Analyzing Autonomous DEs: Spotted Owls

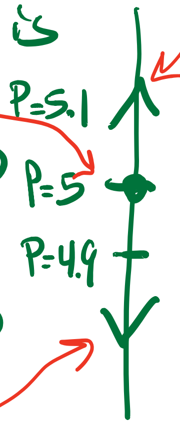
A group of biologists are making predictions about the spotted owl population in a forest in the Pacific Northwest. The autonomous differential equation the scientist use to model the spotted owl population is $\frac{dP}{dt} = \frac{P}{2}\left(1 - \frac{P}{5}\right)\left(\frac{P}{8} - 1\right)$, where P is in hundreds of owls and t is in years. The problem is that the current number of owls is only approximately known.

1. Suppose the scientists estimate that currently P is about 5 (i.e. there are currently about 500 owls in the forest). Since 5 is only an estimate, they make long-term predictions of the owl population for the initial conditions $P = 4.9$, $P = 5.0$, and $P = 5.1$. *Without using a graphing calculator or other software*, determine the long-term predictions for these initial conditions based on the differential equation. Are they similar or different? That is, will slightly different initial conditions yield only slightly different long-term predictions, or will they be radically different? Carry out a similar analysis if the current number of owls is somewhere around 8.

If $P=5$, then $dP/dt = (+)(0)(-) = 0$ means P is constant at $P=5$ for all t.

If $P=5.1$ then $dP/dt = (+)(-)(-) = +$ so P increases as t goes on

If $P=4.9$, then $dP/dt = (+)(+)(-) = -$ so P decreases as t increases



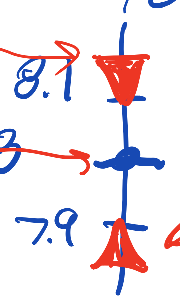
Similar analysis near $P=8$ gives

If $P=8$ $dP/dt = 0$ so constant solution

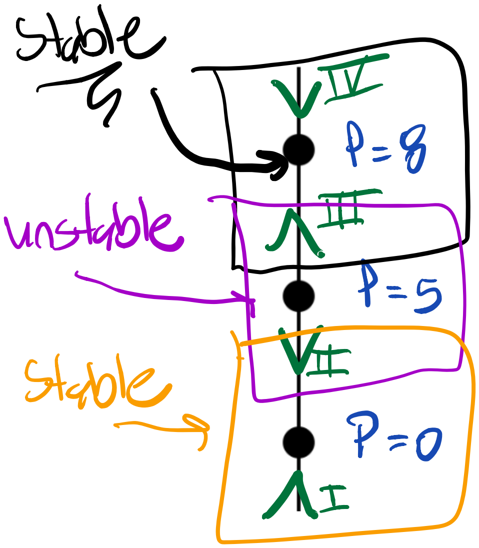
If $P=8.1$ $dP/dt < 0$ so P will decrease

2. Give a one dimensional representation, *without words*, that would describe all solutions to the differential equation.

If $P=7.9$ $dP/dt > 0$ so P will increase.



3. A **phase line** is the standard one-dimensional diagram that depicts the qualitative behavior of solutions to an autonomous differential equation. Label the dots and add arrows to the figure below to represent **all** solutions to the differential equation in Problem 1.



① Find equilibria by solving $dP/dt = 0$.

$dP/dt = \frac{P}{2}\left(\frac{P}{5}-1\right)\left(1-P/8\right) = 0$ if

$\boxed{P = 0, 5, 8 \text{ so we have 3 equilibria.}}$

② Mark each equilibria with a point on phase line

4. For the differential equation in problems 1-3 there are three equilibrium solutions. Recall that equilibrium solutions are constant functions that satisfy the differential equation. How do the other solution functions near each equilibrium solution behave in the long term? If you were to label each of these equilibrium solutions based on the way in which nearby solutions behave, what terms would you use and why?

③ Pick a test value in each interval between equilibria to test sign of dP/dt

In interval I, if $P = -1$, $dP/dt > 0$ so $\wedge$

In interval II, if $P = 4.9$ $dP/dt < 0$ so $\vee$

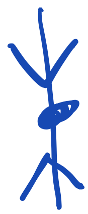
In interval III if $P = 7.9$ $dP/dt > 0$ so $\wedge$

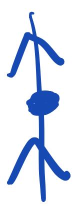
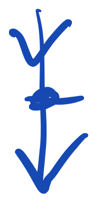
In interval IV if $P = 8.1$ $dP/dt < 0$ so $\vee$

5. Create an autonomous differential equation that has exactly two equilibrium solutions: $y(t) = 3$ is a stable equilibrium and $y(t) = -4$ is an unstable equilibrium.

To summarize Q3 and Q4:

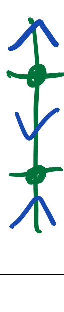
If  ← Equilibrium → $\boxed{\text{unstable}}$

If  ← Equilibrium → $\boxed{\text{stable}}$

If  OR  ← Equilibrium is $\boxed{\text{unstable}}$ sometimes called $\boxed{\text{semi-stable}}$

Q5 $\dfrac{dy}{dt} = (y+4)(y-3)$ X

Has correct equilibrium, is stability correct?


y=3 – unstable
y=-4 – stable ← wrong!!

so $\boxed{\dfrac{dy}{dt} = -(y+4)(y-3)}$