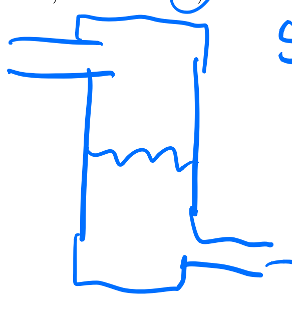
A Salty Tank

1. A very large tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at a rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at a rate of 1 gallon per minute.

$\boxed{1 \text{ lb/gal}, \; 2 \text{ gal/min}}$

(a) Should the rate of change equation for this situation depend just on the amount of salt S in the tank, the time t, or both S and t? Explain your reasoning.



S - How much salt in tank at time t (lb)

$\boxed{1 \text{ gal/min}}$

$$\frac{dS}{dt} = ? \, f(t, S)$$

6 lb / 15 gal

At $t=0$

$\boxed{\text{Solution} = 15 \text{ gal}}$

Salt = 6 lbs

Every minute

2 gal go in

1 gal flows out

As time goes on Volume is changing.

Depend on S too since amt of S in tank changes

(b) The following is a general rule of thumb for setting up rate of change equations for situations like this where there is an input and an output:

$\boxed{\text{rate of change} = \text{rate of change in} - \text{rate of change out}}$

Using the above rule of thumb, figure out a rate of change equation for this situation.
Hint: Think about what the *units* of $\frac{dS}{dt}$ need to be, where S is the amount of salt in the tank in pounds.

Compartmental Analysis

lb/min

$$\frac{dS}{dt} = \begin{pmatrix}\text{Rate of} \\ S \text{ Flowing} \\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{Rate of } S \\ \text{flowing out}\end{pmatrix}$$

t	V
0	15 gal
1	16
2	17

$$\begin{pmatrix}\text{concentration} \\ \text{flowing in}\end{pmatrix}\begin{pmatrix}\text{Rate} \\ \text{of} \\ \text{inflow}\end{pmatrix} - \begin{pmatrix}\text{concentration} \\ \text{of solution} \\ \text{out}\end{pmatrix}\begin{pmatrix}\text{Rate} \\ \text{of} \\ \text{outflow}\end{pmatrix}$$

$$\left(1 \text{ lb/gal}\right)\left(2 \text{ gal/min}\right) - \left(\frac{S(t)}{15+t} \text{ lb/gal}\right)\left(1 \text{ gal/min}\right)$$

$$dS/dt = 2 - \frac{S}{15+t}$$

(c) Use the slope field for this differential equation in the GeoGebra applet, https://ggbm.at/PFRcbkbZ, to sketch a graph of the solution with initial condition $S(0) = 6$. Reproduce this sketch below. Estimate the amount of salt in the tank after 15 minutes.

$S(15) \approx 26$ or 27 lbs

The differential equation you developed for the salty tank is not separable, and therefore using the technique of separation of variables is not appropriate. This differential equation is called **first order linear**, which means it has the form

$$\boxed{a_1(x)\frac{dy}{dx} + a_2(x) \cdot y = b(x),}$$

where $a_1(x)$, $a_2(x)$, and $b(x)$ are all continuous functions of x alone.

Note if we divide both sides by $a_1(x)$, we can rewrite any first order linear differential equation in **standard form**:

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x).}$$

The following technique, which we refer to as the **reverse product rule**, can be used find the general solution to a first-order linear equation.

2. Review the product rule as you remember it from calculus. In general symbolic terms, how do you represent the product rule? How would you describe it in words?

$(fg)' = f'g + fg'$

Consider the differential equation $\boxed{\frac{dy}{dx} + 2y = 3.}$ Note that this is a first order linear differential equation already in **standard form**, where $P(x) = 2$ and $Q(x) = 3$ are both continuous functions. The following illustrates a technique for finding the general solution to linear differential equations. The inspiration for the technique comes from a creative use of the product rule and the Fundamental Theorem of Calculus, as well as use of the previous technique of separation of variables.

Use the product rule to expand $\boxed{(y\mu)'}$. μ = mu text r	$y'\mu + y\mu' = y'\mu + \mu' y$	Box 0
In the equation $\frac{dy}{dx} + 2y = 3$, rewrite $\frac{dy}{dx}$ as y'.	$y' + 2y$	Box 1
Notice that the left-hand side of the equation in Box 1 looks a lot like the expanded product rule but is missing the function μ. So multiply both sides by μ, a function that we will determine shortly.	$\mu y' + \boxed{2\mu} y = 3\mu$	Box 2
Because, so far, μ is an arbitrary function, we can have μ satisfy any differential equation that we want. Use $\mu' = 2\mu$ to rewrite the left-hand side of Box 2 to look like Box 0.	$\mu' = 2\mu$ $\int \frac{1}{\mu} d\mu = \int 2 dx$ $e^{\ln(\mu)} = e^{2x}$ $\mu = e^{2x}$	Box 3

Ignore +C since we need 1 function μ

Use separation of variables to solve $\mu' = 2\mu$.	Done $\mu = e^{2x}$	Box 4
Replace μ in the equation from Box 2 with your solution from Box 4.	$\mu y' + \mu' y = 3\mu$ $e^{2x} y' + 2e^{2x} y = 3e^{2x}$	Box 5
Show that the equation in Box 5 can be rewritten as $\left(ye^{2x}\right)' = 3e^{2x}$ *Hint*: Consider Box 0.	$(\mu y)' = \boxed{(e^{2x} y)' = 3e^{2x}}$	Box 6
Write integrals with respect to x on both sides. Apply the Fundamental Theorem of Calculus.	$\int (e^{2x} y)'\,dx = \int 3e^{2x}\,dx$ $e^{2x} y = \frac{3}{2} e^{2x} + C$	Box 7
Obtain an explicit solution by isolating $y(x)$. $y = \frac{3}{2} + Ce^{-2x}$	$y = \frac{\frac{3}{2} e^{2x}}{e^{2x}} + \frac{C}{e^{2x}}$	Box 8

The key is finding a formula for the function $\mu(x)$ that we multiply on both sides which allowed us to "reverse" the product rule (going from Box 5 to Box 6). The function $\mu(x)$ is called the **integrating factor**.

1 • Check that the differential equation is first order linear, and rewrite it in **standard form**:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

2 • Calculate the integrating factor:

$$\mu(x) = e^{\int P(x)\,dx} \qquad \mu = e^{\int P(x)\,dx}$$

(for convenience, set the abitrary constant $C = 0$.)

3 • Multiply both sides of the standard form by $\mu(x)$, and rewrite the equation as in Box 6:

$$\frac{d}{dx}\left(y\mu(x)\right) = \mu(x)Q(x)$$

.

4 • Integrate both sides with respect to x, and solve for y (if possible).

$(15+t)S = 30t + t^2 + C$ $\boxed{S = \frac{30t+t^2+C}{15+t}}$

3. Use the previous technique, which we refer to as the **method of integrating factors** or **reverse product rule**, to find the general solution for the Salty Tank differential equation from Problem 1.

1. $\boxed{\frac{dS}{dt} = 2 - \frac{S}{15+t}}$

$\frac{dS}{dt} + \underbrace{\frac{1}{15+t}}_{P(x)} S = 2$

2. $\mu = e^{\int \frac{1}{15+t}dt} = e^{\ln(15+t)} = 15+t$

3. $(15+t)S' + (1)S = 2(15+t)$

$\int \frac{d}{dt}\left[(15+t)S\right]dt = \int (30+2t)\,dt$

4. (a) Use the general solution from problem 3 to find the particular solution corresponding to the initial condition $S(0) = 6$ and then use the particular solution to determine the amount of salt in the tank after 15 minutes. That is, compute $S(15)$. Your answer should be close to your estimate from problem 1c. Is it? If not, you likely made an algebraic error.

$S = \frac{30t+t^2+C}{15+t}$ $S(0) = \frac{C}{15} = 6$ $C = 90$

$S(0) = 6$ lbs $\boxed{S = \frac{30t+t^2+90}{15+t}}$

(b) What does your solution predict about the amount of salt in the tank in the long run? How about the concentration?

$\lim_{t\to\infty} \frac{30t+t^2+90}{15+t} = \frac{\infty}{\infty} \overset{L.R.}{=} \lim_{t\to\infty} \frac{30+2t}{1} = \infty$

$\boxed{S(15) = 25.5 \text{ lbs}}$

(c) Explain how you can make sense of the predictions from 4b by using the differential equation itself.

$\frac{dS}{dt} = 2 - \frac{S}{15+t}$ As $t \to \infty$ $\frac{dS}{dt} = 2\,\text{lb/min}$

5. Decide whether each differential equation is linear, and if so write it in standard form.

(a) $x^2y' = x^2 - 3y$

Linear: $x^2y' + 3y = x^2$ $\boxed{y' + \frac{3}{x^2}y = 1}$

(b) $2y\frac{dy}{dx} - 3y = 8$

Not linear since $2y\frac{dy}{dx}$

6. Solve the differential equation:

(a) $z\frac{dw}{dz} + 2w = 5z^3$ $\frac{dw}{dz} + \boxed{\frac{2}{z}}w = 5z^2$ $P(z) = \frac{2}{z}$

① Put into Standard Form $\frac{dw}{dz} + P(z)\,w = Q(z)$

② Find integrating factor $\mu = e^{\int P(z)dz}$ (ignore $+c$)

$\mu = e^{\int \frac{2}{z}dz} = e^{2\ln z} = \left(e^{\ln z}\right)^2$ OR $e^{\ln(z^2)}$ $\mu = z^2$

③ Multiply both sides by μ and express left side as $(w\mu)'$

$z^2w' + 2zw = 5z^4$ $\int (z^2w)'_{dz} = \int 5z^4\,dz$

④ Integrate both sides, solve for w

$z^2w = z^5 + C$ $\boxed{w = z^3 + \frac{C}{z^2}}$

(b) $\sin x\frac{dy}{dx} + y\cos x = x\sin x$, $y\left(\frac{\pi}{2}\right) = 2$

① $y' + \frac{\cos x}{\sin x}y = x$ $P(x) = \frac{\cos x}{\sin x}$

② $\mu = e^{\int \frac{\cos x}{\sin x}dx}$ $\int \frac{\cos x}{\sin x}dx = \int \frac{1}{w}dw = \ln(w)$

$w = \sin x$ $dw = \cos x\,dx$

$\mu = e^{\ln(\sin x)} = \sin x$

③ $\boxed{\sin x\,y' + \cos x\,y} = x\sin x$

④ $\int (\sin x\cdot y)'_{dx} = \int x\sin x\,dx$ $u = x$ $du = dx$ $dv = \sin x\,dx$ $v = -\cos x$

$\sin x\,y = uv - \int v\,du = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$

$y = \frac{-x\cos x}{\sin x} + 1 + \frac{C}{\sin x}$

$y = \frac{-x\cos x}{\sin x} + 1 + \frac{1}{\sin x}$