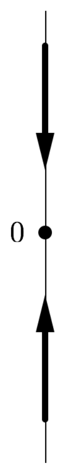
Homework Set 6 Solutions

1. For an autonomous differential equations, it is possible to view all of the solution function graphs in terms of "prototypical" graphs. A prototypical solution graph represents an infinite number of other solution graphs. For example, in part (i) below one can view the entire family of functions that solve the differential equation in terms of two different prototypical solution graphs separated by an equilibrium solution: one prototypical solution graph is above the t-axis and one is below the t-axis. Each is prototypical because it can stand for all other solution graphs (in its respective region) through horizontal translation. Recall the "Making Connections" section of Unit 3.

(i) $\dfrac{dy}{dt} = -y$ (ii) $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{2}\right)$ (iii) $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{2}\right) + 3$ (iv) $\dfrac{dy}{dt} = y^2$

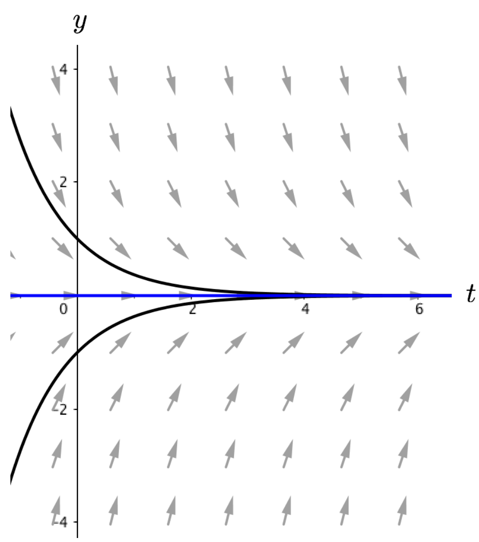
(a) For each differential equation above, draw a phase line and representative graphs of solutions.

(b) For each differential equation above, explain how your response to number 1a can be interpreted in terms of prototypical solutions separated by equilibrium solutions.



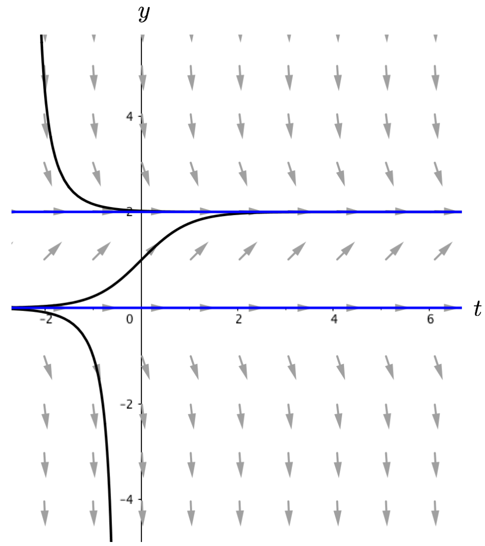
(i) Phase line:

Representative graphs of solutions:

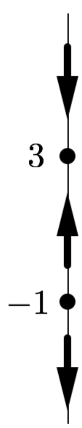


(ii) Phase line:

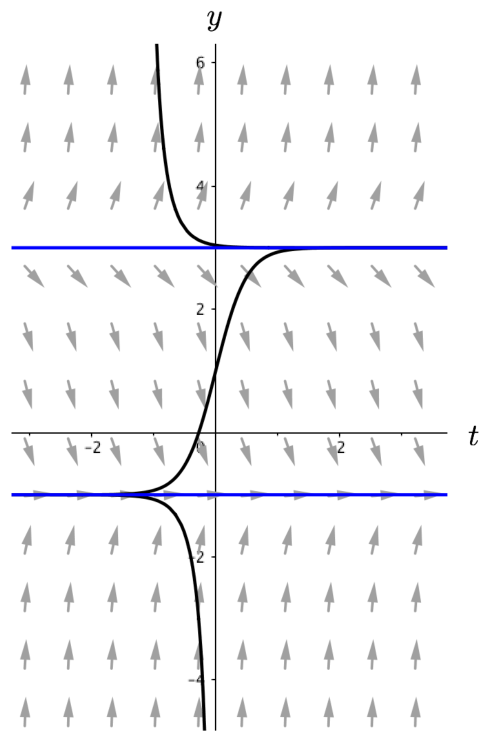
Representative graphs of solutions:



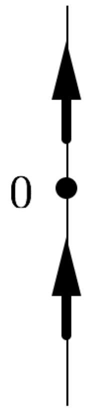
(iii) Phase line:



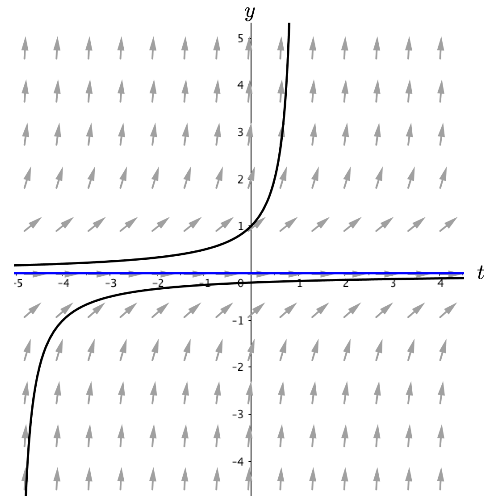
Representative graphs of solutions:



(iv) Phase line:

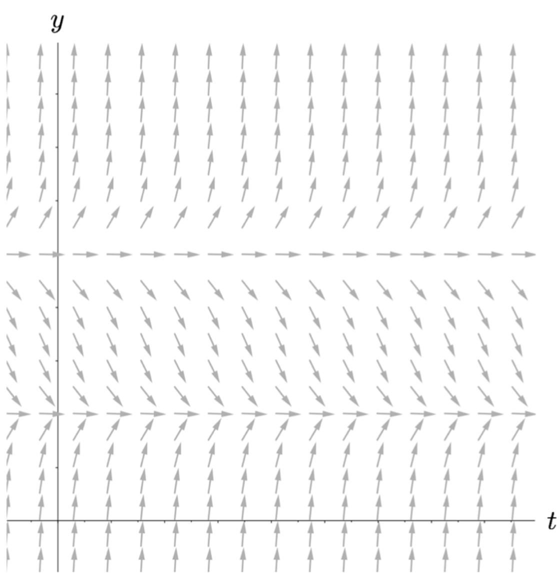


Representative graphs of solutions:

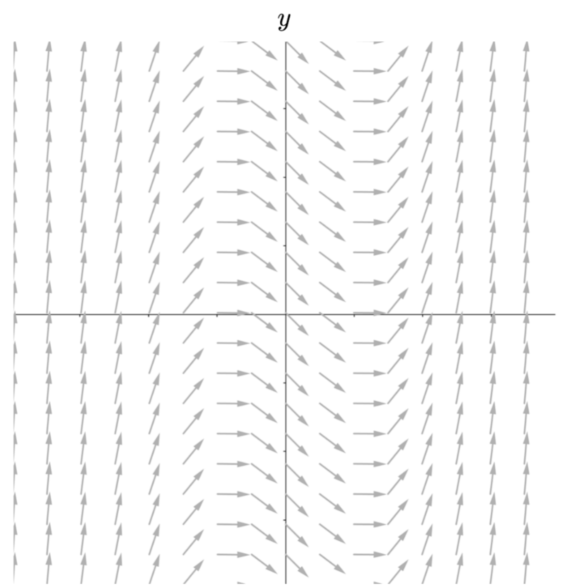


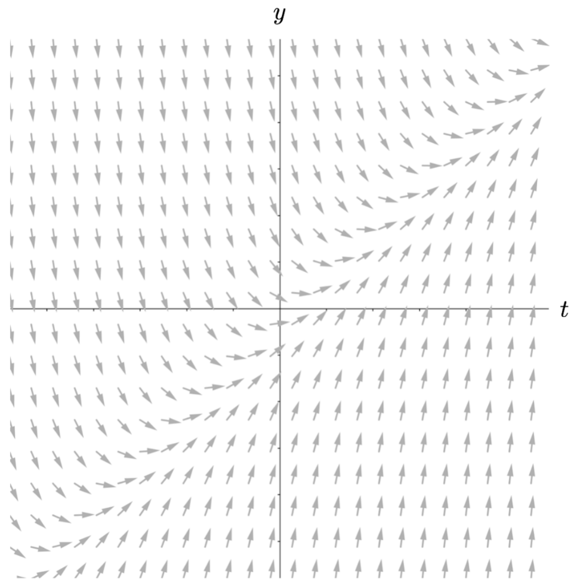
2. For each of the following slope fields, create a differential equation whose slope field would be similar to the one given. Give reasons for why you created the differential equation as you did. You may create whatever scale on the axes that you want.

(a)

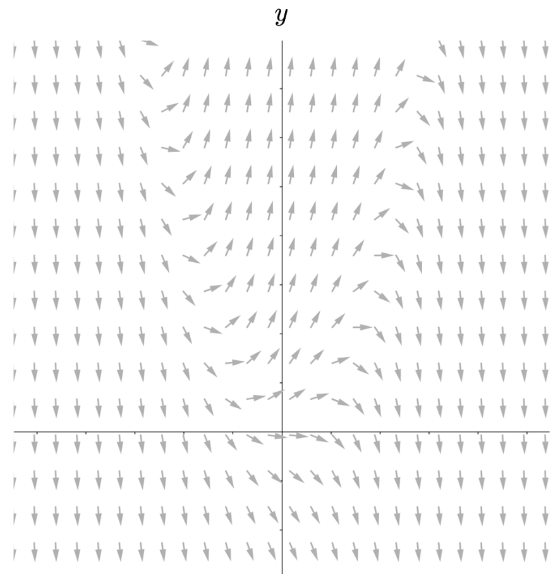


(b)





(c)



(d)

Answers will vary.

(a) The differential equation is autonomous as the slope field indicates horizontal shifts preserve slope. Thus, the differential equation should depend only on y. There are two equilibrium solutions (e.g., 2 and 5). 2 is an stable equilibrium solution and 5 is an unstable equilibrium solution. Therefore between 2 and 5 we need the slope to be negative and elsewhere it needs to be positive (and of course, 0 at the equilibrium solutions). Thus, a possible differential equation is:

$$\frac{dy}{dt} = (y-2)(y-5)$$

(b) The differential equation is not autonomous as the slope field indicates horizontal shifts do not preserve slope. Yet, vertical shifts preserve slope, thus, the differential equation should depend only on t. As the slope field appears cubic the differential equation will be terms of t^2. Thus, a possible differential equation is:

$$\frac{dy}{dt} = (t+2)(t-2)$$

(c) The differential equation is not autonomous with no equilibria and depends on both t and y as neither horizontal nor vertical shifts preserve slope. It appears that the line $y = t$ is a solution so we want some factor of $(y-t)$ or $(t-y)$ in our differential equation. Based on the slopes, one possible differential equation is:

$$\frac{dy}{dt} = t - y$$

(d) The differential equation is not autonomous with no equilibria and depends on both t and y as neither horizontal nor vertical shifts preserve slope. It appears that the curve $y = t^2$ is a solution so we want some factor of $(y-t^2)$ or (t^2-y) in our differential equation. Based on the slopes, one possible differential equation is:

$$\frac{dy}{dt} = y - t^2$$

3. For each part below, create a continuous, autonomous differential equation that has the stated properties (if possible). Explain how you created each differential equation and include all graphs or diagrams you used and how you used them. If it is not possible to come up with a differential equation with the stated properties, provide a justification for why it cannot be done.

   (a) Exactly three constant solution functions, two repellers and one attractor.

   (b) Exactly two constant solution functions, one a repeller and one a node.

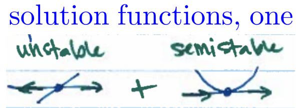
   (c) Exactly two constant solution functions, both attractors.

Answers will vary.

(a) Because we have two unstable and one stable equilibria, one possible way these can be oriented is: 

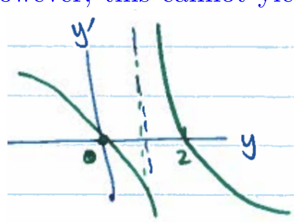
. Thus, we want a positive cubic DE with three unique equilibrium solutions. One possible differential equation is:

$$\frac{dy}{dt} = (y-1)(y-e)(y-\pi^2)$$

(b) Because we have two constant solution functions, one unstable and one semi-stable, one possible way these can be oriented is: 

. Thus, we want a positive cubic DE with two unique equilibrium solutions, one of which touches but does not cross the t-axis. One possible differential equation is:

$$\frac{dy}{dt} = (y-1)(y-2)^2$$

(c) To have exactly two constant solution functions that are both stable these would be oriented like this: , however, this cannot yield a continuous function. For instance, one $\frac{dy}{dt}$ vs. y graph could be this 

. As this is not continuous, it is not possible to have exactly two constant solution functions that are both stable.

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4. For each part below, create an autonomous differential equation that satisfies the stated criteria

   (a) $y(t) = 0$ and $y(t) = -4$ are the only constant solution functions

   (b) $y(t) = e^{-t+1}$ is a solution

   (c) $y(t) = e^{2t-5}$ is a solution

   (d) $y(t) = 10e^{0.3t}$ is a solution

(e) $y(t) = 1 - e^{-t}$ and $y(t) = 1 + e^{-t}$ are solutions

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Answers will vary.

(a) One possible autonomous DE is:
$$\frac{dy}{dt} = y(y+4)$$

(b) We want a function whose derivative is e^{-t+1}. Thus, $y = -e^{-t+1}$ works. Therefore, a possible autonomous DE is:
$$\frac{dy}{dt} = -y$$

(c) Applying the same reason as part (b), one possible autonomous DE is:
$$\frac{dy}{dt} = 2y$$

(d) Applying the same reason as parts (b) and (c), one possible autonomous DE is:
$$\frac{dy}{dt} = 0.3y$$

(e) Let's try $\frac{dy}{dt} = 1 - y$. Then if $y = 1 + e^{-t}$, $\frac{dy}{dt} = 0 - e^{-t} = -e^{-t} = 1 - y$. And if $y = 1 - e^{-t}$, $\frac{dy}{dt} = 0 + e^{-t} = 1 - (1 - e^{-t}) = 1 - y$. Therefore, $\frac{dy}{dt} = 1 - y$.

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5. For a phase line to be a meaningful tool, explain why it is essential for the differential equation to be autonomous.

If the DE is not autonomous, there are not constant function solutions that work for **all** time. Hence, the phase line would not capture equilibrium solutions; it could only possibly record a function that was 0 for some specific time t, which isn't the meaning of equilibrium.

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6. In class you and your classmates continue to develop creative and effective ways of thinking about particular ideas or problems. Discuss at least one idea or way of thinking about a particular problem that has been discussed in class (either in whole class discussion or in small group) that was particularly helpful for enlarging your own thinking and/or that you disagreed with and had a different way of thinking about the idea or problem.

Answers will vary.
