Introduction to the Laplace Transform

One of the basic problem solving techniques in mathematics is to

- transform a difficult problem into an easier one,
- solve the easier problem, and
- then use its solution to obtain a solution of the original problem.

$$a_2(t)\,y'' + a_1(t)\,y' + a_0(t)\,y = g(t)$$

$$3y'' + 2y' + 6y = 0$$

For example. the reverse product rule (method of integrating factor) is used to transform a linear first order differential equation into an easier problem we can solve. In this chapter we study the method of **Laplace transforms**, which is one example of this technique. Like the method of integrating factors, Laplace transforms are **integral operators**. Solving by the method of Laplace transforms:

- Can be used to solve higher order linear differential equations.
- Can be applied for more complicated forcing functions.
- Requires initial conditions.

The **improper integral** of g over $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\ dt = \lim_{N\to\infty} \int_a^N g(t)\ dt.$$

- We say the improper integral **converges** if the limit exists.
- Otherwise we say the improper integral **diverges**.

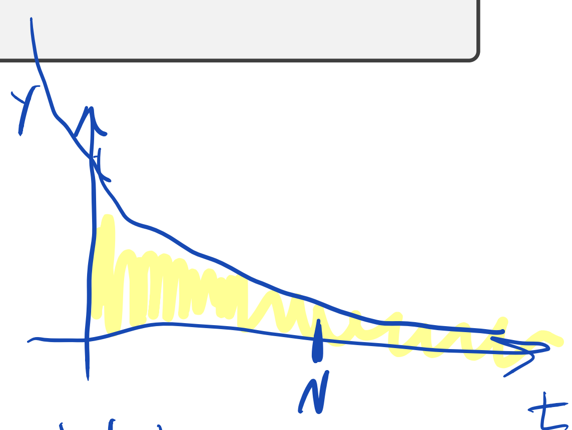
1. Determine whether $\int_0^\infty e^{-2t}\ dt$ converges or diverges.

$$\int_0^\infty e^{-2t}\,dt = \lim_{N\to\infty} \int_0^N e^{-2t}\,dt$$

$$= \lim_{N\to\infty} \left(-\frac{1}{2} e^{-2t} \Big|_{t=0}^{t=N} \right)$$

$$= \lim_{N\to\infty} \left(-\frac{1}{2} e^{-2N} + \frac{1}{2} e^{0} \right) = \lim_{N\to\infty} \left(-\frac{1}{2} e^{-2N} \right) + \frac{1}{2} = \boxed{\frac{1}{2}}$$

Converges to $1/2$



what happens as $N \to \infty$

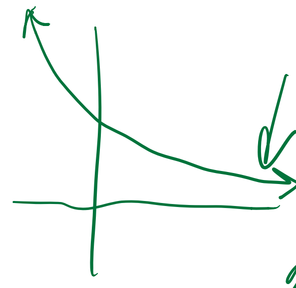
0

$$\lim_{N\to\infty} e^{-Ns} = 0$$

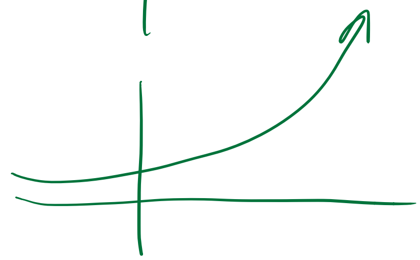
as long as

$s > 0$

e^{-t}



e^{t}



$$\lim_{N\to\infty} e^{-N(s-2)}$$ ← want to make sure

$$-\underbrace{N}_{+}\underbrace{(s-2)}_{+} < 0$$

$s-2 > 0$

$s > 2$

Welcome. Today we'll work on Worksheets 20 and 21.

Definition of the Laplace Transform

Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by

$f(t) \to F(s)$

$$\mathscr{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

- The domain of $F(s)$ is all values of s for which the integral converges.
- The functions f and F form a **transform pair**. $\boxed{f(t)=2 \text{ and } F(s)=\frac{2}{s}, s>0}$

2. Find and state the domain of the Laplace transform $F(s) = \mathscr{L}\{f(t)\}$.

(a) $f(t) = 2$, $t \geq 0$

② Integrate

① $\mathscr{L}\{2\} = \int_0^\infty e^{-st} 2\, dt = \lim_{N\to\infty} \int_0^N 2e^{-st}\, dt$

Substitution $u=-st$

$= \lim_{N\to\infty} \left(-\frac{2}{s} e^{-st}\Big|_{t=0}^{t=N}\right)$

No t's leftover

$= \lim_{N\to\infty} \left(-\frac{2}{s} e^{-sN}\right) - \left(-\frac{2}{s}\right)$

③ Evaluate limit: See next page (include domain)

$\boxed{\mathscr{L}\{2\} = 0 + 2/s = 2/s \text{ for } s>0}$

(b) $f(t) = t$ ① Write out definition as an improper integral

$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = \lim_{N\to\infty} \int_0^N \underbrace{e^{-st} \cdot t}\, dt$

$\lim_{N\to\infty} \left(-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\Big|_{t=0}^{t=N}\right) = \lim_{N\to\infty} \left(\frac{-N}{s} e^{-N\cdot s} - \frac{1}{s^2} e^{-N\cdot s}\right)$

$- \left(-0 - \frac{1}{s^2} e^0\right)$

$$\lim_{N\to\infty}\left(-\frac{2}{s}e^{-sN}\right) = -\frac{2}{s}\left(\lim_{N\to\infty} e^{-sN}\right) = -\frac{2}{s}(0) = 0$$

constant with respect to N

as long as s

$s>0$

The limit will converge for certain values of s.
The values of s for which improper integral converges gives domain of $\mathcal{L}\{f(t)\} = F(s)$

$$\int e^{-st}\cdot t \, dt \qquad u = t \quad du = dt \qquad dv = e^{-st}dt \quad v = -\frac{1}{s}e^{-st}$$

$$= -\frac{t}{s}e^{-st} - \int -\frac{1}{s}e^{-st}\,dt$$

$$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} + C$$

$$= \lim_{N\to\infty}\left(\frac{-N}{s}e^{-N\cdot s} - \frac{1}{s^2}e^{-N\cdot s}\right) - \left(-0 - \frac{1}{s^2}e^{0}\right)$$

$$= \lim_{N\to\infty} \frac{-N}{s}e^{-Ns} - \lim_{N\to\infty}\left(\frac{1}{s^2}e^{-N\cdot s}\right) + \frac{1}{s^2}$$

$-\infty \quad \to 0$

$$0 - 0 + 1/s^2 = 1/s^2$$

$$\lim_{N\to\infty}\frac{-N}{se^{Ns}} = \frac{-\infty}{\infty} \; ? \quad \text{L'Hopital's Rule}$$

$$\lim_{N\to\infty}\frac{-1}{s^2e^{Ns}} = \frac{-1}{\infty} = 0 \qquad \boxed{\mathcal{L}\{t\} = F(s) = 1/s^2}$$

Provided $s > 0$

$$e^{-st+3t}$$

(c) $f(t) = e^{3t}$

$$\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{-st} \cdot e^{3t}\, dt = \lim_{N\to\infty} \int_0^N e^{-t(s-3)}\, dt$$

$$\lim_{N\to\infty} \int_0^N e^{-t(s-3)}\, dt = \lim_{N\to\infty} \left(-\frac{1}{s-3} e^{-t(s-3)} \Big|_{t=0}^{t=N} \right) = \left(\lim_{N\to\infty} e^{-N(s-3)} \right) - \left(-\frac{1}{s-3} \right)$$

$$\lim_{N\to\infty} e^{-N(s-3)} \to 0$$

as long as $s > 3$

$$\boxed{\mathcal{L}\{e^{3t}\} = \frac{1}{s-3},\quad s > 3}$$

(d) $g(t) = \cos(bt)$ where $b \neq 0$ is a constant.

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} \cos(bt)\, dt$$

$u = \cos(bt)$ $\quad u' = -b\sin(bt)$

$v' = e^{-st}$ $\quad v = -\frac{1}{s} e^{-st}$

Then a second to evaluate $\int \frac{b}{s} e^{-st} \sin(bt)\, dt$

$u = \sin(bt)$ $\quad u' = b\cos(bt)$

$v' = e^{-st}$ $\quad v = -\frac{1}{s} e^{-st}$

* Boomerang method

$$\lim_{N\to\infty} \left(\frac{1}{b^2+s^2} \cdot \left[s - \underbrace{s e^{-sN} \cos(bN)}_{\to 0} + \underbrace{b e^{-sN} \sin(bN)}_{\to 0} \right] \right) \quad s>0$$

$$\frac{1}{b^2+s^2}\left(s \right)$$

$$= \frac{s}{b^2+s^2},\quad s>0$$

(e) $f(t) = \begin{cases} 5 & 0 < t < 2 \\ e^{8t} & t > 2 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st} \cdot 5\,dt + \int_2^\infty e^{-st} e^{8t}\,dt$$

$$= \left(-\frac{5}{s} e^{-st} \Big|_{t=0}^{t=2} \right) + \lim_{N\to\infty} \left(\frac{1}{-(s-8)} e^{-t(s-8)} \Big|_{t=2}^{t=N} \right)$$

$$= \left(\frac{-5}{s} e^{-2s} + 5/s \right) + \lim_{N\to\infty} \left(\underbrace{\frac{1}{-(s-8)} e^{-N(s-8)}}_{\to 0 \text{ as long as } s>8} \right) - \left(\frac{1}{-(s-8)} e^{-2(s-8)} \right)$$

$$= \frac{-5}{s} e^{-2s} + 5/s + 0 + \frac{1}{s-8} e^{-2(s-8)} \quad \text{if } s>8$$

$$\boxed{\mathcal{L}\{f\} = F(s) = \frac{-5}{s} e^{-2s} + 5/s + \frac{1}{s-8} e^{-2(s-8)}, \quad s>8}$$

Common Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
$f(t) = 1$	$F(s) = \frac{1}{s},\ s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{s-a},\ s > a$
$f(t) = t^n,\ n = 1, 2, \ldots$	$F(s) = \frac{n!}{s^{n+1}},\ s > 0$
$f(t) = \sin(bt)$	$F(s) = \frac{b}{s^2+b^2},\ s > 0$
$f(t) = \cos(bt)$	$F(s) = \frac{s}{s^2+b^2},\ s > 0$