Homework Set 1 Solutions

1. Consider the following systems of rate of change equations:

System A

$$\frac{dx}{dt} = 3x\left(1 - \frac{x}{10}\right) - 20xy$$
$$\frac{dy}{dt} = -5y + \frac{xy}{20}$$

System B

$$\frac{dx}{dt} = 0.3x - \frac{xy}{100}$$
$$\frac{dy}{dt} = 15y\left(1 - \frac{y}{17}\right) + 25xy$$

In both of these systems, x and y refer to the number of two different species at time t. In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.

Figure out which system is which and explain the reasoning behind your decision.

Answers may vary and this is just one possible example.

We start by considering the relative gain or loss to either Population x or Population y in each system:
System A: The "$-20xy$" term in the $\frac{dx}{dt}$ equation indicates that increasing y (while leaving x fixed) has a large negative impact on x, because this rate of change will (at some point relative to the population sizes) become negative, meaning Population x is in decline. Similarly, looking at the "$+\frac{xy}{20}$" term in the equation $\frac{dy}{dt}$ shows that increasing x will have a small positive impact on Population y.
$\Longrightarrow$ x = small prey (*e.g.*, guinea pigs), y = large predator (*e.g.*, velociraptors).

System B: Similar reasoning applies. In the $\frac{dx}{dt}$ rate of change equation, the term $-\frac{xy}{100}$ results in a small loss to x when y is increased. In the $\frac{dy}{dt}$ rate of change equation, the term $+25xy$ results in a large gain to y when x is increased.
$\Longrightarrow$ x = large prey (*e.g.*, humans), y = small predator (*e.g.*, rabid squirrels).

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2. Consider the rate of change equation

$$\frac{dy}{dt} = 0.5y(2 + y)(y - 8),$$

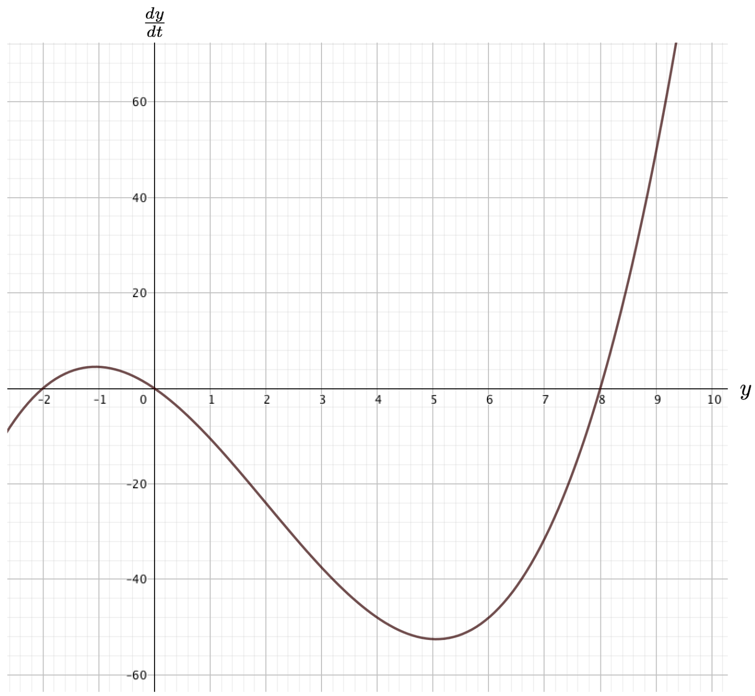
which has been created to provide predictions about the future population of rabbits over time.

(a) For what values of y is $y(t)$ increasing? Explain your reasoning.

(b) For what values of y is $y(t)$ decreasing? Explain your reasoning.

(c) For what values of y is $\dfrac{dy}{dt}$ neither positive nor negative? What does this imply about the solution function $y(t)$?

Note: students have not yet "seen" phase lines or been asked to think about graphing $\frac{dy}{dt}$ versus y, so all of this reasoning should be somewhat informal. They have also probably not heard the term "equilibrium solution" yet; this is meant to prime their thinking about the idea of equilibrium solutions.

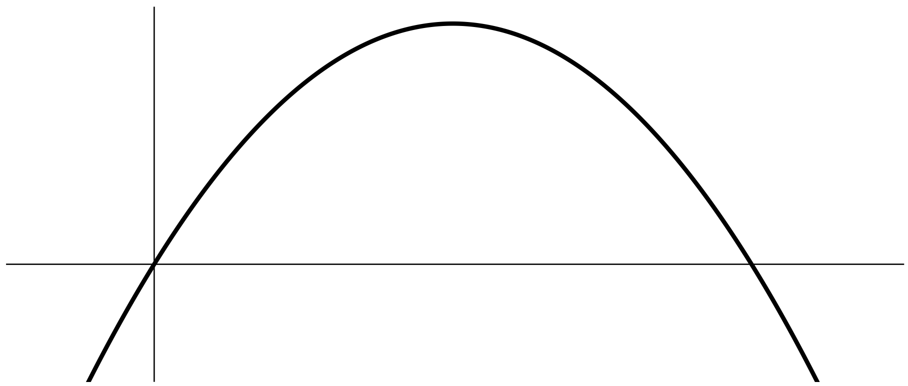


(a) $y(t)$ is increasing whenever $\frac{dy}{dt}$ is positive. The graph above (or some other equivalent sign analysis of the derivative function) shows this happens whenever $y \in (-2, 0) \cup (8, \infty)$. In the rabbit context, ignoring negative y values, the population increases when the rabbit population is above 8.

(b) $y(t)$ is decreasing whenever $\frac{dy}{dt}$ is negative. The graph shows this happens whenever $y \in (0, 8)$. In the rabbit context, ignoring negative y values, the population decreases when the rabbit population is between 0 and 8.

(c) The only time that $\frac{dy}{dt}$ is neither $(+)$ nor $(-)$ is when it equals 0, which happens when $y = -2, 0$, or 8. These constant functions $y(t) = -2, 0, 8$ are (*equilibrium*) solutions because they satisfy the differential equation. In the rabbit context, ignoring negative y values, the population stays at 0 and 8 if that is their initial population.

3. Valeria created the following graph to help her analyze solutions to the differential equation $\dfrac{dy}{dt} = 2y\left(1-\dfrac{y}{10}\right)$. What is this a graph of (*i.e.*, what are the axes for this graph)? What information about solutions can you glean from this graph?



The graph depicts the function $2y - \frac{2y^2}{10} = 2y\left(1-\frac{y}{10}\right)$ and hence the differential equation, so the independent/horizontal/x-axis corresponds to the independent variable y, and the dependent/vertical/y-axis corresponds to the quantity $\frac{dy}{dt}$. In other words, it's a graph of how $\frac{dy}{dt}$ depends on y. This graph can tell you (as above) when $\frac{dy}{dt}$ is $(+)$ or $(-)$, and thus when $y(t)$ is increasing or decreasing. Moreover, it tells you there are two constant (equilibrium) solutions to the differential equation, when $y = 0, 10$ because these clearly satisfy the differential equation.

4. Suppose two students are memorizing the elements on a list according to the rate of change equation
$$\frac{dL}{dt} = 0.5(1-L),$$
where L represents the fraction of the list that is memorized at any time t.
   (a) If one of the students knows one-third of the list at time $t = 0$ and the other student knows none of the list, which student is learning most rapidly at this instant? Why?
   (b) What does the rate of change equation predict for someone who begins with the list completely memorized? Explain.
   (c) Suppose now that the list is infinitely long, like the decimal representation for π. In reality no one can memorize all the digits to π, but what does the rate of change equation predict will happen for a person who starts out not knowing any of the digits? That is, according to the rate of change equation, if $L = 0$ at time $t = 0$, is there ever a value of t for which $L = 1$? Explain.

   (a) This information gives us different initial conditions to test in the DE. If at $t = 0$ a student knows $L = 1/3$ of the list, then the rate of change is $\frac{dL}{dt} = 0.5(1-\frac{1}{3}) = 0.5(\frac{2}{3}) = \frac{1}{3}$. If a student knows none of the list at $t = 0$ then they are learning at a rate of $\frac{dL}{dt} = 0.5(1-0) = \frac{1}{2}$, so the student who has memorized less to begin with is learning the list faster.

(b) If a student knows the entire list at $t = 0$, rather than just a fraction of it, then their value for L is 1, which makes their rate of change 0. So, they couldn't learn any more of the list but they aren't un-learning any of it either! In this way, the rate of change equation predicts that they will always know the entire list.

(c) *Note: as of the first HW assignment we haven't discussed existence or uniqueness yet, so the presence of a "horizontal asymptote" at the solution $L = 1$ is vague intuition at this point. It certainly hasn't been proven yet that other solutions will only approach this value but never reach it. Without Uniqueness, this problem is challenging.*

Answers will vary. Technically, no: the solution curve with initial condition $L = 0$ will never pass through a point at $L = 1$, not for any finite time t. As t grows without bound, we can get arbitrarily close to $L = 1$ but will never actually realize it. (*Looking for language here that captures that L gets "very" close to 1 "in the long run" but "never quite gets there", etc.*)

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5. The letter y appears in two places in the differential equation $\dfrac{dy}{dt} = 0.3y$. Is it appropriate to think of both occurrences of y as function of t? Explain.

Yes, on the left-hand side, the derivative is being taken with respect to t, meaning y is necessarily a function of t. On the right-hand side, though no t is written, it is implicit that y is a function of t (suggested by the LHS already). We are simply suppressing the t-notation on that side to make it easier to look at. We can read this differential equation with meaning. Namely, the derivative with respect to time of the function $y(t)$ is equal to 0.3 times the function of $y(t)$ itself.

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6. In algebra, the goal of solving an equation such as $x^2 + 4x = 2$ is to find the values of x that make a true statement. In differential equations, what is the goal of solving an equation such as $\dfrac{dx}{dt} + 4x = 2$?

The goal here is to find a *family of functions*, $x(t)$, that satisfies the equation (i.e., $x(t)$ makes the equal sign true when plugged into each side of the equation). We can read this differential equation with meaning to help here. Namely, the derivative with respect to time of the function $x(t)$ plus 4 times the function $x(t)$ itself equals 2.

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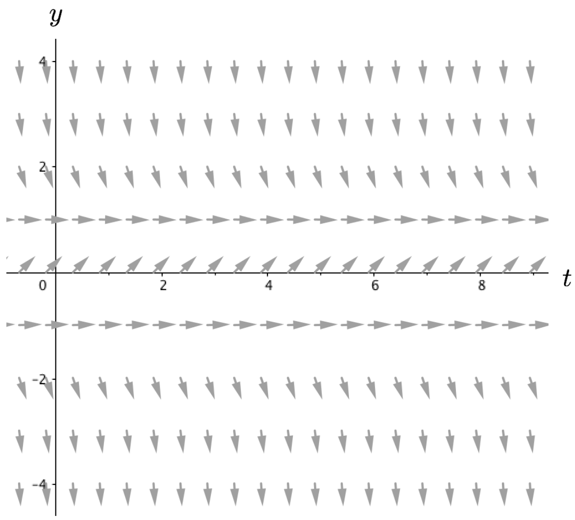
7. For the differential equation $\dfrac{dy}{dt} = 1 - y^2$,

   (a) Sketch a slope field by hand.

   (b) Describe any shortcuts or patterns you used to make the task easier.

   (c) Sketch several $y(t)$ graphs.

(a) 

(b) **Answers will vary.** Since the equation is autonomous, the slope of the vectors at any given y-value will be constant across time (the slopes won't change as you move horizontally). There should also be vertical symmetry above and below $y = 0$ (because of the y^2 term).

(c) **Answers will vary.** There should be multiple solution curves sketched here, preferably with qualitatively different initial conditions: an "S"-curve between $y = -1$ and 1, a curve sloping down to the asymptote at $y = 1$, and a curve sloping away from the asymptote at $y = -1$.

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8. Differential equations are often referred to as mathematical models. Explain what the phrase "mathematical model" means to you, what previous experiences you have had with mathematical models, and how the mathematical use of the word model is similar to and/or different from the everyday use of the word model (*e.g.*, fashion model, model airplane, model student).

Answers will vary.
