Welcome!

Today we are working on Worksheet 07 Population and Mixture Problems

$t^2 x' + 3x = 0$ – Linear ✓

And is separable: $t^2 x' = -3x$

$x' = \frac{-3x}{t^2} = (-3x)\left(\frac{1}{t^2}\right)$

$\int \frac{a}{b-x}\,dx$

$w = b - x$

$dw = -dx$

$-dw = dx$

$\Rightarrow -a\int w^{-1}\,dw = \boxed{-a\ln|b-x| + C}$

$\int \frac{10x+2}{(x+5)(x-3)}\,dx = A\int \frac{1}{x+5}\,dx + B\int \frac{1}{x-3}\,dx$

$\boxed{\frac{A}{x+5} + \frac{B}{x-3}} = \frac{10x+2}{(x+5)(x-3)}$

$\boxed{= 4\ln|x+5| + 6\ln|x-3| + C}$

$A + B = 10$

$-3A + 5B = 2$

$\Rightarrow \boxed{A = 6 \quad B = 4}$

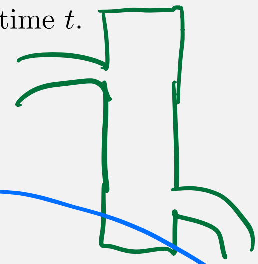
Mixed up order in

Video

Mixture Problems (Application 4.2)

A one-compartment system consists of

- $x(t)$ that represents the amount of a substance (such as salt) at time t.
- an input rate of x.
- an output rate of x.



$$\frac{dx}{dt} = \text{input rate} - \text{output rate}$$

1. A brine solution of salt water that has concentration 0.05 kg per L flows at a constant rate of 6 L per minute into a tank which is initially contains 50 L of a 1% salt solution. The brine solution flows out of the tank at a rate of 4 L per minute. Let $x(t)$ denote the mass of the salt in the tank at time t (in minutes). *Note that 1% salt solution means 1 kg of salt per 100 L of solution.*

initial condition

(a) What is the input rate of x?

$(6\ \text{L/min})(0.05\ \text{kg/L}) = 0.3\ \text{kg/min}$

(b) What is the output rate of x?

$(4\ \text{L/min})\left(\frac{\text{Amt of Salt in tank}}{\text{Volume of Solution in tank}}\right) = (4\ \text{L/min})\left(\frac{x}{50+2t}\right)$

(c) What is the initial mass of the salt in the tank?

$x(0) = \text{Amt Salt at } t=0 = (50\text{L})(1\%) = (50)(0.01\ \text{kg/L}) = 0.5\text{kg}$

(d) Construct an model for this initial value problem (but do not solve it).

$\boxed{\frac{dx}{dt}} = 0.3 - \frac{\boxed{4}x}{\boxed{50+2t}}, \quad x(0) = 0.5\text{ kg}$

$\frac{x}{\text{Volume}}$

(e) What method(s) can we apply to solve the equation in (4) (but don't solve it)?

$\boxed{\text{Not Separable}}$ → Linear ✓ Integrating Factor

Input Rate: (Rate solution Flows in)(concentration solution in) = Rate Salt goes in

depends on t

Output Rate: (Rate Solution Flows out)(concentration solution out ??) = units of salt per unit of time

$\frac{dP}{dt} = -A \cdot P \cdot (P-L) \qquad \frac{1}{P}\frac{1}{P-L}\,dP = -A\,dt$

Population Models (Section 4.1)

The Malthusian law of population growth says the rate of change of the population, $\frac{dP}{dt}$, is directly proportional to the population present, P, at time t:

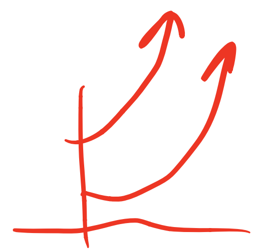
$$\frac{dP}{dt} = kP, \quad P(0) = P_0.$$

$$\boxed{dP/dt = -AP(P-L)}$$

Example 2. Let P denote the population of the world (in billions) t years since 1960. In 1960 the world's population was approximately 3 billion, and the population growth is model by

$$\frac{dP}{dt} = 0.2P \qquad , \; P(0) = 3.$$

7.5 bil



Solving this model gives $P(t) = 3e^{0.02t}$ and predicts the population in 2019 is 9.76 billion.

Why do you think predicted value is different from the actual value?

Not taken into account any death rate.

$$\int \frac{1}{P} \cdot \frac{1}{P-L} dP = \int -A\, dt = -At + C$$

$$\frac{A}{P} + \frac{B}{P-L} = \frac{1}{P(P-L)}$$

$$A = -1/L \quad B = 1/L$$

$$\int -\frac{1}{L} \frac{1}{P} dP + \int \frac{1}{L}\left(\frac{1}{P-L}\right) dP$$

$$= -1/L \ln|P| + 1/L \ln|P-L|$$

$$= \frac{1}{L} \ln\left|\frac{P-L}{P}\right| = -At + C$$

$$1/L\left(-\ln|P| + \ln|P-L|\right)$$

$$\ln\left|\frac{P-L}{P}\right| = -ALt + C$$

$$e \qquad e$$

$$\frac{P-L}{P} = Ce^{-ALt}$$

$$P - L = PCe^{-ALt}$$

$$P(1 - Ce^{-ALt}) = L$$

$$\boxed{P = \frac{L}{1 + Ce^{-ALt}} \qquad P_0 = \frac{L}{1+C} \qquad C = \frac{L - P_0}{P_0}}$$

L = const.

A = const.

C = Initial cond.

The Logistic Models (Section 4.1)

We can construct our population model by considering::

$$\frac{dP}{dt} = \Big(\underbrace{\text{Birth Rate}}_{k_1 P}\Big) - \Big(\underbrace{\text{Death Rate}}_{?}\Big). \quad = -AP(P-L)$$

Competition within the population causes the populations to decrease (disease, murder, natural disasters, war, lack of food/water). If we assume the death rate is proportional to the total number of possible two-party interactions, we get:

$$\text{Death rate} = \boxed{k_2 \binom{P}{2}} = \boxed{k_2\left(\frac{P(P-1)}{2}\right)}.$$

Note: $\binom{P}{2}$ denotes "P choose 2", and in general we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \quad \binom{P}{2} = \frac{P!}{2!(P-2)!} = \frac{P(P-1)}{2}$$

Taking both the birth and death rates into account, we get the Logistic model for population change which we simplify:

$$\frac{dP}{dt} = \Big(k_1 P \Big) - \Big(k_2 \cdot \frac{P(P-1)}{2} \Big).$$

2. Show that the model above can be rewritten in the form $\boxed{\mathbf{\frac{dP}{dt} = -AP(P-L)}}$ where A and L are positive constants.

Equilibrium

$P=0, P=L$

$$\frac{dP}{dt} = k_1 P - k_2\left(\frac{P^2-P}{2}\right) = k_1 P - \frac{k_2}{2}P^2 + \frac{k_2}{2}P$$

$$\frac{dP}{dt} = -\frac{k_2}{2}P^2 + \left(\frac{k_2}{2} + k_1\right)P = -\frac{k_2}{2}P(P-L)$$

$dP/dt =$

$$L = -\frac{2}{k_2}\left(\frac{k_2}{2} + k_1\right)$$

Practice: Population Model for Rabbits

3. A population of rabbits changes over time t (in years) according to the logistic model

Autonomous

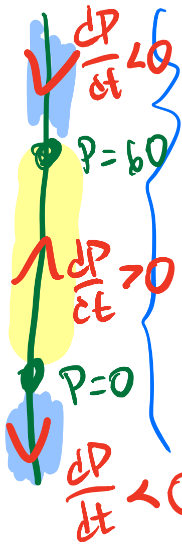
① Factor
② Identify A and L

$$\frac{dP}{dt} = 3P - \frac{1}{20}P^2. = -AP(P-L) =$$

① $\frac{dP}{dt} = 0$ Find Equil. $P=0$ $P=60$

② Sketch Line and mark equilibrium

③ Pick test point to find sign dP/dt



(a) For what initial population sizes P_0 will the population grow at first?

$\frac{dP}{dt} > 0$ $\quad 0 < P_0 < 60$

(b) For what initial population sizes P_0 will the population decrease at first?

$dP/dt < 0$ $\quad P_0 > 60$ or ~~$P_0 < 0$~~

(c) For what initial population sizes P_0 will the population never change?

(d) Explain, in practical terms, why answers in (a)-(c) makes sense.

L carrying capacity

$\boxed{P=60 \text{ Stable Equilibrium}}$

$P=0$ unstable Equilibrium

(e) If the initial rabbit population is $P_0 = P(0) = 50$, find a solution to the initial value problem and find a formula for the population P as a function of time t.

$$\boxed{\frac{dP}{dt} = 3P - \frac{1}{20}P^2 = P\left(3 - \frac{1}{20}P\right) = \boxed{-AP(P-L)} = \boxed{-\frac{1}{20}P(P-60)}}$$

$$\boxed{\frac{dP}{dt} = -\frac{1}{20}P(P-60)}$$

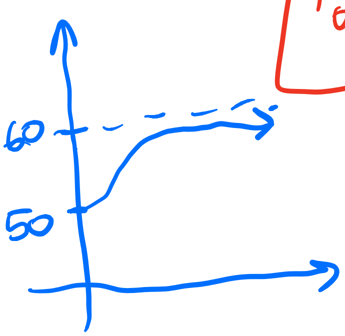
$P=0$, $P=60$

$\boxed{A = \frac{1}{20} \quad L = 60}$

$$\boxed{P = \frac{L}{1+Ce^{-ALt}} \qquad P_0 = \frac{L}{1+C} \qquad \boxed{C = \frac{L-P_0}{P_0}}}$$

where

$$C = \frac{60-50}{50} = 0.2$$



$$P = \frac{60}{1+Ce^{-(1/20)(60)t}} = \frac{60}{1+Ce^{-3t}}$$

$$\boxed{P = \frac{60}{1+0.2e^{-3t}}}$$

$$\lim_{t\to\infty} P = \frac{60}{1+0} = 60$$

0

Practice: Chlorine Levels in Pool

initial condition

Inflow

4. A swimming pool whose volume is 10,000 gallons contains water that is 0.01% chlorine. Starting at $t = 0$, city water containing 0.001% chlorine is pumped into the pool at a rate of 5 gal/min. The pool water flows out at the same rate. Let x denote the amount of chlorine (in pounds) in the pool t minutes since water has begun being pumped into the pool.

Outflow

Note that a concentration of 0.01% *chlorine solution means* 0.01 *pounds of chlorine per 100 gallons of solution.*

*Note: Rate solution flows in = Rate it flows out so volume constant.

(a) Construct a differential equation for rate of change of the mass of chlorine (in pounds) x in the pool at time t.

$$\frac{dx}{dt} = \left(5\,\text{gal}/\text{min}\right)\left(0.00001\,\text{lb}/\text{gal}\right) - \left(5\,\text{gal}/\text{min}\right)\left(\frac{x}{10{,}000}\right)$$

(b) Solve the initial value problem using the differential equation in (a) and the given initial % concentration.

$$\boxed{\frac{dx}{dt} = 0.00005 - 0.0005x \qquad x(0) = (10{,}000)(0.0001) \qquad x(0) = 1\text{kg}}$$

This is separable and linear so we can solve multiple ways!

$$\frac{dx}{dt} = 0.00005(1-10x) \qquad \int \frac{1}{1-10x}\,dx = \int 0.00005\,dt$$

$$w = 1-10x$$
$$dw = -10\,dx$$

$$\frac{-1}{10}\int \frac{1}{w}\,dw = \frac{-1}{10}\ln|1-10x| = 0.00005t + C$$

(c) (Bonus) When will the pool water be 0.002% chlorine?

$$-\frac{1}{10}\ln|1-10x| = 0.00005t + C$$

$$e^{\ln|1-10x|} = e^{-0.0005t + C}$$

$$|1-10x| = e^{C}e^{-0.0005t} = Be^{-0.0005t}$$

$$1-10x = Be^{-0.0005t}$$

* Abs. value not needed since constant B will absorb sign.

$$10x = 1 - Be^{-0.0005t}$$

$$x = 0.1 + Be^{-0.0005t}$$

* $\left(\frac{1}{10}\right)(-B) = \underset{\text{constant}}{\text{Arbitrary}} = B$

If $x(0) = 1$, we have

$$x = 0.1 + Be^{0} = 1 \qquad B = 0.9$$

$$\boxed{x = 0.1 + 0.9e^{-0.0005t}}$$

c) If 10,000 gal is 0.002% chlorine, then

$$x(t) = (10{,}000)(0.00002) = 0.2 \text{ lb chlorine}$$

Solving $0.1 + 0.9e^{-0.0005t} = 0.2$ gives

$t \approx 4394.4$ min or Approx 3 days

Extra Practice with integrating Factor - see end of Zoom recording

$$\frac{1}{2}\frac{dy}{dx} = (y-1)\cos x$$

$$\frac{1}{2}y' = \cos x \, y - \cos x$$

$$\frac{1}{2}y' - (\cos x)y = -\cos x$$

① Put into Standard Form

$$\rightarrow y' - (2\cos x)y = -2\cos x$$

$$P(x) = -2\cos x$$

② Find $\mu(x) = e^{\int P(x)\,dx}$

$$\mu(x) = e^{\int -2\cos x\,dx} = e^{-2\sin x}$$

③ Multiply both sides by μ

$$y'e^{-2\sin x} - 2\cos x\, e^{-2\sin x} y = -2\cos x\, e^{-2\sin x}$$

④ Rewrite Left side as Product Rule $(y\mu)'$

$$\int (y e^{-2\sin x})' = \int -2\cos x\, e^{-2\sin x}$$

⑤ Integrate both sides & Solve for y.

$$y e^{-2\sin x} = \int -2\cos x\, e^{\underbrace{-2\sin x}} \, dx$$

$w = -2\sin x$

$dw = -2\cos x \, dx$

$$= \int e^{w} \, dw = e^{-2\sin x} + C$$

$$y e^{-2\sin x} = e^{-2\sin x} + C$$

$$\boxed{y = 1 + Ce^{2\sin x}}$$

See last page for trig substitution example 6b from HW 2

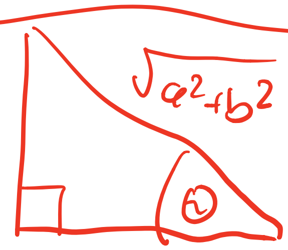
$$y\frac{dz}{dy} = (1+z^2)^{3/2}$$

$$\int \frac{1}{(1+z^2)^{3/2}}\,dz = \int \frac{1}{y}\,dy = \ln|y| + C$$

$$\overset{?}{=}$$

$$\int \left(\frac{1}{\sqrt{1+z^2}}\right)^3 dz$$

$$\int \sqrt{a^2+b^2} \qquad \int \sqrt{a^2-b^2}$$



$$\cos\theta = \frac{adj}{hyp}$$

$$\frac{1}{\sqrt{1+z^2}} = \cos\theta$$

$$\tan\theta = \frac{opp}{adj} = \frac{z}{1} = z \qquad \cos\theta = \frac{1}{\sqrt{1+z^2}}$$

$$\int \cos^3\theta\, dz \qquad \boxed{\tan\theta = z} \longrightarrow \theta = \tan^{-1}(z)$$

$$\sec^2\theta\, d\theta = dz$$

?

$$\int (\cos^3\theta)(\sec^2\theta\, d\theta) = \int \cos\theta\, d\theta = \sin\theta + C$$

$$= \sin(\tan^{-1}(z)) + C$$