Mixture Problems (Application 4.2)

A one-compartment system consists of

- $x(t)$ that represents the amount of a substance (such as salt) at time t.
- an input rate of x.
- an output rate of x.

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}$$

1. A brine solution of salt water that has concentration 0.05 kg per L flows at a constant rate of 6 L per minute into a tank which is initially contains 50 L of a 1% salt solution. The brine solution flows out of the tank at a rate of 4 L per minute. Let $x(t)$ denote the mass of the salt in the tank at time t (in minutes). *Note that 1% salt solution means 1 kg of salt per 100 L of solution.*

   (a) What is the input rate of x?

   (b) What is the output rate of x?

   (c) What is the initial mass of the salt in the tank?

   (d) Construct an model for this initial value problem (but do not solve it).

   (e) What method(s) can we apply to solve the equation in (4) (but don't solve it)?

Population Models (Section 4.1)

The Malthusian law of population growth says the rate of change of the population, $\frac{dP}{dt}$, is directly proportional to the population present, P, at time t:

$$\frac{dP}{dt} = kP, \quad P(0) = P_0.$$

Example 2. Let P denote the population of the world (in billions) t years since 1960. In 1960 the world's population was approximately 3 billion, and the population growth is model by

$$\frac{dP}{dt} = 0.2P \qquad , \ P(0) = 3.$$

Solving this model gives $P(t) = 3e^{0.02t}$, and predicts the population in 2019 is 9.76 billion.

Why do you think predicted value is different from the actual value?

The Logistic Models (Section 4.1)

We can construct our population model by considering::

$$\frac{dP}{dt} = \Big(\text{Birth Rate}\Big) - \Big(\text{Death Rate}\Big).$$

Competition within the population causes the populations to decrease (disease, murder, natural disasters, war, lack of food/water). If we assume the death rate is proportional to the total number of possible two-party interactions, we get:

$$\text{Death rate} = k_2 \binom{P}{2} = k_2 \left(\frac{P(P-1)}{2}\right).$$

Note: $\binom{P}{2}$ denotes "P choose 2", and in general we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Taking both the birth and death rates into account, we get the Logistic model for population change which we simplify:

$$\frac{dP}{dt} = \Bigg(\qquad\qquad\qquad\qquad \Bigg) - \Bigg(\qquad\qquad\qquad\qquad \Bigg).$$

2. Show that the model above can be rewritten in the form $\mathbf{\frac{dP}{dt} = -AP(P - L)}$ where A and L are positive constants.

Practice: Population Model for Rabbits

3. A population of rabbits changes over time t (in years) according to the logistic model

$$\frac{dP}{dt} = 3P - \frac{1}{20}P^2.$$

(a) For what initial population sizes P_0 will the population grow at first?

(b) For what initial population sizes P_0 will the population decrease at first?

(c) For what initial population sizes P_0 will the population never change?

(d) Explain, in practical terms, why answers in (a)-(c) makes sense.

(e) If the initial rabbit population is $P_0 = P(0) = 50$, find a solution to the initial value problem and find a formula for the population P as a function of time t.

Practice: Chlorine Levels in Pool

4. A swimming pool whose volume is 10,000 gallons contains water that is 0.01% chlorine. Starting at $t = 0$, city water containing 0.001% chlorine is pumped into the pool at a rate of 5 gal/min. The pool water flows out at the same rate. Let x denote the amount of chlorine (in pounds) in the pool t minutes since water has begun being pumped into the pool.

   Note that a concentration of 0.01% *chlorine solution means* 0.01 *pounds of chlorine per 100 gallons of solution.*

   (a) Construct a differential equation for rate of change of the mass of chlorine (in pounds) x in the pool at time t.

   (b) Solve the initial value problem using the differential equation in (a) and the given initial % concentration.

   (c) (Bonus) When will the pool water be 0.002% chlorine?