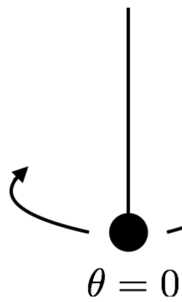
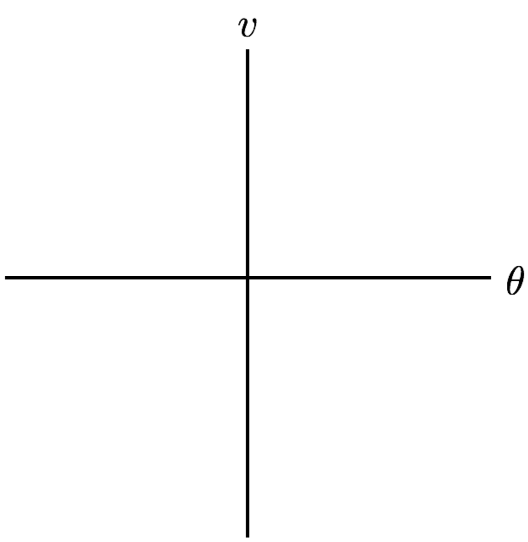
In the Swing of Things

A pendulum is attached to a wall in such a way that it is free to rotate around in a complete circle. Without provocation, Debra takes a baseball bat and hits it, giving it an initial velocity and setting it in motion.



1. If we call θ the angular position of the pendulum (where $\theta = 0$ corresponds to when the pendulum is hanging straight down) and we call the velocity of the pendulum v, what would angular position versus velocity graphs look like for a variety of different initial velocities due to Debra's hit? Provide a brief description of the motion of the pendulum for your graphs.



2. How many equilibrium solutions are there, where are they, and how would you classify them?

Applying Newton's 2nd Law of motion (where $\theta = 0$ corresponds to the downward vertical position and counterclockwise corresponds to positive angles θ) yields the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = 0$$

where b is the coefficient of damping, m is the mass of the pendulum, g is the gravity constant, and l is the length of the pendulum (See homework problem 5 for a derivation of this equation). Estimating the parameter values for the pendulum that Debra hits and changing this second order differential equation to a system of differential equations yields

$$\begin{aligned}\frac{d\theta}{dt} &= v\\ \frac{dv}{dt} &= -0.2v - \sin(\theta)\end{aligned}$$

3. How many equilibrium solutions does this system of differential equations have, where are they, and based on the context what types of equilibrium solutions would you expect them to be? How does this connect with your answer to 2?

4. You might recall that if θ is small, $\sin(\theta) \approx \theta$. Explain why this is true and then use this fact to approximate the above system with a linear system and classify the equilibrium solution at the origin.

5. Classify the equilibrium point at $\theta = \pi$.

6. Use the GeoGebra applet, https://ggbm.at/SpfDSc5Q, to approximate the range of initial velocities with zero initial displacement that will result in the pendulum making exactly one complete rotation before eventually coming to rest.

Linearization and Linear Stability Analysis

In the next several questions we will develop tools to analyze equilibria of nonlinear systems. To do this, we will first build our intuition by studying first order nonlinear equations.

7. Recall from Calculus that the linearization, $L(h)$, of a function around a point of interest, x^*, is given by $L(h) \equiv f(x^*) + hf'(x^*)$. The key feature of the linearization is that, when $x \approx x^*$, that is, $x = x^* + h$ for $h \approx 0$, then $f(x) \approx L(h)$.

Find the linearization of $f(x) = 1 - x^2$ around $x^* = 1$.	
If $x \approx 1$, x can be written as $x = 1 + h$ where $h \approx 0$. Suppose x follows the differential equation $\frac{dx}{dt} = 1 - x^2$. Use the linearization above to write down a linear differential equation for $\frac{dh}{dt}$.	
According to the above differential equation, what is the long term behavior of h?	
If $x(0) \approx 1$, what does the long term behavior of h tell you about the long term behavior of x?	

8. (a) Consider again $\dfrac{dx}{dt} = 1 - x^2$, but this time with $x(0) \approx -1$. Find a new linearization and use it to make a long term prediction about x.

(b) Why was it necessary to construct a **new** linearization to study $x(0) \approx -1$?

(c) Using linearization to determine the stability of a critical point is called "linear stability analysis." Use a phase line to corroborate your linear stability analysis.

(d) For an arbitrary system, $\dfrac{dx}{dt} = f(x)$ with an equilibrium point at $x = x^*$, describe how you can use linear stability analysis to determine the stability of the equilibrium point.

9. Consider the following system:

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

(a) Algebraically find the equilibrium solutions.

(b) Tanesha used the GeoGebra Vector field applet, https://ggbm.at/kkNXUVds, to plot the vector field associated with the differential equation. Based on this vector field, how would you classify the equilibria?

We can also perform linear stability analysis on a system of two or more variables, such as the one in the previous problem. Consider a function $f(x, y)$, then Taylor's theorem states that, if $(x, y) \approx (x^*, y^*)$, that is, if $(x, y) = (x^* + h_1, y^* + h_2)$ where $h_1 \approx 0$ and $h_2 \approx 0$, then

$$f(x, y) \approx L(h_1, h_2) = f(x^*, y^*) + h_1 f_x(x^*, y^*) + h_2 f_y(x^*, y^*)$$

where f_x and f_y are the partial derivatives of f with respect to x and y, respectively.

$L(h_1, h_2)$ is called the linearization of $f(x, y)$ around (x^*, y^*).

10. Consider the system of differential equations:

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

Let's first study the equilibrium at (1,-1).

If the system had $x(0) \approx 1$ and $y(0) \approx -1$, we could write $x = 1 + h_1$ and $y = -1 + h_2$, with $h_1 \approx 0$ and $h_2 \approx 0$. Use the linearization of the original system of equations around $(1, -1)$ to write down a system of differential equations for h_1 and h_2	
What are the long term behaviors of h_1 and h_2?	
What can you conclude about the long term behaviors of x and y?	
Classify the equilibrium point $(1, -1)$, according to your linear stability analysis.	

11. (a) Consider again

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

Use linear stability analysis to classify the equilibrium point at $(-1, 1)$.

(b) Combine your results from question 10 and 11a, to sketch a possible phase plane for the system of differential equations. Does an analysis of the system using nullclines corroborate your linear stability analysis?

12. For a system of differential equations

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with an equilibrium point at (x^*, y^*), the matrix

$$J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$$

is called the **Jacobian matrix**. Explain how you can use the Jacobian matrix to determine the behavior of a the system of differential equations near (x^*, y^*).

13. Use linear stability analysis to classify the critical points you found in the pendulum system.

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -0.2v - \sin(\theta)$$

Homework Set 14

1. Bees and Flowers II. In an earlier problem, we studied systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is both species are harmed by interaction) or cooperative (that is both species benefit from interaction).

$$\begin{array}{cc} \text{(A)} & \text{(B)} \\ \begin{aligned} \frac{dx}{dt} &= -5x + 2xy \\ \frac{dy}{dt} &= -4y + 3xy \end{aligned} & \begin{aligned} \frac{dx}{dt} &= 3x(1 - \frac{x}{3}) - \frac{1}{10}xy \\ \frac{dy}{dt} &= 2y(1 - \frac{y}{10}) - \frac{1}{5}xy \end{aligned} \end{array}$$

   (a) Explain why the second system of rate of change equations describes a situation where the two species are competitive.

   (b) Verify that the equilibrium solutions for system (B) are (0,0), (3, 0), (0, 10), and $(\frac{20}{9}, \frac{70}{9})$.

   (c) Determine the linearized system of differential equations about each equilibrium solution and use the information you gain about the solutions near each of these equilibrium solutions to sketch the phase portrait.

2. Without using technology, use the tools of linearization and nullclines to sketch the phase portrait for the nonlinear system:

$$\begin{aligned} \frac{dx}{dt} &= \cos(y) \\ \frac{dy}{dt} &= y - x \end{aligned}$$

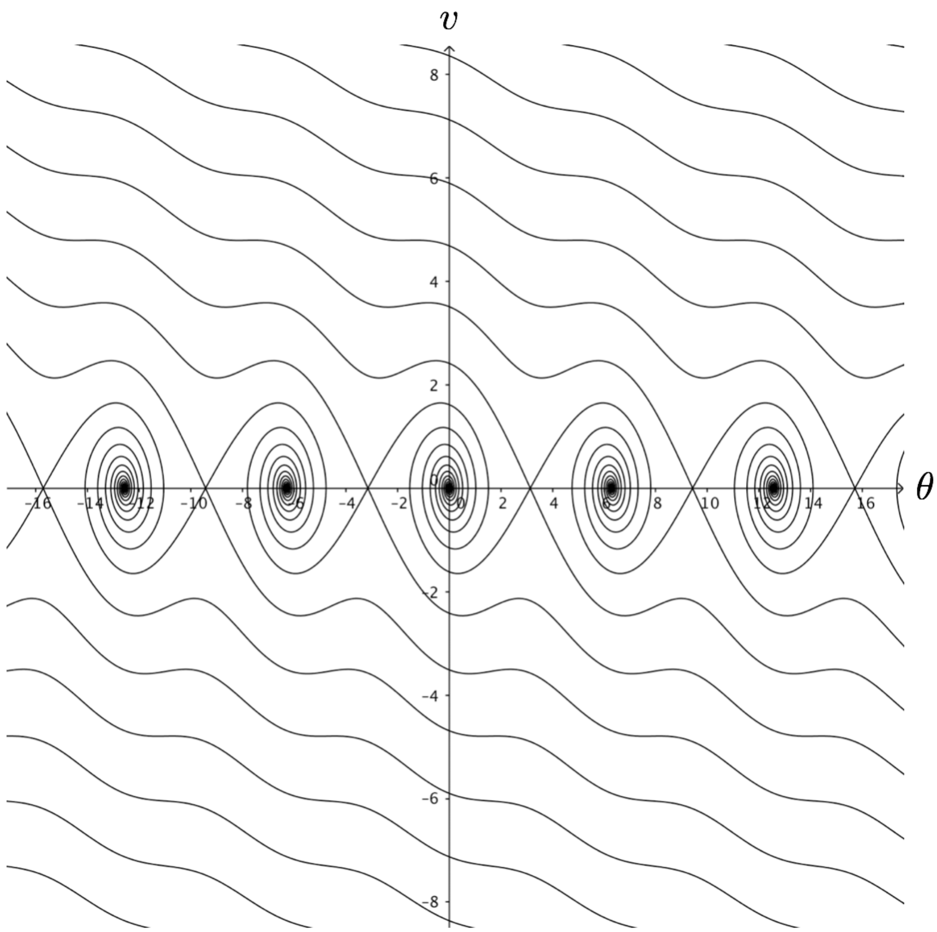
   Be as accurate as possible and show all supporting work.

3. When the John Hancock Building in Boston, MA was first built it tended to sway back and forth so much so that people in the top floors experienced motion sickness. Similar to the spring mass system, we can model the back and forth motion of the building by adding a gravity term to the spring mass model.

   The following system of rate of change equations is a model for helping us make predictions about the motion of a skyscraper swaying in the wind. In this simplified system of rate of change equations, x is the amount of displacement of the building from the vertical position at any time t and y is the horizontal velocity of the building at any time t. Use what you know about linear stability analysis to analyze the behavior of the systems at the critical points and compare to your earlier work. (You might want to use a GeoGebra vector field applet, https://ggbm.at/kkNXUVds, to help understand it first).

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - y + x^3 \end{aligned}$$

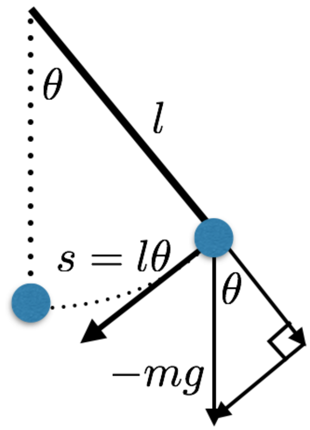
4. Consider the phase plane below for the damped pendulum:



(a) Shade in the region(s) corresponding to initial conditions that will make one full revolution before coming to a stop.

(b) Use a different shading to show the region(s) corresponding to initial conditions that will make two full revolutions before coming to a stop.

5. Consider the diagram below for the pendulum:



(a) The force, due to gravity, on the bob of the pendulum is given by $-mg$. Explain why the proportion of that gravitational force, in the direction tangent to the path of the pendulum's bob, is given by $F = -mg\sin(\theta)$.

(b) In the diagram above, explain why the length of the dotted arc is given by $s = l\theta$, when θ is measured in radians.

(c) The frictional force (due to friction at the fixed point of the pendulum, or due to air resistance, or a combination of these two) opposes the motion of the pendulum. Carefully explain why this force can be represented as $F = -b\dfrac{ds}{dt} = -bl\dfrac{d\theta}{dt}$.

(d) Newton's law states that force is given by mass times acceleration. If m is the mass of the pendulum bob, explain why $F = m\dfrac{d^2s}{dt^2} = ml\dfrac{d^2\theta}{dt^2}$.

(e) Explain how the previous parts of this question can be combined to arrive at a differential equation:

$$ml\frac{d^2\theta}{dt^2} = -bl\frac{d\theta}{dt} - mg\sin(\theta)$$

(f) By defining $v = \dfrac{d\theta}{dt}$, develop a pair of first order differential equations for the (θ, v) system.