Homework Set 11 Solutions

1. The general solution to

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 2y$$

is

$$x(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$$
$$y(t) = c_1 e^{-t}(-\cos(t) - \sin(t)) + c_2 e^{-t}(-\sin(t) + \cos(t))$$

Which part(s) of the general solution accounts for the fact that the differential equations predict that the mass will oscillate about the zero position? Which part(s) of the general solution accounts for the fact that the amplitude of the oscillations decreases over time?

The oscillations occur because of the $\sin(t)$ and $\cos(t)$ terms in the general solution: these alternately take on both positive and negative values that force the solution trajectory to rotate around the origin. The functions e^{-kt} are what force the solutions eventually to 0: as $t \to \infty$ these terms all tend to 0.

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2. Suppose that for a different system of differential equations you got the exact same general solution as homework problem 1 except instead of e^{-t} you got e^{t}. How would this change graphs of solutions in the phase plane? Explain.

If we replaced e^{-t} with e^{t}, all the solution curves would reverse direction. As $t \to \infty$, trajectories would increase outward from the origin, growing without bound, rather than tending toward the origin as they did before.

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3. Find the general solution to the spring mass problem when there is no friction. Sketch these solution in the phase plane and explain how this general solution fits with your expectation for the behavior of the mass over time. Note: when there is no friction, $b = 0$, and the spring constant $k = 2$, we get

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x$$

(Students use the slope-first method here, so general solutions might appear different from those found via the "traditional" eigenvector approach.)
Complex straight line solutions occur along lines with slope $\pm\sqrt{2}i$. Picking an initial condition $(1, \sqrt{2}i)$ on the line $y = \sqrt{2}ix$ we arrive at fundamental solutions

$$x_1(t) = e^{\sqrt{2}it}, y_1(t) = \sqrt{2}ie^{\sqrt{2}it}.$$

Repeating the process for the line $y = -\sqrt{2}ix$ with initial condition $(1, -\sqrt{2}i)$ we get

$$x_2(t) = e^{-\sqrt{2}it}, y_2(t) = -\sqrt{2}ie^{-\sqrt{2}it}.$$

We then use Euler's formula to rewrite these functions in terms of sine and cosine:

$$\begin{aligned} x_1(t) &= e^0(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)) \\ y_1(t) &= \sqrt{2}ie^0(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)) \\ x_2(t) &= e^0(\cos(-\sqrt{2}t) + i\sin(-\sqrt{2}t)) \\ &= e^0(\cos(\sqrt{2}t) - i\sin(\sqrt{2}t)) \\ y_2(t) &= -\sqrt{2}ie^0(\cos(-\sqrt{2}t) + i\sin(-\sqrt{2}t)) \\ &= -\sqrt{2}ie^0(\cos(\sqrt{2}t) - i\sin(\sqrt{2}t)) \end{aligned}$$

To find real-valued solutions we compute:

$$\mathbf{u}_1 = \begin{bmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{bmatrix} = \begin{bmatrix} 2\cos(\sqrt{2}t) \\ -2\sqrt{2}\sin(\sqrt{2}t) \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} x_1(t) - x_2(t) \\ y_1(t) - y_2(t) \end{bmatrix} = \begin{bmatrix} -2\sin(\sqrt{2}t) \\ -2\sqrt{2}\cos(\sqrt{2}t) \end{bmatrix}$$

The general solution is therefore:

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \begin{bmatrix} 2c_1\cos(\sqrt{2}t) - 2c_2\sin(\sqrt{2}t) \\ -2\sqrt{2}c_1\sin(\sqrt{2}t) - 2\sqrt{2}c_2\cos(\sqrt{2}t) \end{bmatrix}$$

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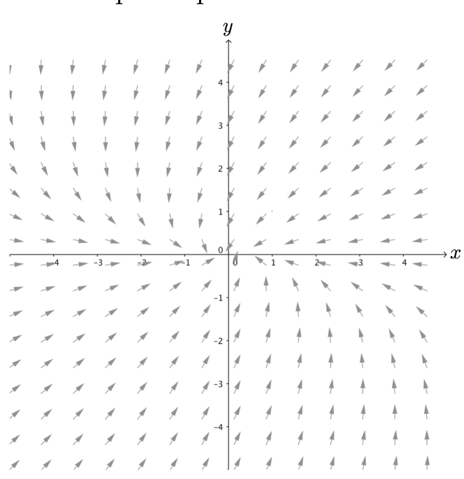
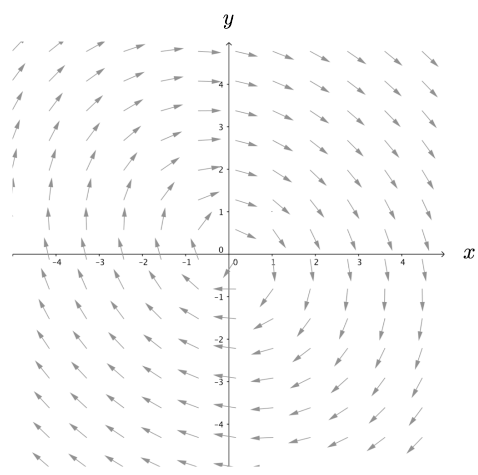
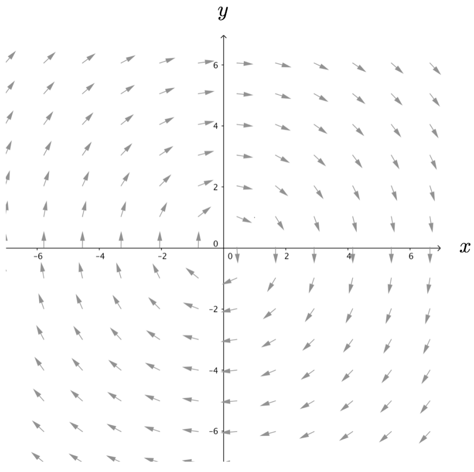
4. Consider the phase planes below:



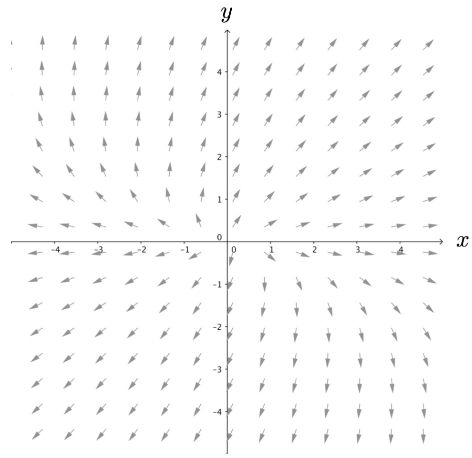
(A)



(B)



(C)



(D)

For each section below, fill in the blank with choices from the following two lists:

Spring System (First Blank)	**Solutions (Second Blank)**
a damped spring	$c_1 \cos(t) + c_2 \sin(t)$
an overdamped spring	$e^{-t}(c_1 \cos(t) + c_2 \sin(t))$
an undamped spring	$e^{t}(c_1 \cos(t) + c_2(\sin(t))$
something other than a spring	$c_1 e^{t} + c_2 e^{2t}$
	$c_1 e^{-t} + c_2 e^{-2t}$
	$c_1 e^{-t} + c_2 e^{2t}$
	$c_1 e^{t} + c_2 e^{-2t}$

Phase plane (A) corresponds to an overdamped spring and the solutions look like $x(t) = c_1 e^{-t} + c_2 e^{-2t}$

Phase plane (B) corresponds to an damped spring and the solutions look like $x(t) = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$

Phase plane (C) corresponds to an undamped spring and the solutions look like $x(t) = c_1 \cos(t) + c_2 \sin(t)$

Phase plane (D) corresponds to something other than a spring and the solutions look like $x(t) =$ $c_1 e^{t} + c_2 e^{2t}$

5. What type of system (undamped, damped, overdamped) do the following best correspond to? Explain your reasoning.
   (a) A car that bounces every time it hits a bump
   (b) A pendulum immersed in a vat of honey
   (c) A bungee jumper

   (a) The bounciness of the car means that it could be undamped, although realistically it must still have some damping.
   (b) The immersion in a fluid as viscous as honey means that it will have a hard time swinging past equilibrium, so this is an overdamped system.
   (c) The bungee jumper is a damped system, since they eventually stop.

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6. In each part, write a differential equation corresponding to the given scenario:
   (a) An undamped spring
   (b) An underdamped spring
   (c) An overdamped spring

   (a) $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -kx$ for any k
   (b) $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -2x - 3y$, as in unit 9
   (c) $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -2x - 2y$, as in this unit

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7. Does Adding Solutions Always Result in Another Solution?

   In deriving the general solution to the spring mass problem, two solutions were added to get another solution. This worked for the particular equations at hand, but does adding two solutions to a system of differential equations of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

   always result in another solution to the same system of differential equations? Below is a proof that this in fact is true.

Claim: If $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are solutions (not necessarily straight line solutions) to a system of differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

then the sum of these two solutions is also a solution. That is, if we call the sum of these two solutions $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ where

$$\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix},$$

then $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ is also a solution to the same system of differential equations.

Proof: In order to show that $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ is a solution, we need to verify it satisfies the system of differential equations. This is, we need to show that

$$\begin{aligned} \frac{d}{dt}x_3(t) &= ax_3(t) + by_3(t) \\ \frac{d}{dt}y_3(t) &= cx_3(t) + dy_3(t) \end{aligned}.$$

Since

$$\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix},$$

we know that

$$\begin{aligned} \frac{d}{dt}x_3(t) &= \frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t) \\ \frac{d}{dt}y_3(t) &= \frac{d}{dt}y_1(t) + \frac{d}{dt}y_2(t) \end{aligned}. \tag{1}$$

Because $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ is a solution, it satisfies the system of differential equations. That is,

$$\begin{aligned} \frac{d}{dt}x_1(t) &= ax_1(t) + by_1(t) \\ \frac{d}{dt}y_1(t) &= cx_1(t) + dy_1(t) \end{aligned} \tag{2}$$

Similarly, since $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ is a solution,

$$\begin{aligned} \frac{d}{dt}x_2(t) &= ax_2(t) + by_2(t) \\ \frac{d}{dt}y_2(t) &= cx_2(t) + dy_2(t) \end{aligned} \tag{3}$$

Substituting (2) and (3) into (1) yields

$$\begin{aligned}\frac{d}{dt}x_3(t) &= ax_1(t) + by_1(t) + ax_2(t) + by_2(t)\\ \frac{d}{dt}y_3(t) &= cx_1(t) + dy_1(t) + cx_2(t) + dy_2(t)\end{aligned}.$$

Rearranging terms yields

$$\begin{aligned}\frac{d}{dt}x_3(t) &= ax_1(t) + ax_2(t) + by_1(t) + by_2(t) = a[x_1(t) + x_2(t)] + b[y_1(t) + y_2(t)]\\ \frac{d}{dt}y_3(t) &= cx_1(t) + cx_2(t) + dy_1(t) + dy_2(t) = c[x_1(t) + x_2(t)] + d[y_1(t) + y_2(t)]\end{aligned}.$$

Finally, using the fact that

$$\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix},$$

yields

$$\begin{aligned}\frac{d}{dt}x_3(t) &= ax_3(t) + by_3(t)\\ \frac{d}{dt}y_3(t) &= cx_3(t) + dy_3(t)\end{aligned}$$

which is what we set out to show. Therefore $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ is also a solution to the system of differential equations.

(a) Suppose that $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are solutions to the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= ax + by + 1\\ \frac{dy}{dt} &= cx + dy + 2\end{aligned}$$

where a, b, c, and d are constants. Josh claims that the sum of these two solutions is also a solution to the same system of differential equations. Do you agree with his claim? Either develop a similar proof as above to support this claim or point to where (and why) the above proof fails.

(b) Suppose that $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are solutions to the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= ax^2 + by\\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

where a, b, c, and d are constants. Angela claims that the sum of these two solutions is also a solution to the same system of differential equations. Do you agree with her claim? Either develop a similar proof as above to support this claim or point to where (and why) the above proof fails.

(a) Suppose (x_1, y_1) and (x_2, y_2) are solutions. Let $(x_3, y_3) = (x_1 + x_2, y_1 + y_2)$. Then $\frac{dx_3}{dt} = \frac{d(x1+x2)}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} = ax_1 + by_1 + 1 + ax_2 + by_2 + 1 = a(x_1 + y_1) + b(y_1 + y_2) + 2 = ax_3 + by_3 + 2$, which does not satisfy the same differential equation. Therefore, Josh's claim that the sum of two solutions to this equation is a solution is false.

(b) Likewise, for the second equation, if $(x_3, y_3) = (x_1 + x_2, y_1 + y_2)$, then $\frac{dx_3}{dt} = \frac{d(x_1+x_2)}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} = ax_1^2 + by_1 + ax_2^2 + by_2$, which cannot be rewritten as $a(x_1 + x_2)^2 + b(y_1 + y_2)$ because of the nonlinearity of the squared exponent. Thus, (x_3, y_3) satisfies a different differential equation than (x_1, y_1) and (x_2, y_2) and so is not a solution to the original differential equation.