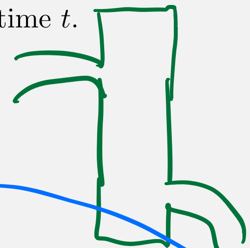
Mixture Problems (Application 4.2)

A one-compartment system consists of

- $x(t)$ that represents the amount of a substance (such as salt) at time t.
- an input rate of x.
- an output rate of x.



$$\frac{dx}{dt} = \text{input rate} - \text{output rate}$$

1. A brine solution of salt water that has concentration 0.05 kg per L flows at a constant rate of 6 L per minute into a tank which is initially contains 50 L of a 1% salt solution. The brine solution flows out of the tank at a rate of 4 L per minute. Let $x(t)$ denote the mass of the salt in the tank at time t (in minutes). *Note that 1% salt solution means 1 kg of salt per 100 L of solution.*

initial condition

(a) What is the input rate of x?

$$(6\ L/min)(0.05\ kg/L) = 0.3\ kg/min$$

(b) What is the output rate of x?

$$(4\ L/min)\left(\frac{\text{Amt of Salt in tank}}{\text{volume of solution in tank}}\right) = (4\ L/min)\left(\frac{x}{50+2t}\right)$$

(c) What is the initial mass of the salt in the tank?

$$x(0) = \text{Amt Salt at } t=0 = (50L)(1\%) = (50)(0.01\ kg/L) = 0.5\ kg$$

(d) Construct an model for this initial value problem (but do not solve it).

$$\boxed{\frac{dx}{dt}} = 0.3 - \frac{\boxed{4}x}{\boxed{50+2t}}, \quad x(0) = 0.5\ kg$$

$\frac{x}{\text{Volume}}$

(e) What method(s) can we apply to solve the equation in (4) (but don't solve it)?

$\boxed{\text{Not Separable}}$ → Linear ✓ Integrating Factor

Input Rate: (Rate solution flows in)(concentration solution in) = Rate Salt goes in — depends on t

Output Rate: (Rate solution flows out)(concentration solution out) = units of salt per unit of time

??

$$\frac{dP}{dt} = -A \cdot P \cdot (P-L) \qquad \frac{1}{P}\frac{1}{P-L}\,dP = -A\,dt$$

Population Models (Section 4.1)

The Malthusian law of population growth says the rate of change of the population, $\frac{dP}{dt}$, is directly proportional to the population present, P, at time t:

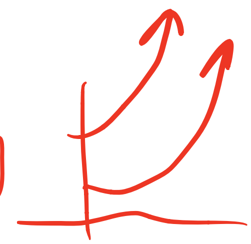
$$\frac{dP}{dt} = kP, \quad P(0) = P_0.$$

$$\boxed{dP/dt = -AP(P-L)}$$

Example 2. Let P denote the population of the world (in billions) t years since 1960. In 1960 the world's population was approximately 3 billion, and the population growth is model by

$$\frac{dP}{dt} = 0.2P \quad , \quad P(0) = 3.$$

7.5 bil



Solving this model gives $P(t) = 3e^{0.02t}$ and predicts the population in 2019 is 9.76 billion.

Why do you think predicted value is different from the actual value?

Not taken into account any death rate.

$$\int \frac{1}{P}\cdot\frac{1}{P-L}\,dP = \int -A\,dt = -At + C$$

$$\frac{A}{P} + \frac{B}{P-L} = \frac{1}{P(P-L)}$$

$$A = -1/L \quad B = 1/L$$

$$\int -\frac{1}{L}\frac{1}{P}dP + \int \frac{1}{L}\left(\frac{1}{P-L}\right)dP$$

$$= -1/L \ln|P| + 1/L \ln|P-L|$$

$$= \frac{1}{L}\ln\left|\frac{P-L}{P}\right| = -At + C$$

$$1/L\,(-\ln|P| + \ln|P-L|)$$

$$\ln\left|\frac{P-L}{P}\right| = -ALt + C$$

$$\frac{P-L}{P} = Ce^{-ALt}$$

$$P - L = PCe^{-ALt}$$

$$P(1 - Ce^{-ALt}) = L$$

L = const.

A = const.

C = Initial cond.

$$P = \frac{L}{1+Ce^{-ALt}}$$

$$P_0 = \frac{L}{1+C} \qquad C = \frac{L-P_0}{P_0}$$

The Logistic Models (Section 4.1)

We can construct our population model by considering::

$$\frac{dP}{dt} = \Big(\underset{k_1 P}{\text{Birth Rate}}\Big) - \Big(\underset{?}{\text{Death Rate}}\Big). = -AP(P-L)$$

Competition within the population causes the populations to decrease (disease, murder, natural disasters, war, lack of food/water). If we assume the death rate is proportional to the total number of possible two-party interactions, we get:

$$\text{Death rate} = \boxed{k_2 \binom{P}{2}} = k_2 \boxed{\left(\frac{P(P-1)}{2}\right)}.$$

Note: $\binom{P}{2}$ denotes "P choose 2", and in general we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \qquad \binom{P}{2} = \frac{P!}{2!(P-2)!} = \frac{P(P-1)}{2}$$

Taking both the birth and death rates into account, we get the Logistic model for population change which we simplify:

$$\longrightarrow \quad \frac{dP}{dt} = \Big(k_1 P \Big) - \Big(k_2 \cdot \frac{P(P-1)}{2} \Big).$$

Equilibrium

2. Show that the model above can be rewritten in the form $\boxed{\frac{\mathbf{dP}}{\mathbf{dt}} = -\mathbf{AP}(\mathbf{P}-\mathbf{L})}$ where A and L are positive constants.

$P=0, P=L$

$$\frac{dP}{dt} = k_1 P - k_2\left(\frac{P^2-P}{2}\right) = k_1 P - \frac{k_2}{2}P^2 + \frac{k_2}{2}P$$

$$\frac{dP}{dt} = -\frac{k_2}{2}P^2 + \left(\frac{k_2}{2}+k_1\right)P = -\frac{k_2}{2}P(P-L)$$

$$dP/dt =$$

$$L = -2/k_2\left(\frac{k_2}{2}+k_1\right)$$