Fluid Simulation for Video Games (part 3)

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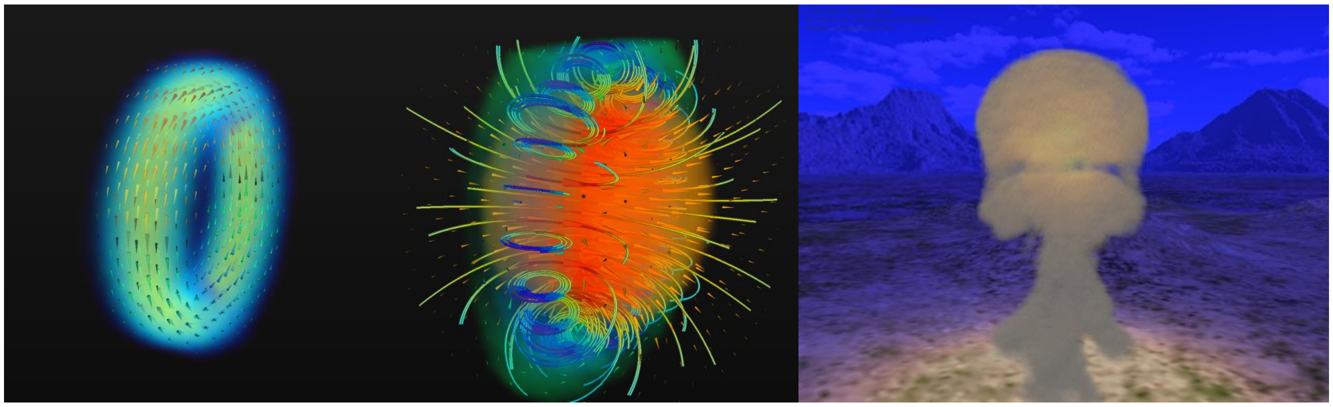

Vortex Particle Fluid Simulation

This article, the third in a series, presents a fluid simulation implemented in C++ that runs in real time using modest and commonly available computer hardware. The first article summarized fluid dynamics; the second surveyed fluid simulation techniques.

The simulation presented here uses vortex particles, called *vortons* by Novikov (1983), to represent the flow field and solves for velocity at each time. This tactic of using vortons preserves vorticity without obvious sources of diffusion, which allows the simulation to retain fine-scaled details. In contrast, other fluid simulation techniques that use primitive variables (velocity and pressure) or grids numerically diffuse vorticity, so the flow tends to look thick and syrupy. When you see the results of *this* simulation, you will be surprised at how much motion detail it preserves, considering how fast it runs.

This simulation also exploits the embarrassingly parallel nature of the algorithms and uses Intel® Threading Building Blocks (TBB) to spread the work across multiple threads.

In the endeavor to achieve real-time fluid motion, some other fluid simulations exploit general-purpose computing on graphics processing unit (GPGPU). However clever, such approaches do not help with current gaming hardware, because in video games, the GPU tends to be busy with *rendering* and has no time left over for simulation. So obtaining real-time speeds using the GPU does not provide a viable solution for current video games. The solution needs to run fast enough to execute in whatever time remains available on the CPU.

Source code accompanies this article, which overviews the concepts behind the simulation, but for the sake of brevity this article stops short of providing a line-by-line commentary of the code. I did, however, strive to comment the code thoroughly, so please see the code for further details.

The sections below describe the steps in implementing a fluid simulation: The first section describes spatial partitioning and discretization, converting from an infinite continuum into a finite discrete problem. The next section describes how to employ that discretization to formulate the fluid dynamics equations into a form suitable for processing on a computer. Finally, you get to see some results aimed at demonstrating the viability of this simulation.

The next article explains two-way fluid-body interactions, and subsequent articles cover performance analysis, optimization, further parallelization, and how to integrate this simulation into a game engine.

Discretization

The second article introduced a taxonomy of simulation techniques that included *grid-based* and *mesh-free* varieties. This simulation uses a mesh-free scheme for representing vorticity and advecting (that is, moving) vortons and a grid-based scheme for interpolating velocity between points, computing spatial derivatives, and searching for nearest neighbors. That makes this simulation a hybrid: vortons (which carry vorticity) and other particles (which facilitate visualization) can move anywhere, but this simulation ascribes other fluid properties (like velocity) only at fixed locations on a uniform grid. In some ways, this simulation resembles a vortex-in-cell simulation as described, for example, by Cottet and Koumoutsakos (2000).

Mesh-free Vortex Particles (Vortons)

Using vortons to represent the flow field provides at least one major benefit over other techniques, such as smoothed particle hydrodynamics (SPH) or Eulerian schemes: The vortons have no obvious unintentional numerical diffusion of vorticity. The motion does not artificially dampen, which allows the flow to have low viscosity and maintain fine-scale details.

SPH and Eulerian schemes suffer dramatically from numerical viscosity, which arises either explicitly (by cranking up viscosity) or implicitly (from the implicit solver), and is a necessary evil that combats numerical instability. (See the discussion of stability in part 2. Also note that those numerically dissipative schemes can use “vorticity confinement” to re-inject motion at small scales, which sometimes yields interesting results. See Steinhoff and Underhill (1994) for more information.)

As described in the part 2, vorticity could be discretized into points (particles, or *vortons*), curves (filaments) or surfaces (sheets or panels). All have their merits, but this article has room for only one, and particles are the simplest. But bear in mind that the other discretizations also have benefits, including dodging numerical diffusion.

Using a Lagrangian (particle) approach also provides the benefit that vorticity can advect outside the original domain, whereas with Eulerian (grid) approaches, allowing fluid to flow out of a domain can be problematic. This means that you do not have to know in advance where the fluid will flow; the vortons simply go where they need to, without any complicated outflow (such as so-called *continuative* or *radiation*) boundary conditions or temperamental regridding schemes.

In this simulation, the C++ class `Vorton` represents a vortex particle. Each vorton has a position and a vorticity. (The `Vorton` class also has other members, explained below.)

Uniform Grid

Although this simulation uses vortex *particles* to carry information about the fluid flow and those particles are free to move anywhere, the simulation also uses a uniform grid to simplify and expedite certain calculations: calculating velocity from vorticity, interpolating velocity values between grid point (or between vortons), and computing spatial derivatives. (Details of those calculations appear in later sections.) This section describes the uniform grid as a kind of container, not entirely unlike a Standard Template Library container.

Most Eulerian simulations rely heavily on uniform grids for storing data and performing computations on that data. Although the simulation presented in this article is primarily Lagrangian, not Eulerian, some of the techniques that apply to Eulerian approaches could also apply to this one. Future articles could explore some ways to hybridize these approaches even further.

This simulation represents uniform grids as a C++ templated `UniformGrid` class, whose template argument is, as is typical for container classes, the data type contained by the class. The `UniformGrid` class inherits from a base class called `UniformGridGeometry`, so let’s start there.

The `UniformGridGeometry` class (which is *not* a templated class—just a regular class) contains information about the shape and size of the region that the uniform grid represents. The constructor provides the means to decide the shape and size automatically given corners of the axis-aligned bounding box and the number of items to be contained. When creating a `UniformGrid` to contain particles, you first need to find the bounding box that contains all particles, and you need to know the total number of particles. Both of those are readily available, so you have what we need. The `UniformGridGeometry` constructor and the `DefineShape` method use that information to create grid geometry suitable to contain information about all the vortons.

A *uniform grid* is a grid in which every cell has the same size as every other. This uniformity makes spatial lookups computationally trivial, because, given a location $\vec{x}$, you can compute the index $\{i_x, i_y, i_z\}$ of the cell that contains that location from

$$i_\alpha = (x_\alpha - X_\alpha)\Delta_\alpha$$

where $\vec{X}$ is the minimum corner of the bounding box, $\vec{\Delta}$ is the size of the bounding box, and the subscript α indicates each of the directions *x, y,* or *z.* The routine `UniformGridGeometry::IndicesOfPosition` computes these indices given a position.

The `UniformGrid` extends `UniformGridGeometry` by associating data with each grid point. Interpolating data values between grid points is straightforward. Given a position, you can find the grid cell that contains it and therefore all grid points that surround that grid cell. Figure 1 shows the relationship between grid points and grid cells. Knowing the eight grid points (and their associated data) that surround a grid cell allows you to linearly interpolate a data value anywhere inside the grid cell. (If you required higher accuracy, you could use higher-order interpolation schemes such as tricubic or even spectral methods that use data from the whole domain, not just adjacent grid points.) The method `UniformGrid::Interpolate` performs this calculation.

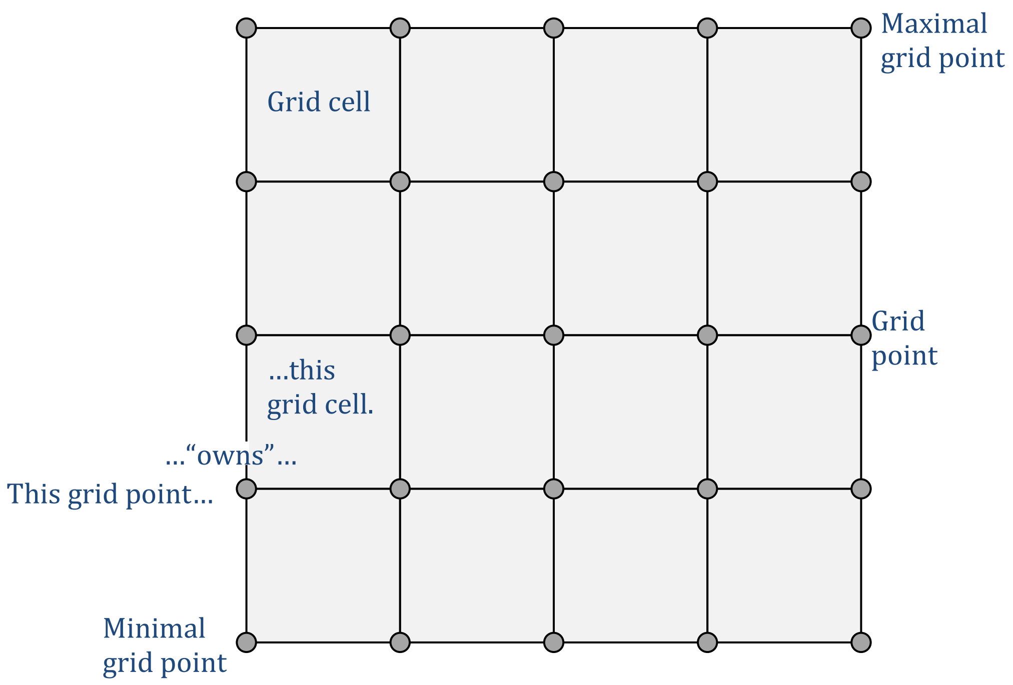

Figure 1. Uniform grid spatial partition

Uniform grids also allow straightforward computation of spatial derivatives using either finite differences or spectral methods. The simulation that accompanies this article uses a centered-difference scheme, as described in the second article, expressed as $\frac{df}{dx} \approx \frac{f(x_{i+1})-f(x_{i-1})}{x_{i+1}-x_{i-1}}$. It turns out that for this simulation, you need to compute all partial derivatives of all components (*x, y,* and *z*) of velocity $\vec{u} \equiv \langle u, v, w \rangle$. In other words, you need each of the following:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \equiv J(\vec{u})$$

This is called the *Jacobian matrix,* and the routine `ComputeJacobian` computes it for any `UniformGrid` that contains a 3-vector quantity, such as velocity.

To compute velocity from vorticity, the simulation effectively integrates vorticity everywhere in space (described in further detail below), but performing that computation in the obvious way would cost too much time. So instead, you use an approximation that relies on structuring data hierarchically. So, in addition to facilitating interpolation and computation of spatial derivatives, `UniformGrid` serves as the basis for a set of nested grids that represents this hierarchy.

Nested Grid

The `NestedGrid` class is another C++ templated container class that builds upon the `UniformGrid` container. This vorton simulation uses `NestedGrid` to aggregate vorticity information to compute velocity from vorticity (as described further below). The concept is straightforward but easier to explain with pictures than words, so look at Figure 2. Think of a `NestedGrid` as a series of layers of `UniformGrid`s, all with the same bounding box. At the finest scale of the `NestedGrid` is the *base layer,* which is the `UniformGrid` that has the largest number of cells and the highest resolution. Above that is another `UniformGrid` that has fewer cells, which I will call the *parent layer.* Each cell in the parent layer represents a cluster of constituent cells in the child layer. Likewise, the parent has another layer above it (and I'll resist the temptation to call it a *grandparent,* but you get the idea), whose cells represent clusters of cells in this parent. *Et cetera.* The top layer in this tree has a single cell, the *root.*

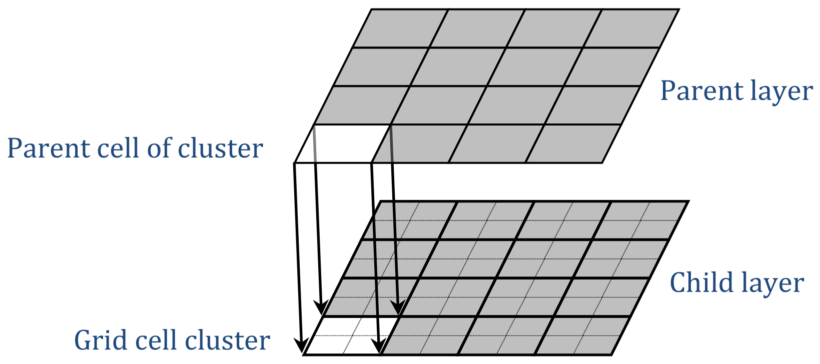

Figure 2. Relationship among layers of UniformGrids in a NestedGrid

The `NestedGrid` has features in common with a complete octree but implemented to reuse the `UniformGrid`. Because this article aims to focus on fluid simulation rather than spatial partitioning algorithms, this implementation seems to introduce fewer distractions, but the intrepid reader might consider experimenting with replacing the `UniformGrid+NestedGrid` combination with other partitions, such as a proper octree.

In this simulation, the `UniformGrid` provides many benefits, whereas `NestedGrid` is a one-trick pony used to compute velocity from vorticity.

The next section digs into the math and algorithms used to evolve the fluid state from one moment to the next.

Fluid Evolution Algorithm

The first article in this series presented the mathematics behind continuum fluid dynamics, and the second article presented how to discretize a continuum so that a computer could simulate fluid motion. The previous section described this discretization strategy: vortons represent vorticity and uniform grids facilitate interpolation and spatial derivatives. This section brings the pieces together and presents a specific algorithm for simulating fluid motion using particles and grids.

This simulation uses a vorticity–velocity formulation, which means it solves the vorticity equation. So, you need to compute each term of the vorticity equation and use a time-integration scheme to evolve the flow. Each subsection below describes how this simulation handles each term in the vorticity equation.

This simulation manifests in the code as the class `VortonSim`.

Velocity from Vorticity

Several terms in the vorticity equation require velocity. Recall from the first article that you can compute flow velocity $\vec{u}$ at position $\vec{x}$ that is a distance $\vec{r} = \vec{x} - \vec{x}'$ from a volume element $d\vec{x}'$ with vorticity $\vec{\omega}$ by integrating over all space:

$$\vec{u}(\vec{x}) = \frac{1}{4\pi}\int \frac{\vec{\omega}(\vec{x}') \times \vec{r}}{r^3} d\vec{x}'$$

If you represented vorticity only at *N* infinitesimal points (that is, the locations of the vortons), the integral becomes a simple summation:

$$\vec{u}(\vec{x}) = \frac{1}{4\pi}\sum_{i=1}^{N} \frac{\vec{\omega_i} \times \vec{r_i}}{{r_i}^3}$$

where the subscript *i* indicates the vorton index. The `VortonSim::ComputeVelocityBruteForce` method implements this direct summation algorithm. Computing this at every vorton requires $O(N^2)$ operations, which would become excessively expensive for more than a very small number of vortons.

This formula has a problem: When the distance from a vorton r_i is very small, the contribution due to that vorton becomes very large, and as r_i approaches zero, the contribution approaches infinity. This is called a *singularity,* and it causes numerical problems. To avoid those problems, give each vortex a tiny size, σ. Outside of that region, vorticity is zero, and the velocity field acts as though it came from a point vorton, resembling the simple formula above. Inside that region, however, vorticity drops so that the velocity approaches zero linearly as the radius approaches zero:

$$\vec{u} = \frac{1}{4\pi}\sum_{i=1}^{N} \vec{\omega_i} \times \vec{r_i}\, \Delta V_i \begin{cases} {r_i}^{-3} : r \geq \sigma \\ {\sigma_i}^{-3} : r \leq \sigma \end{cases}$$

Here, ΔV_i is the volume of the vortex element, which is a tiny blob instead of a zero-sized point, but you still call it a vorton. This model resembles that of a so-called *Rankine vortex.* The resulting vorticity is said to be *mollified,* and avoiding the singularity in this way is called *regularization.* The `Vorton::AccumulateVelocity` method implements the formula inside the summation. Problem solved.

To reduce the computational complexity of this direct summation, take advantage of the fact that when “felt” from far away, the influence of a cluster of vortons “feels” approximately the same as a single “supervorton” whose circulation is the same as the sum of circulation of all the vortons in the cluster. Figure 3 offers a picture of this phenomenon.

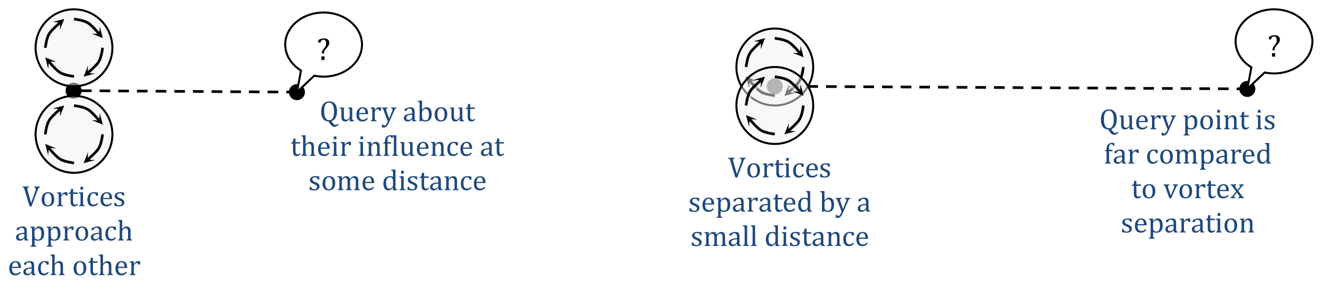
?
Query about their influence at some distance
Vortices approach each other
?
Vortices separated by a small distance
Query point is far compared to vortex separation

Figure 3. Clusters of distant vortons “feel” almost the same as a single supervorton.

You put this idea into practice by creating clusters of vortons, then creating clusters-of-clusters, and so on, until the entire field of vortons is represented by a single cluster, as depicted in Figure 4. I refer to this tree of vorton clusters as the *influence tree,* and this is where the `NestedGrid` comes into play.

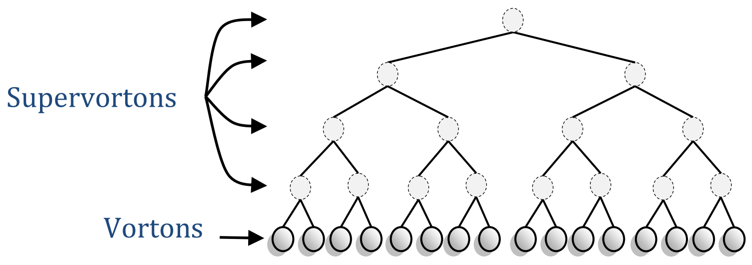

Figure 4. Vorton cluster influence tree. Each ancestor node represents its constituents as a single supervorton.

The influence tree is a `NestedGrid` of vortons, and the `VortonSim` class contains a member named `mInfluenceTree` that is exactly this. The method `CreateInfluenceTree` constructs the influence tree in two phases: building the base layer and aggregating clusters.

Each layer in the tree is a `UniformGrid` of information. Each grid cell contains a vorticity and a position, stored as a representative “supervorton.” The vorticity of a supervorton is the aggregate vorticity of all constituent vortons in that cell, and its position $\vec{x}_\Sigma$ is the “center of vorticity,” which is the average position of all constituent vortons in the cluster, where each vorton position $\vec{x}_i$ is weighted by ω_i, the magnitude of its vorticity:

$$\vec{x}_\Sigma = \sum_{i=1}^{N} \omega_i \vec{x}_i \, .$$

So, stronger constituent vortons have more influence over the position of the supervorton that represents them.

The method `MakeBaseVortonGrid` creates the base layer. It copies information from the array of vortons into a `UniformGrid`, obeying the rules that apply to aggregation. This routine effectively takes an array of vortons and turns them into a uniform grid of supervortons.

The method `AggregateClusters` aggregates vorticity information of clusters of cells in the child layer into cells in the parent layer that represent those clusters as a single supervorton.

Note Astute readers might notice that each layer in the influence tree includes information that is not used, such as the velocity of each cluster. It seems as though we should separate the vorton into two classes: one with velocity and another without. The apparently extraneous velocity member will come in handy, however, in an aggressive optimization that will appear in a future article.

The influence tree represents vortons and clusters of vortons, where each cell contains at most one supervorton. You use that tree to compute velocity from vorticity. That process entails a straightforward recursive tree traversal, making this algorithm a so-called *tree code,* as described by Barnes and Hut (1986), although they formulated their algorithm for gravitational problems like galaxy formation, and they constructed their tree in subtly different ways.

To compute velocity at a given point, traverse the influence tree as follows (and as shown in Figure 5): Start at the root of the influence tree. For each octant in the tree, if the octant does not contain the query point (or if this is a leaf node), then compute the contribution to velocity due to the sole supervorton in that octant as though it were an ordinary vorton. Otherwise, if the octant does contain the query point, descend the tree at that octant. That’s it!

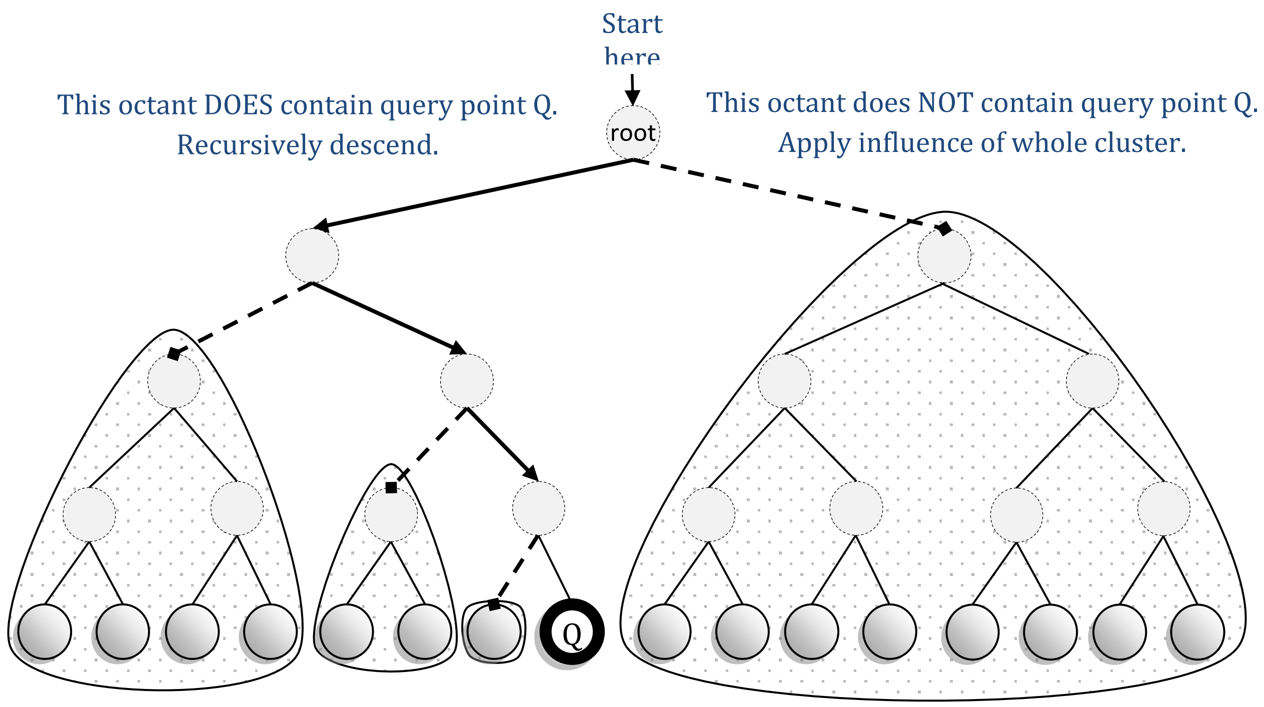
Start here
This octant DOES contain query point Q. Recursively descend.
root
This octant does NOT contain query point Q. Apply influence of whole cluster.
Q

Figure 5. Influence tree traversal for obtaining velocity at query point Q

As a refinement, let us reconsider the descent criterion. The simplest is the one described above: Only descent into octants that contain the query point. You can improve accuracy by augmenting that criterion to be "descend into octants that are within a threshold distance of the query point." Then, you can tune that "threshold distance" to obtain the desired accuracy. If you made the threshold distance very large, then the algorithm effectively devolves into the $O(N^2)$ brute-force direct summation algorithm. For visual effects, however, you can turn that threshold down to zero and obtain acceptable results.

Another detail remains—and this is where you depart from a purely Lagrangian treatment. It would seem natural to evaluate velocity at each vorton and advect the vortons accordingly. In fact, this is a traditional approach, and it works just fine. But this simulation does something different.

The `VortonSim::ComputeVelocityGrid` method evaluates velocity at each point on a grid, and then routines that need velocity interpolate values from the grid. At first glance, this interpolation step might seem extraneous, not to mention that by interpolating you reduce the precision of the velocity calculation. But having velocity on a grid turns out to provide benefits that outweigh this drawback.

Interpolation on a uniform grid is much faster than computing velocity using the tree—perhaps 10 times faster or more. The `UniformGrid::Interpolate` method performs this computation. Because several routines need to compute velocity at specific locations, using the grid as an intermediary speeds up access to velocity compared to computing it using the influence tree.

Also, creating a velocity grid allows you to vary the resolution of the velocity grid independently of the number of vortons. So if you wanted to, you could intentionally reduce the resolution of the velocity grid to gain more speed at the cost of accuracy. Or vice versa.

Finally, computing velocity on a grid simplifies the calculation of spatial derivatives, which you need to compute other terms in the vorticity equation, such as vortex stretching and tilting.

Vortex Stretching and Tilting

The stretching and tilting term of the vorticity equation (described in the first two articles and expressed as $(\vec{\omega} \cdot \vec{\nabla})\vec{u}$) involves spatial derivatives—specifically, the gradients of velocity. You can expand that term to obtain

$$\omega_x \frac{\partial \vec{u}}{\partial x} + \omega_y \frac{\partial \vec{u}}{\partial y} + \omega_z \frac{\partial \vec{u}}{\partial z} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = J(\vec{u})\vec{\omega}\,.$$

In other words, you transform the vorticity of every vorton by the Jacobian matrix of velocity. You compute the Jacobian at every point on the velocity grid (using `ComputeJacobian`), then interpolate the Jacobian at each vorton. Fortunately, because you wrote `UniformGrid` as a template, the `Interpolate` method works identically for both vectors and matrices, so you do not need to write additional code. You simply need to evaluate the stretching and tilting formula at each vorton, and then update the vorticity of each vorton accordingly using some time integration scheme such as the explicit Euler method. The `VortonSim::StretchAndTiltVortons` method accomplishes this.

Note The stretching and tilting term can accumulate errors in vorticity in the form of an irrotational component; vorticity can accumulate divergence. Mathematically, it is impossible for the curl of a field to have divergence. Because vorticity is the curl of velocity, vorticity should not have divergence. But numerically, divergence occurs as a result of round-off and truncation errors. This divergence turns out not to have an immediate impact on the velocity, but it does affect the stretching and tilting term, so it would be nice to eliminate it. There are clever ways to do this by modifying the stretching and tilting computation, and a future article will revisit this topic. For now, suffice it to say that because the goal is to make eye candy, as long as the simulation does not explode, you can squeak by with this inaccuracy—at least for one or two articles.

Diffusion: Particle Strength Exchange

Viscous vortex diffusion effectively smears vorticity. Mathematically, it has the form $\nu \nabla^2 \vec{\omega}$, where ν is viscosity and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator, which is a second spatial derivative. Computing this with finite differences would be problematic, but fortunately there is another way. Effectively, this operator behaves like a blurring operator: It spreads vorticity across space. So, you can implement this as an exchange of vorticity between vortons.

The mathematical derivation, omitted for brevity, entails replacing the differential operator with an integral one. Here, you drastically simplify the approach, boiling it down to its qualitative essentials: Vortons exchange a fraction of their vorticity with other vortons in their vicinity.

This process requires knowing the nearest neighbors for each vorton. A common solution to that problem entails using a spatial partition such as an octree or kD tree and performing a nearest-neighbor search, but that solution costs $O(\log N)$ time per search. In contrast, this simulation simplifies and accelerates this search by reusing `UniformGrid`, where the contained data at each grid point is an array of indices into the array of vortons—that is, `vector<int>`. So, each grid cell can contain references to multiple vortons. After populating the search grid with vorton references, each vorton finds its nearest neighbors as depicted in Figure 6.

When looking for nearest neighbors, each vorton only *seeks* other vortons either in its own cell, or in adjacent cells "in front of" its own cell.

This works out, because vortons "behind" it seek it, and after a vorton finds its neighbors, they talk to each other both ways.

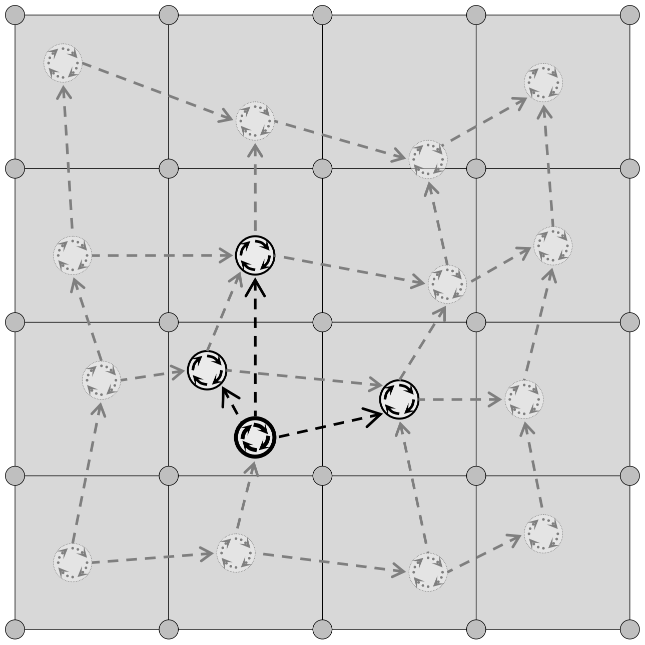

Figure 6. Nearest neighbors of a vorton

In a fluid with viscosity parameter ν, each vorton p then exchanges a portion of its vorticity $\vec{\omega}_p$ with its N neighbors, indexed by *i,* using the formula

$$\frac{d\vec{\omega}_p}{dt} = \nu \sum_{i=1}^{N} (\vec{\omega}_i - \vec{\omega}_p)$$

The method `VortonSim::DiffuseVorticityPSE` implements this formula by considering pairs of vortons, in which case you can rewrite the formula as

$$\frac{d\vec{\omega}_1}{dt} = \nu(\vec{\omega}_2 - \vec{\omega}_1)$$
$$\frac{d\vec{\omega}_2}{dt} = \nu(\vec{\omega}_1 - \vec{\omega}_2) = -\frac{d\vec{\omega}_1}{dt}\,.$$

This form shows that this exchange preserves total circulation (assuming that the particles have the same size, which, in this simulation, they do).

Note To the extent that you care about physical accuracy, note that this formulation oversimplifies the situation. For example, it neglects the distance between vortons and also their size, which is related to the spacing between them. Among other things, that effectively means that the parameter ν is not exactly the same as kinematic viscosity. Readers curious to know more should read a detailed treatment, including a derivation of this approach, in Degond and Mas-Gallic (1989).

Advection

The vorticity equation includes an advective acceleration term, which the first article describes in detail as the nonlinear term that accounts for the characteristic motion that makes a fluid look like a fluid. In the Lagrangian form of the vorticity equation, this term basically means that vortons follow the flow—that is, they move according to the velocity of the fluid, local to their current location. So, you need to know the velocity at the location of each vorton, and the velocity grid gives you that.

Using the velocity grid, the `VortonSim::AdvectVortons` method interpolates to obtain velocity at each vorton and moves the vorton accordingly (for example, using explicit Euler). Fast interpolation using the velocity uniform grid allows you to advect other particles—call them *passive tracers*—with ease. Indeed, `VortonSim::AdvectTracers` does just that. You can then populate the simulation with many of these passive tracers and render them or use them for other purposes, such as approximating aerodynamic drag by colliding them with rigid bodies, which the next article covers.

Parallelization

Many of the routines in this simulation are embarrassingly parallel, meaning that the operations and data of the calculations are independent of each other. This begs for data parallelism. This simulation code uses Intel's Threading Building Blocks (TBB) to parallelize certain routines. The TBB construct "`parallel-for`" is well-suited for the kind of data parallelism that pervades this simulation. TBB makes it easy to parallelize code without having to make numerous sacrifices to the computing gods.

The routines that have a good combination of independence and slow run times include `CreateVelocityGrid` and `FillVertexBuffer`. Each of these therefore uses TBB's `parallel_for` to run across multiple threads. Each of those routines has an associated helper routine that operates on a subset (which the code calls a *slice*) of the total data, and then uses a function object, which is an object that overrides `operator()`, to inform TBB how to execute the process.

In the case of `CreateVelocityGrid`, each slice includes a subset of the *Z* values in the volume of the velocity grid. Because assigning each grid point of the velocity requires no information from any other part of the velocity grid, no synchronization is required.

In the case of `FillVertexBuffer`, each slice includes a subset of the particles. Because this simulation uses a large number of passive tracer particles, this turns out to be a natural fit for data parallelism.

Ironically, the slowest portion of this simulation is rendering the particles – which takes more than half the time. This could be sped up by using programmable shaders or point sprites, but that lies outside the scope of these articles. A future article will, however, use VTune to profile the routine which fills the vertex buffer, which leads to significant optimization.

The total amount of code required to convert these serial routines to a parallel version was around a dozen lines of code. Very little of the original code changed at all, and those changes were trivial in the sense that the algorithm remained the same. The original code, which was a loop, simply changed so that the begin and end points of the loop were driven by formal parameters (i.e. function arguments) instead of being hard-coded. Then, for each of those routines, a very simple class was written to wrap calling those functions, in a manner consistent with the TBB specification. Article 4 provides additional details, and the book *Intel Threading Building Blocks: Outfitting C++ for Multi-core Processor Parallelism* by James Reinders explains this simple process in ample detail.

Results

It is customary to demonstrate the validity of a simulation algorithm by applying it to various well-known problems and comparing the results to expected solutions. For the sake of brevity and following the tradition established in the computer graphics and animation community, this article forgoes rigorous analysis and instead relies on visual inspection to validate the results.

You might think that because this simulation uses vorticity, you should specify the initial conditions directly in the form of a vorticity field. Not so. The preferred way to establish initial conditions for any simulation that uses a vorticity–velocity formulation is to formulate the *velocity* field you want, then take the *curl* of that field and initialize vorticity that way. The routines `ComputeCurlFromJacobian` and `VortonSim::AssignVortonsFromVorticity` facilitate this approach. This extra step prevents the initial vorticity field from having an irrotational component (that is, from having divergence). Be aware, however, that some of the test cases presented here were created by specifying vorticity directly, and the initial vorticity does have divergence. This is intentional, because although I claim that you want to exclude divergence from the initial conditions, the simulation ought to be able to handle its presence.

Let's look at some scenarios:

- A vortex ring, which self-propagates
- A pair of crossed vortex tubes, which causes drastic stretching, allowing you to see stretching occur in the expected place and direction

First, let's look at the vortex ring. If you compute velocity from the vorticity of the vortons, then compute the curl of the velocity, you should recover the original vorticity field. Figure 8 shows the results. To the extent that (a) and (b) resemble each other, the velocity solver has succeeded.

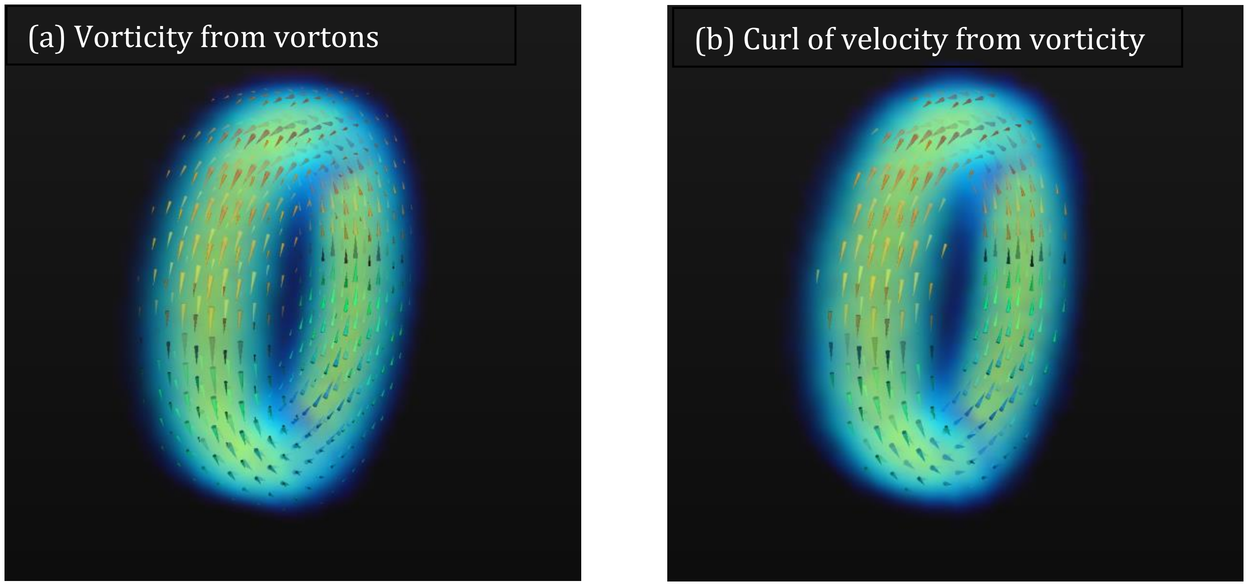
(a) Vorticity from vortons
(b) Curl of velocity from vorticity

Figure 8. Vortex ring (a) vorticity from vortons and (b) curl of velocity field recovered from vorticity using fast approximation. Arrows indicate vorticity. The colored cloud indicates vortex intensity.

Next, look at the velocity field induced by vortons of the vortex ring, computed in two different ways: direct summation (slow, exact) and the influence tree (fast, approximate).

Direct summation makes no approximations: Its only error comes from discretization itself. In contrast, the faster influence tree approximation also contains truncation errors. Once again, the extent to which the images in Figure 9 resemble each other indicates a degree of success for the fast velocity solver.

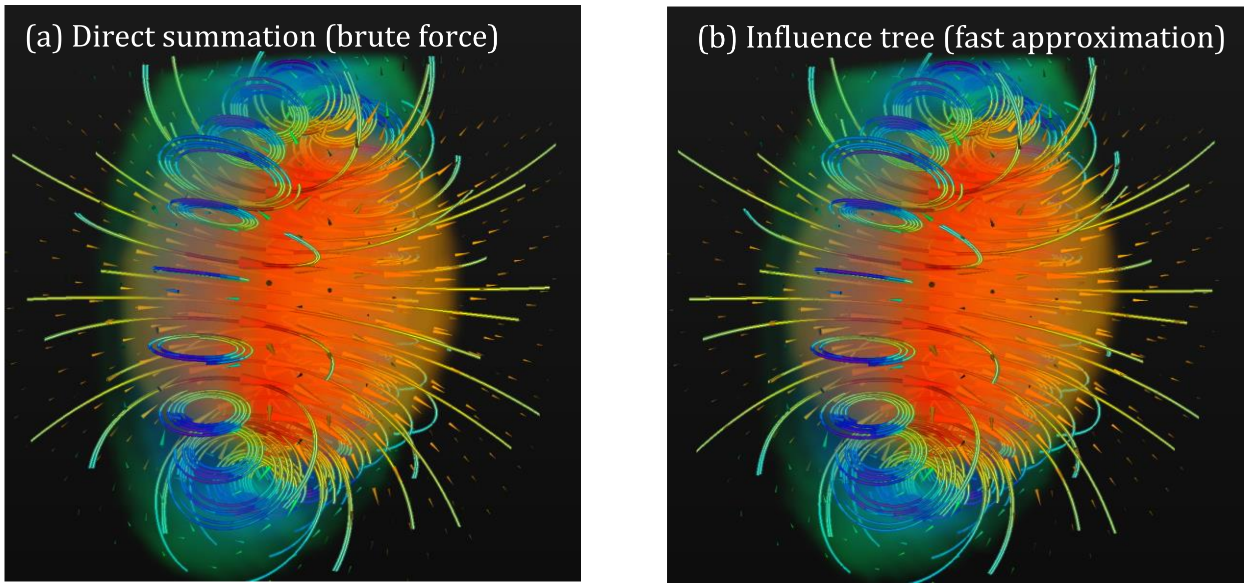
(a) Direct summation (brute force)
(b) Influence tree (fast approximation)

Figure 9. Velocity field recovered from vorticity of vortex ring using (a) direct summation and (b) the influence tree approximation. Arrows indicate velocity flow field. The orange/red cloud indicates velocity moving along +X, and the blue/green cloud indicates velocity moving along –X. Lines indicate streamlines—that is, paths particles would take as they follow the flow.

Figure 10 depicts the vortex ring rendered as particles as it evolves over time.

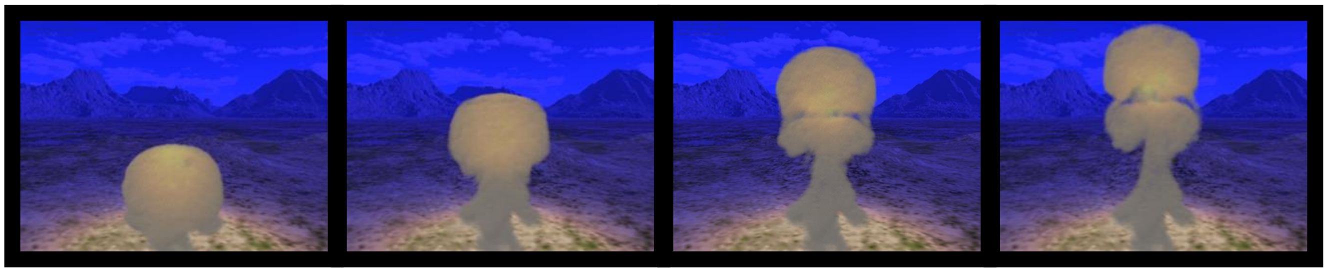

Figure 10. Vortex ring rising, rendered using particles. The simulation runs and renders faster than 30 frames per second.

The next test scenario entails a pair of crossed vortex tubes. These do not show up much in ordinary life, so it's difficult to appeal to them as an immediately obvious validation, but they have the property that the vortex stretching they induce is distinctive. In fact, among computation fluid dynamics researchers, the cross-tubes scenario is quite challenging, because it can result in creating a vortex of infinite strength, in finite time. It's heady stuff, and this article glosses over it, but to the scrutinizing observer, the images in Figure 11 indicate that stretching does indeed occur in the expected configuration.

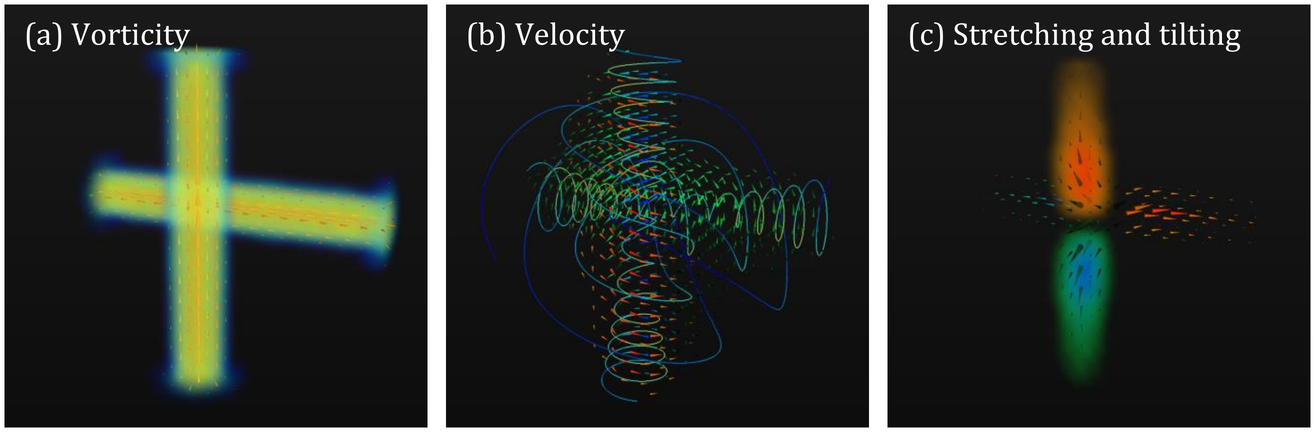

Figure 11. Crossed vortex tubes (a) vorticity, (b) velocity, and (c) vortex stretching and tilting

Although these results do not prove that this simulation is correct, they demonstrate that the code produces plausible and appealing results at interactive speeds, and for visual effects in video games, that can suffice. I encourage you to take a closer look at the routines in the simulation code that compute and track quantities that should be conserved, such as total circulation, hydrodynamic impulse, and angular impulse, and experiment with them.

It would be natural at this stage to report performance and compare that with other techniques, and indeed future articles in this series will focus on performance analysis and comparisons. For now, I simply report that these simulations run and render faster than 30 frames per second on a modest laptop using a single 2.5 GHz core of an Intel® Core™2 Duo processor and faster than 60 frames per second—with only 35% CPU utilization—on a machine with dual 2.33 GHz Quad-Core Intel® Xeon® processors. Most of that time is spent waiting on the GPU to finish rendering, which has nothing to do with the algorithms presented in this article. So, this simulation runs fast. And by the end of this series of articles, it will run even faster.

Summary

This article presents a vortex particle treecode that uses a nested grid to compute velocity from vorticity and uses uniform grids to compute spatial derivatives for stretching and tilting. The uniform grid also expedites finding nearest neighbors to compute viscous diffusion of vorticity using a particle strength exchange scheme. The simulation transports vortex particles to account for advective acceleration, the nonlinear term that accounts for the motion that makes a fluid move like a fluid. Finally, this simulation exploits the embarrassingly data-parallel nature of the algorithms and uses Intel® TBB `parallel-for` construct to spread the computational cost across multiple threads.

The resulting simulation runs and renders all of its demonstration cases faster than 30 frames per second on a modest laptop and faster than 60 frames per second on a multi-core desktop computer.

This simulation code resembles a hybrid between a traditional tree code and a vortex-in-cell scheme. You could modify this code to bridge that gap by using an off-the-shelf Poisson solver to obtain velocity from vorticity. You can also explore using different stretching and tilting formulae to mitigate problems that arise because of a spurious divergence in the vorticity field.

The next article explains how to satisfy boundary conditions and support two-way fluid-body interaction, and subsequent articles cover performance analysis, optimization and integration into a game engine.

[BEGIN SIDEBAR]

Further Reading

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