Common Laplace Transforms and Properties

If $Y(s) = \frac{1}{s^2}$
$\Downarrow$
$y(t) = t$

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s > 0$
$e^{at}t^n$, $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $\cancel{s > 0}$ $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

t^1

$\frac{1}{s^2}$

$n = 1$

Properties:

Linear

L.1 $\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}$, where c is a constant.

L.2 $\mathscr{L}\{f_1(t) + f_2(t)\} = \mathscr{L}\{f_1(t)\} + \mathscr{L}\{f_2(t)\}$

L.3 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$ for all $s > \alpha + a$.

L.4 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then for all $s > \alpha$,

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n \mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

L.5 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$ for all $s > \alpha$.

Explain in Words: L.5 multiply by t^n is equivalent to taking nth derivative of $\mathscr{L}\{f(t)\}$ with a sign adjustment

1. Describe properties 3, 4, and 5 in words. For example in property 3, multiplying $f(t)$ by e^{at} and then taking the Laplace transform has what affect on $\mathscr{L}\{f(t)\}$?

L.3: When we multiply $f(t)$ by e^{at} and then apply $\mathscr{L}$ it is equal to $\mathscr{L}\{f(t)\}$ shifted right a units.

$s > \alpha$

$\mathscr{L}\{e^{at}f(t)\} \Rightarrow s > \alpha + a$

$Y(s)$
$\parallel$
$\mathcal{L}\{y(t)\}$

Solving Diff. Eqs. with Inverse Laplace Transforms

- Find y_h
- Find y_p
- Find $y_p + y_h$
- Use initial conditions

2. Solve $y'' - y = -t$ with $y(0) = 0$ and $y'(0) = 1$.

   (a) Using the properties, apply the Laplace transform to both sides:

   By properties L.1 and L.2
   $$\mathscr{L}\{y'' - y\} = \mathscr{L}\{-t\}.$$

   $\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -\mathcal{L}\{t\}$ Table

   By L.4

   $(s^2 Y(s) - s y(0) - y'(0)) - Y(s) = -\frac{1}{s^2}$

   $(s^2 Y(s) - 1) - Y(s) = -1/s^2$

   Want to solve for $y(t)$.

   (b) Using your answer in 2a, solve for $\mathscr{L}\{y(t)\} = Y(s)$.

   Group $Y(s)$'s on the same

   $s^2 Y(s) - Y(s) = 1 - 1/s^2$

   $Y(s)(s^2 - 1) = \frac{s^2 - 1}{s^2}$

   $Y(s) = \mathcal{L}\{y(t)\} = \frac{1}{s^2}$

   (c) Use the table of common Laplace transforms to identify what function $y(t)$ has $\mathscr{L}\{y(t)\} = Y(s)$.

   what function $y(t)$ has $Y(s) = \frac{1}{s^2}$?

   $\boxed{y(t) = t}$ ← Solution to IVP

   $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

In 2c, we are apply the **Inverse Laplace Transform** to $Y(s)$ in order to identify $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

Section 6.1: Inverse Laplace Transforms

Given $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies $\mathscr{L}\{f\} = F(s)$, then we say $f(t)$ is the **inverse Laplace transform** of $F(s)$ which is denoted by

$$\mathbf{f(t) = \mathscr{L}^{-1}\{F(s)\}}.$$

3. Determine whether the inverse Laplace transform is of the form t^n, $\cos(bt)$, $\sin(bt)$, or e^{at}.

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s > 0$
$e^{at}t^n$, $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, ~~$s > 0$~~ $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

$n = 1$

(a) $F(s) = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$

$f(t) = t$

(b) $F(s) = \frac{2}{s^2 + 4}$ $\leftarrow 2^2$

$f(t) = \sin(2t)$

(c) $F(s) = \frac{4s}{s^2 + 9}$ $\quad b = 3$

$s^2 + 3^2$

$f(t) = 4\cos(3t)$ $\qquad \mathscr{L}\{4\cos(3t)\} = \frac{4s}{s^2+3^2}$

(d) $F(s) = \frac{2}{s + 6}$

$f(t) = e^{at}$

$a = -6$

$\boxed{f(t) = 2e^{-6t}}$ $\qquad \mathscr{L}\{2e^{-6t}\} = \frac{2}{s+6}$

4. Find the inverse Laplace transform of $F(s) = \dfrac{s+2}{s^2+4s+11}$ by answering the questions below.

(a) Complete the square for the expression in the denominator of $F(s)$ to express $s^2 + 4s + 11 = (s-a)^2 + k$

$(s+2)^2 = s^2 + 4s + 4$

$s^2 + 4s + 11 = (s-?)^2 + k$

$4/2$

$= (s+2)^2 + 7$

* If denominator can't be factored

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s > 0$
$e^{at}t^n$, $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

(b) Use the table of common Laplace transforms to identify $\mathscr{L}^{-1}\{F(s)\}$.

$F(s) = \dfrac{s+2}{(s+2)^2+7}$ $\quad a = -2$ $\quad b = \sqrt{7}$

$\boxed{\mathscr{L}^{-1}\{F(s)\} = f(t) = e^{-2t}\cos(\sqrt{7}\,t)}$

If denominator can be factor Partial Fraction Decomp

5. Find the inverse Laplace transform of the function.

(a) $F(s) = \dfrac{5s-10}{s^2-3s-4} = \dfrac{5s-10}{(s-4)(s+1)} = \dfrac{A}{s-4} + \dfrac{B}{s+1} = \dfrac{As + A + Bs - 4B}{(s-4)(s+1)}$

$\boxed{A = 2, \ B = 3}$

$As + Bs = 5s$

$A - 4B = -10$

$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\dfrac{2}{s-4}\right\} + \mathscr{L}^{-1}\left\{\dfrac{3}{s+1}\right\} = 2\mathscr{L}^{-1}\left\{\dfrac{1}{s-4}\right\} + 3\mathscr{L}^{-1}\left\{\dfrac{1}{s+1}\right\}$

$\mathscr{L}^{-1}\left\{\dfrac{1}{s-4}\right\} = e^{4t}$, $\quad \mathscr{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}$

$\boxed{= 2e^{4t} + 3e^{-t}}$

Welcome! Today we'll finish 22 and start 23.

(b) $F(s) = \dfrac{3s-15}{2s^2-4s+10} = \dfrac{3s-15}{2(s^2-2s+5)}$

What function $f(t)$ has $\mathcal{L}\{f(t)\} = F(s)$?

Find $\mathcal{L}^{-1}\{F(s)\}$. $F(s) = \frac{3}{2}\left(\frac{s-5}{s^2-2s+5}\right)$, so $\mathcal{L}^{-1}\{F(s)\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s-5}{s^2-2s+5}\right\}$

Rewrite $(s^2-2s+5) = (s-a)^2+b^2 = (s-1)^2+4$

$$\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s-5}{(s-1)^2+2^2}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2} + \frac{-4}{(s-1)^2+2^2}\right\}$$

$a=1$ $b=2$

$$= \frac{3}{2}\left[\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} + -2\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}\right]$$

$b=2$

$$\frac{3}{2}\left[e^t\cos(2t) - 2e^t\sin(2t)\right] = \boxed{\frac{3}{2}e^t\cos(2t) - 3e^t\sin(2t)}$$

$s-1$

(c) $F(s) = \dfrac{-5s-36}{(s+2)(s^2+9)}$

- $\dfrac{-5s-36}{(s+2)(s^2+9)} = \dfrac{A}{s+2} + \dfrac{Bs+C}{s^2+9}$

$A=-2$
$B=2$
$C=-9$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-9}{s^2+9}\right\}$$

$$= -2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{-9}{s^2+9}\right\}$$

$$= -2e^{-2t} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + -3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{-2e^{-2t} + 2\cos(3t) - 3\sin(3t)}$$

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$, $s>0$
e^{at}	$\frac{1}{s-a}$, $s>a$
t^n, $n=1,2,\ldots$	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s>0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s>0$ ✓
$e^{at}t^n$, $n=1,2,\ldots$	$\frac{n!}{(s-a)^{n+1}}$, ~~s>0~~ $s>$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$

$$\mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\}$$ $p(s)$ and $q(s)$ are polynomials

- Factor $q(s)$ into linear terms, then use partial fraction decomposition

$$\mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = \mathcal{L}^{-1}\left\{\frac{a}{s-r_1}\right\} + \mathcal{L}^{-1}\left\{\frac{b}{s-r_2}\right\} + \mathcal{L}^{-1}\left\{\frac{c}{s-r_3}\right\}$$

$$a e^{r_1 t} + b e^{r_2 t} + c e^{r_3 t}$$

- If $q(s)$ does not factor, then we can express $(s-a)^2 + b^2$

Depending numerator, we'll get $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$