$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = -y$$

x-nullcline: points (x,y) in phase plane $dx/dt = 0$. Vectors point straight up or down

① Solve $\frac{dx}{dt} = 0$. $y - x = 0$ when $y = x$

② Check directions on each x-nullcline

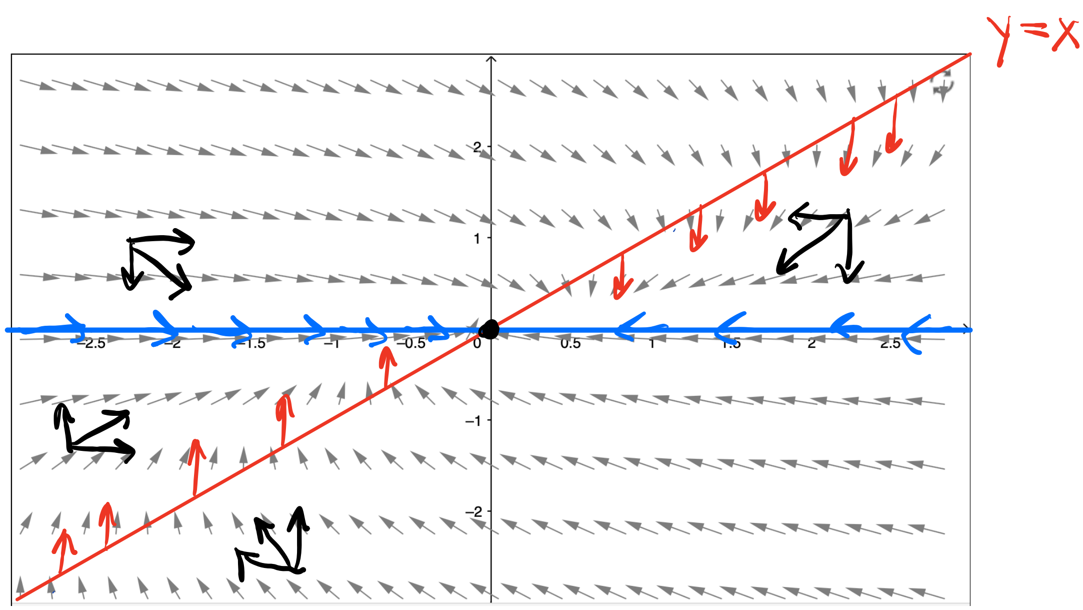
$dy/dt = -y$ >0 when $y<0$; <0 when $y>0$

y-nullcline: points (x,y) in phase plane $dy/dt = 0$. Vectors point directly left or right.

① Solve $dy/dt = 0$ $\quad$ $dy/dt = -y = 0$ when $y = 0$

② Check directions on each y-nullcline

$dx/dt = y - x \approx -x$ >0 when $x<0$; <0 when $x>0$



3. A certain system of differential equations for the variables R and S describes the interaction of rabbits and sheep grazing in the same field. On the phase plane below, dashed lines show the R and S nullclines along with their corresponding vectors.

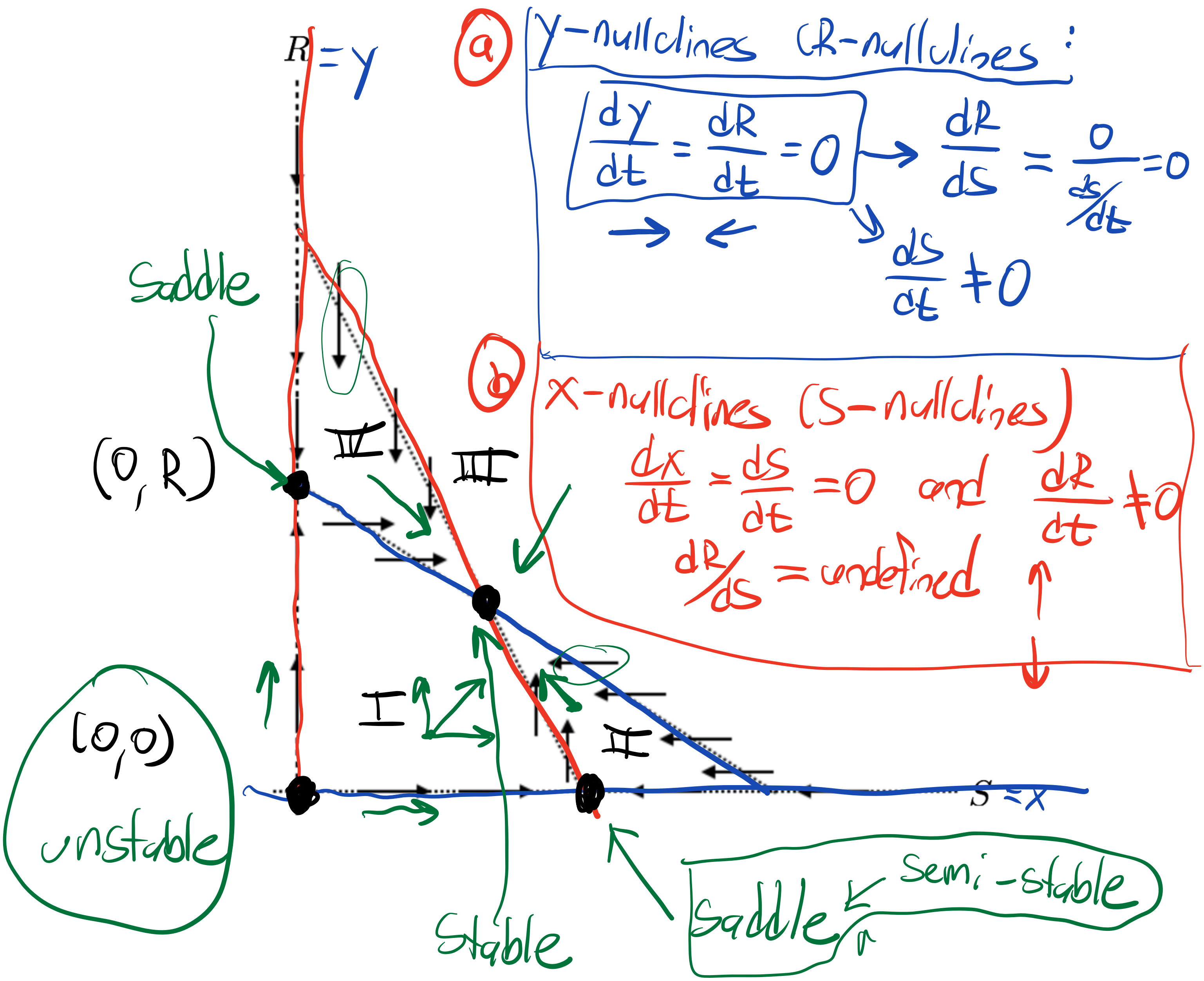
   (a) Identify the R nullclines and explain how you know.

   (b) Identify the S nullclines and explain how you know.

   (c) Identify all equilibrium points.

   (d) Notice that the nullclines carve out 4 different regions of the first quadrant of the RS plane. In each of these 4 regions, add a prototypical-vector that represents the vectors in that region. That is, if you think the both R and S are increasing in a certain region then, draw a vector pointing up and to the right for that region.

   (e) What does this system seem to predict will happen to the rabbits and sheep in this field?



Phase Plane Equations

Consider a system of two differential equations:

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

Recall from the chain rule we have

$$\frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dt},$$

which gives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x,y)}{f(x,y)}.$$

1. Write and solve the corresponding phase plane equation for the system

$$\frac{dx}{dt} = 7y$$
$$\frac{dy}{dt} = -2x$$

next week

$$\boxed{\begin{aligned} x(t) &= f(t) \\ y(t) &= g(t) \end{aligned}}$$

$y = f(x)$

Phase Plane Equation

1st order diff eq

$$\boxed{\frac{dy}{dx} = \frac{-2x}{7y}}$$

Separable:

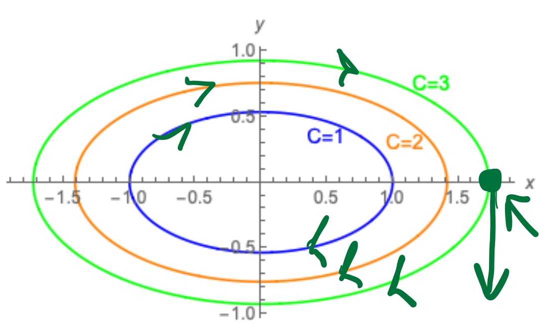
$7y\,dy = -2x\,dx$

$\frac{7}{2}y^2 = -x^2 + C$

$$\boxed{x^2 + \frac{7}{2}y^2 = C}$$

2. Make a sketch of several solutions in the phase plane, include arrows to indicate how solutions behave with respect to time.

$\frac{dy}{dt} = -$



• Direction you move along curves

$(x, 0)$

Equilibrium Solutions

A point (x_0, y_0) is called an **equilibrium** (or critical point) of the system

Solve a system of equations

$$\boxed{\begin{aligned}\frac{dx}{dt} &= f(x,y) = 0\\ \frac{dy}{dt} &= g(x,y) = 0\end{aligned}}$$

Simultaneously

if both $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$.

The corresponding solution $(x(t), y(t)) = (x_0, y_0)$ is called an **equilibrium solution**.

3. Find the equilibrium to the system.

(a) $\frac{dx}{dt} = 2x - y + 8$
$\frac{dy}{dt} = 3x + 6$

① Set both equations equal to 0. $2x - y + 8 = 0$, $3x + 6 = 0$

② Substitute, factor, eliminate etc to solve for (x, y)

using $\frac{dy}{dt} = 3x + 6 = 0$ we get $\underline{x = -2}$

Any equilibrium must have $x = -2$.

To find conditions on y, substitute back into $\frac{dx}{dt} = 2(-2) - y + 8 = 0$ $\quad y = 4$

Thus we have one equilibrium at $(x, y) = (-2, 4)$

(b) $\frac{dx}{dt} = y^2 - xy = 0$
$\frac{dy}{dt} = 2xy - 4 = 0$

$\frac{dx}{dt} = y(y - x) = 0 \rightarrow$ x-nullclines: $y = 0$ $\quad y = x$

If $x = \sqrt{2}$, $y = \sqrt{2}$: $(\sqrt{2}, \sqrt{2})$

$(-\sqrt{2}, -\sqrt{2})$

Case 1: $\cancel{y = 0}$

$\frac{dy}{dt} = 0 - 4 = 0$ no solution

Case 2: $\boxed{y = x}$ $\leftarrow 2x^2 - 4 = 0$ $\quad \boxed{x = \pm\sqrt{2}}$

$$\frac{dy}{dt} = 0 \rightarrow \cancel{x}\, y = \sqrt{2} \qquad 2xy - 4 = 0$$

$$\frac{dx}{dt} = \left(\frac{2}{x}\right)^2 - \left(\frac{2}{x}\right)x = 0 \qquad 2xy = 4$$

$$\boxed{x = \pm\sqrt{2}} \qquad \boxed{y = \frac{4}{2x} = \frac{2}{x}}$$

$$\text{If } x = \sqrt{2} \rightarrow y = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad (\sqrt{2}, \sqrt{2})$$

$$x = -\sqrt{2} \qquad y = -\sqrt{2} \qquad (-\sqrt{2}, -\sqrt{2})$$

4. Find the equilibrium. Then find and solve the phase plane equation.

(a) $\frac{dx}{dt} = 6x$
$\frac{dy}{dt} = 3y$

Equilibrium: $dx/dt = 6x = 0 \Rightarrow x = 0$ if $x=0$ then $dy/dt = 3y = 0 \Rightarrow y = 0$

$\boxed{\text{one equilibrium at } (0,0)}$

$$\frac{dy}{dx} = \frac{3y}{6x} = \frac{y}{2x}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{2x}\,dx$$

$$e^{\ln|y|} = e^{\frac{1}{2}\ln|x| + C}$$

$$|y| = \left(e^{\ln|x|}\right)^{\frac{1}{2}} \cdot e^{C}$$

$$|y| = \sqrt{|x|} \cdot C$$

$$|y| = C\sqrt{|x|}$$

$$\boxed{y = C\sqrt{|x|}}$$

(b) $\frac{dx}{dt} = 4 - 4y$
$\frac{dy}{dt} = -4x$

Equilibrium:

$\frac{dx}{dt} = 4 - 4y = 0$ if $y = 1/4$

Then $\frac{dy}{dt} = -4x = 0$

means $x = 0$.

$\boxed{\text{One equilibrium at } (0, 1/4)}$

Phase Plane Equation

$\frac{dy}{dx} = \frac{-4x}{4-4y}$ separable:

$$\int (4-4y)\,dy = \int -4x\,dx$$

$$4y - 2y^2 = -2x^2 + C$$

$$\boxed{-2y^2 + 4y + 2x^2 = C}$$

(c) $\boxed{\frac{dx}{dt} = 2y^2 - y = 0}$
$\frac{dy}{dt} = x^2 y = 0$

Solving $\frac{dx}{dt} = 2y^2 - y = y(2y-1) = 0$ gives

$y = 0$ OR $y = \frac{1}{2}$

checking dy/dt

(1) If $y = 0$ then $\frac{dy}{dt} = 0$ for all x.

$\boxed{\text{Infinite number of equilibrium}}$

☆ $\boxed{\text{All points of form } \boxed{(x,0)} \text{ are equilibrium.}}$

(2) If $y = \frac{1}{2}$ $\frac{dy}{dt} = \frac{x^2}{2} = 0$ $x = 0$ $\boxed{\left(0, \frac{1}{2}\right)}$ ☆

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x^2 y}{2y^2 - y}$$

$\boxed{\text{Not Linear}}$

$$\boxed{\frac{dy}{dx} = \frac{x^2 y}{2y^2 - y}}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$(2y^2 - y)\frac{dy}{dx} = x^2 y$$

$$(2y^2 - y)\,dy = x^2 y\,dx$$

$$\int (2y - 1)\,dy = \int x^2\,dx$$

$$y^2 - y = \frac{1}{3}x^3 + C \rightarrow \boxed{y^2 - y - \frac{1}{3}x^3 = C}$$