$$\mathcal{L}\{3t^2y''+2\cos(t)y'+3y\}=\mathcal{L}\{e^{5t}\}$$

Properties of the Laplace Transform

1. Let f, f_1, and f_2 be functions whose Laplace transform exists for $s > \alpha$ and let c be a constant. Then for $s > \alpha$, prove the following:

   (a) $\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\}$.

$$\mathcal{L}\{f_1+f_2\} = \int_0^\infty e^{-st}(f_1(t)+f_2(t))\,dt$$
$$= \int_0^\infty \left(e^{-st}f_1(t)+e^{-st}f_2(t)\right)dt$$
$$= \int_0^\infty e^{-st}f_1(t) + \int_0^\infty e^{-st}f_2(t)\,dt$$
$$= \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$$

   (b) $\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$.

$$\mathcal{L}\{cf\} = \int_0^\infty e^{-st}(cf(t))\,dt = c\int_0^\infty e^{-st}f(t)\,dt$$
$$= c\,\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{2\cos(3t)+5-4e^{6t}\} = \frac{6}{s^2+9}+5/s-\frac{4}{s-6}, \quad s>6$$

$$2\mathcal{L}\{\cos(3t)\}+5\mathcal{L}\{1\}-4\mathcal{L}\{e^{6t}\}$$

Referring to Table

$$2\left(\frac{3}{s^2+3^2}\right)+\frac{5}{s}-4\frac{1}{s-6}$$

$s>0 \quad s>0 \quad s>6$

$e^{-st+at} = e^{-t(s-a)}$

Laplace Transform of $g(t) = e^{at} f(t)$

1. If the Laplace transform $\boxed{\mathscr{L}\{f\}(s) = F(s) \text{ exist for } s > \alpha}$, then show that

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a), \quad \text{for } s > \alpha + a.$$

$\boxed{u = (s-a)}$

$$\mathscr{L}\{e^{at} f(t)\} = \int_0^\infty \underbrace{e^{-st} e^{at}} f(t)\, dt = \int_0^\infty e^{-t(\overset{u}{\overset{\shortparallel}{s-a}})} f(t)\, dt$$

$$= \int_0^\infty e^{-t \cdot u} f(t)\, dt \iff \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$= F(u) \qquad u > \alpha \qquad\qquad s > \alpha$$

$$\boxed{= F(s-a) \qquad s - a > \alpha \qquad s > \alpha + a}$$

$\mathscr{L}\{e^{at} f(t)\}$ $\downarrow$ Horizontal Shift of $\mathscr{L}\{f(t)\}$

2. Using the property above and the fact that $\boxed{\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}}$ for $s > 0$, find $\mathscr{L}\{e^{at}\cos(bt)\}$.

$$\boxed{\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0}$$

$$F(s) = \frac{s}{s^2+b^2}$$

$$\mathscr{L}\{e^{at}\cos(bt)\} = F(s-a)$$

$$\boxed{= \frac{s-a}{(s-a)^2+b^2}, \quad s > a}$$

Section 6.2: Laplace Transform of Derivatives

A function is of **exponential order** α if there exists positive constants C and T such that

$$|f(t)| < Ce^{\alpha t} \;\text{ for all } t > T.$$

For example:

- $f(t) = \cos(5t)e^{7t}$ has $\alpha = 7$.
- e^{t^2} does not have an exponential order.

3. If $f(t)$ is continuous on $[0,\infty)$ and $f'(t)$ is piecewise continuous on $[0,\infty)$ with both exponential order α, then prove for $s > \alpha$,

$$\mathscr{L}\{f'\} = s\mathscr{L}\{f\} - f(0) = sF(s) - f(0).$$

4. Using the property from problem 3 and the fact that $\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$ for $s > 0$, find $\mathscr{L}\{\sin(bt)\}$.

5. If $\mathscr{L}\{f(t)\} = F(s)$ for all $s > \alpha$, using the property from problem 3, show that
$$\mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0) \quad \text{for all } s > \alpha.$$

6. Using induction show that
$$\mathscr{L}\{f^{(n)}\} = s^n \mathscr{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

7. Let $F(s) = \mathscr{L}\{f\}$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Prove that for $s > \alpha$ if follows that

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}.$$

8. Using the definition of the Laplace transform, the result that $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$ for $s > a$ and the property above, find a formula for $\mathscr{L}\{t^n e^{at}\}$.

Common Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$

,

Common Properties of Laplace Transforms

L.1 $\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}$, where c is a constant.

L.2 $\mathscr{L}\{f_1(t) + f_2(t)\} = \mathscr{L}\{f_1(t)\} + \mathscr{L}\{f_2(t)\}$

L.3 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$ for all $s > \alpha + a$.

L.4 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then for all $s > \alpha$,

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

L.5 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^nF}{ds^n}$ for all $s > \alpha$.