Properties of the Laplace Transform

1. Let f, f_1, and f_2 be functions whose Laplace transform exists for $s > \alpha$ and let c be a constant. Then for $s > \alpha$, prove the following:

   (a) $\underbrace{\mathscr{L}\{f_1 + f_2\}}_{\text{Start}} = \underbrace{\mathscr{L}\{f_1\} + \mathscr{L}\{f_2\}}_{\text{End}}$.

$$\mathscr{L}\{f_1+f_2\} = \int_0^\infty e^{-st}(f_1(t)+f_2(t))\,dt \quad \text{using def}$$

$$= \int_0^\infty \left(e^{-st}\cdot f_1(t) + e^{-st} f_2(t)\right)dt \quad \text{distributing}$$

$$= \underbrace{\int_0^\infty e^{-st}\cdot f_1(t)\,dt} + \underbrace{\int_0^\infty e^{-st} f_2(t)\,dt} \quad \text{Prop of Integrals}$$

$$= \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\} \quad \blacksquare \quad \text{using def}$$

   (b) $\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$.

Using these properties and table of Laplace Transforms

Find

$$\mathscr{L}\{2\cos(3t) + 5t^2 - 4e^{6t}\}$$

$$= 2\mathscr{L}\{\cos(3t)\} + 5\mathscr{L}\{t^2\} - 4\mathscr{L}\{e^{6t}\} \quad \text{using Linearity}$$

$$F(s) = 2\left(\frac{s}{s^2+9}\right) + 5\left(\frac{2}{s^3}\right) - 4\left(\frac{1}{s-6}\right), \quad \boxed{s > 6}$$

$s>0 \qquad s>0 \qquad s>6$

Laplace Transform of $g(t) = e^{at} f(t)$

1. If the Laplace transform $\boxed{\mathscr{L}\{f\}(s) = F(s)}$ exist for $s > \alpha$, then show that

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a), \quad \boxed{\text{for } s > \alpha + a.}$$

multiplication by e^{at} acts as a translation to the right by a units

$u = s - a$

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} \cdot f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-ut} f(t)\,dt$$

$$= F(u)$$

$$= F(s-a)$$

$$= F(s-a)$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

2. Using the property above and the fact that $\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$ for $\boxed{s > 0}$, find $\mathscr{L}\{e^{at} \underbrace{\cos(bt)}_{f(t)}\}$.

$$\mathcal{L}\{e^{at}\cos(bt)\} = F(s-a) = \frac{(s-a)}{(s-a)^2 + b^2}$$

$s - a > 0$

$\boxed{s > a}$

$g(x) = e^x$

$g(x+h) = e^{x+h}$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\cos(bt)\} = \frac{(s)}{(s)^2 + b^2}$$

$s > 0$

① First find $\mathcal{L}\{\cos(bt)\}$

$\mathcal{L}\{e^{at}\cos(bt)\}$

② Then shift right by a units.

Section 6.2: Laplace Transform of Derivatives

A function is of **exponential order** α if there exists positive constants C and T such that

$$\boxed{|f(t)| < Ce^{\alpha t}} \text{ for all } t > T.$$

For example:

- $f(t) = \cos(5t)e^{7t}$ has $\alpha = 7$. $\quad |\cos(5t)e^{7t}| < 2e^{7t}$
- e^{t^2} does not have an exponential order.

3. If $f(t)$ is continuous on $[0, \infty)$ and $f'(t)$ is piecewise continuous on $[0, \infty)$ with both exponential order α, then prove for $s > \alpha$,

$$\boxed{\mathscr{L}\{f'\} = s\mathscr{L}\{f\} - f(0) = sF(s) - f(0).}$$

Start

$u = e^{-st} \quad u' = -se^{-st}$

$v' = f' \quad v = f$

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \int uv'\,dt$$

$$= uv - \int u'v$$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} \cdot f(t)\,dt$$

$$= e^{-st}f(t) + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= \underbrace{e^{-st}f(t)\Big|_{t=0}^{t=\infty}}_{} + s\mathscr{L}\{f(t)\} = \underbrace{0 - f(0)}_{} + s\mathscr{L}\{f(t)\}$$

$$\boxed{\lim_{N\to\infty} \left(e^{-sN}f(N)\right) - e^{0}f(0) = 0 - f(0) \quad s > \alpha}$$

Since f has exp order α

$$|f(N)| < Ce^{\alpha \cdot N}$$

$$\lim_{N\to\infty} e^{-sN}f(N) < \lim_{N\to\infty} \left(e^{-s\cdot N} \cdot e^{\alpha \cdot N}\right) = C\lim_{N\to\infty} e^{-(s-\alpha)\cdot N} = 0$$

as long as $s - \alpha > 0$ or $s > \alpha$

4. Using the property from problem 3 and the fact that $\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$ for $s > 0$, find $\mathscr{L}\{\sin(bt)\}$.

$$\mathscr{L}\{\underbrace{f(t)}\} = \mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} = F(s) \qquad f(t) = \cos(bt)$$

$$\mathscr{L}\{f'(t)\} = \mathscr{L}\{-b\sin(bt)\} = sF(s) - f(0)$$
$$= s\left(\frac{s}{s^2+b^2}\right) - 1$$
$$= \frac{s^2 - (s^2+b^2)}{s^2+b^2}$$

$$-b\mathscr{L}\{\sin(bt)\} = \mathscr{L}\{-b\sin(bt)\} = \frac{-b^2}{s^2+b^2} \qquad \mathscr{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$$

5. If $\mathscr{L}\{f(t)\} = F(s)$ for all $s > \alpha$, using the property from problem 3, show that

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \quad \text{for all } s > \alpha.$$

$$\boxed{\mathscr{L}\{f'\} = sF(s) - f(0)}$$

$$f''(t) = \frac{d}{dt}(?)$$
$$f''(t) = (f'(t))'$$
Let $g(t) = f'(t)$
Let $\mathscr{L}\{g(t)\} = G(s)$

$$\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\}$$
$$= \mathscr{L}\{g'(t)\}$$
$$= sG(s) - g(0)$$
$$= s\underbrace{\mathscr{L}\{f'(t)\}} - \underbrace{f'(0)}$$
$$= s(\underbrace{sF(s) - f(0)}) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

6. Using induction show that

$$\mathscr{L}\{f^{(n)}\} = s^n\mathscr{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

7. Let $F(s) = \mathscr{L}\{f\}$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Prove that for $s > \alpha$ if follows that

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}.$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}.$$

8. Using the definition of the Laplace transform, the result that $\boxed{\mathscr{L}\{e^{at}\} = \frac{1}{s-a}}$ for $s > a$ and the property above, find a formula for $\mathscr{L}\{t^n e^{at}\}$.

$$\mathcal{L}\{t^n e^{at}\} = (-1)^n \frac{d^n}{ds^n}(F(s))$$

$\mathcal{L}\{e^{at}\}$

By L.5

$n=1$ $\quad -1 \cdot \frac{d}{ds}\left(\frac{1}{s-a}\right) = -1 \frac{d}{ds}\left[(s-a)^{-1}\right] = (s-a)^{-2}$

$n=2$ $\quad (-1)^2 \cdot \frac{d^2}{ds^2}\left[(s-a)^{-1}\right] = \frac{d}{ds}\left[-(s-a)^{-2}\right] = 2(s-a)^{-3}$

$n=3$ $\quad (-1)^3 \cdot \frac{d^3}{ds^3}\left[(s-a)^{-1}\right] = -1 \frac{d}{ds}\left(2(s-a)^{-3}\right) = 3 \cdot 2 (s-a)^{-4}$

$= 3!(s-a)^{-4}$

⋮

General

Case n

Common Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$

Last time

via Property

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Common Properties of Laplace Transforms

L.1 $\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}$, where c is a constant. ✓

L.2 $\mathscr{L}\{f_1(t) + f_2(t)\} = \mathscr{L}\{f_1(t)\} + \mathscr{L}\{f_2(t)\}$ ✓

✓ L.3 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$ for all $s > \alpha + a$. ✓

L.4 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then for all $s > \alpha$, ✓ $n=1,2$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

$n=1$ $\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0)$

$n=2$ $\mathscr{L}\{y''(t)\} = s^2\mathscr{L}\{y(t)\} - s\,y(0) - y'(0)$

L.5 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^nF}{ds^n}$ for all $s > \alpha$.