# Cooling Coffee

A group of students want to develop a rate of change equation to describe the cooling rate for hot coffee in order that they can make predictions about other cups of cooling coffee. Their idea is to use a temperature probe to collect data on the temperature of the coffee as it changes over time and then to use this data to develop a rate of change equation.

The data they collected is shown in the table below. The temperature $C$ (in degrees Fahrenheit) was recorded every 2 minutes over a 14 minute period.

| Time (min) | Temp. (°F) | $\frac{dC}{dt}$ (°F per min) |
|---|---|---|
| 0 | 160.3 | −19.95 |
| 2 | 120.4 | −15.55 |
| 4 | 98.1 | −8.9 |
| 6 | 84.8 | −4.9 |
| 8 | 78.5 | −2.6 |
| 10 | 74.4 | −1.6 |
| 12 | 72.1 | −0.725 |
| 14 | 71.5 | −0.3 |

$C'(0) \approx \dfrac{120.4 - 160.3}{2}$

$C'(2) \approx \dfrac{C(4) - C(0)}{4} = -15.55\ {}^\circ F/min$

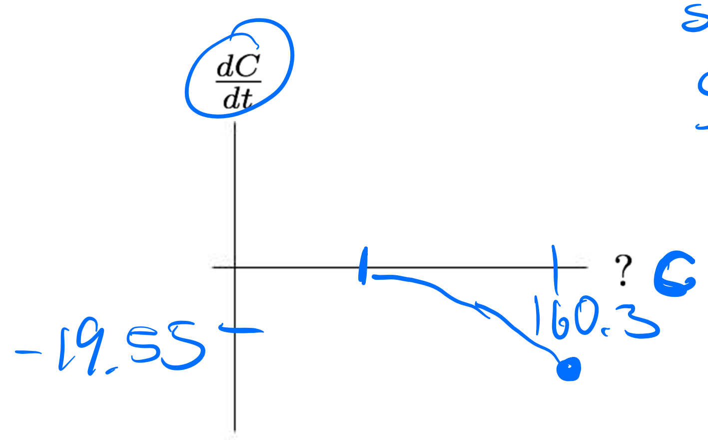
$\frac{dC}{dt}$ is getting less negative as $t$ increases and $C$ is decreasing

1. Figure out a way to use this data to fill in the third column whose values approximate $\dfrac{dC}{dt}$, where $C$ is the temperature of the coffee.

2. Do you expect $\dfrac{dC}{dt}$ to depend on just the temperature $C$, on just the time $t$, or both the temperature $C$ and the time $t$?

→ Closer $C$ is to $A$ smaller in magnitude $\frac{dC}{dt}$

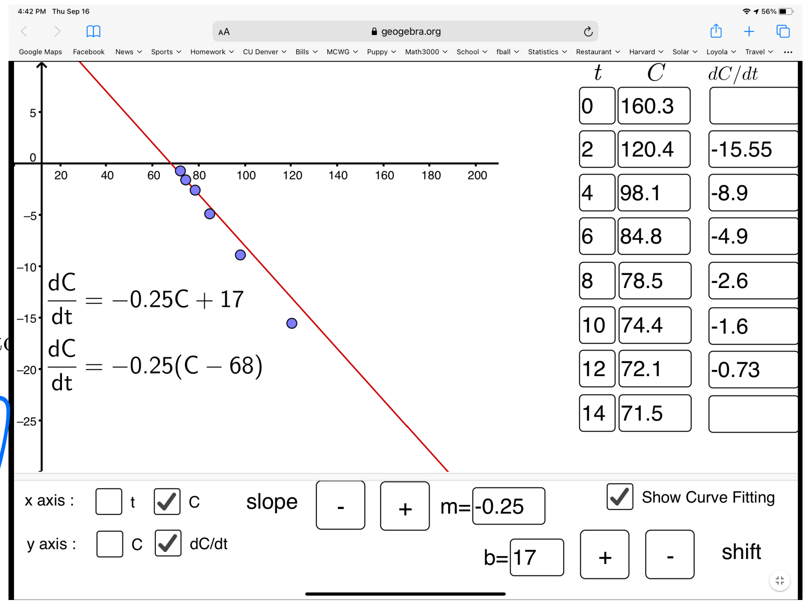
3. Sketch below your best guess for the graph of $\dfrac{dC}{dt}$.



## Newton's Law of Heating and Cooling

4.

(a) Input the data from your extended table in question 1 into the GeoGebra applet https://ggbm.at/uj2gbz3V to plot points for $\frac{dC}{dt}$ vs. $C$. Does this plot confirm or reject your sketch from question 3?




(b) Toggle on the curve fitting t

$\frac{dC}{dt} = -0.25C + 17$

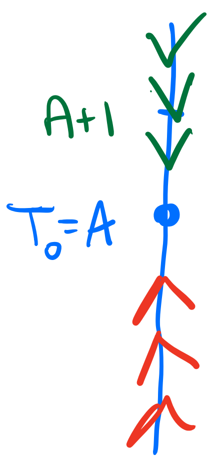
$dC/dt = -0.25(C-68)$

Newton's Law of Heating and Cooling states that the temperature $T$ of an object at time $t$ changes at a rate which is proportional the difference of its temperature and the temperature of its surrounding:

$$\frac{dT}{dt} = -k(T - A)$$

where $A$ is a constant that denotes the ambient temperature and $k > 0$ is a constant that depends on the object.

5. What happens as $\mathbf{t} \to \infty$ if the initial temperature $\mathbf{T_0} > \mathbf{A}$? If $\mathbf{T_0} < \mathbf{A}$?




at $T_0 = A-1$

$\frac{dT}{dt} = (-k)(-1)$

$= k > 0$

Increases if $T_0 < A$ and approaches Ambient Temp as $t \to \infty$

$\frac{dT}{dt} = -k(T-A)$ has one equilibrium at $T=A$

at $T = A+1$, $\frac{dT}{dt} = (-k)(1)$

$= -k < 0$

If $T_0 > A$, Decreases and approaches Ambient Temp as $t \to \infty$

## Practice: Applications to Economics

6. Let $S(p)$ denote the number of units of a particular commodity supplied to the market at a price of $p$ dollars per unit, and let $D(p)$ denote the corresponding number of units demanded by the market at the same price.

- In static circumstances, market equilibrium occurs at the price where demand equals supply.
- However, certain economic models consider a more dynamic economy in which price, supply, and demand are assumed to vary with time.
- One of these, the Evans price adjustment model, assumes that the rate of change of price with respect to time $t$ is proportional to the shortage, which is the difference between the quantity demanded and the quantity supplied..

(a) Write a differential equation for the rate of the change of the price of the good with respect to time.

$$\frac{dp}{dt} = k(D-S) = k[(8-2p)-(2+p)] = k(6-3p)$$
$$= -3k(p-2)$$

(b) If we assume that supply and demand are linear functions given by

$$S(p) = 2 + p \quad \text{and} \quad D(p) = 8 - 2p,$$

Find a general solution to the differential equation in part (a).

$$\int \frac{1}{p-2}\, dp = \int -3k\, dt$$

$$\ln|p-2| = -3kt + C$$

$$p-2 = e^{-3k} \cdot e^{C} = Ce^{-3kt}$$

$$\boxed{P = Ce^{-3kt} + 2}$$

$$P = 3e^{-3(0.183)t} + 2$$

$$P = 3e^{-0.549t} + 2$$

$$\boxed{\lim_{t\to\infty} P(t) = 2}$$

Price approaches \$2.

(c) If the price is \$5 at time $t = 0$ and \$3 at time $t = 2$, determine what happens to $p$ in the long run.

First we solve for $C$ using $P(0)=5$.

$5 = Ce^{0}+2 \quad C=3 \quad P = 3e^{-3kt}+2$

Next we use $P(2)=3$ to solve for $k$:

$3 = 3e^{-3k(2)}+2 \quad e^{-6k} = \frac{1}{3} \quad k = \frac{\ln(\frac{1}{3})}{-6} \approx 0.183$

## Practice: Applications to Forensic Science

7. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3°F. One hour later, the temperature of the body is 89.0°F. The temperature of the room has been maintained at a constant 68°F.

   (a) Assuming the temperature, $T$, of the body obeys Newton's Law of Cooling, write a differential equation for $T$. Your equation will include the constant $k$ (for now).

$$\frac{dT}{dt} = -k(T-A) = -k(T-68)$$

Let $t$ denote hrs since 9Am.

   (b) Solve the differential equation to estimate the time the murder occurred.

Answer is going to depend on $k$ and $+C$

1. Find General solution
2. Use given info to solve for $c$.
3. Finally use given info solve for $k$.
4. Now you have a solution!

① $\int \frac{1}{T-68}\,dT = \int -k\,dt \Rightarrow \ln|T-68| = -kt + C$

$$T-68 = e^{-kt+C} = Ce^{-kt} \Rightarrow \boxed{T = Ce^{-kt} + 68}$$

② $T(0) = 90.3$ so $90.3 = Ce^{-k\cdot 0} + 68 = C + 68$

$C = 22.3°F$ $\boxed{T = 22.3e^{-kt} + 68}$

(3) we also know $T(1) = 89$, so

$$89 = 22.3e^{-k(1)} + 68$$

$$\frac{21}{22.3} = e^{-k} \qquad \ln\left(\frac{21}{22.3}\right) = -k \quad k = -\ln\left(\frac{21}{22.3}\right) \approx 0.06$$

(4) Thus $\boxed{T = 22.3e^{-0.06t} + 68}$

To time the death we use 98.6°F as the temperature of a healthy adult. When was the body temp 98.6°F?

$$98.6 = 22.3e^{-0.06t} + 68$$ solving for $t \approx -5.27$ hrs since 9 AM, which is approx 3:44 AM.