Common Laplace Transforms and Properties

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$

Properties:

L.1 $\underline{\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}}$, where c is a constant.

shifts to right by a units

L.2 $\mathscr{L}\{f_1(t) + f_2(t)\} = \underline{\mathscr{L}\{f_1(t)\} + \mathscr{L}\{f_2(t)\}}$

L.3 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$ for all $s > \alpha + a$.

L.4 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then for all $s > \alpha$,

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

L.5 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^nF}{ds^n}$ for all $s > \alpha$.

Explain in Words:

1. Describe properties 3, 4, and 5 in words. For example in property 3, multiplying $f(t)$ by e^{at} and then taking the Laplace transform has what affect on $\mathscr{L}\{f(t)\}$?

y_H $\qquad y_p = At + B$

$r^2 - 1 = 0$

Solving Diff. Eqs. with Inverse Laplace Transforms

$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}},\ s > 0$

2. Solve $y'' - y = -t$ with $y(0) = 0$ and $y'(0) = 1$.

(a) Using the properties, apply the Laplace transform to both sides: $Y(s)$

$$\mathscr{L}\{y'' - y\} = \mathscr{L}\{-t\}.$$

By Table

By L.1 and L.2 $\quad \mathscr{L}\{y''-y\} = \mathscr{L}\{y''\} - \mathscr{L}\{y\} = -\mathscr{L}\{t\}$

By L.4 $\quad = s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0) - Y(s) = -\left(\frac{1}{s^2}\right)$

$= s^2 Y(s) - s(0) - 1 - Y(s) = -\frac{1}{s^2}$

$s^2 Y(s) - 1 - Y(s) = -1/s^2$

Algebra

(b) Using your answer in 2a, solve for $\mathscr{L}\{y(t)\} = Y(s)$.

Group Like Terms

$s^2 Y(s) - Y(s) = -1/s^2 + 1$

$Y(s)(s^2 - 1) = -\frac{1}{s^2} + 1$

$Y(s) = \left(-\frac{1}{s^2} + 1\right)\left(\frac{1}{s^2-1}\right) = \left(\frac{-1+s^2}{s^2}\right)\left(\frac{1}{s^2-1}\right)$

$\boxed{Y(s) = \frac{1}{s^2}}$

(c) Use the table of common Laplace transforms to identify what function $y(t)$ has $\mathscr{L}\{y(t)\} = Y(s)$.

$Y(s) = \mathscr{L}\{y(t)\} = 1/s^2 \quad$ what is $y(t)$?

Take the inverse Laplace transform.

If $Y(s) = 1/s^2$, then $\boxed{y(t) = t}$ Initial Value Problem Solution

In 2c, we are apply the **Inverse Laplace Transform** to $Y(s)$ in order to identify $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

$\mathcal{L}$

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}, \; s > 0$
e^{at}	$\frac{1}{s-a}, \; s > a$
$t^n, \; n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \; s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \; s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \; s > 0$
$e^{at}t^n, \; n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \; s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \; s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \; s > a$

$n=1$

$$\frac{1}{s^2} = \frac{n!}{s^{n+1}}$$

$n=1$

$$Y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

Section 6.1: Inverse Laplace Transforms

Given $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies $\mathscr{L}\{f\} = F(s)$, then we say $f(t)$ is the **inverse Laplace transform** of $F(s)$ which is denoted by

$$\mathbf{f(t)} = \mathscr{L}^{-1}\{\mathbf{F(s)}\}.$$

3. Determine whether the inverse Laplace transform is of the form t^n, $\cos(bt)$, $\sin(bt)$, or e^{at}.

(a) $F(s) = \frac{1}{s^2}$ $\rightarrow$ $f(t) = t$

$\boxed{t^n}$

(b) $F(s) = \frac{2}{s^2+4}$

$\sin(bt)$ $\quad b^2 = 4$

$\boxed{f(t) = \sin(2t)}$

(c) $F(s) = \frac{4s}{s^2+9} = 4\left(\frac{s}{s^2+9}\right)$

$b = 3$

$\boxed{f(t) = 4\cos(3t)}$

(d) $F(s) = \frac{2}{s+6} = 2\left(\frac{1}{s-(-6)}\right)$ $\quad a = -6$

$\boxed{f(t) = 2e^{-6t}}$

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, $n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s > 0$ constant
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s > 0$ variable s
$e^{at}t^n$, $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

4. Find the inverse Laplace transform of $F(s) = \dfrac{s+2}{s^2+4s+11}$ by answering the questions below.

   (a) Complete the square for the expression in the denominator of $F(s)$ to express $s^2+4s+11 = (s-a)^2+b$.

   (b) Use the table of common Laplace transforms to identify $\mathscr{L}^{-1}\{F(s)\}$.

5. Find the inverse Laplace transform of the function.

   (a) $F(s) = \dfrac{5s-10}{s^2-3s-4}$

(b) $F(s) = \dfrac{3s-15}{2s^2-4s+10}$

(c) $F(s) = \dfrac{-5s-36}{(s+2)(s^2+9)}$