For any $1 \leq k \leq L$

Assumptions 1 For some $s \geq 1$, the entries of $\sqrt{d}V_k$ are symmetric i.i.d., s^2 sub-Gaussian random variable, independent of d and L, with unit variance.

Assumptions 2 For some $C > 0$, independent of d and L, and for any $h \in \mathbb{R}^D$

$$\frac{\|h\|^2}{2} \leq \mathbb{E}[\|g(h, \theta_k)\|^2] \leq \|h\|^2 .$$
$$\mathbb{E}[\|g(h, \theta_k)\|^8] \leq C\,\|h\|^8 .$$

Proposition 2 [Admited ?] Consider a ResNet (4) such that Assumptions (A1) and (A2) are satisfied. If $L\alpha_L^2 \leq 1$, then, for any $\delta \in (0,1)$, with probability at least $1-\delta$,

$$\frac{\|h_L - h_0\|^2}{\|h_0\|^2} \leq \frac{2L\alpha_L^2}{\delta}.$$

Proposition 3 [Admited] Consider a ResNet (4) such that Assumptions (A1) and (A2) are satisfied.

(i) Assume that $d \geq 64$ and $\alpha_L^2 \leq \frac{2}{(\sqrt{C}s^4+4\sqrt{C}+16s^4)d}$. Then, for any $\delta \in (0,1)$, with probability at least $1-\delta$,

$$\frac{\|h_L - h_0\|^2}{\|h_0\|^2} > \exp\left(\frac{3L\alpha_L^2}{8} - \sqrt{\frac{11L\alpha_L^2}{d\delta}}\right) - 1,$$

provided that

$$2L\exp\left(-\frac{d}{64\alpha_L^2 s^2}\right) \leq \frac{\delta}{11}.$$

(ii) Assume that $\alpha_L^2 \leq \frac{1}{\sqrt{C}(d+128s^4)}$. Then, for any $\delta \in (0,1)$, with probability at least $1-\delta$,

$$\frac{\|h_L - h_0\|^2}{\|h_0\|^2} < \exp\left(L\alpha_L^2 + \sqrt{\frac{5L\alpha_L^2}{d\delta}}\right) + 1.$$

Corollaire (4). *Consider a ResNet (4) such that Assumptions (A1) and (A2) are satisfied, and let $\alpha_L = 1/L^\beta$, with $\beta > 0$.*

(i) If $\beta > \frac{1}{2}$, then

$$\frac{\|h_L - h_0\|}{\|h_0\|} \xrightarrow{\mathbb{P}} 0 \text{ as } L \to \infty.$$

(ii) If $\beta < \frac{1}{2}$ and $d \geq 9$, then

$$\frac{\|h_L - h_0\|}{\|h_0\|} \xrightarrow{\mathbb{P}} \infty \text{ as } L \to \infty.$$

(iii) If $\beta = \frac{1}{2}$, $d \geq 64$, $L \geq \left(\frac{1}{2}\sqrt{C}s^4 + 2\sqrt{C} + 8s^4)d + 96\sqrt{C}s^4\right)$, then, for any $\delta \in (0,1)$, with probability at least $1-\delta$,

$$\exp\left(\frac{3}{8} - \sqrt{\frac{22}{d\delta}}\right) - 1 < \frac{\|h_L - h_0\|^2}{\|h_0\|^2} < \exp\left(1 + \sqrt{\frac{10}{d\delta}}\right) + 1,$$

provided that

$$2L\exp\left(-\frac{Ld}{64s^2}\right) \leq \frac{\delta}{11}.$$

Proof: Statement (i) is a consequence of Proposition 2. We have $L\alpha_L^2 = \frac{L}{L^{2\beta}} = L^{1-2\beta}$, as $\beta > 1/2 \Leftrightarrow 1-2\beta < 0$ we have $L^{1-2\beta} = \frac{1}{L^{2\beta-1}} \underset{L\to+\infty}{\longrightarrow} 0$. Thus

$$\frac{\|h_L - h_0\|^2}{\|h_0\|^2} \leq \frac{2L\alpha_L^2}{\delta}. \underset{L\to+\infty}{\xrightarrow{\mathbb{P}}} 0$$

Statement (ii) is a consequence of Proposition 3. ☐