Common Laplace Transforms and Properties

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$

Properties:

L.1 $\mathscr{L}\{cf(t)\} = c\mathscr{L}\{f(t)\}$, where c is a constant.

L.2 $\mathscr{L}\{f_1(t) + f_2(t)\} = \mathscr{L}\{f_1(t)\} + \mathscr{L}\{f_2(t)\}$

L.3 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$ for all $s > \alpha + a$.

L.4 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then for all $s > \alpha$,

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

L.5 If $F(s) = \mathscr{L}\{f(t)\}$ exists for all $s > \alpha$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$ for all $s > \alpha$.

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y(t)\} - s^0 y(0) = s Y(s) - y(0)$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y(t)\} - s y(0) - y'(0) = s^2 Y(s) - s y(0) - y'(0)$$

Section 6.2: Solving ODE's

Step 1 Take the Laplace transform of both sides. Refer to properties.

Step 2 Rearrange and group like terms to solve for $\mathcal{L}\{y(t)\} = Y(s)$

Step 3 Take the inverse Laplace transform and solve for $y(t) = \mathcal{L}^{-1}\{Y(x)\}$.

L.1 and L.2

1. Solve the initial value problem using Laplace Transforms (not previous methods).

   (a) $y'' - 2y' + 5y = 0$ with $y(0) = 2$ and $y'(0) = 4$.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) - 2\left(s Y(s) - y(0)\right) + 5 Y(s) = 0 \quad \text{using L.4}$$

$$\left(s^2 Y(s) - 2s - 4\right) - 2s Y(s) + 4 + 5 Y(s) = 0$$

Solve for $Y(s)$:

$$s^2 Y - 2s Y + 5 Y = 2s \qquad Y(s)\left(s^2 - 2s + 5\right) = 2s$$

$$Y(s) = \frac{2s}{s^2 - 2s + 5}$$

$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

Take $\mathcal{L}^{-1}$ to solve $y(t)$.

$s - 1 + \frac{1}{2}(2)$

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4} + \frac{1}{2}\frac{2}{(s-1)^2+4}\right\}$$

$$\boxed{2e^{t}\cos(2t) + e^{t}\sin(2t) = y(t)}$$

(b) $y'' - y' - 2y = 0$ with $y(0) = -2$ and $y'(0) = 5$.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left(s^2\bar{Y} - sy(0) - y'(0)\right) - \left(s\bar{Y} - y(0)\right) - 2\bar{Y} = 0$$

$$s^2\bar{Y} + 2s - 5 - s\bar{Y} - 2 - 2\bar{Y} = 0$$

$$s^2\bar{Y} - s\bar{Y} - 2\bar{Y} = -2s + 7 \qquad \bar{Y} = \frac{-2s+7}{s^2 - s - 2} = \frac{-2s+7}{(s-2)(s+1)}$$

$\frac{A}{s-2} + \frac{B}{s+1} = \frac{-2s+7}{(s-2)(s+1)}$ has solution $A = 1$ and $B = -3$ thus

$$\mathcal{L}^{-1}\{\bar{Y}\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{3}{s+1}\right\}$$

$$\boxed{y(t) = e^{2t} - 3e^{-t}}$$

(c) $y'' - 4y' - 5y = 4e^{3t}$ with $y(0) = 2$ and $y'(0) = 7$.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = 4\mathcal{L}\{e^{3t}\}$$

$$s^2Y - 4sY - 5Y - 2s - 7 + 14 = 4\left(\frac{1}{s-3}\right)$$

$$\bar{Y} = \frac{4}{(s-3)(s^2-4s-5)} + \frac{2s-1}{s^2-4s-5}\cdot\frac{s-3}{s-3} = \frac{2s^2-7s+7}{(s-3)(s-5)(s+1)}$$

$(s-5)(s+1)$

$$= \frac{A}{s-3} + \frac{B}{s-5} + \frac{C}{s+1} = \frac{2s^2-7s+7}{(s-3)(s-5)(s+1)}$$

$A = -\frac{1}{2}$

$B = \frac{11}{6}$

$C = \frac{2}{3}$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{-\frac{1}{2}\left(\frac{1}{s-3}\right) + \frac{11}{6}\left(\frac{1}{s-5}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)\right\}$$

$$\boxed{y(t) = -\frac{1}{2}e^{3t} + \frac{11}{6}e^{5t} + \frac{2}{3}e^{-t}}$$

(d) $ty'' - ty' + y = 2$ with $y(0) = 2$ and $y'(0) = -1$.

(e) $y'' + ty' - y = 0$ with $y(0) = 0$ and $y'(0) = 3$.