Common Laplace Transforms and Properties

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$

Properties:

L.1 $\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}$, where c is a constant.

L.2 $\mathscr{L}\{f_1(t) + f_2(t)\} = \mathscr{L}\{f_1(t)\} + \mathscr{L}\{f_2(t)\}$

L.3 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$ for all $s > \alpha + a$.

L.4 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then for all $s > \alpha$,

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

L.5 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$ for all $s > \alpha$.

Explain in Words:

1. Describe properties 3, 4, and 5 in words. For example in property 3, multiplying $f(t)$ by e^{at} and then taking the Laplace transform has what affect on $\mathscr{L}\{f(t)\}$?

Solving Diff. Eqs. with Inverse Laplace Transforms

2. Solve $y'' - y = -t$ with $y(0) = 0$ and $y'(0) = 1$.

   (a) Using the properties, apply the Laplace transform to both sides:

   $$\mathscr{L}\{y'' - y\} = \mathscr{L}\{-t\}.$$

   (b) Using your answer in 2a, solve for $\mathscr{L}\{y(t)\} = Y(s)$.

   (c) Use the table of common Laplace transforms to identify what function $y(t)$ has $\mathscr{L}\{y(t)\} = Y(s)$.

In 2c, we are apply the **Inverse Laplace Transform** to $Y(s)$ in order to identify $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

Section 6.1: Inverse Laplace Transforms

Given $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies $\mathscr{L}\{f\} = F(s)$, then we say $f(t)$ is the **inverse Laplace transform** of $F(s)$ which is denoted by

$$\mathbf{f(t)} = \mathscr{L}^{\mathbf{-1}}\{\mathbf{F(s)}\}.$$

3. Determine whether the inverse Laplace transform is of the form t^n, $\cos(bt)$, $\sin(bt)$, or e^{at}.

   (a) $F(s) = \dfrac{1}{s^2}$

   (b) $F(s) = \dfrac{2}{s^2+4}$

   (c) $F(s) = \dfrac{4s}{s^2+9}$

   (d) $F(s) = \dfrac{2}{s+6}$

4. Find the inverse Laplace transform of $F(s) = \dfrac{s+2}{s^2+4s+11}$ by answering the questions below.

   (a) Complete the square for the expression in the denominator of $F(s)$ to express $s^2+4s+11 = (s-a)^2+b$.

   (b) Use the table of common Laplace transforms to identify $\mathscr{L}^{-1}\{F(s)\}$.

5. Find the inverse Laplace transform of the function.

   (a) $F(s) = \dfrac{5s-10}{s^2-3s-4}$

(b) $F(s) = \dfrac{3s - 15}{2s^2 - 4s + 10}$

(c) $F(s) = \dfrac{-5s - 36}{(s+2)(s^2+9)}$