# Eighty Years of Ramsey $R(3, k)$ . . . and Counting!

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How frequently does an intriguing problem come up over lunchtime, only to have it solved the next morning? How many mathematical problems are seemingly intractable? Decades go by without a hint of progress. What a delight when a problem is worked on over many many years with progress occurring incrementally until it finally succumbs. Fermat's Last Theorem is perhaps the best example. Hilbert's Tenth Problem is another marvellous story. In discrete mathematics, my vote is for the asymptotics of the Ramsey number $R(3, k)$. The story begins in 1931, is resolved in 1995, with a coda in 2008, and with the final story perhaps not yet told.

In this chapter we consider only the *asymptotics*, the behavior of $R(3, k)$ for $k$ large. There has been a great deal of work on the values $R(3, k)$ for $k$ small, with the exact values known for $3 \leq k \leq 9$. The frequently updated survey of Radziszowski [9], together with his paper appearing in this volume, gives these results and much much more.

## 1 Basics

But first, the problem. We deal throughout with graphs that are undirected and have neither loops nor multiple edges. We write $G = (V, E)$ where $V, E$ are the sets of vertices and edges of $G$, respectively.

**Definition 1.** A set $I \subseteq V$ is *independent* if for no $v, w \in I$ is $\{v, w\} \in E$. The *independence number* of a graph, denoted $\alpha(G)$, is the maximal size $|I|$ of an independent set in $G$.

The study of $R(3, k)$ splits into upper bounds and lower bounds so we define it in a slightly unusual way.

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**Definition 2.** $R(3,k) \leq n$ if for every triangle-free graph $G$ on $n$ vertices there *exists* an independent set $I$, $|I| \geq k$.

**Definition 3.** $R(3,k) > n$ if there *exists* a triangle-free graph $G$ on $n$ vertices which does not have an independent set $I$, $|I| \geq k$.

## 2 George, Esther, Paul

The story begins in late 1931. Three youngsters, full of mathematical promise, walk the beautiful hills around Budapest. George Szekeres had completed his studies in chemical engineering. His interest in mathematics was already very strong, but it would take 15 years of global uproar before he would take on that subject as his profession in Australia. Esther Klein was a talented mathematics student who had just returned from Gottingen with an intriguing problem. The youngest was only 18-years old, not yet *die Zauberer von Budapest*, but he was already well known in Hungarian circles for his mathematical abilities. This was Paul Erdős.

Some 50 years later, George Szekeres wrote about those times:

> The origins of the paper go back to the early thirties. We had a very close circle of young mathematicians, foremost among them Erdős, Turán and Gallai; friendships were forged which became the most lasting that I have ever known and which outlived the upheavals of the thirties, a vicious world war and our scattering to the four corners of the world. I [...] often joined the mathematicians at weekend excursions in the charming hill country around Budapest and (in the summer) at open air meetings on the benches of the city park.

Klein proposed a geometry problem that they all set out to solve. In short order, Szekeres gave a solution. But, in fact, he had rediscovered Ramsey's theorem. While Ramsey's paper had been published in 1927, Ramsey himself was interested in a problem in logic and none of the three had been aware of his work. Erdős presented an independent proof and their results appeared in a joint paper [7]. Erdős always called this paper the "Happy Ending Paper" as George Szekeres and Esther Klein were soon married. After spending the war years as refugees in Shanghai they emigrated to Australia where they were leaders in the development of Australian mathematics, particularly in introducing the Hungarian style of mathematical contests to generations of Australian students.

The Szekeres argument gave the existence of $R(3,k)$ (and much much more), but just how big is it? We give the basic upper bound.

**Theorem 2.1.** $R(3,k) \leq k^2$.

*Proof.* Here, and throughout, we aim for a computer science perspective on the proofs. These were certainly not the ways the original proofs were framed!

Let $G$ have $k^2$ vertices. Consider the program:

IF some $v \in V$ has degree $\geq k$

  Neighbors of $v$ form independent set $I$

ELSE
  $I \leftarrow \emptyset$
  WHILE $G$ is nonempty
    Select any $v$.
    Add $v$ to $I$
    Delete $v$ and neighbors from $G$
    End WHILE

The neighbors of any vertex $v$ of a triangle-free graph form an independent set. Thus in the IF case we find $I$ of the desired size. ELSE, for each $v$ added to $I$ at most $k$ vertices (it and its neighbors) are deleted. Having begun with $k^2$ vertices, the final $I$ has size at least $k$. $\square$

## 3 Erdős Magic

In April 1946 Erdős [4] made a conceptual breakthrough whose effects we are still feeling.

**Theorem 3.1.** *If*

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

*there* exists *a graph G on n vertices with neither clique nor independent set of size k.*

*Proof.* Consider the *random graph* $G \sim G(n, \frac{1}{2})$. Technically, we have a probability space whose elements are the labelled graphs on $n$ vertices. Probabilities are determined by saying $\Pr[\{i, j\} \in E] = \frac{1}{2}$ and that these events are independent. For each set $S$ of $k$ vertices we have the "bad" event $B_S$ that $S$ is either complete or independent. Then $\Pr[B_S] = 2^{1-\binom{k}{2}}$. The probability of a disjunction is at most the sum of the probabilities so that ($\vee$ over all $S \subset V$, $|S| = k$)

$$\Pr[\bigvee B_S] \le \binom{n}{k} 2^{1-\binom{k}{2}} < 1 \quad \square$$

Let *GOOD* denote the completement event $\wedge \overline{B_S}$. Then *GOOD* has positive probability. That is, there is positive probability that the random graph has the desired property. The probabilistic method, or Erdős Magic, is now born. As the event is nonempty there must be a point in the probability space for which it holds. That is, there absolutely positively must exist a graph $G$ with the desired properties. Our book [1] is one of many to cover the many applications of this methodology.

Let $G(n, p)$, as usual, denote the random graph with $n$ vertices with $p$ the probability of adjacency. In studying $R(3, k)$ one is led to the study of sparse random graphs. The probability that $G(n, p)$ has an independent set of size $k$ is at most $\binom{n}{k}$, the number of such sets, times $(1-p)^{\binom{k}{2}}$, the probability of no internal edges. We bound the first term from above by $n^k$ (it being rather remarkable how effective such gross bounds can be) and the second by $e^{-pk^2/2}$. When $k = 2.01((\ln n)/p)$ the second term dominates and so $G(n, p)$ almost surely has no such independent set.

Following Definition 2, attempts to find a lower bound on $R(3, k)$ start with a random graph $G(n, p)$. Immediately there is a problem. To avoid triangles one needs $p = O(n^{-1})$, but in this range there are independent sets of size $\Omega(n)$ and this would yield only a linear lower bound on $R(3, k)$. All successful approaches use a larger $p$, one for which $G(n, p)$ will have triangles, and then somehow fix the triangles. We begin with a rather weak result.

**Theorem 3.2.** $R(3, k) = \Omega((k/\ln k)^{3/2})$. *That is, there* exists *a graph $G$ on $n$ vertices with no triangle and no independent set of size $cn^{2/3}\ln n$.*

Here we begin with $2n$ vertices and consider a random graph with edge probability $p = n^{-2/3}$. There will be an expected number $\sim n^3p^3/6 = n/6$ triangles. Setting $k = 2.01((\ln n)/p)$, the expected number of independent sets of size $k$ is less than one. By a problem, let us mean either a triangle or an independent set of size $k$. Thus the expected number of problems is around $n/6$. From Erdős Magic, there is a graph $G$ for which the number of problems is less than around $n/6$, and so certainly there is one where the number of problems is less than $n$. Take that graph $G$ and eliminate one vertex from each triangle and one vertex from each independent set of size $k$. The remaining graph, call it $G^*$, has no problems, and it has at least $n$ vertices.

The bulk of lower-bound arguments for $R(3, k)$ examine $G(n, p)$ with $p = cn^{-1/2}$ with $c$ an appropriate small constant. Here the expected number of triangles is roughly $c^3n^{3/2}/6$. The expected number of edges is roughly $c^2n^{3/2}/2$ which will be considerably more. One wants to make the graph triangle-free by somehow eliminating the relatively small number of edges in triangles, but doing this in a way that keeps the size of the independent set around $K((\ln n)/p)$. It is not so easy!

## 4 An Erdős Gem

Erdős was one of, possibly the, most prolific mathematicians in history. With the passage of time we can look at certain of his papers and recognize their depth and importance. In that light, the following 1961 Erdős gem [3] is a personal favorite of this author. It would be an outstanding paper in any time period, but that it was done when the probabilistic method was still in its infancy is truly a testament to Erdős's genius.

**Theorem 4.1.** $R(3, k) = \Omega((k/\ln k)^2)$. *That is, there* exists *a graph $G$ on $n$ vertices with no triangle and no independent set of size $cn^{1/2}\ln n$.*

Erdős considers $G(n, p)$ with $p = \epsilon n^{-1/2}$, $\epsilon$ a small constant. Set $x = cn^{1/2}\ln n$, $c$ a large constant. Call an $x$-set $I$ a failure if every edge $\{u, v\} \in G$ with $u, v \in I$ can be extended to a triangle $\{u, v, z\}$ where the third vertex $z$ lies outside of $I$. He shows that with high probability there are no failures. This takes quite some doing, but we can give a heuristic explanation. An edge in $I$ has, on average, $np^2 \sim \epsilon^2$ extensions to a triangle outside of $I$ thus probability less than one half (taking $\epsilon$ small) of being so extendable. Each pair $u, v \in I$ would then have probability at least $p/2$ of being an edge and not being so extendable. Now suppose these events were independent over the pairs. Then the chance that $I$ is a failure, that is, that no pair had this property, would be around $1 - (p/2)$ to the power $\binom{x}{2}$. This is basically $\exp(-px^2/4)$ which is smaller than $n^{-x}$, smaller than one over the total number of $x$-sets. Then the expected number of failures $I$ would be much less than one and almost surely there would be none of them. The actual proof is more complicated as these events are very definitely not independent.

Let's assume the claim. Now we can put on our computer science hats (definitely not the original Erdős style!) and complete the proof. Take a $G$ with no failures. We apply a greedy algorithm to find a triangle-free subgraph of $G$. Order the edges of $G$ arbitrarily and consider them in that order. Accept an edge if it does not create a triangle along with the edges previously accepted. Let $H$ denote the final graph created. We tautologically do not have a triangle in $H$. Now consider any $x$-set $I$. As $I$ is not a failure for $G$ it has an edge $\{u, v\}$ which is not extendable to a triangle outside of $I$. When we reached this edge in determining $H$ it was either accepted or rejected. If it was accepted then $I$ was not independent in $H$. If it was rejected it was only because it would have created a triangle in $H$. But that triangle $\{u, v, w\}$ must lie entirely inside $I$ since the edge is not extendable to a triangle outside of $I$. That would mean that edges $\{u, w\}$ and $\{v, w\}$ would already be in $H$. So in this case too $I$ would not be independent. That is, $H$ does not contain any independent $x$-sets $I$.

## 5 Upper Bounds

We return to the upper bound and improvements on Theorem 2.1. In 1968, Graver and Yackel improved this result to

$$R(3, k) = O\left(k^2 \frac{\ln \ln k}{\ln k}\right) \tag{1}$$

That result held for 12 years, until it was supplanted in 1980 by Ajtai, Komlós and Szemerédi.

**Theorem 5.1.**

$$R(3, k) = O\left(\frac{k^2}{\ln k}\right) \tag{2}$$

At the time, the improvement was not considered so significant, but events proved otherwise. Ajtai, Komlós and Szemerédi actually proved a general theorem about independent sets in triangle-free graphs.

**Theorem 5.2.** *Let $G$ be a triangle-free graph on $n$ vertices in which the average degree is at most $k$. Then there exists an independent set $I$ with*

$$|I| \geq c\frac{n}{k}\ln k \tag{3}$$

Here $c$ is an absolute positive constant. Turán's theorem gives that a graph with $n$ vertices and average degree at most $k$ (and hence at most $nk/2$ edges) has independence number at least $n/(k+1)$, the extreme case occurring when $G$ is the union of disjoint cliques of size $k+1$. In this context Theorem 5.2 can be understood as saying that $\alpha(G)$ is increased when one requires that $G$ is triangle-free. The Ramsey bound Theorem 5.1 follows immediately from Theorem 5.2. For let $G$ be any triangle free graph on $n = c^{-1}(k^2/\ln k)$ vertices. If any vertex has degree at least $k$ its neighbors form an independent set of size at least $k$. Otherwise all vertices have degree less than $k$, hence the average degree is less than $k$, hence there is an independent set of size at least $c(n/k)\ln k$, which is $k$. In either case there is an independent set of size at least $k$.

We give a rough idea of the argument for Theorem 3. The key is a lemma which we do not prove. Let $G$ be a triangle-free graph with average degree $u$. The lemma states, roughly, that there is a vertex $v$ of degree about $u$ such that removing it and its neighbors yields a graph $G^-$ whose edge density is not more than that of $G$. Now begin with a triangle-free $G$ on $n$ vertices with average degree $k$ or less and consider a process where at each step we select a vertex $v$ as above, add it to the independent set, and remove $v$ and all of its neighbors. We parametrize time $t$ saying that at time $t$ the number of vertices $v$ so selected is $(n/k)t$. Let $S(t)n$ be the number of vertices remaining in the graph at that time. Under the assumption that density has not increased, at time $t$ the average degree would then be at most $S(t)k$. When $v$ is now selected $1 + S(t)k \sim S(t)k$ vertices are removed. Parametrized time $t$ has increased by $k/n$. This gives a difference equation

$$S\left(t + \frac{k}{n}\right) - S(t) \sim -\frac{S(t)k}{n} \tag{4}$$

which turns into a differential equation

$$S'(t) = -S(t) \tag{5}$$

with the simple solution

$$S(t) = ke^{-t} \tag{6}$$

The procedure continues until $t \sim \ln k$, giving an independent set of size $\sim (n/k)\ln k$.

## 6 The Lovász Local Lemma

One of the great advances in the probabilistic method was the Lovász local lemma, which first appeared in [5]. We give a formulation here that is not the most general, but will suffice for our application and indeed for almost all known applications. We are given a set $\Omega$ and for each $e \in \Omega$ a random variable $X_e$. We assume that the $X_e$ are mutually independent. We let $\Gamma$ index a set of events. For each $\alpha \in \Gamma$ we have a set $A_\alpha \subset \Omega$ and a "bad" event $B_\alpha$. The event $B_\alpha$ can depend only on the values $X_e$ with $e \in A_\alpha$.

In our example, the $X_e$ describe the random graph $G(n,p)$. We let $\Omega$ be the set of potential edges $e$ (that is, two element sets of vertices) on a vertex set $\{1,\ldots,n\}$. For each $e$ we let $X_e$ have values zero and one with $\Pr[X_e = 1] = p$ and the $X_e$ mutually independent. Then the edge set of $G(n,p)$ is those $e$ for which $X_e = 1$. Now the bad events will be of two types. $\Gamma$ is indexed by the three element subsets $S$ of vertices and the $k$ element subsets $T$ of vertices. For each triple $S = \{i,j,h\}$ of vertices we have the event $B_S$ that $S$ is a triangle. That is, $X_{ij} = X_{jh} = X_{ih} = 1$. For each $k$-set $T$ of vertices we have the event $B_T$ that $T$ is an independent set. That is, $X_{ij} = 0$ for all $i,j \in T$.

We write $\alpha \sim \beta$ if $\alpha \neq \beta$ (a technicality) and, critically, $A_\alpha \cap A_\beta \neq \emptyset$. Note that when a family of $\alpha$ have no $\alpha \sim \alpha'$ the corresponding events $A_\alpha$ are mutually independent. In our example, two events $B_S, B_{S'}$ are $S \sim S'$ if $S \neq S'$ and $S, S'$ overlap in at least two vertices, and hence in at least one edge $e$.

**Theorem 6.1.** *Let $B_\alpha$, $\alpha \in \Gamma$ be events as described above. Suppose there exist real numbers $x_\alpha$, $\alpha \in \Gamma$, with $0 \le x_\alpha < 1$ and*

$$\Pr(B_\alpha) \le x_\alpha \prod_{\beta \sim \alpha} (1 - x_\beta) \tag{7}$$

*Then*

$$\Pr(\wedge_{\alpha \in \Gamma} \overline{B_\alpha}) \ge \prod_{\alpha \in \Gamma} (1 - x_\alpha) \tag{8}$$

*In particular, with positive probability no event $B_\alpha$ holds.*

Our object now is to show $R(3,k) > n$ for $n$ as large as possible. We look at $G(n,p)$. If the conditions of Theorem 6.1 hold then with positive (albeit small!) probability $G(n,p)$ will have neither triangle nor independent set of size $k$. Erdős Magic then implies that there exists a specific $G$ on $n$ vertices with this property, so that $R(3,k) > n$. Suppose that for each 3-set $S$ we select the same value for $x_S$; call it $y$. Suppose that for each $k$-set $T$ we select the same value for $x_T$; call it $z$. Let's put an upper bound, for $\alpha$ of each type, of the number of $\beta$ of each type with $\alpha \sim \beta$. For each 3-set $S$ there are $3(n-3) \le 3n$ other $S'$ with $S \sim S'$. For each $k$-set $T$ there are $\binom{k}{2}(n-k) + \binom{k}{3} \le k^2 n/2$ $k$-sets $T$ with $S \sim T$. For each $k$-set $T$ there are at most $\binom{n}{k}$ (that is, all) $k$-sets $T'$ with $T \sim T'$. For each 3-set $S$ there are at most $\binom{n}{k}$ (that is, all) $k$-sets $T$ with $S \sim T$.

In application, Theorem 6.1 becomes:

**Theorem 6.2.** *If there exist* $p \in [0,1]$ *and* $y, z \in [0,1)$ *with*

$$p^3 \le y(1-y)^{3n}(1-z)^{\binom{n}{k}} \tag{9}$$

*and*

$$(1-p)^{\binom{k}{2}} \le z(1-y)^{k^2 n/2}(1-z)^{\binom{n}{k}} \tag{10}$$

*then* $R(3,k) > n$.

Theorem 6.2 leads to a problem in what we like to call asymptotic calculus. What is the largest $n$, as an asymptotic function of $k$ such that there exist $p \in [0,1]$, $y, z \in [0,1)$ satisfying 9 and 10. This is not an easy problem but it is an elementary problem.

Here this author can add a personal note. Some three decades ago I was able to show that the largest such $n$ was of the order $\Theta(k^2/\ln^2 k)$. This gave an alternate proof to Theorem 4.1, the gem of Erdős. One needed merely the analytic skills of a reasonable graduate student; one did not need the brilliance of Erdős, nor of Lovász, to find the bound. Sometimes in mathematics one's deepest work, even when successful, receives little attention. In this case the opposite was true and my applications [10] of Lovász local lemma to improve Ramsey bounds (also on $R(k,k)$ and on $R(l,k)$ for $l \ge 3$ fixed and $k \to \infty$) have been frequently quoted.

## 7 Random Greedy Triangle-Free

In 1995 [6] Paul Erdős, along with coauthors Peter Winkler and Stephen Suen, returned to the asymptotics of $R(3,k)$, a problem he had first considered some 63 or 64 years before. Erdős had a great faith, albeit unspoken, in his nose for interesting beautiful mathematics. In 1946 he "invented" the probabilistic method. For the next quarter century he published many papers in that area. While others appreciated the beauty of the results, he had few followers during that time. But he continued, convinced of the intrinsic interest in that area, and his convictions were borne out. Today (thanks, in part, to the development of probabilistic algorithms in computer science), the use of the probabilistic techniques he developed is widespread and is, in this author's opinion, one of his enduring legacies. He had an equal faith in Ramsey theory, to which he returned again and again, always coming up with new questions, new conjectures, new theorems, and new methodologies. It would be a perfect end to this narrative to say that in 1995 Erdős resolved the asymptotics of $R(3,k)$. Alas, that was not the case.

But Erdős did open the door to the final assault.

Erdős, with Winkler and Suen, examined the *random greedy triangle-free algorithm*. The algorithm itself is trivial. We begin with the empty graph $G$ on vertex set $\{1,\ldots,n\}$. Order the $\binom{n}{2}$ potential edges randomly. Now consider these potential edges sequentially. When considering edge $e$, add $e$ to $G$ (we say *accept* $e$)

if doing so will not create a triangle in $G$. Otherwise we *reject* $e$ and $G$ stays the same. Continue until all of the potential edges are considered. (Equivalently, we may begin with the empty graph and at each stage add an edge selected uniformly from those that would not create a triangle.) This yields a graph $G^{final}$ which tautologically has no triangle. Erdős and his coauthors looked at its independent sets. They were able to give a partial analysis until the time when $Kn^{3/2}$ potential edges had been considered, $K$ a particular absolute constant. With that analysis, they could show that there was no independent set of size $c_1\sqrt{n}\ln n$. By Erdős Magic there therefore existed such a triangle-free graph. Setting $k = c_1\sqrt{n}\ln n$ this yielded $R(3,k) > n = c_2k^2\ln^{-2}k$, giving yet another new proof of Erdős 1961 gem. Indeed, they found a better constant $c_2$ than previously known.

This author was able to push their methods and analyze the random greedy triangle-free algorithm until $Kn^{3/2}$ edges had been examined, where $K$ could be an arbitrarily large constant. With that analysis, I could show that there was no independent set of size $c_1\sqrt{n}\ln n$ where $c_1$ could be made arbitrarily small. By Erdős Magic there therefore existed such a triangle-free graph. Setting $k = c_1\sqrt{n}\ln n$ this yielded

$$R(3,k) \gg k^2\ln^{-2}k \tag{11}$$

This argument was never published. For in a matter of months there was a stunning breakthrough.

## 8 *R(3, k)* Resolved!

When the asymptotics of $R(3,k)$ were finally resolved there was (so the joke goes) a great surprise. The mathematician finding the solution was not a Hungarian! Rather it was the Korean Jeong-Han Kim [8]. Kim had recently received his Ph.D. from Jeff Kahn at Rutgers and was one of the stars of the new generation, using more advanced and sophisticated probabilistic methods.

**Theorem 8.1.**

$$R(3,k) = \Theta\left(\frac{k^2}{\ln^2 k}\right) \tag{12}$$

Kim's proof was an extension in spirit of the methods of Erdős, Winkler and Suen. Rather than the pure random greedy triangle-free algorithm he used a nibble method. At each stage a small but carefully chosen number of edges were added to the graph. Sophisticated use of martingales played a key role. There were underlying differential equations with careful error analysis. And there was a lot of just plain cleverness. It was a masterwork, resolving a 64-year-old problem. Kim was awarded the Fulkerson Prize for this achievement.

Let me add a personal note, what was for this author a most memorable moment. In January to April 1998 I was in Australia, working with Nick Wormald at

University of Melbourne. On March 6 I gave an invited talk at the University of Sydney. My title was "60 years of Ramsey $R(3, k)$," covering much of the material in this paper. In the front row were George Szekeres and Esther Klein Szekeres. Though in their late 80s they were enjoying an active retirement. They had enjoyed a life full of mathematics and good cheer. They lived several more years, both passing away on the same day, August 28, 2005.

## 9 Random Greedy Triangle-Free Redux

This author thought at that time that the story of $R(3, k)$ was completed. (The value of a constant $c$ so that $R(3, k) \sim ck^2 \ln^{-2} k$ remains open to this day, but this problem seems beyond our reach.) But a coda, or perhaps a new beginning would be a more appropriate term, was added in April 2008. This author received an email from Tom Bohman [2]. He had been able to analyze the random greedy triangle-free algorithm. (By coincidence, I was at ETH (Zurich) and just the day before had been speaking about the algorithm with Angelica Steger and how an analysis had proven so elusive.) He was able to show that algorithm gave a final $G$ with $\Theta(n^{3/2}\sqrt{\ln n})$ edges and that the largest independent set would have size $\Theta(\sqrt{n}/\sqrt{\ln n})$. This gave another and, at least to this author, more natural proof of Kim's result.

We can give a natural heuristic for these results. Suppose that after $un^{3/2}$ potential edges have been considered $xn^{3/2}$ have been accepted and think of $x = f(u)$. What about the next potential edge $e = \{i, j\}$? Suppose we think of the $xn^{3/2}$ accepted edges as a random graph on that many edges, or $G(n, p)$ where $p = 2xn^{-1/2}$. (Not only is this supposition a stretch but it is clearly false as $G(n, p)$ would have many triangles but the algorithm tautologically gives a graph with no triangles. Nonetheless, this is a most useful heuristic.) We would accept $e$ if for no $k \neq i, j$ are both $\{i, k\}, \{j, k\}$ in $G(n, p)$. They are both in with probability $p^2$ and so the probability of acceptance would be $(1 - p^2)^{n-2} \sim e^{-4x^2}$. Under this heuristic the rate of acceptance is now $e^{-4x^2}$. This leads to the differential equation

$$f'(u) = \frac{dx}{du} = e^{-4x^2} \tag{13}$$

We have an initial condition $f(0) = 0$ as initially the graph is empty. We solve this to get

$$u = \int_0^x e^{4t^2} dt \tag{14}$$

While this integral does not have a closed form, for large $x$ we would have

$$u = e^{4x^2(1+o(1))} dt \tag{15}$$

If we can continue this heuristic to the end of the process we would have $u = n^{1/2+o(1)}$ and therefore $x = \Theta(\sqrt{\ln n})$. This would argue that the process ends with $\Theta(n^{3/2}\sqrt{\ln n})$ edges. Making another stretch and thinking of the final $G$ as a random graph on this many edges it would have $\alpha(G) = \Theta(\sqrt{n \ln n})$ as desired.

Well, it's not so easy. Bohman parametrizes time $t$ as above, when $tn^{3/2}$ edges have been accepted. The pairs $e = \{i, j\}$ are now in three categories. Some $e$ are `in`, meaning they are already in $G$. Some $e$ are `open`, meaning that there is no $k \neq i, j$ with $\{i,k\}, \{j,k\}$ already in the graph. Open edges can be added to $G$. (Tautologically they have not already been considered, for then they would be in $G$.) The other $e$ are `closed`, meaning their addition would cause a triangle, and so they cannot ever be added.

Bohman sets $Q$ equal the number of open edges and parametrizes $Q = q(t)n^2$. (Already there is a notion that with high probability $Q$ will be concentrated and this requires substantiation.) For each pair $i, j$ of vertices with $\{i, j\}$ not in $G$ he sets $X_{ij}$ equal the number of $k \neq i, j$ with both $\{i,k\}$ and $\{j,k\}$ open. He parametrizes $X_{ij} = x(t)n$. (Now there is a further notion that with high probability all the $X_{ij}$ are asymptotically the same.) He further lets $Y_{ij}$ equal the number of $k \neq i, j$ such that one of $\{i,k\}, \{j,k\}$ is open and the other is in. He parametrizes $Y_{ij} = y(t)n^{1/2}$.

Bohman now looks at the expected change in $X_{uv}$ when a random open edge is added to the graph. The main picture is the following. Consider a $w$ for which $\{u, w\}$ and $\{v, w\}$ are open. Now pick either $u$ or $v$, say $u$. Now consider a $z$ with one of $\{z, u\}, \{z, w\}$ open and the other in. Say $\{z, u\}$ is open and suppose it is selected and added to $G$. As $\{z, u\}$ and $\{z, w\}$ are now in, $\{u, w\}$ changes from open to closed. This decrements $X_{uv}$ be one. There are $x(t)n$ choices of $w$, two choices of $u$ or $v$ and then $y(t)n^{1/2}$ choices of $z$, giving $2x(t)y(t)n^{3/2}$ choices that decrement $X_{uv}$. As there are $q(t)n^2$ open edges in total and the next edge is chosen uniformly from them, the expected decrease in $X_{uv}$ is $[2x(t)y(t)/q(t)]n^{-1/2}$. Selection of one edge increased parametrized time by $n^{-1/2}$. This leads to the difference equation

$$x(t + n^{-1/2}) - x(t) = -[2x(t)y(t)/q(t)]n^{-1/2} \tag{16}$$

which in turn leads to the differential equation

$$x'(t) = -2x(t)y(t)/q(t) \tag{17}$$

By similar arguments one gets differential equations for $q$ and $y$:

$$y'(t) = -\frac{y^2(t) + 2x(t)}{q(t)} \tag{18}$$

$$q'(t) = -y(t) \tag{19}$$

This system of differential equations has the very clean solution:

$$x(t) = e^{-8t^2} \quad y(t) = 4te^{-4t^2} \quad q(t) = \frac{1}{2}e^{-4t^2} \tag{20}$$

It is intriguing to note that these values are the same, appropriately interpreted, as if $G$ were a random graph with $tn^{3/2}$ edges, or for $G(n, p)$ with $p = 2tn^{-1/2}$.

For example, the probability that a pair $\{i, j\}$ is not joined by a path of length two (effective, that it is open) would be $(1-p^2)^{n-2} \sim e^{-4t^2}$ and so the number of such pairs would be $\sim e^{-4t^2}\binom{n}{2} \sim q(t)n^2$. A direct argument, even a strong heuristic, that the values of $x, y, q$ mirror those of the random graph remains elusive.

The difficult part of Bohman's argument is to show that in the actual random process the values of $X_{uv}, Y_{uv}, Q$ remain close to the expected values $x(t)n, y(t)n^{1/2}, q(t)n^2$. Some careful analysis with strong use of martingales (actually super and submartingales) is used. The larger $t$ gets the more difficult this becomes as an instability in the random process could conceivably lead to further instability. If one needed only to get this result for a large constant $t$ there would be results on approximating a random process by a differential equation over a compact space. Of particular difficulty is that to get the full result Bohman carries the analysis out to $t = \epsilon\sqrt{\ln n}$ for some small positive absolute constant $\epsilon$. This corresponds to the consideration of $n^{3/2+\epsilon_1}$ potential edges for some small absolute constant $\epsilon_1$. (He doesn't take the analysis out to when $n^{2-o(1)}$ edges have been considered as the instabilities overwhelm the analysis before that. For that reason the constants he achieves are subject to improvement.) To do this he gives somewhat ad hoc error bounds on the variables at $t$ as a function of $t$. These bounds increase as a function of $t$ but remain small through $t = \epsilon\sqrt{\ln n}$. Last, but not least, this analysis gives the number of edges accepted and one needs further study to show that $\alpha(G)$ is appropriately small.

## 10 Epilogue

Is the story of $R(3, k)$ over? I think not. I think there is plenty of room for a consolidation of the results. My dream is a ten-page paper which gives $R(3, k) = \Theta(k^2/\ln k)$. The upper bound Theorem 2 can be nicely described with proof in a few pages. The Kim–Bohman lower bound is not quite there yet. But it seems that the Bohman approach could be greatly simplified, making use of some known stability results for random processes. When I speak on this topic I keep waiting for someone in the audience to say "Of course, this follows easily from the well-known results of XYZ." It hasn't happened yet.

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