Rabbits and Foxes

Most species live in interaction with other species. For example, perhaps one species preys on another species, like foxes and rabbits. Below is a **system of rate of change equations** intended to predict future populations of rabbits and foxes over time, where R is the population (in hundreds) of rabbits at any time t and F is the population of foxes (in tens) at any time t (in years).

Rabbits pop grows if no foxes

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

Interactions with fox cause Rabbit pop. to decrease

1. (a) In earlier work with the rate of change equation $\frac{dP}{dt} = kP$ we assumed that there was only one species, that the resources were unlimited, and that the species reproduced continuously. Which, if any, of these assumptions is modified and how is this modification reflected in the above system of differential equations?

More than one species. Not unlimited resources. Rabbits population is affected by interaction with foxes. Without any rabbits, foxes will decrease.

(b) Interpret the meaning of *each* term in the rate of change equations (e.g., how do you interpret or make sense of the $-1.4RF$ term) and what are the implications of this term on the future predicted populations? Similarly for $3R$, $-F$, and $0.8RF$.

See above for dR/dt. For dF/dt, $-F$ means fox pop. decreases without rabbits. $0.8RF$ means fox pop. increases when rabbits present

2. Scientists studying a rabbit-fox population estimate that the current number of rabbits is 100 ($R = 1$) and that the number of foxes is 10 ($F = 1$). Use two steps of Euler's method with step size of $\Delta t = 0.5$ to get numerical estimates for the future number of rabbits and foxes as predicted by the differential equations.

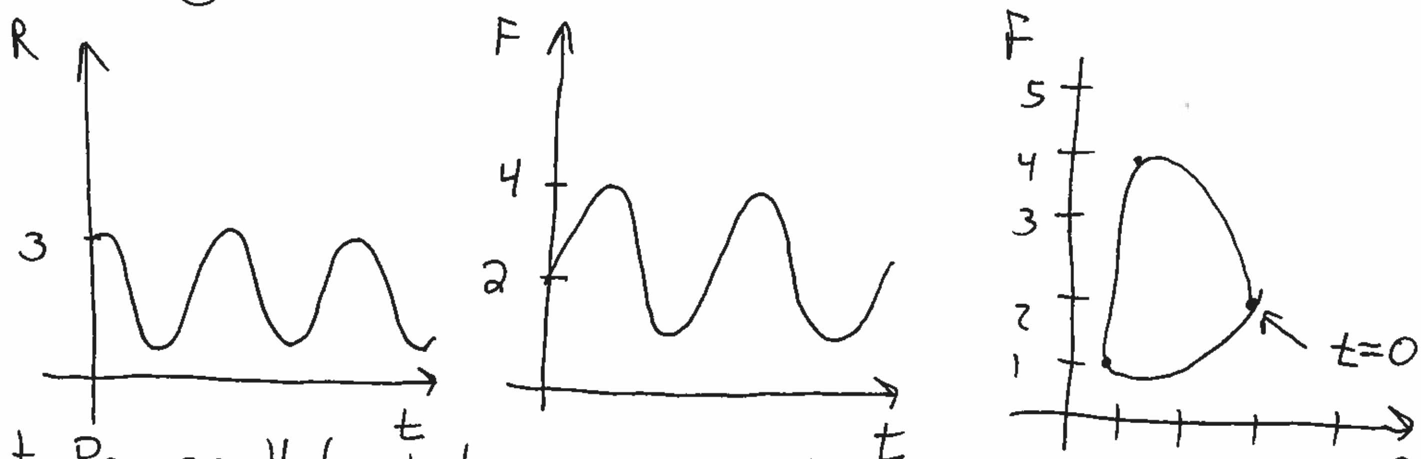
t	R (in hundreds)	F (in tens)	dR/dt	ΔR	dF/dt	ΔF
0	1	1	1.6	0.8	-0.2	-0.1
0.5	1.8	0.9	3.132	1.566	0.396	0.198
1.0	3.366	1.098				

3. (a) For the system of differential equations from problem 1,

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

use the GeoGebra applet https://ggbm.at/U3U6MsyA to generate predictions for the future number of rabbits and foxes if at time 0 we initially have 300 rabbits ($R = 3$) and 20 foxes ($F = 2$). Sketch graphs of R vs t, F vs t, and F vs R.

Using Runga-Kutta



Rabbit Pop oscillates between 100 and about 300 rabbits.
Fox Pop. oscillates between about 10 and 40 foxes.

(b) Using initial condition $R = 3$ (300 rabbits) and $F = 2$ (20 foxes), tell the story of what happens to the rabbit and fox population as time continues.

At first rabbit pop decreases while fox population increases due to over abundance of food (rabbits) for foxes. Then as rabbit pop. decreases, foxes compete for limited resources and begin to die off. Decrease in foxes causes rabbit pop. to increase again and we cycle back through this behavior.

4. (a) Suppose the current number of rabbits is $R = 3$ (300 rabbits) and the number of foxes is 0. Without using any technology and without making any calculations, what does the system of rate of change equations (same one as problem 3a) predict for the future number of rabbits and foxes? Explain your reasoning.

If $F_0 = 0$, there will never be any foxes, $F(t) = 0$.
Without predator, Rabbit pop. has $dR/dt = 3R$.
Rabbits increase exponentially indefinitely

(b) Use the same GeoGebra applet from problem 3b to generate the 3D plot and all three different views or projections of the 3D plot. Show each graph and explain how each illustrates your conclusion in problem 4a.

(c) Using initial condition $R = 3$ (300 rabbits) and $F = 0$ (no foxes), tell the story of what happens to the rabbit and fox population as time continues.

Never any foxes. Since no predators, Rabbit population grows forever.

5. (a) What would it mean for the rabbit-fox system to be in equilibrium? Are there any equilibrium solutions to this system of rate of change equations? If so, determine all equilibrium solutions and generate the 3D and other views for each equilibrium solution.

Both R and F remain constant for all t.

$\frac{dR}{dt} = 0$, $dF/dt = 0$ solving $\begin{cases} 3R - 1.4RF = 0 \\ -F + 0.8RF = 0 \end{cases}$ Gives $\boxed{R = 1.25,\ F = 2.143 \text{ OR } R = 0,\ F = 0}$

(b) For single differential equations, we classified equilibrium solutions as stable (attractors), unstable repellers, and semi-stable (nodes). For each of the equilibrium solutions in the previous problem, create your own terms to classify the equilibrium solutions in 5a and briefly explain your reasons behind your choice of terms.

$(R,F) = (0,0)$ is an unstable equilibrium since once $R > 0$ or $F > 0$, Populations would head away from $(0,0)$

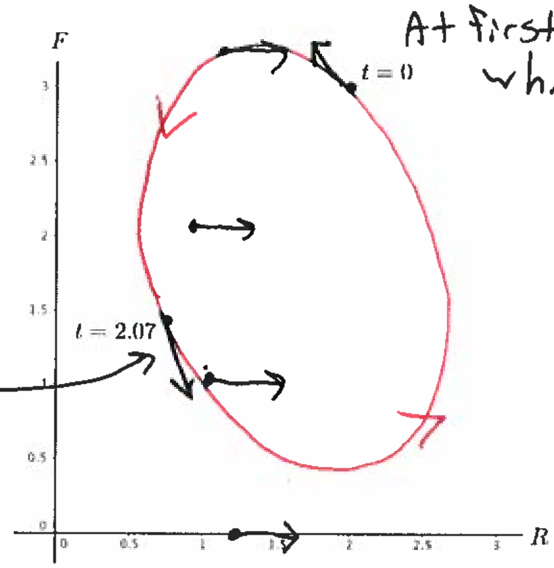
$(R,F) = (2.143, 1.25)$ is stable since solutions near this equilibrium stay near it.

6. One view of solutions for studying solutions to systems of autonomous differential equations is the xy-plane, called the **phase plane**. The phase plane is the analog to the phase line for a single autonomous differential equation.

   (a) Consider the rabbit-fox system of differential equations and a solution graph, as viewed in the phase plane (that is, the RF-plane), and the two points in the table below. These two points are on the same solution curve. Recall that the solutions we've seen in the past are closed curves, but notice that the solution could be moving clockwise / counterclockwise. Fill in the following table and decide which way the solution should be moving, and explain your reasoning.

t	R	F	dR/dt	dF/dt	dF/dR
0	2	3	-2.4	1.8	$\frac{1.8}{-2.4} = -0.75$
2.07	0.756	1.431	0.753	-0.566	$\frac{-0.566}{0.753} = -0.752$

At first R goes down while F goes up



$\frac{dR}{dt} = 3R - 1.4RF$

$\frac{dF}{dt} = -F + 0.8RF$

At t=2.07 R is increasing while F is decreasing.

   (b) On the same set of axes from problem 6a plot additional vectors at the following points and state what is unique about these vectors.

R	F	dR/dt	dF/dt	dF/dR
1.25	0	3.75	0	0
1.25	1	2	0	0
1.25	2	0.25	0	0
1.25	3	-1.5	0	0

All dF/dR vectors have slope 0.