Basic question 1: Complexity

Here is a simple function that finds the pair of elements in the input list *xs* whose sum is closest to *n*:

```
function find(n: int, xs: list of ints) -> pair of ints:
    pairs = new list of (pair of ints)

    for i from 0 to length(xs):
        for j from i+1 to length(xs):
            pairs.add((xs[i], xs[j]))

    closest = pairs[0]

    for each p in pairs:
        if abs(n - sum(p)) < abs(n - sum(closest)):
            closest = p

    return closest
```

What is the asymptotic complexity of the function `find` in the number of elements *N* of `xs`? You may assume that adding an element to a list and all arithmetic operations (including calculating the sum of a pair) are O(1).

Write your answer in O-notation. Be as exact and simple as possible. Justify why the complexity of the function has this order of growth.

Answer

Complexity: ______________________________

Justification (you can also add notes directly in the code above):

Basic question 2: Sorting

Perform a quicksort partitioning of the following array, using the median of the first, middle and last element as pivot:

0	1	2	3	4	5	6	7	8
63	65	35	5	30	10	47	75	4

Note: quicksorting the left and right parts of the partition is not part of this question.

Answer

Write down how the array looks after the partitioning:

0	1	2	3	4	5	6	7	8

What sequence of swaps did you make when partitioning the array?

If you used a different algorithm from that of your course, explain it here:

Basic question 3: Bug disinfection

A dynamic array should have *amortised* constant complexity, O(1), for adding and removing elements at the end. Here is an attempt at implementing a dynamic array (of strings):

```
class DynamicArray<Elem>:
    internal: array of strings
    size: int

    const sizeIncrease = 100

    method addLast(x: Elem):
        if this.size >= length(this.internal):
            this.resize()
        this.internal[this.size] = x
        this.size += 1

    method resize():
        maxSize = this.size + this.sizeIncrease
        oldInternal = this.internal
        this.internal = new array with maxSize cells
        for k from 0 to (but not including) maxSize:
            this.internal[k] = oldInternal[k]
```

(Note that the loop for k includes the starting index 0, but not the ending index maxSize.)

Unfortunately, two bugs have sneaked into the code somewhere, so it doesn't behave as it should.

Answer

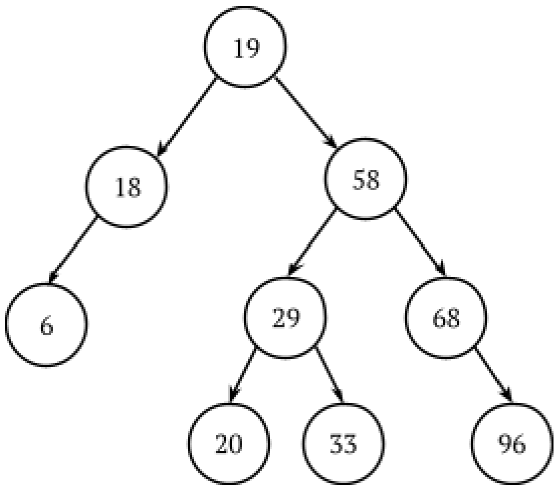
Mark the places in the code where the bugs are. Explain what the problem is and how to fix it:

Bug 1: ____

Bug 2: ____

Basic question 4: Search trees

Consider the AVL tree to the right. In this question you are going to insert 25 into the tree using the AVL insertion algorithm. Additionally, you should annotate every node in the resulting tree with its AVL *balance factor* (height of the right subtree minus the height of the left subtree).



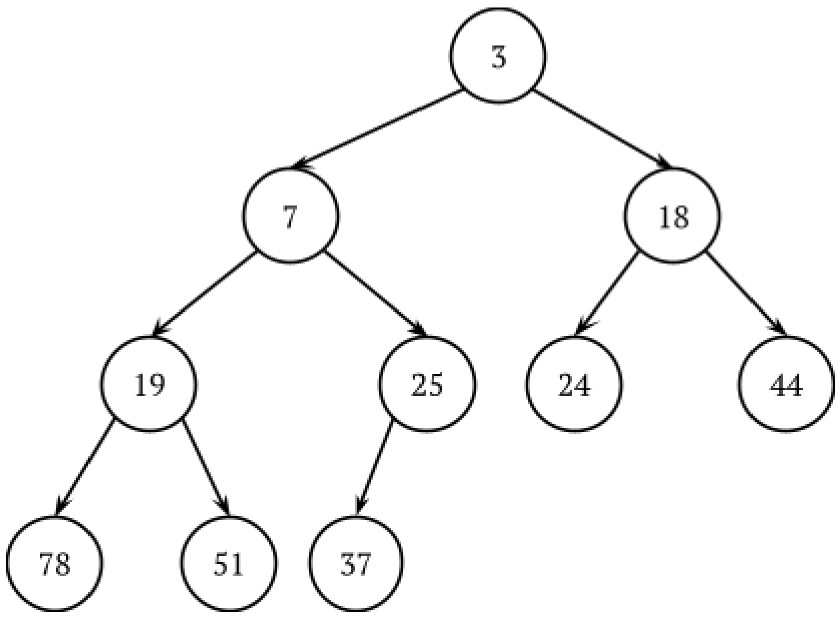
Answer

How does the tree look after you have inserted 25, but before rebalancing? Annotate the nodes in the resulting tree with their AVL balance factor.

How does the tree look after rebalancing? Again, annotate the nodes with their balance factor.

Basic question 5: Priority queues

You are given a minimum priority queue implemented as the following binary heap:



Answer

How does the above heap look when represented as an array?
You can choose whether to start at index 0 or index 1.

0	1	2	3	4	5	6	7	8	9	10

Now remove the minimum element from the priority queue. How does the array look afterwards?

0	1	2	3	4	5	6	7	8	9	10

Basic question 6: Hash tables

Suppose we have the following *open-addressing* hash table using *linear probing*. We are going to store strings (array of characters) consisting of the letters `A`, `C`, and `T` in the hash table. The hash code for the characters is: h(`A`) = 1, h(`C`) = 3, and h(`T`) = 5. The hash code for a string is the sum of the hash code of all characters in the string. For example, the hash code of `ATT` is:

$$h(\text{ATT}) = 1+5+5 = 11$$

The hash table uses the hash code to calculate an index by taking the remainder of the hash code divided by the size of the array (also called modular hashing):

$$index(\text{ATT}) = h(\text{ATT}) \% 10 = 11 \% 10 = 1$$

0	1	2	3	4	5	6	7	8	9
ACT	TT		C	CA	TTT	CC	AT		CCC

The hash table was created by adding the elements `C`, `AT`, `CA`, `CC`, `TT`, `ACT`, `CCC`, `TTT` **in an unknown order**. The array has never been resized, and no elements have been deleted. In what orders could the elements have been added?

A) `CC, CA, AT, C, CCC, TTT, ACT ,TT`

B) `TT, CCC, CC, TTT, AT, ACT, C, CA`

C) `TT, CC, CA, TTT, AT, CCC, ACT, C`

D) `C, ACT, TTT, TT, CC, CCC, AT, CA`

E) `CA, CCC, ACT, C, TTT, CC, TT, AT`

Determine which of these orders are possible (there may be several, or even none). For the others, explain why they are impossible.

Answer

Which orders are possible?

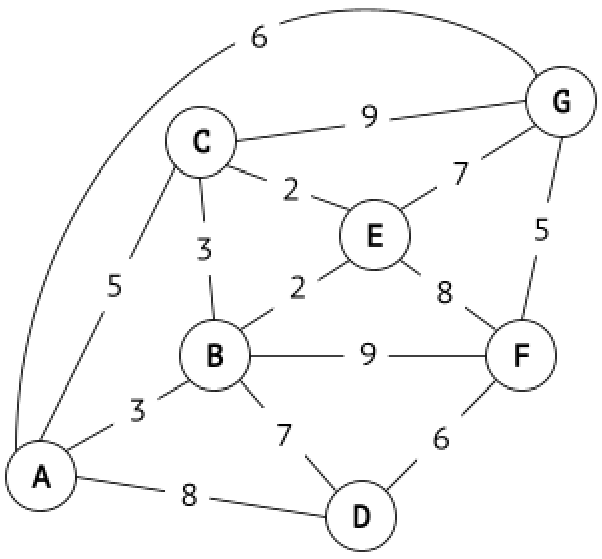
____

Explain why the others are impossible:

Basic question 7: Graphs

Perform Dijkstra's algorithm on the graph to the right, starting in node **A**. Show each step of the algorithm: which node is removed from the priority queue, which node(s) are added to the priority queue, and what the priority queue looks like after each iteration.

Write the priority queue like this: "`X:4, Y:6, Z:8`", where the numbers are the priorities.



Answer

Removed node	**Added node(s)**	**Priority queue after adding new nodes**
–	`A`	`A:0`

Note that you will not fill all rows in the table.

Basic question 8: Fill-in the blanks

Here is an iterative implementation of insertion into a set, represented as a binary search tree. However, someone has stolen parts of the code. Please repair it by filling in the blanks.

```
class BSTNode:
    value : number
    left  : BSTNode
    right : BSTNode

class BSTSet:
    root : BSTNode

    method add(value : number):
        parent = NULL
        node = ________________
        while node is not NULL:
            parent = node
            if value < node.value:
                node = ________________
            else if value > node.value:
                node = ________________
            else:  // the value is already in the set
                return
        newnode = new BSTNode(value, NULL, NULL)
        if parent is NULL:
            ________________ = newnode
        else if value < parent.value:
            ________________ = newnode
        else:  // now we know that value > parent.value
            ________________ = newnode
```

Answer

Fill in all blanks so that the resulting code is correct.

Advanced question 9: Complexity

The following function indexes takes an integer *x* as input and returns a list of integers:

```
function indexes(x: int) -> list of ints:
    l = new list of ints; i = 0; n = 1

    while n <= x:
        if (x & n) > 0:    // bitwise and-operation to check
            l.addLast(i)     // if i-th bit is set, takes constant time
        i = i + 1
        n = n * 2

    return l
```

The function converts the decimal number x to a binary number (a binary number uses 2 as a base instead of 10, and consists of ones and zeros), and checks at which places (indexes) the binary digit is one. For example, the decimal number 13 is 1101 in the binary number system, and our function would then return the list [0,2,3], because the 0-th, 2-nd, and 3-rd places are one.

We use the indexes function in the following powerset function:

```
function powerset(xs: list of ints) -> list of list of ints:
    ps = new list of list of ints

    for i from 0 to 2^(length of xs):
        p = new list of ints

        for each j in indexes(i):
            p.add(xs[j])

        ps.add(p)

    return ps
```

that returns the 'power set' of the input list xs, that is all sub-lists including the empty list and the input list itself. For example, calling powerset on [1,2,3] gives:

```
[[],[1],[2],[1,2],[3],[1,3],[2,3],[1,2,3]]
```

Asymptotically, what are the **time complexities** of the indexes and powerset functions, related to the size of their input? Assume that adding to a list and indexing is O(1). The bitwise 'and' (&) operator is O(1) as well.

Write your answers in O-notation. Be as exact and simple as possible. Explain the key reasons why each complexity has the order of growth you give.

Answer

Please answer on a separate page.

Advanced question 10: Disjoint arrays

Design an algorithm that takes two arrays, and returns true if the arrays are disjoint, i.e. have no elements in common.

You may freely use standard data structures and algorithms from the course in your solution, without explaining how they are implemented.

Your algorithm should take O(n log(m)) time, where n is the size of the larger array and m is the size of the smaller array.

Write down your algorithm as pseudocode (but Python, Java or Haskell are also fine). Explain which standard data structures and algorithms from the course that you have used.

Answer

Advanced question 11: Median Set

Your task is to create a data structure that can store a set of comparable elements, with special support for retrieving the *median* element in constant time. Note that we don't have any duplicates in a set. If the set is non-empty and the n elements in the set are:

$$x_0, x_1, \ldots, x_{n-1} \text{ with } x_{i-1} < x_i \text{ for all } 0 < i < n$$

then the median element is:

$$x_{\frac{n-1}{2}}$$

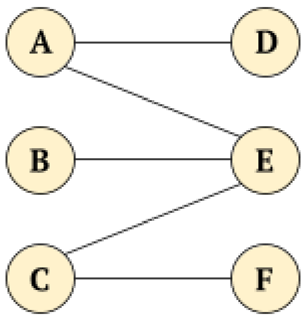
The other standard operations on a set should be supported as well. To summarise, the data structure should support the following operations, with time complexities as stated:

- add, O(log n)
- remove, O(log n)
- contains, O(log n)
- getMedian, O(1) (precondition: the set is non-empty)

You may freely use any standard data structure or algorithm from the course. Write down your data structure and the implementation of the operations using pseudo code, but you may use Java, Python or Haskell if you are more comfortable with that. Justify why your implementation of the operations has the required complexity.

Answer

Advanced question 12: Bipartite graphs



An undirected graph is *bipartite* iff the nodes can be divided into two disjoint subsets V_0 and V_1 such that all edges have one endpoint in V_0 and another in V_1. To the right is an example of a bipartite graph with V_0 = {A,B,C} and V_1 = {D,E,F}.

Design an efficient algorithm that takes a connected graph as input and returns true if and only if the graph is bipartite. By efficient we mean that its runtime complexity should be *better* than quadratic, O(N^2). Also describe which data structure(s) you use to implement the graph.

What is the asymptotic runtime complexity of your algorithm, in terms of the size N? (The size of the graph is the total number of vertices and edges, or $N = |V| + |E|$). Explain your reasoning.

Answer