2η Σειρά Ασκήσεων

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$a = AM \bmod 10 + 1 = 6 + 1 = 7$

2.1) $x_1(t) = \sin(20a\pi t)\dfrac{\cos(300\pi t)}{\pi t} = \sin(140\pi t)\dfrac{\cos(300\pi t)}{\pi t}$

$x_2(t) = 2\dfrac{\sin(500\pi t)}{\pi t}$

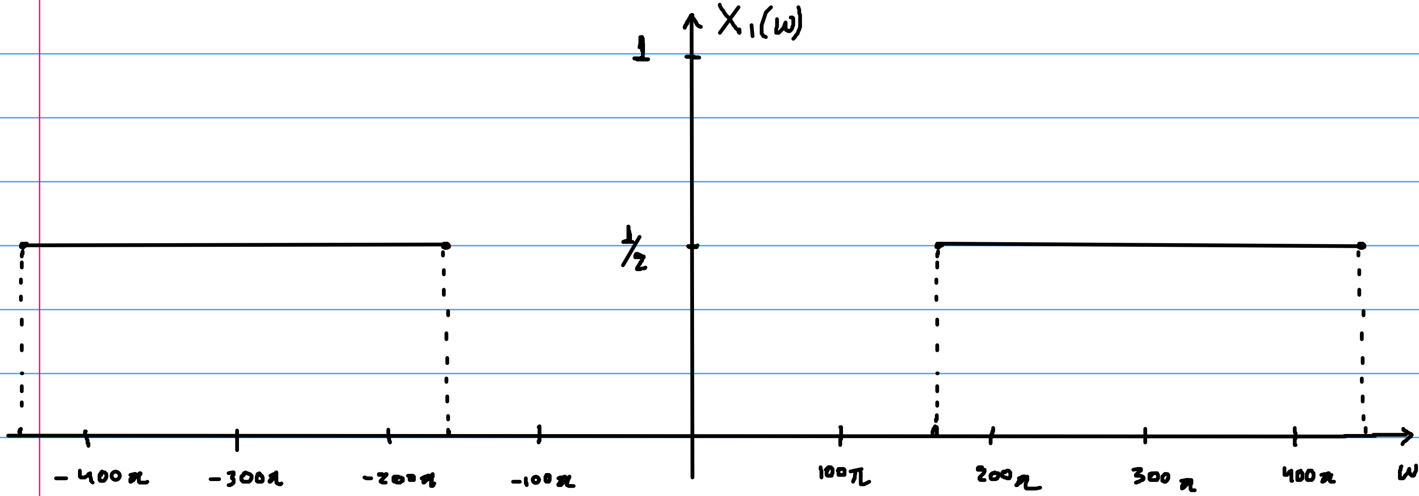
$x_1(t) = 2\sin(140\pi t)\cos(300\pi t)\cdot\dfrac{1}{2\pi t} =$

$= \left[\sin\left(140\pi t + 300\pi t\right) + \sin\left(140\pi t - 300\pi t\right)\right]\cdot\dfrac{1}{2\pi t} =$

$= \dfrac{\sin 440\pi t}{2\pi t} - \dfrac{\sin 160\pi t}{2\pi t}$

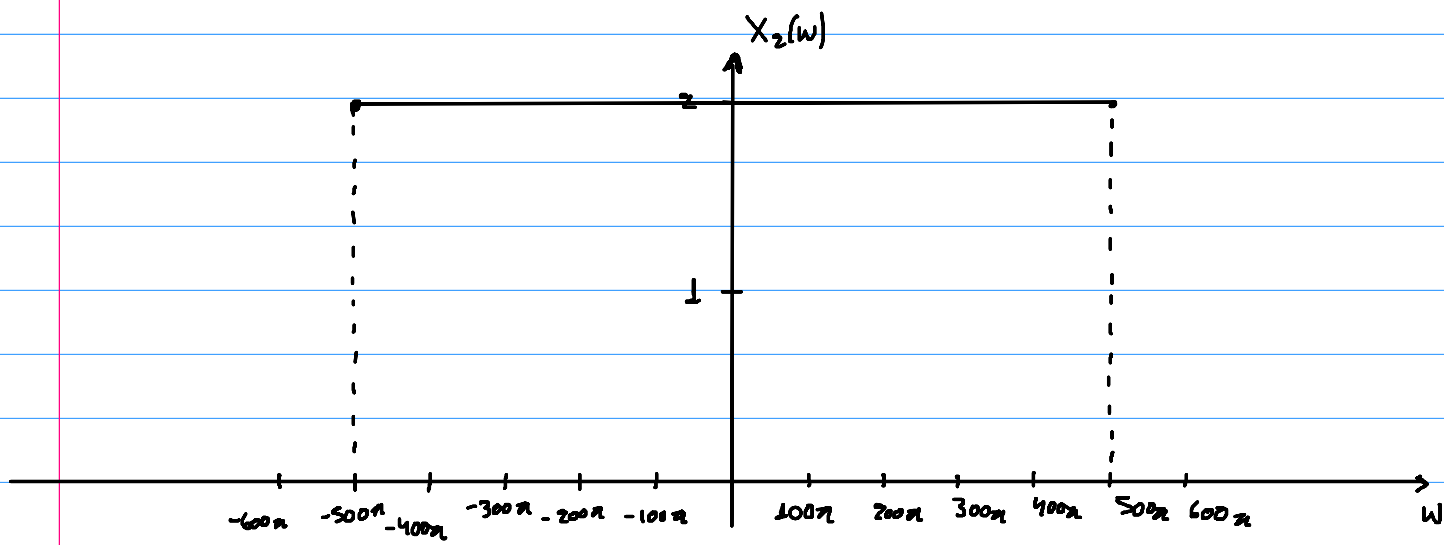
$X_1(\omega) = F[x_1(t)] = \dfrac{1}{2}F\left[\dfrac{\sin 440\pi t}{\pi t}\right] - \dfrac{1}{2}F\left[\dfrac{\sin 160\pi t}{\pi t}\right]$

$$X_1(\omega) = \begin{cases} 0, & \omega \in (-\infty, -440\pi) \cup (-160\pi, 160\pi) \cup (440\pi, +\infty) \\ \dfrac{1}{2}, & \omega \in (-440\pi, -160\pi) \cup (160\pi, 440\pi) \end{cases}$$



$$X_2(w) = F[x_2(t)] = 2F\left[\frac{\sin(500\pi t)}{\pi t}\right] =$$

$$= \begin{cases} 0, & w \in (-\infty, -500\pi) \cup (500\pi, +\infty) \\ 2, & -500\pi < w < 500\pi \end{cases}$$

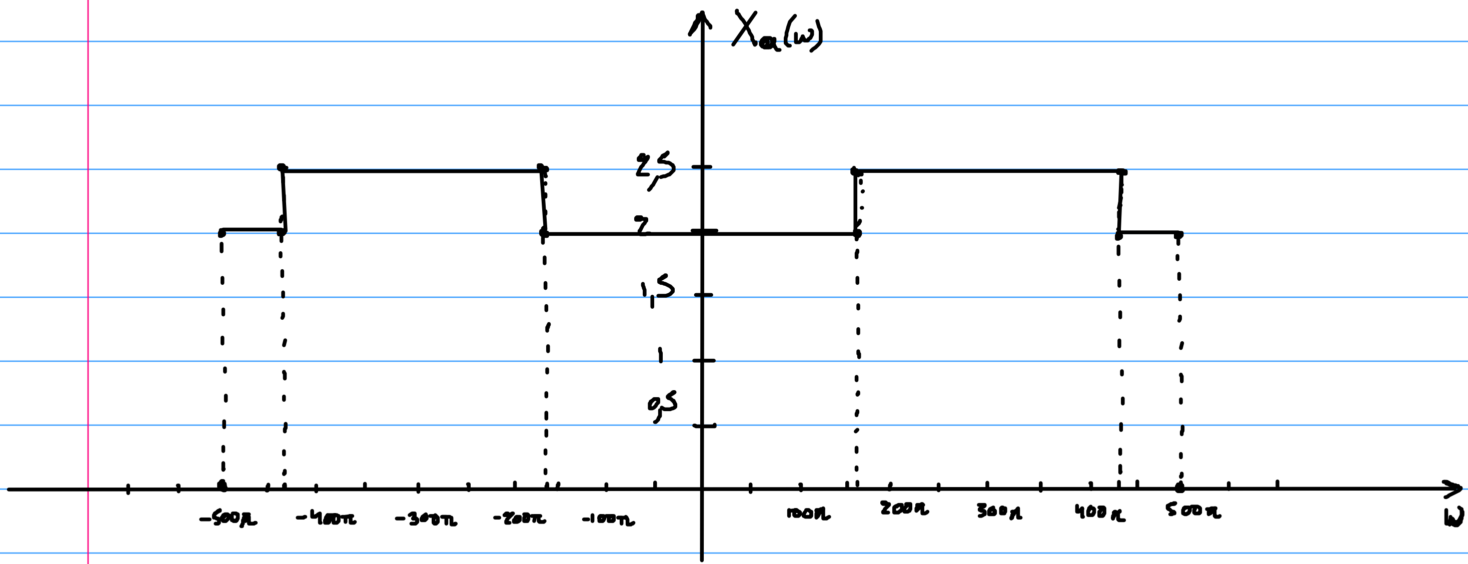


a) $x_1(t) + x_2(t)$

$$X_a(w) = F[x_1(t) + x_2(t)] = F[x_1(t)] + F[x_2(t)] =$$

$$= X_1(w) + X_2(w) =$$

$$\begin{cases} 0, & w \in (-\infty, -500\pi) \cup (500\pi, +\infty) \\ 2, & w \in (-500\pi, -440\pi) \cup (-160\pi, 160\pi) \cup (440\pi, 500\pi) \\ \frac{5}{2}, & w \in (-440\pi, -160\pi) \cup (160\pi, 440\pi) \end{cases}$$



$$w = 1000\pi \Rightarrow w_s > 2w \Rightarrow \frac{\cancel{2}\pi}{T_s} > \cancel{2}w \Rightarrow T_s < \frac{\pi}{w} = \frac{\pi}{1000\pi} = 10^{-3} s$$

b) $x_b(t) = x_1^3(t) = x_1(t) \cdot x_1^2(t)$

$$x_1^2(t) \Rightarrow F[x_1^2(t)] = X_1(w) * X_1(w),$$

$$w_{x_1^2} = w_{x_1} + w_{x_1} = 880\pi + 880\pi = 1760\pi$$

$$X_b(w) = X_1(w) * F[x_1^2(t)],$$

$$w = w_{x_1} + w_{x_1^2} = 880\pi + 1760\pi = 2640\pi$$

$$w_s > 2w \Rightarrow \frac{\cancel{2}\cancel{\pi}}{T_s} > \cancel{2}w \Rightarrow T_s < \frac{2640\cancel{\pi}}{\cancel{\pi}} = 2640\,s$$

γ) $x_\gamma(t) = x_1(t) * x_2(t)$

$X_\gamma(w) = X_1(w) \cdot X_2(w)$

$w = \min\{w_1, w_2\} = \{880\pi, 1000\pi\} = 880\pi$

$$w_s > 2w \Rightarrow \frac{\cancel{2}\cancel{\pi}}{T_s} > \cancel{2}\cdot 880\cancel{\pi} \Rightarrow T_s < 880\,s$$

δ) $x_\delta(t) = x_1(t)\cdot[x_2(t)]^2 + [x_1(t)]^2 * x_2(t)$

$$X_\delta(w) = F\left[x_1(t)\cdot[x_2(t)]^2\right] + F\left[[x_1(t)]^2 * x_2(t)\right]$$

$$w = \max\{w_1 + 2w_2, \min\{2w_1, w_2\}\} =$$

$$= \max\{880\pi + 2000\pi, \min\{1760\pi, 1000\pi\}\} =$$

$$= \max\{2880\pi, 1000\pi\} = 2880\pi$$

$$w_s > 2w \Rightarrow \frac{\cancel{2}\pi}{T_s} > \cancel{2}w \Rightarrow T_s < 2880\,s$$

2.9) α) $\omega_0 = \frac{1}{2}(\omega_1+\omega_2)$

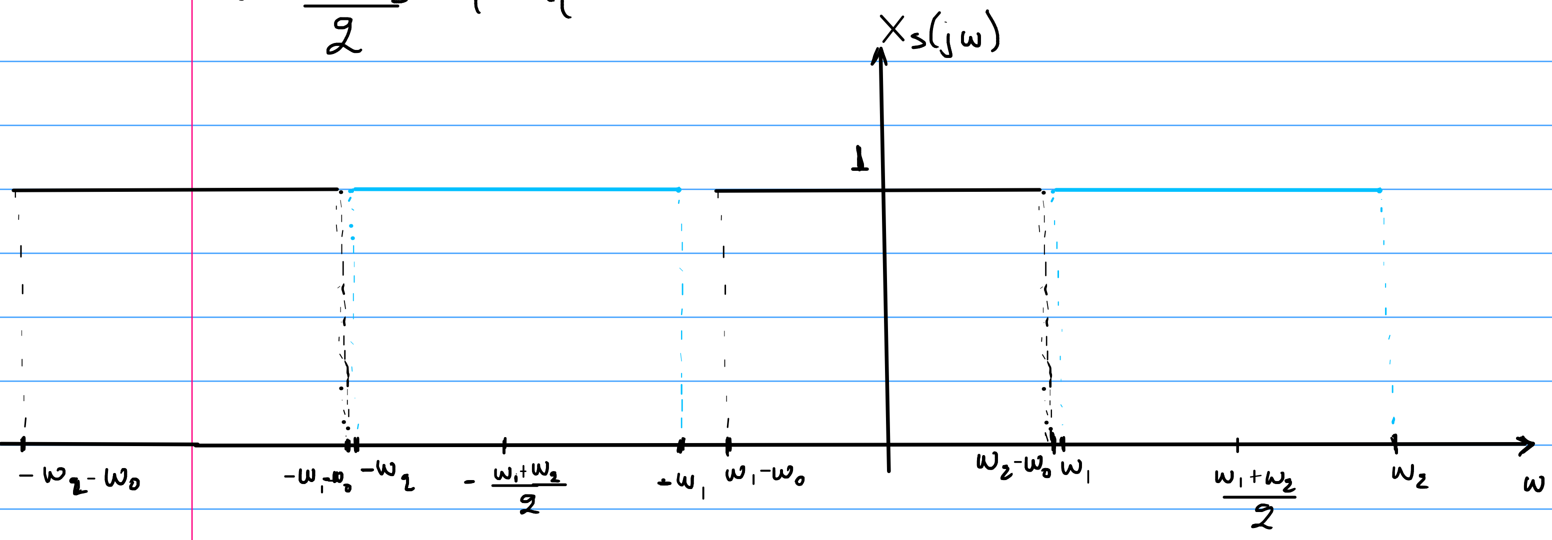
$$H(j\omega) = \begin{cases} 1, & |\omega| \le \frac{1}{2}(\omega_2-\omega_1) \\ 0, & \text{αλλού} \end{cases} = -\frac{1}{2}(\omega_2-\omega_1) \le \omega \le \frac{1}{2}\omega_2-\omega_1 \Rightarrow$$

$$\frac{\omega_1-\omega_2}{2} \le \omega \le \frac{\omega_2-\omega_1}{2}$$

$$X(j\omega) = \begin{cases} 1, & \omega \in [-\omega_2,-\omega_1] \cup [\omega_1,\omega_2] \\ 0, & \text{αλλού} \end{cases}$$

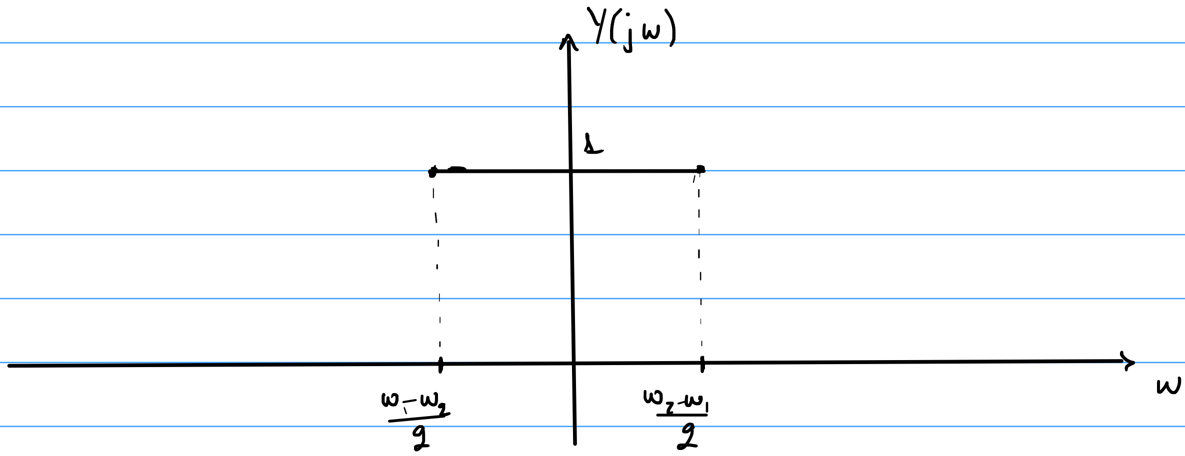
$$x_s(t) = x(t)e^{-j\omega_0 t} \xleftrightarrow{F} X_s(j\omega) = X(j\omega + j\omega_0) = X(j(\omega+\omega_0))$$

Άρα το $X_s(j\omega)$ είναι το $X(j\omega)$ μετατοπισμένο κατά $\omega_0 = \frac{\omega_1+\omega_2}{2}$ αριστερά.



$$X_s(j\omega) = \begin{cases} 1, & \omega \in \left(-\frac{\omega_1+3\omega_2}{2}, -\frac{3\omega_1+\omega_2}{2}\right) \cup \left(\frac{\omega_1-\omega_2}{2}, \frac{\omega_2-\omega_1}{2}\right) \\ 0, & \text{αλλού} \end{cases}$$

$$Y(j\omega) = X_s(j\omega)\cdot H_s(j\omega) = \begin{cases} 1, & \omega \in \left(\frac{\omega_1-\omega_2}{2}, \frac{\omega_2-\omega_1}{2}\right) \\ 0, & \text{αλλού} \end{cases}$$



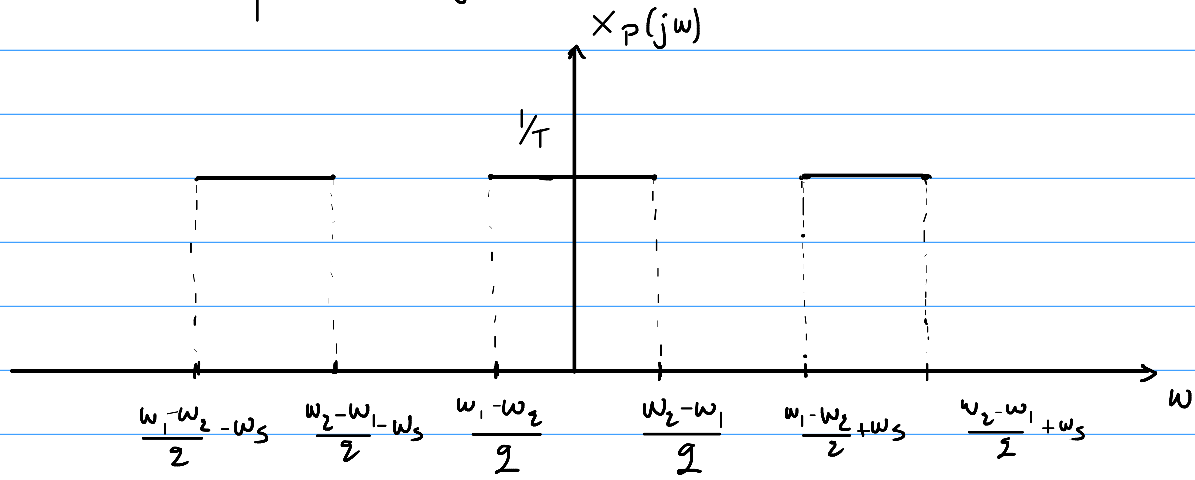
$$x_p(t) = y(t) \cdot p(t)$$

$$P(jw) = F\left[\sum_{n=-\infty}^{+\infty} \delta(t-nT)\right] = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(jw - kjw_s)$$

$$X_p(jw) = Y(jw) * P(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\theta)\cdot \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\theta - kjw_s)\,d\theta =$$

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} Y(jw - kjw_s)$$

Άρα το $X_p(jw)$ αποτελείται από επαναλαμβανόμενα $Y(jw)$ με πλάτος $\frac{1}{T}$ του $Y(jw)$.



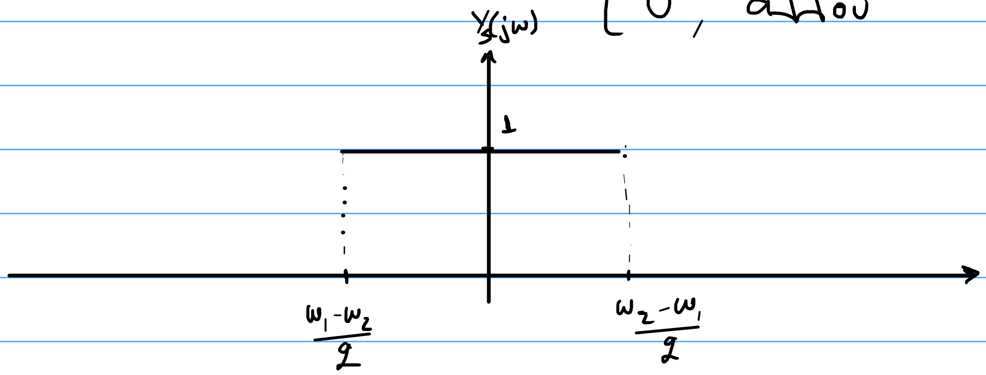
β) Πρέπει $\omega_s \geq \frac{\omega_2-\omega_1}{2} - \frac{\omega_1-\omega_2}{2} = \frac{2\omega_2-2\omega_1}{2} = \omega_2-\omega_1$

$\frac{2\pi}{T_s} \geq \omega_2-\omega_1 \Rightarrow T_s \leq \frac{2\pi}{\omega_2-\omega_1}$ άρα $T_{s_{max}} = \frac{2\pi}{\omega_2-\omega_1}$

γ) Έχουμε $H_s(j\omega) = \begin{cases} T_{s_{max}}, & |\omega| \leq \frac{\omega_2-\omega_1}{2} \\ 0, & \text{αλλού} \end{cases}$

$x_p(t) \longrightarrow \boxed{h_s(t)} \longrightarrow y_s(t)$

$Y_s(j\omega) = X_p(j\omega)\cdot H(j\omega) = \begin{cases} 1, & |\omega| \leq \frac{\omega_2-\omega_1}{2} \\ 0, & \text{αλλού} \end{cases}$



$Y_s(j\omega) = H(j\omega)\cdot X_s(j\omega) \Rightarrow X_s(j\omega) = \frac{Y_s(j\omega)}{H(j\omega)} = \begin{cases} 1, & |\omega| \leq \frac{\omega_2-\omega_1}{2} \\ \text{αυθαίρετο αλλού} \end{cases}$

Όμως $x_s(t) = e^{-j\omega_0 t}x(t) \Rightarrow x(t) = e^{j\omega_0 t}x_s(t) \overset{F}{\longleftrightarrow}$

$X(j\omega) = X_s(j(\omega-\omega_0))$

Άρα $X(j\omega) = \begin{cases} 1, & |\omega-\omega_0| \leq \frac{\omega_2-\omega_1}{2} \Rightarrow \omega_1 \leq \omega \leq \omega_2 \\ \text{αυθαίρετο αλλού} \end{cases}$

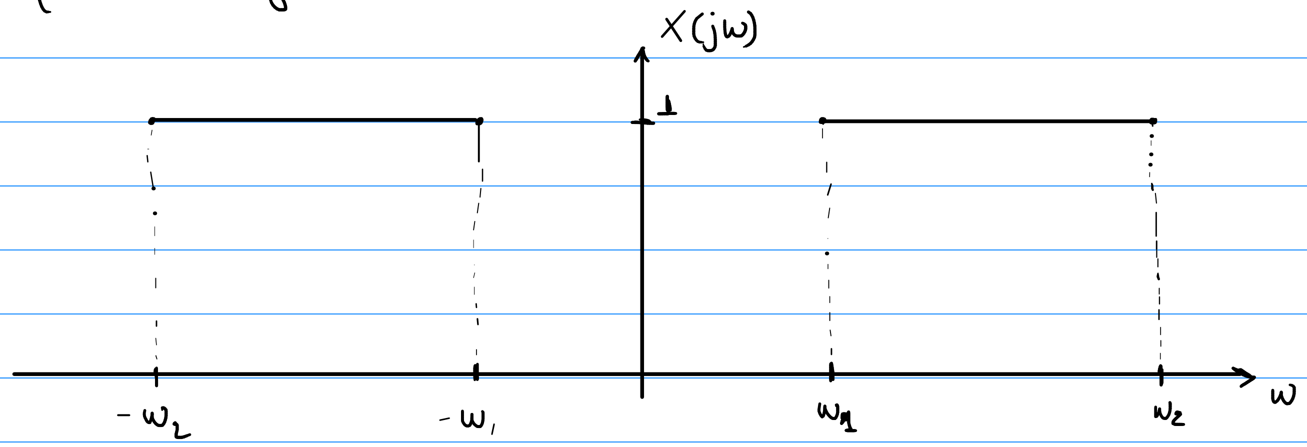
Θεωρούμε $X(j\omega) \neq 0$ για $\omega_1 \leq |\omega| \leq \omega_2$

$|\omega| \geq \omega_1$ και $|\omega| \leq \omega_2$

$\omega \leq -\omega_1$ ή $\omega \geq \omega_1$ | $-\omega_2 \leq \omega \leq \omega_2$

Έχουμε τις περιοχές $[-\omega_2, -\omega_1]$ και $[\omega_1, \omega_2]$

Για τη δεύτερη ξέρουμε $X(j\omega) = 1$ άρα και για την πρώτη $X(j\omega) = 1$.



2.3) $\omega_s = (2 + 7 \bmod 3)\omega_M = (2+1)\omega_M = 3\omega_M$

a) $$y_d[n] = \frac{x[n-1] - 2x[n] + x[n+1]}{4} = \frac{1}{4}x[n-1] - \frac{1}{2}x[n] + \frac{1}{4}x[n+1]$$

$$Y_d[\Omega] = \frac{1}{4}e^{-j\Omega} X_d[\Omega] - \frac{1}{2}X_d[\Omega] + \frac{1}{4}e^{j\Omega}X_d[\Omega] \Rightarrow$$

$$\frac{Y_d[\Omega]}{X_d[\Omega]} = \frac{1}{2}\left(\frac{e^{-j\Omega} + e^{j\Omega}}{2} - 1\right) = \frac{1}{2}\left(\cos\Omega - 1\right)$$

$$H_d[\Omega] = \frac{1}{2}\left(\cos\Omega - 1\right)$$

$$h_d[n] = F^{-1}[H_d[\Omega]] = \frac{1}{4}F^{-1}[e^{-j\Omega} + e^{j\Omega}] - \frac{1}{2}F^{-1}[1] =$$

$$= \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n+1] - \frac{1}{2}\delta[n] =$$

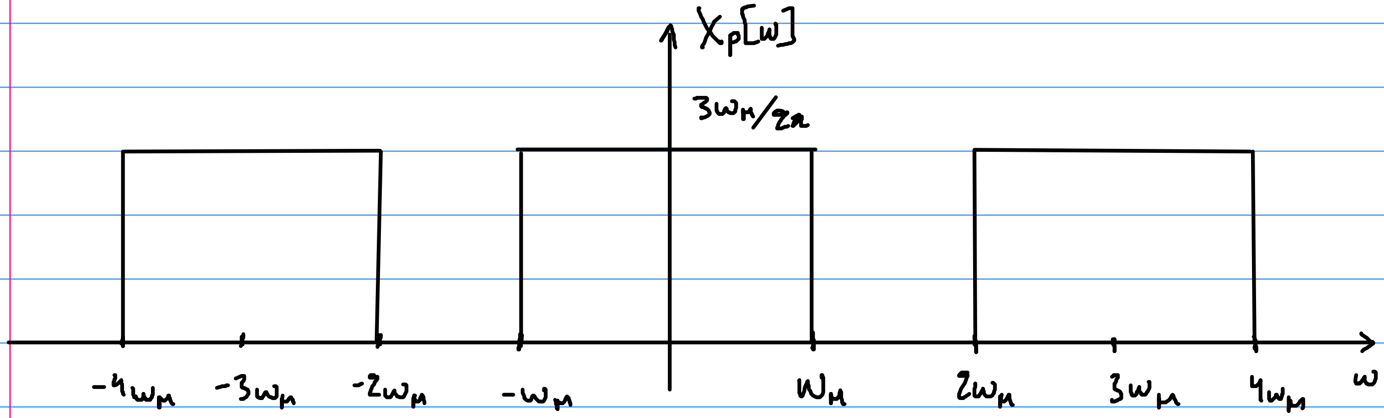
$$= \frac{\delta[n-1] - 2\delta[n] + \delta[n+1]}{4}$$

b) $x_c(t) = \dfrac{\sin(\omega_M t)}{\pi t}$, $\omega_s = \dfrac{2\pi}{T_s} \Rightarrow T_s = \dfrac{2\pi}{\omega_s} = \dfrac{2\pi}{3\omega_M}$

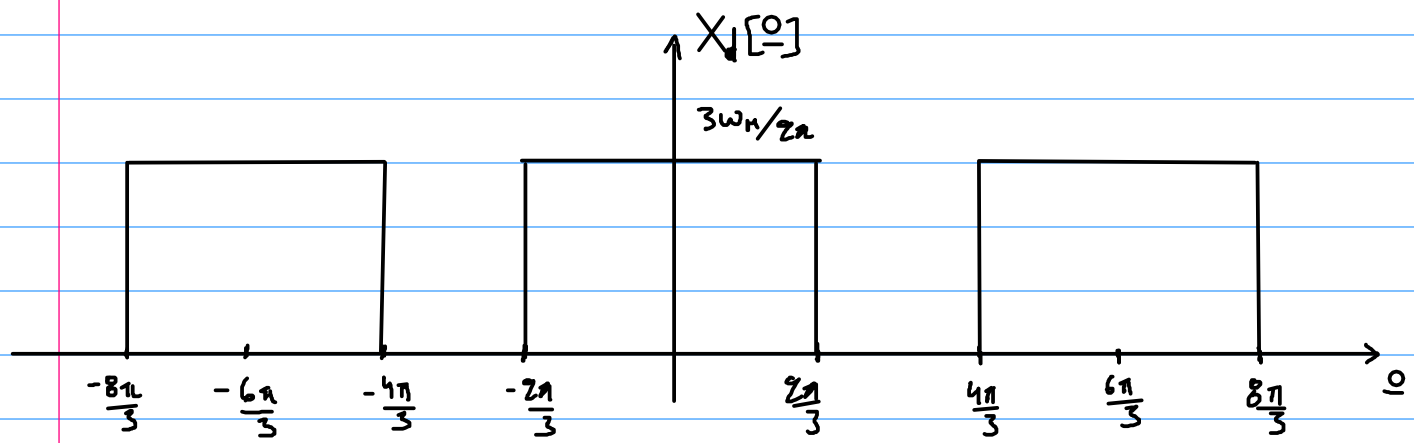
$$x_p(t) = x_c(t) \cdot p(t) = \frac{\sin(\omega_M t)}{\pi t} \sum_n \delta(t - nT)$$

$$X_p[\omega] = X_c(\omega) * F\left[\sum_n \delta(t - nT_s)\right] = X_c * \left[\frac{1}{T_s}\sum_{m=-\infty}^{+\infty}\delta(\omega - m\omega_s)\right] =$$

$$= \frac{1}{T_s}\sum_{m=-\infty}^{+\infty} X_c(\omega - m\omega_s) = \frac{3\omega_M}{2\pi}\sum_{m=-\infty}^{+\infty} X_c(\omega - m\omega_s)$$

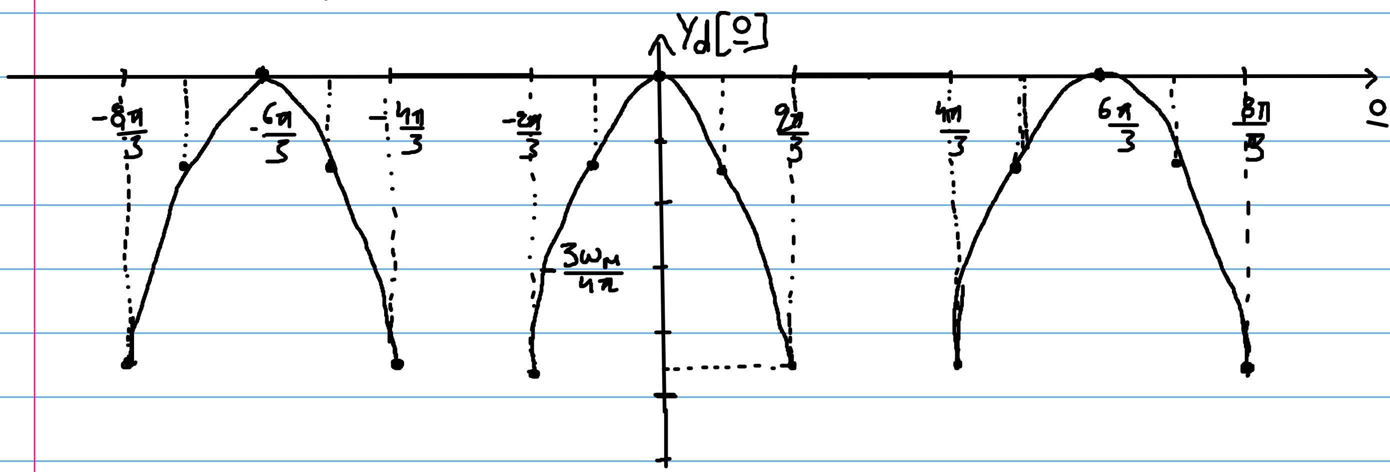


$$X_d(\Omega) = X_p\left(\frac{\Omega}{T_s}\right) = X_p\left(\frac{3\omega_M}{2\pi}\cdot\Omega\right)$$

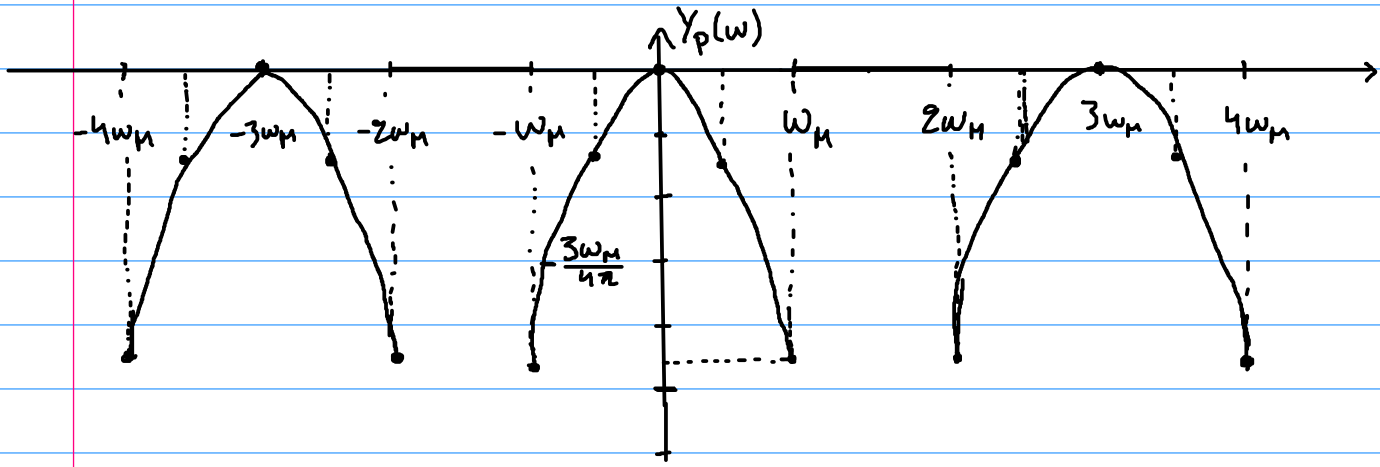


$$Y_d[\Omega] = \frac{1}{4}e^{j\Omega}X_d[\Omega] - \frac{1}{2}X_d[\Omega] + \frac{1}{4}e^{-j\Omega}X_d[\Omega] =$$

$$= \frac{X_d[\Omega]}{2}\left(\frac{e^{-j\Omega}+e^{j\Omega}}{2} - 1\right) = \frac{X_d[\Omega]}{2}\left(\cos\Omega - 1\right)$$

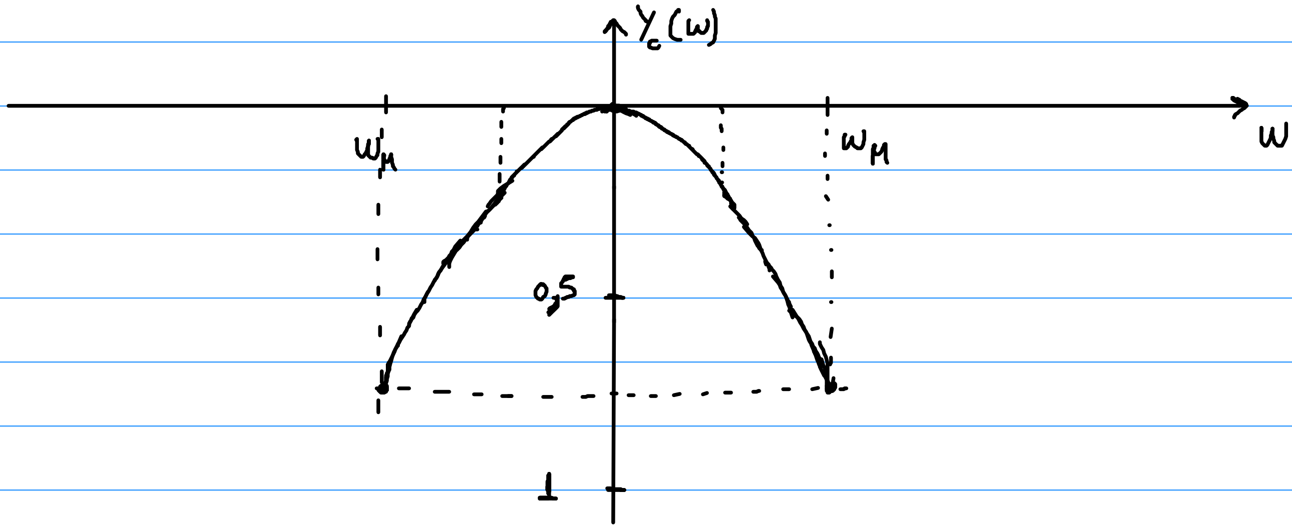


$$Y_d[\Omega] = Y_p\left(\frac{\Omega}{T_s}\right) \Rightarrow Y_p(\omega) = Y_d[\omega T_s] = Y_d\left[\omega \cdot \frac{2\pi}{3\omega_M}\right]$$



$$H_r(\omega) = \begin{cases} T, & |\omega| < \frac{\omega_s}{2} = \frac{3\omega_M}{2} \\ 0, & |\omega| > \frac{\omega_s}{2} = \frac{3\omega_M}{2} \end{cases}$$

$$Y_c(\omega) = Y_p(\omega)\cdot H_r(\omega) = \begin{cases} \frac{2\pi}{3\omega_M}\cdot Y_p(\omega), & |\omega| < \frac{3\omega_M}{2} \\ 0, & |\omega| > \frac{3\omega_M}{2} \end{cases}$$



γ) $$H_c(\omega) = \frac{Y_c(\omega)}{X_c(\omega)} = \frac{Y_p(\omega)\cdot H_r(\omega)}{X_c(\omega)} = \frac{Y_d(\omega T_s) H_r(\omega)}{X_c(\omega)} =$$

$$= \frac{X_d(\omega T_s) H_d(\omega T_s) H_r(\omega)}{X_c(\omega)} = \frac{X_p(\omega) H_d(\omega T_s) H_r(\omega)}{X_c(\omega)} = \frac{H_d(\omega T_s)\cdot \cancel{X_c(\omega)}}{\cancel{X_c(\omega)}} = H_d(\omega T_s)$$

Επομένως $H_c(\omega) = \begin{cases} H_d(\omega T_s) & , |\omega| \leq \frac{\pi}{T_s} = \frac{\pi}{\frac{2\pi}{3\omega_M}} = \frac{3\omega_M}{2} \\ 0 & , |\omega| > \frac{3\omega_M}{2} \end{cases} \Rightarrow$

$$H_c(\omega) = \begin{cases} \frac{1}{2}\left(\cos\left(\frac{2\pi}{3}\cdot\frac{\omega}{\omega_M}\right) - 1\right) & , |\omega| \leq \frac{3\omega_M}{2} \\ 0 & , |\omega| > \frac{3\omega_M}{2} \end{cases}$$

2.4) a) $x_1[n] = 7^n \sin[\pi(n+1)/3]\cos[\pi(n+1)/6]u[n] =$

$$= \frac{7^n}{2}\cdot\left(\sin\frac{\pi}{2}(n+1) + \sin\frac{\pi}{6}(n+1)\right)u[n] =$$

$$= \frac{7^n}{2}\left[\frac{e^{j\frac{\pi}{2}(n+1)} - e^{-j\frac{\pi}{2}(n+1)}}{2j} + \frac{e^{j\frac{\pi}{6}(n+1)} - e^{-j\frac{\pi}{6}(n+1)}}{2j}\right]u[n] =$$

$$= \frac{7^n}{4}\left(\frac{\cancel{j}e^{j\frac{\pi}{2}n} + \cancel{j}e^{-j\frac{\pi}{2}n}}{\cancel{j}} + \frac{\left(\frac{\sqrt{3}}{2}+j\frac{1}{2}\right)e^{j\frac{\pi}{6}n} - \left(\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)e^{-j\frac{\pi}{6}n}}{j}\right)u[n] =$$

$$= \frac{7^n}{4}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n} + \frac{\sqrt{3}}{2j}\left(e^{j\frac{\pi}{6}n} - e^{-j\frac{\pi}{6}n}\right) + \frac{1}{2}\left(e^{j\frac{\pi}{6}n} + e^{-j\frac{\pi}{6}n}\right)\right)u[n]$$

$$= \frac{7^n}{4}\left(2\cos\frac{\pi}{2}n + \sqrt{3}\sin\frac{\pi}{6}n + \cos\frac{\pi}{6}n\right)u[n] =$$

$$= \frac{1}{2}7^n\cos\left(\frac{\pi}{2}n\right)u[n] + \frac{\sqrt{3}}{4}7^n\sin\left(\frac{\pi}{6}n\right)u[n] + \frac{1}{4}7^n\cos\left(\frac{\pi}{6}n\right)u[n]$$

$$X_1(z) = \frac{1}{2}\cdot\frac{1-\left[7\cos\frac{\pi}{2}\right]z^{-1}}{1-2\left[7\cos\frac{\pi}{2}\right]z^{-1} + 7^2z^{-2}} + \frac{\sqrt{3}}{4}\cdot\frac{\left[7\sin\frac{\pi}{6}\right]z^{-1}}{1-\left[2\cdot 7\cos\frac{\pi}{6}\right]z^{-1} + 7^2z^{-2}} +$$

$$+\frac{1}{4}\cdot\frac{1-\left[7\cos\frac{\pi}{6}\right]z^{-1}}{1-\left[2\cdot 7\cos\frac{\pi}{6}\right]z^{-1}+7^2z^{-2}}=\frac{1}{2}\cdot\frac{1}{1+\left(\frac{7}{z}\right)^2}+\frac{\sqrt{3}}{\cancel{4}2}\cdot\frac{\frac{7}{\cancel{2}z}}{1-7\sqrt{3}z^{-1}+\left(\frac{7}{z}\right)^2}+$$

$$+\frac{1}{4}\cdot\frac{1-\frac{\sqrt{3}}{2z}}{1-\frac{7\sqrt{3}}{z}+\left(\frac{7}{z}\right)^2}=\frac{z^2}{2(z^2+7^2)}+\frac{\sqrt{3}}{2}\cdot\frac{7z}{z^2-7\sqrt{3}z+7^2}+\frac{1}{4}\cdot\frac{\frac{2z-7\sqrt{3}}{2\cancel{z}}}{\frac{z^2-7\sqrt{3}z+7^2}{z^{\cancel{2}}}}=$$

$$=\frac{z^2}{2(z^2+7^2)}+\frac{\sqrt{3}}{2}\cdot\frac{7z}{z^2-z\sqrt{3}+7^2}+\frac{1}{2}\cdot\frac{z(2z-7\sqrt{3})}{z^2-7z\sqrt{3}+7^2}=$$

$$=\frac{z^2}{2(z^2+7^2)}+\frac{\cancel{7z\sqrt{3}}+\cancel{2}z^2-\cancel{7z\sqrt{3}}}{\cancel{2}(z^2-7z\sqrt{3}+7^2)}=z^2\cdot\frac{z^2+7^2-7z\sqrt{3}+2z^2+2\cdot 7^2}{2(z^2+7^2)(z^2-7z\sqrt{3}+7^2)}=$$

$$=z^2\cdot\frac{3z^2-7\cdot z\sqrt{3}+3\cdot 7^2}{2(z^2+7^2)(z^2-7z\sqrt{3}+7^2)}=z^2\cdot\frac{3z^2-7\sqrt{3}z+147}{2(z^2+49)(z^2-7\sqrt{3}z+49)},\ |z|>7$$

b) $x_2[n]=n7^{-|n|}=n7^{-n}u[n]+n7^{n}u[-n]$

$$X_2(z)=Z[x_2[n]]=Z[n7^{-n}u[n]]+Z[n7^{n}u[-n]]=$$

$$=Z\left[n\left(\tfrac{1}{7}\right)^{n}u[n]\right]-\underbrace{Z\left[-n\left(\tfrac{1}{7}\right)^{-n}u[-n]\right]}_{\text{Χρονική αντιστροφή}}=$$

$$=\frac{\frac{1}{7}\cdot z^{-1}}{\left(1-\frac{1}{7}z^{-1}\right)^2}-\frac{\frac{1}{7}(z^{-1})^{-1}}{\left(1-\frac{1}{7}(z^{-1})^{-1}\right)^2}=\frac{\frac{1}{7z}}{\left(1-\frac{1}{7z}\right)^2}-\frac{\frac{z}{7}}{\left(1-\frac{z}{7}\right)^2}=$$

$$=\frac{\frac{1}{\cancel{7z}}}{\frac{(7z-1)^2}{(7z)^2}}-\frac{\frac{z}{\cancel{7}}}{\frac{(7-z)^2}{7^2}}=\frac{7z}{(7z-1)^2}-\frac{7z}{(7-z)^2}=$$

$$=7z\cdot\frac{(7-z-7z+1)(7-z+7z-1)}{(7-z)^2(7z-1)^2}=7z\cdot\frac{(7+1)(1-z)(7-1)(1+z)}{(7-z)^2(7z-1)^2}=7z\,\frac{48(1-z^2)}{(7-z)^2(7z-1)^2}=$$

$$= \frac{336z(1-z^2)}{(7-z)^2(7z-1)^2}, |z| > \frac{1}{7}, \quad \left|\frac{1}{z}\right| > \frac{1}{7} \Rightarrow |z| < 7$$

Άρα $\frac{1}{7} < |z| < 7$

γ) $$x_3[n] = \frac{n}{7^n} \sin\left[\frac{\pi n}{3} + \frac{\pi}{4}\right] u[n] =$$

$$= \frac{n}{7^n}\left[\sin\left(\frac{\pi n}{3}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi n}{3}\right)\sin\left(\frac{\pi}{4}\right)\right] u[n] =$$

$$= \frac{n}{7^n}\left[\frac{\sqrt{2}}{2}\sin\left(\frac{\pi n}{3}\right) + \frac{\sqrt{2}}{2}\cos\left(\frac{\pi n}{3}\right)\right] u[n] =$$

$$= \frac{\sqrt{2}}{2}\left[\frac{n}{7^n}\left(\sin\left(\frac{\pi n}{3}\right) + \cos\left(\frac{\pi n}{3}\right)\right)\right] u[n] =$$

$$= \frac{\sqrt{2}}{2}\left[n(7^{-1})^n \sin\left(\frac{\pi n}{3}\right) u[n] + n(7^{-1})^n \cos\left(\frac{\pi n}{3}\right) u[n]\right]$$

$$X_3(z) = \frac{\sqrt{2}}{2} Z\left[n(7^{-1})^n \sin\frac{\pi n}{3} \cdot u[n] + n(7^{-1})^n \cos\left(\frac{\pi n}{3}\right) u[n]\right] =$$

$$= \frac{\sqrt{2}}{2}\left[-z\frac{d Z\left[(7^{-1})^n \sin\frac{\pi n}{3} u[n]\right]}{dz} - z\frac{d Z\left[(7^{-1})^n \cos\left(\frac{\pi n}{3}\right) u[n]\right]}{dz}\right] =$$

$$= \frac{\sqrt{2}}{2}\left[-z \cdot \frac{d}{dz}\left(\frac{7^{-1}\sin\frac{\pi}{3} z^{-1}}{1 - 2 \cdot 7^{-1}\cos\frac{\pi}{3} z^{-1} + 7^{-2} z^{-2}}\right) - z\frac{d}{dz}\left(\frac{1 - 7^{-1}\cos\frac{\pi}{3} z^{-1}}{1 - 2 \cdot 7^{-1}\cos\frac{\pi}{3} z^{-1} + 7^{-2} z^{-2}}\right)\right] =$$

$$= \frac{\sqrt{2}}{2}\left[-z\frac{d}{dz}\left(\frac{\frac{\sqrt{3}}{2 \cdot 7 \cdot z}}{1 - \frac{1}{7z} + \frac{1}{7^2 z^2}}\right) - z\frac{d}{dz}\left(\frac{1 - \frac{1}{2 \cdot 7 \cdot z}}{1 - \frac{1}{7 \cdot z} + \frac{1}{7^2 \cdot z^2}}\right)\right] =$$

$$= \frac{\sqrt{2}}{2}\left[-z\frac{d}{dz}\left(\frac{\sqrt{3}}{2} \cdot \frac{7z}{49z^2 - 7z + 1}\right) - z\frac{d}{dz}\left(\frac{\frac{14z - 1}{2 \cdot \cancel{7z}}}{\frac{7^2z^2 - 7z + 1}{7^2 z^{\cancel{2}}}}\right)\right] =$$

$$= \frac{\sqrt{2}}{2}\left[-z\frac{d}{dz}\left(\frac{\sqrt{3}}{2}\cdot\frac{7z}{49z^2-7z+1}\right) - z\frac{d}{dz}\left(\frac{1}{2}\cdot\frac{7z(14z-1)}{49z^2-7z+1}\right)\right] =$$

$$= -\frac{7\sqrt{2}}{4}\left[\sqrt{3}\,z\cdot\frac{49z^2-7z+1-z(98z-7)}{(49z^2-7z+1)^2} +\right.$$

$$\left. + z\cdot\frac{(28z-1)(49z^2-7z+1)-(14z^2-1)(98z-7)}{(49z^2-7z+1)^2}\right] =$$

$$= -\frac{7\sqrt{2}}{4}\cdot z\left[\sqrt{3}\,\frac{-49z^2+1}{(49z^2-7z+1)^2} +\right.$$

$$\left. + \frac{\cancel{4\cdot 7^3 z^3} - 4\cdot 7^2 z^2 + 4\cdot 7z - 7^2z^2 + 7z - 1 - \cancel{4\cdot 7^3 z^3} + 2\cdot 7^2 z^2 + 2\cdot 7^2 z - 7}{(49z^2-7z+1)}\right] =$$

$$= -\frac{7\sqrt{2}}{4} z\cdot\frac{-49\sqrt{3}z^2+\sqrt{3}-147z^2+133z-8}{(49z^2-7z+1)} =$$

$$= \frac{7\sqrt{2}}{4}\cdot\frac{156z^3-133z^2+(8-\sqrt{3})z}{(49z^2-7z+1)}, \quad |z| > \frac{1}{7}$$

δ)

$$x_4[n] = \begin{cases} 7^n, & n<0 \Rightarrow n \le -1 \\ \left(\frac{1}{2}\right)^n, & n = 0, 2, 4, \dots \\ \left(\frac{1}{3}\right)^n, & n = 1, 3, 5, \dots \end{cases}$$

$$x_4[n] = 7^n u[-n-1] + \left(\frac{1}{2}\right)^n \frac{\cos(n\pi)+1}{2} u[n] + \left(\frac{1}{3}\right)^n \left[\frac{\cos(n\pi)+1}{2} - \cos(n\pi)\right] u[n] =$$

$$= 7^n u[-n-1] + \left(\frac{1}{2}\right)^n \cdot \frac{\cos(n\pi)+1}{2} u[n] + \left(\frac{1}{3}\right)^n \cdot \frac{1-\cos(n\pi)}{2} u[n] =$$

$$= -\left(-7^n u[-n-1]\right) + \left(\frac{1}{2}\right)^n \frac{\cos(n\pi)+1}{2} u[n] + \left(\frac{1}{3}\right)^n \cdot \frac{1-\cos(n\pi)}{2} u[n] =$$

$$= -\left(-7^n u[-n-1]\right) + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n \cos(n\pi) u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^n u[n] +$$

$$+ \frac{1}{2}\left(\frac{1}{3}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{3}\right)^n \cos(n\pi) u[n]$$

$$X_4(z) = -Z\left[-7^n u[-n-1]\right] + \frac{1}{2} Z\left[\left(\frac{1}{2}\right)^n \cos(\pi n) u[n]\right] + \frac{1}{2} Z\left[\left(\frac{1}{2}\right)^n u[n]\right] +$$

$$+ \frac{1}{2} Z\left[\left(\frac{1}{3}\right)^n u[n]\right] - \frac{1}{2} Z\left[\left(\frac{1}{3}\right)^n \cos(\pi n) u[n]\right] =$$

$$= -\frac{1}{1-7z^{-1}} + \frac{1}{2} \cdot \frac{1 - \frac{1}{2}\cos\pi \cdot z^{-1}}{1 - 2\cdot\frac{1}{2}\cos\pi \cdot z^{-1} + \frac{1}{4}z^{-2}} + \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{2} \cdot \frac{1}{1-\frac{1}{3}z^{-1}} -$$

$$- \frac{1}{2} \cdot \frac{1-\frac{1}{3}\cos\pi z^{-1}}{1 - 2\cdot\frac{1}{3}\cos\pi z^{-1} + \frac{1}{9}z^{-2}} = -\frac{z}{z-7} + \frac{1}{2} \cdot \frac{\frac{2z+1}{2z}}{1 + \frac{1}{z} + \frac{1}{4z^2}} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}z}{2z-1} +$$

$$+ \frac{1}{2} \cdot \frac{3z}{3z-1} - \frac{1}{2} \cdot \frac{\frac{3z+1}{3z}}{1 + \frac{2}{3z} + \frac{1}{9z^2}} =$$

$$= \frac{z}{7-z} + \frac{z(2z+1)}{4z^2+4z+1} + \frac{z}{2z-1} + \frac{1}{2} \cdot \frac{3z}{3z-1} - \frac{1}{2} \cdot \frac{3z(3z+1)}{9z^2+6z+1} =$$

$$= \frac{z}{7-z} + \frac{z}{2z+1} + \frac{z}{2z-1} + \frac{1}{2}\left(\frac{3z}{3z-1} - \frac{3z}{3z+1}\right) =$$

$$= \frac{z}{7-z} + z \cdot \frac{2z\cancel{-1}+2z\cancel{+1}}{4z^2-1} + \frac{3z}{\cancel{2}}\left(\frac{\cancel{3z}+1\cancel{-3z}+1}{9z^2-1}\right) =$$

$$= \frac{z}{7-z} + \frac{4z^2}{4z^2-1} + \frac{3z}{9z^2-1}, \quad |z|<7 \text{ και } |z|>\frac{1}{2} \text{ και } |z|>\frac{1}{3} \Rightarrow$$

$$\boxed{\frac{1}{2} < |z| < 7}$$

2.5) a) $X_1(z) = \dfrac{z-3}{z^2-2z+2}$, $x_1[n]$ αιτιατό

$$X_1(z) = \frac{z-3}{z^2 - 2\frac{\sqrt{2}}{2}\cdot\sqrt{2}\, z + (\sqrt{2})^2}$$

$\Omega_0 = \dfrac{\pi}{4}$, $r = \sqrt{2}$ $\quad x[n] = (\sqrt{2})^n \cos\left(\frac{\pi}{4}n\right)u[n]$

$$X(z) = \frac{1 - \sqrt{2}\cdot\frac{\sqrt{2}}{2}\cdot z^{-1}}{1 - 2\cdot\sqrt{2}\cdot\frac{\sqrt{2}}{2}z^{-1} + (\sqrt{2})^2\cdot z^{-2}} = \frac{\frac{z-1}{z}}{\frac{z^2-2z+2}{z^2}} = \frac{z^2-z}{z^2-2z+2} =$$

$$= \frac{z^2-2z+2+z-2}{z^2-2z+2} = 1 + \frac{z-2}{z^2-2z+2}$$

$x[n] = (\sqrt{2})^n \sin\left(\frac{\pi}{4}n\right)u[n]$

$$X(z) = \frac{\sqrt{2}\cdot\frac{\sqrt{2}}{2}z^{-1}}{1 - 2\cdot\sqrt{2}\cdot\frac{\sqrt{2}}{2}z^{-1} + 2z^{-2}} = \frac{z^{-1}}{\frac{z^2-2z+2}{z^2}} = \frac{z}{z^2-2z+2}$$

$$\frac{z-3}{z^2-2z+2} = a\left(1 + \frac{z-2}{z^2-2z+2}\right) + b\cdot\frac{z}{z^2-2z+2} + \gamma \Rightarrow$$

$z-3 = az^2 - 2az + \cancel{2a} + az - \cancel{2a} + bz + \gamma z^2 - 2\gamma z + 2\gamma \Rightarrow$

$z-3 = (a+\gamma)z^2 + (b-a-2\gamma)z + 2\gamma$

$a+\gamma=0 \Rightarrow$, $2\gamma = -3 \Rightarrow$, $b-a-2\gamma = 1 \Rightarrow$

$a = -\gamma = \dfrac{3}{2}$ $\quad \gamma = -\dfrac{3}{2}$ $\quad b = 2\gamma + a + 1 = -3 + \dfrac{3}{2} + 1 = -\dfrac{1}{2}$

Λόγω γραμμικότητας:

Άρα $x_1[n] = \frac{3}{2}(\sqrt{2})^n \cos\left(\frac{\pi}{4}n\right)u[n] - \frac{1}{2}(\sqrt{2})^n \sin\left(\frac{\pi}{4}n\right)u[n] - \frac{3}{2}\delta[n]$

β) $X_2(z) = \dfrac{z + 2z^{-1}}{z^2 + 4}$, $|z|>2$

$$X_2(z) = \frac{z^{-1} + 2z^{-3}}{1 + 4z^{-2}} = \frac{z^{-1}}{1+4z^{-2}} + \frac{2z^{-3}}{1+4z^{-2}} =$$

$$= \frac{1}{2} \cdot \frac{2z^{-1}}{1+4z^{-2}} + \frac{2z^{-1}}{1+4z^{-2}} \cdot z^{-2} \quad , \Omega_0 = \frac{\pi}{2} , r=2 , m=2$$

$$x_2[n] = \frac{1}{2} \cdot 2^n \sin\left(\frac{\pi}{2}n\right)u[n] + 2^n \sin\left(\frac{\pi}{2}n\right)u[n] * \delta[n-2] =$$

$$= \frac{1}{2} \cdot 2^n \sin\left(\frac{\pi}{2}n\right)u[n] + 2^{n-2} \cdot \sin\left(\frac{\pi}{2}(n-2)\right)u[n-2] =$$

$$= 2^{n-1} \sin\left(\frac{\pi}{2}n\right)u[n] - 2^{n-2} \sin\left(\frac{\pi}{2}n\right)u[n-2] =$$

$$= 2^{n-1} \sin\left(\frac{\pi}{2}n\right)\left[u[n] - \frac{u[n-2]}{2}\right] =$$

$$= 2^{n-1} \sin\left(\frac{\pi}{2}n\right) \cdot \frac{2u[n] - u[n-2]}{2} =$$

$$= 2^{n-2} \sin\left(\frac{\pi}{2}n\right)\left(2u[n] - u[n-2]\right)$$

γ) $X_3(z) = \dfrac{7z^2 + 4z}{z^3 + z^2 - 4z - 4}$, $|z| < 1$

$$X_3(z) = \frac{7z^2 + 4z}{z^2(z+1) - 4(z+1)} = \frac{7z^2 + 4z}{(z+1)(z-2)(z+2)}$$

$$\frac{7z^2+4z}{(z+1)(z+2)(z-2)} = \frac{A}{z+1} + \frac{B}{z+2} + \frac{\Gamma}{z-2} \Rightarrow$$

$$7z^2+4z = A(z^2-4) + B(z+1)(z-2) + \Gamma(z+1)(z+2) \Rightarrow$$
$$7z^2+4z = Az^2 - 4A + Bz^2 - Bz - 2B + \Gamma z^2 + 3\Gamma z + 2\Gamma \Rightarrow$$
$$7z^2+4z = (A+B+\Gamma)z^2 + (3\Gamma - B)z + 2\Gamma - 4A - 2B$$

$$A+B+\Gamma = 7 \Rightarrow \cancel{A}+2-3A+2-\cancel{A} = 7 \Rightarrow \boxed{A=-1}$$
$$3\Gamma - B = 4 \Rightarrow B = 3\Gamma - 4 \Rightarrow B = 6A+3B-4 \Rightarrow 2B = 4-6A \Rightarrow$$
$$2\Gamma - 4A - 2B = 0 \Rightarrow \Gamma = 2A+B = 2A+2-3A = 2-A \qquad B = 2-3A$$

$$\boxed{\begin{array}{l} B=5 \\ \Gamma=3 \end{array}}$$

$$X_3(z) = -\frac{1}{z+1} + 5\cdot\frac{1}{z+2} + 3\cdot\frac{1}{z-2} =$$

$$= -\frac{z^{-1}}{1+z^{-1}} + 5\cdot\frac{z^{-1}}{1+2z^{-1}} + 3\cdot\frac{z^{-1}}{1-2z^{-1}} =$$

$$= -\left(1 - \frac{1}{1+z^{-1}}\right) + \frac{5}{2}\left(1 - \frac{1}{1+2z^{-1}}\right) + \frac{3}{2}\left(-1 + \frac{1}{1-2z^{-1}}\right) =$$

$$= -\cancel{1} + \cancel{\frac{5}{2}} - \cancel{\frac{3}{2}} + \frac{1}{1+z^{-1}} - \frac{5}{2}\cdot\frac{1}{1+2z^{-1}} + \frac{3}{2}\cdot\frac{1}{1-2z^{-1}} =$$

$$x_3[n] = -(-1)^n u[-n-1] - \frac{5}{2}\cdot(-2)^n u[-n-1] + \frac{3}{2}\cdot 2^n u[-n-1] =$$
$$= u[-n-1]\left(\frac{3}{2}\cdot 2^n - \frac{5}{2}(-2)^n - (-1)^n\right)$$

Πράγματι, $|z| < |-1| = 1$ και $|z| < |-2| = 2$ και $|z| < |2| = 2$

Άρα $|z| < 1$

δ) $X_4(z) = \dfrac{z^3(5z+1)}{(z+1)(z^2-2z-3)}$, $1<|z|<3$

$$X_4(z) = \frac{5z^4+z^3}{z^3-z^2-5z-3}$$

$$\begin{array}{l|l}
\cancel{5z^4}+z^3+0z^2+0z+0 & z^3-z^2-5z-3 \\ \cline{2-2}
-\cancel{5z^4}+5z^3+25z^2+15z & 5z+6 \\ \cline{1-1}
\cancel{6z^3}+25z^2+15z & \\
-\cancel{6z^3}+6z^2+30z+18 & \\ \cline{1-1}
31z^2+45z+18 &
\end{array}$$

$$X_4(z) = \frac{(5z+6)(z^3-z^2-5z-3)+31z^2+45z+18}{z^3-z^2-5z-3} = 5z+6+\frac{31z^2+45z+18}{(z+1)^2(z-3)}$$

$$\frac{31z^2+45z+18}{(z+1)^2(z-3)} = \frac{A}{z+1}+\frac{B}{(z+1)^2}+\frac{\Gamma}{z-3} \Rightarrow$$

$$31z^2+45z+18 = Az^2-2Az-3A+Bz-3B+\Gamma z^2+2\Gamma z+\Gamma \Rightarrow$$

$$31z^2+45z+18 = (A+\Gamma)z^2+(B+2\Gamma-2A)z-3A-3B+\Gamma$$

$$\left.\begin{array}{l}
A+\Gamma=31 \Rightarrow \Gamma=31-A \\
B+2(\Gamma-A)=45 \Rightarrow B+2(31-2A)=45 \Rightarrow 4A=B+62-45=B+17 \\
\Gamma-3(A+B)=18 \Rightarrow 31-A-3A-3B=18 \Rightarrow 4A=-3B+13
\end{array}\right\} \Rightarrow$$

$$B+17=-3B+13 \Rightarrow 4B=-4 \Rightarrow \boxed{B=-1}$$

$$A=\frac{-1+17}{4}=\frac{16}{4}=\boxed{4}$$

$$\Gamma=31-4=\boxed{27}$$

$$X_4(z) = 5z + 6 + \frac{4}{z+1} - \frac{1}{(z+1)^2} + \frac{27}{z-3} =$$

$$= 5z^{-(-1)} + 6 + \frac{4}{z-(-1)} + \frac{-1}{(z-(-1))^2} + 27 \cdot \frac{z^{-1}}{1-3z^{-1}} =$$

$$= 5z^{-(-1)} + 6 + \frac{4}{z-(-1)} + \frac{-1}{(z-(-1))^2} + 9 \cdot \left(-1 + \frac{1}{1-3z^{-1}}\right) =$$

$$= 5z^{-(-1)} - 3 + \frac{4}{z-(-1)} + \frac{-1}{(z-(-1))^2} + 9 \cdot \frac{1}{1-3z^{-1}}$$

$$x_4[n] = 5\delta[n+1] - 3\delta[n] + 4(-1)^{n-1}u[n-1] + (n-1)(-1)^{n-1}u[n-1] -$$

$$- 9 \cdot 3^n u[-n-1] =$$

$$= 5\delta[n+1] - 3\delta[n] + (n+3)(-1)^{n-1}u[n-1] - 3^{n+2}u[-n-1]$$

Πράγματι $\left.\begin{array}{l} |z| > |-1| = 1 \\ |z| < |3| = 3 \end{array}\right\} \Rightarrow 1 < |z| < 3$

2.6) a) $y[n] - \frac{1}{4}y[n-2] = x[n] + 2x[n-1] + x[n-2] \Rightarrow$

$$Y(z) - \frac{1}{4}z^{-2}Y(z) = X(z) + 2z^{-1}X(z) + z^{-2}X(z) \Rightarrow$$

$$Y(z) \cdot \frac{4 - z^{-2}}{4} = X(z)\left(1 + 2z^{-1} + z^{-2}\right) \Rightarrow$$

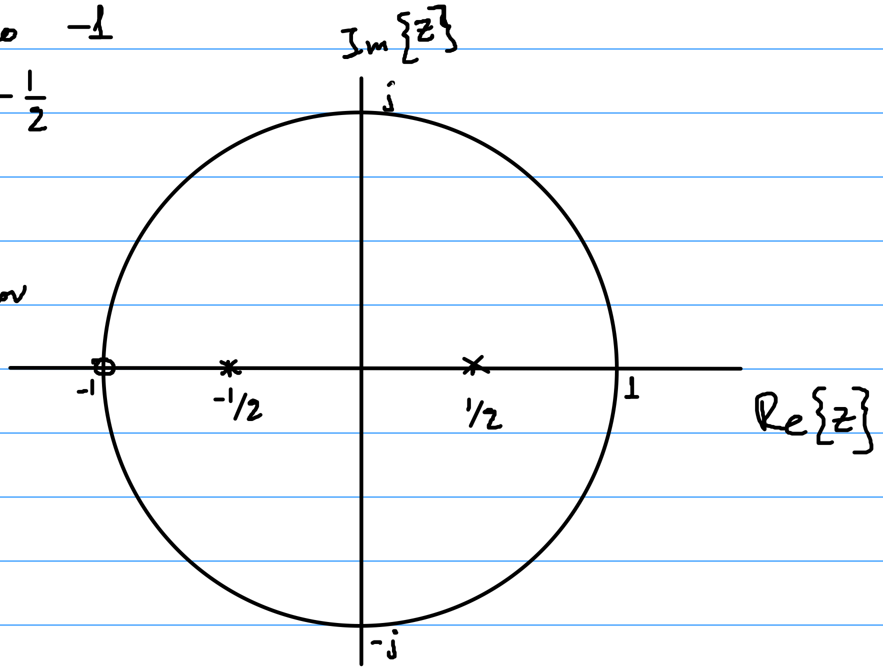
$$\frac{Y(z)}{X(z)} = \frac{4 + 8z^{-1} + 4z^{-2}}{4 - z^{-2}} \Rightarrow \boxed{H(z) = \frac{4 + 8z^{-1} + 4z^{-2}}{4 - z^{-2}}}$$

$$H(z) = \frac{4z^2 + 8z + 4}{4z^2 - 1} = \frac{4(z+1)^2}{(2z-1)(2z+1)}$$

Διπλό μηδενικό στο -1

Πόλοι στο $\frac{1}{2}, -\frac{1}{2}$

και οι δύο πόλοι βρίσκονται μέσα στον μοναδιαίο κύκλο, επομένως το σύστημα είναι ευσταθές.



β) $y[n] - \frac{1}{4}y[n-2] = x[n] + 2x[n-1] + x[n-2]$

Για να βρούμε την $H(\underline{\Omega})$ θα πάρουμε το DFT της εξίσωσης διαφορών:

$$Y[\Omega] - \frac{1}{4} e^{-j2\Omega} Y[\Omega] = X[\Omega] + 2e^{-j\Omega} X[\Omega] + e^{-j2\Omega} X[\Omega] \Rightarrow$$

$$Y[\Omega] \cdot \frac{4 - e^{-j2\Omega}}{4} = X[\Omega] \cdot \left(1 + 2e^{-j\Omega} + e^{-j2\Omega}\right) \Rightarrow$$

$$H[\Omega] = \frac{Y[\Omega]}{X[\Omega]} = 4 \cdot \frac{1 + 2e^{-j\Omega} + e^{-j2\Omega}}{4 - e^{-j2\Omega}} = 4 \cdot \frac{e^{j2\Omega} + 2e^{j\Omega} + 1}{4e^{j2\Omega} - 1} =$$

$$= 4 \cdot \frac{\left(e^{j\Omega} + 1\right)^2}{\left(2e^{j\Omega} - 1\right)\left(2e^{j\Omega} + 1\right)}$$

$$|H[\Omega]| = 4 \cdot \frac{\left|e^{j\Omega} + 1\right|^2}{\left|2e^{j\Omega} - 1\right| \cdot \left|2e^{j\Omega} + 1\right|} = 4 \cdot \frac{(\cos\Omega + 1)^2 + \sin^2\Omega}{\sqrt{\left[(2\cos\Omega - 1)^2 + 4\sin^2\Omega\right]\left[(2\cos\Omega + 1)^2 + 4\sin^2\Omega\right]}} =$$

$$= 4 \cdot \frac{\cancel{\cos^2\Omega} + 2\cos\Omega + 1 + 1 - \cancel{\cos^2\Omega}}{\sqrt{(5 - 4\cos\Omega)(5 + 4\cos\Omega)}}$$

$$= 4 \cdot \frac{2(\cos\Omega + 1)}{\sqrt{25 - 16\cos^2\Omega}} = \frac{16\cos^2(\Omega/2)}{\sqrt{25 - 16\cos^2\Omega}}$$

$|H(\Omega)| = 0$ για $\frac{\Omega}{2} = (2k+1)\frac{\pi}{2} \Rightarrow \Omega = (2k+1)\pi$

για $\frac{\Omega}{2} = k\pi \Rightarrow \Omega = 2k\pi$

Έχουμε $|H(\Omega)| = \frac{16}{\sqrt{25-16}} = \frac{16}{3}$

Με τη βοήθεια του Matlab βρίσκουμε την περίοδο 6,27 γ) $H(z) = \frac{4(z+1)^2}{(2z-1)(2z+1)} = \frac{4(z^2+2z+1)}{4z^2-1} = \frac{4z^2+8z+4}{4z^2-1}$

$$\begin{array}{r|l} 4z^2+8z+4 & 4z^2-1 \\ \hline -\cancel{4z^2}+0+1 & 1 \\ \hline 8z+5 & \end{array}$$

$$H(z) = \frac{8z+5+4z^2-1}{4z^2-1} = 1 + \frac{8z+5}{(2z-1)(2z+1)}$$

$$\frac{8z+5}{(2z-1)(2z+1)} = \frac{A}{2z-1} + \frac{B}{2z+1} \Rightarrow 8z+5 = 2Az + A + 2Bz - B \Rightarrow$$

$$8z+5 = 2(A+B)z + A - B$$

$$\left.\begin{array}{l} A+B=4 \\ A-B=5 \end{array}\right\} (+) \Rightarrow 2A = 9 \Rightarrow A = \frac{9}{2}, \quad B = -\frac{1}{2}$$

$$H(z) = 1 + \frac{9}{2} \cdot \frac{1}{2z-1} - \frac{1}{2} \cdot \frac{1}{2z+1} = 1 + \frac{9}{4} \cdot \frac{1}{z-\frac{1}{2}} - \frac{1}{4} \cdot \frac{1}{z+\frac{1}{2}}$$

$$h[n] = \delta[n] + \frac{9}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1] - \frac{1}{4}\left(-\frac{1}{2}\right)^{n-1} u[n-1] =$$

$$= \delta[n] + \left(\frac{1}{2}\right)^{n+1} u[n-1]\left(9 + (-1)^n\right)$$

δ) $x[n] = \left(-\frac{1}{2}\right)^n u[n]$, $y[-1] = \frac{1}{2}$, $y[-2] = 1$

Για την απόκριση μηδενικής κατάστασης έχουμε:

$$y_{zs}[n] = x[n] * h[n] \longrightarrow Y_{zs}(z) = X(z) \cdot H(z)$$

$$X(z) = Z^{-1}\left[\left(-\frac{1}{2}\right)^n u[n]\right] = \frac{1}{1 + \frac{1}{2z}} = \frac{2z}{2z+1}$$

$$H(z) = \frac{4z^2 + 8z + 4}{4z^2 - 1}$$

$$Y_{zs}(z) = \frac{2z}{2z+1} \cdot \frac{4z^2+8z+4}{4z^2-1} = \frac{8z^3 + 16z^2 + 8z}{8z^3 + 4z^2 - 2z - 1}$$

$$\begin{array}{l|l} \cancel{8z^3} + 16z^2 + 8z + 0 & 8z^3 + 4z^2 - 2z - 1 \\ \hline \cancel{-8z^3} - 4z^2 + 2z + 1 & 1 \\ \hline 12z^2 + 10z + 1 & \end{array}$$

$$Y(z) = 1 + \frac{12z^2 + 10z + 1}{(2z-1)(2z+1)^2}$$

$$\frac{12z^2 + 10z + 1}{(2z-1)(2z+1)^2} = \frac{A}{2z-1} + \frac{B}{2z+1} + \frac{\Gamma}{(2z+1)^2} \Rightarrow$$

$$12z^2 + 10z + 1 = 4Az^2 + 4Az + A + 4Bz^2 - B + 2\Gamma z - \Gamma \Rightarrow$$

$$4(A+B) = 12 \Rightarrow A + B = 3 \Rightarrow A = 3 - B$$

$$4A + 2\Gamma = 10 \Rightarrow 2A + \Gamma = 5 \Rightarrow \Gamma = 5 - 2A = 5 - 6 + 2B = 2B - 1$$

$$A - B - \Gamma = 1 \Rightarrow 3 - B - B - 2B + \cancel{1} = \cancel{1} \Rightarrow 4B = 3 \Rightarrow B = \frac{3}{4}$$

$$A = \frac{9}{4}, \quad \Gamma = \frac{1}{2}$$

$$Y_{zs}(z) = 1 + \frac{9}{4} \cdot \frac{1}{2z-1} + \frac{3}{4} \cdot \frac{1}{2z+1} + \frac{1}{2} \cdot \frac{1}{(2z+1)^2} =$$

$$= 1 + \frac{9}{8} \cdot \frac{1}{z-\frac{1}{2}} + \frac{3}{8} \cdot \frac{1}{z+\frac{1}{2}} + \frac{1}{4} \cdot \frac{1/2}{(z+\frac{1}{2})^2}$$

$$y_{zs}[n] = Z^{-1}\left[Y_{zs}(z)\right] = Z^{-1}[1] + \frac{9}{8} Z^{-1}\left[\frac{1}{z-1/2}\right] + \frac{3}{8} Z^{-1}\left[\frac{1}{z+1/2}\right] -$$

$$- \frac{1}{4} Z^{-1}\left[\frac{-1/2}{(z+1/2)^2}\right] =$$

$$= \delta[n] + \frac{9}{8} \cdot \left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{3}{8} \cdot \left(-\frac{1}{2}\right)^{n-1} u[n-1] - \frac{1}{4}(n-1)\left(-\frac{1}{2}\right)^{n-1} u[n-1] =$$

$$= \delta[n] + \left(\frac{1}{2}\right)^{n} u[n-1] \left(\frac{9}{4} - \frac{3}{4}(-1)^n + \frac{n-1}{2}(-1)^n\right) =$$

$$= \delta[n] + \left(\frac{1}{2}\right)^{n+2} u[n-1] \left(9 - (2n-5)(-1)^n\right)$$

Για την απόκριση μηδενικής εισόδου:

$$y_{zi}[n] - \frac{1}{4} y_{zi}[n-2] = 0 \xleftrightarrow{Z}$$

$$Y_{zi}(z) - \frac{1}{4}\left(z^{-2} Y_{zi}(z) + z^{-1} y(-1) + y(-2)\right) = 0 \Rightarrow$$

$$Y_{zi}(z) - \frac{1}{4z^2} Y_{zi} - \frac{1}{4}\left(\frac{1}{2z} + 1\right) = 0 \Rightarrow$$

$$Y_{zi}(z)\left(\frac{4z^2-1}{\cancel{4}z^{\cancel{2}}}\right) = \frac{1}{\cancel{4}} \cdot \frac{2z+1}{2\cancel{z}} \Rightarrow Y_{zi}(z) = \frac{1}{2} \cdot \frac{z}{2z-1} = \frac{1}{4} \cdot \frac{2z}{2z-1} =$$

$$= \frac{1}{4}\left(1 + \frac{1}{2z-1}\right) = \frac{1}{4}\left(1 + \frac{1}{2}\cdot\frac{1}{z-\frac{1}{2}}\right)$$

$$y_{zi}[n] = Z^{-1}\left[Y_{zi}(z)\right] = \frac{1}{4}\left(\delta[n] + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1} u[n-1]\right) =$$

$$= \frac{1}{4}\delta[n] + \frac{1}{4}\left(\frac{1}{2}\right)^{n} u[n-1]$$

Άρα $y[n] = y_{zs}[n] + y_{zi}[n] =$

$$= \delta[n] + \left(\frac{1}{2}\right)^{n+2} u[n-1]\left(9 - (2n-5)(-1)^{n}\right) + \frac{1}{4}\delta[n] + \left(\frac{1}{2}\right)^{n+2} u[n-1] =$$

$$= \frac{5}{4}\delta[n] + \left(\frac{1}{2}\right)^{n+2} u[n-1]\left(9 + 1 - (2n-5)(-1)^{n}\right) =$$

$$= \frac{5}{4}\delta[n] + \left(\frac{1}{2}\right)^{n+2} u[n-1]\left(10 - (2n-5)(-1)^{n}\right)$$

2.7) $x[n] = \delta[n] + 2\delta[n-2]$

a) $X[k] = \sum_{n=0}^{5} x[n] e^{-\frac{j2\pi}{6} \cdot kn} =$

$= \sum_{n=0}^{4} e^{-j\frac{2\pi}{6}kn} \left(\delta[n] + 2\delta[n-2]\right) =$

$= 1 + 0 + 2e^{-j\frac{4\pi}{6}k} + 0 + 0 + 0 = 1 + 2e^{-j\frac{2\pi}{3}k}$

$X[0] = 1 + 2 = 3$

$X[1] = 1 + 2e^{-j\frac{2\pi}{3}} = 1 + 2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -j\sqrt{3}$

$X[2] = 1 + 2e^{-j\frac{4\pi}{3}} = 1 + 2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = j\sqrt{3}$

$X[3] = 1 + 2e^{-j2\pi} = 1 + 2 = 3$

$X[4] = 1 + 2e^{-j\frac{8\pi}{3}} = 1 + 2e^{-j\frac{2\pi}{3}} = -j\sqrt{3}$

$X[5] = 1 + 2e^{-j\frac{10\pi}{3}} = 1 + 2e^{-j\frac{4\pi}{3}} = j\sqrt{3}$

b) $Z[k] = e^{jk\frac{2\pi}{6}} \cdot X[k] = e^{jk\frac{2\pi}{6}} \left[1 + 2e^{-j\frac{4\pi}{6}k}\right] =$

$= e^{j\frac{2\pi k}{6}} + 2e^{-j\frac{2\pi k}{6}}$

$z[n] = \frac{1}{N} \sum_{k=0}^{N-1} Z[k] e^{j\frac{2\pi}{N}kn} =$

$= \frac{1}{6} \sum_{k=0}^{5} \left[e^{j\frac{2\pi k}{6}} + 2e^{-j\frac{2\pi k}{6}}\right] e^{j\frac{2\pi k}{6}n} =$

$$=\frac{1}{6}\sum_{k=0}^{5}\left[e^{j\frac{\pi k}{3}}+2e^{-j\frac{\pi k}{3}}\right]e^{j\frac{\pi k}{3}n}=$$

$$=\frac{1}{6}\Big[3+\left(e^{j\frac{\pi}{3}}+2e^{-j\frac{\pi}{3}}\right)e^{j\frac{\pi}{3}n}+\left(e^{j\frac{2\pi}{3}}+2e^{-j\frac{2\pi}{3}}\right)e^{j\frac{2\pi}{3}n}+$$

$$+\left(e^{j\pi}+2e^{-j\pi}\right)e^{j\pi n}+\left(e^{j\frac{4\pi}{3}}+2e^{-j\frac{4\pi}{3}}\right)e^{j\frac{4\pi}{3}n}+\left(e^{j\frac{5\pi}{3}}+2e^{-j\frac{5\pi}{3}}\right)e^{j\frac{5\pi}{3}n}\Big]=$$

$$=\frac{1}{6}\Big[3+\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}+2\cdot\frac{1}{2}-2\cdot j\frac{\sqrt{3}}{2}\right)e^{j\frac{\pi}{3}n}+\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}-2\cdot\frac{1}{2}-j2\cdot\frac{\sqrt{3}}{2}\right)e^{j\frac{2\pi}{3}n}+$$

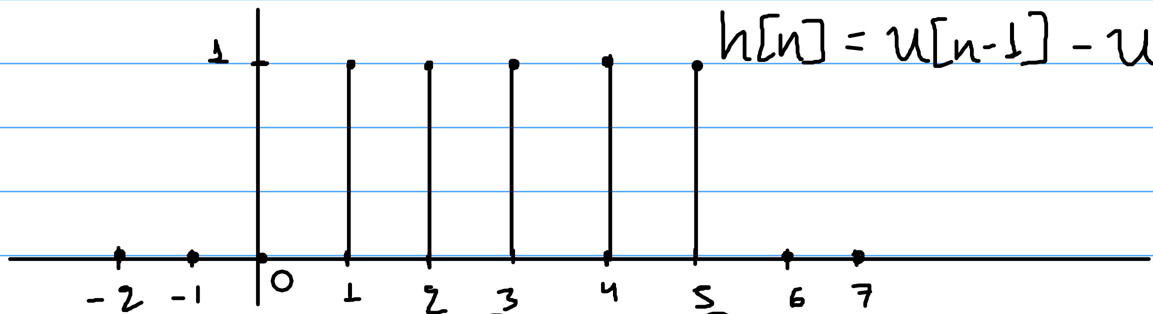
$$+(-1-2)(-1)^{n}+\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}-2\cdot\frac{1}{2}+j\frac{\sqrt{3}}{2}\cdot 2\right)e^{j\frac{4\pi}{3}n}+\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}+2\cdot\frac{1}{2}+2j\frac{\sqrt{3}}{2}\right)e^{j\frac{5\pi}{3}n}\Big]=$$

$$=\frac{1}{6}\Big[3\left(1+(-1)^{n+1}\right)+\left(\frac{3}{2}-j\frac{\sqrt{3}}{2}\right)e^{j\frac{\pi}{3}n}-\left(\frac{3}{2}+j\frac{\sqrt{3}}{2}\right)e^{j\frac{2\pi}{3}n}+\left(\frac{3}{2}-j\frac{\sqrt{3}}{2}\right)(-1)^{n+1}e^{j\frac{\pi}{3}n}+$$

$$-\left(\frac{3}{2}+j\frac{\sqrt{3}}{2}\right)(-1)^{n+1}e^{j\frac{2\pi}{3}n}\Big]=\frac{1}{6}\Big[3\left(1+(-1)^{n+1}\right)+\left(\frac{3}{2}-j\frac{\sqrt{3}}{2}\right)e^{j\frac{\pi}{3}n}\left(1+(-1)^{n}\right)-$$

$$-\left(\frac{3}{2}+j\frac{\sqrt{3}}{2}\right)e^{j\frac{2\pi}{3}n}\left(1+(-1)^{n+1}\right)\Big]=\frac{\left(1+(-1)^{n+1}\right)}{6}\Big[3+\left(\frac{3}{2}-j\frac{\sqrt{3}}{2}\right)e^{j\frac{\pi}{3}n}-\left(\frac{3}{2}+j\frac{\sqrt{3}}{2}\right)e^{j\frac{2\pi}{3}n}\Big]$$

γ)



$h[n]=u[n-1]-u[n-6]$

$$h[n]=\delta[n-1]+\delta[n-2]+\delta[n-3]+\delta[n-4]+\delta[n-5]$$

$$H[k]=\sum_{n=0}^{N-1}h[n]e^{-\frac{2j\pi}{N}kn}=$$

$$=\sum_{n=0}^{5}\left(\delta[n-1]+\delta[n-2]+\delta[n-3]+\delta[n-4]+\delta[n-5]\right)e^{-\frac{j\pi}{3}kn}=$$

$$= e^{-j\frac{\pi k}{3}} + e^{-j\frac{2\pi}{3}k} + e^{-j\pi k} + e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k} =$$

$$H[0] = 5 \quad , \quad H[1] = e^{-j\frac{\pi}{3}} + e^{j\frac{\pi}{3}} + e^{-j\frac{2\pi}{3}} + e^{j\frac{2\pi}{3}} + e^{-j\pi} = 1 - 1 - 1 = -1$$

$$H[2] = e^{-j\frac{2\pi}{3}} - e^{-j\frac{\pi}{3}} + e^{-j2\pi} + e^{-j\frac{2\pi}{3}} - e^{-j\frac{\pi}{3}} = 2\left(e^{-j\frac{2\pi}{3}} - e^{-j\frac{\pi}{3}}\right) + 1 = 2\left(-\frac{1}{2} - j\cancel{\frac{\sqrt{3}}{2}} - \frac{1}{2} + j\cancel{\frac{\sqrt{3}}{2}}\right) + 1 = -1$$

$$H[3] = e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi} + e^{-j4\pi} + e^{-j5\pi} = \cancel{-1} + \cancel{1} \cancel{-1} + \cancel{1} - 1 = -1$$

$$H[4] = -e^{-j\frac{\pi}{3}} + e^{-j\frac{2\pi}{3}} + e^{-j4\pi} - e^{-j\frac{\pi}{3}} + e^{-j\frac{2\pi}{3}} = 2\left(e^{-j\frac{2\pi}{3}} - e^{-j\frac{\pi}{3}}\right) + 1 = -1$$

$$H[5] = e^{j\frac{2\pi}{3}} + e^{j\frac{\pi}{3}} + e^{-j5\pi} + e^{-j\frac{2\pi}{3}} +^{-j\frac{\pi}{3}} = \cancel{2 \cdot \frac{1}{2}} - \cancel{2 \cdot \frac{1}{2}} - 1 = -1$$

$$F[k] = X[k] \cdot H[k]$$

$F[0] = 3 \cdot 5 = 15$

$F[1] = -j\sqrt{3} \cdot (-1) = j\sqrt{3}$

$F[2] = j\sqrt{3} \cdot (-1) = -j\sqrt{3}$

$F[3] = 3 \cdot (-1) = -3$

$F[4] = -j\sqrt{3} \cdot (-1) = j\sqrt{3}$

$F[5] = j\sqrt{3} \cdot (-1) = -j\sqrt{3}$

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{6} \sum_{k=0}^{5} F[k] e^{j\frac{\pi}{3}kn} =$$

$$= \frac{1}{6}\left[15 + j\sqrt{3}e^{j\frac{\pi}{3}n} - j\sqrt{3}e^{j\frac{2\pi}{3}n} - 3e^{j\pi n} + j\sqrt{3}e^{j\frac{4\pi}{3}n} - j\sqrt{3}e^{j\frac{5\pi}{3}n}\right] =$$

$$= \frac{1}{6}\left[15 + 3(-1)^{n+1} + j\sqrt{3}\left(e^{j\frac{\pi}{3}n} + e^{j\pi n}e^{j\frac{\pi}{3}n} - e^{j\frac{2\pi}{3}n} - e^{j\pi n}e^{j\frac{2\pi}{3}n}\right)\right] =$$

$$= \frac{1}{6}\left[15 + 3(-1)^{n+1} + j\sqrt{3}\left(e^{j\frac{\pi}{3}n}\left(1 + (-1)^n\right) - e^{j\frac{2\pi}{3}n}\left(1 + (-1)^n\right)\right)\right] =$$

$$= \frac{1}{6}\left[3\left(5 + (-1)^{n+1}\right) + j\sqrt{3}\left(1 + (-1)^n\right)\left(e^{j\frac{\pi}{3}n} - e^{j\frac{2\pi}{3}n}\right)\right]$$

$$f[0] = \frac{1}{6}\left[3 \cdot 4 + j2\sqrt{3} \cdot (\cancel{1} - \cancel{1})\right] = 2$$

$$f[1] = \frac{1}{6}\left[3 \cdot 6 + j\sqrt{3} \cdot 0\right] = 3$$

$$f[2] = \frac{1}{6}\left[3\cdot 4 + j2\sqrt{3}\left(e^{j\frac{2\pi}{3}} - e^{j\frac{4\pi}{3}}\right)\right] = 2 + j\frac{\sqrt{3}}{3}\left(-\cancel{\frac{1}{2}} + j\frac{\sqrt{3}}{2} \cancel{+\frac{1}{2}} + j\frac{\sqrt{3}}{2}\right) =$$

$$= 2 - \frac{\sqrt{3}\cdot\sqrt{3}}{3} = 1$$

$$f[3] = \frac{1}{6}\left[3\cdot 6 + j\sqrt{3}\cdot 0\right] = 3$$

$$f[4] = \frac{1}{6}\left[3\cdot 4 + j2\sqrt{3}\left(e^{j\frac{4\pi}{3}} - e^{j\frac{8\pi}{3}}\right)\right] = 2 + j\frac{\sqrt{3}}{3}\left(-\cancel{\frac{1}{2}} - j\frac{\sqrt{3}}{2} \cancel{+\frac{1}{2}} - j\frac{\sqrt{3}}{2}\right) =$$

$$= 2 + \frac{\sqrt{3}\cdot\sqrt{3}}{3} = 3$$

$$f[5] = \frac{1}{6}\left[3\cdot 6 + j\sqrt{3}\cdot 0\right] = 3$$

δ) $G[k] = X[k]H[k] \Rightarrow g[n] = x[n] \circledast h[n]$

$$g[n] = x[n] \circledast h[n] = \sum_{m=0}^{N-1} x[m]\, h\left([n-m]\right)_N =$$

$$= \sum_{m=0}^{7} x[m]\, h\left([n-m]\right)_8 =$$

$$h[n] = \begin{cases} 1, & n = 1,2,3,4,5 \\ 0, & \text{αλλού} \end{cases}$$

$$g[0] = \sum_{m=0}^{7} x[m]h[-m] = 0$$

$$g[1] = \sum_{m=0}^{7} x[m]h[1-m] = x[0] = 1$$

$$g[2] = \sum_{m=0}^{7} x[m]h[2-m] = x[0] + x[1] = 1 + 0 = 1$$

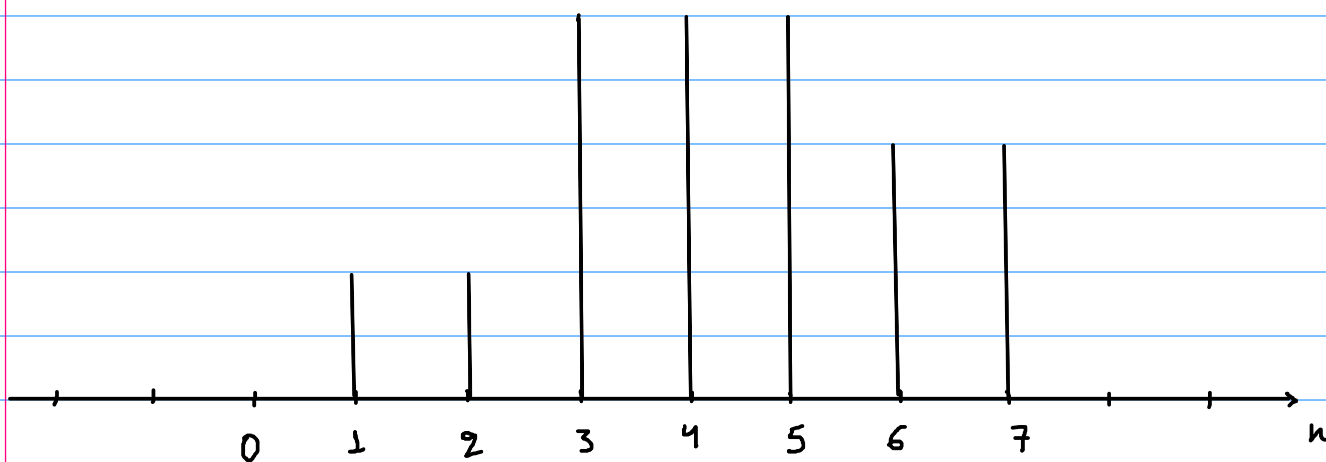
$$g[3] = \sum_{m=0}^{7} x[m]h[3-m] = x[0] + x[1] + x[2] = 1 + 0 + 2 = 3$$

$$g[4] = \sum_{m=0}^{7} x[m]h[4-m] = X[0]+X[1]+X[2]+X[3] = 1+0+2+0 = 3$$

$$g[5] = \sum_{m=0}^{7} x[m]h[5-m] = X[0]+\cancel{X[1]}+X[2]+\cancel{X[3]}+\cancel{X[4]} = 3$$

$$g[6] = \sum_{m=0}^{7} x[m]h[6-m] = \cancel{X[1]}+X[2]+\cancel{X[3]}+\cancel{X[4]}+\cancel{X[5]} = 2$$

$$g[7] = \sum_{m=0}^{7} x[m]h[7-m] = X[2]+\cancel{X[3]}+\cancel{X[4]}+X[5]+\cancel{X[6]} = 2$$



e) $y[n] = x[n] * h[n] = \left(\delta[n] + 2\delta[n-2]\right) * \left(u[n-1] - u[n-6]\right) =$

$= \delta[n]*u[n-1] - \delta[n]*u[n-6] + 2\delta[n-2]u[n-1] - 2\delta[n-2]u[n-6] =$

$= u[n-1] - u[n-6] + 2u[n-3] - 2u[n-8] =$

$= \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + 2\big(\delta[n-3] + \delta[n-4] +$
$+ \delta[n-5] + \delta[n-6] + \delta[n-7]\big) =$

$= \delta[n-1] + \delta[n-2] + 3\delta[n-3] + 3\delta[n-4] + 3\delta[n-5] + 2\delta[n-6] + 2\delta[n-7]$

$y[n] \equiv g[n] \neq f[n]$