Homogeneous Second Order Linear Differential Equations

A second order linear differential equation with constant coefficients has the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

Welcome!

where a, b, and c are constants and f is a continuous function of t.

- If $f(t) = 0$, then the equation is called **homogeneous**.
- If $f(t) \neq 0$, then the equation is called **nonhomogeneous**.

Today we'll work on Worksheet 10 Complex roots

We have shown that to find solutions to the homogeneous case $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$, we can:

1. Set up the corresponding characteristic polynomial, $ar^2 + br + c = 0$.
2. Find solutions $r = r_1$ and $r = r_2$ to the characteristic equation.
3. Quadratic equations may have real or complex solutions:
   - If r_1 and r_2 are distinct real numbers, then the general solution is
   $$x(t) = C_1e^{r_1t} + C_2e^{r_2t}.$$
   - If there is one repeated root, r_1, then the general solution is
   $$x(t) = C_1e^{r_1t} + C_2te^{r_1t}.$$
   - **If the solutions are of the form $\mathbf{r} = \alpha \pm \mathbf{i}\beta$, then what?**

$$y'' + 5y' + 6y = 0$$
$$r^2 + 5r + 6 = 0$$
$$(r+3)(r+2) = 0$$
$$r = -3, -2$$
$$\boxed{y = c_1e^{-3t} + c_2e^{-2t}}$$

$$y'' - 6y' + 10y = 0$$
$$r^2 - 6r + 10 = 0$$
Doesn't Factor
$$r = \frac{6 \pm \sqrt{36-40}}{2}$$
$$r = \frac{6}{2} \pm \frac{\sqrt{-4}}{2} \rightarrow 2i$$
$$\boxed{r = 3 \pm i}\ ??$$
$$e^{(3+i)t}$$

Complex Solutions

beta = β

$e^{(3+i)t} = e^{3t} e^{it}$?

1. Let $f(t) = e^{i\beta t}$ and answer the questions below.

   (a) Find a formula for f', f'', f''', f^{iv}, and f^{v}.

$$f'(t) = i\beta e^{i\beta t} = i\beta e^{i\beta t} \Rightarrow i\beta$$
$$f''(t) = i^2\beta^2 e^{i\beta t} = -\beta^2 e^{i\beta t} = -\beta^2$$
$$f'''(t) = i^3\beta^3 e^{i\beta t} = -i\beta^3 e^{i\beta t}$$
$$f^{iv}(t) = i^4\beta^4 e^{i\beta t} = \beta^4 e^{i\beta t}$$
$$f^{v}(t) = i^5\beta^5 e^{i\beta t} = i\beta^5 e^{i\beta t}$$
$$\vdots$$

   (b) Express $f(t) = e^{i\beta t}$ using as a Taylor series at $t = 0$:

$$f(t) = f(0) + \frac{f'(0)}{1!}t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{iv}(0)}{4!}t^4 + \frac{f^{v}(0)}{5!}t^5 + \dots$$

$$e^{i\beta t} = 1 + i\beta t - \frac{\beta^2}{2!}t^2 - \frac{i\beta^3}{3!}t^3 + \frac{\beta^4}{4!}t^4 + \frac{i\beta^5}{5!}t^5 - \frac{\beta^6}{6!}t^6 + \dots$$

   (c) Group the real and imaginary parts of the first several terms in the Taylor series together.

$$e^{i\beta t} = \boxed{\left(1 - \frac{\beta^2}{2!}t^2 + \frac{\beta^4}{4!}t^4 - \frac{\beta^6}{6!}t^6 + \dots\right)} + i\boxed{\left(\beta t - \frac{\beta^3}{3!}t^3 + \frac{\beta^5}{5!}t^5 \dots\right)}$$

   (d) Do you recognize these are Taylor series of common functions?

$$\boxed{e^{i\beta t} = \cos(\beta t) + i\sin(\beta t)}$$

Euler's Formula

The previous question is a proof of **Euler's formula** which allows us to write exponentials in **polar form**,

$e^{\alpha t}$ $e^{i\beta t}$

$$e^{(\alpha+i\beta)t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right).$$

$y = e^{3t}\left[\cos(t) + i\sin(t)\right]$

2. If $z(t) = P(t) + iQ(t)$ is complex solution to a differential equation of the form $az'' + bz' + cz = 0$, prove that the real part $P(t)$ is a solution itself and the imaginary part $Q(t)$ (not including the i) is also a solution itself. Note the derivative of a complex function is the sum of the derivatives of the real and imaginary parts of the complex function:

$$z'(t) = P'(t) + iQ'(t).$$

$ay'' + by' + cy = 0$

y is a real valued function

Given $\boxed{z(t) = P(t) + iQ(t)}$ is a solution to $az'' + bz' + cz = 0$. Show $\boxed{P(t)}$ and $\boxed{Q(t)}$ are each solutions.

$z' = P'(t) + iQ'(t)$

$z'' = P''(t) + iQ''(t)$

$* \quad a(P'' + iQ'') + b(P' + iQ') + c(P + iQ) = 0$

$\underbrace{(aP'' + bP' + cP)} + i\underbrace{\left[aQ'' + bQ' + cQ\right]} = 0$

Complex number $= 0 = 0 + 0i$

So $aP'' + bP' + cP = 0$ ✓

$aQ'' + bQ' + cQ = 0$ ✓

$z = P(t)$ is a real solution

$z = Q(t)$ is a real solution.

$e^{(3\pm i)t}$

Euler's Formula

$$e^{(3+i)t} \implies e^{3t}\left[\cos(t) + i\sin(t)\right]$$

$$y = e^{3t}\cos(t) + i\, e^{3t}\sin(t)$$

is a complex solution

From previous result

$P(t) = c_1 e^{3t}\cos t$ is a real solution.

$Q(t) = c_2 e^{3t}\sin t$ "

General Solution

$$\boxed{y = c_1 e^{3t}\cos t + c_2 e^{3t}\sin t}$$

$\cos(t) = \cos(-t)$

$\sin(-t) = -\sin(t)$

$$y = B_1 e^{3t}\cos t + B_2 e^{3t}\sin(-t)$$

3. Find the general solution to the homogeneous differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 17x = 0$$

Use the above results on exponentiation of complex numbers to find the general solution to the differential equation.

① Set up Characteristic Equation.

$r^2 + 2r + 17 = 0$

② Solve characteristic Eq.

$r = -1 \pm 4i$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2}$

③ Depending on the roots, plug solutions ② into general form of solution

$X = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$

$$\boxed{X = C_1 e^{-t} \cos(4t) + C_2 e^{-t} \sin(4t)}$$

To summarize our results, when solving a homogeneous second order differential with constant coefficients, we can find the zeros of the corresponding characteristic equation. Then

- If r_1 and r_2 are distinct real numbers, then the general solution is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- If there is one repeated root, r_1, then the general solution is

$$y = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$$

- If the solutions are of the form $\mathbf{r} = \alpha \pm \mathbf{i}\beta$, then the general solution is

$$\boxed{y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)}$$

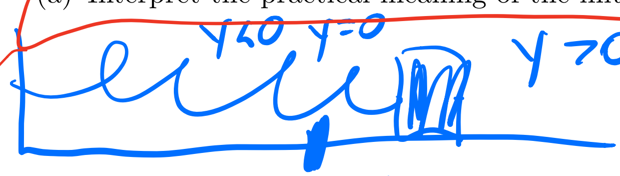
Section 2.4: Mass-Spring Oscillator

4. Consider the mass-spring oscillator that has mass $m = 1$ kg, stiffness $k = 4$ kg/sec^2, and damping b kg/sec. The displacement y from equilibrium position at time t seconds satisfies the initial value problem

$$y'' + by' + 4y = 0; \quad y(0) = 1 \quad y'(0) = 0.$$

y=0 means at equi'l position
y>0 right
y<0 left

(a) Interpret the practical meaning of the initial conditions.



mass Friction Equilibrium

$$\boxed{ay'' + by' + cy = 0}$$

Stiffness

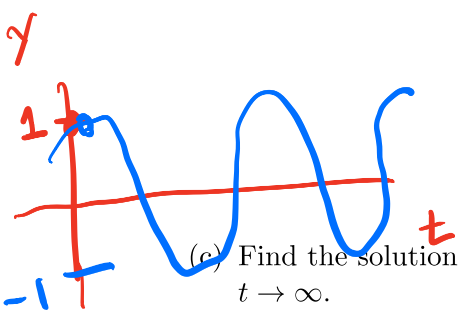
$\boxed{a, b, c \geq 0}$

$\boxed{\text{undamped}}$

(b) Find the solution if the damping coefficient is $b = 0$ and describe what happens to the mass as $t \to \infty$.

$y(0)=1$ At time 0, (initially) mass is 1 unit to right
$y'(0)=0$ Initially at $t=0$, mass is let go (not pushed or pulled)

$$y'' + 4y = 0$$
$$r^2 + 4 = 0$$
$$r^2 = -4 \quad r = \pm\sqrt{-4} = \pm 2i$$



(c) Find the solution if the damping coefficient is $b = 5$ and describe what happens to the mass as $t \to \infty$.

$y = C_1 \cos(2t) + C_2 \sin(2t)$ $\quad y = \cos(2t)$
$y(0) = 1 \to C_1 = 1$
$y'(0) = 0 \to C_2 = 0$

As $t \to \infty$ mass oscillate back and forth forever.

(d) Find the solution if the damping coefficient is $b = 4$ and describe what happens to the mass as $t \to \infty$.

(e) Find the solution if the damping coefficient is $b = 2$ and describe what happens to the mass as $t \to \infty$.

$r = 2 \pm 3i$

$\hookrightarrow \quad e^{(2+3i)t} \qquad e^{(2-3i)t}$

$$\downarrow$$

$$c_1\left[e^{2t}\cos(3t) + e^{2t}\sin(3t)\right]$$

$$+c_2\left[e^{2t}\cos(-3t) + e^{2t}\sin(-3t)\right]$$

$$c_1e^{2t}\cos(3t) + c_2e^{2t}\cos(-3t) = \underbrace{(c_1+c_2)}_{B_1}e^{2t}\cos(3t)$$

$$c_1e^{2t}\sin(3t) - c_2e^{2t}\sin(3t) = \underbrace{(c_1-c_2)}_{B_2}e^{2t}\sin(3t)$$

Justification
for $c_1e^{(\alpha+i\beta)t} + c_2e^{(\alpha-\beta i)t}$
leading to
$c_1e^{\alpha t}\cos(\beta t) + c_2e^{\alpha t}\sin(\beta t)$