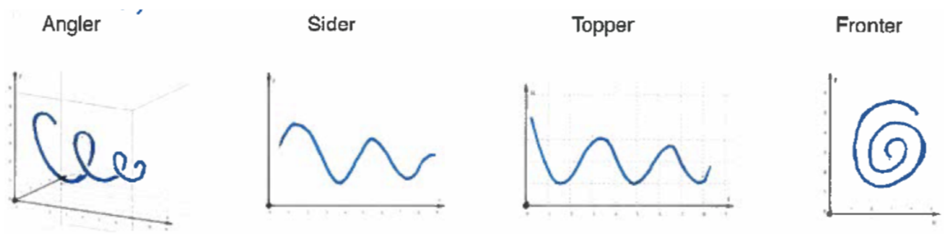
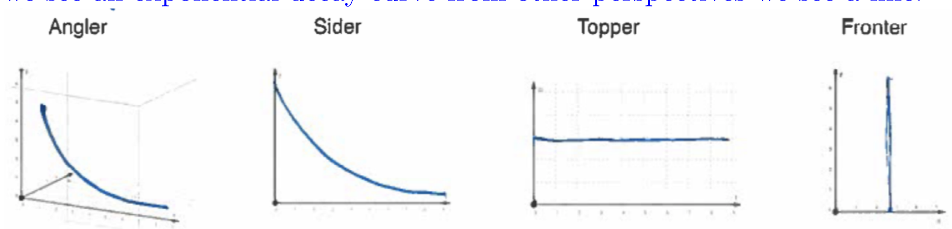
Homework Set 9 Solutions

1. (a) Consider again the crop duster plane problem but this time the red mark slowly drifts toward the center as the propellers rotate as the plane rolls along the runway. Sketch what the four observers see this time.

   (b) What do the four observers ideally see if the propellers are not rotating and the red mark drifts toward the center at a rate proportional to its distance from the center as the plane rolls along the runway?

(a) 

(b) With no rotation, the red mark drifts toward the center of the propellor at a rate proportional to its distance from the center, or in other words, along a path described by $x\prime = kx$ where x is the distance from the center. This describes exponential decay so (from certain perspectives) we see an exponential decay curve from other perspectives we see a line.



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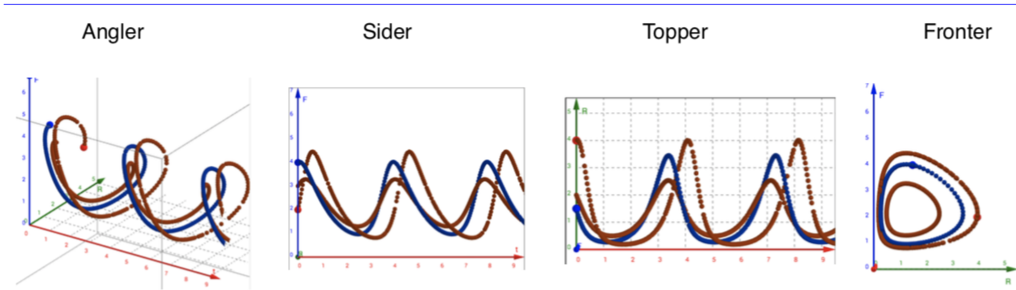
2. Consider the same system of differential equations from problem 1. Use the GeoGebra applet https://ggbm.at/U3U6MsyA to generate predictions for the future number of rabbits and foxes if at time 0 we initially have the following different initial conditions: (i) 2 rabbits and 3 foxes, (ii) 1.5 rabbits and 4 foxes, and (iii) 4 rabbits and 2 foxes. For each of the different views, graph all three solutions on the same set of axes.



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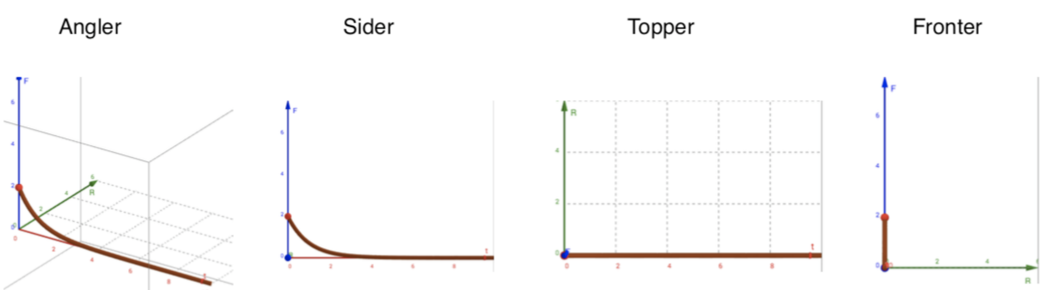
3. (a) Referring back to the rabbit and fox system of differential equations, suppose the current number of rabbits is 0 and the number of foxes is 2. Without using any technology and without making any calculations, what does the system of rate of change equations predict for the future number of rabbits and foxes? Explain your reasoning.

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

(b) Use the GeoGebra applet to generate the 3D plot and all three different views or projections of the 3D plot. Show each graph and explain how each illustrates your conclusion in problem 3a).

(c) Suppose the current number of rabbits is 0 and the number of foxes is 6. What does the system of rate of change equations predict for the future number of rabbits and foxes? How and why is this prediction related to the prediction when the initial number of rabbits is 0 and the number of foxes is 2?

(a) If $R = 0$ and $F = 2$ the equation for $\frac{dR}{dt}$ becomes identically 0, which means the rabbit population does not change and stays at 0. The other equation becomes $\frac{dF}{dt} = -F$ which is always negative, and thus predicts the fox population will die out (via exponential decay, which we can observe just from the form of this DE). Thus, both populations end up at 0.

(b) 

(c) This is very similar to the initial condition in (a): both populations end up at 0 (rabbits starting there), but here the foxes start with a larger initial population so they take a bit longer to die off.

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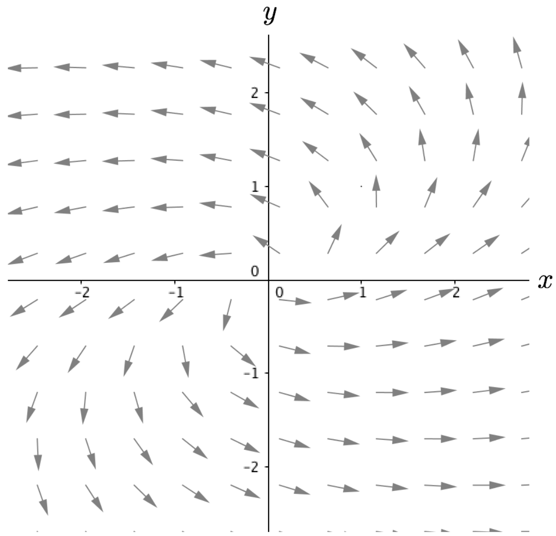
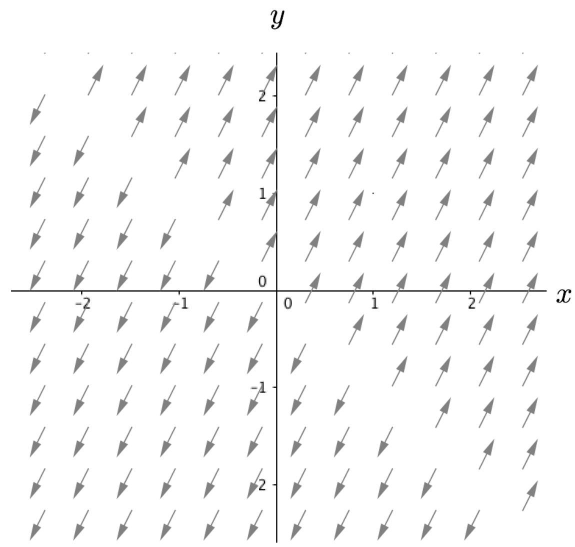
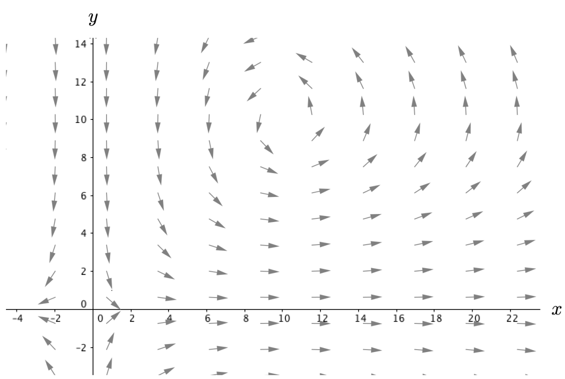
4. Here are three vector fields, A, B, and C. Below the vector fields are some pairs of rate of change equations. Determine which of the pairs match each of the vector fields. Write an explanation of each.



(a)



(b)



(c)

(i) $$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = -x + y$$

(ii) $$\frac{dx}{dt} = x - 0.1xy \qquad \frac{dy}{dt} = -y + 0.1xy$$

(iii) $$\frac{dx}{dt} = 2x - 3y \qquad \frac{dy}{dt} = x + y$$

(iv) $$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = 2x + 2y$$

(a) → (iii), (b) → (iv), (c) → (ii) – Explanations vary.

5. In previous problems dealing with two species, one of the animals was the predator and the other was the prey. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is both species are harmed by interaction) or cooperative (that is both species benefit from interaction).

   (a) Which system of rate of change equations describes a situation where the two species compete and which system describes competitive species? Explain your reasoning.

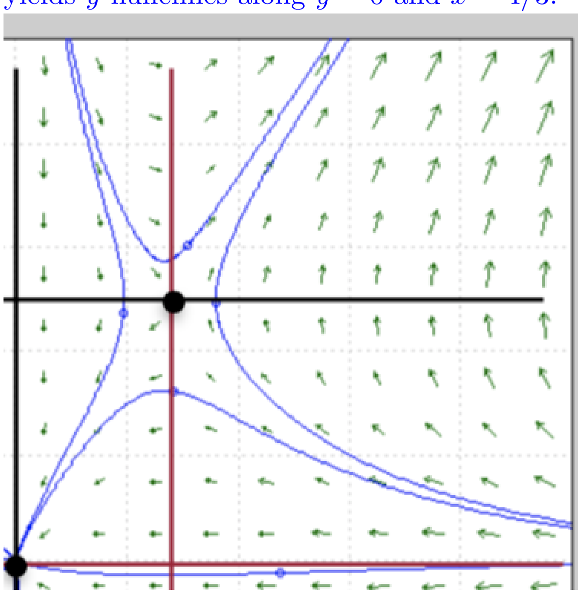
$$\begin{array}{cc} \text{(A)} & \text{(B)} \\ \dfrac{dx}{dt} = -5x + 2xy & \dfrac{dx}{dt} = 3x(1 - \dfrac{x}{3}) - \dfrac{1}{10}xy \\ \dfrac{dy}{dt} = -4y + 3xy & \dfrac{dy}{dt} = 2y(1 - \dfrac{y}{10}) - \dfrac{1}{5}xy \end{array}$$

   (b) For system (A), plot all nullclines and use this plot to determine all equilibrium solutions. Verify your equilibrium solutions algebraically.

   (c) Use your results from 5b to sketch in the long-term behavior of solutions with initial conditions anywhere in the first quadrant of the phase plane. For example, describe the long-term behavior of solutions if the initial condition is in such and such region of the first quadrant. Provide a sketch of your analysis in the x-y plane and write a paragraph summarizing your conclusions and any conjectures that you have about the long-term outcome for the two populations depending on the initial conditions.

(a) System A is cooperative: the interaction term with both quantities xy is positive here, meaning they add to the growth of each individual population when either one grows. System B is competitive because each population decreases with respect to the interaction term, meaning as one population increases the other decreases.

(b) Setting $\frac{dx}{dt} = 0$ and solving yields x-nullclines along $x = 0$ and $y = 5/2$. Similarly, setting $\frac{dy}{dt} = 0$ yields y-nullclines along $y = 0$ and $x = 4/3$.



(c) (See above graph). Behavior of solutions in the first quadrant depends on the position of initial conditions relative to the two equilibria at $(0, 0)$ and $(4/3, 5/2)$. As shown, some arc in toward the origin (particularly if starting to the left of the nullclines at $x = 4/3$ or below the nullclines at $y = 5/2$. These quantities are changing briefly, one population growing, until both populations head toward extinction. Others (with initial conditions on opposite sides of the two nullclines mentioned) arc toward the other equilibrium solution and then away again. In context, this means populations rise or dip toward $(4/3, 5/2)$ and then increase away from it with both populations growing without bound.

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6. Consider the following systems of rate of change equations:

$$\begin{array}{ll} \underline{\textbf{System A}} & \underline{\textbf{System B}} \\ \dfrac{dx}{dt} = 3x(1-\dfrac{x}{10}) - \dfrac{1}{20}xy & \dfrac{dx}{dt} = 3x - \dfrac{xy}{100} \\ \dfrac{dy}{dt} = -5y + \dfrac{xy}{20} & \dfrac{dy}{dt} = 15y(1-\dfrac{y}{17}) + 25xy \end{array}$$

In both of these systems, x and y refer to the number of two different species at time t. In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.

(a) For both systems of differential equations, what does x represent? The predator or the prey? Explain.

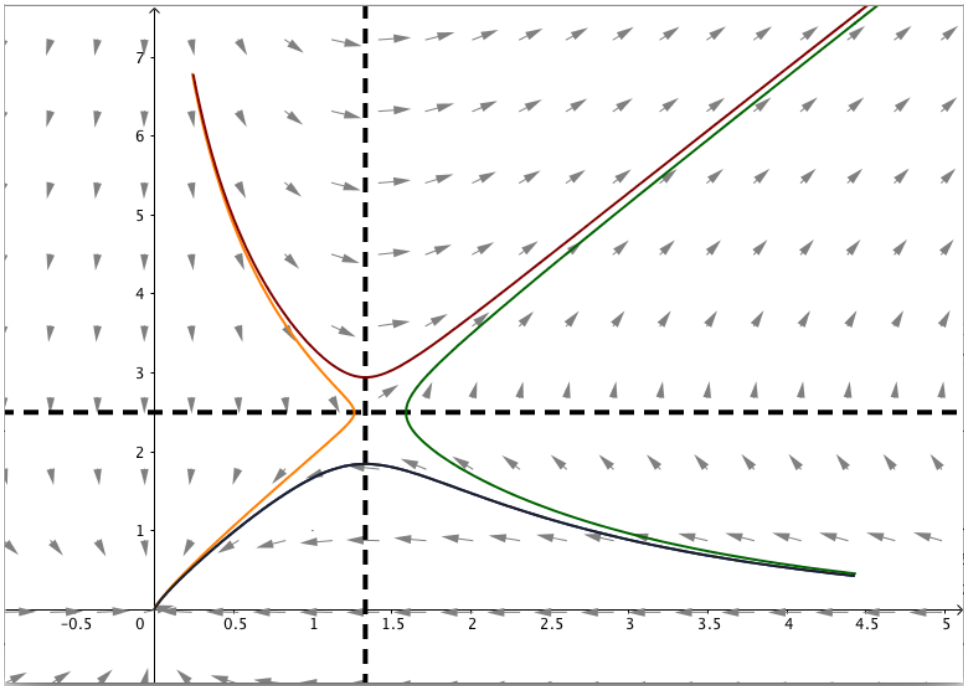
(b) What system represents predator and prey that are relatively the same size? Explain.

(c) For system (A), plot all nullclines and use this plot to determine all equilibrium solutions. Verify your equilibrium solutions algebraically.

(d) Use your results from 6c to sketch in the long-term behavior of solutions with initial conditions anywhere in the first quadrant of the phase plane. For example, describe the long-term behavior of solutions if the initial condition is in such and such region of the first quadrant. Provide a sketch of your analysis in the x-y plane and write a paragraph summarizing your conclusions and any conjectures that you have about the long-term outcome for the two populations depending on the initial conditions.

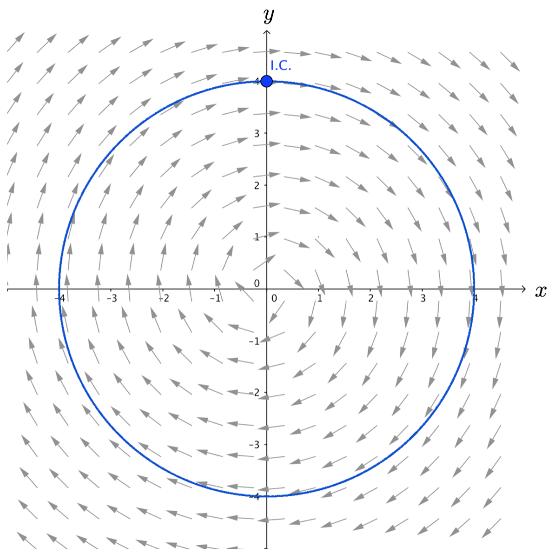
(a) In both systems x represents prey. This is because the population of x decreases with respect to the interaction term: so if y increases, x decreases.

(b) In system A, the predator and prey are relatively the same size. Focusing on the interaction (xy) terms, we see that a small increase in x, say $x \to x + 1$, is scaled down by a factor of 20 (the $\frac{1}{20}$) term in the $\frac{dy}{dt}$ equation, so increasing x has a relatively small impact on y. Likewise, increasing y has a relatively small impact on x. Since both species have relatively the same impact on each other, one conclusion is that they are relatively the same size, but there are alternative interpretations as well.

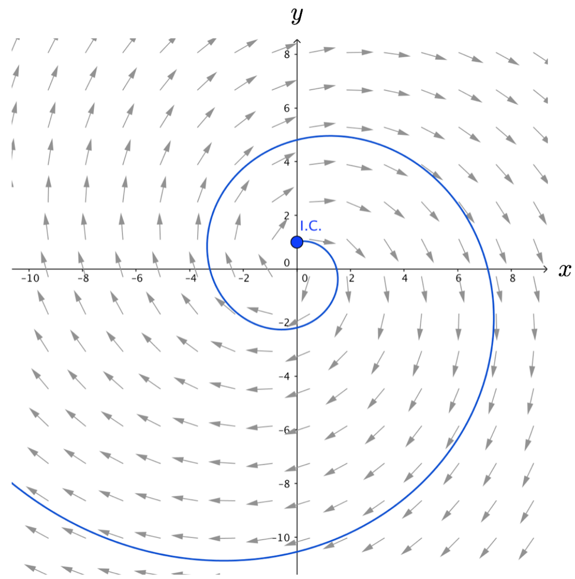
(c) 

(d) (See graph in part (c)). In the first quadrant, curves depict a y population staying relatively constant (small predators) while the large prey population x decreases slowly. At some point we see a tipping point (around $x = 100$) where the large prey finally has a low enough population to succumb to the small predators, and both populations decrease rapidly to 0.

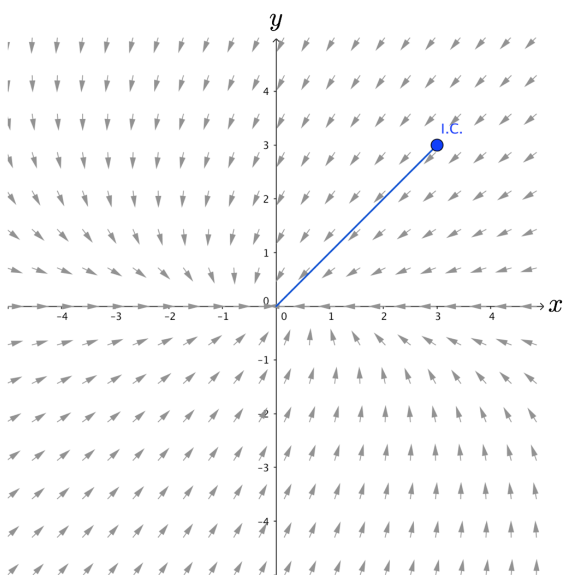
7. Provide sketches of x vs t and y vs t for each of the following phase planes and solution curves.



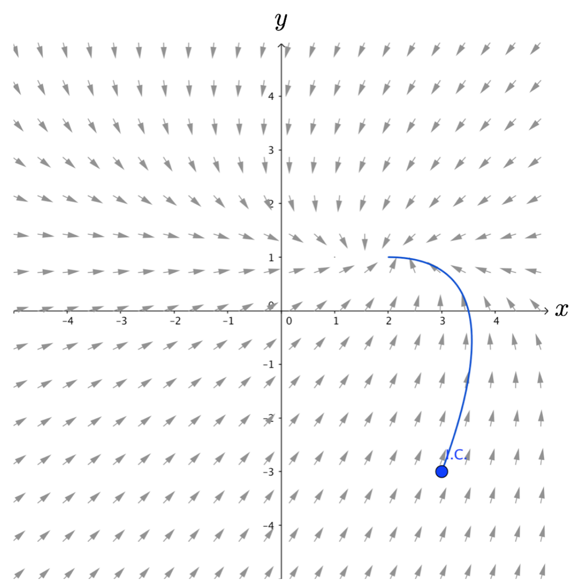
(a)



(b)

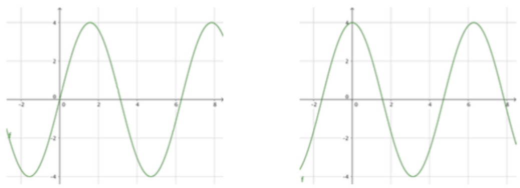


(c)

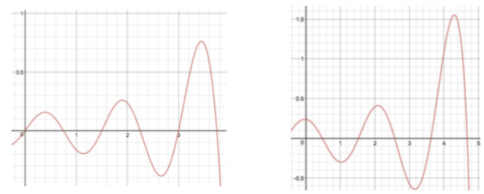


(d)

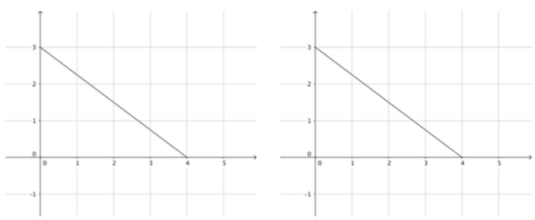
(a) x vs. t y vs. t



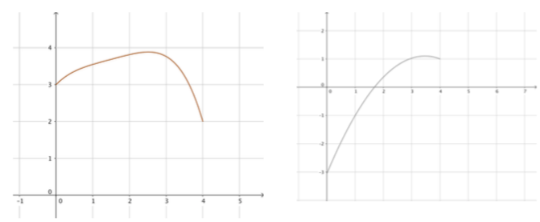
(b) x vs. t y vs. t



(c) x vs. t y vs. t



(d) x vs. t y vs. t



Note: (c) and (d) have slightly more curvature than shown to represent the exponential slowdown near the critical point. These graphs will soon be updated to reflect this.