Bees and Flowers

Often scientists use rate of change equations in their study of population growth for one or more species. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are *harmed by* interaction) or cooperative (that is, both species *benefit from* interaction).

1. Which system of rate of change equations below describes a situation where the two species compete and which system describes cooperative species? Explain your reasoning.

(i) $$\frac{dx}{dt} = -5x + 2xy \qquad \frac{dy}{dt} = -4y + 3xy$$

(ii) $$\frac{dx}{dt} = 4x - 2xy \qquad \frac{dy}{dt} = 2y - xy$$

The equations in (ii) correspond to competitive species. One way to see this is the sign of the coefficient in front of the "mixed" terms xy is negative. When x and y interact, both population rates of change decrease.

The equations in (i) are cooperative. Similar to equations (ii), the mixed terms xy have a positive sign. When x and y interact, population rates of change increase.

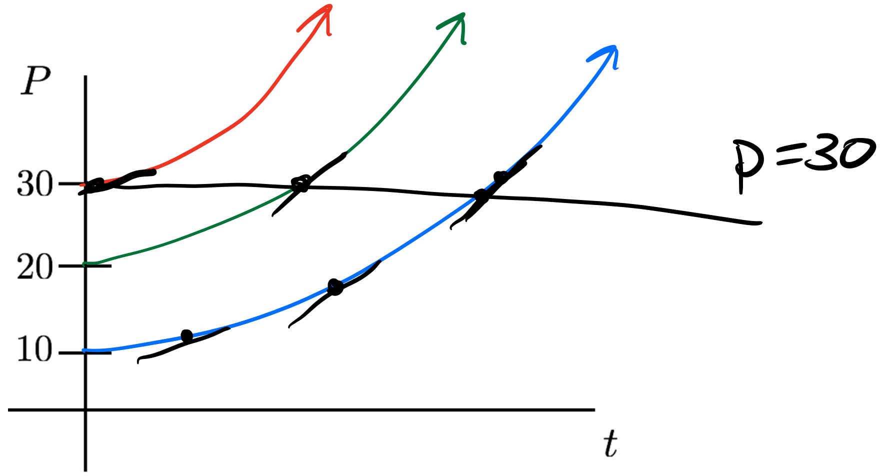
Other explanations are possible.

A Simplified Situation

The previous problem dealt with a complex situation with two interacting species. To develop the ideas and tools that we will need to further analyze complex situations like these, we will simplify the situation by making the following assumptions:

- There is only one species (*e.g.*, fish)
- The species has been in its habitat (*e.g.*, a lake) for some time prior to what we call $t = 0$
- The species has access to unlimited resources (*e.g.*, food, space, water)
- The species reproduces continuously

2. Given these assumptions for a certain lake containing fish, sketch three possible population versus time graphs: one starting at $P = 10$, one starting at $P = 20$, and the third starting at $P = 30$.



(a) For your graph starting with $P = 10$, how does the slope vary as time increases? Explain.

From the figure above we can see the slope is increasing as $t \to \infty$ since as the population of fish grows, there are more interactions, and population grows more and more rapidly

(b) For a set P value, say $P = 30$, how do the slopes vary across the three graphs you drew?

As you move to the right along the line $P=30$ the slopes get steeper.

3. This situation can also be modeled with a rate of change equation, $\frac{dP}{dt} = \textit{something}$. What should the "something" be? Should the rate of change be stated in terms of just P, just t, or both P and t? Make a conjecture about the right hand side of the rate of change equation and provide reasons for your conjecture.

Answers may vary. But we observed in 2b that as P increases dP/dt increases. So should depend on P.

What Exactly is a Differential Equation and What are Solutions?

A **differential equation** is an equation that relates an unknown function to its derivative(s). Suppose $y = y(t)$ is some unknown function, then a differential equation, would express the rate of change, $\frac{dy}{dt}$, in terms of y and/or t. For example, all of the following are *first order* differential equations.

$$\frac{dP}{dt} = kP, \qquad \frac{dy}{dt} = y + 2t, \qquad \frac{dy}{dt} = t^2 + 5, \qquad \frac{dy}{dt} = \frac{6y-2}{ty}, \qquad \frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$$

Given a differential equation for some unknown function, **solutions** to this rate of change equation are *functions* that satisfies the rate change equation.

One way to read the differential equation $\frac{dy}{dt} = y + 2t$ aloud you would say, "dee y dee t equals y plus two times t." However, this does **not** relate to the *meaning* of the solution.

4. (a) Is the function $y = 1 + t$ a solution to the differential equation $\dfrac{dy}{dt} = \dfrac{y^2-1}{t^2+2t}$? How about the function $y = 1 + 2t$? How about $y = 1$? Explain your reasoning.

$y=1+t$ Left side $dy/dt = 1$. Right side $\dfrac{(1+t)^2-1}{t^2+2t} = \dfrac{t^2+2t}{t^2+2t} = 1$. Yes!

$y=1+2t$ Left side $dy/dt = 2$. Right side $\dfrac{(1+2t)^2-1}{t^2+2t} = \dfrac{4t^2+4t}{t^2+2t} \neq 2$. No!

$y=1$. Left side $\dfrac{dy}{dt} = 0$. Right side $\dfrac{(1)^2-1}{t^2+2t} = \dfrac{0}{t^2+2t} = 0$. Yes!

(b) Is the function $y = t^3 + 2t$ a solution to the differential equation $\dfrac{dy}{dt} = 3y^2 + 2$? Why or why not?

Left side: $dy/dt = 3t^2+2$

Right side: $3(t^3+2t)+2 = 3t^3+6t+2$.

Since Left $\neq$ Right, No!

OR. If y is a polynomial, then dy/dt must have degree less than y.

5. Figure out all the functions that satisfy the rate of change equation $\dfrac{dP}{dt} = 0.3P$.

What functions have derivatives that are constant multiples of the original function? Exponentials. $d/dt\left(e^{kt}\right) = k e^{kt}$

In this case, $k = 0.3$. So $P = e^{0.3t}$ is a solution.

So is any function of the form $P = Ce^{0.3t}$.

6. Figure out all of the solutions to the differential equation $\dfrac{dy}{dt} = t^2 + 5$.

We can integrate both sides with respect to t.

$\int\left(\frac{dy}{dt}\right) dt = y + A$

$\int(t^2+5)dt = \frac{1}{3}t^3 + 5t + B$

Thus

$y + A = \frac{1}{3}t^3 + 5t + B$

$y = \frac{1}{3}t^3 + 5t + (B-A)$

$\boxed{y = \frac{1}{3}t^3 + 5t + C}$