Common Laplace Transforms and Properties

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$

Properties:

L.1 $\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}$, where c is a constant.

L.2 $\mathscr{L}\{f_1(t) + f_2(t)\} = \mathscr{L}\{f_1(t)\} + \mathscr{L}\{f_2(t)\}$

L.3 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$ for all $s > \alpha + a$.

L.4 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then for all $s > \alpha$,

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

L.5 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^nF}{ds^n}$ for all $s > \alpha$.

$$\mathscr{L}\{y'\} = s\mathscr{L}\{y(t)\} - s^0 y(0) = sY(s) - y(0)$$

$$\mathscr{L}\{y''\} = s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0) = s^2 Y(s) - sy(0) - y'(0)$$

Section 6.2: Solving ODE's

Step 1 Take the Laplace transform of both sides. Refer to properties.

Step 2 Rearrange and group like terms to solve for $\mathcal{L}\{y(t)\} = Y(s)$

Step 3 Take the inverse Laplace transform and solve for $y(t) = \mathcal{L}^{-1}\{Y(x)\}$.

L.1 and L.2

1. Solve the initial value problem using Laplace Transforms (not previous methods).

   (a) $y'' - 2y' + 5y = 0$ with $y(0) = 2$ and $y'(0) = 4$.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) - 2\left(s Y(s) - y(0)\right) + 5Y(s) = 0 \quad \text{using L.4}$$

$$\left(s^2 Y(s) - 2s - 4\right) - 2s Y(s) + 4 + 5Y(s) = 0$$

Solve for $Y(s)$:

$$s^2 Y - 2sY + 5Y = 2s \qquad Y(s)\left(s^2 - 2s + 5\right) = 2s$$

$$Y(s) = \frac{2s}{s^2 - 2s + 5}$$

$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$, $s > a$

Take $\mathcal{L}^{-1}$ to solve $y(t)$.

$s - 1 + \frac{1}{2}(2)$

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2 - 2s + 5}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)^2 + 4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2 + 4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 4} + \frac{1}{2}\frac{2}{(s-1)^2 + 4}\right\}$$

$$\boxed{2e^{t}\cos(2t) + e^{t}\sin(2t) = y(t)}$$

(b) $y'' - y' - 2y = 0$ with $y(0) = -2$ and $y'(0) = 5$.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left(s^2 Y - s y(0) - y'(0)\right) - \left(sY - y(0)\right) - 2Y = 0$$

$$s^2 Y + 2s - 5 - sY - 2 - 2Y = 0$$

$$s^2 Y - sY - 2Y = -2s + 7 \qquad Y = \frac{-2s+7}{s^2 - s - 2} = \frac{-2s+7}{(s-2)(s+1)}$$

$\frac{A}{s-2} + \frac{B}{s+1} = \frac{-2s+7}{(s-2)(s+1)}$ has solution $A = 1$ and $B = -3$ thus

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{3}{s+1}\right\}$$

$$\boxed{y(t) = e^{2t} - 3e^{-t}}$$

(c) $y'' - 4y' - 5y = 4e^{3t}$ with $y(0) = 2$ and $y'(0) = 7$.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = 4\mathcal{L}\{e^{3t}\}$$

$$s^2 Y - 4sY - 5Y - 2s - 7 + 14 = 4\left(\frac{1}{s-3}\right)$$

$$Y = \frac{4}{(s-3)(s^2 - 4s - 5)} + \frac{2s-1}{s^2 - 4s - 5} \cdot \frac{s-3}{s-3} = \frac{2s^2 - 7s + 7}{(s-3)(s-5)(s+1)}$$

$(s-5)(s+1)$

$$= \frac{A}{s-3} + \frac{B}{s-5} + \frac{C}{s+1} = \frac{2s^2 - 7s + 7}{(s-3)(s-5)(s+1)}$$

$A = -\frac{1}{2}$

$B = \frac{11}{6}$

$C = \frac{2}{3}$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{-\frac{1}{2}\left(\frac{1}{s-3}\right) + \frac{11}{6}\left(\frac{1}{s-5}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)\right\}$$

$$\boxed{y(t) = -\frac{1}{2}e^{3t} + \frac{11}{6}e^{5t} + \frac{2}{3}e^{-t}}$$

Welcome! Today we'll finish Worksheet 23.

Solve $y'' + 4y' - 5y = te^t$ with $y(0)=1$ and $y'(0)=0$.

1. Take $\mathcal{L}$ of both sides

$e^{at}t^n, \; n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \; s > a$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

$$\left(s^2 Y_{(s)} - s y(0) - y'(0)\right) + 4\left(s Y_{(s)} - y(0)\right) - 5\left(Y_{(s)}\right) = \frac{1}{(s-1)^2}$$

2. Solve for $Y_{(s)} = \mathcal{L}\{y(t)\}$

$$\left(s^2 Y_{(s)} + 4s Y_{(s)} - 5Y_{(s)}\right) - s - 4 = \frac{1}{(s-1)^2}$$

$$Y_{(s)}\left(s^2 + 4s - 5\right) = \frac{1}{(s-1)^2} + s + 4 = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2}$$

$$Y_{(s)} = \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3}$$

3. Find $\mathcal{L}^{-1}\{Y_{(s)}\} = y(t)$.

$$\mathcal{L}^{-1}\left\{\frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}\right\}$$

$$A = \frac{35}{216}, \quad B = \frac{181}{216}, \quad C = -\frac{1}{36} \quad D = \frac{1}{6}$$

Thus

$$\mathcal{L}^{-1}\{Y_{(s)}\} = \frac{35}{216}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} + \frac{181}{216}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{36}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= \frac{35}{216}e^{-5t} + \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$$

$$= \frac{35}{216}e^{-5t} + \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}t^2e^t$$

(d) $ty'' - ty' + y = 2$ with $y(0) = 2$ and $y'(0) = -1$.

1. Take $\mathcal{L}$ of both sides

$$\mathcal{L}\{t^1 y''\} - \mathcal{L}\{ty'\} + \mathcal{L}\{y\} = \mathcal{L}\{2\}$$

$$(-1)^1 \frac{d}{ds}\underbrace{\mathcal{L}\{y''\}}_{\text{by L.5}} - (-1)^1 \frac{d}{ds}\mathcal{L}\{y'\} + Y(s) = \frac{2}{s}$$

$$(-1)\frac{d}{ds}\left(s^2 Y(s) - sy(0) - y'(0)\right) + \frac{d}{ds}\left(sY(s) - y(0)\right) + Y(s) = \frac{2}{s}$$

$$(-1)\left(2sY(s) + s^2 Y'(s) - y(0)\right) + \left(Y(s) + sY'(s)\right) + Y(s) = \frac{2}{s}$$

$$-2sY(s) - s^2 Y'(s) + 2 + Y(s) + sY'(s) + Y(s) = \frac{2}{s}$$

$$Y'(s)\left(-s^2 + s\right) + Y(s)\left(-2s + 1 + 1\right) = \frac{2}{s} - 2 = \frac{2-2s}{s}$$

$$Y'(s)\left(-s^2 + s\right) + Y(s)(-2s+2) = \frac{2-2s}{s}$$

2. Solve for $Y(s)$

$$x'(-t^2+t) + x(-2t+2) = \frac{2-2t}{t}$$

Linear 1st order

Integrating Factor

$$Y' + Y\left(\frac{-2s+2}{-s^2+s}\right) = \frac{2-2s}{s(-s^2+s)}$$

simplify

$$\frac{2\cancel{(-s+1)}}{s\cancel{(-s+1)}}$$

$$Y' + Y\left(\frac{2}{s}\right) = \frac{2}{s^2}$$

$$y' + y\left(\frac{2}{s}\right) = \frac{2}{s^2} \qquad \mu = e^{\int p(x)\,dx}$$

$$\mu = e^{\int \frac{2}{s}\,ds} = e^{2\ln s} = \left(e^{\ln s}\right)^2 = s^2$$

$$\boxed{y' s^2 + y\cdot(2s)} = 2 \qquad (s^2 y)' = y' s^2 + 2s\,y$$

$$\int \underbrace{(s^2 y)'}\,ds = \int 2\,ds$$

$$s^2 y = 2s + C \qquad \boxed{Y(s) = 2/s + C/s^2}$$

3. Take $\mathcal{L}^{-1}(Y(s))$ to solve for $y(t)$

$$\mathcal{L}^{-1}\left\{2/s + C/s^2\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} + C\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{y(t) = 2 + Ct}$$

We have initial conditions.

$y(0) = 2$ ✓

$y'(0) = -1$

$y'(t) = C$

$y'(0) = C = -1$

$$\boxed{y(t) = 2 - t}$$

(e) $y'' + ty' - y = 0$ with $y(0) = 0$ and $y'(0) = 3$.

1. Take the Laplace transform of both sides.

$$\mathcal{L}\{y''\} + \mathcal{L}\{ty'\} - \mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$(s^2 Y(s) - s\cdot 0 - 3) + (-1)\frac{d}{ds}(sY(s) - 0) - Y(s) = 0$$

$$(s^2 Y(s) - 3) + (-1)(sY'(s) + Y(s)) - Y(s) = 0$$

$-sY'(s) + (s^2 - 2)Y(s) = 3$ which in standard form is $Y'(s) + \left(\frac{-s^2+2}{s}\right)Y(s) = -\frac{3}{s}$

2. Solve for $Y(s)$

$$Y'(s) + \left(\frac{-s^2+2}{s}\right)Y(s) = -\frac{3}{s}$$

$$\mu = e^{\int(-s + \frac{2}{s})\,ds} = e^{-\frac{1}{2}s^2 + 2\ln(s)} = e^{-s^2/2}\cdot s^2$$

$$Y'\cdot\left(e^{-\frac{s^2}{2}}\cdot s^2\right) + \left(se^{-s^2/2}(-s^2+2)\right)\cdot Y = -3se^{-s^2/2}$$

$$\int\left(Y(s)\cdot\left(e^{-s^2/2}s^2\right)\right)' ds = \int -3se^{-s^2/2}\,ds$$

Integration by substitution with $w = -s^2/2$ $dw = -s$

$\vdots$

$$Y(s)\cdot\left(e^{-s^2/2}\cdot s^2\right) = 3e^{-s^2/2} + C$$

$Y(s) = 3/s^2 + \frac{Ce^{s^2/2}}{s^2}$ since for any Laplace transform $F(s)$, $\lim_{s\to\infty} F(s) = 0$, then $\lim_{s\to\infty}\left(\frac{3}{s^2} + \frac{Ce^{s^2/2}}{s^2}\right) = 0$ only if $C = 0$.

Thus $Y(s) = 3/s^2$

3. Find $\mathcal{L}^{-1}\{Y(s)\}$

$$\boxed{y(t) = \mathcal{L}^{-1}\{3/s^2\} = 3\,\mathcal{L}^{-1}\{1/s^2\} = 3t}$$