop a differential equation that fits the basic features.

(a) There are two different phase lines.



(b) 



(c) Left phase line / graph: $\frac{dy}{dt} = -y^2(y-15)^2(y-60)$
Right phase line / graph: $\frac{dy}{dt} = -y(y-15)(y-60)$

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7. In what ways is the letter y in the differential equation $\frac{dy}{dt} = .3y$ both a variable and a function? In what ways is $\frac{dy}{dt}$ a function?

There are numerous explanations here. For example, one is: The letter y is implied to be a function of t, but $\frac{dy}{dt}$ depends explicitly on y as a variable.

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8. Newton's law of cooling is an empirical law that states that an object immersed in a constant, ambient temperature will have its temperature change at a rate proportional to the difference between the its temperature and the ambient temperature. Explain how the cooling coffee problem reflects Newton's law of cooling.

The differential equation we discovered that modeled the situation with the coffee cup was a proportion of the difference between the temperature of the coffee and room temperature. This lets us know that the rate of change of the temperature of the coffee cup with respect to time is proportional to the difference between the temperature of the coffee and the room temperature. This is Newton's Law of Cooling.

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9. A body was found in a temperature controlled environment (i.e., you know the room temperature) and is subject to Newton's law of cooling. Explain why you only need the room temperature and the measurement of the body's temperature at two different times to give an estimate of the time of death.

Because of Newton's law of cooling, the body cools at a rate proportional to the difference from the body and room temperature, and so the room temperature gives the equilibrium temperature for the body. Thus, we might write, using B for body and R for room temperature, $\frac{dB}{dt} = k(B - R)$, where k is a constant to be determined. Separating the variables yields $\frac{dB}{B-R} = \frac{k}{dt}$, and integration gives $\ln|B - R| = kt + C$. Assuming the body initially exceeds room temperature we can write $B - R = e^{kt+C} = Ce^{kt}$. Knowing the body temperature at two different times means that we know that $B_1 - R = Ce^{kt_1}$ and $B_2 - R = Ce^{kt_2}$. Taking the ratio, we see that the C's cancel, and the only variable is k, which is readily solved for. With k in hand, we can solve for C using $B_1 - R = Ce^{kt_1}$. Finally, now that we know R, C, k, we can solve for the time of death by finding the time such that $B(t) = 98.6$.