Second Order Linear Differential Equations

We have developed methods for finding the general homogeneous solution, $y_h(t)$, to a second order differential equation of the form

$$ay'' + by' + cy = 0, \qquad \text{for constants } a, b, \text{ and } c:$$

1. Write the differential equation in standard form, $ay'' + by' + cy = 0$
2. Set up the corresponding characteristic equation: $ar^2 + br + c = 0$ and find the roots.
   - If two distinct real roots r_1 and r_2, the general solution is $y_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
   - If one repeated real root r_1, the general solution is $y_h(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$.
   - If complex solutions $r = \alpha \pm i\beta$, the general solution is $y_h(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$.

We have also discussed how to find the particular solution, $y_p(t)$ to a nonhomogeneous second order differential equation of the form

$$ay'' + by' + cy = f(t).$$

1. Based on the form of $f(t)$, we guess $y_p(t)$ will have a similar form.

$f(t)$	$y_p(t)$
Ct^n (for n a nonnegative integer)	$A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0$
Ce^{rt}	Ae^{rt}
$Ce^{\alpha t}\cos(\beta t)$	$Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t)$
$Ce^{\alpha t}\sin(\beta t)$	$Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t)$

   - You have **resonance** when your initial guess for the particular solution is a homogeneous solution.
   - When there is resonance, multiply the initial guess by t (or t^2 if y_h is a solution with a repeated real root).

2. Plug your guess into the differential equation, and solve for the undetermined coefficients.

Finding the General Solution to the Nonhomogeneous Case

1. Consider the nonhomogeneous differential equation

$ay_h'' + by_h' + cy_h = 0$ $\quad$ $ay_p'' + by_p' + cy = f(t)$

$$ay'' + by' + cy = f(t).$$

Let $y_h(t)$ denote the general solution to corresponding homogeneous equation $ay'' + by' + cy = 0$ and let $y_p(t)$ denote the particular solution for the nonhomogeneous equation. **Show that the function** Plug in $\boxed{y(t) = y_h(t) + y_p(t)}$ **is a solution to the nonhomogeneous differential equation.**

$\boxed{ay'' + by' + cy} = a(y_h'' + y_p'') + b(y_h' + y_p') + c(y_h + y_p)$

$= (ay_h'' + by_h' + cy_h) + (ay_p'' + by_p' + cy_p)$

$0 + f(t) = \boxed{f(t)}$ ✓

$y = y_h + y_p$

2. Consider the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 12x = e^{4t}.$$

(a) Find the general homogeneous solution.

$x'' - x' - 12x = 0$

$r^2 - r - 12 = 0$

$(r-4)(r+3) = 0$

$r = 4, -3$

$\boxed{x_h = c_1 e^{4t} + c_2 e^{-3t}}$

(b) Find the particular solution to the nonhomogeneous differential equation.

$f(t) = e^{4t}$ $\quad$ $\cancel{y_p = Ae^{4t}}$ $\quad$ $y_p = Ate^{4t}$

Solve for A so y_p is a solution

(c) Give the general solution to nonhomogeneous differential equation.

4b) $x_p = \boxed{A}te^{4t}$ $\qquad x_p' = Ae^{4t} + 4Ate^{4t}$

$x'' - x' - 12x = e^{4t}$ $\qquad x_p'' = \underbrace{4Ae^{4t} + 4Ae^{4t}} + 16Ate^{4t}$

$(8Ae^{4t} + \cancel{16Ate^{4t}}) - (Ae^{4t} + \cancel{4Ate^{4t}}) - \cancel{12Ate^{4t}}$

$7Ae^{4t} = e^{4t}$ $\qquad 7A = 1$ $\qquad A = 1/7$

$x_n = C_1 e^{4t} + C_2 e^{-3t}$ $\qquad x_p = \frac{1}{7} te^{4t}$

$$\boxed{x = C_1 e^{4t} + C_2 e^{-3t} + \underbrace{\tfrac{1}{7} te^{4t}}}$$

$x(0) = 5$

$x'(0) = 10$

$x(0) = C_1 + C_2 + 0 = 5$

$x'(t)$

Breaking Up Sums in the Nonhomogeneous Function

To find the general solution to the nonhomogeneous differential equation you simply add the particular solution to the general solution to the corresponding homogeneous equation. This 3-step strategy:

- Find the general solution to the corresponding homogeneous equation
- Find the particular solution to the nonhomogeneous equation
- Add the previous results

is called the **Method of Undetermined Coefficients**.

3. Let $y_1(t)$ denote the particular solution to $ay'' + by' + cy = f_1(t)$ and $y_2(t)$ denote the particular solution to $ay'' + by' + cy = f_2(t)$. Show that $y_p(t) = y_1(t) + y_2(t)$ is the particular solution to $ay'' + by' + cy = f_1(t) + f_2(t)$.

Exercise:

Solution: Plugging in $y_p(t)$ gives

$$a(y_1'' + y_2'') + b(y_1' + y_2') + c(y_1 + y_2)$$

$$= (ay_1'' + by_1' + cy_1) + (ay_2'' + by_2' + cy_2) = f_1(t) + f_2(t) \quad \checkmark$$

Worksheet 9 & 11

4. Consider the differential equation

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 85\sin(2t) + 18.$$

Recall that we have already worked on parts of this problem. We have found that $x_h(t) = C_1e^{-9t} + C_2e^{-t}$ and the particular solution of $f_1(t) = 85\sin(2t)$ is $x_1(t) = -4\cos(2t) + \sin(2t)$. Using these results, finish solving the differential equation and give the general solution.

$\to x_h: \; x'' + 10x' + 9x = 0 \quad r = -1, -9 \quad x_h = C_1e^{-9t} + C_2e^{-t}$

$x_{p_1}: \; x'' + 10x' + 9x = 85\sin(2t) \quad x_{p_1} = -4\cos(2t) + \sin(2t)$

$x_{p_2}: \; x'' + 10x' + 9x = 18 \quad x_{p_2} = 2$

$$\boxed{x = x_h + x_{p_1} + x_{p_2} = C_1e^{-9t} + C_2e^{-t} - 4\cos(2t) + \sin(2t) + 2}$$

Products of Different Forms

We can use the method of undetermined coefficients when the forcing function $f(t)$ is:

- A power function, $f(t) = Ct^n$ (for n a nonnegative integer).
- A function of the form $f(t) = Ce^{\alpha t}\cos(\beta t)$ or $f(t) = Ce^{\alpha t}\sin(\beta t)$.
- For a product of a power function t^n and an exponential such as $f(t) = Ct^n e^{rt}$, we guess

$$y_p(t) = (A_n t^n + A_{n_1} t^{n-1} + \ldots A_1 t + A_0)e^{rt}$$

  - Multiply by t if r is a real root (not repeated) of the the characteristic equation.
  - Multiply by t^2 if r is a repeated real root of the the characteristic equation.

$r^2 - 5r - 6 = 0$

$(r-6)(r+1) = 0$

$r = 6, -1$

5. Give a guess for the particular solution to

$At^2 + Bt + C$ De^{6t}

$$y'' - 5y' - 6y = 4t^2 e^{6t}.$$

$y_h = C_1 e^{6t} + C_2 e^{-t}$

$y_p = (At^2 + Bt + C)e^{6t} \to At^2 e^{6t} + Bte^{6t} + Ce^{6t}$

$y_p = t\left[At^2 + Bt + C\right]e^{6t}$

Resonance

- For a product of a power function, an exponential, and either a sine or cosine such as $f(t) = Ct^n e^{\alpha t}\cos(\beta t)$ or $f(t) = Ct^n e^{\alpha t}\sin(\beta t)$, we guess

$$y_p(t) = (A_n t^n + A_{n_1} t^{n-1} + \ldots A_1 t + A_0)e^{\alpha t}\cos(\beta t) + (B_n t^n + B_{n_1} t^{n-1} + \ldots B_1 t + B_0)e^{\alpha t}\sin(\beta t)$$

  - Multiply by t if $\alpha \pm i\beta$ are the complex roots of the the characteristic equation.

6. Give a guess for the particular solution to

$y_h = C_1 e^{6t} + C_2 e^{-t}$

$$y'' - 5y' - 6y = 4t^2 e^{6t}\sin t.$$

No Resonance

$$y_p = (At^2 + Bt + C)e^{6t}\sin(t) + (Dt^2 + Et + F)e^{6t}\cos(t)$$

7. Consider the differential equation $x'' - 8x' + 12x = f(t)$. For each $f(t)$, what would be your guess for the particular solution? Do not solve for the undetermined coefficients. If the method of undetermined coefficients cannot be applied, explain why not.

$r^2 - 8r + 12 = 0$

$y_h = c_1 e^{6t} + c_2 e^{2t}$

X (a) $f(t) = 10\sin(2t)$

$x(t) = A\sin(2t) + B\cos(2t)$

X (b) $f(t) = 10e^{6t}\sin(2t)$

$x(t) = Ae^{6t}\sin(2t) + Be^{6t}\cos(2t)$

X (c) $f(t) = 10\tan(2t)$?

Doesn't work → Variation of parameters

✓ (d) $f(t) = 10te^{2t}$ → $x(t) = (At + B)e^{2t} \cdot t$

$x(t) = (At^2 + Bt)e^{2t}$

X (e) $f(t) = 8t^{-2}$?

variation of parameters → skip

X (f) $f(t) = 6te^{5t}\cos(3t)$

$x = (At + B)e^{5t}\cos(3t) + (Ct + D)e^{5t}\sin(3t)$