Homogeneous Second Order Linear Differential Equations

A second order linear differential equation with constant coefficients has the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

Welcome!

where a, b, and c are constants and f is a continuous function of t.

- If $f(t) = 0$, then the equation is called **homogeneous**.
- If $f(t) \neq 0$, then the equation is called **nonhomogeneous**.

Today we'll work on Worksheet 10 Complex roots

We have shown that to find solutions to the homogeneous case $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$, we can:

1. Set up the corresponding characteristic polynomial, $ar^2 + br + c = 0$.
2. Find solutions $r = r_1$ and $r = r_2$ to the characteristic equation.
3. Quadratic equations may have real or complex solutions:
   - If r_1 and r_2 are distinct real numbers, then the general solution is
     $$x(t) = C_1e^{r_1 t} + C_2e^{r_2 t}.$$
   - If there is one repeated root, r_1, then the general solution is
     $$x(t) = C_1e^{r_1 t} + C_2te^{r_1 t}.$$
   - **If the solutions are of the form $\mathbf{r} = \alpha \pm \mathbf{i}\beta$, then what?**

$y'' + 5y' + 6y = 0$

$r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r = -3, -2$

$\boxed{y = C_1e^{-3t} + C_2e^{-2t}}$

$y'' - 6y' + 10y = 0$

$r^2 - 6r + 10 = 0$

Doesn't factor

$r = \frac{6 \pm \sqrt{36-40}}{2}$

$r = \frac{6}{2} \pm \frac{\sqrt{-4}}{2}$ $\sqrt{-4} \to 2i$

$\boxed{r = 3 \pm i}$??

$e^{(3+i)t}$

Complex Solutions

beta $= \beta$

$e^{(3+i)t} = e^{3t} e^{it}$?

1. Let $f(t) = e^{i\beta t}$ and answer the questions below.

   (a) Find a formula for f', f'', f''', f^{iv}, and f^{v}.

$f'(t) = i\beta e^{i\beta t} = i\beta e^{i\beta t} \Rightarrow i\beta$

$f''(t) = i^2\beta^2 e^{i\beta t} = -\beta^2 e^{i\beta t} = -\beta^2$

$f'''(t) = i^3\beta^3 e^{i\beta t} = -i\beta^3 e^{i\beta t}$

$f^{iv}(t) = i^4\beta^4 e^{i\beta t} = \beta^4 e^{i\beta t}$

$f^{v}(t) = i^5\beta^5 e^{i\beta t} = i\beta^5 e^{i\beta t}$

$\vdots$

   (b) Express $f(t) = e^{i\beta t}$ using as a Taylor series at $t = 0$:

$$f(t) = f(0) + \frac{f'(0)}{1!}t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{iv}(0)}{4!}t^4 + \frac{f^{v}(0)}{5!}t^5 + \dots$$

$$e^{i\beta t} = 1 + i\beta t - \frac{\beta^2}{2!}t^2 - \frac{i\beta^3}{3!}t^3 + \frac{\beta^4}{4!}t^4 + \frac{i\beta^5}{5!}t^5 - \frac{\beta^6}{6!}t^6 + \dots$$

   (c) Group the real and imaginary parts of the first several terms in the Taylor series together.

$$e^{i\beta t} = \boxed{\left(1 - \frac{\beta^2}{2!}t^2 + \frac{\beta^4}{4!}t^4 - \frac{\beta^6}{6!}t^6 + \dots\right)} + i\boxed{\left(\beta t - \frac{\beta^3}{3!}t^3 + \frac{\beta^5}{5!}t^5 \dots\right)}$$

   (d) Do you recognize these are Taylor series of common functions?

$$\boxed{e^{i\beta t} = \cos(\beta t) + i\sin(\beta t)}$$

Euler's Formula

The previous question is a proof of **Euler's formula** which allows us to write exponentials in **polar form**,

$e^{\alpha t}$ $e^{i\beta t}$

$$e^{(\alpha+i\beta)t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right).$$

$y = e^{3t}[\cos(t) + i\sin(t)]$

2. If $z(t) = P(t) + iQ(t)$ is complex solution to a differential equation of the form $az'' + bz' + cz = 0$, prove that the real part $P(t)$ is a solution itself and the imaginary part $Q(t)$ (not including the i) is also a solution itself. Note the derivative of a complex function is the sum of the derivatives of the real and imaginary parts of the complex function:

$$z'(t) = P'(t) + iQ'(t).$$

$ay'' + by' + cy = 0$
y is a real valued function

Given $z(t) = P(t) + iQ(t)$ is a solution to $az'' + bz' + cz = 0$. Show $P(t)$ and $Q(t)$ are each solutions.

$z' = P'(t) + iQ'(t)$
$z'' = P''(t) + iQ''(t)$

$* \; a(P'' + iQ'') + b(P' + iQ') + c(P + iQ) = 0$

$(aP'' + bP' + cP) + i[aQ'' + bQ' + cQ] = 0$

Complex number $= 0 = 0 + 0i$
so $aP'' + bP' + cP = 0$ ✓
$aQ'' + bQ' + cQ = 0$ ✓

$z = P(t)$ is a real solution

$z = Q(t)$ is a real solution.

$e^{(3\pm i)t}$

Euler's Formula

$$e^{(3+i)t} \implies e^{3t}\left[\cos(t) + i\sin(t)\right]$$

$$y = e^{3t}\cos(t) + i\,e^{3t}\sin(t)$$

is a complex solution

From previous result

$P(t) = c_1 e^{3t}\cos t$ is a real solution.

$Q(t) = c_2 e^{3t}\sin t$ " .

General Solution

$$\boxed{y = c_1 e^{3t}\cos t + c_2 e^{3t}\sin t}$$

$\cos(t) = \cos(-t)$

$\sin(-t) = -\sin(t)$

$$y = B_1 e^{3t}\cos t + B_2 e^{3t}\sin(-t)$$

3. Find the general solution to the homogeneous differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 17x = 0$$

Use the above results on exponentiation of complex numbers to find the general solution to the differential equation.

① Set up Characteristic Equation.

$r^2 + 2r + 17 = 0$

② Solve characteristic Eq.

$\frac{-b \pm \sqrt{b^2-4ac}}{2}$

$r = -1 \pm 4i$

③ Depending on the roots, plug solutions ② into general form of solution

$x = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$

$$\boxed{x = c_1 e^{-t}\cos(4t) + c_2 e^{-t}\sin(4t)}$$

To summarize our results, when solving a homogeneous second order differential with constant coefficients, we can find the zeros of the corresponding characteristic equation. Then

- If r_1 and r_2 are distinct real numbers, then the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

- If there is one repeated root, r_1, then the general solution is

$$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

- If the solutions are of the form $\mathbf{r} = \alpha \pm \mathbf{i}\beta$, then the general solution is

$$\boxed{y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)}$$

mass Friction spring stiffness

Section 2.4: Mass-Spring Oscillator

4. Consider the mass-spring oscillator that has mass $m = 1$ kg, stiffness $k = 4$ kg/sec^2, and damping b kg/sec. The displacement y from equilibrium position at time t seconds satisfies the initial value problem

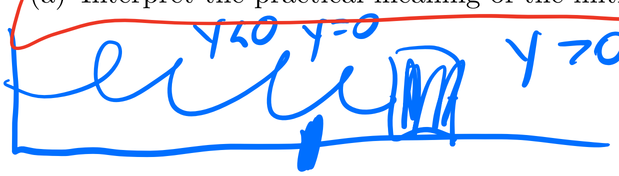
$$y'' + by' + 4y = 0; \quad y(0) = 1 \quad y'(0) = 0.$$

$y=0$ means at equi'l position

$y>0$ right

$y<0$ left

(a) Interpret the practical meaning of the initial conditions.



$y(0)=1$ At time 0, (initially) mass is lent to right

$y'(0)=0$ Initially at $t=0$, mass is let go (not pushed or pulled)

(b) Find the solution if the damping coefficient is $b = 0$ and describe what happens to the mass as $t \to \infty$.

undamped

Purely imaginary roots

$y''+4y=0$

$r^2+4=0$

$r^2=-4 \quad r=\pm\sqrt{-4} = \pm 2i$

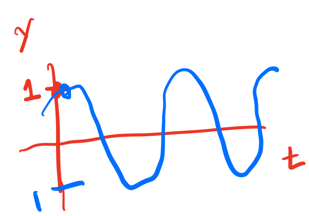
$y = C_1\cos(2t) + C_2\sin(2t)$

$y(0)=1 \rightarrow C_1=1$

$y'(0)=0 \rightarrow C_2=0$

$\boxed{y=\cos(2t)}$

As $t\to\infty$ mass oscillate back and forth forever



overdamped

(c) Find the solution if the damping coefficient is $b = 5$ and describe what happens to the mass as $t \to \infty$.

$y''+5y'+4y=0$

$r^2+5r+4=0$

$(r+4)(r+1)=0$

$r=-4,-1$ Two distinct real roots

$y = C_1e^{-4t}+C_2e^{-t}$

$y(0)=C_1+C_2=1$

$y'(t) = -4C_1e^{-4t}-C_2e^{-t}$

$y'(0) = -4C_1-C_2$

so we solve the system

$C_1+C_2=1$

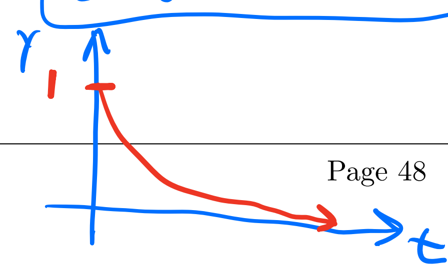
$-4C_1-C_2=0$

and $C_1=-\frac{1}{3}$ $C_2=\frac{4}{3}$

so

$\boxed{y = -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t}}$

$\boxed{\lim_{t\to\infty} y(t) = 0}$



critically damped

(d) Find the solution if the damping coefficient is $b = 4$ and describe what happens to the mass as $t \to \infty$.

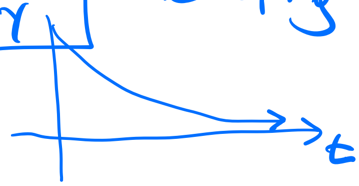
we have $y'' + 4y' + 4y = 0$ so $r^2 + 4r + 4 = (r+2)^2 = 0$ has one repeated real root, $r = -2$. The general solution is thus

$y = C_1 e^{-2t} + C_2 t e^{-2t}$

$y(0) = \boxed{C_1 = 1}$ and $y'(t) = -2C_1 e^{-2t} - 2C_2 t e^{-2t} + C_2 e^{-2t}$. Thus

$y'(0) = -2C_1 + C_2 = -2 + C_2 = 0$ $\boxed{C_2 = 2}$. Our solution is therefore $\boxed{y = e^{-2t} + 2te^{-2t}}$

$\boxed{\lim_{t\to\infty} y(t) = 0}$ The spring will go back to its equilibrium.



underdamped

(e) Find the solution if the damping coefficient is $b = 2$ and describe what happens to the mass as $t \to \infty$.

we have $y'' + 2y' + 4y = 0$ so $r^2 + 2r + 4 = 0$. $r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$

Since we have two complex conjugate roots, then the general solution is $y(t) = C_1 e^{-t}\cos(\sqrt{3}t) + C_2 e^{-t}\sin(\sqrt{3}t)$.

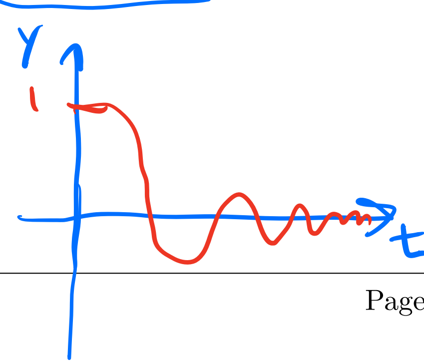
For $y(0) = C_1 + 0 = 1$ so $\boxed{C_1 = 1}$. We have

$y'(t) = -\sqrt{3}C_1 e^{-t}\sin(\sqrt{3}t) - C_1 e^{-t}\cos(\sqrt{3}t) + \sqrt{3}C_2 e^{-t}\cos(\sqrt{3}t) - C_2 e^{-t}\sin(\sqrt{3}t)$

$y'(0) = -C_1 + \sqrt{3}C_2 = -1 + \sqrt{3}C_2 = 0$ so $\boxed{C_2 = 1/\sqrt{3}}$

$\boxed{y = e^{-t}\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t).}$

$\lim_{t\to\infty} y = 0$ so it settles back at equilibrium after some oscillations



$r = 2 \pm 3i$

$\hookrightarrow e^{(2+3i)t} \qquad e^{(2-3i)t}$

$\downarrow$

$C_1\left[e^{2t}\cos(3t) + e^{2t}\sin(3t)\right]$

$+C_2\left[e^{2t}\cos(-3t) + e^{2t}\sin(-3t)\right]$

$$C_1e^{2t}\cos(3t) + C_2e^{2t}\cos(-3t) = \underbrace{(C_1+C_2)}_{B_1}e^{2t}\cos(3t)$$

$$C_1e^{2t}\sin(3t) - C_2e^{2t}\sin(3t) = \underbrace{(C_1-C_2)}_{B_2}e^{2t}\sin(3t)$$

Justification

for $C_1e^{(\alpha+i\beta)t} + C_2e^{(\alpha-\beta i)t}$ leading to

$C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t)$