

$$\mathcal{L}\{3t^2 y'' + 2\cos(t) y' + 3y\} = \mathcal{L}\{e^{5t}\}$$

IODE

Sectoin 6.1-6.2: Properties of Laplace Transforms

## Properties of the Laplace Transform

1. Let  $f$ ,  $f_1$ , and  $f_2$  be functions whose Laplace transform exists for  $s > \alpha$  and let  $c$  be a constant. Then for  $s > \alpha$ , prove the following:

(a)  $\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$ .

$$\begin{aligned} \mathcal{L}\{f_1 + f_2\} &= \int_0^\infty e^{-st} (f_1(t) + f_2(t)) dt \\ &= \int_0^\infty (e^{-st} f_1(t) + e^{-st} f_2(t)) dt \\ &= \int_0^\infty e^{-st} f_1(t) dt + \int_0^\infty e^{-st} f_2(t) dt \\ &= \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\} \end{aligned}$$

(b)  $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ .

$$\begin{aligned} \mathcal{L}\{cf\} &= \int_0^\infty e^{-st} (cf(t)) dt = c \int_0^\infty e^{-st} f(t) dt \\ &= c \mathcal{L}\{f(t)\} \end{aligned}$$

$$\mathcal{L}\{2\cos(3t) + 5 - 4e^{6t}\} = \frac{6}{s^2+9} + \frac{5}{s} - \frac{4}{s-6}, \quad s > 6$$

$$2\mathcal{L}\{\cos(3t)\} + 5\mathcal{L}\{1\} - 4\mathcal{L}\{e^{6t}\}$$

Referring  
to Table

$$2\left(\frac{3}{s^2+9}\right) + \frac{5}{s} - 4 \frac{1}{s-6}$$

$s > 0$

$s > 0$

$s > 6$

$$e^{-st+at} = e^{-t(s-a)}$$

Laplace Transform of  $g(t) = e^{at} f(t)$

1. If the Laplace transform  $\mathcal{L}\{f\}(s) = F(s)$  exist for  $s > \alpha$ , then show that

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a), \text{ for } s > \alpha + a.$$

$$u = s - a$$

$$\begin{aligned} \mathcal{L}\{e^{at} f(t)\} &= \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-t(s-a)} f(t) dt \\ &= \int_0^\infty e^{-t \cdot u} f(t) dt \iff \int_0^\infty e^{-st} f(t) dt = F(u) \\ &= F(u) \quad u > \alpha \\ &= F(s-a) \end{aligned}$$

$\downarrow$   
 $s-a > \alpha$   
 $s > \alpha + a$

$\mathcal{L}\{e^{at} f(t)\}$        $s > \alpha$   
 $\downarrow$   
Horizontal Shift  
of  $\mathcal{L}\{f(t)\}$

2. Using the property above and the fact that  $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$  for  $s > 0$ , find  $\mathcal{L}\{e^{at} \cos(bt)\}$ .

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0$$

$$F(s) = \frac{s}{s^2+b^2}$$

$$\mathcal{L}\{e^{at} \cos(bt)\} = F(s-a)$$

$$= \frac{s-a}{(s-a)^2+b^2}, \quad s > a$$

## Section 6.2: Laplace Transform of Derivatives

A function is of **exponential order  $\alpha$**  if there exists positive constants  $C$  and  $T$  such that

$$|f(t)| < Ce^{\alpha t} \text{ for all } t > T.$$

For example:

- $f(t) = \cos(5t)e^{7t}$  has  $\alpha = 7$ .
- $e^{t^2}$  does not have an exponential order.

$$\boxed{f(t) = t} < e^t$$

$\alpha = 1$

3. If  $f(t)$  is continuous on  $[0, \infty)$  and  $f'(t)$  is piecewise continuous on  $[0, \infty)$  with both exponential order  $\alpha$ , then prove for  $s > \alpha$ ,

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0) = sF(s) - f(0).$$

$$u = e^{-st} \quad u' = -s e^{-st}$$

$$v' = f'(t)dt \quad v = f(t)$$

I.B.P.

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t) dt = \lim_{N \rightarrow \infty} \int_0^N e^{-st} f'(t) dt \\ &= \left( \lim_{N \rightarrow \infty} \int_0^N e^{-st} f(t) dt \right) - \lim_{N \rightarrow \infty} \int_0^N -s e^{-st} f(t) dt \\ &\quad \boxed{\left( \lim_{N \rightarrow \infty} e^{-sN} f(N) \right) - f(0)} + s \boxed{\int_0^\infty e^{-st} f(t) dt} = F(s) \end{aligned}$$

$$\lim_{N \rightarrow \infty} f(N) e^{-sN} \leq \lim_{N \rightarrow \infty} |e^{-sN} f(N)| < \lim_{N \rightarrow \infty} e^{-sN} \cdot C e^{\alpha N} = \lim_{N \rightarrow \infty} C e^{-N(s-\alpha)} \rightarrow 0$$

$f(t)$  is exponential order  $\alpha$

$$|f(N)| < C e^{\alpha t}$$

$$s - \alpha > 0$$

$$\boxed{\left( \lim_{N \rightarrow \infty} e^{-sN} f(N) \right) - f(0)} + s \boxed{\int_0^\infty e^{-st} f(t) dt} = F(s)$$

$$s > \alpha$$

$$\mathcal{L}\{f'(t)\} = 0 \quad -f(0) + s F(s) = \boxed{sF(s) - f(0)} \quad \checkmark$$

4. Using the property from problem 3 and the fact that  $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$  for  $s > 0$ , find  $\mathcal{L}\{\sin(bt)\}$ .

$$-b\sin(bt) = f'(t)$$

$$\cos(bt) = f(t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} \quad s > 0$$

$$\mathcal{L}\{\sin(bt)\} = ?$$

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \mathcal{L}\{-b\sin(bt)\} = sF(s) - f(0) = s\left(\frac{s}{s^2+b^2}\right) - 1 \\ &= \frac{s^2 - (s^2+b^2)}{s^2+b^2} = \frac{-b^2}{s^2+b^2} \end{aligned}$$

\* we have

$$\begin{aligned} \mathcal{L}\{-b\sin(bt)\} &= \frac{-b}{s^2+b^2} \\ -\frac{1}{b} \mathcal{L}\{-b\sin(bt)\} &= -\frac{1}{b} \left( \frac{-b}{s^2+b^2} \right) \\ \mathcal{L}\{\sin(bt)\} &= \frac{b}{s^2+b^2} \end{aligned}$$

5. If  $\mathcal{L}\{f(t)\} = F(s)$  for all  $s > \alpha$ , using the property from problem 3, show that

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \quad \text{for all } s > \alpha.$$

$$\begin{aligned} g(t) &= f'(t) \\ g'(t) &= f''(t) \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{g'(t)\} &= s\mathcal{L}\{g(t)\} - g(0) \\ &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \end{aligned}$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

6. Using induction show that

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0).$$

7. Let  $F(s) = \mathcal{L}\{f\}$  and assume  $f(t)$  is piecewise continuous on  $[0, \infty)$  and of exponential order  $\alpha$ . Prove that for  $s > \alpha$  it follows that

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}.$$

8. Using the definition of the Laplace transform, the result that  $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$  for  $s > a$  and the property above, find a formula for  $\mathcal{L}\{t^n e^{at}\}$ .

# Welcome! Today we'll start with Worksheet 21.

ODE

Sectoin 6.1-6.2: Properties of Laplace Transforms

## Common Laplace Transforms

$$f(t) = e^{7t} \quad F(s) = \frac{1}{s-7}, \quad s > 7$$

$$f(t) = \sin(3t)$$

$$F(s) = \frac{3}{s^2 + 9}, \quad s > 0$$

$$f(t) = \cos(3t)$$

$$F(s) = \frac{s}{s^2 + 9}, \quad s > 0$$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
$e^{at}$	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2 + b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \dots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$

$$f(t) = t^2$$

$$n=2$$

$$F(s) = \frac{2!}{s^3} = \frac{2}{s^3}, \quad s > 0$$

$$f(t) = t^3$$

$$n=3$$

$$F(s) = \frac{3!}{s^4}, \quad s > 0$$

$$F(s) = \frac{6}{s^5}, \quad s > 0$$

## Common Properties of Laplace Transforms

L.1  $\mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}$ , where  $c$  is a constant.

L.2  $\mathcal{L}\{f_1(t) + f_2(t)\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\}$

L.3 If  $F(s) = \mathcal{L}\{f(t)\}$  exists for all  $s > \alpha$ , then  $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$  for all  $s > \alpha + a$ .

✓ L.4 If  $F(s) = \mathcal{L}\{f(t)\}$  exists for all  $s > \alpha$ , then for all  $s > \alpha$ ,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0).$$

⊗ L.5 If  $F(s) = \mathcal{L}\{f(t)\}$  exists for all  $s > \alpha$ , then  $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$  for all  $s > \alpha$ .

Monday  
Reading

Chapters 3 and 8 : Worksheets 14-19 HW 6-8

Worksheet 14 : Intro to Systems

Reading Equations

Approximating Solutions Visually in Phase Plane

Numerically Approximating Solutions

Worksheet 15 :

Reading and Sketching Vector Fields

Finding x and y nullclines

Using nullclines to describe long-term behavior

Worksheet 16 :  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$  ← separation, Integrating Factor

Setting up and solving phase plane equation.

Sketching solutions in phase plane Implicit

Finding Equilibrium solutions

Worksheet 17 : Linear Systems

Finding eigenvalues of matrix of coefficients

Solve for  $y$  by substituting into  $x'$  equation

No eigenvectors needed to find general solution

## Worksheet 18: Stability of Equilibrium Linear System

For two distinct real eigenvalues

(0,0)

→ Find eigenvectors

→ Give vector form of solution

→ Sketch phase plane near (0,0)

Based on eigenvalues determine stability  
and sketch approximate phase plane.

## Worksheet 19: Non-linear Systems

Find Equilibrium

Find Jacobian at each equilibrium

Classify stability

Sketch approx phase plane near equilibrium.

No Laplace Transform !! (Yet)