Triangle-Free Graphs

Ramsey Numbers

David Stalfa & Adam Plumer

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- an x clique, or
- y independent vertices

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An (x,y) graph = a graph with

- no x-clique
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$R(x,y) \leq n$ iff there is no (x,y) graph of size n

(3,k) = set of triangle free graphs

$R(3,4) \leq 9$

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Proof by contradiction

Suppose there exists a (3,4) graph G of size 9

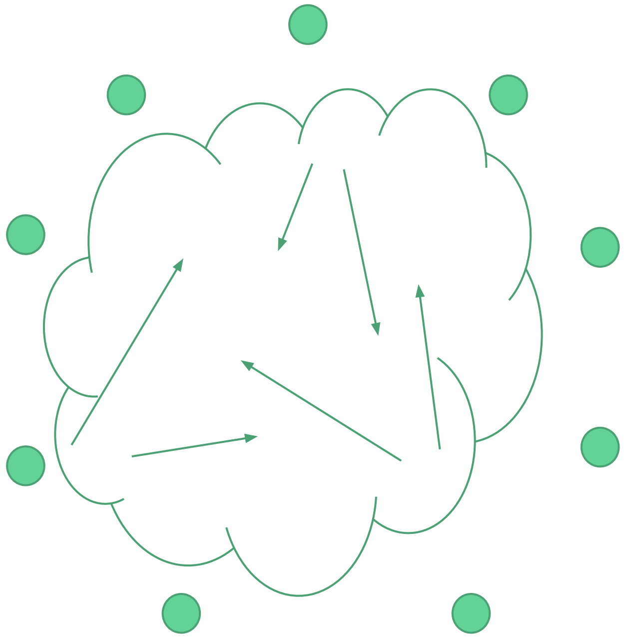
- G has a certain structure

- removing vertices from G would result in a graph from which it is
 impossible to reconstruct G

Therefore, there is no (3,4) graph of size 9

$R(3,4) \leq 9$

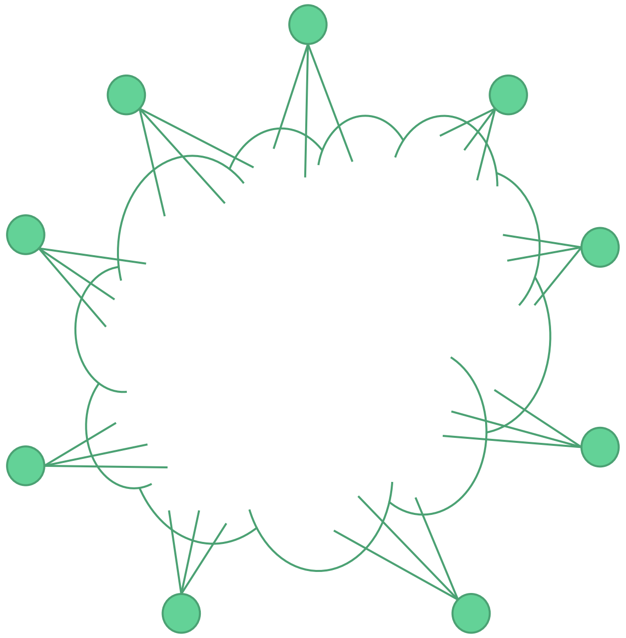
Suppose there is some (3,4) graph G of size 9



$R(3,4) \leq 9$

G is regular degree 3

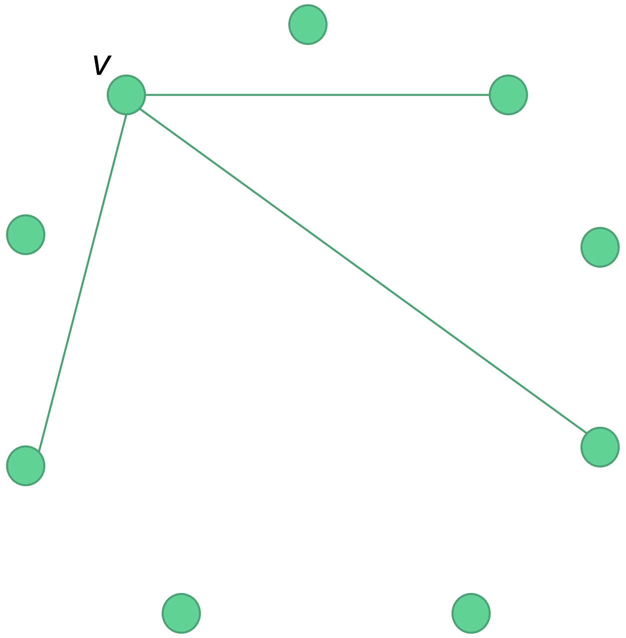
Max degree of G = min degree of G = 3



R(3,4) $\leq$ 9

G is regular degree 3

Remove one vertex and its neighborhood



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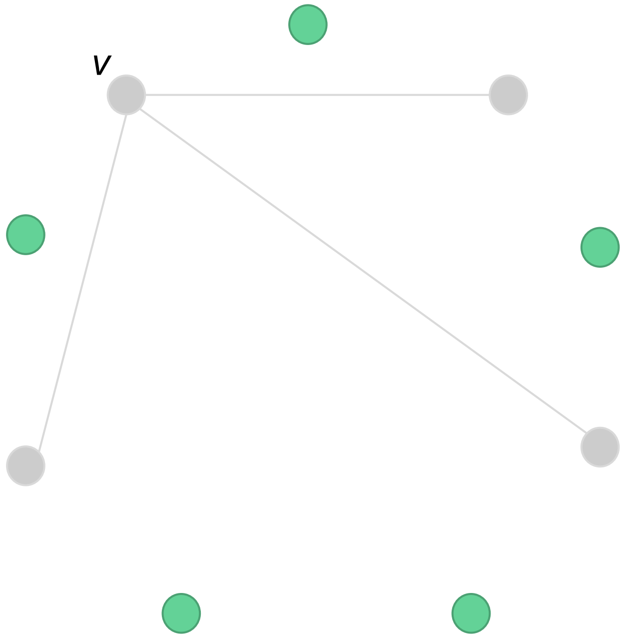

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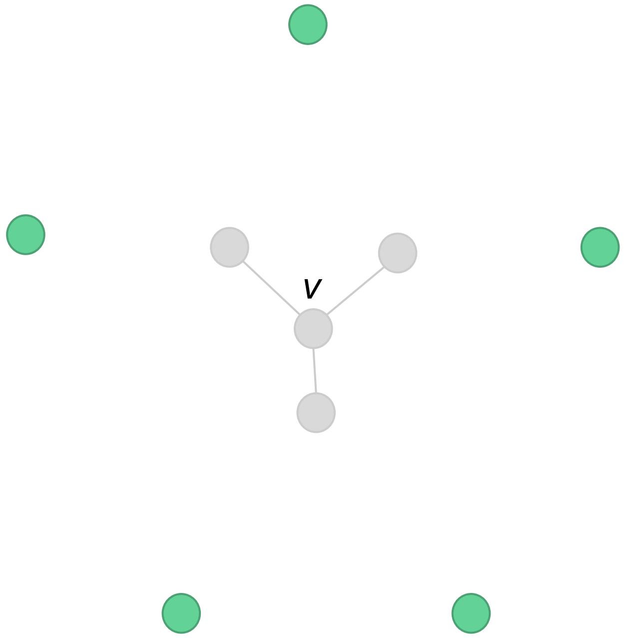
Leaves a size 5 graph



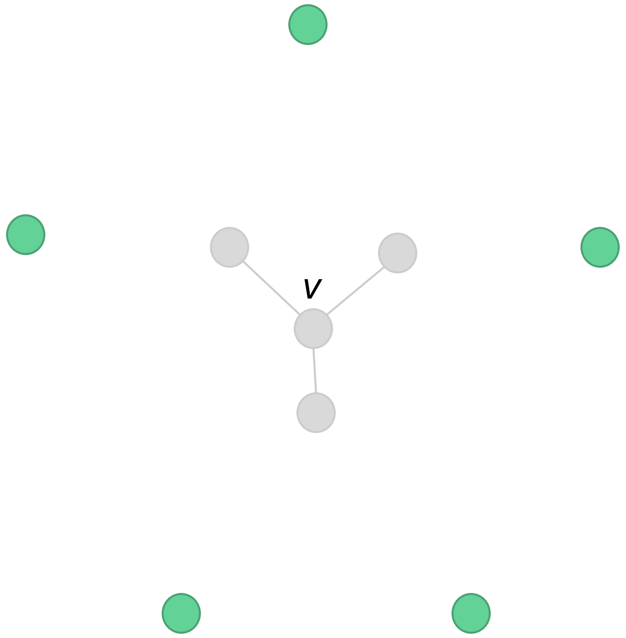
$R(3,4) \leq 9$

G is regular degree 3

Leaves a size 5 graph

- no 3-clique
- no 3 independent

  (all are independent of *v* in G)



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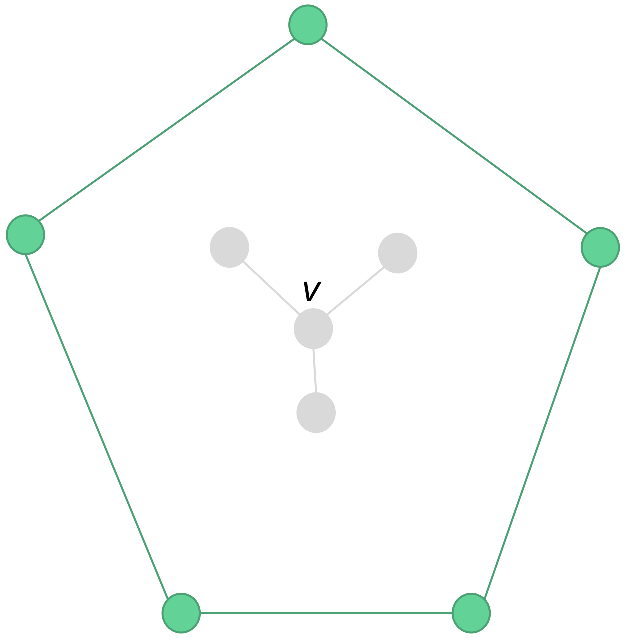
G is regular degree 3

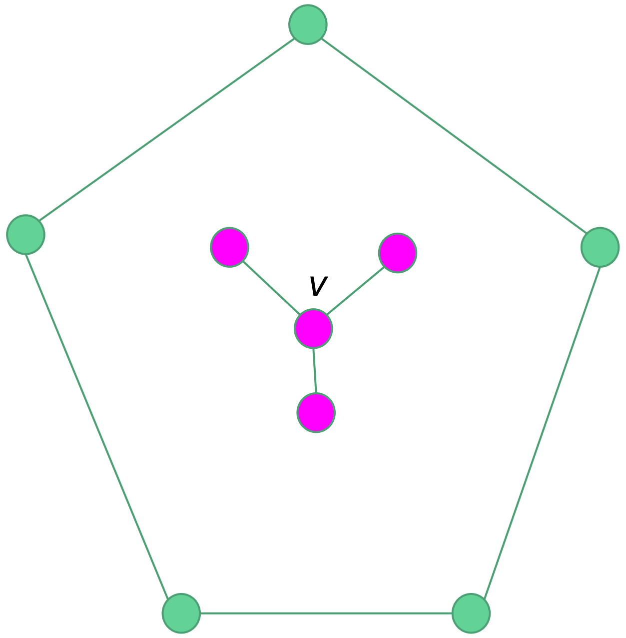
There is only one (3,3) graph of size 5



$R(3,4) \leq 9$ G is regular degree 3

There is only one (3,3) graph of size 5

Add v and its neighbors back



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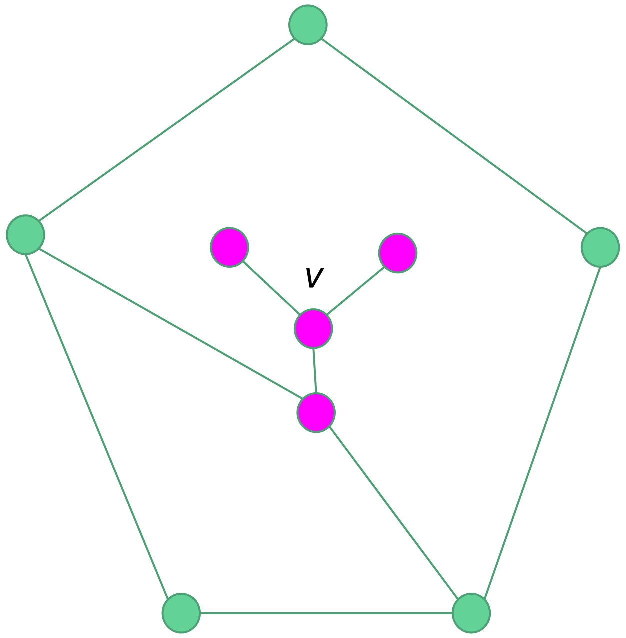

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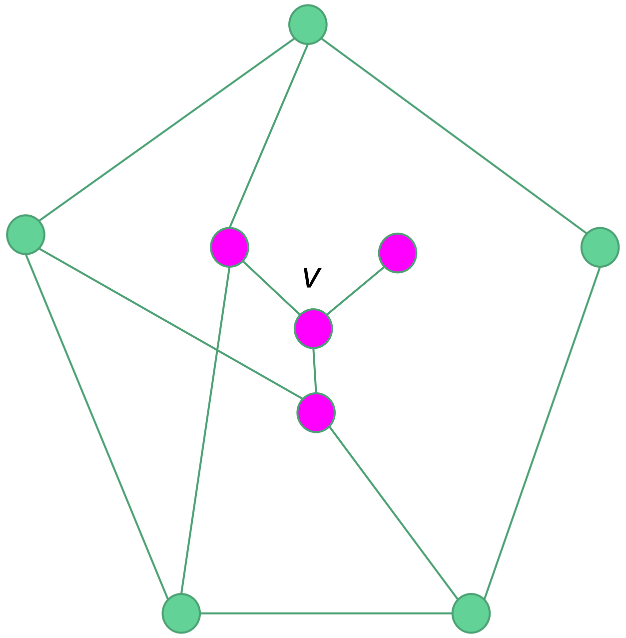

R(3,4) ≤ 9

Suppose there exists a (3,4) graph G of size 9

- G must be regular degree 3

- removing a neighborhood from G results in a (3,3) graph of size 5

- G cannot be constructed from a (3,3) graph of size 5

So, there is no (3,4) graph of size 9

Therefore, R(3,4) ≤ 9

$R(3,10) \leq 42$

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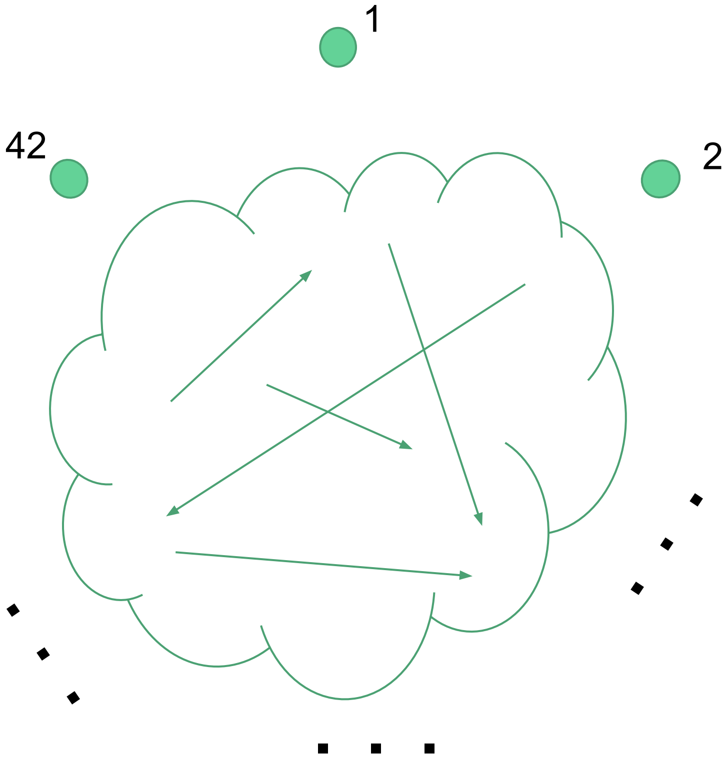
Suppose there exists a (3, 10) graph G of size 42



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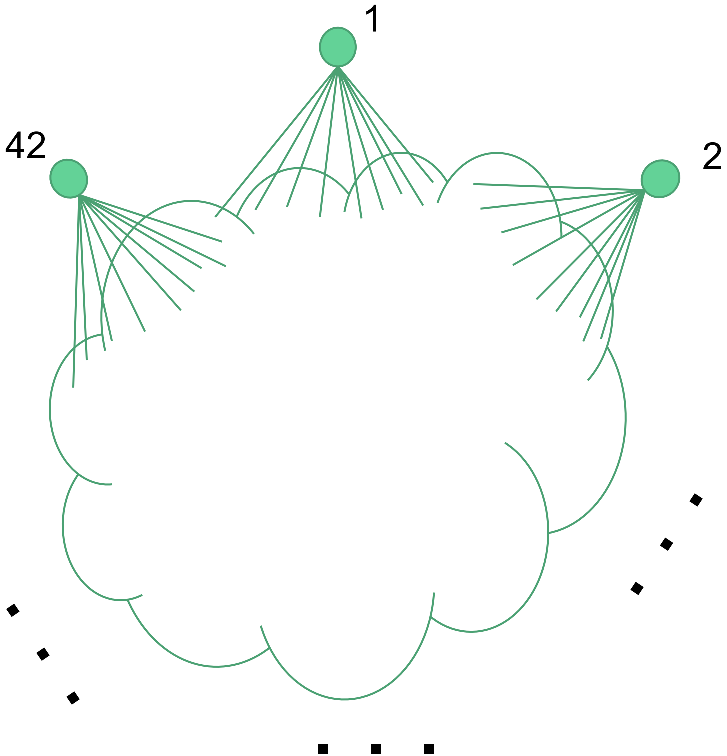

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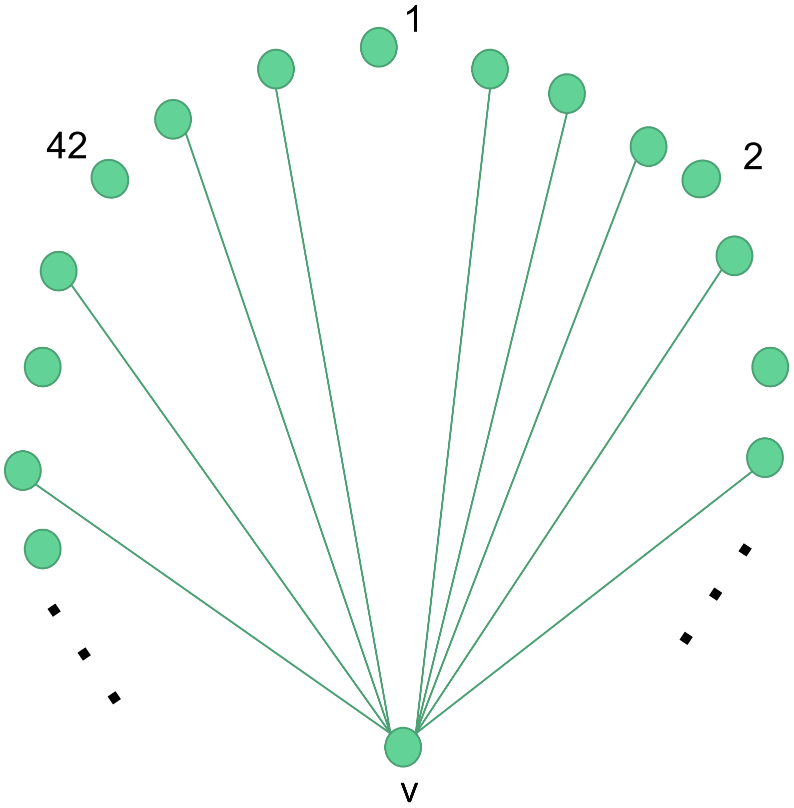
- Remove a vertex v and its neighborhood



$R(3,10) \leq 42$

Suppose there exists a (3, 10) graph G of size 42

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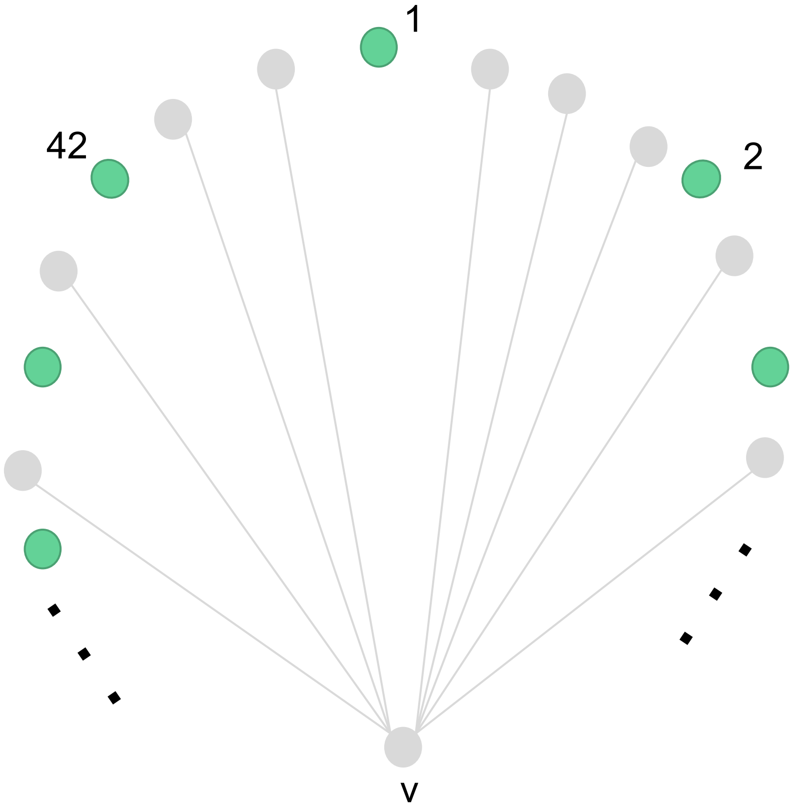
- Remove a vertex v and its neighborhood

- Results in a (3, 9) graph H of size 32



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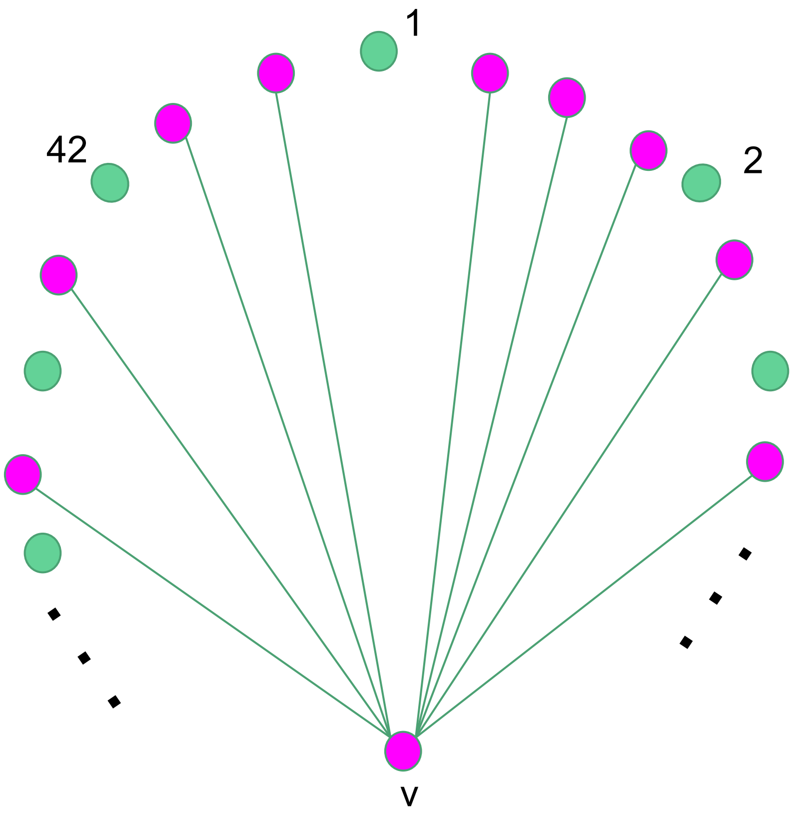
- G cannot be reconstructed from any such H



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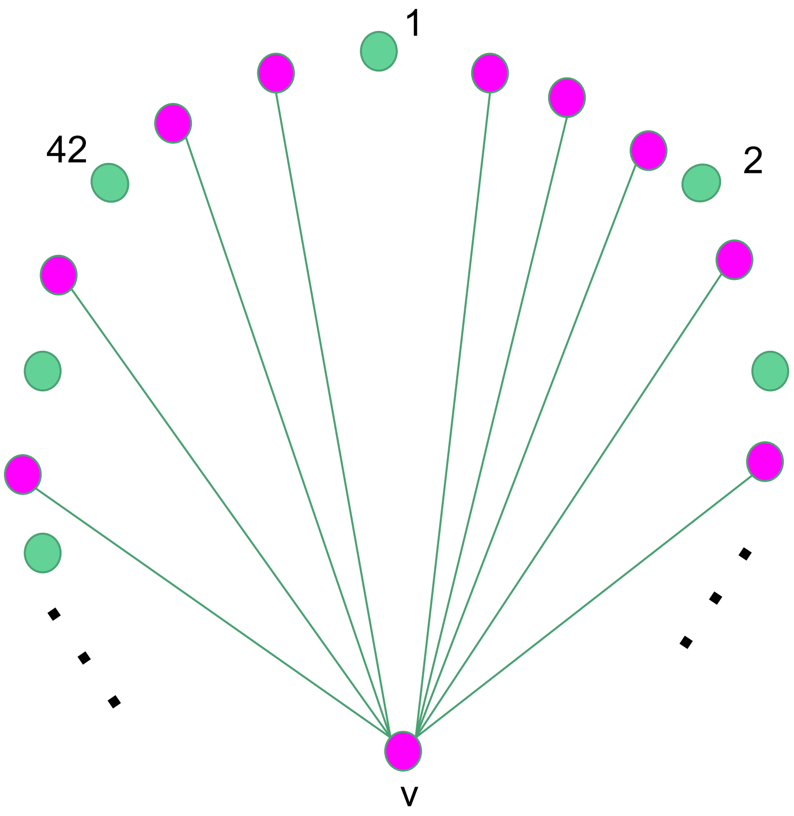

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These steps are computationally hard

Maximal triangle free (mtf) method

mtf graph = a graph in which the addition of any edge yields a 3-clique

There is some (x,y) graph of size n iff there is some mtf (x,y) graph of size n

Maximal triangle free (mtf) method

mtf graph = a graph in which the addition of any edge yields a 3-clique

There is some (x,y) graph of size n iff there is some mtf (x,y) graph of size n

There are often far fewer mtf (3, k) graphs of size n
than there are (3, k) graphs of size n

mtf method Find (3,4) graphs of size 6

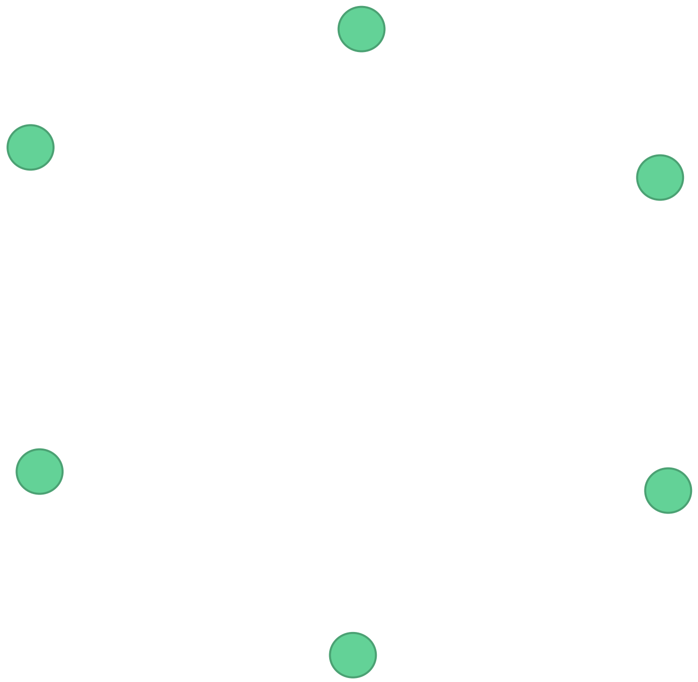
Build up all mtf graphs of size 6

If an mtf graph is a (3,4) graph

Recursively remove edges and check for new 4-independent sets

mtf method

Find (3,4) graphs of size 6



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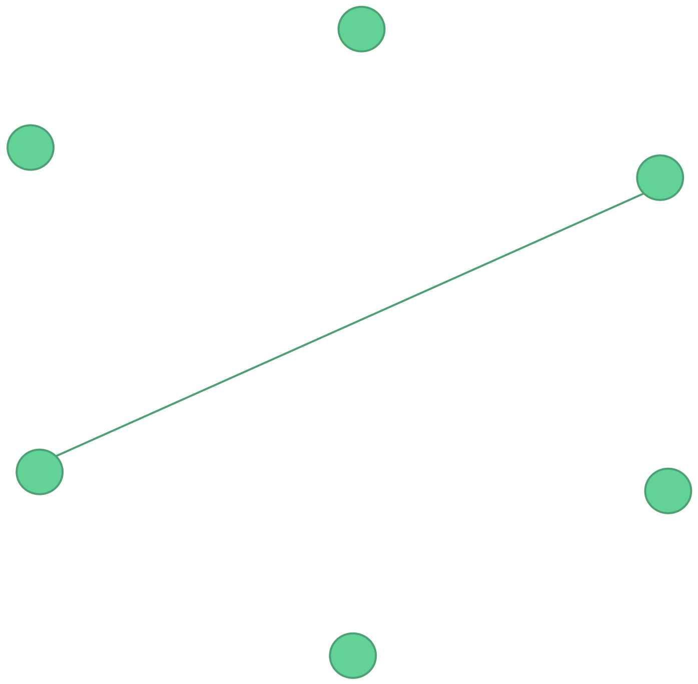

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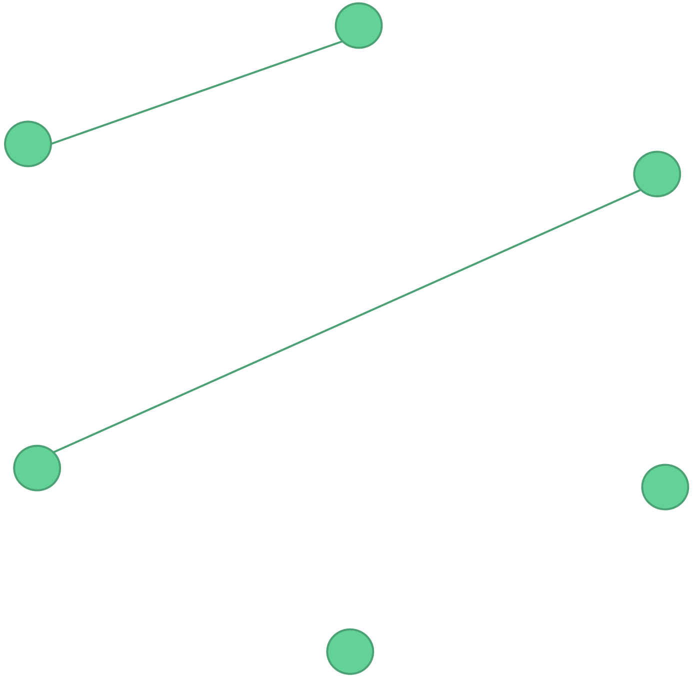

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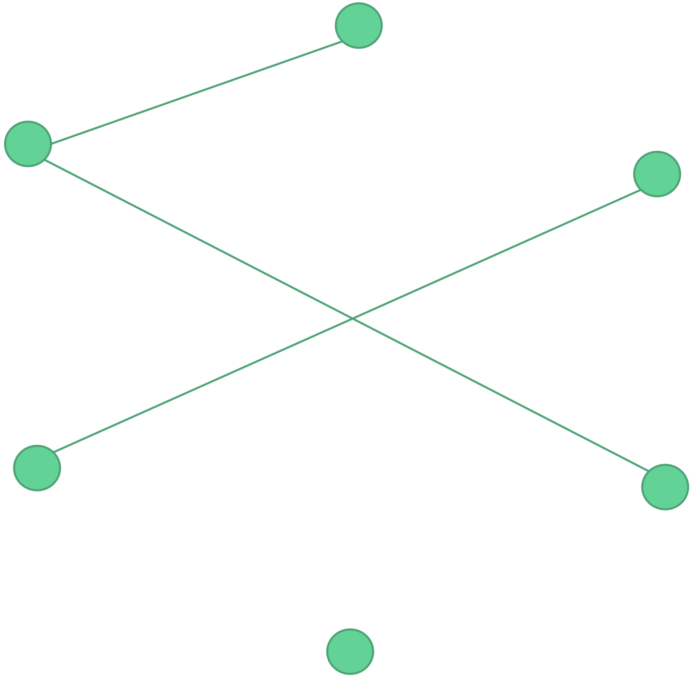

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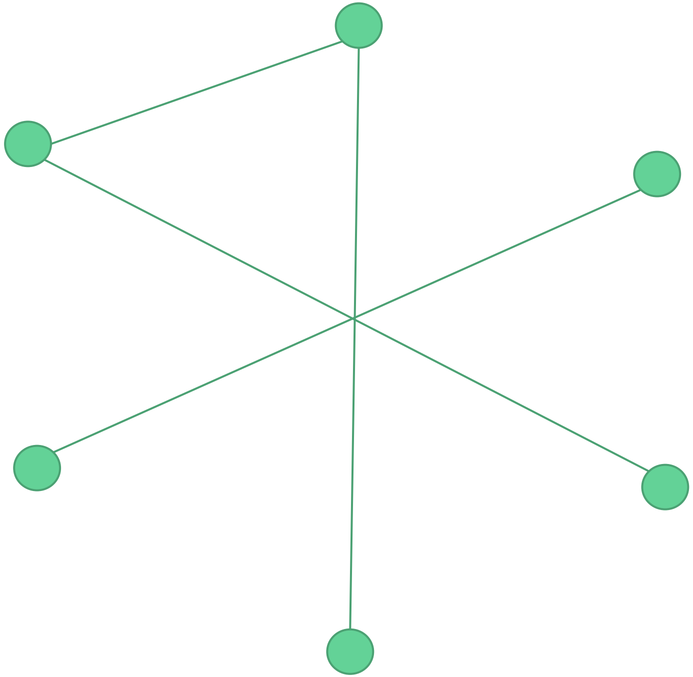

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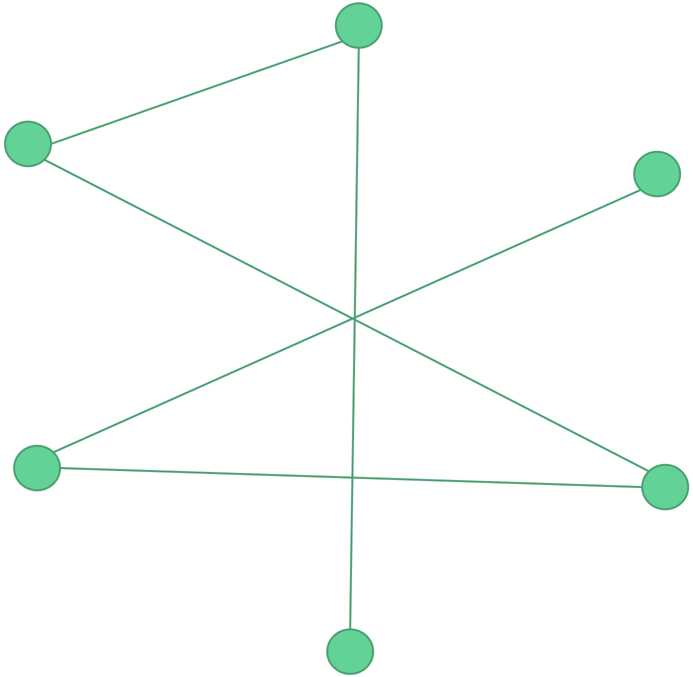

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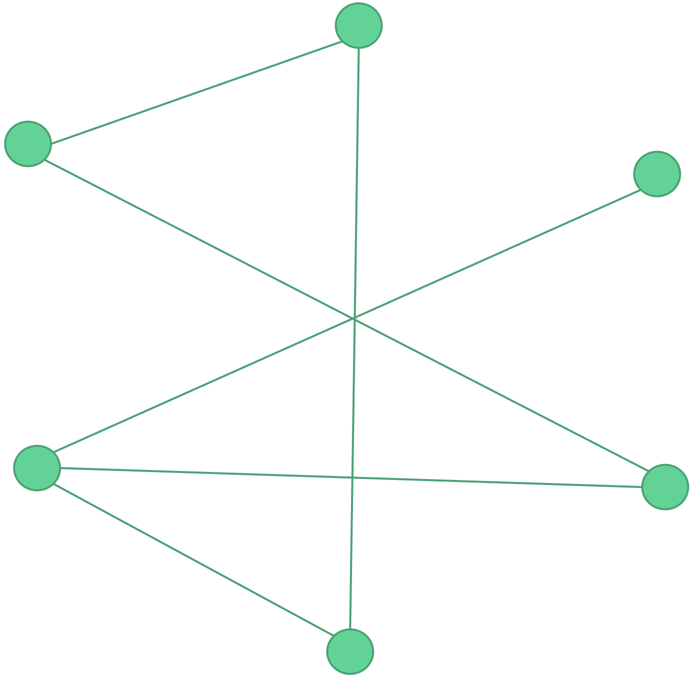
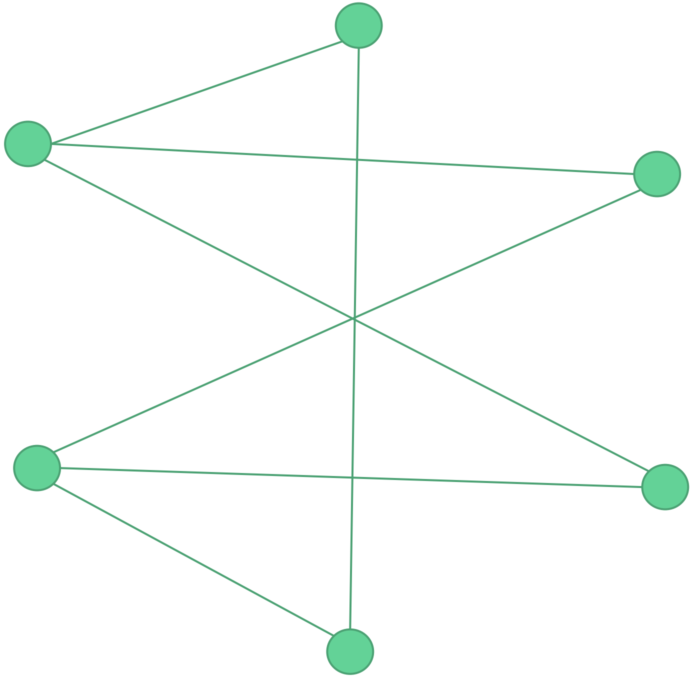


mtf method Find (3,4) graphs of size 6

This is an mtf graph



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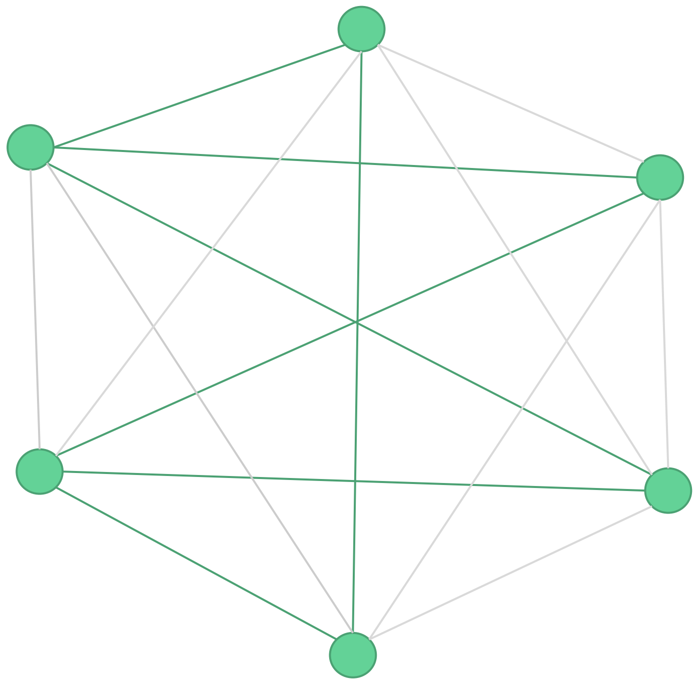

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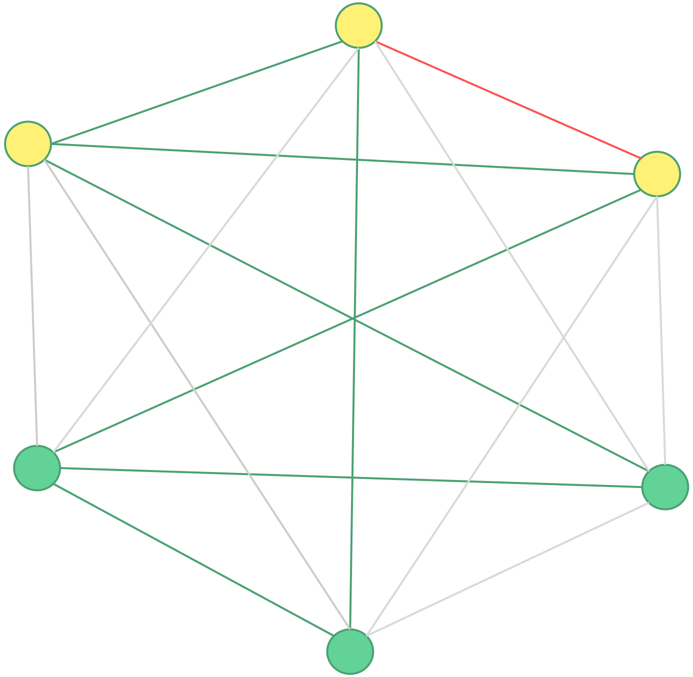

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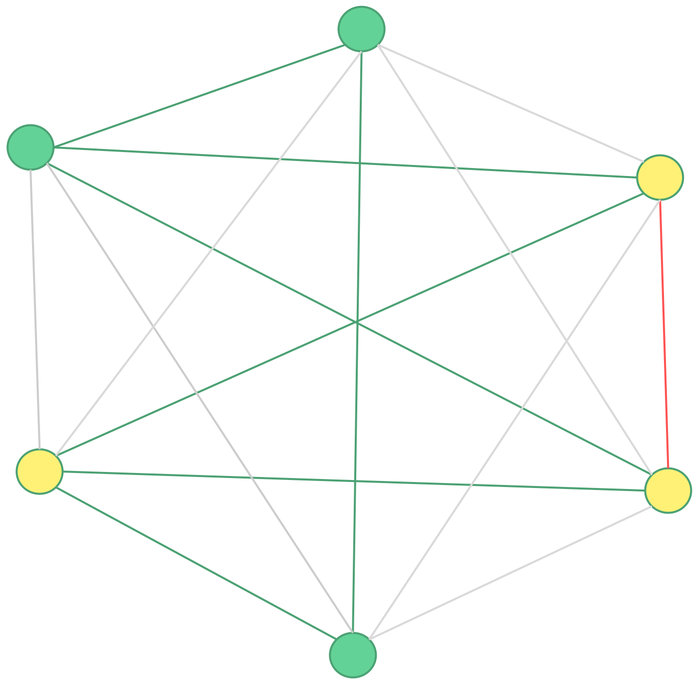

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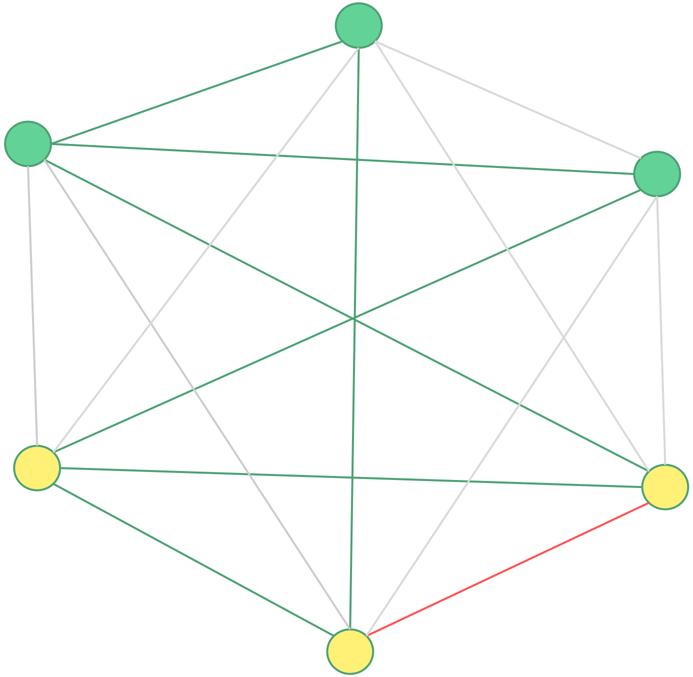

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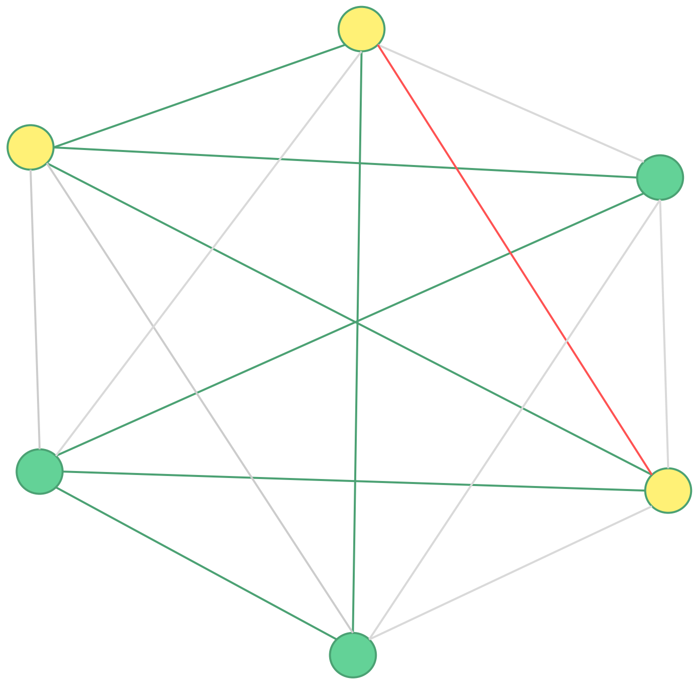

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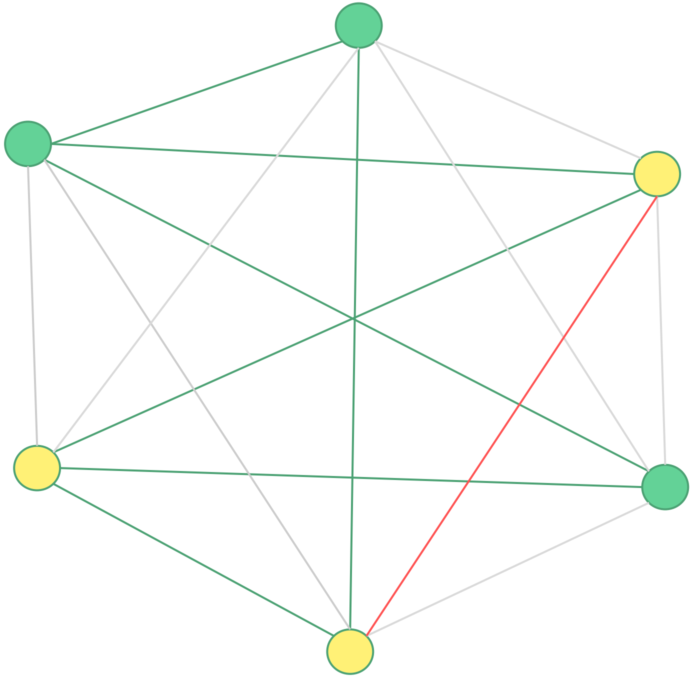

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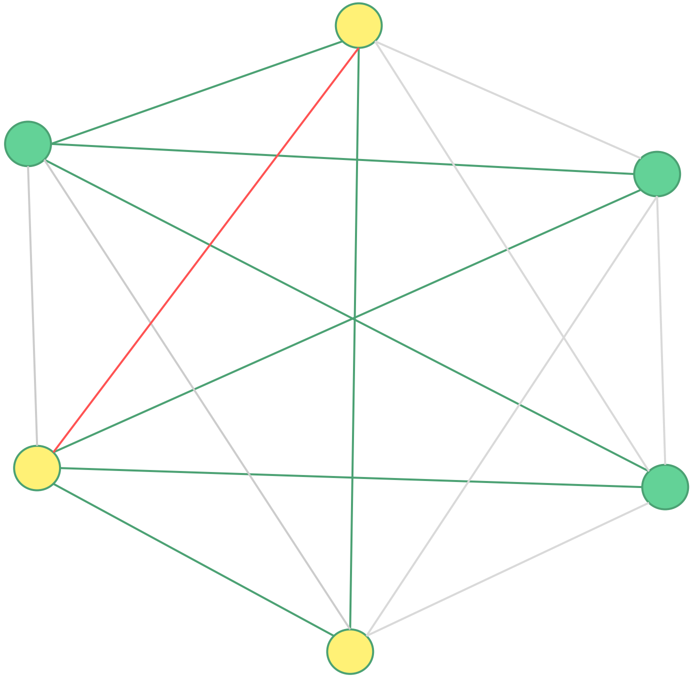

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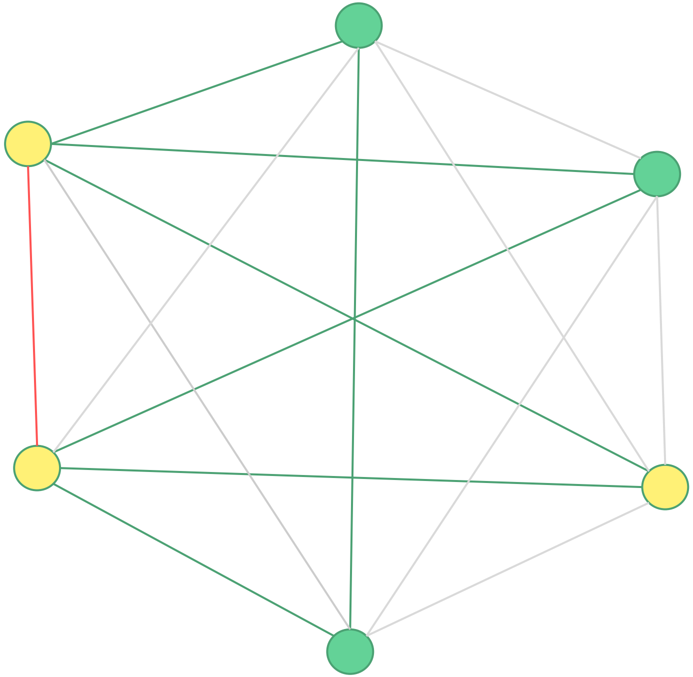

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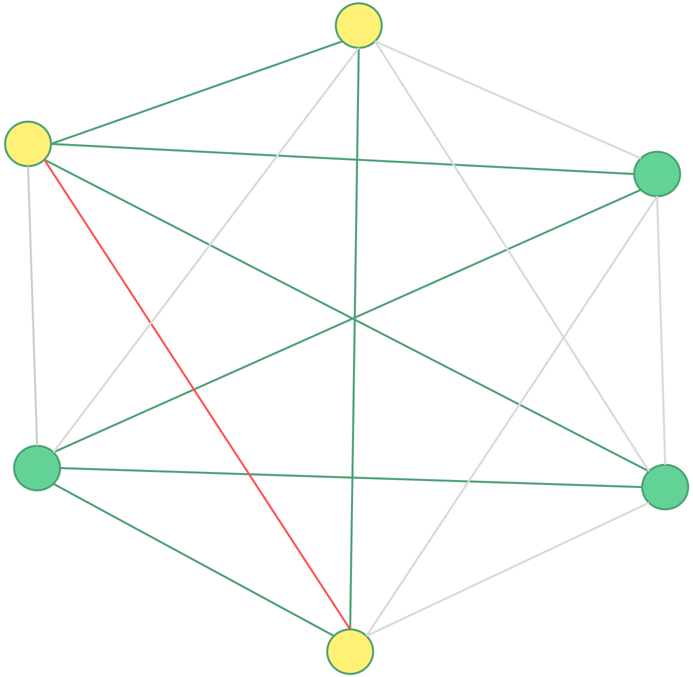

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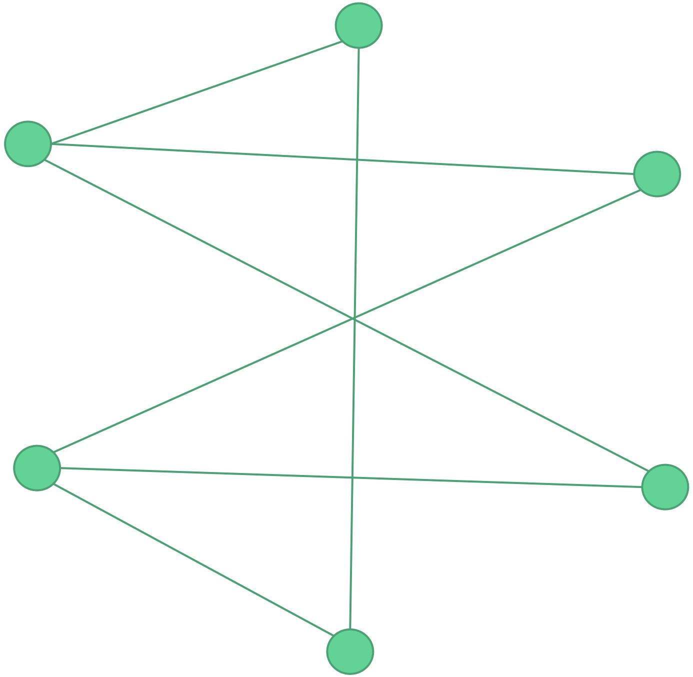

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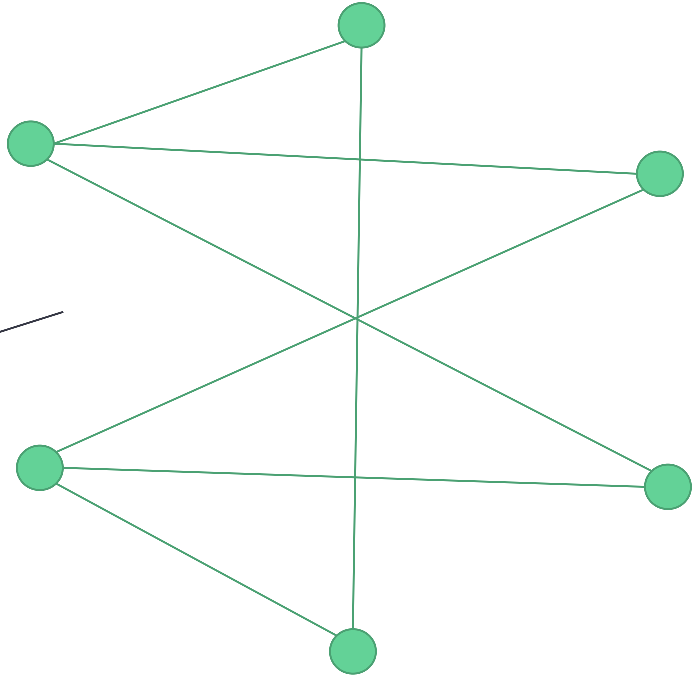
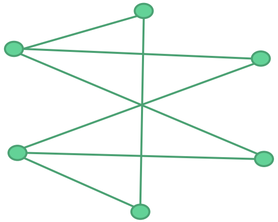
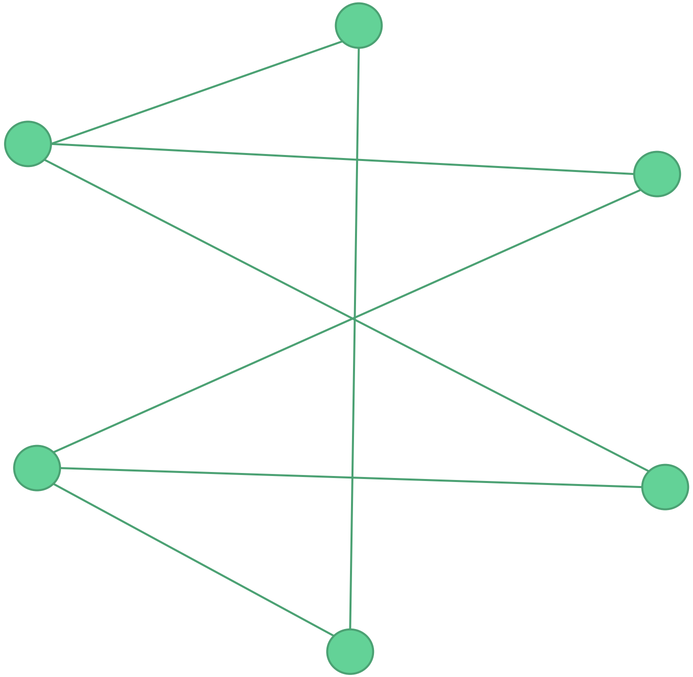
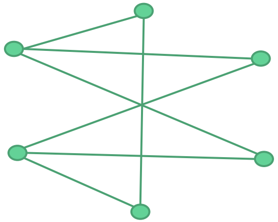




mtf method   Find (3,4) graphs of size 6

Recursively remove edges to try to find more (3,4) graphs





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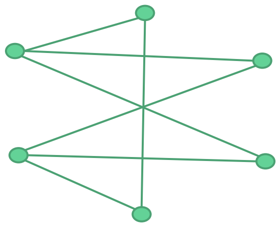

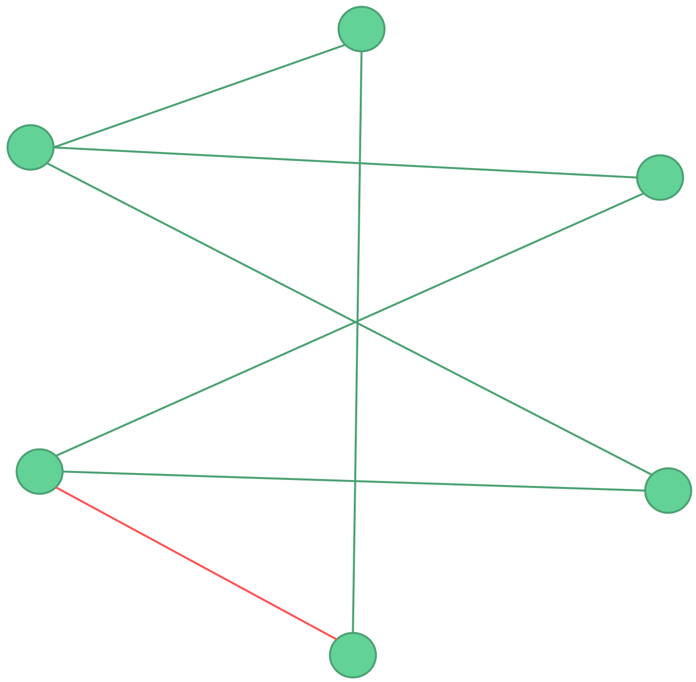

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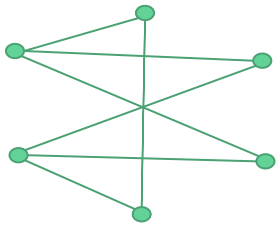

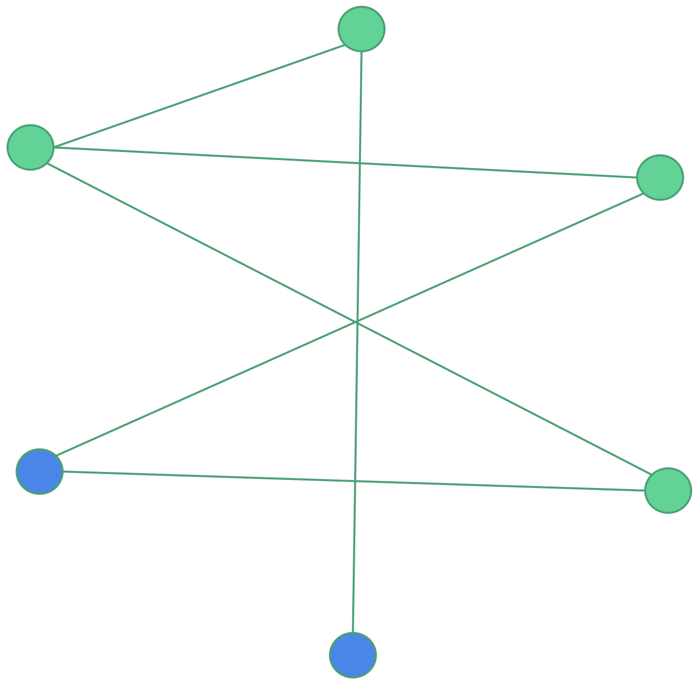

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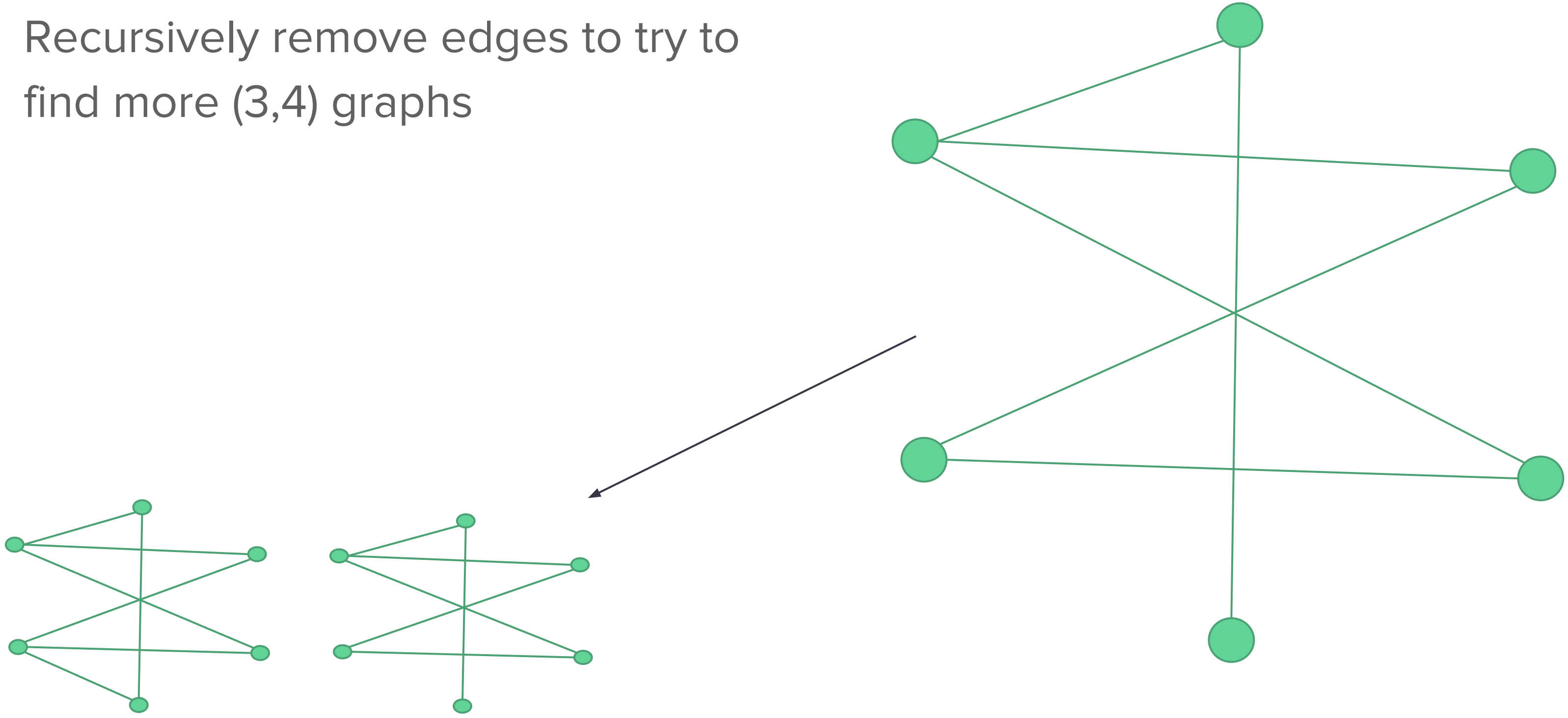

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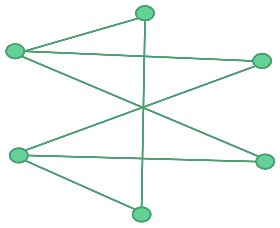

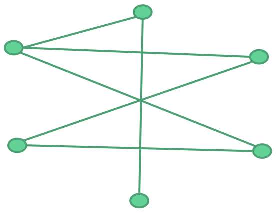

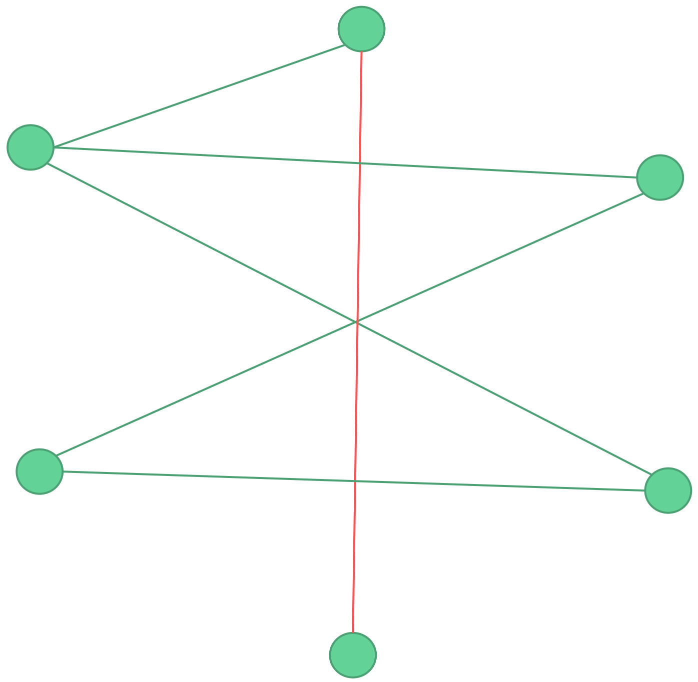

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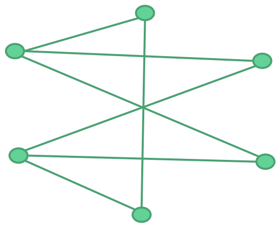

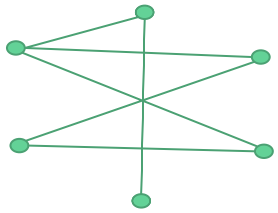

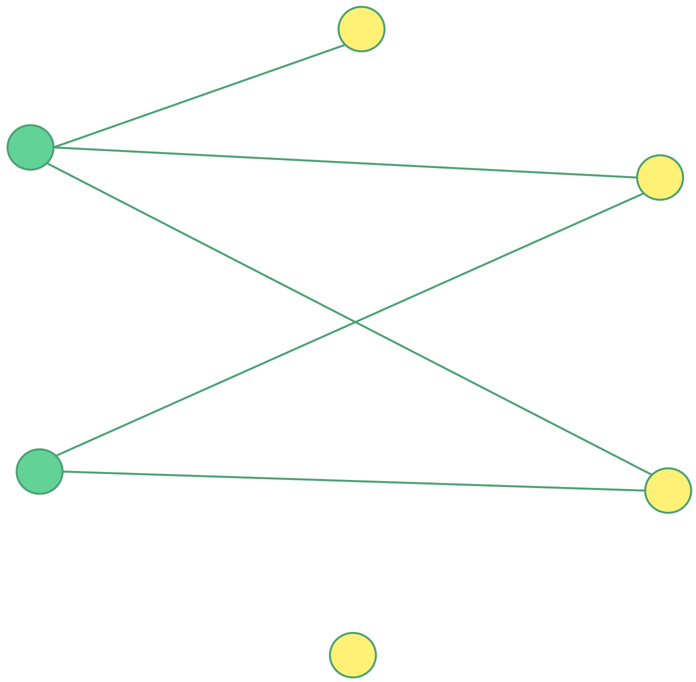

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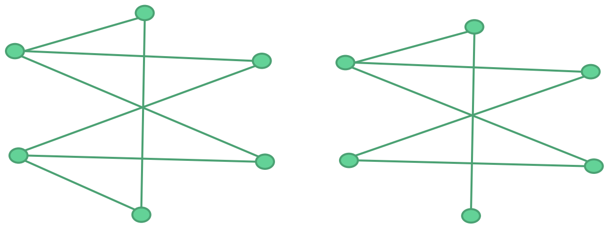

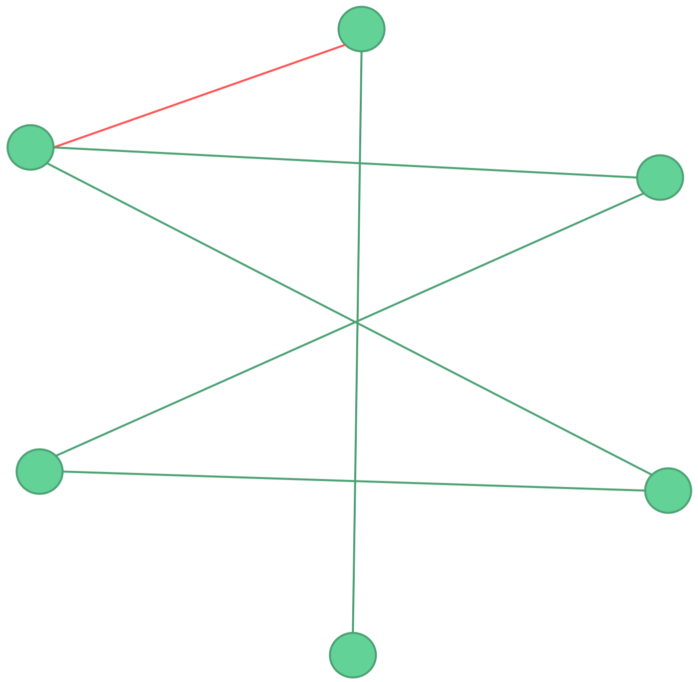

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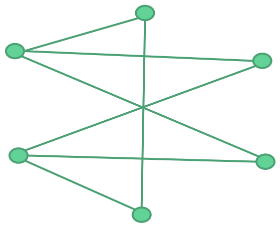

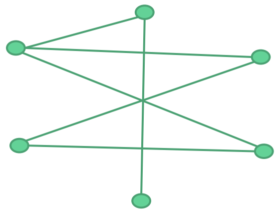

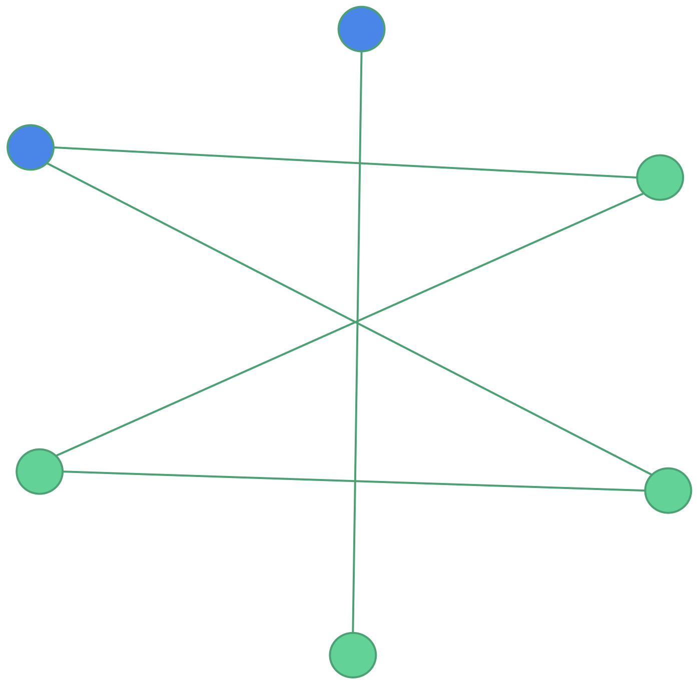

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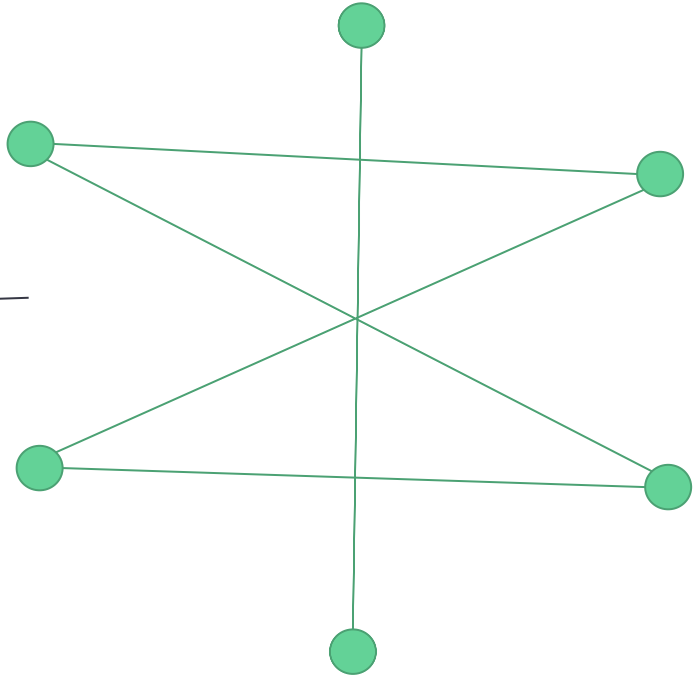

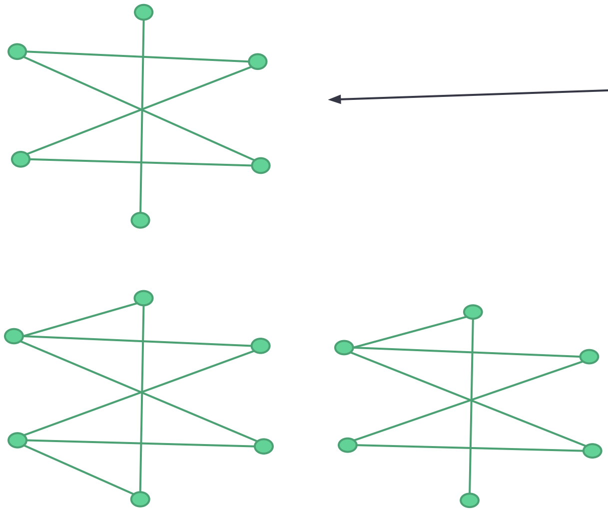

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Build up all mtf graphs of size 6

If an mtf graph is a (3,4) graph

Recursively remove edges and check for new 4-independent sets

Fun Facts

Naive upper bound gets pretty close -> R (2, 10) + R (3, 9) = 10 + 36 = 46

Lower bound for R (3, 10) is 40, found in 1987

Since 1987, found 4 x 10^7 different r(3, 10, 39) graphs

Tight general lower bound found in 1995 (t^2/log t)

Methods for finding lower bound efficiently:

- Artificial bee colony algorithm
- Simulated annealing

New Computational Upper Bounds for Ramsey Numbers R(3,k)

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