Guess and Test for Nonhomogeneous Cases

So far we have been using patterns recognized in wisely guessing the form of solutions to homogeneous second order differential equations of the form $ay'' + by' + cy = 0$. Can we adjust our guesses to handle nonhomogeneous differential equations as well?

1. Find a solution to the following **nonhomogeneous** differential equation:

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18.$$

forcing function is constant

Guess $x_p = 2$ is a constant

What is your best guess for a function whose second derivative plus 10 times its first derivative plus 9 times the function itself sum to 18? Test out your guess to see if it works. If it doesn't work keep trying.

Test $x = 18$:

$0 + 10(0) + 9(18) = 36 \neq 18$

$0 + 0 + 9(2) = 18 = 18$ ✓

$x_p = 18$ not a solution

Homogeneous: $x'' + 10x' + 9x = 0$

$r^2 + 10r + 9 = 0$

$(r+9)(r+1) = 0$

$x_h = C_1 e^{-9t} + C_2 e^{-t} + 18$

a particular solution

$x'' + 10x' + 9x = 18$

$\cancel{x_p = 18}$, $\cancel{x_p' = 0}$, $\cancel{x_p'' = 0}$

$x_p = 2$

$x_p' = 0$

$x_p'' = 0$

The solution you found in the previous problem is called the **particular solution** to the nonhomogeneous differential equation it is not the general solution. For now we will focus on how we can find the particular solution by wisely guessing their general form based on the nonhomogeneous part of the differential equation. We will soon combine what we know about the homogeneous case and particular solutions to find general solutions.

2. Find a solution to the following nonhomogeneous differential equation:

$$\boxed{\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18t.}$$

Linear

Guess $\boxed{x = At + B}$

What is your best guess? Test out your guess to see if it works. If it doesn't work keep trying.

$\boxed{x = 2t}$ $x' = 2$ $x'' = 0$

$0 + 10(2) + 9(2t) = 18t$

$18t + \underbrace{20}_{} = 18t$

$x = 2t + B$

$x = 2t - 20$

$\boxed{x = 2t + B}$ →

Plug and find value of B that works

$x' = 2$

$x'' = 0$

$0 + 10(2) + 9(2t + B) = 18t$

$20 + 18t + 9B = 18t$

$20 + 9B = 0$

$B = -20/9$

$\boxed{x_p = 2t - 20/9}$

3. Based on the previous examples, what would be a good guess for the general form of the particular solution to

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18t^3.$$

New

Your guess should have depend on constants whose values you do not need to determine for this example.

$\boxed{x_p = At^3 + Bt^2 + Ct + D}$

1. Reasonable Guess
2. Plug it in
3. Solve for undetermined coefficients

Method of undetermined coefficients

4. Sean and Phil are trying to find the particular solution to $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = \boxed{85\sin(2t)}$. Sean guesses $x(t) = A\sin(2t)$ for the particular solution and Phil guesses $x(t) = B\cos(2t)$.

   (a) Do you think these are reasonable guesses? Explain why or why not.

x_p = polynomial, not reasonable

$x_p = A\sin(2t)$ seems reasonable since taking derivative of $\sin(2t)$ gives $\cos(2t)$ and $\sin(2t)$ back

$x_p = B\cos(2t)$

   (b) For each of their guesses, can you find a value of A or B such that their guess is a solution? If yes, write down the general solution. If no, come up with a different guess for the particular solution and show that your guess is correct.

Try $A\sin(2t) = x_p$ $\quad x_p' = 2A\cos(2t)$

can't work $\quad x_p'' = -4A\sin(2t)$

$x_p = B\cos(2t)$ doesn't work either

$$\boxed{x_p = A\sin(2t) + B\cos(2t)}$$

$x'' + 10x' + 9x = 85\sin(2t)$

⓪ Find homogeneous solution

① Guess that x_p has same form as forcing function

② Plug it in check.

$9 \quad x_p = A\sin(2t) + B\cos(2t)$

$10 \quad x_p' = 2A\cos(2t) - 2B\sin(2t)$

$1 \quad x_p'' = -4A\sin(2t) - 4B\cos(2t)$

$$\underbrace{\left(-4A\sin(2t) - 4B\cos(2t)\right)}_{x_p''} + \underbrace{\left(20A\cos(2t) - 20B\sin(2t)\right)}_{10x_p'}$$

$$+ \underbrace{\left(9A\sin(2t) + 9B\cos(2t)\right)}_{9x_p} = 85\sin(2t) + 0\cos(2t)$$

③ Solve for undetermined coefficients

$(\boxed{-4A - 20B + 9A})\sin(2t) = 85\sin(2t)$

$(-4B + 20A + 9B)\cos(2t) = 0\cos(2t)$

$5A - 20B = 85$ $\quad A = 1$

$20A + 5B = 0$ $\quad B = -4$

$$\boxed{x_p = \sin(2t) - 4\cos(2t)}$$ is one solution

General Solution: $x = \boxed{x_{homogeneous}} + x_p$