A Rate of Change Equation for Limited Resources

In a previous problem we saw that the rate of change equation $\frac{dP}{dt} = 0.3P$ can be used to model a situation where there is one species, continuous reproduction, and unlimited resources. In most situations, however, the resources are not unlimited, so to improve the model one has to modify the rate of change equation $\frac{dP}{dt} = 0.3P$ to account for the fact that resources are limited.

1. (a) In what ways does the modified rate of change equation

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{10}\right)$$

account for limited resources? (Think of 10 as scaled to mean 10,000 or 100,000)

If P is large, say $P_0 = 20$, $\frac{dP}{dt} < 0$, so P is decreasing as Expected if P large then have competition for limited resources.

(b) How do you interpret the solution with initial condition $P(0) = 10$?

If $P(0) = 10$, $\frac{dP}{dt} = 0$, so P remains constant. Thus if start with 10,000 fish when $t=0$, Population will hold constant for all t

(c) Open the Slope Field Viewer, https://ggbm.at/ZGeeGQbp, and plot the slope field for

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{10}\right).$$

(Note: In the Slope Field Viewer you will need to use the variable y instead of P, and you may want to change the viewing window using the button on the right of the applet.) In what ways are your responses to parts 1a and 1b visible in the slope field?

(d) In this problem, negative P values do not make sense, but we can still mathematically make sense of the slope field for negative P values. Explain why the slope field looks the way it does below the t-axis.

If $P(0) < 0$, then $\frac{dP}{dt}$ will be negative. All such solutions will be decreasing

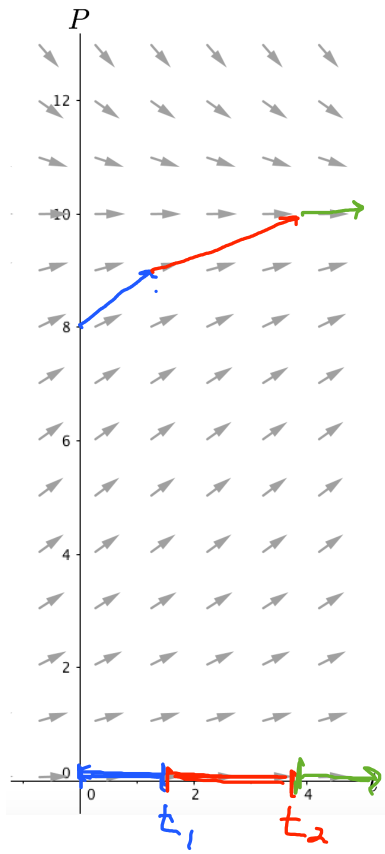
2. If there are initially $P(0) = 2$ fish in the lake, approximately how many fish are in the lake at time $t = 2$? How did you arrive at your approximation? (Hint: Initially $\frac{dP}{dt} = 0.48$, but what meaning does 0.48 have?)

At $(0,2)$, $\frac{dP}{dt} = 0.3(2)(1 - \frac{2}{10}) = 0.48$. Each year P increases by about 0.48. So when $t=2$,

$P(2) \approx 2 + 2(0.48) = 2.96$ thousand Fish

Using a Slope Field to Predict Future Fish Populations

Below is a slope field for the rate of change equation $\dfrac{dP}{dt} = 0.3P\left(1 - \dfrac{P}{10}\right)$.



3. (a) On the slope field above, stitch together in a tip to tail manner several tangent vectors to produce a graph of the population versus time if at time $t = 0$ we know there are 8 fish in the lake (again, think of 8 as scaled for say, 8000 or 80,000).

   (b) Reproduce your technique as much as possible using the Slope Field Stitcher applet, https://ggbm.at/FZn4WHeU. You can use the arrow buttons to move the initial vector around, and then create subsequent vectors to stitch on using the appropriate button.

4. Explain how you are thinking about rate of change **in your method**. For example, is the rate of change constant over some increment? If yes, over what increment? If no, is the rate of change always changing?

In the picture above, we assume the rate of change is constant over time $0 \le t < t_1$ (blue line) then gets flatten and is constant over $t_1 \le t < t_2$ (red line) Finally rate of change is 0 for all $t \ge t_2$ (green line)
Rate of change is not always changing, nor is it always the same.

5. Using the differential equation $\frac{dP}{dt} = P\left(1 - \frac{P}{20}\right)$ and initial condition $P(0) = 10$, José and Julie started the following table to numerically keep track of their tip-to-tail method for connecting tangent vectors. Explain José's and Julie's approach and complete their table. **Round to two decimal places.**

t	P	$\frac{dP}{dt}$
0	10	5
0.5	12.5	4.688
1.0	14.844	3.827
1.5	16.757	

First they found the slope at (0, 10)

$\frac{dP}{dt} = 10\left(1 - \frac{10}{20}\right) = 5$. If we assume $P' = 5$ over $0 \le t < 0.5$, $\Delta P \simeq 5(0.5) = 2.5$ ($0.5 = \Delta t$)

then $P(0.5) \simeq P(0) + \Delta P$

$\simeq 10 + 2.5 = 12.5$

Now we recalibrate slope at (0.5, 12.5) and continue.

6. Using the same differential equation and initial condition as José and Julie, Derrick and Delores started their table as shown below. Explain how Derrick and Delores' approach is different from José and Julie's and then complete their table. **Round to two decimal places.**

$\frac{dP}{dt} = P\left(1 - \frac{P}{20}\right)$

t	P	$\frac{dP}{dt}$
0	10	5
.25	11.25	4.922
.5	12.48	4.692
.75	13.654	

Approach is the same just they update slope more frequently ($\Delta t = 0.25$ instead of 0.5)

7. Which approach do you think is more accurate and why?

The approach in #6 since they use smaller Δt and update slope more frequently.

8. (a) Consider the differential equation $\frac{dy}{dt} = y + t$ and initial condition $y(0) = 4$. Use José and Julie's approach to find $y(1.5)$. Show your work graphically and in a table of values.

t	y	dy/dt	Δy
0	4	4	2
0.5	6	6.5	3.25
1	9.25	10.25	5.125
1.5	14.375		

$\boxed{y(1.5) \approx 14.375}$

(b) Is your value for $y(1.5)$ the exact value or an approximate value? Explain.

Approx since in actual solution $\frac{dp}{dt}$ should change continuously.

9. **Generalizing your tip-to-tail approach**. Create an equation-based procedure/algorithm that would allow you to predict future y-values for any differential equation $\frac{dy}{dt}$, any given initial condition, and any time increment.

$$y_{next} = y_{now} + \left(\frac{dy}{dt}\right)_{now} \cdot \Delta t$$

OR

$$\boxed{y_{n+1} = y_n + y_n' \cdot \Delta t}$$

Euler's Method