$$\mathcal{L}\{3t^2y'' + 2\cos(t)\,y' + 3y\} = \mathcal{L}\{e^{5t}\}$$

Properties of the Laplace Transform

1. Let f, f_1, and f_2 be functions whose Laplace transform exists for $s > \alpha$ and let c be a constant. Then for $s > \alpha$, prove the following:

   (a) $\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\}$.

$$\mathcal{L}\{f_1+f_2\} = \int_0^\infty e^{-st}\left(f_1(t)+f_2(t)\right)dt$$

$$= \int_0^\infty \left(e^{-st}f_1(t) + e^{-st}f_2(t)\right)dt$$

$$= \int_0^\infty e^{-st} f_1(t) + \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$$

   (b) $\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$.

$$\mathcal{L}\{cf\} = \int_0^\infty e^{-st}\left(cf(t)\right)dt = c\int_0^\infty e^{-st} f(t)\,dt$$

$$= c\,\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{2\cos(3t) + 5 - 4e^{6t}\} = \frac{6}{s^2+9} + \frac{5}{s} - \frac{4}{s-6}, \quad s>6$$

$$2\mathcal{L}\{\cos(3t)\} + 5\mathcal{L}\{1\} - 4\mathcal{L}\{e^{6t}\}$$

Referring to Table

$$2\left(\frac{3}{s^2+3^2}\right) + \frac{5}{s} - 4\,\frac{1}{s-6}$$

$s>0 \qquad s>0 \qquad s>6$

$e^{-st+at} = e^{-t(s-a)}$

Laplace Transform of $g(t) = e^{at}f(t)$

1. If the Laplace transform $\boxed{\mathscr{L}\{f\}(s) = F(s) \text{ exist for } s > \alpha}$, then show that
$$\mathscr{L}\{e^{at}f(t)\} = F(s-a), \quad \text{for } s > \alpha + a.$$

$\boxed{u = (s-a)}$

$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-t(s-a)}f(t)\,dt$, with $u = (s-a)$

$= \int_0^\infty e^{-t\cdot u} f(t)\,dt \iff \int_0^\infty e^{-st}f(t)\,dt = F(s)$

$= F(u) \quad u > \alpha \quad\quad s > \alpha$

$\boxed{= F(s-a) \quad s-a > \alpha \quad s > \alpha + a}$

$\mathscr{L}\{e^{at}f(t)\}$ → Horizontal Shift of $\mathscr{L}\{f(t)\}$

2. Using the property above and the fact that $\boxed{\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}}$ for $s > 0$, find $\mathscr{L}\{e^{at}\cos(bt)\}$.

$\boxed{\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0}$

$F(s) = \frac{s}{s^2+b^2}$

$\mathscr{L}\{e^{at}\cos(bt)\} = F(s-a)$

$\boxed{= \frac{s-a}{(s-a)^2+b^2}, \quad s > a}$

Section 6.2: Laplace Transform of Derivatives

A function is of **exponential order** α if there exists positive constants C and T such that

$$|f(t)| < Ce^{\alpha t} \quad \text{for all } t > T.$$

$\alpha = 1$

For example:

- $f(t) = \cos(5t)e^{7t}$ has $\alpha = 7$.
- e^{t^2} does not have an exponential order.

$\boxed{f(t) = t} < e^t$

3. If $f(t)$ is continuous on $[0, \infty)$ and $f'(t)$ is piecewise continuous on $[0, \infty)$ with both exponential order α, then prove for $s > \alpha$,

$$\mathscr{L}\{f'\} = s\mathscr{L}\{f\} - f(0) = sF(s) - f(0).$$

$u = e^{-st}$ $\quad u' = -se^{-st}$

$v' = f'(t)dt$ $\quad v = f(t)$

I.B.P.

$$\mathscr{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st} f'(t)}_{uv} dt = \lim_{N\to\infty} \int_0^N e^{-st} f'(t)\,dt$$

$$= \left(\lim_{N\to\infty} e^{-st} f(t)\Big|_0^N\right) - \underbrace{\lim_{N\to\infty}\int_0^N -se^{-st} f(t)\,dt}_{-\int u'v}$$

?

$$\boxed{\left(\lim_{N\to\infty} e^{-sN} f(N)\right)} - \big(f(0)\big) + s\boxed{\int_0^\infty e^{-st} f(t)\,dt}$$

$\mathscr{L}\{f(t)\} = F(s)$

$$\lim_{N\to\infty} f(N)e^{-sN} \le \lim_{N\to\infty} \left|e^{-sN} f(N)\right| < \lim_{N\to\infty} e^{-sN} \cdot Ce^{\alpha N} = \lim_{N\to\infty} Ce^{-N(s-\alpha)}$$

$\to 0$

$s - \alpha > 0$

$f(t)$ is exponential order α

$|f(N)| < Ce^{\alpha t}$

$\mathscr{L}\{f(t)\} = F(s)$

$$\boxed{\left(\lim_{N\to\infty} e^{-sN} f(N)\right)} - \big(f(0)\big) + s\boxed{\int_0^\infty e^{-st} f(t)\,dt}$$

$s > \alpha$

$$\mathscr{L}\{f'(t)\} = 0 \quad -f(0) + sF(s) = \boxed{sF(s) - f(0)} \quad \checkmark$$

4. Using the property from problem 3 and the fact that $\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$ for $s > 0$, find $\mathscr{L}\{\sin(bt)\}$.

$-b\sin(bt) = f'(t)$

$\cos(bt) = f(t)$

$\mathscr{L}\{f(t)\} = \mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} \quad s > 0$

$\mathscr{L}\{\sin(bt)\} = ?$

$\mathscr{L}\{f'(t)\} = \mathscr{L}\{-b\sin(bt)\} = sF(s) - f(0) = s\left(\frac{s}{s^2+b^2}\right) - 1$

$= \frac{s^2 - (s^2+b^2)}{s^2+b^2} = \frac{-b^2}{s^2+b^2}$

$\star$ We have

$\mathscr{L}\{-b\sin(bt)\} = \frac{-b^2}{s^2+b^2}$

$-\frac{1}{b}\mathscr{L}\{-b\sin(bt)\} = -\frac{1}{b}\left(\frac{-b^2}{s^2+b^2}\right)$

$\boxed{\mathscr{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}}$

5. If $\mathscr{L}\{f(t)\} = F(s)$ for all $s > \alpha$, using the property from problem 3, show that

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \quad \text{for all } s > \alpha.$$

$g(t) = f'(t)$

$g'(t) = f''(t)$

$\mathscr{L}\{g'(t)\} = s\mathscr{L}\{g(t)\} - g(0)$

$= s\underbrace{\mathscr{L}\{f'(t)\}} - f'(0)$

$= s\left(sF(s) - f(0)\right) - f'(0)$

$\boxed{\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$

6. Using induction show that

$$\mathscr{L}\{f^{(n)}\} = s^n\mathscr{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

Welcome! Today we'll finish 21 and work on 22.

7. Let $F(s) = \mathscr{L}\{f\}$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Prove that for $s > \alpha$ if follows that

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}.$$

<u>See Leibniz Rule</u>: If $I(x) = \int_{u(x)}^{v(x)} f(t,x)\,dt$, then

$$I'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(t,x)\,dt + f(v(x),x)v'(x) - f(u(x),x)u'(x).$$

For $n=1$, $\mathscr{L}\{t f(t)\} = \int_0^\infty e^{-st} t f(t)\,dt$. Let $I(s) = \int_0^\infty e^{-st} f(t)\,dt$. By Leibniz Rule,

$$I'(s) = \int_0^\infty \partial/\partial s\left(e^{-st} f(t)\right)dt + \lim_{N\to\infty}\left(e^{-s\cdot N} f(s)\cdot d/ds(N)\right) - e^{-s\cdot 0} f(s)\cdot d/ds(0)$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt + 0 - 0$$

$= -\mathscr{L}\{t f(t)\}$. Since $I(s) = \mathscr{L}\{f(t)\}$, we have $\mathscr{L}\{t f(t)\} = (-1)^1 \frac{d}{ds}\mathscr{L}\{f(t)\}$.

Then do induction on n.

8. Using the definition of the Laplace transform, the result that $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$ for $s > a$ and the property above, find a formula for $\mathscr{L}\{t^n e^{at}\}$.

For $n=1$: $\mathscr{L}\{t e^{at}\} = (-1)^1 d/ds\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$

$\mathscr{L}\{t^n e^{at}\}$

For $n=2$: $\mathscr{L}\{t^2 e^{at}\} = \mathscr{L}\{t \cdot t e^{at}\} = -1\, d/ds\left(\frac{1}{(s-a)^2}\right)$

$t \cdot f(t)$

$$= \frac{2}{(s-a)^3} = \frac{n!}{(s-a)^{n+1}},\ s > a$$

For $n = k+1$

$$\mathscr{L}\{t^{k+1} e^{at}\} = \mathscr{L}\{t\, t^k e^{at}\} = -1\, d/ds\left(\frac{k!}{(s-a)^{k+1}}\right) = \frac{(k+1)!}{(s-a)^{k+2}}$$

Find $\mathcal{L}\{t^2 \sin(2t)\}$ By Property L.5

We know $\mathcal{L}\{t^2 f(t)\} = (-1)^2 \cdot \frac{d^2}{ds^2}\left(\mathcal{L}\{f(t)\}\right)$

$n=2$

$f(t) = \sin(2t)$ $\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$, $s>0$

$\frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$

$$\mathcal{L}\{t^2 \sin(2t)\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$n=2$

Quotient Rule

Welcome! Today we'll start with worksheet 21.

Common Laplace Transforms

$f(t)=e^{7t}$ $F(s)=\frac{1}{s-7}, s>7$

$f(t)=\sin(3t)$

$F(s)=\frac{3}{s^2+9}$ $s>0$

$f(t)=\cos(3t)$

$F(s)=\frac{s}{s^2+9}$ $s>0$

$f(t)$	$F(s)=\mathscr{L}\{f(t)\}$
1	$\frac{1}{s},\ s>0$
e^{at}	$\frac{1}{s-a},\ s>a$
$t^n,\ n=1,2,\ldots$	$\frac{n!}{s^{n+1}},\ s>0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s>0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s>0$
$e^{at}t^n,\ n=1,2,\ldots$	$\frac{n!}{(s-a)^{n+1}},\ \cancel{s>0}$ $s>a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s>a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s>a$

$f(t)=t^2$

$n=2$

$F(s)=\frac{2!}{s^3}=\frac{2}{s^3},$ $s>0$

$f(t)=t^3$

$n=3$

$F(s)=\frac{3!}{s^{3+1}},\ s>0$

$F(s)=\frac{6}{s^4},\ s>0$

Common Properties of Laplace Transforms

L.1 $\mathscr{L}\{cf(t)\}=c\mathscr{L}\{f(t)\}$, where c is a constant.

L.2 $\mathscr{L}\{f_1(t)+f_2(t)\}=\mathscr{L}\{f_1(t)\}+\mathscr{L}\{f_2(t)\}$

L.3 If $F(s)=\mathscr{L}\{f(t)\}$ exists for all $s>\alpha$, then $\mathscr{L}\left\{e^{at}f(t)\right\}=F(s-a)$ for all $s>\alpha+a$.

L.4 If $F(s)=\mathscr{L}\{f(t)\}$ exists for all $s>\alpha$, then for all $s>\alpha$,

$$\mathscr{L}\left\{f^{(n)}(t)\right\}=s^n\mathscr{L}\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0).$$

L.5 If $F(s)=\mathscr{L}\{f(t)\}$ exists for all $s>\alpha$, then $\mathscr{L}\{t^nf(t)\}=(-1)^n\frac{d^nF}{ds^n}$ for all $s>\alpha$.

Monday Reading