CS4261 Cheatsheet
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Nash Equilibrium

A **Game** is a general abstract framework for strategic interactions, with usually 1) A set of **Players** $N = \{1, \dots, n\}$, and subsequently for each player $i \in N$, 2) A set of possible **Actions** $A_i = \{a_{i1}, a_{i2} \dots\}$, and 3) a **Utility Function** $u_i : A \mapsto \mathbb{R}$, which indicates the utility Player i can get from an action profile, then lastly 4) a **(Pure) Action Profile** that denotes the actions taken by all the players $\vec{a} \in A_1 \times A_2 \times \dots \times A_n = A$.
A **Normal Form Game** is a Matrix Representation of the player utilities for a 2-Player game. Conventionally, each element in the Normal Form Game is a pair of real values $C_{ij} = (u, v) \in \mathbb{R}^2$ where u and v are the utility of the row and column player given an action profile $\vec{a} = (a_{1,i}, a_{2,j})$.

Pure Nash Equilibrium

Given actions taken by everyone else $\vec{a}_{-i}$, the **Best Response** set of Player i is defined

$$BR_i(\vec{a}_{-i}) = \{b \in A_i | b \in \operatorname{argmax} u_i(\vec{a}_{-i}, b)\}$$

An action profile is a **Pure Nash Equilibrium** if it is one best response for everyone given what others have chosen, i.e.

$$\forall i \in N, a_i \in BR_i(\vec{a}_{-i})$$

Analysis: Not all games have Pure Nash Equilibria.

Mixed Nash Equilibrium

Let $\vec{p} \in \Delta(A_i)$ be the probability distribution over Player i's actions. A **Mixed / Randomised Strategy Profile** is given by $\vec{p} = (\vec{p}_1, \dots, \vec{p}_n) \in \Delta(A_1) \times \dots \times \Delta(A_n)$. Player utility is $u_i(\vec{p}) = \sum_{\vec{a} \in A} u_i(\vec{a}) P(\vec{a}) = \mathbb{E}_{\vec{a} \sim \vec{p}}[u_i(\vec{a})]$.
A mixed profile is a **Mixed Nash Equilibrium** if

$$\forall i \in N, \vec{q}_i \in \Delta(A_i),\ u_i(\vec{p}) \geq u_i(\vec{p}_{-1}, \vec{q}_i)$$

Analysis: All games have Mixed Nash Equilibria.

In a Mixed Nash Equilibrium, a player is **Indifferent** if he gets the same **expected utility** from choosing any action (as a result of the other player playing a mixed strategy).

Compute Nash Equilibria in 2 Player Games

- Compute all NE in which at least one player plays a pure strategy.
- Compute all NE in which both players play mixed strategies. In this case, each player must be indifferent between the two strategies.

Dominant Strategies

A strategy $\vec{p} \in \Delta(A_i)$ **Dominates** $\vec{q} \in \Delta(A_i)$ if

$$\forall \vec{p}_{-i} \in \Delta(A_{-i}),\ u_i(\vec{p}_{-i}, \vec{p}) \geq u_i(\vec{p}_{-i}, \vec{q})$$

There are similar definitions for **Strictly Domination**.
Intuitively, no mather what others do, playing $\vec{p}$ is always better than $\vec{q}$.

Dominant Strategy Theorem

If an action $a \in A_i$ is **strictly dominated** by some strategy $\vec{p} \in \Delta(A_i)$, then action a is never played with any positive probability in any Nash Equilibrium.
Note: The theorem enables us to prune actions that will not occur in any Nash Equilibria.

Auction

Single-Item Auction

Types of Single-Item Auction

- **English Auction** - Auctioneer sets a starting price. Bidders take turn raising their bids. The bidder makes the last bid wins and pay his bid.
- **Japanese Auction** - Auctioneer sets a starting price and raises it. A bidder can drop out and not return once dropped. The last standing bidder gets the item and pays the current price.
- **Vickrey / Second-Price Auction** - All bidders submit bids simultaneously. The highest bidder wins and pays the second highest price.

Vickrey Auction Problem Specification

There are n players $N = \{1, 2, \dots, n\}$, each with a valuation of the item v_i. The actions are to place a bid at different prices. The payoff for a player is $v - p$ if getting the item, and 0 otherwise.

Analysis

Vickrey Auctions are **Truthful**, i.e. bidding according to one's true valuation is a dominant strategy.
First-Price Auctions are **Not Truthful**.
Note: Dominant strategies are Nash Equilibria in Auction games, but they are not necessarily the only Nash Equilibria.

Multi-Unit Auction

There are n players $N = \{1, 2, \dots, n\}$, each with a valuation of the item v_i. There are $k \leq n$ identical copies of the item.

The objective is to design a mechanism where

- Truthful bidding is a dominant strategy
- Items are allocated to the k highest bidders

Vickrey Clarke Groves (VCG) Mechanism

1. Choose some outcome o^* that maximises social welfare $\sum_i v_i(o^*)$
2. Calculate the payment that Player j must take with $p_j = \sum_{i \neq j} v_i(o^*_{-j}) - \sum_{i \neq j} v_i(o^*)$, where o^*_{-j} is the outcome that maximises $\sum_{i \neq j} v_i(o_{-j})$.

Note: The payment for each Player is essentially the **Externality** that he imposes on other players, which is the difference in the max welfare of others between if he is absent and if present.
Analysis: VCG is truthful. Vickrey Auction is a special case of VCG.

Combinatorial Auction

There are n Players and m possibly distinct items for sale. Each player has a valuation for each subset of the m objects.

VCG is truthful, but it can be computationally intensive, and suffers from **Revenue Non-Monotonicty**, a paradox where adding more players in the bidding game may lead to a decrease in the **Revenue**, i.e. sum of all players' payment $R = \sum_{i=1}^{n} p_i$.
Note: Single-Item Auctions have no revenue non-monotonicity.

Facility Location

Overview

There are n players $N = \{1, \dots, n\}$, each with a location $x_i \in \mathbb{R}$, assuming $x_1 \leq x_2 \leq \dots \leq x_n$ for convenience.

The objective is to design $f : \mathbb{R}^n \mapsto \mathbb{R}$ that minimises either of

- **Total Cost** - $\sum_{i \in N} |f(\vec{x}) - x_i|$
- **Max Cost** - $\max_{i \in N} |f(\vec{x}) - x_i|$

Analysis: OPT for Total Cost is **Not Truthful**. OPT for Max Cost is **Truthful** if it always "snaps" to a median player.

Max Cost Approximation Theorems

Deterministic Case

Any deterministic truthful mechanism for facility location has a worst-case approximation ratio ≤ 2 to the maximum cost.

Randomised Case

Any randomised truthful mechanism for facility location has a worst-case approximation ratio $\leq \frac{3}{2}$ to the maximum cost.

Routing Games

In a traffic network, players are drivers trying to find a route that minimises their total traffic time.

- **Proportion Version** There is 1 unit of traffic to allocate in total. Drivers are considered proportion of the total traffic.
- **Atomic Version** The traffic consists of $k \in \mathbb{N}$ drivers, each being an atomic entity.

Price of Anarchy (PoA) is the ratio of the social cost under the worst case Nash Equilibrium and under socially optimal solution

$$PoA = \frac{\text{WorstNash}(G)}{OPT(G)}$$

Analysis: 1) $PoA \geq 1$ with the smaller being the better. 2) All Nash Equilibria in a Routing Game have the same social cost.

Atomic Version

Atomic Routing Game Theorem

In an atomic routing game, a pure NE flow always exists.

Higher Level Idea

Every atomic routing game is a potential game, where all players are inadvertently and collectively optimising a potential function, $\Phi(f)$,

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

Analysis: When a player deviates (changes path), change in the deviator's individual cost is equal to $\triangle\Phi$. "Alignment in individual and social objective".

Cooperative Games

Definitions

A **Cooperative Game** $\mathcal{G}(N, v)$ consists of a set of players $N = \{1, \dots, n\}$, and a valuation function $v : 2^n \to \mathbb{R}_{\geq 0}$ for each player. A **Coalition Structure** (CS) is a partition of N while $\mathrm{OPT}(G) = \max_{\mathrm{CS}} \sum_{S \in \mathrm{CS}} v(S)$ is the optimal.

Properties, Cooperative Games

Monotone - For all $S \subseteq T \subseteq N$, $v(S) \leq v(T)$.
Simple - Monotone and for all $S \subseteq N$, $v(S) \in \{0, 1\}$.
Super-additive - For any disjoint $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T)$.
Convex - For $S \subseteq T \subseteq N$ and any $i \in N \backslash T$, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

Properties, Game Payoff

Imputation - An efficient & individual rational payoff allocation $\vec{x}$.
$\vec{x}$ is **Efficient** if satisfying $\sum_{i \in N} x_i = v(N)$.
$\vec{x}$ is **Individual Rational** if $x_i \geq v(i)$ for all $i \in N$.

The Core

An imputation $\vec{x}$ is in the **Core** if it satisfies for all $S \subseteq N$,

$$\sum_{i \in S} x_i = x(s) \geq v(S)$$

Note: A core is a set of vectors, not a set of players.

Properties, the Core

Assume the game $\mathcal{G} = (N, v)$ is simple:
Winning / **Losing Coalition** - A coalition with value 1 / 0.
Veto Player - a player that is in every winning coalition.
Analysis: For p veto, any coalition without p cannot win, and any wit p does not necessarily win.

Veto Player Theorem

For a simple game, $\mathrm{Core}(\mathcal{G}) \neq \emptyset$ iff. $\mathcal{G}$ has veto players.
The core only distribute payoffs among the veto players.

Shapley Value

For Player i and $S \subseteq N$, the **Marginal Contribution** of i to S is

$$m_i(S) = v(S \cup \{i\}) - v(S)$$

Given a permutation $\sigma \in \Pi(N)$, the **Predecessors** of i in σ are

$$P_i(\sigma) = \{j \in N | \sigma(j) < \sigma(i)\}$$

We can write $m_i(\sigma) = m_i(P_i(\sigma))$ for marginal contribution.
Shapley Value of a player in a Coalition is his expected marginal contribution.

$$Sh_i = \mathbb{E}\big[m_i(\sigma)\big] = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i(\sigma)$$

Shapley Value Theorem

Shapley Value is the only payoff allocation value satisfying efficiency, linearity, dummy and symmetry.
Note: [1] **Symmetry**: symmetric players are paid equally. [2] **Dummy**: Dummy players are not paid.

Induced Subgraph Games

An example of Cooperative Games $\mathcal{G} = (V, E)$ where players are vertices and $e = (u, v) \in E$ with weight w_e describing the utility of u and v in a coalition.

Induced Subgraph Game Core Theorem

The core of an induced subgraph game is non-empty iff. the graph has no negative cut.
Note: A negative cut is a cut (set of edges that parition the graph into two) where the sum of edge weights is negative.

Shapley Value, Induced Subgraph Game

The payoff for each player i will be

$$\phi_i = \frac{1}{2} \sum_{j \in N} w(i, j)$$

Nash Bargaining Solution

Definitions

A **Bargaining Game** is a pair $(S, \vec{d})$ for $S \subseteq \mathbb{R}^2$ and $\vec{d} \in \mathbb{R}^2$ with at least one point $(x_1, y_1) \in S$ such that $x_1 \geq d_1$ and $y_1 \geq d_2$ for $\vec{d} = (d_1, d_2)$. **Two players** chooses $x, y \in \mathbb{R}$ respectively. If $(x, y) \in S$, the two players receives x and y respectively. Otherwise, they receive d_1 and d_2. A **solution** is a function $\vec{f} = (f_1, f_2)$ that takes in $(S, \vec{d})$ and outputs a value for the two players each.

Pareto Optimality

An outcome (x_1, x_2) **Pareto Dominates** another outcome (x_2, y_2) if $x_1 \geq x_2$ and $y_1 \geq y_2$ and at least one of these two inequalities is strict. The dominating one is the **Pareto Improvement**. An outcome without a Pareto Improvement is **Pareto Optimal**.
In a Bargaining game, S's top right boundary is the Pareto Frontier.

Properties, Bargaining Game Solutions

Efficiency - No outcome (v_1, v_2) dominates $\big(f_1(S, \vec{d}), f_2(S, \vec{d})\big)$.

Symmetry - Let $S^T = \{(y, x) : (x, y) \in S\}$ and $\vec{d}^T = (d_2, d_1)$, then

$$\big(f_1(S^T, \vec{d}^T), f_2(S^T, \vec{d}^T)\big) = \big(f_1(S, \vec{d}), f_2(S, \vec{d}\big)$$

Independence of Irrelevant Alternative (IIA) - Let $S' \subseteq S$ such that $\big(f_1(S, \vec{d}), f_2(S, \vec{d}\big) \in S'$, then,

$$\big(f_1(S', \vec{d}), f_2(S', \vec{d}\big) = \big(f_1(S, \vec{d}), f_2(S, \vec{d}\big)$$

Invariance under Equivalent Representations (IER) - For any $\alpha_1, \alpha_2 \in \mathbb{R}, \vec{\beta} \in \mathbb{R}^2$,

$$f_i((\alpha_1, \alpha_2)S + \vec{\beta},\ (\alpha_1, \alpha_2)\vec{d} + \vec{\beta}) = \alpha_i f_i(S, \vec{d}) + \beta_i,\ i = 1, 2$$

Nash Bargaining Solution

The **Nash Bargaining Solution** for a bargaining game is the solution

$$\underset{(v_1, v_2) \in S}{\operatorname{argmax}}\ (v_1 - d_1)(v_2 - d_2)$$

The Nash Bargaining Solution is the only solution that satisfies Efficiency, Symmetry, IIA and IER.

Utility Maximises

- **Utilitarian**: $\max \sum_i u_i(A)$.
- **Nash**: $\max \prod_i u_i(A)$.
- **Egalitarian**: $\max \min_i u_i(A)$.

Stable Matching

Definitions

In the case of matching medical students S to hospitals H, with $|S| = n, |H| = m$: Each student s has a strict preference order over H, denoted by $\succ_s$. Similarly for $\succ_h$. A matching $M : S \to H$ is a one-to-one mapping.

The goal is to design a market that **Thick**, **Safe** (truthful, fair and encouraging participation), and **Timely**.

Stable Matching

A pair $(s, h) \in S \times H$ **Blocks** matching M, if

$$h \succ_s M(s) \land s \succ_h M^{-1}(h)$$

A matching M is **Stable** if there are no blocking pairs.
Analysis: [1] A stable matching always exists. [2] A stable matching can always be found in polynomial time. e.g. GS Algorithm.

Gale-Shapley Deferred Acceptance Algorithm

Procedures:

1. Start with all students unassigned.
2. While there are unassigned students: Each unassigned student proposes to favourite **not-yet-proposed-to** hospital. Then each hospital looks at **students proposed to it in this round and whoever currently assigned** to it, and picks most preferred; all others remain unassigned.
3. Repeat until all matched.

Analysis: [1] Terminate in $\leq n^2$ iterations with a stable matching.

Fairness for GS Algorithm

Given $s \in S$, a **Valid** hospital $h \in H$ is one that there exists some stable matching M such that $M(s) = h$. **best(s)** and **worse(s)** are the most and least highly ranked valid hospital for s.

Theorem

GS Algorithm (with student proposing) assigns each student $s \in S$ to hospital $\text{best}(s)$, and each hospital $h \in H$ to student $\text{worst}(h)$.

Fair Allocation of Indivisible Goods

Definitions

With players $N = \{1, 2, ..., \}$ and indivisible goods $G = \{g_1, g_2, ..., g_m\}$, Player i has value $v_i(g)$ for good g. An **Allocation**, $A = (A_1, ..., A_n)$ is a partition, with **Bundle** A_i is allocated to Player i.

Valuation is **Additive** if for all $G' \subseteq G$, $v_i(G') = \sum_{g \in G'} v_i(g)$

Fairness

Proportionality - $v_i(A_i) \geq \frac{1}{n} \cdot v_i(G)$ for all $i \in N$.

Envy-freeness - $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$.

Envy-freeness Up to One Good (EF1) - For any $i, j \in N$, if $A_j \neq \emptyset$, there exists $g \in A_j$ such that

$$v_i(A_i) \geq v_i(A_j \backslash \{g\})$$

Analysis: EF1 allocations always exist.

Envy-freeness Up to Any Good (EFX) - For any $i, j \in N$ and any $g \in A_j$, we have

$$v_i(A_i) \geq v_i(A_j \backslash \{g\})$$

Analysis: [1] EFX is stronger than EF1. [2] EFX allocations always for $n = 2$ (Cut-and-Choose protocol) and $n = 4$, and its existence for $n \geq 4$ remains an open problem.

Maximum Nash Welfare

An allocation that maximises the Nash welfare, known as **Maximum Nash Welfare** (MNW) allocation, satisfies EF1.

Note: If $MNW = 0$, maximising the number of players with positive utility, then maximise Nash welfare among these players.

Analysis: The allocation is also Pareto Optimal.

Envy-free

Round-Robin

Procedures: Let players take turns choosing their favourite good from the remaining, in the order $1, 2, ..., n, 1, 2, ..., n, 1, 2, ...$ until goods run out.

Note: Require **additive** valuations.

Envy-Cycle Elimination

Procedures: [1] Allocate one good at a time in an arbitrary order. [2] Maintain an **envy graph** with $e : i \to j$ for each i envying j. [3] At each step, the next good is allocated to a player with no incoming edges. [3'] Any cycle that arises is eliminated by giving j's entire bundle to any i for $i \to j$.

Note: Require **Monotone** valuation, not necessarily **additive**.

Proportionality

Maximin Share (MMS) of Player i: Player i divides all goods into n bundles so as to maximise the values of the value of the minimum-value bundle. An relaxation of proportionality:

$$\text{MMS}_i \leq \frac{v_i(G)}{n}$$

Analysis: [1] Solution achieving MMS for every player exists for $n = 2$ (Cut-and-Choose), but might not exist for $n \geq 3$. [2] Solution with at least $\frac{3}{4}\text{MMS}_i$ for each always achievable for all n.

Query Complexity

The Envy-Cycle Elimination Algoritm can be implemented using $O(nm)$ queries, even with monotonic valuations.

For EF1 solution for two agents with monotonic valuations, $O(\log m)$ queries suffice.

Any deterministic EF1 algorithm needs $\Omega(\log m)$ queries.

Any deterministic EFX algorithm needs queries **exponential** in m.

Cake Cutting

Definitions

Given a cake represented as interval $[0, 1]$, players $N = \{1, ..., n\}$, and valuation $v_1, ..., v_n$ over the cake, the goal is to find an allocation $A = (A_1, ..., A_n)$ where each A_i is a union of **finitely many** intervals.

Valuation Function

Each valuation function v_i is defined with a density function f_i

$$v_i(B) = \int_B f_i(x), \ \forall B \subseteq [0, 1]$$

Note: $v_i(x, y) = v_i([x, y])$ denotes the utility of Player i getting $[x, y]$.

Valuation function is [1] Non-negative, [2] Additive (over disjoint pieces), [3] Non-atomic (where value of a point is 0), and [4] Normalised (where value of entire cake is 1).

Robertson-Webb Model

Robertson-Webb Model defines two types of queries:

- `Eval_i(x, y)` - Return $v_i(x, y)$.
- `Cut_i(x, a)` - Return the leftmost point y s.t. $v_i(x, y) = a$ or such points do not exist.

Complexity of a model can be determined based on numbers of RW Model queries needed.

Proportionality

Cut-and-Choose Protocol (for $n = 2$ players) - One player cuts, the other player chooses first.

Analysis: It is always proportional and envy-free.

Dubin-Spanier Protocol - 1) Referee moves a knife over the cake from the left. 2) Whenever the current piece is worth $1/n$ to some player, that player leaves the procedure with that piece. 3) The last player gets the remaining cake.

Analysis: [1] Proportional for any n. [2] $O(n^2)$ queries.

Even-Paz Protocol □for $n = 2^k$ players) - 1□Each player marks the point that divides the cake into two of equal value according to his valuation. 2) Let t be the $n/2$-th mark from the left, recurse on $[0, t], [t, 1]$ each with $n/2$ agents. 3) Terminate with one player getting the whole cake.

Analysis: [1] Proportional for any n. [2] Require $O(n \log n)$ queries (optimal query number in all proportional protocols).

Envy-freeness

Envy-free protocols are complicated. **Cut-and-Choose** is EF for $n = 2$. **Selfridge-Conway Protocol** is EF for $n = 3$.

Envy-Free Approximation

1) Referee moves a knife over the cake from the left. 2) Whenever the current piece is worth $1/3$ to some player, that player leaves the procedure with that piece. 3) When the knift reaches the right end, 3a) if there are still agents left, the remaining is given to any of them; 3b) otherwise the remaining is given to the last agent.

Analysis: EF up to $1/3$, i.e. any player envies any other by at most a numeric value of $1/3$.

Truthfulness

Cut-and-Choose is truthful for the choose not for the cutter.

Rent Division

Definitions

Given players (roommates) $N = \{1, ..., n\}$ and n rooms of an apartment, and rental price for the entire apartment, r, each Player i has a value for room j, v_{ij} subject to $\sum_j v_{ij} = r$.

The objective is to output an outcome $(\sigma, \vec{p})$

- Room allocation function $\sigma : N \to N$, mapping player to room
- Rent division $\vec{p} = (p_1, ..., p_n)$ s.t. $\sum_j p_j = r$

Note: p_j is the rent for Room j, not for Player j by convention.

Allocation Algorithms

Envy-free Outcomes

An outcome $(\sigma, \vec{p})$ is **Envy-Free** if for all $i, j \in N$,

$$v_{i,\sigma(i)} - p_{\sigma(i)} \geq v_{ij} - p_j$$

Intuitively, it means no one prefers anyone's room for the price that person is charged.

Analysis: EF Outcomes can be computed using Linear Programming, but it is not always fair.

Equitability & Maximin Outcomes

Equitability - Maximum difference in utility of any two players. To maximise equitability is to minimise the following expression

$$D(\vec{p}) = \max_{i,j \in N} (u_i(\vec{p}) - u_j(\vec{p}))$$

Maximin - Minimum utility any player derives. It is the **Egalitarian Utility**, defined as

$$U(\vec{p}) = \min_{i \in N} u_i(\vec{p})$$

Theorem: There is a unique Maximin price vector and it is also equitable. (Yet there are still equitable, non-Maximin price vectors).

Committee Voting

Definitions

Approval Voting is the game where each voter submits a subset of candidates he / she approves. It consists of voters $N = \{1, ..., n\}$, candidates $C = \{1, ..., m\}$, and set of candidates $A_i \subseteq C$ that Voter i approves for $i = 1, ..., n$. The goal is to choose a committee $W \subseteq C$ of size $|W| = k$ given.
Voter i's utility is number of committee members he approves:

$$u_i(W) = |A_i \cap W|$$

Social Welfare & Coverage

For a given committee,
Social Welfare - $\sum_{i \in N} u_i(W)$, total number of approvals committee members receive.
Coverage - Total number of voters who approves at least one committee member. Note:A candidate **Covers** a voter if the voter approves the candidate.
Approval Voting (AC) - Select a committee that maximises social welfare,

$$W_{AC} = \max_{W \subseteq C, |W|=k} \sum_{i \in N} u_i(W)$$

Chamberlin-Courant (CC) - Select a committee that maximises coverage,

$$W_{CC} = \max_{W \subseteq C, |W|=k} |\{i \in N | A_i \cap W \neq \emptyset\}|$$

Note: Both AV and CC solutions are not unique.

Justified Representation

A group of voters $S \subseteq N$ is a **Cohesive Group** if $|S| \geq n/k \wedge |\cap_{i \in S} A_i| \geq 1$
Justified Representation (JR) - For any cohesive group of voters $S \subseteq N$, there exists $i \in S$ such that $A_i \cap W \neq \emptyset$

GreedyCC

Find a CC committee is NP-hard, equivalent to SET-COVER.
GreedyCC is one greedy solution: 1) Start with an empty set of candidates. 2) In each step, choose a candidate that covers as many uncovered voters as possible. 3) Repeat this until k candidates have been chosen.
Analysis: GreedyCC satisfies JR.

Extended JR

For a positive integer $t \in \mathbb{Z}_{>0}$, a group of voters $S \subseteq N$ is a **t-Cohesive Group** if $|S| \geq t \times n/k \wedge |\cap_{i \in S} A_i| \geq t$
Extended Justified Representation (EJR) - For any positive integer $t \in \mathbb{Z}_{>0}$ and any t-cohesive group of voters $S \subseteq N$, there exists $i \in S$ such that $|A_i \cap W| \geq t$
Note: A cohesive group is a 1-cohesive group. JR is only EJR of the case $t = 1$.

Thiele Method

Thiele Method - A generalised method for finding out a group of candidates that satisfies a specific criterion.

- Choose an infinite non-increasing sequence $\{s_n\}_{n=1}^{\infty}$, based on the criterion of interest.
- Choose a committee W that maximises

$$\sum_{i \in N} \sum_{j \in u_i(W)} s_j = \sum_{i \in N} (s_1 + s_2 + ... + s_{u_i(W)})$$

Note: If a voter approves r candidates, he contributes a value of $s_1 + ... + s_r$ to the score.
Analysis: [1] $s_i = 1, \forall i$ for AV. [2] $s_1 = 1,\ s_2 = s_3 = ... = 0$ for CC.

Proportional Approval Voting

Proportional Approval Voting (PAV) - The solution when we set $s_n = 1/n$, i.e. the Harmonic Sequence, then apply Thiele Method. PAV satisfies EJR.
Note: PAV is NP-Hard. Greedy variant of PAV is not EJR or even JR.

Method of Equal Shares

Method of Equal Shares (MES) - Another procedure for finding a EJR solution

1. Each voter has a budget of k/n. Each candidate costs 1. The voters who approve this candidate have to "pool" their money to add this candidate to the committee.
2. Start with an empty committee.
3. In each round, we want to add a candidate whose approved voters have a total budget of ≥ 1 left.
4. If there are several such candidates, choose one such that the maximum amount that any agent has to pay is minimised.
5. If no more candidate can be afforded but the committee still has size $< k$, fill in the rest of the committee using some tie-breaking criterion.

Note: We do not require all players to pool a candidate must provide the equal amount. Analysis: MES satisfies EJR, and can be implemented in polynomial time.

Summary

	JR	EJR	Poly-time
AV	No	No	Yes
CC	Yes	No	No
GreedyCC	Yes	No	Yes
PAV	Yes	Yes	No
MES	Yes	Yes	Yes

Tournaments

Definitions

A **Tournament** $T = (A, \succ)$ consists of a set of **Alternatives** (players), A, and a dominance relation $\succ$ such that $\forall i \neq j \in A, i \succ j \vee j \succ i$.
Note: [1] By convention, we draw a direct edge from i to j for $i > j$.
Outdegree of an alternative $x \in A$ is number of alternatives dominated by x. A **Condorcet Winner** / **Loser** is an alternative that dominates / is dominated by all others (i.e. outdegree $n - 1$ / 0).
A **Tournament Solution**, S, is a method for choosing tournament winners, satisfying conditions: [1] It returns a **non-empty** subset of alternatives for any tournament (can be A trivially), $\forall T = (A, \succ), S(T) \subseteq A$. [2] **Invariant Under Isomorphism**: for $h : A \to A'$ an isomorphism between $T = (A, \succ)$ and $T' = (A', \succ')$, satisfying

$$S(T') = h(S(T))$$

Analysis: By symmetric, in a cyclic tournament of $|A| = 3$, every tournament solution must select all alternatives.

Tournament Solutions

A summary of types of tournament solutions:

- **Copeland Set** (CO) - Alternatives with highest outdegrees.
- **Top Cycle** (TC) - Alternatives that can reach every other alternative via a directed path (of any length)
- **Uncovered Set** (UC) - Alternatives that can reach every other alternative via a directed path of length ≤ 2
- **Banks Set** (BA) - Alternatives that appear as the maximal (i.e., strongest) element of some transitive sub-tournament that cannot be extended

Note: **Transitive Tournament** - A tournament where alternatives can be ordered a sequence $a_1, ..., a_k$ s.t. $\forall 1 \leq i < j \leq k, a_i \succ a_j$.
Containment Relations: $UC \subseteq TC, CO \subseteq UC, BA \subseteq UC$.

Top Cycle

Equivalent Definition for TC: The unique smallest nonempty set B of alternatives such that all in B dominates all outside B.

Uncovered Set

x **Covers** y if [1] $x \succ y$, and [2] $\forall z \in A, y \succ z \implies x \succ z$.
Equivalent Definition fot UC: The set of all uncovered alternatives.

Axioms

CO, TC, UC, and BA all satisfy Condorcet-Consistency and Monotonicity

- **Condorcet-Consistency** - If there is a Condorcet winner x, then x is uniquely chosen.
- **Monotonicty** - If x is chosen, then it should remain chosen when it is strengthened against another alternative y (and everything else stays the same).

For $y \succ x$, **Strengthen** x against y is to change $y \succ x$ to $x \succ y$.

Tournament Fixing Problem

In a **balanced knockout tournament**, given alternatives A, (probabilistic) dominance relation $\succ$ and our favourite alternative $x \in A$, the goal is find a tournament bracket that maximises win-rate for x. If such a bracket exists, such a winner is called **Knockout Winner**.
Analysis: Any alternative who dominates $< \log_2 n$ others are not knockout winners.

String King Theorem

An alternative is x is a **King** if it is in UC. Note: it beats any other directly or via a third alternative.
For x is a king, suppose $x \in A$ beats $P \subseteq A$ and loses to $Q \subseteq A$. if $|P| \geq n/2$, x is a knockout winner.
Analysis: Any alternative in CO is a knockout winner.
Note: Players not in UC, or those in UC with $|P| < n/2$ can still be knockout winners.

Intermediate Results

From Assignments

Assignment5 Qn3 For a cooperative game with non-empty core, the Shapley vector is not necessarily in the core.