CS4261 Cheatsheet

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Nash Equilibrium

A **Game** is a general abstract framework for strategic interactions, with usually 1) A set of **Players** $N = \{1, \ldots, n\}$, and subsequently for each player $i \in N$, 2) A set of possible **Actions** $A_i = \{a_{i1}, a_{i2} \ldots\}$, and 3) a **Utility Function** $u_i : A \mapsto \mathbb{R}$, which indicates the utility Player i can get from an action profile, then lastly 4) a **(Pure) Action Profile** that denotes the actions taken by all the players $\vec{a} \in A_1 \times A_2 \times \ldots \times A_n = A$.

A **Normal Form Game** is a Matrix Representation of the player utilities for a 2-Player game. Conventionally, each element in the Normal Form Game is a pair of real values $C_{ij} = (u, v) \in \mathbb{R}^2$ where u and v are the utility of the row and column player given an action profile $\vec{a} = (a_{1,i}, a_{2,j})$.

Pure Nash Equilibrium

Given actions taken by everyone else $\vec{a}_{-i}$, the **Best Response** set of Player i is defined

$$BR_i(\vec{a}_{-i}) = \{b \in A_i | b \in \text{argmax } u_i(\vec{a}_{-i}, b)\}$$

An action profile is a **Pure Nash Equilibrium** if it is one best response for everyone given what others have chosen, i.e.

$$\forall i \in N, a_i \in BR_i(\vec{a}_{-i})$$

Analysis: Not all games have Pure Nash Equilibria.

Mixed Nash Equilibrium

Let $\vec{p} \in \Delta(A_i)$ be the probability distribution over Player i's actions. A **Mixed / Randomised Strategy Profile** is given by $\vec{p} = (\vec{p}_1, \ldots, \vec{p}_n) \in \Delta(A_1) \times \ldots \times \Delta(A_n)$. Player utility is $u_i(\vec{p}) = \sum_{\vec{a} \in A} u_i(\vec{a}) P(\vec{a}) = \mathbb{E}_{\vec{a} \sim \vec{p}}[u_i(\vec{a})]$.

A mixed profile is a **Mixed Nash Equilibrium** if

$$\forall i \in N, \vec{q}_i \in \Delta(A_i),\ u_i(\vec{p}) \geq u_i(\vec{p}_{-1}, \vec{q}_i)$$

Analysis: All games have Mixed Nash Equilibria.

In a Mixed Nash Equilibrium, a player is **Indifferent** if he gets the same **expected utility** from choosing any action (as a result of the other player playing a mixed strategy).

Compute Nash Equilibria in 2 Player Games

- Compute all NE in which at least one player plays a pure strategy.
- Compute all NE in which both players play mixed strategies. In this case, each player must be indifferent between the two strategies.

Dominant Strategies

A strategy $\vec{p} \in \Delta(A_i)$ **Dominates** $\vec{q} \in \Delta(A_i)$ if

$$\forall \vec{p}_{-i} \in \Delta(A_{-i}),\ u_i(\vec{p}_{-i}, \vec{p}) \geq u_i(\vec{p}_{-i}, \vec{q})$$

There are similar definitions for **Strictly Domination**.

Intuitively, it means no mather what others do, playing $\vec{p}$ is always better than $\vec{q}$.

Dominant Strategy Theorem

If an action $a \in A_i$ is **strictly dominated** by some strategy $\vec{p} \in \Delta(A_i)$, then action a is never played with any positive probability in any Nash Equilibrium.

Note: The theorem enables us to prune actions that will not occur in any Nash Equilibria.

Auction

Single-Item Auction

Types of Single-Item Auction

- **English Auction** - Auctioneer sets a starting price. Bidders take turn raising their bids. The bidder makes the last bid wins and pay his bid.
- **Japanese Auction** - Auctioneer sets a starting price and raises it. A bidder can drop out and not return once dropped. The last standing bidder gets the item and pays the current price.
- **Vickrey / Second-Price Auction** - All bidders submit bids simultaneously. The highest bidder wins and pays the second highest price.

Vickrey Auction Problem Specification

There are n players $N = \{1, 2, \ldots, n\}$, each with a valuation of the item v_i. The actions are to place a bid at different prices. The payoff for a player is $v - p$ if getting the item, and 0 otherwise.

Analysis

Vickrey Auctions are **Truthful**, i.e. bidding according to one's true valuation is a dominant strategy.

First-Price Auctions are **Not Truthful**.

Note: Dominant strategies are Nash Equilibria in Auction games, but they are not necessarily the only Nash Equilibria.

Multi-Unit Auction

Game Specification

There are n players $N = \{1, 2, \ldots, n\}$, each with a valuation of the item v_i. There are $k \leq n$ identical copies of the item.

The objective is to design a mechanism where

- Truthful bidding is a dominant strategy
- Items are allocated to the k highest bidders

Vickrey Clarke Groves (VCG) Mechanism

Procedures:

1. Choose some outcome o^* that maximises social welfare $\sum_i v_i(o^*)$
2. Calculate the payment that Player j must take with $p_j = \sum_{i \neq j} v_i(o^*_{-j}) - \sum_{i \neq j} v_i(o^*)$, where o^*_{-j} is the outcome that maximises $\sum_{i \neq j} v_i(o_{-j})$.

Note: The payment for each Player is essentially the **Externality** that he imposes on other players, which is the difference in the max welfare of others between if he is absent and if present.

Analysis: VCG is truthful. Vickrey Auction is a special case of VCG.

Combinatorial Auction

Problem Specification

There are n Players and m possibly distinct items for sale. Each player has a valuation for each subset of the m objects.

VCG is truthful, but it can be computationally intensive, and suffers from **Revenue Non-Monotonicty**, a paradox where adding more players in the bidding game may lead to a decrease in the **Revenue**, i.e. sum of all players' payment $R = \sum_{i=1}^{n} p_i$.

Note: Single-Item Auctions have no revenue non-monotonicity.

Facility Location

Problem Specification

There are n players $N = \{1, \ldots, n\}$, each with a location $x_i \in \mathbb{R}$ assuming $x_1 \leq x_2 \leq \ldots \leq x_n$ for convenience.

The objective is to design $f : \mathbb{R}^n \mapsto \mathbb{R}$ that minimises either of

- **Total Cost** - $\sum_{i \in N} |f(\vec{x}) - x_i|$
- **Max Cost** - $\max_{i \in N} |f(\vec{x}) - x_i|$

Analysis: OPT for Total Cost is **Not Truthful**. OPT for Max Cost is **Truthful** if it always "snaps" to a median player.

Max Cost Approximation Theorems

Deterministic Case

Any deterministic truthful mechanism for facility location has a worst-case approximation ratio ≤ 2 to the maximum cost.

Randomised Case

Any randomised truthful mechanism for facility location has a worst-case approximation ratio $\leq \frac{3}{2}$ to the maximum cost.

Routing Games

In a traffic network, players are drivers trying to find a route that minimises their total traffic time.

- **Proportion Version** There is 1 unit of traffic to allocate in total. Drivers are considered proportion of the total traffic.
- **Atomic Version** The traffic consists of $k \in \mathbb{N}$ drivers, each being an atomic entity.

Price of Anarchy (PoA) is the ratio of the social cost under the worst case Nash Equilibrium and under socially optimal solution

$$PoA = \frac{\text{WorstNash}(G)}{OPT(G)}$$

Analysis: 1) $PoA \geq 1$ with the smaller being the better. 2) All Nash Equilibria in a Routing Game have the same social cost.

Atomic Version

Atomic Routing Game Theorem

In an atomic routing game, a pure Nash Equilibrium flow always exists.

Higher Level Idea

Every atomic routing game is a potential game, where all players are inadvertently and collectively striving to optimise a potential function, $\Phi(f)$,

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

Analysis: When a player deviates (changes path), change in the deviator's individual cost is equal to $\triangle\Phi$. "Alignment in individual and social objective".

Cooperative Games

The Core

Induced Subgraph Games

Problem Specification