Welcome!! Today we'll work on worksheet 09.

Second Order Linear Differential Equations

2nd Order Linear

A second order linear differential equation has the form

$$\boxed{P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t)}$$

where P, Q, R, and G are continuous functions. There are many applications for which this type of differential equation is a useful model. Here are some examples.

Glass Breaking: You probably have all seen in cartoons or on Mythbusters where a wineglass is broken by singing a particular high-pitched note. The phenomenon that makes this possible is called *resonance*. Resonance results from the fact that the crystalline structures of certain solids have natural frequencies of vibration. An external force of the same frequency will "resonate" with the object and create a huge increase in energy. For instance, if the frequency of a musical note matches the natural vibration of a crystal wineglass, the glass will vibrate with increasing amplitude until it shatters. The following is one model for understanding resonance:

$$\frac{d^2x}{dt^2} + k^2x = \cos(kt)$$

Tacoma Narrows Bridge: The Tacoma Narrows Bridge in Washington State was one of the largest suspended bridges built at the time. The bridge connecting the Tacoma Narrows channel collapsed in a dramatic way on Thursday November 7, 1940. Winds of 35-46 miles/hours produced an oscillation which eventually broke the construction. The bridge began first to vibrate torsionally, giving it a twisting motion. Later the vibrations entered a natural resonance (same term as in the glass breaking) with the bridge. Here is a simplified second order differential equation that models the situation of the Tacoma Bridge:

$$\frac{d^2y}{dt^2} + 4y = 2\sin(2.1t)$$

Sometimes resonance is a good thing! Violins, for instance, are designed so that their body resonates at as many different frequencies as possible, which allows you to hear the vibrations of the strings!

There are many other situations that can be modeled with second order differential equations, including mass-spring systems, RLC circuits, pendulums, car springs bouncing, etc. In this section you will learn how to solve second order linear differential equations with constant coefficients. That is, equations where P, Q, and R are constant. If G is zero, then the equation is called **homogeneous**. When G is nonzero then the equation is called **nonhomogeneous**. As you will discover in the sections that follow, the distinction between homogeneous and non-homogeneous equations will be quite useful.

Guess and Test

1. (a) Read the following equations *with meaning*, by completing the following sentence, "$x(t)$ is a function for which its second derivative ..." (try saying "itself" instead of "x").

   i. $\frac{d^2x}{dt^2} = -x$ ii. $\frac{d^2x}{dt^2} + x = 0$ iii. $\frac{d^2x}{dt^2} + 4x = 0$ iv. $\frac{d^2x}{dt^2} = x$ $x'' - x = 0$

   (b) For each differential equation above, based on your readings *with meaning*, find two different solution functions.

$x = \cos(t)$
$x = \sin(t)$

$x'' + kx = 0$

$x'' = -4x$
$x = \cos(bt)$
$x = \sin(bt)$

$x = \cos(2t)$
$x = \sin(2t)$

$\boxed{x = e^t \quad x = e^{-t}}$

$x' = -e^{-t}$
$x'' = e^{-t}$
$= x$

2. Your task in this problem is to use the "guess and test" approach to find a solution to the linear second order, homogeneous differential equation

$$\boxed{\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 0}$$

$ax'' + bx' + cx = 0$

By now you know very well that solutions are functions. What is your best guess for a function whose second derivative plus 10 times its first derivative plus 9 times the function itself sum to zero? Explain briefly the rationale for your guess and then test it out to see if it works. If it doesn't work keep trying.

Adding multiples of x'', x' and x cancel out giving 0?

constant coefficients homogeneous

$x = e^{rt}$
$x = \cos(bt)$
$x = \sin(bt)$

$\boxed{x = e^{-t}}$ $\times 9$ $x' = -e^{-t}$ $\times 10$ $x'' = e^{-t}$ ✓

3. Determine if a constant multiple of your solution is also a solution.

$x = e^{-t}$ is a solution to $x'' + 10x' + 9x = 0$.

$\boxed{x_2 = Ce^{-t}}$ $x_2' = -Ce^{-t}$ $x_2'' = Ce^{-t}$

$Ce^{-t} + 10(-Ce^{-t}) + 9(Ce^{-t}) = C\left[e^{-t} - 10e^{-t} + 9e^{-t}\right] = 0$

4. Try and find a different solution, one that is not a constant multiple of your solution to problem 2.

General Fact:

$ax'' + bx' + cx = 0$

If x is a solution,

Then cx is a solution.

$x'' + 10x' + 9x = 0$

$\boxed{x = ce^{-t}}$

$r^2 e^{rt} + 10 r e^{rt} + 9 e^{rt} = 0$

$x = e^{rt}$ For what values of r is $x = e^{rt}$ a solution?

$x' = r e^{rt}$

$x'' = r^2 e^{rt}$

$e^{rt}[r^2 + 10r + 9] = 0$

$e^{rt}(r+1)(r+9) = 0$

$r = -1$ $\quad$ $r = -9$

Another solution

5. Determine the *general solution* to $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 9x = 0$.

$x = e^{-t}$

$\boxed{x = e^{-9t}}$

so $x_1 = c_1 e^{-t}$ ✓ is a solution.

$x_2 = c_2 e^{-9t}$ ✓

$\boxed{x = c_1 e^{-t} + c_2 e^{-9t}}$

$\boxed{x = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$

$r_1 = ?$

$r_2 = ?$

~~A~~ General solution

Has 2 constants.

$ax''+bx'+cx=0$

$ar^2+br+c=0$

6. Consider again the differential equation $\frac{d^2x}{dt^2}+10\frac{dx}{dt}+9x=0.$

By guessing $x(t)=e^{rt}$, show that this guess yields a solution to the differential equation precisely when $r^2+10r+9=0$. "Characteristic polynomial" Linear Algebra ↳ Eigenvalues	$x=c_1e^{r_1t}+c_2e^{r_2t}$ $r^2+10r+9=0$
Solve this quadratic equation to find two different values of r.	$(r+1)(r+9)=0$ $r=-1,-9$
State two different solutions for the differential equation, one for each value of r.	$x=e^{-t}$ and $x=e^{-9t}$
Form the general solution by multiplying your two solutions by constants c_1 and c_2, and adding the results.	$\boxed{x=c_1e^{-t}+c_2e^{-9t}}$ ☆
Congratulate yourself :)	

7. Find the general solution to the following differential equation: $\frac{d^2x}{dt^2}+\frac{dx}{dt}-6x=0.$

① Characteristic Polynomial: $r^2+r-6=0$

② Solve: $(r+3)(r-2)=0$ $\boxed{r=-3,2}$

③ General Solution $x=c_1e^{r_1t}+c_2e^{r_2t}$

$\boxed{x=c_1e^{-3t}+c_2e^{2t}}$

Prod Rule

$$\boxed{y = C_1 e^{2t} + C_2 t e^{2t}}$$

8. Show $y = Cte^{2t}$ is a solution to $y'' - 4y' + 4y = 0$.

$$y' = (Ct)' e^{2t} + (Ct)(e^{2t})' = Ce^{2t} + 2Cte^{2t}$$

$y'' =$

Check at Home:

$$y = Cte^{2t}$$

① $\boxed{r^2 - 4r + 4 = 0}$

② $(r-2)^2 = 0$

$\boxed{r = 2}$

Repeated real root

9. Why do you think the previous example has a "different" looking solution?

Special Case: If characteristic equation has one repeated real root, r_1

$$\boxed{x = c_1 e^{r_1 t} + c_2 t e^{r_1 t}}$$

General Case

$$\boxed{x = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

2 Distinct Roots

Solve each of the differential equations below. If initial conditions are given, find the particular solution.

10. $z'' + z' = z$

$z'' + z' - z = 0$

$r^2 + r - 1 = 0$

$(r - \ )(r - \ )$?

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\boxed{z = c_1 e^{\left(\frac{-1+\sqrt{5}}{2}\right)t} + c_2 e^{\left(\frac{-1-\sqrt{5}}{2}\right)t}}$$

11. $3w'' + 18w' + 27w = 0$

$3(w'' + 6w' + 9) = 0$

$3(r^2 + 6r + 9) = 0$

$3(r+3)(r+3) = 0$

$3(r+3)^2 = 0$

one repeated real root $r = -3$

$\boxed{w = c_1 e^{-3t} + c_2 t e^{-3t}}$

12. $w'' + 8w' + 16w = 0$; $w(0) = -2$; $w'(0) = 12$.

$r^2 + 8r + 16 = 0$ $\quad (r+4)^2 = 0$

$r = -4$

$\boxed{w = c_1 e^{-4t} + c_2 t e^{-4t}}$

$\boxed{w = -2e^{-4t} + 4te^{-4t}}$

$w(0) = c_1 e^0 + c_2(0) e^0 = -2 \Rightarrow c_1 = -2$

$w(t) = -2e^{-4t} + c_2 t e^{-4t}$

$w' = 8e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$

$w'(0) = 8 + c_2 - 0 = 12$

$c_2 = 4$

13. $y'' + 3y' - 10y = 0$; $y(0) = 8$, $y'(0) = 6$.

① $r^2 + 3r - 10 = 0$

② $(r+5)(r-2) = 0$

$r = -5, 2$

③ $\boxed{y = c_1 e^{-5t} + c_2 e^{2t}}$

$\boxed{y(0) = c_1 + c_2 = 8}$

$y' = -5c_1 e^{-5t} + 2c_2 e^{2t}$

$y'(0) = -5c_1 + 2c_2 = 6$

2 Eq. in two unknowns

$c_1 + c_2 = 8$ $\qquad c_1 = 10/7$

$-5c_1 + 2c_2 = 6$ $\qquad c_2 = 46/7$

$\boxed{y = \frac{10}{7} e^{-5t} + \frac{46}{7} e^{2t}}$