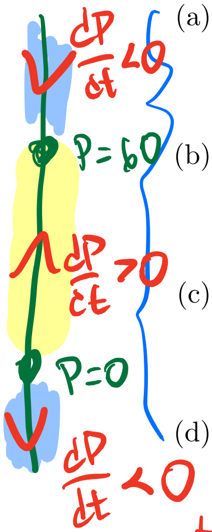
Practice: Population Model for Rabbits

3. A population of rabbits changes over time t (in years) according to the logistic model

$$\frac{dP}{dt} = 3P - \frac{1}{20}P^2. = -AP(P-L) =$$

① Factor
② Identify A and L

Autonomous
① $\frac{dP}{dt} = 0$
Find Equil.
$P=0$ $P=60$
② Sketch Line and mark equilibrium
③ Pick test point to find sign dP/dt



(a) For what initial population sizes P_0 will the population grow at first?

$\frac{dP}{dt} > 0$ $\quad 0 < P_0 < 60$

(b) For what initial population sizes P_0 will the population decrease at first?

$dP/dt < 0$ $\quad P_0 > 60$ or $\cancel{P_0 < 0}$

(c) For what initial population sizes P_0 will the population never change?

(d) Explain, in practical terms, why answers in (a)-(c) makes sense.

L carrying capacity
$\boxed{P=60 \text{ Stable Equilibrium}}$
$P=0$ unstable Equilibrium

(e) If the initial rabbit population is $P_0 = P(0) = 50$, find a solution to the initial value problem and find a formula for the population P as a function of time t.

$$\boxed{\frac{dP}{dt} = 3P - \frac{1}{20}P^2 = P\left(3 - \frac{1}{20}P\right) = \boxed{-AP(P-L)} = -\frac{1}{20}P(60 + P)}$$

$$\boxed{\frac{dP}{dt} = -\frac{1}{20}P(P-60)}$$

$$\boxed{A = \frac{1}{20} \quad L = 60}$$

$P=0$
$P=60$

$$\boxed{P = \frac{L}{1+Ce^{-ALt}} \qquad P_0 = \frac{L}{1+C} \qquad \boxed{C = \frac{L-P_0}{P_0}}}$$

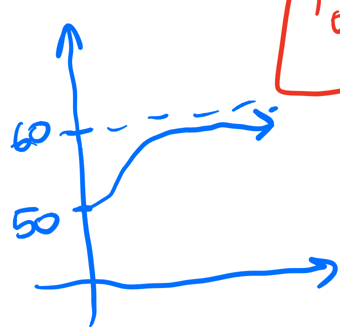
where

$$C = \frac{60-50}{50} = 0.2$$

$$P = \frac{60}{1+Ce^{-(1/20)(60)t}} = \frac{60}{1+Ce^{-3t}}$$

$$\boxed{P = \frac{60}{1+0.2e^{-3t}}}$$

$$\lim_{t\to\infty} P = \frac{60}{1+0} = 60$$



Practice: Chlorine Levels in Pool

Inflow

initial condition

4. A swimming pool whose volume is 10,000 gallons contains water that is 0.01% chlorine. Starting at $t = 0$, city water containing 0.001% chlorine is pumped into the pool at a rate of 5 gal/min. The pool water flows out at the same rate. Let x denote the amount of chlorine (in pounds) in the pool t minutes since water has begun being pumped into the pool.

Outflow

Note that a concentration of 0.01% *chlorine solution means* 0.01 *pounds of chlorine per 100 gallons of solution.*

*Note: Rate solution flows in = Rate it flows out so volume constant.

(a) Construct a differential equation for rate of change of the mass of chlorine (in pounds) x in the pool at time t.

$$\frac{dx}{dt} = (5\ \text{gal/min})(0.00001\ \text{lb/gal}) - (5\ \text{gal/min})\left(\frac{x}{10{,}000}\right)$$

(b) Solve the initial value problem using the differential equation in (a) and the given initial % concentration.

$$\boxed{\frac{dx}{dt} = 0.00005 - 0.0005x}$$

$$x(0) = (10{,}000)(0.0001)$$

$$\boxed{x(0) = 1\ \text{kg}}$$

This is separable and linear so we can solve multiple ways!

$$\frac{dx}{dt} = 0.00005(1-10x) \qquad \int \frac{1}{1-10x}\,dx = \int 0.00005\,dt$$

$$w = 1-10x$$
$$dw = -10\,dx$$

$$-\frac{1}{10}\int \frac{1}{w}\,dw = -\frac{1}{10}\ln|1-10x| = 0.00005t + C$$

(c) (Bonus) When will the pool water be 0.002% chlorine?

$$-\frac{1}{10}\ln|1-10x| = 0.00005t + C$$

$$e^{\ln|1-10x|} = e^{-0.0005t + C}$$

$$|1-10x| = e^{C}e^{-0.0005t} = Be^{-0.0005t}$$

$$1-10x = Be^{-0.0005t}$$

* Abs. value not needed since constant B will absorb sign.

$$10x = 1 - Be^{-0.0005t}$$

$$x = 0.1 + Be^{-0.0005t}$$

* $\left(\frac{1}{10}\right)(-B) = \text{Arbitrary constant} = B$

If $x(0) = 1$, we have

$$x = 0.1 + Be^{0} = 1 \qquad B = 0.9$$

$$\boxed{x = 0.1 + 0.9e^{-0.0005t}}$$

c) If 10,000 gal is 0.002% chlorine, then

$$x(t) = (10{,}000)(0.00002) = 0.2 \text{ lb chlorine}$$

Solving $0.1 + 0.9e^{-0.0005t} = 0.2$ gives

$t \approx 4394.4$ min or Approx 3 days