Simulating Traffic Is Our Jam

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Abstract

We explore the behavior of traffic flow in various situations using the framework of PDEs. We aim to more deeply understand the traffic equations developed in Volume 4, as well as to develop a tangible model for traffic at an intersection. We first look at the behavior of a one-dimensional traffic flow when restricted by a traffic light. We use the classic traffic flow equation with a maximum density term, then proceed by the method of characteristics to analyze and quantify behavior. In the second half of our report, we develop a multi-dimensional SIR model to examine a four-way traffic light for an intersection. In each case we find results that align with intuitive understanding of traffic, and with our real-life observations. We conclude that these models, while not completely accurate or representative, are useful in modeling the phenomena in question.

1 Background/Motivation

There have been numerous papers[1] written modeling traffic flow with cellular automata. Essentially, this means that instead of modeling the flow as a fluid, one defines the behavior of individual "cells" (cars in this case), and runs simulations to determine the behavior of the entire system. There are notably fewer papers using differential equations to simulate traffic flow, especially in traffic four-ways. Because of this lopsidedness in research, we decided to model traffic flow using PDEs. We analyze a specific version of the traffic equation given in volume 4, and develop and analyze it further than the textbook did.
We later develop an SIR model to simulate a four-way stop. This choice is motivated largely by the versatility and simplicity of the SIR model, and we find some interesting results. In anticipation of further development of controls next semester, we purposefully included several alterable variables that can help the model simulate various situations.

2 Modeling Traffic Flow – PDE Approach

To begin our first foray into modelling traffic flow, we will pull from Volume 4 and work with the given equation for traffic flow $u_t + V_\infty \left(1 - \frac{2u}{u_\infty}\right) u_x = 0$. We want to consider three cases:

1. What happens to the left of a light after a red light turns green?
2. What happens before a light when a light turns from green to red?
3. What happens both before and after a light when it turns from red to green?

2.1 Method of Characteristics

To begin, we will try to find an explicit solution using the method of characteristics.

Suppose the solution u is some curve parameterized by s as $u(x(s), t(s))$. Taking the derivative of u with respect to s gives

$$u'(x(s), t(s)) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s} = u_x\frac{\partial x}{\partial s} + u_t\frac{\partial t}{\partial s}$$

and matching terms with the original differential equation gives us that

$$\frac{\partial x}{\partial s} = V_\infty\left(1 - \frac{2}{u_\infty}u\right)$$
$$\frac{\partial t}{\partial s} = 1$$
$$\frac{\partial u}{\partial s} = 0.$$

Using the fact that $\frac{\partial t}{\partial s} = 1$, we have that

$$\begin{aligned}\frac{\partial x}{\partial t} &= V_\infty(1 - \frac{2}{u_\infty}) \\ x(t) - x(0) &= \int_0^t V_\infty(1 - \frac{2}{u_\infty}u)dt \\ x &= V_\infty(1 - \frac{2}{u_\infty}u)t + x(0) \\ t &= \frac{1}{V_\infty(1 - \frac{2}{u_\infty}u)}x - x(0)\end{aligned}$$

where to evaluate the integral, we used the fact that $\frac{\partial u}{\partial s}$ is 0, and so the characteristics are constant along the plane. This yields a few cases for the characterstic curves depending on what value of u the curve takes. Based on the physical properties of our model, we have that $u \in [0, u_\infty]$. The slope of the characteristics changed depending on the value of u, specifically the slope $m \Rightarrow 0$ when $u \in [0, \frac{u_\infty}{2})$, the slope is vertical or undefined when $u = \frac{u_\infty}{2}$, and $m < 0$ when $u \in (\frac{u_\infty}{2}, u_\infty]$.

Next, suppose that there are some values x_0, x_1 from which we draw characteristic curves, and suppose $u(x_0) < u(x_1)$. This will result in shocks, (assuming the shocks happen inside the domain of relevance). See Figures 1, 2, and 3 for examples.

Conversely, if the density of the initial condition is decreasing (i.e. when $x_0 < x_1 \implies u(x_0) < u(x_1)$), there are no shocks. See Figures 4, 5, and 6 for examples.

As a result of the pattern of characteristics, there are a lot of constraints as to the auxiliary conditions that we can feed our model given that we want well-defined solutions. As demonstrated above, no initial condition that is increasing will be well defined on any unbounded domain. Furthermore, note that if $u(x_0) \in (0, \frac{u_\infty}{2})$, the slope of the characteristic will be positive, and in the case that $u(x_0) \in (\frac{u_\infty}{2}, u_\infty)$ the slope will be negative. As a result, any left boundary that contains values in $(\frac{u_\infty}{2}, u_\infty)$ will overdetermine the initial condition, and likewise on the right boundary if there are values in $(0, \frac{u_\infty}{2})$ the initial condition will also be overdetermined, and so we will not have a well-defined solution.

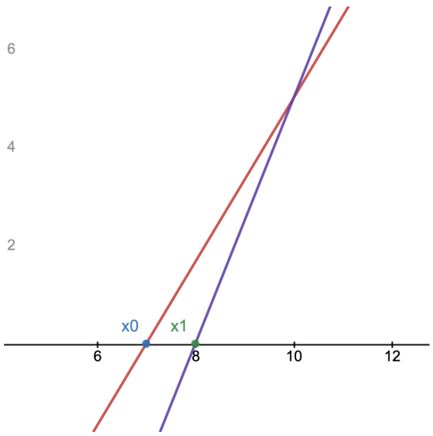


Figure 1: $u(x_0) < u(x_1), u(x_0), u(x_1) \in (0, \frac{u_\infty}{2})$

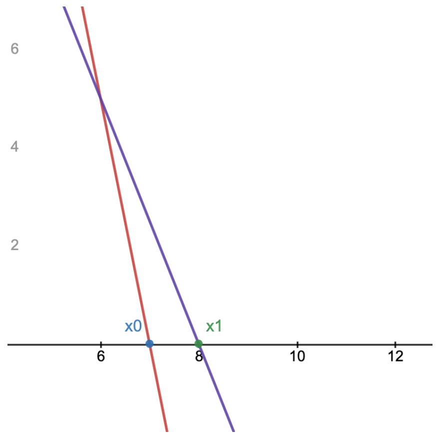


Figure 2: $u(x_0) < u(x_1), u(x_0), u(x_1) \in (\frac{u_\infty}{2}, u_\infty)$

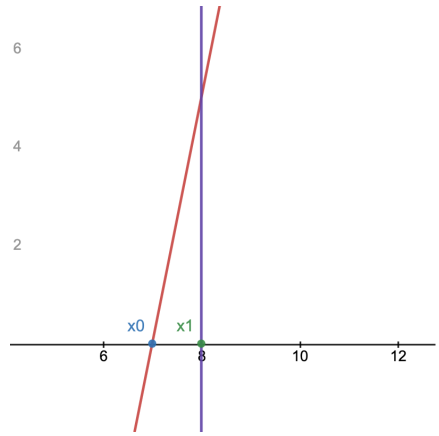


Figure 3: $u(x_0) < u(x_1), u(x_0) \in (\frac{u_\infty}{2}, u_\infty), u(x_1) = \frac{u_\infty}{2}$

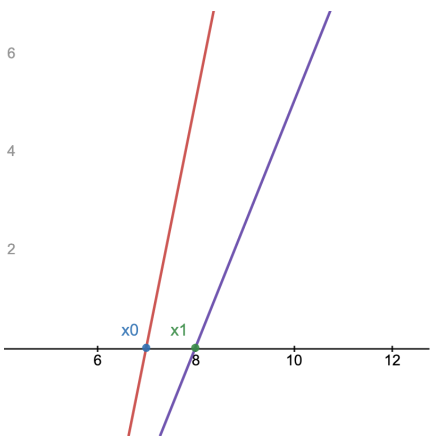


Figure 4: $u(x_0) > u(x_1), u(x_0), u(x_1) \in (0, \frac{u_\infty}{2})$

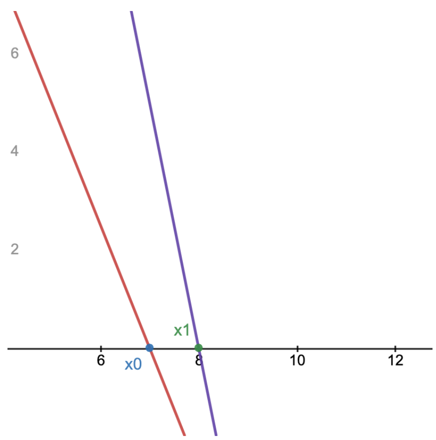


Figure 5: $u(x_0) > u(x_1), u(x_0), u(x_1) \in (\frac{u_\infty}{2}, u_\infty)$

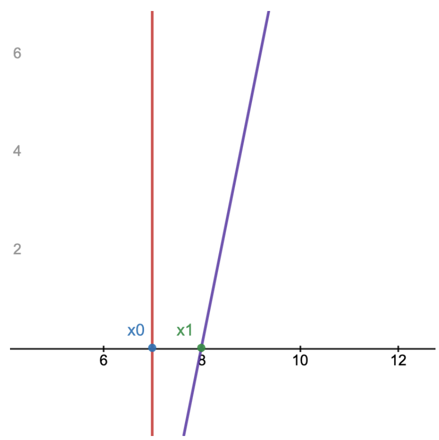


Figure 6: $u(x_0) < u(x_1), u(x_0) = \frac{u_\infty}{2}, u(x_0) \in (\frac{u_\infty}{2}, u_\infty)$

2.2 Introduction of Diffusion Term

The introduction of the diffusion term then helps us work around some of the problems the characteristics run into. In addition, we were able to use boundary conditions that had discontinuities. If we had these with only an advection term, these discontinuities would simply be carried through the solution. However, with diffusion, these jumps tend to smooth out over time.

2.3 Finite Difference Scheme

We create a finite difference scheme of the equation by taking a forward time difference and a centered approximation of the spatial derivative. This gives us

$$\begin{aligned}
u(x,t+\Delta t) &= u(x,t) + \Delta t[f(u(x,t))] \\
&= u(x,t) + V_\infty \Delta t \left[\epsilon u_{xx} - u_x\left(1 - \frac{2u(x,t)}{u_\infty}\right)\right] \\
&= u(x,t) + \left[\frac{V_\infty \Delta t \epsilon}{(\Delta x)^2}(u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t))\right] \\
&\qquad - \left[\frac{V_\infty \Delta t}{2\Delta x}\left(1 - \frac{2u(x,t)}{u_\infty}\right)(u(x+\Delta x,t) - u(x-\Delta x,t)\right]
\end{aligned}$$

3 Results of PDE Approach

For our first case we applied our finite difference scheme to the equation with auxilliary conditions

$$u(x,0) = f(x) = u_\infty e^{-10x}, u(0,t) = u_\infty e^{-2t}, u(2,t) = 0$$

Note that our right homogenous dirichlet boundary condition is at $x = 2$ however our plot will finish at $x = 1$. This is because we didn't actually want a right boundary condition based on the physical interpretation of the situation (we were forced to include a second boundary condition as a result of the second order diffusive term). In an attempt to discount the boundary condition, we only considered the domain $[0, 1]$ such that the boundary condition affects our solution minimally within a short time frame.

The results of our finite difference scheme are in Figure 7. As we can see, the initial condition evolves rapidly to the right with the car density settling around $u_\infty/2$. Furthermore you can see somewhat of a wave front that has formed at $t = .1$. Finally, considerable instability can be noted along the left boundary (with some of the instability advecting toward the right). One possible reason for this instability is the Forward Euler finite difference approximation applied to time.

In addition to this case, we also desired to examine one other case examining how cars pile up behind a light once it turns red. This problem has auxilliary conditions

$$u(x,0) = \frac{u_\infty}{2}, u(0,t) = \frac{u_\infty}{2}, u_x(1,t) = 0.$$

However, upon implementing the right homogenous boundary condition we found that the condition $u_x(1,t) = 0$ actually just deletes the advection term, making to so the curve doesn't "pile up" on the right hand side, rather the curve settles to whatever value is coming in on the right hand side. This situation matches the intuition gained in studying the diffusion term in the context of the heat equation, this behavior makes sense, but the model does not match our expected physical interpration of cars piling up at a stop light.

4 Modeling Traffic Flow – SIR Approach

To better approximate a more complicated system with multiple interacting streams, we employed an SIR styled model.

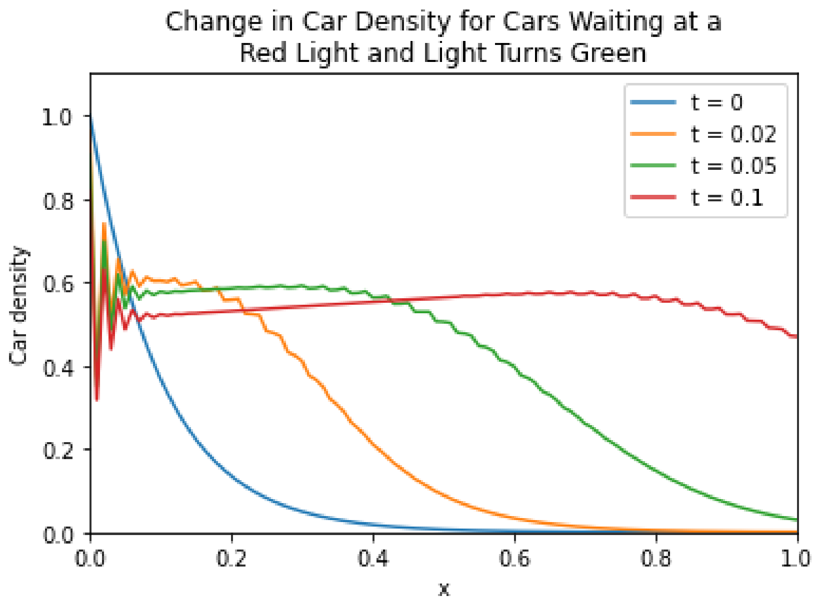


Figure 7

4.1 Structure

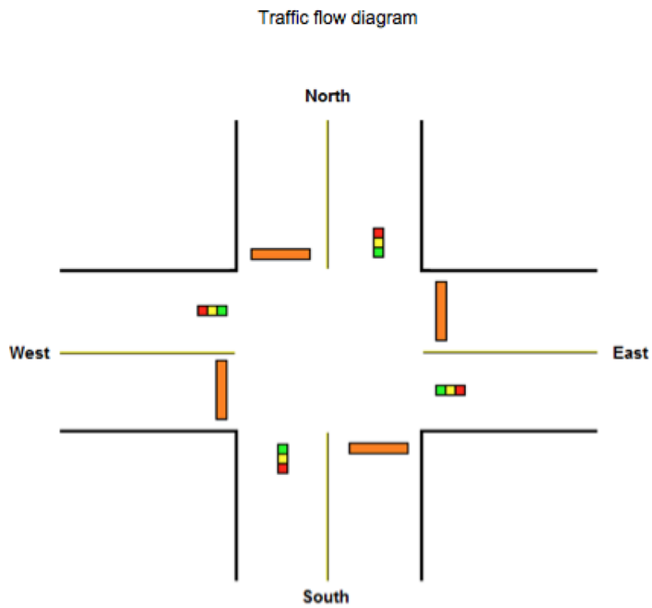
While there are many different traffic light intersections and variants, this model assumes a standard 4 way intersection with traffic coming from each of the cardinal directions (north, east, south, and west). Each direction has 3 streams of traffic, a turning left stream, a straight stream, and a right stream.

The change in the straight stream is only dependent on whether the light is red or green. The left stream is "unprotected", meaning that the change in the left stream depends on whether the light is green and how much traffic is coming from the opposite straight stream. The right stream is free to turn at any time, and is only dependent on how much traffic is coming from the perpendicular straight stream.

4.2 Implementation

First, we define a light function. We want this function to be periodic and to move between 0 (red) and 1 (green). Rather than having one function that returns red for one direction and green for the other direction, we created two correlated functions that define the state of the light in the north-south direction and the east-west direction:

$$\text{lightNS(t)} = \begin{cases} \sin(3t) & \frac{t}{2\pi} \mod 2\pi \in [0, \frac{\pi}{6}) \\ 1 & \frac{t}{2\pi} \mod 2\pi \in [\frac{\pi}{6}, \frac{5\pi}{6}) \\ \sin(3t) & \frac{t}{2\pi} \mod 2\pi \in [\frac{5\pi}{6}, \pi) \\ 0 & \text{otherwise} \end{cases}$$



$$\text{lightEW(t)} = \begin{cases} -\sin(3t) & \frac{t}{2\pi} \mod 2\pi \in [\pi, \frac{7\pi}{6}) \\ 1 & \frac{t}{2\pi} \mod 2\pi \in [\frac{7\pi}{6}, 11\frac{5\pi}{6}) \\ -\sin(3t) & \frac{t}{2\pi} \mod 2\pi \in [\frac{11\pi}{6}, 2\pi) \\ 0 & \text{otherwise} \end{cases}$$

Next, we define 12 different "bins", one for each stream of our model (eg. North Forward), and then define the derivative of each bin:

$$
\begin{aligned}
N'_F(t) &= -1 * \text{LightNS}(t) \\
E'_F(t) &= -1 * \text{LightEW}(t) \\
S'_F(t) &= -1 * \text{LightNS}(t) \\
W'_F(t) &= -1 * \text{LightEW}(t)
\end{aligned}
$$

$$
\begin{aligned}
N'_L(t) &= -1 * \text{LightNS}(t) * (1 + S'_F(t)) \\
E'_L(t) &= -1 * \text{LightEW}(t) * (1 + W'_F(t)) \\
S'_L(t) &= -1 * \text{LightNS}(t) * (1 + N'_F(t)) \\
W'_L(t) &= -1 * \text{LightEW}(t) * (1 + E'_F(t))
\end{aligned}
$$

$$
\begin{aligned}
N'_R(t) &= -1 * W'_F(t) \\
E'_R(t) &= -1 * S'_F(t) \\
S'_R(t) &= -1 * E'_F(t) \\
W'_R(t) &= -1 * N'_F(t)
\end{aligned}
$$

The definitions above assume constant base changes between all streams, that no new cars are entering the system, and that left turns have zero change whenever the cars coming straight are coming at full capacity. To account for these, we add some constants that allow for configuring the system:

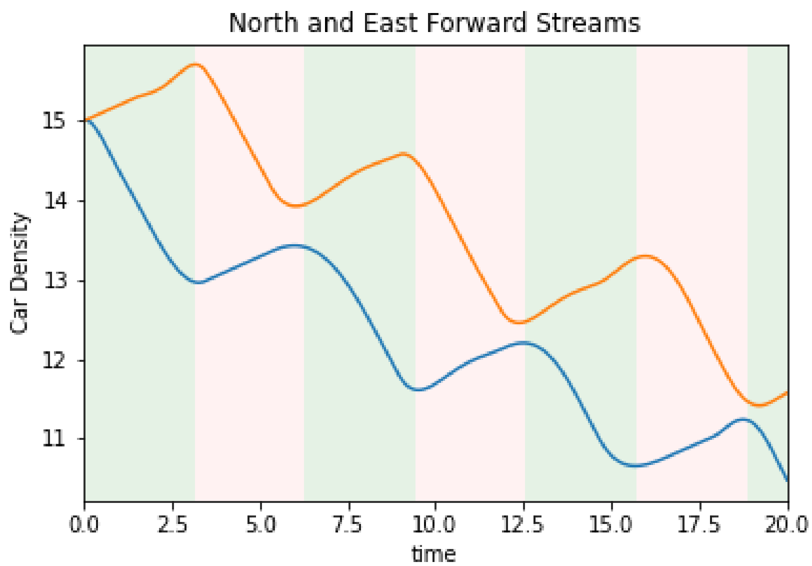
$$
\begin{aligned}
N'_F(t) &= -1 * c_f * \text{LightNS}(t) + add_f \\
E'_F(t) &= -1 * c_f * \text{LightEW}(t) + add_f \\
S'_F(t) &= -1 * c_f * \text{LightNS}(t) + add_f \\
W'_F(t) &= -1 * c_f * \text{LightEW}(t) + add_f
\end{aligned}
$$

$$
\begin{aligned}
N'_L(t) &= -1 * c_l * \text{LightNS}(t) * ((left\_turn\_gap\_c * c_f) + S'_F(t)) + add_l \\
E'_L(t) &= -1 * c_l * \text{LightEW}(t) * ((left\_turn\_gap\_c * c_f) + W'_F(t)) + add_l \\
S'_L(t) &= -1 * c_l * \text{LightNS}(t * ((left\_turn\_gap\_c * c_f) + N'_F(t)) + add_l \\
W'_L(t) &= -1 * c_l * \text{LightEW}(t) * ((left\_turn\_gap\_c * c_f) + E'_F(t)) + add_l
\end{aligned}
$$

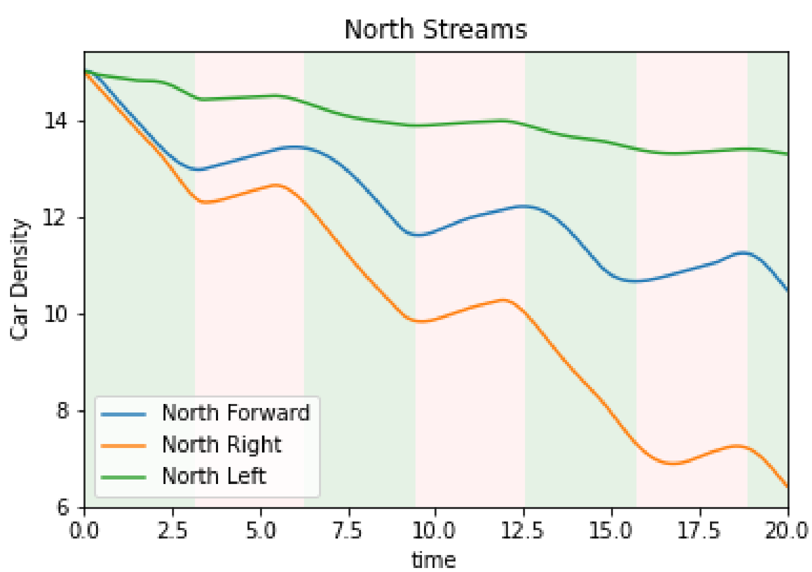
$$
\begin{aligned}
N'_R(t) &= -1 * c_r * W'_F(t) + add_r \\
E'_R(t) &= -1 * c_r * S'_F(t) + add_r \\
S'_R(t) &= -1 * c_r * E'_F(t) + add_r \\
W'_R(t) &= -1 * c_r * N'_F(t) + add_r
\end{aligned}
$$

5 Results of SIR Approach

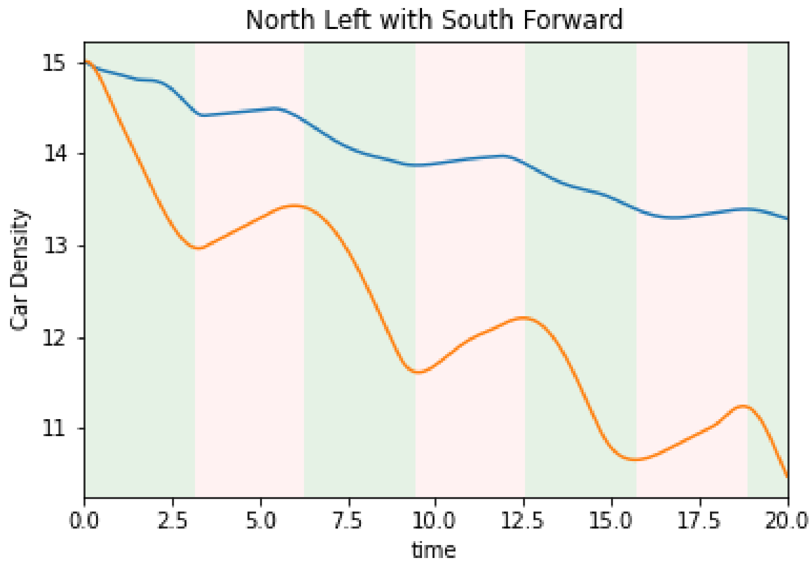
Solutions to this system were calculated numerically with scipy.integrate.solve_ivp.



Below we see the North and East forward streams, with the north-south light function shown with the alternating green and red zones. This model aligns with our expectations, with the number of cars waiting to go straight decreasing from the north when the light is green while the east stream accumulates more cars.



Next, we see the three different north streams. We see that while the light is green the forward and right streams change identically, however, the right stream continues to decrease during the transition from green to red. This again lines up with what we would expect from the rules of the system, as the right turn can continue to turn right up until cars start to flow forward from the east.



Next we examine how a left turn interacts with the forward blocking stream. We see that the left turn has the sharpest decrease right before the light turns red. This is an interesting attribute of the model that lines up with how left turns behave in day to day life. Often, people waiting to turn left will creep into the intersection, then turn left as the light turns red and the oncoming traffic stops.

6 Conclusions

Through our various endeavors, we consider our models successful at shedding light on both the limitations of the PDE model from the book and on the behavior of traffic. In expanding this project next semester, we can relax these assumptions and see what other patterns we can find. We can also exercise the flexibility built into the SIR model in order to determine the optimal behavior of the traffic lights. We determine that in moving forward, our models will become even more useful in helping us understand various traffic patterns.

References

[1] Ruskin H.J., Wang R. (2002) Modeling Traffic Flow at an Urban Unsignalized Intersection. In: Sloot P.M.A., Hoekstra A.G., Tan C.J.K., Dongarra J.J. (eds) Computational Science — ICCS 2002. ICCS 2002. Lecture Notes in Computer Science, vol 2329. Springer, Berlin, Heidelberg. https://doi.org/10.1007/3-540-46043-8_38