Stability: Long Run Behavior of Solutions

Recall model of the bacteria populations in Colony 1 and Colony 2 given by system of differential equations

$$\frac{dx}{dt} = 3x + 10y$$
$$\frac{dy}{dt} = -2y$$

We previously found the general solution for this system, which can be expressed as

$$x(t) = C_1e^{3t} + C_2e^{-2t}$$
$$y(t) = -\frac{1}{2}C_2e^{-2t}$$

Solutions that start on x-axis remain on x-axis

1. Give the solutions if in addition we have the initial condition $(x(0), y(0)) = (3, 0)$.

$x = 3 = c_1 + c_2 \rightarrow c_1 = 3$ $\quad x(t) = 3e^{3t}$

$y = 0 = c_2$ $\quad y(t) = 0$

$(x(t), y(t)) = (3e^{3t}, 0)$

2. Give the solutions if in addition we have the initial condition $(x(0), y(0)) = (2, -1)$.

$\boxed{y = -1 = -\frac{1}{2}c_2 \rightarrow c_2 = 2 \quad x = 2 = c_1 + c_2 = c_1 + 2 \quad c_1 = 0}$

$\boxed{x = 2e^{-2t} \quad y = -e^{-2t}}$

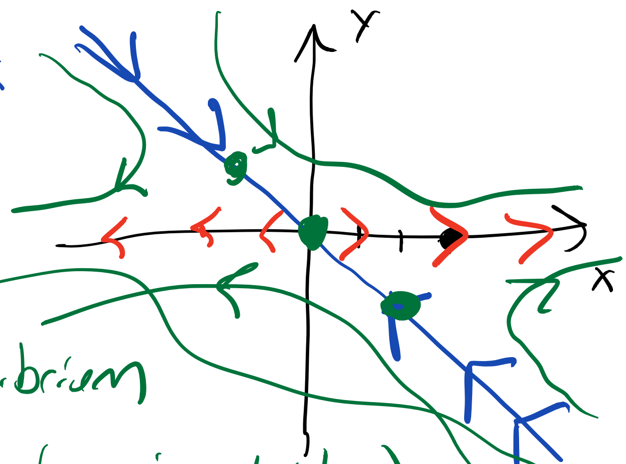
$\boxed{(2e^{-2t}, -e^{-2t})}$

3. Sketch the graphs (in the phase plane) of the solution with initial condition $(x(0), y(0)) = (3, 0)$ and the solution with initial condition $(x(0), y(0)) = (2, -1)$.

$$\frac{y}{x} = \frac{-e^{-2t}}{2e^{-2t}} = -\frac{1}{2}$$

$y = -\frac{1}{2}x$

$y = -\frac{1}{2}x$



(0,0) equilibrium

Saddle (semi-stable)

4. Using your graph in problem 3, make a rough sketch of the solution corresponding to the initial conditions $(x(0), y(0)) = (1, 1)$ and $(x(0), y(0)) = (1, -1)$.

Eigenvectors

For the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$\mathbf{v}_1 = \langle 3, 0 \rangle$ is an **eigenvector** corresponding to the eigenvalue $r_1 = 3$ and $\mathbf{v}_2 = \langle -2, 1 \rangle$ is an eigenvector of $r_2 = -2$. From phase plane graph in problem 4, we see that any solution that starts on the line passing through the eigenvector:

- $\mathbf{v}_1 = \langle 3, 0 \rangle$ points directly away from the equilibrium at the origin.
- $\mathbf{v}_2 = \langle 2, -1 \rangle$ points directly towards the origin.
- The sign of the eigenvalue determines whether solutions move towards or away from the equilibrium.

Using the eigenvectors we can express the solutions in vector form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1 e^{3t} + & C_2 e^{-2t} \\ & -\frac{1}{2} C_2 e^{-2t} \end{bmatrix} = C_1 e^{r_1 t} \mathbf{v}_1 + C_2 e^{r_2 t} \mathbf{v}_2.$$

In general, $\mathbf{v}$ is an **eigenvector** for the eigenvalue λ of a square matrix $\mathbf{A}$ if and only if

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}.$$

$3x = 3x$ $\quad$ x is arbitrary

$$A = \begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$3x + 10y = 3x$

$0x - 2y = 3y$

$0 = 5y \Rightarrow \boxed{y = 0}$

$\langle x, y \rangle = \langle x, 0 \rangle$ For example $\boxed{\langle 2, 0 \rangle = \vec{v}_1}$ $\quad \lambda_1 = 3$

$$\begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$3x + 10y = -2x$

$-2y = -2y$

$\boxed{y \text{ is arbitrary}}$ $\quad y = -\frac{1}{2}x$

$10y = -5x$ $\quad \langle x, -\frac{1}{2}x \rangle = \langle 1, -\frac{1}{2} \rangle$ Same direction as $\langle 2, -1 \rangle$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$\lambda_1 = 3 \qquad \vec{V}_1 = \langle 1, 0 \rangle$

$\lambda_2 = -2 \qquad \vec{V}_2 = \langle 1, -\tfrac{1}{2} \rangle$

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \vec{V}_1 e^{\lambda_1 t} + C_2 \vec{V}_2 e^{\lambda_2 t} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ -\tfrac{1}{2} \end{bmatrix} e^{-2t}$$

$$x(t) = C_1 e^{3t} + C_2 e^{-2t}$$

$$y(t) = -\frac{1}{2} C_2 e^{-2t}$$

Last Class:

Eigenvectors: We say $\vec{v}$ is an eigenvector corresponding to eigenvalue λ of matrix A if

$$A\vec{v} = \lambda \vec{v}$$

Example: $x' = 3x + 10y$ $\quad A = \begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix}$ $\boxed{\lambda = 3, -2}$

$y' = -2y$

Eigenvectors for $\lambda_1 = 3$:

$$\begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 3\begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{aligned} 3a + 10b &= 3a \\ -2b &= 3b \end{aligned}$$

$-2b = 3b$ implies $b = 0$ so $3a = 3a$ implies a is arbitrary once we set $b = 0$.

$$\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$ Thus $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector

Eigenvectors for $\lambda_2 = -2$:

$$\begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -2\begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{aligned} 3a + 10b &= -2a \\ -2b &= -2b \end{aligned}$$

$-2b = -2b$ is true for all b, so b is arbitrary

$3a + 10b = -2a$ implies $-5a = 10b$ or $b = -\frac{1}{2}a$

$\vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -\frac{1}{2}a \end{bmatrix}$. Thus $\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ is an eigenvector

Vector Form of General Solution:

If system has real, distinct eigenvalues λ_1 and λ_2. Then

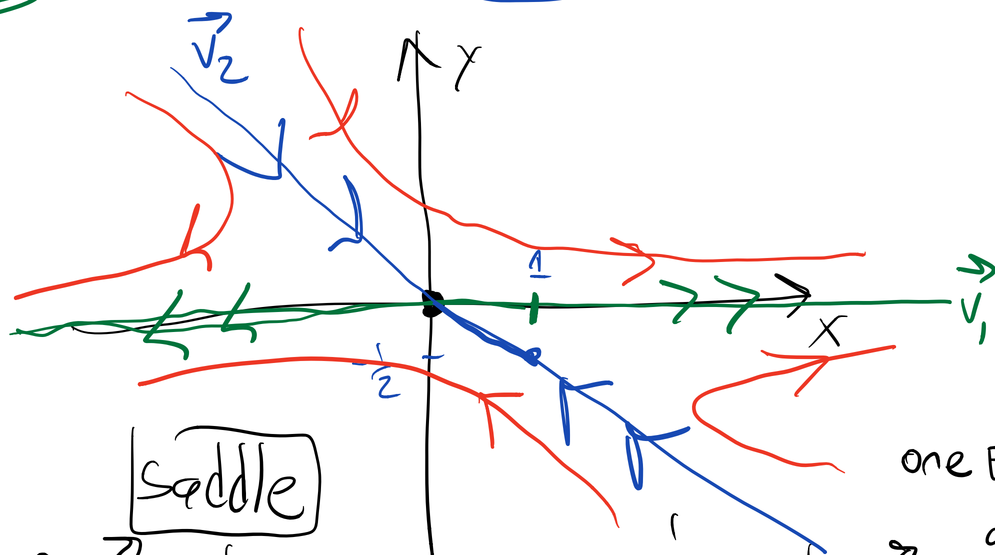
1. Find one possible eigenvector for λ_1, $\vec{v}_1$.
2. Find one possible eigenvector for λ_2, $\vec{v}_2$
3. The vector form of the solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2$$

$$\boxed{\begin{array}{l} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \\ x(t) = C_1 e^{3t} + C_2 e^{-2t} \\ y(t) = C_2 e^{-2t}\left(-\frac{1}{2}\right) = -\frac{1}{2} C_2 e^{-2t} \end{array}}$$

Using $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$ to sketch phase plane

$\lambda_1 = 3$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\lambda_2 = -2$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$



$\boxed{\text{Saddle}}$

Is $(0,0)$ stable or unstable?

↳ semi-stable

$x' = ax + by$
$y' = cx + dy$

one Equil at $(0,0)$

① Sketch straight line through $\vec{v}_1$ and $\vec{v}_2$

② Indicate direction solutions move as $t \to \infty$

- If $\lambda > 0$, then move away from origin
- If $\lambda < 0$, then move in to the origin

Vector Form of Solutions

5. Consider the system of differential equations:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find the eigenvalues and eigenvectors for the system and give the general solution in vector form. Then make a sketch of several solutions to this system in the phase plane.

① Find eigenvalues $\lambda = 7, -1$

② Find an eigenvector for each eigenvalue

③ Give vector form of solution

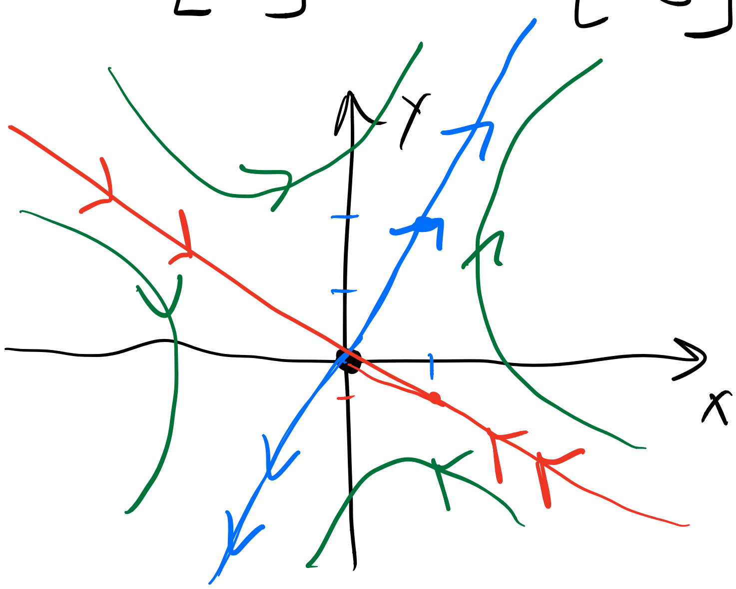
④ Make a phase plane sketch.

$\lambda_1 = 7 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\lambda_2 = -1 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$ or $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$$

④



$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_1 = 7$

Eigenvalues and Solutions in the Phase Plane

6. Match the vector fields labeled A-F (on the next page) with a system of differential equations whose matrix of coefficients has the given eigenvalues.

Eigenvalues of matrix of coefficients	Label of corresponding phase plane
$\lambda_1 = 4$ and $\lambda_2 = 1$	B
$\lambda_1 = -4$ and $\lambda_2 = -2$	C
$\lambda = 9$ (repeated)	F
$\lambda = \pm 2i$	A
$\lambda = 3 \pm 2i$	E
$\lambda = -3 \pm 2i$	D

spirals in $\to$ Spiral sink

$\frac{dx}{dt} = 3x+7y \leftarrow dy/dt = 6$

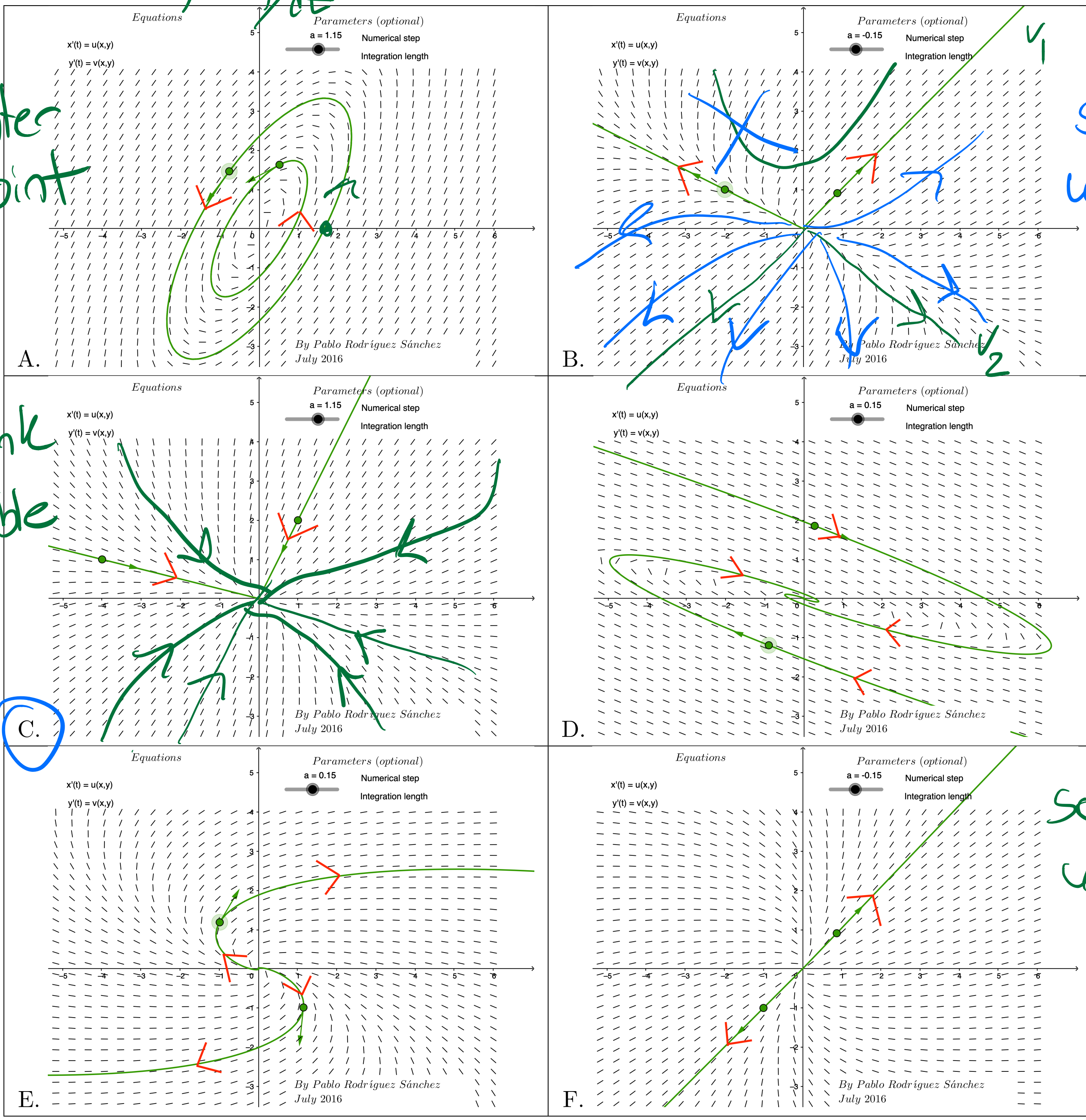
dy/dt

center point

source unstable

sink stable

source unstable



Repeated Root one eigen vector

Stability of the Equilibrium

7. Based on your answers in problem 6, explain how the eigenvalues can be used to determine whether the equilibrium at the origin is stable or unstable? What happens when the matrix of coefficients has complex eigenvalues?

If real eigenvalues

- Distinct λ_1 and λ_2 both positive, then unstable source
- Distinct λ_1 and λ_2 both negative, then stable sink
- Distinct λ_1 and λ_2 with opposite signs, then semi-stable saddle

If complex eigenvalues $\lambda = \alpha \pm \beta i$

- Real part $\alpha = 0$, then $(0,0)$ is a center point, stable
- Real part $\alpha > 0$, then $(0,0)$ is a spiral source, unstable
- Real part $\alpha < 0$, then $(0,0)$ is a spiral sink, unstable

The sign of the real parts of the eigenvalues determine stability of the equilibrium at $(0,0)$.