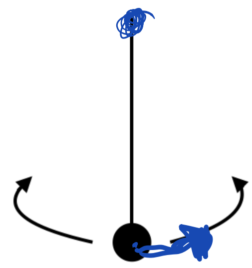
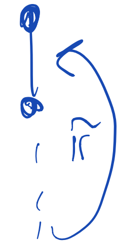
Welcome! Today we'll work on worksheet 19.

In the Swing of Things

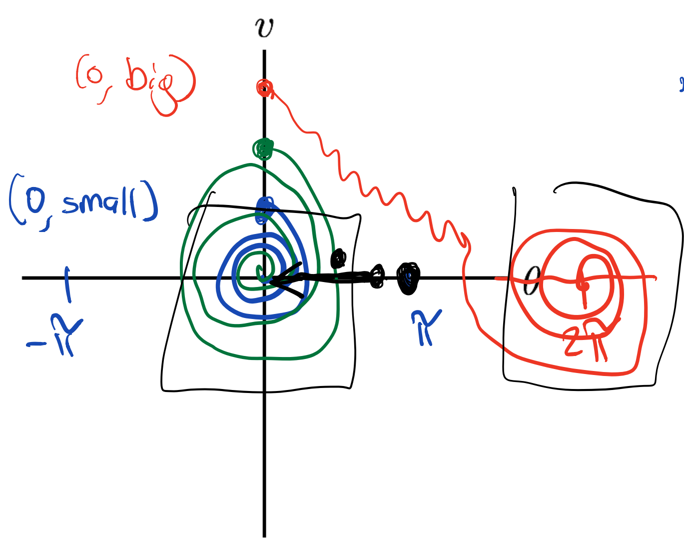
A pendulum is attached to a wall in such a way that it is free to rotate around in a complete circle. Without provocation, Debra takes a baseball bat and hits it, giving it an initial velocity and setting it in motion.

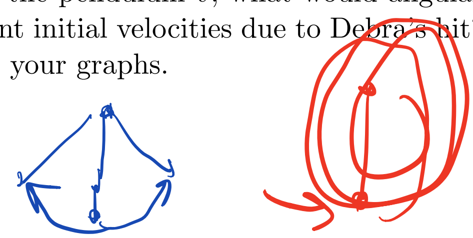


$\theta = 0$



1. If we call θ the angular position of the pendulum (where $\theta = 0$ corresponds to when the pendulum is hanging straight down) and we call the velocity of the pendulum v, what would angular position versus velocity graphs look like for a variety of different initial velocities due to Debra's hit? Provide a brief description of the motion of the pendulum for your graphs.





$x' = ax + by$

$y' = cx + dy$

$(0,0)$

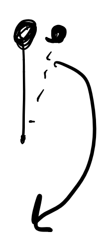
2. How many equilibrium solutions are there, where are they, and how would you classify them?

$(0,0)$ no velocity, will stay at $(0,0)$ as $t \to \infty$.

stable

$(\pi, 0) = (\theta, v)$

unstable



Applying Newton's 2nd Law of motion (where $\theta = 0$ corresponds to the downward vertical position and counterclockwise corresponds to positive angles θ) yields the differential equation

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = 0}$$

$v = \frac{d\theta}{dt}$ $\quad \frac{dv}{dt} = \frac{d^2\theta}{dt^2}$

where b is the coefficient of damping, m is the mass of the pendulum, g is the gravity constant, and l is the length of the pendulum (see video for a derivation of this equation). Estimating the parameter values for the pendulum that Debra hits and changing this second order differential equation to a system of differential equations yields

$$\boxed{\frac{d\theta}{dt} = v}$$
$$\frac{dv}{dt} = -0.2v - \sin(\theta)$$

$\frac{dv}{dt} + \frac{b}{m}v + \frac{g}{l}\sin\theta = 0$

$dv/dt = -b/m\, v - g/l \sin\theta$

3. How many equilibrium solutions does this system of differential equations have, where are they, and based on the context what types of equilibrium solutions would you expect them to be? How does this connect with your answer to 2?

$\theta' = v = 0 \rightarrow \boxed{v = 0}$

$v' = -0.2v - \sin\theta = 0$

$\boxed{(k\pi, 0)}$ ← infinite number of equilibrium

$v' = -\sin\theta = 0 \quad \theta = k\pi$ for k integer

4. You might recall that if θ is small, $\boxed{\sin(\theta) \approx \theta.}$ Explain why this is true and then use this fact to approximate the above system with a linear system and classify the equilibrium solution at the origin.

$\theta' = v$

$v' = -0.2v - \theta$

$\rightarrow$ $\theta' = 0\theta + v$

$v' = -\theta - 0.2v$

$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix}$

5. Classify the equilibrium point at $\theta = \pi$.

Is $(0,0)$ stable?

$\det \begin{bmatrix} 0-\lambda & 1 \\ -1 & -0.2-\lambda \end{bmatrix} = 0$

$-\lambda(-0.2-\lambda) + 1 = 0$

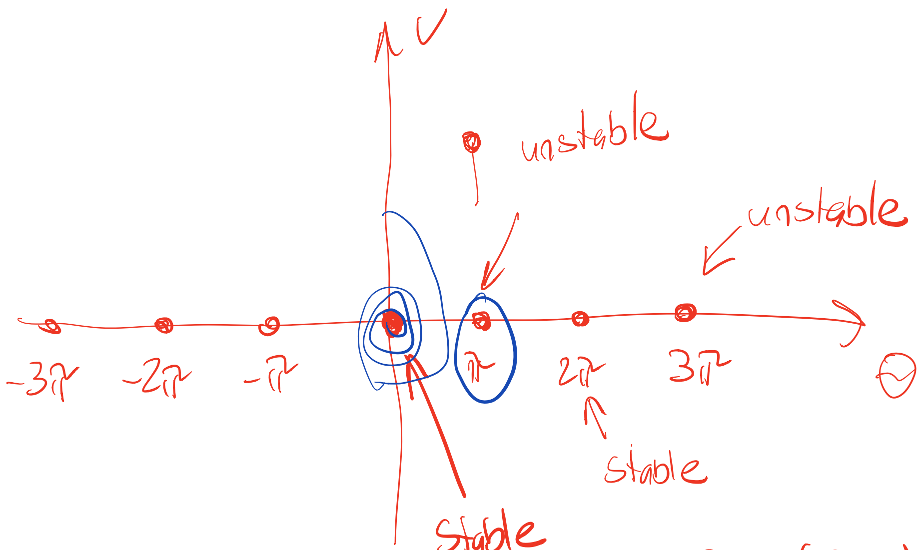
$\lambda^2 + 0.2\lambda + 1 = 0$

$\lambda = \frac{-0.2 \pm \sqrt{0.04 - 4}}{2}$

$\lambda = \alpha \pm \beta i$ for $\alpha < 0$

6. Use the GeoGebra applet, https://ggbm.at/SpfDSc5Q, to approximate the range of initial velocities with zero initial displacement that will result in the pendulum making exactly one complete rotation before eventually coming to rest.

Spiral sink

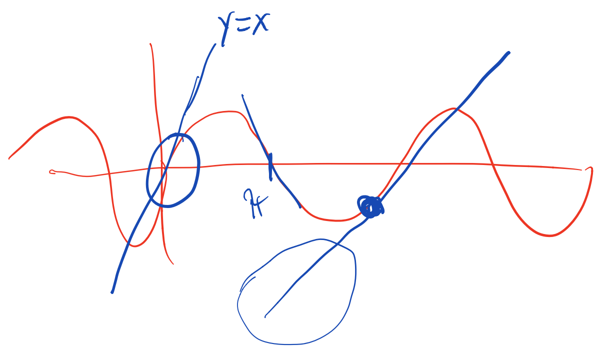


$(\theta, v) \approx (0,0)$

Do initial conditions $\boxed{\text{very close}}$ to the equilibrium head towards the equilibrium or away?

Stable

unstable



$y = \sin x$

$L(x) = f'(a)(x-a) + f(a)$

Linearization and Linear Stability Analysis

We can perform **linear stability analysis** on a system of two or more variables, such as the one in the previous pendulum problem. Consider a function $f(x, y)$, then recall from multivariable calculus we find a formula for the linearization of $f(x, y)$ around the point (x_0, y_0) using the formula:

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus for points $(x, y) = (x_0 + u, y_0 + v)$ close to (x_0, y_0) we can rewrite the linearization as

$$f(x, y) \approx L(u, v) = f(x_0, y_0) + f_x(x_0, y_0)u + f_y(x_0, y_0)v.$$

① Find Equilibrium

7. Consider the following system:

$f(x,y) = 1 - x^2$

$f_x(x,y) = -2x$

$f_y(x,y) = 0$

$$\frac{dx}{dt} = \underbrace{1 - x^2}_{f(x,y)} = 0$$
$$\frac{dy}{dt} = \underbrace{-3x - 3y}_{g(x,y)} = 0$$

Note the system has two equilibria: $(1, -1)$ and $(-1, 1)$. Let's first study the equilibrium at (1,-1).

Near this equilibrium we can write $x = 1 + u$ and $y = -1 + v$ for small u and v. Use the linearization of the original system of equations around $(1, -1)$ to write down a system of differential equations for u and v $g(x,y) = -3x - 3y$ $g_x(x,y) = -3$ $g_y(x,y) = -3$ $g(1,-1) = 0$	$L_f(x,y) = f_x(1,-1)(x-1) + f_y(1,-1)(y+1) + f(1,-1)$ $= -2(x-1) + 0(y+1) + 0$ $L_f(x,y) = -2x + 2 = -2(1+u) + 2 = -2 - 2u + 2$ $\boxed{L_f = -2u}$ $L_g(x,y) = -3(x-1) - 3(y+1)$ $= -3x + 3 - 3y - 3$ $= -3x - 3y$ $\boxed{L_g = -3(1+u) - 3(-1+v) = -3u - 3v}$
Classify the equilibrium point $(1, -1)$, using techniques from linear algebra. $x = 1 + u \rightarrow \frac{dx}{dt} = \frac{du}{dt}$ $y = -1 + v$ $\frac{dy}{dt} = \frac{dv}{dt}$	$\frac{du}{dt} = -2u$ $\frac{dv}{dt} = -3u - 3v$ $\rightarrow$ $\begin{bmatrix} -2 & 0 \\ -3 & -3 \end{bmatrix}$

$\boxed{\text{Ex}}$ $\frac{dx}{dt} = 1 - x^2 = f(x,y)$

$dy/dt = -3x - 3y = g(x,y)$

$$\text{Jacobian} = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} -2x & 0 \\ -3 & -3 \end{bmatrix}$$

at $(1,-1)$

$$\begin{bmatrix} -2 & 0 \\ -3 & -3 \end{bmatrix}$$

at $(-1,1)$

$$\begin{bmatrix} 2 & 0 \\ -3 & -3 \end{bmatrix}$$

$$\frac{dx}{dt} = 1 - x^2$$

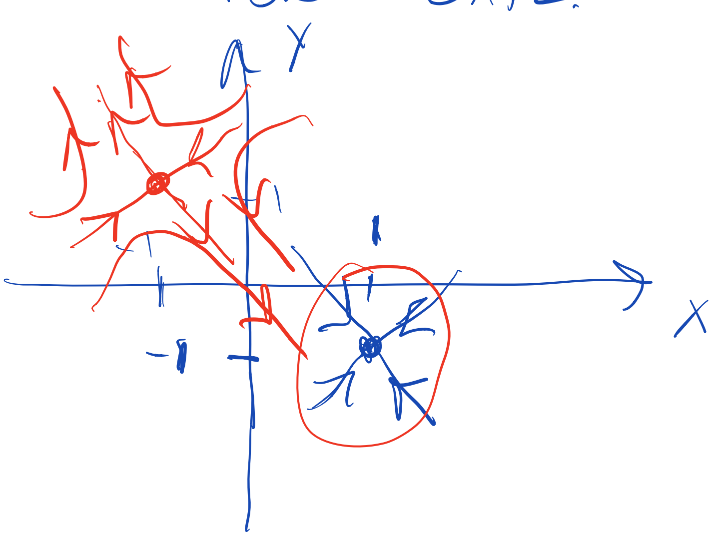
$$\frac{dy}{dt} = -3x - 3y$$

$$\Longrightarrow \boxed{\begin{aligned} \frac{du}{dt} &= -2u \\ \frac{dv}{dt} &= -3u - 3v \end{aligned}}$$

Good Approx as long as $(u,v) \approx (0,0)$

$$A = \begin{bmatrix} -2 & 0 \\ -3 & -3 \end{bmatrix} \rightarrow \boxed{\lambda = -3, -2}$$

So Equilibrium at $(1,-1)$ is a stable sink.



8. (a) Consider again

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

Use linear stability analysis to classify the equilibrium point at $(-1, 1)$.

① Find Equil

② Compute Jacobian

③ Evaluate Jacobian at each equilibrium

④ Find eigenvalues

② Introduce $\boxed{x = -1 + u \quad y = 1 + v}$

③ Find L_f and L_g at $(-1,1)$

$f(x,y) = 1 - x^2 \qquad f_x(x,y) = -2x \qquad f_y(x,y) = 0$

$f_x(-1,1) = 2 \qquad f_y(-1,1) = 0$

$f(-1,1) = 0$

$L_f(x,y) = 2(x+1) + 0(y-1) + 0 = 2x + 2$

$L_g(x,y) = -3(x+1) - 3(y-1) + 0$

④ Express L_f and L_g in terms of u and v.

(b) Combine your results from question 7 and 8a, to sketch a possible phase plane for the system of differential equations. Does an analysis of the system using nullclines corroborate your linear stability analysis?

$L_f = 2(-1+u) + 2 = 2u$

$L_g = -3((-1+u)+1) - 3((1+v)-1) + 0 = -3u - 3v$

⑤ $\frac{du}{dt} = 2u$ $\qquad A = \begin{bmatrix} 2 & 0 \\ -3 & -3 \end{bmatrix}$ $\qquad \det \begin{bmatrix} 2-\lambda & 0 \\ -3 & -3-\lambda \end{bmatrix} = 0$

$\frac{dv}{dt} = -3u - 3v$

$\lambda = 2, -3 \leftarrow$ saddle

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

- Find Equilibrium
- Linearize at each equilibrium
- Identify type of stability

① Find equilibrium: Solve $x' = 1 - x^2 = 0$, $y' = -3x - 3y = 0$ $\boxed{(1,-1) \text{ and } (-1,1)}$

② Linearize at each equilibrium:

Calc way: $x' = f(x,y) = 1 - x^2$, $f_x = -2x$ $f_y = 0$; | $y' = g(x,y) = -3x - 3y$ $g_x = -3$, $g_y = -3$

At $(1,-1)$, $f_x(1,-1) = -2$, $f_y(1,-1) = 0$, $f(1,-1) = 0$ | At $(1,-1)$ $g_x(1,-1) = g_y(1,-1) = -3$ $g(1,-1) = 0$

$L_f(x,y) = -2(x-1) + 0(y+1) + 0 = -2x + 2$ | $L_g(x,y) = -3(x-1) - 3(y+1) + 0 = -3x - 3y$

Let $x = 1 + u$, $y = -1 + v$ and rewrite linearized system in terms of u and v.

At $(1,-1)$: $dx/dt = 1 - x^2 \approx -2x + 2 \Longrightarrow du/dt = -2(1+u) + 2 = -2u + 0v$

$dy/dt = -3x - 3y \approx -3x - 3y \Longrightarrow dv/dt = -3(1+u) - 3(-1+v) = -3u - 3v$

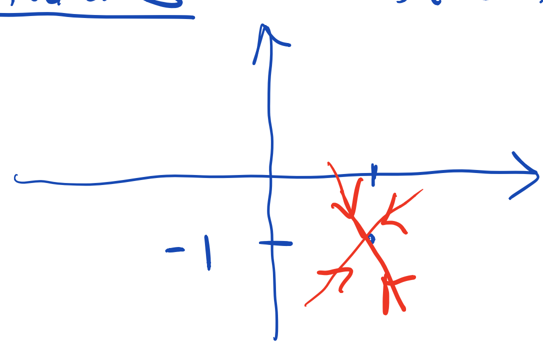
$$\boxed{\begin{aligned} du/dt &= -2u \\ dv/dt &= -3u - 3v \end{aligned}}$$

OR using Jacobian $J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$

$$J = \begin{bmatrix} -2 & 0 \\ -3 & -3 \end{bmatrix}$$

$\boxed{(1,-1) \text{ is a stable sink}}$

③ Find eigenvalues: $(-2-\lambda)(-3-\lambda) = 0$ $\lambda = -3, -2$



$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

at $(-1,1)$

$f_x(x,y) = -2x \qquad f_x(-1,1) = 2$

$f_y(x,y) = 0 \qquad f_y(-1,1) = 0$

$g_x(x,y) = -3 \qquad g_x(-1,1) = -3$

$g_y(x,y) = -3 \qquad g_y(-1,1) = -3$

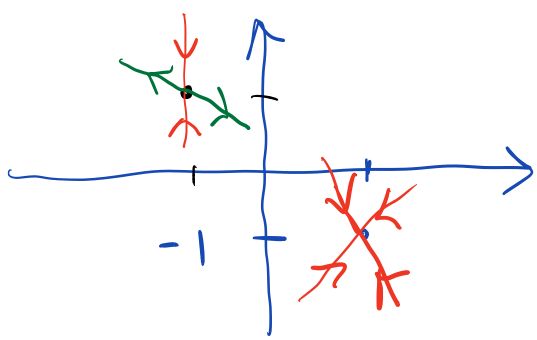
$$J(-1,1) = \begin{bmatrix} 2 & 0 \\ -3 & -3 \end{bmatrix}$$

$p(\lambda) = (2-\lambda)(-3-\lambda) = 0$

$\boxed{\lambda = 2, -3 \quad \text{saddle}}$

$\lambda = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -3/5 \end{bmatrix}$

$\lambda = -3 \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



9. For a system of differential equations

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with an equilibrium point at (x^*, y^*), the matrix

$$J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$$

is called the **Jacobian matrix**. Explain how you can use the Jacobian matrix to determine the behavior of a the system of differential equations near (x^*, y^*).

In #3 we found pendulum equilibrium $(k\pi, 0)$ for $k \in \mathbb{Z}$.
To linearize, we find the Jacobian

$f_\theta(\theta, v) = 0$

$f_v(\theta, v) = 1$

$g_\theta(\theta, v) = -\cos\theta$

$g_v(\theta, v) = -0.2$

$$J(\theta, v) = \begin{bmatrix} 0 & 1 \\ -\cos\theta & -0.2 \end{bmatrix}$$

10. Use linear stability analysis to classify the critical points you found in the pendulum system.

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -0.2v - \sin(\theta)$$

If $k = 0, \pm 2, \pm 4, \ldots$ then

$$J(k\pi, 0) = \begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -\theta - 0.2v \end{aligned}}$$

$\frac{\sqrt{0.04-4}}{2}$

$p(\lambda) = (-\lambda)(-0.2-\lambda) + 1 = 0$

$\lambda = -0.1 \pm \frac{\sqrt{3.96}}{2} i$

If $k = \pm 1, \pm 3, \ldots$

$$J(k\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= \theta - 0.2v \end{aligned}}$$

$p(\lambda) = (-\lambda)(-0.2-\lambda) - 1 = 0$

$\lambda \approx -1.1$ and 0.9

- If k is even, then the equilibrium $(k\pi, 0)$ is a spiral sink

- If k is odd, then the equilibrium $(k\pi, 0)$ is a saddle point.

Example: $\frac{dx}{dt} = 4x - 2xy = 0$ $\quad x(4-2y)=0$ $\quad \boxed{x=0}$ or $\boxed{y=2}$

$\frac{dy}{dt} = 6y + 2xy = 0$

① Find equilibrium

② Give the linearization of the system at each equilibrium

③ Classify each equilibrium point.

$$4x - 2xy = 2x(2-y) = 0$$

$x=0$ or $y=2$

If $x=0$, $\frac{dy}{dt} = 6y = 0$ if $y=0$

If $y=2$, $\frac{dy}{dt} = 12 + 4x = 0$ if $x=-3$

$$\boxed{\begin{matrix}(0,0)\\(-3,2)\end{matrix}}$$

② $f_x(x,y) = 4-2y$

$f_y(x,y) = -2x$

$g_x(x,y) = 2y$

$g_y(x,y) = 6+2x$

$$J(x,y) = \begin{bmatrix} 4-2y & -2x \\ 2y & 6+2x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \qquad \lambda = 4, 6 \qquad \boxed{\begin{matrix}\frac{du}{dt} = 4u \\ \frac{dv}{dt} = 6v\end{matrix}}$$

$(0,0)$ is an unstable source

$$J(-3,2) = \begin{bmatrix} 0 & 6 \\ 4 & 0 \end{bmatrix} \qquad (-\lambda)(-\lambda) - 24 = 0$$

$$\lambda^2 - 24 = 0 \qquad \lambda = \pm\sqrt{24}$$

$$\boxed{\begin{aligned} \frac{du}{dt} &= 6v \\ \frac{dv}{dt} &= 4u \end{aligned}}$$

$\boxed{(-3,2) \text{ is a saddle}}$