

1. Poisson Processes

To **homogenize a non-homogen.** ~: [1] Define $\Lambda(t) = \int_0^t \lambda(u) du$
[2] $Y(s) = X(t)$ where $s = \Lambda(t)$ homogen. PoisProc. rate 1.

Given $X(t) = n$, the joint distribution of waiting time W_1, \dots, W_n is

$$f(w_1, w_2, \dots, w_n | X(t) = n) = \frac{n!}{t^n}, 0 \leq w_1 \leq w_2 \leq \dots \leq w_n \leq t$$

It is the joint distribution of n **ranked** independent $Unif(0, t)$ r.v.'s.

Given $X(t) = n$, the distribution of the k -th waiting time has the same distribution as that of the k -th order statistic of n independent $Unif(0, t)$ r.v.'s.

$$f_k(x) = \frac{n!}{(n-k)!(k-1)!} \frac{1}{t} \left(\frac{x}{t}\right)^{k-1} \left(1 - \frac{x}{t}\right)^{n-k}, 0 \leq x \leq t$$

Consider $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ two independent Poisson Processes with rates λ_1 and λ_2 . Define W_n^X and W_m^Y as the waiting time of the n -th and m -th waiting time of $X(t)$ and $Y(t)$ respectively.

$$P(W_n^X < W_m^Y) = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n+m-1-k}$$

Analysis: It is equivalent as getting n or more heads in $n + m - 1$ tosses where getting a head has probability $\lambda_1 / (\lambda_1 + \lambda_2)$.

Compound Poisson Process

$$E[X(t)] = \lambda t E[Y_i] \text{ and } Var[X(t)] = \lambda t (E[Y_i]^2 + Var(Y_i))$$

$$\lambda = \lambda_1 + \lambda_2 \quad F(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} F_1(x) + \frac{\lambda_2}{\lambda_1 + \lambda_2} F_2(x)$$

Conditional Poisson Process

$E[N(t)] = E(L)t$ and $Var(N(t)) = tE(L) + t^2 Var(L)$
Conditional probability of L given $N(t) = n$ (posterior),

$$P(L \leq x | N(t) = n) = \frac{\int_0^x e^{-\lambda t} (\lambda t)^n g(\lambda) d\lambda}{\int_0^\infty e^{-\lambda t} (\lambda t)^n g(\lambda) d\lambda}$$

2. Continuous Time Markov Chains

Overview

Note: For an absorbing state i , we may set $v_i = 0$.

Define the **Transition Probabilities** of a CTMC $X(t)$ as

$$P_{ij}(t) := P(X(t+s) = j | X(s) = i)$$

Note: [1] $P(t)$ uniquely specifies a CTMC. [2] $P_{ij} \neq P_{ji}(t)$.

Chapman-Kolmogorov Equation

$$P_{ij}(t+s) = \sum_{k \in S} P_{ik}(t) P_{kj}(s)$$

Discretisation of CTMC

For a CTMC $\{X(t), t \geq 0\}$, $\{Y_1(n)\}_{n \geq 0}$ discretises it at equal intervals if for some constant $l > 0$,

$$Y_1(n) = X(nl), n = 0, 1, 2, \dots$$

Analysis: $Y_1(n)$ has state space S and transition matrix $P(l)$.

For a CTMC $\{X(t), t \geq 0\}$, $\{Y_2(n)\}_{n \geq 0}$ is the **Embedded Chain** if it only considers the states visited by $X(t)$.

Analysis: $Y_2(n)$ has state space S and transition matrix P .

Infinitesimal Generator

Lemma: Transition Rates, for a CTMC,

- $\lim_{h \rightarrow 0} \frac{P_{ii}(h) - P_{ii}(0)}{h} = -v_i$
- $\lim_{h \rightarrow 0} \frac{P_{ij}(h) - P_{ij}(0)}{h} = v_i P_{ij}$, for all $i \neq j$

The **Infinitesimal Generator** G of a CTMC is defined as

$$G = (G_{ij})_{S \times S} = P'(0), \text{ where } G_{ij} = \begin{cases} -v_i, & i = j \\ v_i P_{ij}, & i \neq j \end{cases}$$

Kolmogorov's Forward & Backward Equations respectively:

$$P'(t) = P(t)G \Leftrightarrow P'_{ij}(t) = \sum_{k \neq i} P_{ik}(t) q_{kj} - v_j P_{ij}(t)$$

$$P'(t) = GP(t) \Leftrightarrow P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

CTMC Long-Term Properties

Stationary Distribution

For a CTMC $\{X(t), t \geq 0\}$, a row vector $\pi = (\pi_i)_{i \in S}$ with $\pi_i \geq 0$ and $\sum_i \pi_i = 1$ is a **Stationary Distribution** if for all $t \geq 0$,

$$\pi = \pi P(t),$$

Global Balancing Equations

$$\pi G = 0 \Leftrightarrow \sum_{j \neq i} \pi_i q_{ij} = v_j \pi_j, \forall j$$

Limiting Distribution

For a CTMC $\{X(t), t \geq 0\}$, its **Limiting Distribution**, $\{P_j, j \in S\}$, is:

$$P_j = \lim_{t \rightarrow \infty} P_{ij}(t)$$

Note: [1] For each j , the limit needs to exist and be the same for all i . [2] When both π and P exists, $\pi = P$.

If $X(t)$ satisfies conditions below, it is **Ergodic** (converse not true):

[1] All states of $X(t)$ **communicate**. [2] $X(t)$ is **positive recurrent**, i.e. for all $i, j \in S$, $\mathbb{E}[\min_{t \geq 0} \{X(t) = j | X(0) = i\}] < \infty$

An ergodic chain has stationary & limiting distributions & equal.

Suppose embedded chain stationary distribution ψ . Then $\forall i, j \in S$,

$$\psi_i = \frac{\pi_i v_i}{\sum_j \pi_j v_j} \Leftrightarrow \pi_i = \frac{\psi_i / v_i}{\sum_j \psi_j / v_j}$$

Time Reversibility

For an **ergodic** CTMC $\{X(t), t \geq 0\}$ and a sufficiently large t , define reversed process $\{Y(t), t \geq 0\}$

$$Y(0) = X(t), Y(s) = X(t-s), 0 < s < t$$

$\{X(t), t \geq 0\}$ is **Time-Reversible** if $X(t)$ and $Y(t)$ has the same probability structure: [1] Same v , and [2] Same jump matrix.

Local Balanced Equations

$$\pi_j q_{ji} = \pi_i q_{ij}, \forall i, j$$

If it is satisfied, $X(t)$ is time reversible with limiting distribution π .

Proposition: Time Reversibility Subset

Truncate a time-reversible CTMC $X(t)$ from S to $A \subseteq S$, then it remains time-reversible and has limiting distribution

$$\pi_j^A = \frac{\pi_j}{\sum_{i \in A} \pi_i}, \forall j \in A$$

Proposition: Time Reversibility Vectors

For CTMCs $\{X_i(t), t \geq 0\}$, $i = 1, 2, \dots, n$ time reversible, the vector process $\{(X_1(t), \dots, X_n(t)), t \geq 0\}$ is also time reversible.

CTMC Techniques

Uniformization

For a CTMC $\{X(t), t \geq 0\}$, where $\exists v \in \mathbb{R}$ s.t. $v_i \leq v, \forall i \in S$,

$$P_{ij}(t) = \sum_{n=0}^{\infty} (P^*)_{ij} \frac{(vt)^n}{n!} e^{-vt}, \text{ where } P^*_{ij} = \begin{cases} 1 - v_i/v, & i = j \\ (v_i/v)P_{ij}, & i \neq j \end{cases}$$

Intuition: P^* is the jump matrix after **Uniformisation**. A CTMC with identical v_i is a Poisson Process with rate v_i .

CTMC with Absorbing States

For a CTMC $\{X(t), t \geq 0\}$, if there is a state i s.t. $\forall t > 0, s \geq 0$,

$$P(X(t+s) = i | X(s) = i) = 1$$

, (or $P_{ii}(t) = 1, \forall t > 0$), then we call i an **Absorbing State**.

Assume state 0 is absorbing, **probability of absorbing** $u_i = \lim_{t \rightarrow \infty} P(X(t) = 0 | X(0) = i)$ from state i by CTMC is the same as that based on its embedded chain.

Define **expected time of absorption** w_i for starting at state i . **Case 1** When $i = 0, w_i = 0$. **Case 2a** When $u_i < 1, w_i = \infty$. **Case 2b** When $u_i = 1, w_i = \mathbb{E}[\text{time till 1st jump}] + \sum_{j \neq i} P_{ij} w_j$.

3. Renewal Process

Overview

A **Renewal Process** is a counting process $\{N(t), t \geq 0\}$ for a sequence of non-negative r.v.s $\{X_1, X_2, \dots\}$ that are iid. with a distribution F .

- $F(x) = P(X_k \leq x), k = 1, 2, \dots$, CDF of sojourn time
- $F_k(x) = P(W_k \leq x), k = 1, 2, \dots$, CDF of waiting time W_k .
- $M(t) = E[N(t)]$, **Renewal Function**, expected # of renewals

Properties

- $N(t) \geq k \Leftrightarrow W_k \leq t$
- $W_{N(t)} \leq t < W_{N(t)+1}$
- $P(N(t) = k) = F_k(t) - F_{k+1}(t)$
- $F_k(t) = \int_0^t F_{k-1}(t-y) dF(y)$, one-step analysis

The Renewal Equation - For a renewal process with sojourn times distributed as F , let $M(t) = E[X(t)]$, then

$$M(t) = F(t) + \int_0^t M(t-x) f(x) dx$$

Waiting time in Renewal Process:

$$\mathbb{E}[W_{N(t)+1}] = \mathbb{E}[X_1](M(t) + 1)$$

Note: Not a Random Sum as $N(t) + 1$ not independent with X_i .

Special Random Variables

- **Excess Time / Residual Time:** $\gamma_t = W_{N(t)+1} - t$
- **Current Life / Age:** $\delta_t = t - W_{N(t)} \geq 0$
- **Total Life:** $\beta_t = \gamma_t + \delta_t$

Special Case: Poisson Distribution

- $\gamma_t \sim \text{Exp}(\lambda)$
- δ_t follows $\text{Exp}(\lambda)$ truncated at t .
- $\mathbb{E}[\beta_t] = 1/\lambda + (1 - \exp(-\lambda t))/\lambda$

Limiting Behaviours

Elementary Renewal Theorem

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X_k]}$$

Central Limit Theorem for Renewal Process

Let $\mu = E(X_k)$, $\sigma^2 = \text{Var}(X_k)$, then as $t \rightarrow \infty$, $\frac{\text{Var}(N(t))}{t} \rightarrow \frac{\sigma^2}{\mu^3}$ and

$$N(t) \sim \mathcal{N}\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right) \text{ approximately}$$

Generalisation

Renewal Reward Process

Given a renewal process $N(t)$ with interarrival times $X_n, n \geq 1$ and suppose there is a reward for each renewal R_n that are i.i.d., the **Renewal Reward Process** $\{R(t), t \geq 0\}$ is

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

Note: [1] R_n can depend on X_n , time. [2] Rewards can occur between/along renewals.

Limiting Theorems for Renewal Reward Process

For $E[R_n] < \infty$ and $E[X_n] < \infty$,

$$\lim_{t \rightarrow \infty} \frac{E[R(t)]}{t} = \frac{E[R_n]}{E[X_n]}$$

Example: Avg. Current Life $\lim_{t \rightarrow \infty} (\int_0^t \delta(s) ds) / t = E[X^2] / 2E[X]$.

Regenerative Process

A stochastic process $\{X(t), t \geq 0\}$ is a **Regenerative Process** if there exists time pts when the process probabilistically restarts itself.

Note: [1] "Restart" includes both same transition & whether current state is same as initial. [2] Neither of MC & RegenProc \subseteq the other.

Delayed Renewal Process

A **Delayed Renewal Process** is one when the component in operation at $t = 0$ is not new, but all subsequent ones are.

Analysis: Same set of parameters & limiting behaviours.

4. Brownian Motion

Multi-Normal Distribution

A k -dim random vector $\mathbf{X} = (X_1, \dots, X_k)'$ with mean vector $\mu \in \mathbb{R}_{k \times 1}$ and covariance matrix $\Sigma \in \mathbb{R}_{k \times k}$ is **multivariate normally distributed** $\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$ if the joint density function is

$$f(x_1, \dots, x_k) = \frac{1}{\sqrt{2\pi\|\Sigma\|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)' \Sigma^{-1}(\mathbf{x} - \mu)\right)$$

For any $\mathbf{a} \in \mathbb{R}_{1 \times k}$, $\mathbf{a}\mathbf{X} \sim \mathcal{N}(\mathbf{a}\mu, \mathbf{a}\Sigma\mathbf{a}')$.

For any matrix $\mathbf{Q} \in \mathbb{R}_{m \times k}$ with rank $m \leq k$, $\mathbf{Q}\mathbf{X} \sim \mathcal{N}(\mathbf{Q}\mu, \mathbf{Q}\Sigma\mathbf{Q}')$.

For any partition,

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$$

And the conditional distribution is still normal:

$$\mathbf{X}_1 | \mathbf{X}_2 \sim \mathcal{N}(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}(\mathbf{X}_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

Overview

(Standard) Brown Motion

Process $\{X(t), t \geq 0\}$ is a **Brownian Motion** with parameter σ if: [1] $X(0) = 0$, [2] $\{X(t), t \geq 0\}$ has stationary & independent increments, and [3] For every $t > 0$, $X(t) \sim \mathcal{N}(0, \sigma^2 t)$.

A **Standard Brownian Motion** $\{B(t), t \geq 0\}$ has $\sigma = 0$.

For time $t_1 \leq t_2$,

$$\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \begin{pmatrix} \sigma^2 t_1 & \sigma^2 t_1 \\ \sigma^2 t_1 & \sigma^2 t_2 \end{pmatrix})$$

For any time $s, t > 0$,

- If $s \geq t$, $X(s)|X(t) \sim \mathcal{N}(X(t), \sigma^2(s - t))$.
- If $s < t$, $X(s)|X(t) \sim \mathcal{N}(\frac{s}{t}X(t), \sigma^2 s^2 - \sigma^2 s^2/t^2)$

Brownian Motion with Drift

A **Brownian Motion with Drift** $\{X(t), t \geq 0\}$ with parameters μ and σ : $X(t) = \sigma B(t) + \mu t$.

For any time $s, t > 0$,

$$X(s)|X(t) \sim \mathcal{N}\left(\mu, \frac{\min(s, t)}{t}[X(t) - \mu t], \sigma^2 s - \sigma^2 [\min(s, t)]^2/t\right)$$

Geometric Brownian Motion

A **Geometric Brownian Motion** $Y(t)$ with parameters μ and σ is:

$$Y(t) = e^{\sigma B(t) + \mu t}$$

Note: $Y(0) = 1$ and $Y(t) \geq 0, \forall t$.

For any time $s < t$,

$$\mathbb{E}[Y(t)] = M_{X(t)}(1) = e^{\mu t + \sigma^2 t/2}$$

$$\text{Var}(Y(t)) = M_{X(t)}(2) - (M_{X(t)}(1))^2 = e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1)$$

$$\mathbb{E}[Y(t)|Y(s)] = Y(s) \exp(\mu(t - s) + \sigma^2(t - s)/2)$$

$$\text{Cov}(Y(s), Y(t)) = \exp(\mu(t + s) + \sigma^2(t + s)/2)(\exp(\sigma^2 s) - 1)$$

Intermediate Results & Others

From Lecture & Tutorials

For **Birch & Death Process** with $+1 \lambda_i$ & $-1 \mu_i$, limiting distribution:

$$\pi_n = \pi_0 \prod_{i=1}^n \frac{\lambda_{i-1}}{\mu_i}, \text{ subject to } \sum_{n=0}^{\infty} \pi_n = 1$$

Delay Renewal Example: Consider Y_1, Y_2, \dots iid. A **Pattern** is a r -dim vector (y_1, \dots, y_r) . Every time $(Y_{n-r+1}, \dots, Y_n) = (y_1, \dots, y_r)$, a renewal occurs at n , denoted as $I(n) = 1$. The counting process $N(n)$ is a Delayed Renewal Process.

Define **Overlapping** $k = \max\{j < r : (y_{r-j+1}, \dots, y_r) = (y_1, \dots, y_j)\}$ how much two renewals overlap. Let $p = P(I(n) = 1)$.

When $k = 0$: $E[X_1] = 1/p$, $\text{Var}(X_1) = 1/p^2 - (2r - 1)/p$.

When $k > 0$: $E[X_1] = E[X_{y_1, \dots, y_k}] + 1/p$, and $\text{Var}(X_1) = \text{Var}(X_{y_1, \dots, y_k}) + p^{-2} - (2r - 1)/p + 2p^{-3} \sum_{j=r-k}^{r-1} E[I(r)I(r + j)]$.

(Tut5Qn2) For $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ independent, we have $\min(X_1, X_2) \sim \text{Exp}(\lambda_1 + \lambda_2)$.

Appendix: Probability Theory

Gamma Distribution

$X \sim \text{Gamma}(\alpha, \lambda)$ for shape α , and rate $\lambda > 0$. ($1/\lambda$ scale param.)

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0$$

Statistics: $E(X) = \frac{\alpha}{\lambda}$, $\text{var}(X) = \frac{\alpha}{\lambda^2}$

MGF: $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$, $t < \lambda$

Special case: $\text{Exp}(\lambda) = \text{Gamma}(1, \lambda)$, $\chi_n^2 = \text{Gamma}(\frac{n}{2}, \frac{1}{2})$

Properties: $\text{Gamma}(a, \lambda) + \text{Gamma}(b, \lambda) = \text{Gamma}(a + b, \lambda)$, and $cX \sim \text{Gamma}(\alpha, \frac{\lambda}{c})$

Gamma function $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$

$\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(n) = (n - 1)!$, $n \in \mathbb{Z}^+$

Beta Distribution

$X \sim B(a, b)$ where $a > 0, b > 0$ has support $[0, 1]$

$$f(X) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1 - x)^{b-1}, 0 \leq x \leq 1$$

Statistics: $E(X) = \frac{1}{1 + b/a}$, $\text{var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$

Special case: $\text{Unif}(0, 1) = B(1, 1)$

Beta function $B(a, b) = \int_0^1 t^{a-1} (1 - t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$

Common MGFs

Binomial: $M_X(t) = (1 - p + pe^t)^n$

Poisson: $M_X(t) = \exp(\lambda(e^t - 1))$

Exponential: $M_X(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$

Normal: $M_X(t) = \exp(\mu t + \sigma^2 t^2/2)$

Others

$E[(X - \mu)^4] = 3\sigma^4$ for $X \sim \mathcal{N}(\mu, \sigma^2)$

$E[(\int_0^T B(s) ds)^2] = \int_0^T \int_0^T E[B(s)B(t)] dt ds$