Welcome! Worksheet 17 today.

Interconnected Populations

Let $x(t)$ denote the number of bacteria in a colony (labeled Colony 1) at time t hours since noon. Let $y(t)$ denote the number of bacteria in a separate colony (labeled Colony 2) at time t hours since noon. The size of populations x and y can be modeled by system of differential equations

Independent of t

Autonomous

$$\frac{dx}{dt} = 3x \longrightarrow$$ x is increasing by three times each hour

$$\frac{dy}{dt} = -2y$$

1. Explain in practical terms how each of the populations is changing over time. Are the two populations interacting? What happens to the size of each colony in the long run?

y will decrease to 0. Every hour y decreases by 1/2.

These two populations are not interconected.

2. Solve each of the differential equations and give general solutions for x and y.

$\frac{dx}{dt} = 3x$ $\quad x = C_1 e^{3t}$ $\quad C_1 = x(0)$

$dy/dt = -2y$ $\quad y = C_2 e^{-2t}$ $\quad C_2 = y(0)$

3. Now imagine Colony 1 and Colony 2 are arranged such that the size of populations x and y can be modeled by system of differential equations

$$\frac{dx}{dt} = 3x + 10y$$
$$\frac{dy}{dt} = -2y$$

(a) Explain in practical terms how each of the populations is changing over time. Are the two populations interacting? What happens to the size of each colony in the long run?

y is the same behavior as previous model. $y = C_2 e^{-2t}$

If y is present, then x increases more rapidly.

(b) Notice the solution for y is the same for this modified setup. Give an educated guess for the general form of the expression for the new solution for $x(t)$.

$\frac{dx}{dt} = 3x \quad x = C_1 e^{3t}$

$dy/dt = -2y \quad y = C_2 e^{-2t}$ → Same

$x = \frac{C_1 e^{3t}}{10}$

$\boxed{y = C_2 e^{-2t}, \quad x = C_1 e^{3t} - 2C_2 e^{-2t}}$

(c) Substitute your answer in 3b as well as the original solution for $y(t)$ in 2 into the differential equation for x,

$$\frac{dx}{dt} = 3x + 10y,$$

and find the general solution for x.

Guess $\boxed{x(t) = Ae^{3t} + Be^{-2t}}$

$y(t) = C_2 e^{-2t}$

$x = C_1 e^{3t} - 2C_2 e^{-2t}$

$dx/dt =$

$3Ae^{3t} - 2Be^{-2t} = 3(Ae^{3t} + Be^{-2t}) + 10(C_2 e^{-2t})$

$3Ae^{3t} - 2Be^{-2t} = 3Ae^{3t} + (3B + 10C_2)e^{-2t}$

$3A = 3A \qquad$ A can be any thing

$-2B = 3B + 10C_2 \qquad B = -2C_2$

A More General Case

If we consider a general system of differential equations of the form

$$x' = ax + by$$
$$y' = cx + dy,$$

Linear system with constant coefficients.

then we can generalize our approach in the previous example by guessing solutions of the form

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad \text{and} \quad y(t) = C_3 e^{r_1 t} + C_4 e^{r_2 t},$$

where we note there is some dependence between constant C_1 and C_3 and constants C_2 and C_4.

4. Explain how we might interpret the meaning of the constants r_1 and r_2 in practical terms.

5. By plugging the guess for x and y into the differential equation $x' = ax + by$ you can derive two equations that the values r_1 and r_2 must satisfy. Fill in the blanks to express these two equations.

See next page

$C_1 r_1 =$ ______________ $C_2 r_2 =$ ______________

6. Similarly, by plugging the guess for x and y into the differential equation $y' = cx + dy$ you can derive two equations that the values r_1 and r_2 must satisfy. Fill in the blanks to express these two equations.

See next page

$C_3 r_1 =$ ______________ $C_4 r_2 =$ ______________

7. From 5 and 6, we get a system of two equations that r_1 must simultaneously satisfy. Write the resulting system of two equations below.

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad \text{and} \quad y(t) = C_3 e^{r_1 t} + C_4 e^{r_2 t},$$

$$x' = ax + by$$
$$y' = cx + dy,$$

$$x' = r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t} = a\left(C_1 e^{r_1 t} + C_2 e^{r_2 t}\right) + b\left(C_3 e^{r_1 t} + C_4 e^{r_2 t}\right)$$
$$= aC_1 e^{r_1 t} + aC_2 e^{r_2 t} + bC_3 e^{r_1 t} + bC_4 e^{r_2 t}$$
$$= (aC_1 + bC_3)e^{r_1 t} + (aC_2 + bC_4)e^{r_2 t}$$

$$\boxed{\begin{aligned} r_1 C_1 &= aC_1 + bC_3 \\ r_2 C_2 &= aC_2 \;\; bC_4 \end{aligned}}$$

$$y' = r_1 C_3 e^{r_1 t} + r_2 C_4 e^{r_2 t} = c\left(C_1 e^{r_1 t} + C_2 e^{r_2 t}\right) + d\left(C_3 e^{r_1 t} + C_4 e^{r_2 t}\right)$$
$$= cC_1 e^{r_1 t} + cC_2 e^{r_2 t} + dC_3 e^{r_1 t} + dC_4 e^{r_2 t}$$
$$= (cC_1 + dC_3)e^{r_1 t} + (cC_2 + dC_4)e^{r_2 t}$$

$$\boxed{\begin{aligned} r_1 C_3 &= cC_1 + dC_3 \\ r_2 C_4 &= cC_2 + dC_4 \end{aligned}}$$

Conditions for r_1:

$$\left.\begin{aligned} aC_1 + bC_3 &= r_1 C_1 \\ cC_1 + dC_3 &= r_1 C_3 \end{aligned}\right\}$$

Condition for r_2:

$$aC_2 + bC_4 = r_2 C_2$$
$$cC_2 + dC_4 = r_2 C_4$$

$$r_1 \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} aC_1 + bC_3 \\ cC_1 + dC_3 \end{bmatrix}$$

Section 3.2: Essentials of Linear Algebra

An $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns. For example

$$\begin{bmatrix} 3 & -2 & 7 \\ -12 & 0 & 5 \end{bmatrix}$$

is a 2 by 3 matrix.
We can multiple a scalar (a regular number) and a matrix by simply multiplying each value in the matrix by the scalar. For example:

$$2\begin{bmatrix} 3 & -2 & 7 \\ -12 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 14 \\ -24 & 0 & 10 \end{bmatrix} \quad \text{and} \quad \lambda\begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} \lambda C_1 \\ \lambda C_3 \end{bmatrix}.$$

We can add two m by n matrices A and B by adding values in the same row and column, for example

$$\begin{bmatrix} 3 & -2 & 7 \\ -12 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -4 \\ 10 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ -2 & -3 & 9 \end{bmatrix}.$$

Note as a result of this definition we cannot add two matrices if they have different dimensions.
Multiplication of two matrices is a little more complicated. Let $\mathbf{A}$ denote an m by n matrix and $\mathbf{B}$ denote an n by p matrix, then we have

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \dots & \mathbf{a_{1n}} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \mathbf{b_{12}} & \dots & b_{1p} \\ b_{21} & \mathbf{b_{22}} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & \mathbf{b_{n2}} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \mathbf{c_{12}} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & b_{mp} \end{bmatrix}$$

where entry c_{ij} in the i^{th} row and j^{th} column of the product is found by multiplying and adding entries of the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$ as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Note that as a result of this definition $\mathbf{AB}$ is only defined if the number of columns of $\mathbf{A}$ matches the number of rows of $\mathbf{B}$.

8. Compute the product **AB** for the matrices:

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ -7 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 \\ -12 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} C_1 \\ C_3 \end{bmatrix}$$

$$AB = \begin{bmatrix} aC_1 + bC_3 \\ cC_1 + dC_3 \end{bmatrix}$$

Some important terminology when working with matrices:

- A matrix that has only one column is often called a **column vector**.
- A column vector that is a column of all zeros is called the **zero vector** and denoted **0** or $\vec{0}$ in order to distinguish itself from the scalar value 0.
- An $n \times n$ matrix is called a **square matrix**.

9. Give the matrix **A** such that the system of equations for r_1 in 7 can be written in matrix form as

$$\mathbf{A} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = r_1 \begin{bmatrix} C_1 \\ C_3 \end{bmatrix}.$$

$$\left.\begin{aligned} aC_1 + bC_3 &= r_1 C_1 \\ cC_1 + dC_3 &= r_1 C_3 \end{aligned}\right\}$$

$aC_2 + bC$

$cC_2 + d$

$$r_1 \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix}$$

Section 3.3: Expressing Systems in Matrix Form

10. Let $\mathbf{x}'$ denote a column vector of derivatives such as

$$\mathbf{x}' = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix}.$$

Give the matrix $\mathbf{A}$ such that the system of differential equations below can be written in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_1 + 7x_2 - x_3 \\ \frac{dx_2}{dt} &= -2x_1 - 11x_3 \\ \frac{dx_3}{dt} &= 8x_3 - x_2 \end{aligned}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 4 & 7 & -1 \\ -2 & 0 & -11 \\ 0 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

11. Express the system below in matrix form.

$$\begin{aligned} x' &= \cos(2t)x + \sin(2t)z \\ y' &= e^t y \\ z' &= \sin(2t)x + \cos(2t)z \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(2t) & 0 & \sin(2t) \\ 0 & e^t & 0 \\ \sin(2t) & 0 & \cos(2t) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Section 3.4: Eigenvalues of a 2×2 Matrix

We say a scalar λ is an **eigenvalue** of a square matrix $\mathbf{A}$ if there exists a nonzero vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. The eigenvalues of matrix have many useful applications and interpretations. We will see in the context of differential equations, they can tell us very important information about how the solutions behave.

We will mostly be working with a system of two differential equations, so it will suffice to restrict our attention to the case where $\mathbf{A}$ is a 2×2 matrix, but the discussion below can be generalized to deal with much larger systems. In the 2×2 case, λ is an eigenvalue of $\mathbf{A}$ if there exists a nonzero vector $\mathbf{v}$ such that

$$r_1 \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} \qquad \lambda = r_1 \qquad \vec{v} = \begin{bmatrix} c_1 \\ c_3 \end{bmatrix}$$

$$\begin{aligned} \lambda\mathbf{v} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{v} \\ \mathbf{0} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{v} - \lambda\mathbf{v} \\ \mathbf{0} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{v} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \mathbf{v} \\ \mathbf{0} &= \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \mathbf{v} \end{aligned}$$

Using linear algebra, we can show that λ is an eigenvalue of $\mathbf{A}$ if and only if

$$\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \right) = (a-\lambda)(d-\lambda) - bc = 0.$$

Note the resulting quadratic equation (in λ) is called the **characteristic equation** for $\mathbf{A}$.

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}$$

Find the eigenvalues of A

① Subtract λ from each diagonal entry

$$\begin{bmatrix} 3-\lambda & 2 \\ -1 & 6-\lambda \end{bmatrix}$$

② Set up characteristic equation

$$\det \begin{bmatrix} 3-\lambda & 2 \\ -1 & 6-\lambda \end{bmatrix} = 0$$

③ Solve quadratic equation for λ.

$$(3-\lambda)(6-\lambda) - (2)(-1) = 0$$

$$18 - 9\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 9\lambda + 20 = 0$$

$$(\lambda - 5)(\lambda - 4) = 0$$

$$\boxed{\lambda = 5, 4}$$

$$x = c_1 e^{5t} + c_2 e^{4t}$$

12. Recall the model for bacteria populations x and y in Colony 1 and Colony 2 in problem . Follow the steps below to find general solutions to the system

$$y = \frac{\frac{dx}{dt}}{10} - \frac{3x}{10}$$

$$\frac{dx}{dt} = 3x + 10y$$
$$\frac{dy}{dt} = -2y.$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Write the system in matrix form and identify the matrix **A**.	$A = \begin{bmatrix} 3 & 10 \\ 0 & -2 \end{bmatrix}$
Determine the characteristic equation.	$\det \begin{bmatrix} 3-\lambda & 10 \\ 0 & -2-\lambda \end{bmatrix} = 0$ $(3-\lambda)(-2-\lambda) - (10)(0) = 0$
Solve the characteristic equation.	$(3-\lambda)(-2-\lambda) = 0$ $\lambda = 3, -2$
Give a general solution for $x(t)$.	$\boxed{x = c_1 e^{3t} + c_2 e^{-2t}}$ $y = c_3 e^{3t} + c_4 e^{-2t}$
Plug your solution for $x(t)$ into the differential equation x' and solve for $y(t)$. Should have 2 general constants	$3c_1 e^{3t} - 2c_2 e^{-2t} = 3(c_1 e^{3t} + c_2 e^{-2t}) + 10y$ $3c_1 e^{3t} - 3c_1 e^{3t} - 2c_2 e^{-2t} - 3c_2 e^{-2t} = 10y$ $-5c_2 e^{-2t} = 10y$ $\boxed{y = -\tfrac{1}{2} c_2 e^{-2t}}$

Welcome! Today we'll
Finish Worksheet 17 and work on Worksheet 18.

Last Class:

Solving $x' = ax + by$
$y' = cx + dy$

1. Set up matrix
2. Find eigenvalues
3. Give general solution for x (or y).
   - If distinct real eigenvalues then $x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
   - If one repeated real eigenvalue, then $x = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$
   - If complex eigenvalues $\lambda = \alpha \pm \beta i$, then
     $x = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$
4. Plug general solution for x into equation for x' and solve for y (or plug solution for y into equation for y' and solve for x.)

Practice

13. Find a general solution to the system of differential equations.

(a) $x' = -\frac{20}{9}x - \frac{8}{9}y$
$y' = -\frac{4}{9}x - \frac{34}{9}y$

① $A = \begin{bmatrix} -20/9 & -8/9 \\ -4/9 & -34/9 \end{bmatrix}$

② $\det \begin{bmatrix} -20/9 - \lambda & -8/9 \\ -4/9 & -\frac{34}{9} - \lambda \end{bmatrix} = 0$

$\left(-\frac{20}{9} - \lambda\right)\left(-\frac{34}{9} - \lambda\right) - \frac{32}{81} = 0$

$P(\lambda) = \frac{648}{81} + \frac{54}{9}\lambda + \lambda^2 = 0$

$P(\lambda) = (\lambda+4)(\lambda+2) = 0$

$\lambda = -4, -2$

$\boxed{x = c_1 e^{-4t} + c_2 e^{-2t}}$

$-4c_1 e^{-4t} - 2c_2 e^{-2t} = -\frac{20}{9}\left(c_1 e^{-4t} + c_2 e^{-2t}\right) - \frac{8}{9}y$

solving for y gives

$\boxed{y = 2c_1 e^{-4t} - \frac{1}{4} c_2 e^{-2t}}$

(b) $x' = x - y$
$y' = 2x - y$

① $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

② $\det \begin{bmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{bmatrix} = 0$

$p(\lambda) = (1-\lambda)(-1-\lambda) - (-1)(2) = \lambda^2 - 1 + 2 = \lambda^2 + 1 = 0$

$\lambda = \pm i$

③ $x = c_1 \cos(t) + c_2 \sin(t)$

④ $x' = -c_1 \sin t + c_2 \cos t = c_1 \cos t + c_2 \sin t - y$

$$\boxed{\begin{aligned} y &= (c_1 + c_2)\sin t + (c_1 - c_2)\cos t \\ x &= c_1 \cos t + c_2 \sin t \end{aligned}}$$

(c) $x' = 12x - 3y$
$y' = 3x + 6y$

$A = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$ $p(\lambda) = \lambda^2 - 18\lambda + 81 = 0$ $\lambda = 9$

$x = C_1 e^{9t} + C_2 t e^{9t}$

$9C_1 e^{9t} + C_2 e^{9t} + 9C_2 t e^{9t} = 12C_1 e^{9t} + 12C_2 t e^{9t} - 3y$

$3y = 3C_1 e^{9t} - C_2 e^{9t} + 3C_2 t e^{9t}$

$y = C_1 e^{9t} - \frac{C_2}{3} e^{9t} + C_2 t e^{9t}$

$$\boxed{\begin{aligned} y &= \left(C_1 - \frac{C_2}{3}\right) e^{9t} + C_2 t e^{9t} \\ x &= C_1 e^{9t} + C_2 t e^{9t} \end{aligned}}$$

(d) $x' = 4x + 5y$
$y' = -x + 2y$

$A = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$ $p(\lambda) = \lambda^2 - 6\lambda + 13$ $\lambda = 3 \pm 2i$

$x(t) = C_1 e^{3t} \cos(2t) + C_2 e^{3t} \sin(2t)$

$x' = 3C_1 e^{3t} \cos(2t) - 2C_1 e^{3t} \sin(2t) + 3C_2 e^{3t} \sin(2t) + C_2 e^{3t} \cos(2t)$

$\quad = 4C_1 e^{3t} \cos(2t) + 4C_2 e^{3t} \sin(2t) + 5y$

Solving for y

$$\boxed{\begin{aligned} y(t) &= \left(\frac{-C_1 + 2C_2}{5}\right) e^{3t} \cos(2t) + \left(\frac{-2C_1 - C_2}{5}\right) e^{3t} \sin(2t) \\ x(t) &= C_1 e^{3t} \cos(2t) + C_2 e^{3t} \sin(2t) \end{aligned}}$$

(e) $x' = 4x + 21y$
$y' = -2x - 8y$

$A = \begin{bmatrix} 4 & 21 \\ -2 & -8 \end{bmatrix}$, $p(\lambda) = \lambda^2 + 4\lambda + 10 = 0$, $\lambda = -2 \pm \sqrt{6}\, i$

$$\boxed{x = C_1 e^{-2t}\cos(\sqrt{6}\, t) + C_2 e^{-2t}\sin(\sqrt{6}\, t)}$$

Solving for y gives

$$\boxed{y = \frac{-6C_1 + \sqrt{6}\, C_2}{21} e^{-2t}\cos(\sqrt{6}\, t) + \left(\frac{-\sqrt{6}\, C_1 - 6C_2}{21}\right) e^{-2t}\sin(\sqrt{6}\, t)}$$

(f) $x' = 2x + 2y$
$y' = x + 3y$

$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, $p(\lambda) = \lambda^2 - 5\lambda + 4 = 0$

$\lambda = 4, 1$

$$\boxed{\begin{aligned} x &= C_1 e^{4t} + C_2 e^{t} \\ y &= C_1 e^{4t} - \tfrac{1}{2} C_2 e^{t} \end{aligned}}$$

solving for y gives