A

Mathematics of Finite Rotations

TABLE OF CONTENTS

§A.1. Introduction

This Appendix provides a compendium of formulas and results concerning the mathematical treatment of finite rotations in 3D space. Emphasis is placed on matrix representations of use in the corotational formulation of finite elements.

The material is taken from a survey paper: C. A. Felippa and B. Haugen, A unified formulation of small-strain corotational finite elements - I. Theory, *Comp. Meths. Appl. Mech. Engrg.*, **194**, 2285–2336, 2005.

§A.2. Plane vs. Spatial Rotations

Plane rotations make up an easy topic. A rotation in e.g., the xy plane, is defined by just a scalar: the rotation angle θ about z. Plane rotations commute: $\theta_1 + \theta_2 = \theta_2 + \theta_1$, because the θs are numbers.

The study of spatial rotations is more difficult. The subject is dominated by the fundamental theorem of Euler:

> *The general displacement of a rigid body with one point fixed is a rotation about some axis which passes through that point.*

Consequently 3D rotations have both magnitude: the angle of rotation, and direction: the axis of rotation. These are nominally the same two attributes that categorize vectors. Not coincidentally, rotations are often depicted as vectors but with a double arrow.

Finite spatial rotations, however, do not obey the laws of vector calculus, although infinitesimal rotations do. Most striking is failure of commutativity: switching two successive rotations does not yield the same answer unless the axis of rotation is kept fixed. Within the framework of matrix algebra, finite rotations can be represented as either 3×3 real orthogonal matrices $\mathbf{R}$ called *rotators* or as real 3×3 skew-symmetric matrices $\boldsymbol{\Omega}$ called *spinors*.

The spinor representation is important in physical modeling and theoretical derivations, because the matrix entries are closely related to the two foregoing attributes. The rotator representation is important in numerical computations, as well as being naturally related to polar factorizations of a transformation matrix. The two representations are related by various transformations illustrated in Figure A.1. Of these the Cayley transform (1857) is the oldest although not the most important one.

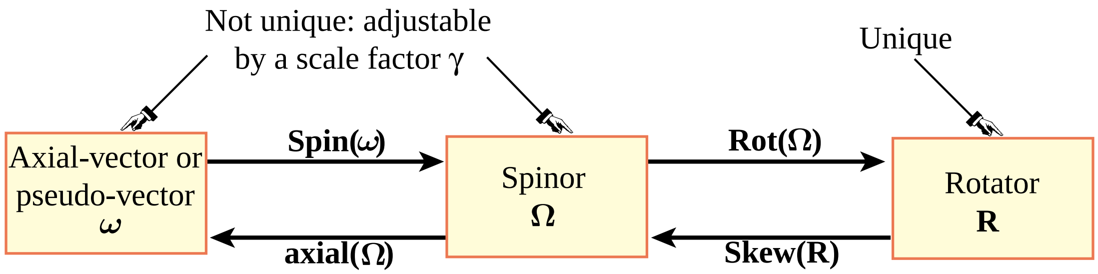

FIGURE A.1. Representations of finite space rotations and mapping operations.

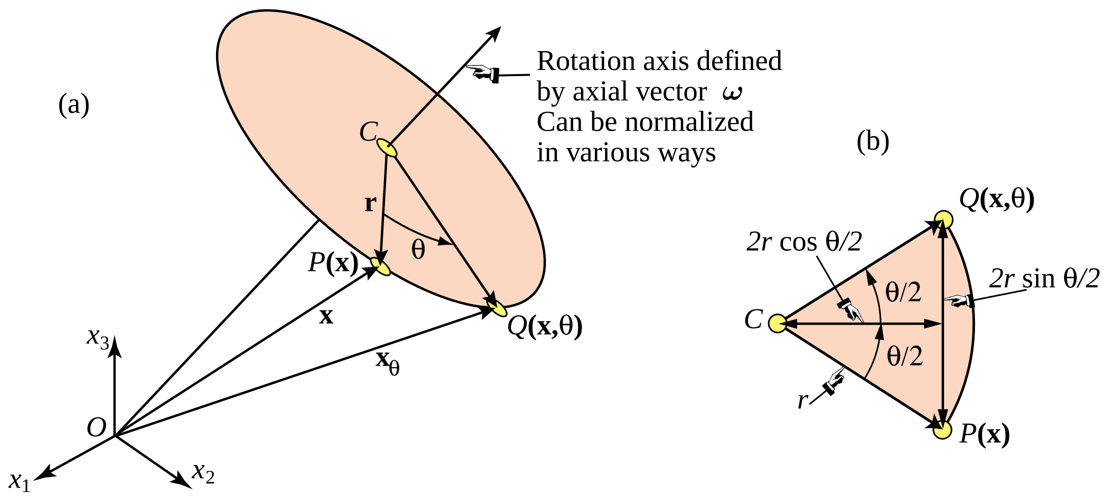

FIGURE A.2. Attributes of rotation in 3D.

Now a 3×3 skew-symmetric matrix is defined by three scalar parameters. These three numbers can be arranged as components of an *axial vector* $\boldsymbol{\omega}$. Although $\boldsymbol{\omega}$ looks like a vector, it does not obey certain properties of classical vectors such as the composition rule. Therefore the term *pseudo-vector* is sometimes used for $\boldsymbol{\omega}$.

This overview is intended to stress that finite 3D rotations can appear in various mathematical representations, as depicted in Figure A.1. The exposition that follows expands on this topic, and studies the connector links shown in Figure A.1.

§A.3. Spinors

Figure A.2(a) depicts a 3D rotation in space (x_1, x_2, x_3) by an angle θ about an axis of rotation $\vec{\boldsymbol{\omega}}$. For convenience the origin of coordinates O is placed on $\vec{\boldsymbol{\omega}}$. The rotation axis is defined by three directors: $\omega_1, \omega_2, \omega_3$, at least one of which must be nonzero. These numbers may be scaled by an nonzero factor γ through which the vector may be normalized in various ways as discussed later. The positive sense of θ obeys the RHS screw rule.

The rotation takes an arbitrary point $P(\mathbf{x})$, located by its position vector $\mathbf{x}$, into $Q(\mathbf{x}, \theta)$, located by its position vector $\mathbf{x}_\theta$. The center of rotation C is defined by projecting P on the rotation axis. The plane of rotation CPQ is normal to that axis at C. The radius of rotation is vector $\mathbf{r}$ of magnitude r from C to P. As illustrated in Figure A.2(b), the distance between P and Q is $2r \sin \frac{1}{2}\theta$.

§A.3.1. Spin Matrix and Axial Vector

Given the three directors ω_1, ω_2, and ω_3 of the axis $\boldsymbol{\omega}$, we can associate with it a 3×3 skew-symmetric matrix $\boldsymbol{\Omega}$, called a *spin tensor* or *spin matrix*, by the rule

$$\boldsymbol{\Omega} = \mathbf{Spin}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_3 & 0 \end{bmatrix} = -\boldsymbol{\Omega}^T. \tag{A.1}$$

Premultiplication of a vector $\mathbf{v}$ by $\boldsymbol{\Omega}$ is equivalent to the cross product of $\boldsymbol{\omega}$ and $\mathbf{v}$:

$$\vec{\boldsymbol{\theta}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}} \quad \Rightarrow \quad \boldsymbol{\theta} = \boldsymbol{\Omega}\mathbf{v}. \tag{A.2}$$

In particular: $\boldsymbol{\Omega}\,\boldsymbol{\omega} = \mathbf{0}$, as may be directly verified.

The converse operation to (A.1) extracts the 3-vector $\boldsymbol{\omega}$, called *pseudovector*, or *axial vector*, from a given spin tensor:

$$\boldsymbol{\omega} = \mathbf{axial}(\boldsymbol{\Omega}) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{A.3}$$

The length of this vector is denoted by ω:

$$\omega = |\boldsymbol{\omega}| = +\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}. \tag{A.4}$$

As general notational rule, we will use corresponding upper and lower case symbols for the spin matrix and its axial vector, respectively. For example, $\mathbf{N}$ and $\mathbf{n}$, $\boldsymbol{\Theta}$ and $\boldsymbol{\theta}$, $\mathbf{B}$ and $\mathbf{b}$.

§A.3.2. Normalizations

As noted, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ can be multiplied by a nonzero scalar factor γ to obtain various *normalizations*. In general γ has the form $g(\theta)/\omega$, where $g(.)$ is a function of the rotation angle θ. The purpose of normalizations is to simplify the connections **Rot** and **Skew** to the rotator, to avoid singularities for special angles, and to connect the components ω_1, ω_2 and ω_3 closely to the rotation amplitude. This section review some normalizations that have practical or historical importance.

Taking $\gamma = 1/\omega$ we obtain the unit axial-vector and unit spinor, which are denoted by $\mathbf{n}$ and $\mathbf{N}$, respectively:

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \omega_1/\omega \\ \omega_2/\omega \\ \omega_3/\omega \end{bmatrix} = \frac{\boldsymbol{\omega}}{\omega}, \quad \mathbf{N} = \mathbf{Spin}(\mathbf{n}) = \frac{\boldsymbol{\Omega}}{\omega} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_3 & 0 \end{bmatrix}. \tag{A.5}$$

Taking $\gamma = \tan\frac{1}{2}\theta/\omega$ is equivalent to multiplying the n_i by $\tan\frac{1}{2}\theta$. We thus obtain the Rodrigues parameters $b_i = n_i \tan\frac{1}{2}\theta$, $i = 1, 2, 3$. These are collected in the Rodrigues axial-vector $\mathbf{b}$ with associated spinor $\mathbf{B}$:

$$\mathbf{b} = \tan\tfrac{1}{2}\theta\,\mathbf{n} = \frac{\tan\frac{1}{2}\theta}{\omega}\,\boldsymbol{\omega}, \quad \mathbf{B} = \mathbf{Spin}(\mathbf{b}) = \tan\tfrac{1}{2}\theta\,\mathbf{N} = \frac{\tan\frac{1}{2}\theta}{\omega}\,\boldsymbol{\Omega}. \tag{A.6}$$

This representation permits an elegant formulation of the rotator via the Cayley transform studied later. It collapses, however, as θ nears 180° since $\tan\frac{1}{2}\theta \to \pm\infty$ as $\theta \leftrightarrow 180^\circ$. One way to circumvent the singularity is through the use of the four Euler-Rodrigues parameters, also called *quaternion coefficients*:

$$p_0 = \cos\tfrac{1}{2}\theta, \quad p_i = n_i \sin\tfrac{1}{2}\theta = \omega_i/\omega \sin\tfrac{1}{2}\theta, \quad i = 1, 2, 3. \tag{A.7}$$

under the constraint $p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$. This set is often used in multibody dynamics, robotics and control. It comes at the cost of carrying along an extra parameter and an additional constraint.

A related singularity-free normalization, introduced by Fraeijs de Veubeke [243], takes $\gamma = \sin\frac{1}{2}\theta/\omega$. It is equivalent to using only the last three parameters of (A.7):

$$\mathbf{p} = \sin\tfrac{1}{2}\theta\,\mathbf{n} = (\sin\tfrac{1}{2}\theta/\omega)\;\boldsymbol{\omega}, \quad \mathbf{P} = \mathbf{Spin}(\mathbf{p}) = \sin\tfrac{1}{2}\theta\,\mathbf{N} = (\sin\tfrac{1}{2}\theta/\omega)\;\boldsymbol{\Omega}. \tag{A.8}$$

Finally, an important normalization that preserves three parameters while avoiding singularities is that associated with the exponential map. Introduce a *rotation vector* $\boldsymbol{\theta}$ defined as

$$\boldsymbol{\theta} = \theta\mathbf{n} = (\theta/\omega)\,\boldsymbol{\omega}, \qquad \boldsymbol{\Theta} = \mathbf{Spin}(\boldsymbol{\theta}) = \theta\,\mathbf{N} = (\theta/\omega)\,\boldsymbol{\Omega}. \tag{A.9}$$

For this normalization the angle is the length of the rotation vector: $\theta = |\boldsymbol{\theta}| = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$. The selection of the sign is a matter of convention.

§A.3.3. Spectral Properties

Study of the spinor eigensystem $\boldsymbol{\Omega}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ is of interest for various developments. Begin by forming the characteristic equation

$$\det(\boldsymbol{\Omega} - \lambda\mathbf{I}) = -\boldsymbol{\lambda}^3 - \omega^2\lambda = 0, \tag{A.10}$$

where $\mathbf{I}$ denotes the identity matrix of order 3. It follows that the eigenvalues of $\boldsymbol{\Omega}$ are $\lambda_1 = 0$, $\lambda_{2,3} = \pm\omega i$. Consequently $\boldsymbol{\Omega}$ is singular with rank 2 if $\omega \neq 0$ whereas if $\omega = 0$, $\boldsymbol{\Omega}$ is null.

The eigenvalues are collected in the diagonal matrix $\boldsymbol{\Lambda} = \mathbf{diag}(0,\ \omega i,\ -\omega i)$ and the corresponding right eigenvectors $\mathbf{v}_i$ in columns of $\mathbf{V} = [\,\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\,]$, so that $\boldsymbol{\Omega}\mathbf{V} = \mathbf{V}\boldsymbol{\Lambda}$. A cyclic-symmetric expression of $\mathbf{V}$, obtained through *Mathematica*, is

$$\mathbf{V} = \begin{bmatrix} \omega_1 & \omega_1 s - \omega^2 + i(\omega_2 - \omega_3)\omega & \omega_1 s - \omega^2 - i(\omega_2 - \omega_3)\omega \\ \omega_2 & \omega_2 s - \omega^2 + i(\omega_3 - \omega_1)\omega & \omega_2 s - \omega^2 - i(\omega_3 - \omega_1)\omega \\ \omega_3 & \omega_3 s - \omega^2 + i(\omega_1 - \omega_2)\omega & \omega_3 s - \omega^2 - i(\omega_1 - \omega_2)\omega \end{bmatrix} \tag{A.11}$$

where $s = \omega_1 + \omega_2 + \omega_3$. Its inverse is

$$\mathbf{V}^{-1} = \frac{1}{\omega^2}\begin{bmatrix} \omega_1 & -\frac{1}{2}\dfrac{\omega_2^2 + \omega_3^2}{\omega_1 s - \omega^2 - i(\omega_2 - \omega_3)\omega} & -\frac{1}{2}\dfrac{\omega_2^2 + \omega_3^2}{\omega_1 s - \omega^2 + i(\omega_2 - \omega_3)\omega} \\ \omega_2 & -\frac{1}{2}\dfrac{\omega_3^2 + \omega_1^2}{\omega_2 s - \omega^2 - i(\omega_3 - \omega_1)\omega} & -\frac{1}{2}\dfrac{\omega_3^2 + \omega_1^2}{\omega_2 s - \omega^2 + i(\omega_3 - \omega_1)\omega} \\ \omega_3 & -\frac{1}{2}\dfrac{\omega_1^2 + \omega_2^2}{\omega_3 s - \omega^2 - i(\omega_1 - \omega_2)\omega} & -\frac{1}{2}\dfrac{\omega_1^2 + \omega_2^2}{\omega_3 s - \omega^2 + i(\omega_1 - \omega_2)\omega} \end{bmatrix} \tag{A.12}$$

The real and imaginary part of the eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are orthogonal. This is a general property of skew-symmetric matrices; cf. Bellman [67, p. 64].

Because the eigenvalues of $\boldsymbol{\Omega}$ are distinct if $\omega \neq 0$, an arbitrary matrix function $\mathbf{F}(\boldsymbol{\Omega})$ can be explicitly obtained [314,309] as

$$\mathbf{F}(\boldsymbol{\Omega}) = \mathbf{V}\begin{bmatrix} f(0) & 0 & 0 \\ 0 & f(\omega i) & 0 \\ 0 & 0 & f(-\omega i) \end{bmatrix}\mathbf{V}^{-1}, \tag{A.13}$$

in which $f(.)$ is the scalar version of $\mathbf{F}(.)$. One important application of (A.13) is the matrix exponential: $f(.) \to e^{(.)}$.

The square of $\boldsymbol{\Omega}$, computed through direct multiplication, is

$$\boldsymbol{\Omega}^2 = -\begin{bmatrix} \omega_2^2 + \omega_3^2 & -\omega_1\omega_2 & -\omega_1\omega_3 \\ -\omega_1\omega_2 & \omega_3^2 + \omega_1^2 & -\omega_2\omega_3 \\ -\omega_1\omega_3 & -\omega_2\omega_3 & \omega_1^2 + \omega_2^2 \end{bmatrix} = \boldsymbol{\omega}\boldsymbol{\omega}^T - \omega^2\mathbf{I} = \omega^2(\mathbf{n}\mathbf{n}^T - \mathbf{I}). \tag{A.14}$$

This is a symmetric matrix of trace $-2\omega^2$ whose eigenvalues are 0, $-\omega^2$ and $-\omega^2$.

By the Cayley-Hamilton theorem [67,309,314,636] $\mathbf{\Omega}$ satisfies its own characteristic equation (A.10)

$$\mathbf{\Omega}^3 = -\omega^2\mathbf{\Omega}, \quad \mathbf{\Omega}^4 = -\omega^2\mathbf{\Omega}^2, \ \ldots \quad \text{and generally} \quad \mathbf{\Omega}^n = -\omega^2\mathbf{\Omega}^{n-2},\ n \geq 3. \tag{A.15}$$

Hence if $n = 3, 5, \ldots$ the odd powers $\mathbf{\Omega}^n$ are skew-symmetric with distinct purely imaginary eigenvalues, whereas if $n = 4, 6 \ldots$, the even powers $\mathbf{\Omega}^n$ are symmetric with repeated real eigenvalues.

The eigenvalues of $\mathbf{I} + \gamma\mathbf{\Omega}$ and $\mathbf{I} - \gamma\mathbf{\Omega}$, are $(1, 1 \pm \gamma\omega i)$ and $(-1, 1 \pm \gamma\omega i)$, respectively. Hence those two matrices are guaranteed to be nonsingular. This has implications in the Cayley transform.

Example A.1. Consider the pseudo-vector $\boldsymbol{\omega} = [\,6 \quad 2 \quad 3\,]^T$, for which $\omega = \sqrt{6^2 + 2^2 + 3^2} = 7$. The associated spin matrix and its square are

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}, \qquad \mathbf{\Omega}^2 = -\begin{bmatrix} 13 & -12 & -18 \\ -12 & 45 & -6 \\ -18 & -6 & 40 \end{bmatrix}. \tag{A.16}$$

The eigenvalues of $\mathbf{\Omega}$ are $(0, 7i, -7i)$ while those of $\mathbf{\Omega}^2$ are $(0, -7, -7)$.

§A.4. From Spinors To Rotators

Referring to Figure A.2, a rotator is an operator that maps a generic point $P(\mathbf{x})$ to $Q(\mathbf{x}_\theta)$ given the rotation axis $\vec{\boldsymbol{\omega}}$ and the angle θ. We consider only rotator representations in the form of *rotation matrices* $\mathbf{R}$, defined by

$$\mathbf{x}_\theta = \mathbf{R}\mathbf{x}. \tag{A.17}$$

This 3×3 matrix is *proper orthogonal*, that is, $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = +1$. It must reduce to $\mathbf{I}$ if the rotation vanishes. Another important attribute is the *trace property*

$$\mathbf{trace}(\mathbf{R}) = 1 + 2\cos\theta, \tag{A.18}$$

proofs of which may be found for example in Goldstein [267, p.124] or Hammermesh [285, p.325]. The problem considered in this section is the construction of $\mathbf{R}$ from the rotation data. The inverse problem: given $\mathbf{R}$, extract $\boldsymbol{\omega}$ and θ, is treated in the next section. Now if $\mathbf{R}$ is assumed to be analytic in $\mathbf{\Omega}$ it must have the Taylor expansion $\mathbf{R} = \mathbf{I} + c_1\mathbf{\Omega} + c_2\mathbf{\Omega}^2 + c_3\mathbf{\Omega}^3 + \ldots$, where all c_i must vanish if $\theta = 0$. But in view of the Cayley-Hamilton theorem (A.15) , all powers of order 3 or higher may be eliminated, and so $\mathbf{R}$ must be a linear function of $\mathbf{I}$, $\mathbf{\Omega}$ and $\mathbf{\Omega}^2$. For convenience this will be written

$$\mathbf{R} = \mathbf{I} + \alpha(\gamma\mathbf{\Omega}) + \beta(\gamma\mathbf{\Omega})^2, \tag{A.19}$$

in which γ is the scaling factor discussed above, whereas α and β are scalar functions of θ and of the invariants of $\mathbf{\Omega}$ or $\boldsymbol{\omega}$. Since the only invariant of the latter is ω, we can anticipate that $\alpha = \alpha(\theta, \omega)$ and $\beta = \beta(\theta, \omega)$, both vanishing if $\theta = 0$. Two techniques to determine those coefficients for $\gamma = 1$ are discussed next. Table A.1 summarizes the most important representations of the rotator in terms of the scaled $\mathbf{\Omega}$.

Table A.1. Rotator Forms for Various Spinors

Parametrization	γ	α	β	Spinor	Rotator $\mathbf{R}$
None	1	$\dfrac{\sin\theta}{\omega}$	$\dfrac{2\sin^2\frac{1}{2}\theta}{\omega^2}$	$\boldsymbol{\Omega}$	$\mathbf{I}+\dfrac{\sin\theta}{\omega}\boldsymbol{\Omega}+\dfrac{2\sin^2\frac{1}{2}\theta}{\omega^2}\boldsymbol{\Omega}^2$
Unit axial-vector	$\dfrac{1}{\omega}$	$\sin\theta$	$2\sin^2\frac{1}{2}\theta$	$\mathbf{N}=\gamma\boldsymbol{\Omega}$	$\mathbf{I}+\sin\theta\,\mathbf{N}+2\sin^2\frac{1}{2}\theta\,\mathbf{N}^2$,
Rodrigues-Cayley	$\dfrac{\tan\frac{1}{2}\theta}{\omega}$	$2\cos^2\frac{1}{2}\theta$	$2\cos^2\frac{1}{2}\theta$	$\mathbf{B}=\gamma\boldsymbol{\Omega}$	$\mathbf{I}+2\cos^2\frac{1}{2}\theta\,(\mathbf{B}+\mathbf{B}^2)=(\mathbf{I}+\mathbf{B})(\mathbf{I}-\mathbf{B})^{-1}$
DeVeubeke	$\dfrac{\sin\frac{1}{2}\theta}{\omega}$	$2\cos\frac{1}{2}\theta$	2	$\mathbf{P}=\gamma\boldsymbol{\Omega}$	$\mathbf{I}+2\cos\frac{1}{2}\theta\,\mathbf{P}+2\mathbf{P}^2$
Exponential map	$\dfrac{\theta}{\omega}$	$\dfrac{\sin\theta}{\theta}$	$\dfrac{2\sin^2\frac{1}{2}\theta}{\theta^2}$	$\boldsymbol{\Theta}=\gamma\boldsymbol{\Omega}$	$\mathbf{I}+\dfrac{\sin\theta}{\theta}\boldsymbol{\Theta}+\dfrac{2\sin^2\frac{1}{2}\theta}{\theta^2}\boldsymbol{\Theta}^2=e^{\boldsymbol{\Theta}}=e^{\theta\mathbf{N}}$

§A.4.1. The Algebraic Approach

This approach finds α and β for $\gamma = 1$ (the unscaled spinor) directly from algebraic conditions. Taking the trace of $\mathbf{R}$ given by (A.19) and applying the trace property (A.18) requires

$$3-2\beta\omega^2 = 1+2\cos\theta, \quad \rightarrow \quad \beta = \frac{1-\cos\theta}{\omega^2} = \frac{2\sin^2\frac{1}{2}\theta}{\omega^2}. \tag{A.20}$$

The orthogonality condition $\mathbf{I}=\mathbf{R}^T\mathbf{R}=(\mathbf{I}-\alpha\boldsymbol{\Omega}+\beta\boldsymbol{\Omega}^2)(\mathbf{I}+\alpha\boldsymbol{\Omega}+\beta\boldsymbol{\Omega}^2)=\mathbf{I}+(2\beta-\alpha^2)\boldsymbol{\Omega}^2+\beta^2\boldsymbol{\Omega}^4=\mathbf{I}+(2\beta-\alpha^2-\beta^2\omega^2)\boldsymbol{\Omega}^2$ leads to

$$2\beta-\alpha^2-\beta^2\omega^2=0 \quad \rightarrow \quad \alpha=\frac{\sin\theta}{\omega}. \tag{A.21}$$

Hence

$$\mathbf{R}=\mathbf{I}+\frac{\sin\theta}{\omega}\boldsymbol{\Omega}+\frac{1-\cos\theta}{\omega^2}\boldsymbol{\Omega}^2=\mathbf{I}+\frac{\sin\theta}{\omega}\boldsymbol{\Omega}+\frac{2\sin^2\frac{1}{2}\theta}{\omega^2}\boldsymbol{\Omega}^2. \tag{A.22}$$

From a computational viewpoint the sine-squared form should be preferred for small angles to avoid the $1-\cos\theta$ cancellation. Replacing the components of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^2$ gives the explicit rotator form

$$\mathbf{R}=\frac{1}{\omega^2}\begin{bmatrix} \omega_1^2+(\omega_2^2+\omega_3^2)\cos\theta & 2\omega_1\omega_2\sin^2\frac{1}{2}\theta-\omega_3\omega\sin\theta & 2\omega_1\omega_3\sin^2\frac{1}{2}\theta+\omega_2\omega\sin\theta \\ 2\omega_1\omega_2\sin^2\frac{1}{2}\theta+\omega_3\omega\sin\theta & \omega_2^2+(\omega_3^2+\omega_1^2)\cos\theta & 2\omega_2\omega_3\sin^2\frac{1}{2}\theta-\omega_1\omega\sin\theta \\ 2\omega_1\omega_3\sin^2\frac{1}{2}\theta-\omega_2\omega\sin\theta & 2\omega_2\omega_3\sin^2\frac{1}{2}\theta+\omega_1\omega\sin\theta & \omega_3^2+(\omega_1^2+\omega_2^2)\cos\theta \end{bmatrix}. \tag{A.23}$$

This is invariant to scaling of the ω_i, and consequently (A.23) is unique.

§A.4.2. The Geometric Approach

The vector representation of the rigid motion pictured in Figure A.2 is

$$\vec{\mathbf{x}}_\theta=\vec{\mathbf{x}}\cos\theta+(\vec{\mathbf{n}}\times\vec{\mathbf{x}})\sin\theta+\vec{\mathbf{n}}(\vec{\mathbf{n}}\cdot\vec{\mathbf{x}})(1-\cos\theta)=\vec{\mathbf{x}}+(\vec{\mathbf{n}}\times\vec{\mathbf{x}})\sin\theta+\left[\vec{\mathbf{n}}\times(\vec{\mathbf{n}}\times\vec{\mathbf{x}})\right](1-\cos\theta), \tag{A.24}$$

where $\vec{\mathbf{n}}$ is $\vec{\boldsymbol{\omega}}$ normalized to unit length as per (A.6). This can be recast in matrix form by substituting $\vec{\mathbf{n}}\times\vec{\mathbf{x}}\rightarrow\mathbf{N}\mathbf{x}=\boldsymbol{\Omega}\mathbf{x}/\omega$ and $\mathbf{x}_\theta=\mathbf{R}\mathbf{x}$. Cancelling $\mathbf{x}$ we get back (A.22).

If $\boldsymbol{\omega}$ is unit-length-normalized to $\mathbf{n}$ as per (A.6), $\gamma = 1/\omega$ and $\mathbf{R} = \mathbf{I} + \sin\theta\,\mathbf{N} + (1-\cos\theta)\,\mathbf{N}^2$. This is the matrix form of (A.24). Because $\mathbf{N}^2 = \mathbf{n}\mathbf{n}^T - \mathbf{I}$, an ocassionally useful variant is

$$\mathbf{R} = \mathbf{W} + (1-\cos\theta)\,\mathbf{n}\,\mathbf{n}^T, \qquad \mathbf{W} = \cos\theta\,\mathbf{I} + \sin\theta\,\mathbf{N}. \tag{A.25}$$

In terms of the three Rodrigues-Cayley parameters b_i introduced in(A.6), $\alpha = \beta = 2\cos^2\frac{1}{2}\theta$ and $\mathbf{R} = \mathbf{I} + 2\cos^2\frac{1}{2}\theta\,(\mathbf{B}+\mathbf{B}^2)$. This can be explicitly worked out to be

$$\mathbf{R} = \frac{1}{1+b_1^2+b_2^2+b_3^2}\begin{bmatrix} 1+b_1^2-b_2^2-b_3^2 & 2(b_1b_2-b_3) & 2(b_1b_3+b_2) \\ 2(b_1b_2+b_3) & 1-b_1^2+b_2^2-b_3^2 & 2(b_2b_3-b_1) \\ 2(b_1b_3-b_2) & 2(b_2b_3+b_1) & 1-b_1^2-b_2^2+b_3^2 \end{bmatrix} \tag{A.26}$$

This form was derived by Rodrigues [540] and used by Cayley [110] to study rigid body motions. It has the advantage of being obtainable through an algebraic matrix expression: the Cayley transform, which is presented below. It becomes indeterminate, however, as $\theta \to 180^\circ$, since all terms approach $0/0$. This indeterminacy is avoided by using the four Euler-Rodrigues parameters, which are also the quaternion coefficients, defined in (A.7). In terms of these the rotator becomes

$$\mathbf{R} = 2\begin{bmatrix} p_0^2+p_1^2-\frac{1}{2} & p_1p_2-p_0p_3 & p_1p_3+p_0p_2 \\ p_1p_2+p_0p_3 & p_0^2+p_2^2-\frac{1}{2} & p_2p_3-p_0p_1 \\ p_1p_3-p_0p_2 & p_2p_3+p_0p_1) & p_0^2+p_3^2-\frac{1}{2} \end{bmatrix} \tag{A.27}$$

This rotator cannot become singular, but this advantage is paid at the cost of carrying along an extra parameter plus the constraint $p_0^2+p_1^2+p_2^2+p_3^2 = 1$.

The normalization of DeVeubeke, given in (A.8) as $p_i = (\omega_i/\omega)\sin\frac{1}{2}\theta$, leads to $\alpha = 2\cos\frac{1}{2}\theta$ and $\beta = 2$. Hence $\mathbf{R} = \mathbf{I} + 2\cos\frac{1}{2}\theta\,\mathbf{P} + 2\mathbf{P}^2$.

§A.4.3. The Cayley Transform

Given any skew-symmetric real matrix $\mathbf{S} = -\mathbf{S}^T$, we can apply the transformation

$$\mathbf{Q} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1} \tag{A.28}$$

Then $\mathbf{Q}$ is a proper orthogonal matrix, that is $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\det\mathbf{Q} = +1$. This is stated in several textbooks, e.g., Gantmacher [254] but none gives a proof. Here is the proof of orthogonality: $\mathbf{Q}^T\mathbf{Q} = (\mathbf{I}+\mathbf{S})^{-1}(\mathbf{I}-\mathbf{S})(\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1} = (\mathbf{I}+\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1} = \mathbf{I}\mathbf{I} = \mathbf{I}$ because $\mathbf{I}+\mathbf{S}$ and $\mathbf{I}-\mathbf{S}$ commute. The property $\det\mathbf{Q} = +1$ can be easily proven from the spectral properties. The inverse transformation

$$\mathbf{S} = (\mathbf{Q}-\mathbf{I})(\mathbf{Q}+\mathbf{I})^{-1} \tag{A.29}$$

produces skew-symmetric matrices from a source orthogonal matrix $\mathbf{Q}$. Equations (A.28) and (A.29) are called the *Cayley transforms* after Cayley [110]. These formulas are ocassionally useful in the construction of approximations for moderate rotations.

An interesting question is: given $\boldsymbol{\Omega}$ and θ, can (A.28) be used to produce the exact $\mathbf{R}$? The answer is: yes, if $\boldsymbol{\Omega}$ is scaled by a factor $\gamma = \gamma(\theta,\omega)$. We thus investigate whether $\mathbf{R} = (\mathbf{I}+\gamma\boldsymbol{\Omega})(\mathbf{I}-\gamma\boldsymbol{\Omega})^{-1}$ exactly for some γ. Premultiplying both sides by $\mathbf{I}-\gamma\boldsymbol{\Omega}$ and representing $\mathbf{R}$ by (A.19) we require

$$(\mathbf{I}-\gamma\boldsymbol{\Omega})(\mathbf{I}+\alpha\boldsymbol{\Omega}+\beta\boldsymbol{\Omega}^2) = \mathbf{I} + (\alpha-\gamma+\gamma\beta\omega^2)\boldsymbol{\Omega} + (\beta-\alpha\gamma)\boldsymbol{\Omega}^2 = \mathbf{I}+\gamma\boldsymbol{\Omega} \tag{A.30}$$

Identifying we get the conditions $\beta = \alpha\gamma$ and $\alpha - \gamma + \gamma\beta\omega^2 = \gamma$. The first one gives $\gamma = \beta/\alpha$, which inserted in the second requires $\alpha^2 - 2\beta + \beta\omega^2 = 0$. Fortunately this is identically satisfied by (A.21). Thus the only solution is $\gamma = \beta/\alpha = (1 - \cos\theta)/(\omega \sin\theta) = \tan\frac{1}{2}\theta/\omega$. This is precisely the Rodrigues normalization (A.7) or (A.27). Consequently

$$\mathbf{R} = (\mathbf{I} + \mathbf{B})(\mathbf{I} - \mathbf{B})^{-1}, \quad \mathbf{B} = \frac{\tan\frac{1}{2}\theta}{\omega}\mathbf{\Omega}. \tag{A.31}$$

The explicit calculation of **R** in terms of the b_i leads to (A.26).

§A.4.4. Exponential Map

This is a final representation of **R** that has both theoretical and practical importance. Given a skew-symmetric real matrix **S**, the matrix exponential

$$\mathbf{Q} = e^{\mathbf{S}} = \mathbf{Exp}(\mathbf{S}) \tag{A.32}$$

is proper orthogonal. Here is the simple proof of Gantmacher [254, p 287]: $\mathbf{Q}^T = \mathbf{Exp}(\mathbf{S}^T) = \mathbf{Exp}(-\mathbf{S}^T) = \mathbf{Q}^{-1}$. If the eigenvalues of **S** are λ_i, then $\sum_i \lambda_i = \mathbf{trace}(\mathbf{S}) = 0$. The eigenvalues of **Q** are $\mu_i = \exp(\lambda_i)$; thus $\det(\mathbf{Q}) = \Pi_i \mu_i = \exp(\sum_i \lambda_i) = \exp(0) = +1$. The transformation (A.32) is called an *exponential map*. The converse is of course $\mathbf{S} = \mathbf{Log}(\mathbf{Q})$.

As in the case of the Cayley transform, one may pose the question of whether we can get the rotator $\mathbf{R} = \mathbf{Exp}(\gamma\mathbf{\Omega})$ exactly for some factor $\gamma = \gamma(\theta, \omega)$. To study this question we need an explicit form of the exponential. This can be obtained from (A.13) in which the function **Exp** so that the diagonal matrix entries are 1 and $\exp(\pm\gamma\omega i) = \cos\gamma\omega \pm i \sin\gamma\omega$. The following approach is more instructive and leads directly to the final result. Start from the definition of the matrix exponential

$$\mathbf{Exp}(\gamma\mathbf{\Omega}) = \mathbf{I} + \gamma\mathbf{\Omega} + \frac{\gamma^2}{2!}\mathbf{\Omega}^2 + \frac{\gamma^3}{3!}\mathbf{\Omega}^3 + \ldots \tag{A.33}$$

and use the Cayley-Hamilton theorem (A.15) to eliminate all powers of order 3 or higher in $\mathbf{\Omega}$. Identify the coefficient series of $\mathbf{\Omega}$ and $\mathbf{\Omega}^2$ with those of the sine and cosine, to obtain

$$\mathbf{Exp}(\gamma\mathbf{\Omega}) = \mathbf{I} + \frac{\sin(\gamma\omega)}{\omega}\mathbf{\Omega} + \frac{1 - \cos(\gamma\omega)}{\omega^2}\mathbf{\Omega}^2. \tag{A.34}$$

Comparing to (A.22) requires $\gamma\omega = \theta$, or $\gamma = \theta/\omega$. Introducing $\theta_i = \theta\omega_i/\omega$ and $\mathbf{\Theta} = \mathbf{Spin}(\boldsymbol{\theta}) = \theta\mathbf{N} = (\theta/\omega)\mathbf{\Omega}$ as in (A.9), one gets

$$\mathbf{R} = \mathbf{Exp}(\mathbf{\Theta}) = \mathbf{I} + \frac{\sin\theta}{\theta}\mathbf{\Theta} + \frac{1 - \cos\theta}{\theta^2}\mathbf{\Theta}^2 = \mathbf{I} + \frac{\sin\theta}{\theta}\mathbf{\Theta} + \frac{2\sin^2\frac{1}{2}\theta}{\theta^2}\mathbf{\Theta}^2. \tag{A.35}$$

On substituting $\mathbf{\Theta} = \theta\mathbf{N}$ this recovers $\mathbf{R} = \mathbf{I} + \sin\theta\,\mathbf{N} + (1 - \cos\theta)\,\mathbf{N}^2$, as it should.

This representation has several advantages: (i) it is singularity free, and (ii) the θ_i are exactly proportional to the angle, and (ii) it simplifies differentiation. Because of these favorable properties the exponential map has become a favorite of implementations where large angles may occur, as in orbiting structures and robotics.

Example A.2.

Take again $\boldsymbol{\omega} = [\,3 \quad 2 \quad 6\,]^T$ so $\omega = 7$. Consider the 3 angles $\theta = 2^\circ$, $\theta = 90^\circ$ and $\theta = 180^\circ$. The rotators calculated from (A.22) are, to 8 figures:

$$\mathbf{R} = \tag{A.36}$$

§A.5. From Rotators to Spinors

If $\mathbf{R}$ is given, the extraction of the rotation amount θ and the unit pseudo-vector $\mathbf{n} = \boldsymbol{\omega}/\omega$ is often required. The former is easy using the trace property (A.19):

$$\cos\theta = \tfrac{1}{2}\left(\mathbf{trace}(\mathbf{R}) - 1\right). \tag{A.37}$$

Recovery of $\mathbf{n}$ is also straightforward using the particular form $\mathbf{R} = \mathbf{I} + \sin\theta\,\mathbf{N} + (1-\cos\theta)\mathbf{N}^2$ since $\mathbf{R} - \mathbf{R}^T = 2\sin\theta\,\mathbf{N}$, whence

$$\mathbf{N} = \frac{\mathbf{R} - \mathbf{R}^T}{2\sin\theta}, \qquad \mathbf{n} = \mathbf{axial}(\mathbf{N}). \tag{A.38}$$

One issue is the sign of θ since (A.37) is satisfied by $\pm\theta$. If the sign is reversed, so is $\mathbf{n}$. Thus in principle it is possible to select $\theta \geq 0$ if no constraints are placed on the direction of the rotation axis.

The above formulas are prone to numerical instability for θ near 0° and 180° since $\sin\theta$ vanishes. A robust algorithm is that given by Spurrier [579] in the language of quaternions. Choose the algebraically largest of $\mathbf{trace}(\mathbf{R})$ and R_{ii}, $i = 1, 2, 3$. If $\mathbf{trace}(\mathbf{R})$ is the largest, compute

$$p_0 = \cos\tfrac{1}{2}\theta = \tfrac{1}{2}\sqrt{1 + \mathbf{trace}(\mathbf{R})}, \qquad p_i = n_i \sin\tfrac{1}{2}\theta = \tfrac{1}{4}(R_{kj} - R_{jk})/p_0, \qquad i = 1, 2, 3, \tag{A.39}$$

in which j and k are the cyclic permutations of i. Otherwise let R_{ii} be the algebraically largest diagonal entry, and again denote i, j, k the cyclic permutation. Then use

$$\begin{aligned} &p_i = n_i \sin\tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}R_{ii} + \tfrac{1}{4}(1 - \mathbf{trace}(\mathbf{R}))}, \qquad p_0 = \cos\tfrac{1}{2}\theta = \tfrac{1}{4}(R_{kj} - R_{jk})/p_i, \\ &p_i = \tfrac{1}{4}(R_{l,i} + R_{i,l})/p_i, \qquad l = j, k. \end{aligned} \tag{A.40}$$

From p_0, p_1, p_2, p_3 it is easy to pass to $\theta, n1, n2, n3$ once the sign of θ is chosen as discussed above.

Remark A.1. The theoretical formula for the matrix logarithm, which is applicable to any matrix size, is

$$\boldsymbol{\Omega} = \log\mathbf{R} = \frac{\arcsin\tau}{2\tau}\,\mathbf{axial}(\mathbf{R} - \mathbf{R}^T), \tag{A.41}$$

in which $\tau = \frac{1}{2}|\mathbf{axial}(\mathbf{R} - \mathbf{R}^T)|$. But for numerical computations this expression is largely useless.

§A.6. Rotator Derivatives

The derivatives and differentials of the rotator with respect to angle and rotation axis direction changes are required in many developments.

As independent parameters we will take θ and $\boldsymbol{\omega} = \omega\mathbf{n}$ (the n_i are not good choices because they are linked by a constraint). For convenience, define

$$\begin{aligned} &d\boldsymbol{\omega} = \begin{bmatrix} d\omega_1 \\ d\omega_2 \\ d\omega_3 \end{bmatrix}, \quad d\boldsymbol{\Omega} = \mathbf{Spin}(d\boldsymbol{\omega}), \quad d^2\boldsymbol{\omega} = \begin{bmatrix} d^2\omega_1 \\ d^2\omega_2 \\ d^2\omega_3 \end{bmatrix}, \quad d^2\boldsymbol{\Omega} = \mathbf{Spin}(d^2\boldsymbol{\omega}), \\ &\mathbf{W} = \cos\theta\,\mathbf{I} + \sin\theta\,\mathbf{N} = \mathbf{R} - (1-\cos\theta)\,\mathbf{n}\mathbf{n}^T. \end{aligned} \tag{A.42}$$

It is convenient to depart from $\mathbf{R} = \mathbf{Exp}(\theta\mathbf{N}) = \mathbf{I} + \sin\theta\,\mathbf{N} + (1-\cos\theta)\,\mathbf{N}^2 = \mathbf{W} + (1-\cos\theta)\mathbf{n}\mathbf{n}^T$. Succesive differentiation gives

$$d\mathbf{R} = \mathbf{R}\;d(\theta\mathbf{N}) = \mathbf{R}\,(\mathbf{N}\,d\theta + \theta\,d\mathbf{N}), \tag{A.43}$$

and

$$d^2\mathbf{R} = \mathbf{R}\,d(\theta\mathbf{N})\,d(\theta\mathbf{N}) + \mathbf{R}\;d^2(\theta\mathbf{N}) = \mathbf{R}\left[(\mathbf{N}\,d\theta + \theta\,d\mathbf{N})^2 + \mathbf{N}\,d^2\theta + 2\,d\theta\,d\mathbf{N} + \theta\,d^2\mathbf{N}\right] \tag{A.44}$$

These expressions can be simplified using the following identities:

$$\begin{gathered}
\mathbf{RN} = \mathbf{NR} = \mathbf{N} + \sin\theta\,\mathbf{N}^2 + (1-\cos\theta)\mathbf{N}^3 = \cos\theta\,\mathbf{N} + \sin\theta\,\mathbf{N}^2 = \mathbf{WN} = \mathbf{NW},\\
\mathbf{NRN} = \mathbf{N}^2\,\mathbf{R} = \mathbf{R}\,\mathbf{N}^2 = -\sin\theta\,\mathbf{N} + \cos\theta\,\mathbf{N}^2 = \mathbf{R} - \mathbf{I} - \mathbf{N}^2 = \mathbf{R} - \mathbf{n}\mathbf{n}^T,\\
d\mathbf{N} = \frac{d\boldsymbol{\Omega}}{\omega} - \mathbf{N}\,\frac{\boldsymbol{\omega}^T d\boldsymbol{\omega}}{\omega^2} \quad \mathbf{N}\,d\mathbf{N} = \frac{d\boldsymbol{\omega}\,\boldsymbol{\omega}^T - (\boldsymbol{\omega}^T d\boldsymbol{\omega})\,\mathbf{I}}{\omega^2}, \quad \mathbf{N}^2\,d\mathbf{N} = \frac{\mathbf{N}\,d\boldsymbol{\omega}\,\boldsymbol{\omega}^T - (\boldsymbol{\omega}^T d\boldsymbol{\omega})\,\mathbf{N}}{\omega^2},\\
\mathbf{R}\,d\mathbf{N} = \mathbf{R}\frac{d\Omega}{\omega} - \mathbf{WN}\frac{\boldsymbol{\omega}^T d\boldsymbol{\omega}}{\omega^2}.
\end{gathered} \tag{A.45}$$

Here $\mathbf{Nn} = \mathbf{0}$ has been used to eliminate some terms.

Inserting into (A.43) gives

$$d\mathbf{R} = \mathbf{W}\,\mathbf{N}\,d\theta + \theta\,\mathbf{R}\,d\mathbf{N} \tag{A.46}$$

This is a compacted form verified through *Mathematica*.

The expression of $d^2\mathbf{R}$, which is required for stiffness level and acceleration computations, will be worked out later.

§A.7. Spinor and Rotator Transformations

Suppose $\boldsymbol{\Omega}$ is a spinor and $\mathbf{R} = \mathbf{Rot}(\boldsymbol{\Omega})$ the associated rotator, referred to a Cartesian frame $\mathbf{x} = \{x_i\}$. It is required to transform $\mathbf{R}$ to another Cartesian frame $\bar{\mathbf{x}} = \{\bar{x}_i\}$ related by $T_{ij} = \partial\bar{x}_i/\partial x_j$, where T_{ij} are entries of a 3×3 orthogonal matrix $\mathbf{T} = \partial\bar{\mathbf{x}}/\partial\mathbf{x}$. Application of (**?**) yields

$$\bar{\mathbf{R}} = \mathbf{TRT}^T, \quad \bar{\mathbf{R}}^T = \mathbf{TR}^T\mathbf{T}^T, \qquad \mathbf{R} = \mathbf{T}^T\bar{\mathbf{R}}\mathbf{T}, \quad \mathbf{R} = \mathbf{T}^T\bar{\mathbf{R}}^T\mathbf{T}. \tag{A.47}$$

More details may be found in [267, Chapter 4]. Pre and post-multiplying (**?**) by $\mathbf{T}$ and $\mathbf{T}^T$, respectively, yields the transformed spinor $\bar{\mathbf{N}} = \mathbf{T}\,\mathbf{N}\,\mathbf{T}^T$, which is also skew-symmetric because $\bar{\mathbf{N}}^T = \mathbf{T}\,\mathbf{N}^T\,\mathbf{T}^T = -\bar{\mathbf{N}}$. Likewise for the other spinors listed in Table A.5. Relations (**?**) with $\mathbf{Q}\to\mathbf{T}$ show how axial vectors transform.

§A.8. Axial Vector Jacobian

The Jacobian matrix $\mathbf{H}(\boldsymbol{\theta}) = \partial\boldsymbol{\theta}/\partial\boldsymbol{\omega}$ of the rotational axial vector $\boldsymbol{\theta}$ with respect to the spin axial vector $\boldsymbol{\omega}$, and its inverse $\mathbf{H}(\boldsymbol{\theta})^{-1} = \partial\boldsymbol{\omega}/\partial\boldsymbol{\theta}$, appear in the EICR. The latter was first derived by Simo [564] and Szwabowicz [602], and rederived by Nour-Omid and Rankin [433, p. 377]:

$$\mathbf{H}(\boldsymbol{\theta})^{-1} = \frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{\theta}} = \frac{\sin\theta}{\theta}\mathbf{I} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Theta} + \frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\boldsymbol{\theta}^T = \mathbf{I} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Theta} + \frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\Theta}^2. \quad \text{(A.48)}$$

The last expression in (A.48), not given by the cited authors, is obtained on replacing $\boldsymbol{\theta}\boldsymbol{\theta}^T = \theta^2\mathbf{I} + \boldsymbol{\Theta}^2$. Use of the inversion formula (**?**) gives

$$\mathbf{H}(\boldsymbol{\theta}) = \frac{\partial\boldsymbol{\theta}}{\partial\boldsymbol{\omega}} = \mathbf{I} - \tfrac{1}{2}\boldsymbol{\Theta} + \eta\boldsymbol{\Theta}^2, \quad \text{(A.49)}$$

in which

$$\eta = \frac{1 - \frac{1}{2}\theta\cot(\frac{1}{2}\theta)}{\theta^2} = \tfrac{1}{12} + \tfrac{1}{720}\theta^2 + \tfrac{1}{30240}\theta^4 + \tfrac{1}{1209600}\theta^6 + \ldots \quad \text{(A.50)}$$

The η given in (A.50) results by simplifying the value $\eta = [\sin\theta - \theta(1+\cos\theta)]/[\theta^2\sin\theta]$ given by previous investigators. Care must be taken on evaluating η for small angle θ because it approaches $0/0$. If $|\theta| < 1/20$, say, the series given above may be used, with error $< 10^{-16}$ when 4 terms are retained. If θ is a multiple of 2π, η blows up since $\cot(\frac{1}{2}\theta) \to \infty$, and a modulo-$2\pi$ reduction is required.

In the formulation of the tangent stiffness matrix, the spin derivative of $\mathbf{H}(\boldsymbol{\theta})^T$ contracted with a nodal moment vector $\mathbf{m}$ is required:

$$\begin{aligned}\mathbf{L}(\boldsymbol{\theta},\mathbf{m}) &= \frac{\partial\mathbf{H}(\boldsymbol{\theta})^T}{\partial\boldsymbol{\omega}}:\mathbf{m} = \frac{\partial}{\partial\boldsymbol{\theta}}[\mathbf{H}(\boldsymbol{\theta})^T\mathbf{m}]\,\mathbf{H}(\boldsymbol{\theta}) \\ &= \left\{\eta\,[(\boldsymbol{\theta}^T\mathbf{m})\,\mathbf{I} + \boldsymbol{\theta}\,\mathbf{m}^T - 2\mathbf{m}\,\boldsymbol{\theta}^T] + \mu\boldsymbol{\Theta}^2\,\mathbf{m}\,\boldsymbol{\theta}^T - \tfrac{1}{2}\mathbf{Spin}(\mathbf{m})\right\}\mathbf{H}(\boldsymbol{\theta}).\end{aligned} \quad \text{(A.51)}$$

in which

$$\mu = \frac{d\eta/d\theta}{\theta} = \frac{\theta^2 + 4\cos\theta + \theta\sin\theta - 4}{4\theta^4\sin^2(\frac{1}{2}\theta)} = \tfrac{1}{360} + \tfrac{1}{7560}\theta^2 + \tfrac{1}{201600}\theta^4 + \tfrac{1}{5987520}\theta^6 + \ldots \quad \text{(A.52)}$$

This expression of μ was obtained by simplifying results given in [433, p. 378].

§A.9. Spinor and Rotator Differentiation

Derivatives, differentials and variations of axial vectors, spinors and rotators with respect to various choices of independent variables appear in applications of finite rotations to mechanics. In this section we present only expressions that are useful in the CR description. They are initially derived for dynamics and then specialized to variations. Several of the formulas are new.

§A.9.1. Angular Velocities

We assume that the rotation angle $\boldsymbol{\theta}(t) = \theta(t)\mathbf{n}(t)$ is a given function of time t, which is taken as the independent variable. The time derivative of $\boldsymbol{\Theta}(t)$ is $\dot{\boldsymbol{\Theta}} = \mathbf{axial}(\dot{\boldsymbol{\theta}})$. To express rotator differentials in a symmetric manner we introduce an axial vector $\dot{\phi}$ and a spinor $\dot{\boldsymbol{\Phi}} = \mathbf{Spin}(\dot{\phi})$, related to $\dot{\boldsymbol{\Theta}}$ congruentially through the rotator:

$$\dot{\boldsymbol{\Theta}} = \mathbf{Spin}(\dot{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ \dot{\theta}_3 & 0 & -\dot{\theta}_1 \\ -\dot{\theta}_2 & \dot{\theta}_1 & 0 \end{bmatrix} = \mathbf{R}\dot{\boldsymbol{\Phi}}\mathbf{R}^T, \quad \dot{\boldsymbol{\Phi}} = \mathbf{Spin}(\dot{\phi}) = \begin{bmatrix} 0 & -\dot{\phi}_3 & \dot{\phi}_2 \\ \dot{\phi}_3 & 0 & -\dot{\phi}_1 \\ -\dot{\phi}_2 & \dot{\phi}_1 & 0 \end{bmatrix} = \mathbf{R}^T\dot{\boldsymbol{\Theta}}\mathbf{R}. \tag{A.53}$$

From (**?**) it follows that the axial vectors are linked by

$$\dot{\phi} = \mathbf{R}^T\dot{\boldsymbol{\theta}}, \qquad \dot{\boldsymbol{\theta}} = \mathbf{R}\dot{\phi}. \tag{A.54}$$

Corresponding relation between variations or differentials, such as $\delta\boldsymbol{\Theta} = \mathbf{R}\,\delta\boldsymbol{\Phi}\,\mathbf{R}^T$ or $d\boldsymbol{\Theta} = \mathbf{R}\,d\boldsymbol{\Phi}\,\mathbf{R}^T$ are incorrect as shown in below. In CR dynamics, $\dot{\boldsymbol{\theta}}$ is the vector of *inertial angular velocities* whereas $\dot{\phi}$ is the vector of *dynamic angular velocities*.

Repeated temporal differentiation of $\mathbf{R} = \mathbf{Exp}(\boldsymbol{\Theta})$ gives the rotator time derivatives

$$\dot{\mathbf{R}} = \dot{\boldsymbol{\Theta}}\,\mathbf{R}, \quad \ddot{\mathbf{R}} = (\ddot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Theta}}^2)\mathbf{R}, \quad \dddot{\mathbf{R}} = (\dddot{\boldsymbol{\Theta}} + 2\ddot{\boldsymbol{\Theta}}\dot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Theta}}\ddot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Theta}}^3)\mathbf{R}, \; \ldots \tag{A.55}$$

$$\dot{\mathbf{R}} = \mathbf{R}\,\dot{\boldsymbol{\Phi}}, \quad \ddot{\mathbf{R}} = \mathbf{R}(\ddot{\boldsymbol{\Phi}} + \dot{\boldsymbol{\Phi}}^2), \quad \dddot{\mathbf{R}} = \mathbf{R}(\dddot{\boldsymbol{\Phi}} + 2\ddot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Phi}} + \dot{\boldsymbol{\Phi}}\ddot{\boldsymbol{\Phi}} + \dot{\boldsymbol{\Phi}}^3), \; \ldots \tag{A.56}$$

The following four groupings appear often: $\mathbf{R}\,\dot{\mathbf{R}}^T$, $\dot{\mathbf{R}}\,\mathbf{R}^T$, $\mathbf{R}^T\,\dot{\mathbf{R}}$ and $\dot{\mathbf{R}}^T\,\mathbf{R}$. From the identities $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ and $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ it can be shown that they are skew-symmetric, and may be associated to axial vectors with physical meaning. For example, taking time derivatives of $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ yields $\mathbf{R}\,\dot{\mathbf{R}}^T + \dot{\mathbf{R}}\,\mathbf{R}^T = \mathbf{0}$, whence $\mathbf{R}\,\dot{\mathbf{R}}^T = -\dot{\mathbf{R}}\,\mathbf{R}^T = -(\mathbf{R}\,\dot{\mathbf{R}})^T$. Pre and postmultiplication of $\dot{\mathbf{R}}$ in (A.55) and (A.56) by $\mathbf{R}$ and $\mathbf{R}^T$ furnishes

$$\dot{\mathbf{R}}\,\mathbf{R}^T = -\mathbf{R}\,\dot{\mathbf{R}}^T = \dot{\boldsymbol{\Theta}}, \qquad \mathbf{R}^T\,\dot{\mathbf{R}} = -\dot{\mathbf{R}}^T\mathbf{R} = \dot{\boldsymbol{\Phi}}. \tag{A.57}$$

Note that the general integral of $\dot{\mathbf{R}} = \dot{\boldsymbol{\Theta}}\,\mathbf{R}$ aside from a constant, is $\mathbf{R} = \mathbf{Exp}(\boldsymbol{\Theta})$, from which $\theta(t)$ can be extracted. On the other hand there is no integral relation defining $\phi(t)$; only the differential equation $\dot{\mathbf{R}} = \mathbf{R}\,\dot{\boldsymbol{\Phi}}$.

§A.9.2. Angular Accelerations

Postmultiplying the second of (A.55) by $\mathbf{R}^T$ yields

$$\ddot{\mathbf{R}}\mathbf{R}^T = \ddot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Theta}}^2 = \dot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\theta}}\,\dot{\boldsymbol{\theta}}^T - (\dot{\theta})^2\mathbf{I} = \begin{bmatrix} 0 & -\ddot{\theta}_3 & \ddot{\theta}_2 \\ \ddot{\theta}_3 & 0 & -\ddot{\theta}_1 \\ -\ddot{\theta}_2 & \ddot{\theta}_1 & 0 \end{bmatrix} + \begin{bmatrix} -\dot{\theta}_2^2 - \dot{\theta}_3^2 & \dot{\theta}_1\dot{\theta}_2 & \dot{\theta}_1\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_2 & -\dot{\theta}_3^2 - \dot{\theta}_1^2 & \dot{\theta}_2\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_3 & \dot{\theta}_2\dot{\theta}_3 & -\dot{\theta}_1^2 - \dot{\theta}_2^2 \end{bmatrix}, \tag{A.58}$$

in which $(\dot{\theta})^2 = \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2$. When applied to a vector $\mathbf{r}$, $(\ddot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Theta}}\dot{\boldsymbol{\Theta}})\mathbf{r} = \ddot{\boldsymbol{\theta}} \times \mathbf{r} + \dot{\boldsymbol{\theta}}\,\dot{\boldsymbol{\theta}}^T\mathbf{r} - (\dot{\theta})^2\mathbf{r}$. This operator appears in the expression of particle accelerations in a moving frame. The second and third term give rise to the Coriolis and centrifugal forces, respectively. Premultiplying the second derivative in (A.56) by $\mathbf{R}$ yields $\mathbf{R}\ddot{\mathbf{R}} = \ddot{\boldsymbol{\Phi}} + \dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Phi}}$, and so on.

§A.9.3. Variations

Some of the foregoing expressions can be directly transformed to variational and differential forms while others cannot. For example, varying $\mathbf{R} = \mathbf{Exp}(\mathbf{\Theta})$ gives

$$\delta\mathbf{R} = \delta\mathbf{\Theta}\,\mathbf{R}, \qquad \delta\mathbf{R}^T = -\mathbf{R}^T\,\delta\mathbf{\Theta}. \tag{A.59}$$

This matches $\dot{\mathbf{R}} = \dot{\mathbf{\Theta}}\,\mathbf{R}$ and $\mathbf{R}^T = -\mathbf{R}^T\dot{\mathbf{\Theta}}$ from (A.55) on replacing $(\dot{\,})$ by δ. On the other hand, the counterparts of (A.56): $\dot{\mathbf{R}} = \mathbf{R}\,\dot{\mathbf{\Phi}}$ and $\dot{\mathbf{R}}^T = -\dot{\mathbf{\Phi}}\,\mathbf{R}$ are *not* $\delta\mathbf{R} = \mathbf{R}\,\delta\mathbf{\Phi}$ and $\delta\mathbf{R}^T = -\delta\mathbf{\Phi}\,\mathbf{R}^T$, a point that has tripped authors unfamiliar with moving frame dynamics. Correct handling requires the introduction of a third axial vector $\boldsymbol{\psi}$:

$$\delta\mathbf{R} = \mathbf{R}\,\delta\mathbf{\Psi}, \quad \delta\mathbf{R}^T = -\delta\mathbf{\Psi}\,\mathbf{R}, \quad \text{in which} \quad \delta\mathbf{\Psi} = \mathbf{Spin}(\delta\psi) = \begin{bmatrix} 0 & -\delta\psi_3 & \delta\psi_2 \\ \delta\psi_3 & 0 & -\delta\psi_1 \\ -\delta\psi_2 & \delta\psi_1 & 0 \end{bmatrix}. \tag{A.60}$$

Axial vectors $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ can be linked as follows. Start from $\delta\mathbf{R} = \mathbf{R}\,\delta\mathbf{\Psi}$ and $\dot{\mathbf{R}} = \mathbf{R}\,\dot{\mathbf{\Phi}}$. Time differentiate the former: $\delta\dot{\mathbf{R}} = \mathbf{R}\,\delta\dot{\mathbf{\Psi}} + \dot{\mathbf{R}}\,\delta\mathbf{\Psi}$, and vary the latter: $\delta\dot{\mathbf{R}} = \delta\mathbf{R}\,\dot{\mathbf{\Phi}} + \mathbf{R}\,\delta\dot{\mathbf{\Phi}}$, equate the two right-hand sides, premultiply by $\mathbf{R}^T$, replace $\mathbf{R}^T\dot{\mathbf{R}}$ and $\mathbf{R}^T\delta\mathbf{R}$ by $\dot{\mathbf{\Phi}}$ and $\delta\mathbf{\Psi}$, respectively, and rearrange to obtain

$$\delta\dot{\mathbf{\Phi}} = \delta\dot{\mathbf{\Psi}} + \dot{\mathbf{\Phi}}\,\delta\mathbf{\Psi} - \delta\mathbf{\Psi}\,\dot{\mathbf{\Phi}}, \quad \text{or} \quad \delta\dot{\phi} = \delta\dot{\psi} + \dot{\mathbf{\Phi}}\,\delta\psi = \delta\dot{\psi} - \delta\mathbf{\Psi}\,\dot{\phi}. \tag{A.61}$$

The last transformation is obtained by taking the axial vectors of both sides, and using **(?)**. It is seen that $\delta\dot{\mathbf{\Psi}}$ and $\delta\dot{\mathbf{\Phi}}$ match if and only if $\dot{\mathbf{\Phi}}$ and $\delta\mathbf{\Psi}$ commute.

Higher variations and differentials can be obtained through similar techniques. The general rule is: if there is a rotator integral such as $\mathbf{R} = \mathbf{Exp}(\mathbf{\Theta})$, time derivatives can be directly converted to variations. If no integral exists, utmost care must be exerted. Physically quantities such as $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are related to a moving frame, which makes differential relations rheonomic.