Introduction to the Laplace Transform

One of the basic problem solving techniques in mathematics is to

- transform a difficult problem into an easier one,
- solve the easier problem, and
- then use its solution to obtain a solution of the original problem.

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$$

$$3y'' + 2y' + 6y = 0$$

For example. the reverse product rule (method of integrating factor) is used to transform a linear first order differential equation into an easier problem we can solve. In this chapter we study the method of **Laplace transforms**, which is one example of this technique. Like the method of integrating factors, Laplace transforms are **integral operators**. Solving by the method of Laplace transforms:

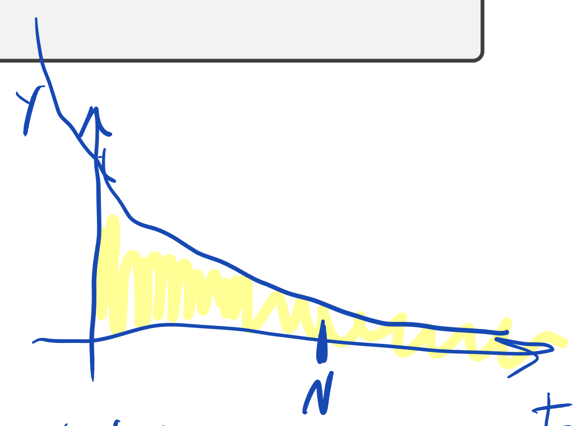
- Can be used to solve higher order linear differential equations.
- Can be applied for more complicated forcing functions.
- Requires initial conditions.

The **improper integral** of g over $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\ dt = \lim_{N\to\infty} \int_a^N g(t)\ dt.$$

- We say the improper integral **converges** if the limit exists.
- Otherwise we say the improper integral **diverges**.

1. Determine whether $\int_0^\infty e^{-2t}\ dt$ converges or diverges.



$$\int_0^\infty e^{-2t}\,dt = \lim_{N\to\infty} \int_0^N e^{-2t}\,dt$$

$$= \lim_{N\to\infty} \left(-\frac{1}{2} e^{-2t} \Big|_{t=0}^{t=N} \right)$$

$$= \lim_{N\to\infty} \left(-\frac{1}{2} e^{-2N} + \frac{1}{2} e^{0} \right) = \lim_{N\to\infty} \left(-\frac{1}{2} e^{-2N} \right) + \frac{1}{2} = \boxed{\frac{1}{2}}$$

what happens as $N \to \infty$

0

Converges to 1/2

Definition of the Laplace Transform

Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by

$f(t) \to F(s)$

$$\mathscr{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

- The domain of $F(s)$ is all values of s for which the integral converges.
- The functions f and F form a **transform pair**.

2. Find and state the domain of the Laplace transform $F(s) = \mathscr{L}\{f(t)\}$.

   (a) $f(t) = 2$, $t \geq 0$

   (b) $f(t) = t$ ① write out definition as an improper integral

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = \lim_{N\to\infty} \int_0^N \underbrace{e^{-st} \cdot t}\, dt$$

$$\lim_{N\to\infty}\left(-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \Big|_{t=0}^{t=N} \right) = \lim_{N\to\infty}\left(\frac{-N}{s} e^{-N\cdot s} - \frac{1}{s^2} e^{-N\cdot s} \right)$$

$$- \left(-0 - \frac{1}{s^2} e^0 \right)$$

$$\int e^{-st} \cdot t \, dt \qquad u = t \quad du = dt \qquad dv = e^{-st} dt \quad v = -\frac{1}{s} e^{-st}$$

$$= -\frac{t}{s} e^{-st} - \int -\frac{1}{s} e^{-st} \, dt$$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} + C$$

$$= \lim_{N \to \infty} \left(\frac{-N}{s} e^{-N \cdot s} - \frac{1}{s^2} e^{-N \cdot s} \right) - \left(-0 - \frac{1}{s^2} e^{0} \right)$$

$$= \lim_{N \to \infty} \frac{-N}{s} e^{-Ns} - \lim_{N \to \infty} \left(\frac{1}{s^2} e^{-N s} \right) + \frac{1}{s^2}$$

$-\infty$ $\to 0$

$$0 - 0 + 1/s^2 = 1/s^2$$

$$\lim_{N \to \infty} \frac{-N}{s e^{Ns}} = \frac{-\infty}{\infty} \; ? \quad \text{L' Hopital's Rule}$$

$$\lim_{N \to \infty} \frac{-1}{s^2 e^{Ns}} = \frac{-1}{\infty} = 0$$

$$\boxed{\mathcal{L}\{t\} = F(s) = 1/s^2}$$

Provided $s > 0$

(c) $f(t) = e^{3t}$

(d) $g(t) = \cos(bt)$ where $b \neq 0$ is a constant.

(e) $f(t) = \begin{cases} 5 & 0 < t < 2 \\ e^{8t} & t > 2 \end{cases}$

Common Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
$f(t) = 1$	$F(s) = \frac{1}{s},\ s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{s-a},\ s > a$
$f(t) = t^n,\ n = 1, 2, \dots$	$F(s) = \frac{n!}{s^{n+1}},\ s > 0$
$f(t) = \sin(bt)$	$F(s) = \frac{b}{s^2 + b^2},\ s > 0$
$f(t) = \cos(bt)$	$F(s) = \frac{s}{s^2 + b^2},\ s > 0$