Homogeneous Second Order Linear Differential Equations

A second order linear differential equation with constant coefficients has the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

where a, b, and c are constants and f is a continuous function of t.

- If $f(t) = 0$, then the equation is called **homogeneous**.
- If $f(t) \neq 0$, then the equation is called **nonhomogeneous**.

We have shown that to find solutions to the homogeneous case $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$, we can:

1. Set up the corresponding characteristic polynomial, $ar^2 + br + c = 0$.
2. Find solutions $r = r_1$ and $r = r_2$ to the characteristic equation.
3. Quadratic equations may have real or complex solutions:
   - If r_1 and r_2 are distinct real numbers, then the general solution is
     $$x(t) = C_1e^{r_1t} + C_2e^{r_2t}.$$
   - If there is one repeated root, r_1, then the general solution is
     $$x(t) = C_1e^{r_1t} + C_2te^{r_1t}.$$
   - **If the solutions are of the form $\mathbf{r} = \alpha \pm \mathbf{i}\beta$, then what?**

Complex Solutions

1. Let $f(t) = e^{i\beta t}$ and answer the questions below.

   (a) Find a formula for f', f'', f''', f^{iv}, and f^{v}.

   (b) Express $f(t) = e^{i\beta t}$ using as a Taylor series at $t = 0$:

   $$f(t) = f(0) + \frac{f'(0)}{1!}t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{iv}(0)}{4!}t^4 + \frac{f^{v}(0)}{5!}t^5 + \ldots$$

   (c) Group the real and imaginary parts of the first several terms in the Taylor series together.

   (d) Do you recognize these are Taylor series of common functions?

Euler's Formula

The previous question is a proof of **Euler's formula** which allows us to write exponentials in **polar form**,

$$e^{(\alpha+i\beta)t} = e^{\alpha t}\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right).$$

2. If $z(t) = P(t) + iQ(t)$ is complex solution to a differential equation of the form $az'' + bz' + cz = 0$, prove that the real part $P(t)$ is a solution itself and the imaginary part $Q(t)$ (not including the i) is also a solution itself. Note the derivative of a complex function is the sum of the derivatives of the real and imaginary parts of the complex function:

$$z'(t) = P'(t) + iQ'(t).$$

3. Find the general solution to the homogeneous differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 17x = 0$$

Use the above results on exponentiation of complex numbers to find the general solution to the differential equation.

To summarize our results, when solving a homogeneous second order differential with constant coefficients, we can find the zeros of the corresponding characteristic equation. Then

- If r_1 and r_2 are distinct real numbers, then the general solution is

- If there is one repeated root, r_1, then the general solution is

- If the solutions are of the form $\mathbf{r} = \alpha \pm \mathbf{i}\beta$, then the general solution is

Section 2.4: Mass-Spring Oscillator

4. Consider the mass-spring oscillator that has mass $m = 1$ kg, stiffness $k = 4$ kg/sec^2, and damping b kg/sec. The displacement y from equilibrium position at time t seconds satisfies the initial value problem

$$y'' + by' + 4y = 0; \quad y(0) = 1 \;\; y'(0) = 0.$$

   (a) Interpret the practical meaning of the initial conditions.

   (b) Find the solution if the damping coefficient is $b = 0$ and describe what happens to the mass as $t \to \infty$.

   (c) Find the solution if the damping coefficient is $b = 5$ and describe what happens to the mass as $t \to \infty$.

(d) Find the solution if the damping coefficient is $b = 4$ and describe what happens to the mass as $t \to \infty$.

(e) Find the solution if the damping coefficient is $b = 2$ and describe what happens to the mass as $t \to \infty$.