6. One view of solutions for studying solutions to systems of autonomous differential equations is the xy-plane, called the **phase plane**. The phase plane is the analog to the phase line for a single autonomous differential equation.

Phase-Plane
↓
Vector Field ←
SS
Slope Field

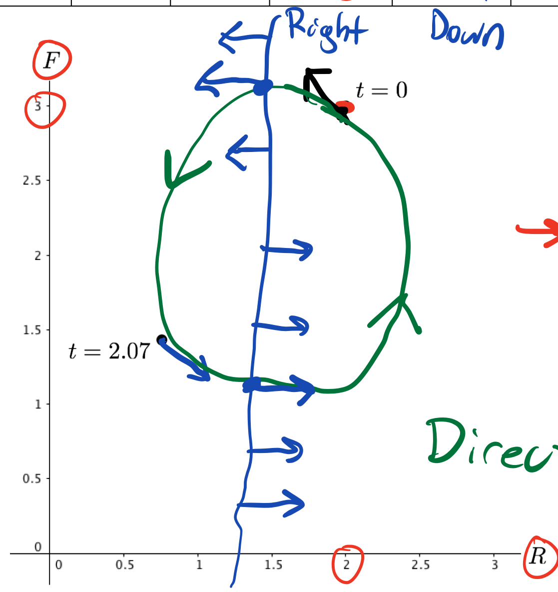
(a) Consider the rabbit-fox system of differential equations and a solution graph, as viewed in the phase plane (that is, the RF-plane), and the two points in the table below. These two points are on the same solution curve. Recall that the solutions we've seen in the past are closed curves, but notice that the solution could be moving clockwise / counterclockwise. Fill in the following table and decide which way the solution should be moving, and explain your reasoning.

t	R	F	dR/dt	dF/dt	dF/dR
0	2	3	-2.4	1.8	$1.8/-2.4 = -0.75$
2.07	0.756	1.431	0.753	-0.566	-0.752

$= \frac{dF/dt}{dR/dt}$

R↓ F↑
Left Up

Right Down



$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

Rabbit
Fox
Predator-Prey

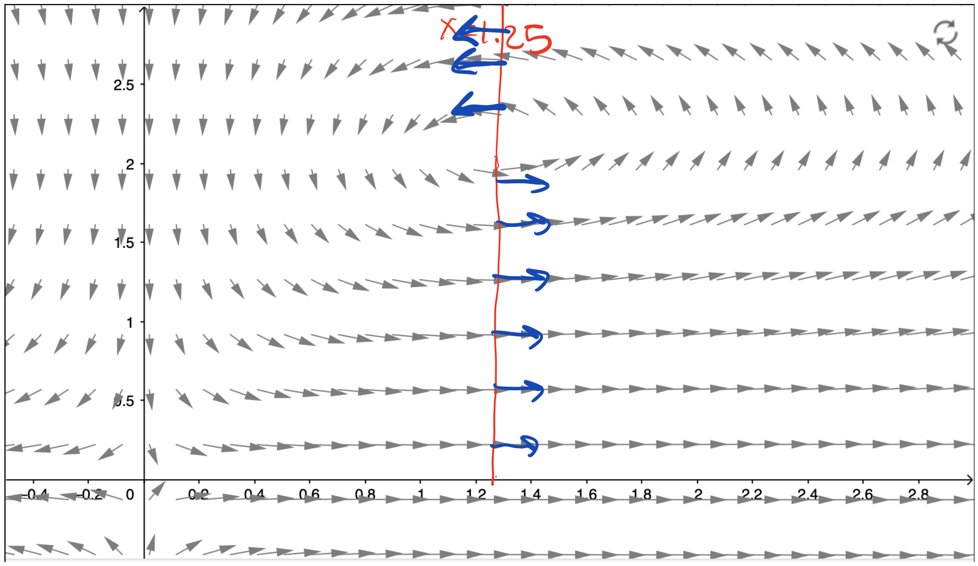
(b) On the same set of axes from problem 6a plot additional vectors at the following points and state what is unique about these vectors.

	R	F	dR/dt	dF/dt	dF/dR
Right	1.25	0	3.75	0	0
"	1.25	1	2	0	0
"	1.25	2	0.25	0	0
Left	1.25	3	-1.5	0	0

$= \frac{dF/dt}{dR/dt}$

$\frac{dF}{dR} = 0$

Horizontal Vectors Point directly Left or Right



$$\frac{dR}{dt} = 3R - 1.4RF$$

$$\frac{dF}{dt} = -F + 0.8RF$$

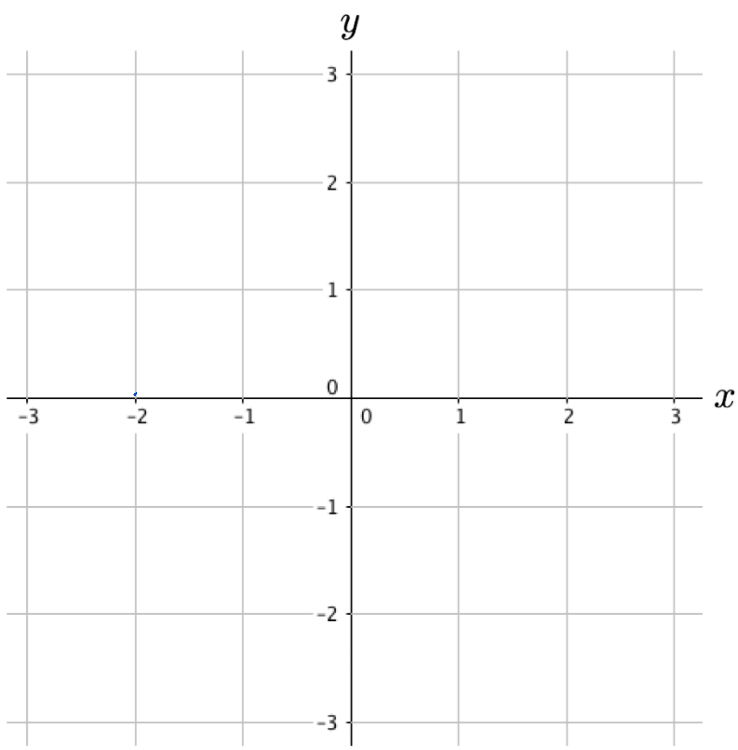
Vector Fields

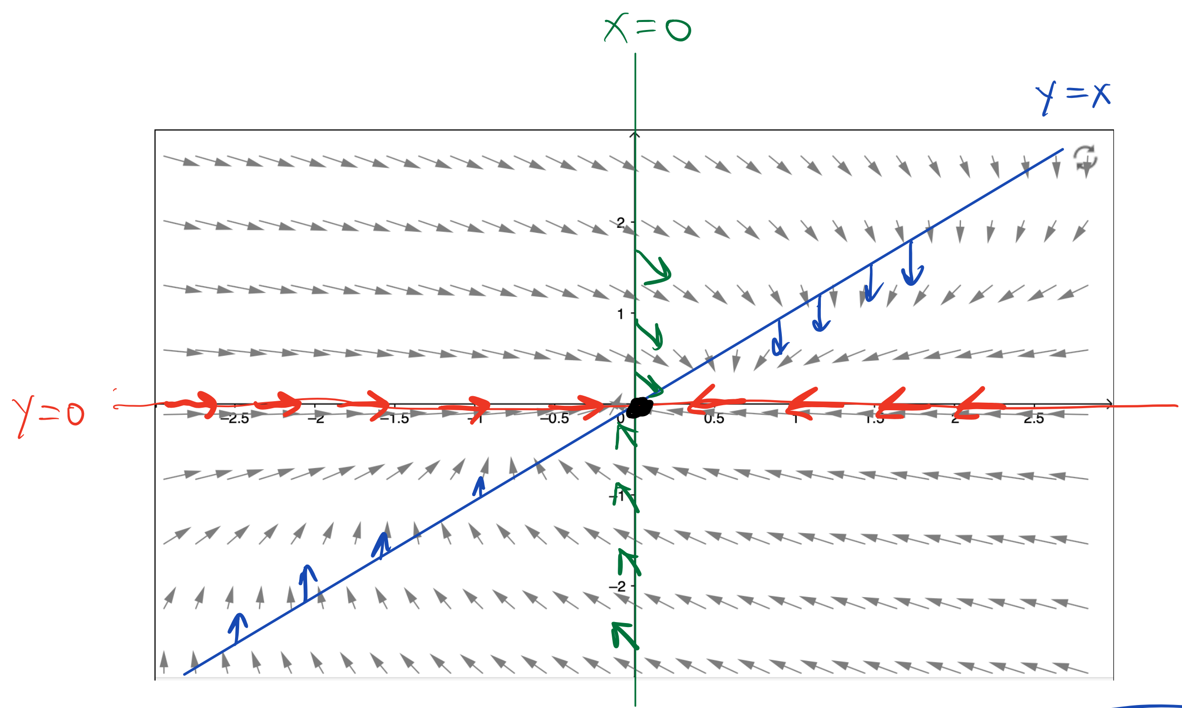
Slope fields are a convenient way to visualize solutions to a single differential equation. For systems of autonomous differential equations the equivalent representation is a **vector field**. Similar to a slope field, a vector field shows a selection of vectors with the correct slope but with a normalized length. In the previous problem you plotted a few such vectors but typically more vectors are needed to be able to visualize the solution in the phase plane.

1. On a grid where x and y both range from -3 to 3, plot by hand a vector field for the system of differential equations

$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = -y$$

   and sketch in several solution graphs in the phase plane.





Vertical Vectors = x-nullcline

$$\frac{dx}{dt} = y - x$$

Isoclines

$$\frac{dy}{dt} = -y$$

Vertical

Any point on line $y=x$ has $\boxed{dx/dt = 0}$

Any point on line $x=0$ has slope -1

Any point on $y=0$ has $dy/dt = 0$ → ←

y-nullcline
Horizontal Vectors

At (0,0) special case $dx/dt = 0$ and $dy/dt = 0$

Equilibrium.

2. (a) You may have noticed in problem 1 that along $x = 0$ all the vectors have the same slope. Similarly for vectors along the $y = x$. Any line or curve along which vectors all have the same slope is called an **isocline**. An isocline where $dx/dt = 0$ is called an **x-nullcline** because there is the horizontal component to the vector is zero and hence the vector points straight up or down. An isocline where $dy/dt = 0$ is called a **y-nullcline** because the vertical component of the vector is zero and hence the vector points left or right. On a grid from -4 to 4 for both axes, plot all nullclines for the following system:

$$\boxed{\frac{dx}{dt} = 3x - 1.4xy}$$
$$\frac{dy}{dt} = -y + 0.8xy$$

To Find x-nullclines

① Set $dx/dt = 0$ solve.

$3x - 1.4xy = 0$

$x(3 - 1.4y) = 0$

$\boxed{x = 0}$

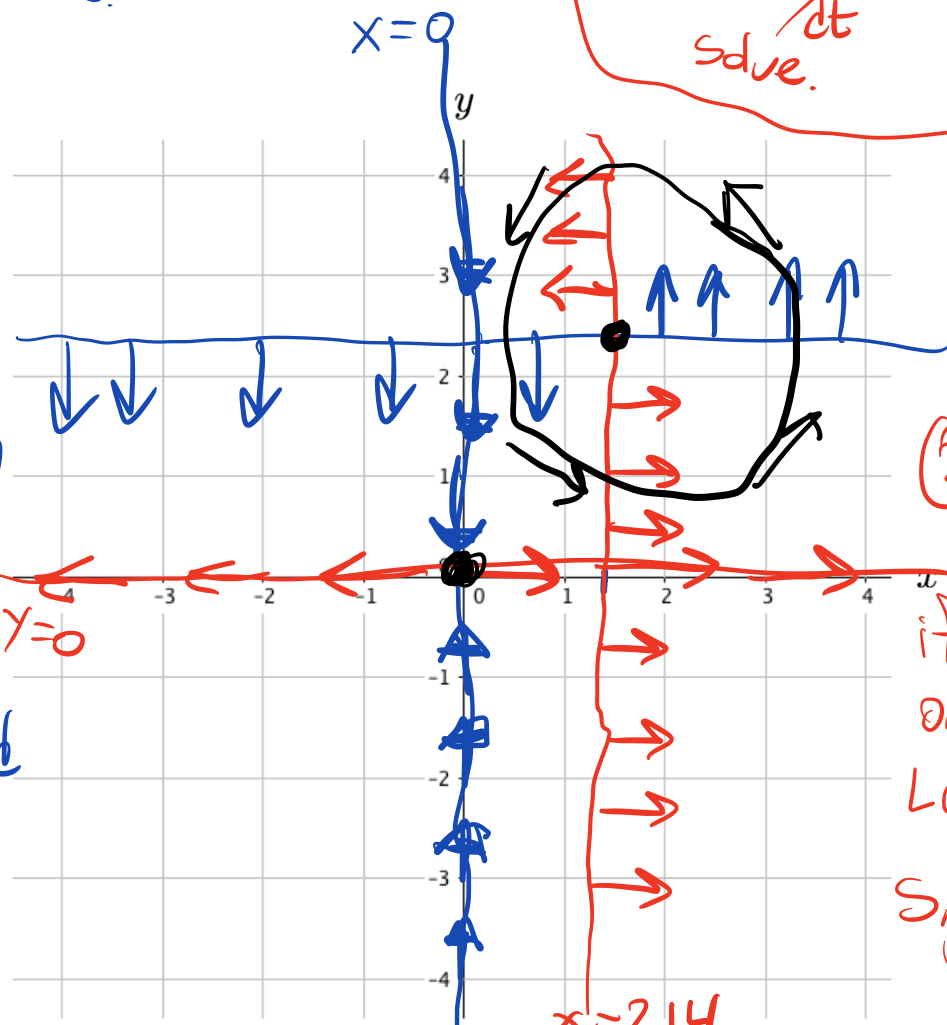
$3 - 1.4y = 0$

$\boxed{y \approx 2.14}$

② Determine whether direction is ↑ or ↓ based on sign $\frac{dy}{dt}$

① To find y-nullclines set $dy/dt = 0$ and solve.

② Determine if left or right Looking at sign of dx/dt.



(b) How do these nullclines point to the cyclic nature of the Rabbit-Fox system?

Y = 0 X = 1.25

$\boxed{\text{Equilibrium occur at points where different nullclines intersect}}$

(2) Determine whether direction is ↑ or ↓ based on sign $\frac{dy}{dt}$

$$\frac{dy}{dt} = -y + 0.8xy$$

$x = 0$

$y \approx 2.14$

On $x = 0$ we have

$$dy/dt = -y$$

$dy/dt > 0$ when $y < 0$

$dy/dt < 0$ when $y > 0$

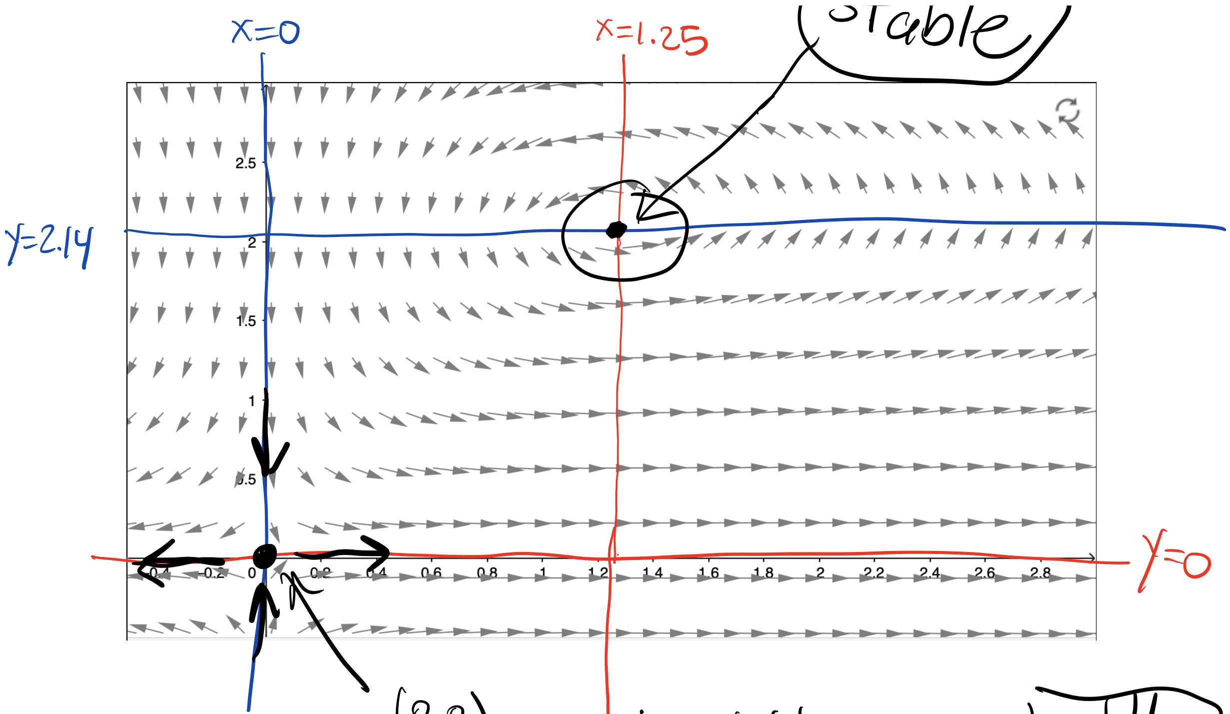
On $y \approx 2.14$ we have

$$dy/dt = -2.14 + 0.8x(2.14)$$

$$\boxed{dy/dt = -2.14 + 1.712x > 0}$$

$x > 1.25$ ↑

$dy/dt < 0$ $x < 1.25$ ↓



semi-stable or a saddle

unstable

$$\frac{dR}{dt} = 3R - 1.4RF$$

$$\frac{dF}{dt} = -F + 0.8RF$$

$x = 1.25$

$$dx/dt = 3.75 - 1.75y = 1.75(2.14 - y)$$

$dx/dt = 1.75(2.14 - y) > 0$ $\quad$ $2.14 - y > 0$

$y > 2.14$ $\quad$ $dx/dt < 0$

$y < 2.14$ $\quad$ $dx/dt > 0$

3. A certain system of differential equations for the variables R and S describes the interaction of rabbits and sheep grazing in the same field. On the phase plane below, dashed lines show the R and S nullclines along with their corresponding vectors.

   (a) Identify the R nullclines and explain how you know.

   (b) Identify the S nullclines and explain how you know.

   (c) Identify all equilibrium points.

   (d) Notice that the nullclines carve out 4 different regions of the first quadrant of the RS plane. In each of these 4 regions, add a prototypical-vector that represents the vectors in that region. That is, if you think the both R and S are increasing in a certain region then, draw a vector pointing up and to the right for that region.

   (e) What does this system seem to predict will happen to the rabbits and sheep in this field?