ST4238 Cheatsheet
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1. Poisson Processes

Overview

Definition 1

A **Poisson Process** with rate $\lambda > 0$ is an integer-valued stochastic process $\{X(t), t \geq 0\}$ for which

- for any time points $t_0 = 0 < t_1 < t_2 < ... < t_n$, increments
$$X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$$
are independent r.v.
- for $s \geq 0$ and $t > 0$, the r.v. $X(s+t) - X(s) \sim Pois(\lambda t)$
- $X(0) = 0$

Note: The process above is **homogeneous**. If $\lambda = \lambda(t)$ varies with time, then the process is **non-homogeneous**, and replace the second property above with

$$X(s+t) - X(s) \sim Pois\Big(\int_s^{s+t} \lambda(u)du\Big)$$

Note: A **Cox Process** is where $\lambda(t)$ is a stochastic process itself.

To **homogeniz a non-homogeneous Poisson Process**:

1. Define $\Lambda(t) = \int_0^t \lambda(u)du$
2. Define a new process $Y(s) = X(t)$ where $s = \Lambda(t)$
3. Now $Y(s)$ is a homogeneous Poisson Process with rate 1.

Definition 2 - LRE

Let $N((s,t])$ be a r.v. counting number of occurrences in the interval $(s,t]$. Then $N((s,t])$ is a **Poisson Point Process** of intensity $\lambda > 0$ if

- for any time points $t_0 = 0 < t_1 < ... < t_n$, increments
$$N((t_0,t_1]), N((t_1,t_2]), ..., N((t_{n-1},t_n])$$
are independent r.v.
- $\exists \lambda > 0$, s.t. as $h \to 0$, $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$.
- as $h \to 0$, $P(N((t,t+h]) \geq 2) = \lambda + o(h)$.

Note: For non-homogeneous, $P(N((t,t+h]) \geq 1) = \lambda(t)h + o(h)$.

Law of Rare Events

Let $\epsilon_1, \epsilon_2, ...$ be independent Ber. r.v.'s with $P(\epsilon_i = 1) = p_i$ and let $S_n = \sum i = 1^n \epsilon_i$. The exact probability for S_n and Poisson probability with $\lambda = \sum_{i=1}^n p_i$ differ by at most

$$|P(S_n = k) - e^{-\lambda}\frac{\lambda^k}{k!}| \leq \sum i = 1^n p_i^2$$

Note: In the case of $p_1 = ... = \lambda/n$, RHS becomes λ^2/n.

Definition 3 - Sojourn Time

Consider a sequence $\{S_n, n \geq 0\}$ of i.i.d. $Exp(\lambda)$. Define a counting process by specifying the occurrence time of n-th event

$$W_n = S_0 + S_1 + ... + S_{n-1}$$

The new counting process will be a Poisson Process with rate λ.

Waiting & Sojourn Time

The **Waiting Time**, W_n of a Poisson Process $X(t)$ is the time of n-th occurrence, for $n \in \mathbb{N}$. We set $W_0 = 0$.

The **Sojourn Time**, $S_n = W_{n+1} - W_n$ is the time where the process sojourns in state n, for $n \in \mathbb{Z}_{\geq 0}$.

For homogeneous Poisson Processes, $S_n \sim Exp(\lambda)$ and $W_n \sim Gamma(n, \lambda)$

Properties

Arrival Time

Given $X(t) = n$, the joint distribution of waiting time $W_1, ..., W_n$ is

$$f(w_1, w_2, ..., w_n | X(t) = n) = \frac{n!}{t^n}, 0 \leq w_1 \leq w_2 \leq ... \leq 2_n \leq t$$

It is the joint distribution of n **ranked** independent $Unif(0,t)$ r.v.'s.

Given $X(t) = n$, the distribution of the k-th waiting time has the same distribution as that of the k-th order statistic of n independent $Unif(0,t)$ r.v.'s.

$$f_k(x) = \frac{n!}{(n-k)!(k-1)!}\frac{1}{t}(\frac{x}{t})^{k-1}(1-\frac{x}{t})^{n-k}, 0 \leq k \leq n$$

Merging & Splitting Processes

For $\{N_1(t), t \geq 0\}, \{N_2(t), t \geq 0\}, ..., \{N_m(t), t \geq 0\}$ independent Poisson Processes with rates $\lambda_1, \lambda_2, ..., \lambda_m$, let $N(t) = \sum_{i=1}^m N_i(t), t \geq 0$ be the **Merging Process**. Then $\{N(t), t \geq 0\}$ is also a Poisson Process with rate $\lambda = \lambda_1 + \lambda_2 + ... + \lambda_m$.

For $\{N(t), t \geq 0\}$ a Poisson Process with rate λ, if each event occurred can be of type A and B with probability p and $1-p$ independently, then let $X(t)$ and $Y(t)$ be the **Splitting Processes** counting number of type A and B occurrences. Then $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are Poisson Processes with rates λp and $\lambda(1-p)$, and they are independent.

Comparision of Two Processes

Consider $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ two independent Poisson Processes with rates λ_1 and λ_2. Define W_n^X and W_m^Y as the waiting time of the n-th and m-th waiting time of $X(t)$ and $Y(t)$ respectively.

$$P(W_n^X < W_m^Y) = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} (\frac{\lambda_1}{\lambda_1+\lambda_2})^k (\frac{\lambda_2}{\lambda_1+\lambda_2})^{n+m-1-k}$$

Analysis: It is equivalent as getting n or more heads in $n+m-1$ tosses where getting a head has probability $\lambda_1/(\lambda_1+\lambda_2)$.

Variants

Compound Poisson Process

A stochastic process $\{X(t), t \geq 0\}$ is a **Compound Poisson Process** if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \ t \geq 0$$

, where $\{N(t), t \geq 0\}$ is a Poisson Process with rate λ and $Y_i \sim F$ is a family of i.i.d. r.v.'s independent of $\{N(t), t \geq 0\}$.
Note: We need [1] rate λ, and [2] distribution F to specify a compound Poisson Process.

$E[X(t)] = \lambda t E[Y_i]$
$Var[X(t)] = \lambda t\big(E[Y_i]^2 + Var(Y_i)\big)$

To merge two compound Poisson Processes $X(t)$ and $Y(t)$ with parameters (λ_1, F_1) and (λ_2, F_2) as $N(t) = X(t) + Y(t)$, the result is a compound Poisson Process with parameters

$$\lambda = \lambda_1 + \lambda_2$$
$$F(x) = \frac{\lambda_1}{\lambda_1+\lambda_2}F_1(x) + \frac{\lambda_2}{\lambda_1+\lambda_2}F_2(x)$$

Conditional Poisson Process

A stochastic process $\{N(t), t \geq 0\}$ is a **Conditional Poisson Process** if there is a positive r.v. L such that $\{N(t)|L = \lambda, t \geq 0\}$ is a Poisson Process with rate λ.
If L has pdf $g(\cdot)$, then pdf for increment of $N(t)$ is

$$P(N(t+s) - N(s) = n) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}g(\lambda)d\lambda$$

$E[N(t)] = E(L)t$
$Var(N(t)) = tE(L) + t^2 Var(L)$
Conditional probability of L given $N(t) = n$ (posterior),

$$P(L \leq x | N(t) = n) = \frac{\int_0^x e^{-\lambda t}(\lambda t)^n g(\lambda)d\lambda}{\int_0^\infty e^{-\lambda t}(\lambda t)^n g(\lambda)d\lambda}$$

Multi-Dimensional Poisson Process

Let S be a subset of $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$. Let $\mathcal{A}$ be the set of subsets of S and for any set $A \in \mathcal{A}$, define $|A|$ as the size of A. Then $\{N(A) : A \in \mathcal{A}\}$ is a homogeneous Poisson process with $\lambda > 0$ if,

- for each $A \in \mathcal{A}$, $N(A) \sim Pois(\lambda|A|)$
- for every finite collection $\{A_1, ..., A_n\}$ of disjoint subsets of S, r.v.'s $N(A_1), ..., N(A_n)$ are independent.

2. Continusous Time Markov Chains

Overview

For a stochastic process $\{X(t), t \geq 0\}$, if for all $s > u \geq 0, t > 0$,

$$P(X(s+t) = j|X(s) = i, X(u) = k) = P(X(s+t) = j|X(s) = i)$$

, then we call $\{X(t), t \geq 0\}$ a **Continuous-Time Markov Chain**, and the property **Markovian Property**.

A CTMC $\{X(t), t \geq 0\}$ has **Stationary / Homogeneous Transition Probabilities** if for all $s \geq 0, t > 0$ and states i, j,

$$P(X(s+t) = j|X(s) = i) \text{ is independent of } s$$

A CTMC $\{X(t), t \geq 0\}$ can be specified with: [1] **State Space** S, [2] **Waiting Time Rate Vector** $\vec{\nu}$, where the time $X(t)$ stays in state $i \in S$ follows $Exp(\nu_i)$, & [3] **Jump Probabilities** P_{ij}, the probability of $X(t)$ currently in state $i \in S$ and moves to $j \in S$ at first transition.
Note: By definition, $P_{ii} = 0$ and $\sum_{j \neq i} P_{ij} = 1$ for all $i \in S$.
Note: For an absorbing state i (e.g. $i = 0$ in Birth & Death Process), we may set $\nu_i = 0$.

Transition Probability Function

Define the **Transition Probabilities** of a CTMC $X(t)$ as

$$P_{ij}(t) := P(X(t+s) = j|X(s) = i)$$

Note: A transition probability matrix can uniquely specify a CTMC.
Note: $P_{ij} \neq P_{ij}(t)$ since there might not be exactly one transition.

Chapman-Kolmogorov Equation

$$P_{ij}(t+s) = \sum_{k \in S} P_{ik}(t) P_{kj}(s)$$

Discretisation of CTMC

For a CTMC $\{X(t), t \geq 0\}$, $\{Y_1(n)\}_{n \geq 0}$ discretises it at equal intervals if for some constant $l > 0$,

$$Y_1(n) = X(nl), n = 0, 1, 2, \ldots$$

Analysis: $Y_1(n)$ has state space S and transition matrix $P(l)$.

For a CTMC $\{X(t), t \geq 0\}$, $\{Y_2(n)\}_{n \geq 0}$ is the **Embedded Chain** if it only considers the states visited by $X(t)$.
Analysis: $Y_2(n)$ has state space S and transition matrix P.

Infinitesimal Generator

Lemma: Transition Rates, for a CTMC,

- $\lim_{h \to 0} \frac{P_{ii}(h) - P_{ii}(0)}{h} = -\nu_i$
- $\lim_{h \to 0} \frac{P_{ij}(h) - P_{ij}(0)}{h} = \nu_i P_{ij}$, for all $i \neq j$

The **Infinitesimal Generator** G of a CTMC is defined as

$$G = (G_{ij})_{S \times S} = P'(0), \text{ where } G_{ij} = \begin{cases} -\nu_i, & i = j \\ \nu_i P_{ij}, & i \neq j \end{cases}$$

Kolmogorov's Forward & Backward Equations respectively:

$$P'(t) = P(t)G \iff P'_{ij}(t) = \sum_{k \neq i} P_{ik}(t) q_{kj} - \nu_j P_{ij}(t)$$

$$P'(t) = GP(t) \iff P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

CTMC Long-Term Properties

Stationary Distribution

For a CTMC $\{X(t), t \geq 0\}$, a row vector $\boldsymbol{\pi} = (\pi_i)_{i \in S}$ with $\pi_i \geq 0$ and $\sum_i \pi_i = 1$ is a **Stationary Distribution** if for all $t \geq 0$,

$$\boldsymbol{\pi} = \boldsymbol{\pi} P(t),$$

Global Balancing Equations

$$\boldsymbol{\pi} G = \mathbf{0} \iff \equiv \sum_{j \neq i} \pi_i q_{ij} = v_j \pi_j, \ \forall j$$

Limiting Distribution

For a CTMC $\{X(t), t \geq 0\}$, its **Limiting Distribution**, $\{P_j, j \in S\}$, is:

$$P_j = \lim_{t \to \infty} P_{ij}(t)$$

Note: [1] For each j, the limit needs to exist and be the same for all i. [2] When both $\boldsymbol{\pi}$ and P exists, $\boldsymbol{\pi} = P$.

If $X(t)$ satisfies conditions below, it is **Ergodic** (converse not true):

- All states of $X(t)$ **communicate**.
- $X(t)$ is **positive recurrent**, i.e. for all $i, j \in S$,

$$\mathbb{E}[\min_{t \geq 0}\{X(t) = j|X(0) = i\}] < \infty$$

An ergodic chain has stationary & limiting distributions & equal.

Suppose the embedded chain of $\{X(t), t \geq 0\}$, $\{E(n)\}_{n=0}^{\infty}$ has stationary distribution $\boldsymbol{\psi}$. Then for all state i,

$$\psi_i = \frac{\pi_i \nu_i}{\sum_j \pi_j \nu_j} \iff \pi_i = \frac{\psi_i / \nu_i}{\sum_j \psi_j / \nu_j}$$

Time Reversibility

For an **ergodic** CTMC $\{X(t), t \geq 0\}$ and a sufficiently large t, define reversed process $\{Y(t), t \geq 0\}$

$$Y(0) = X(t), \ Y(s) = X(t-s), 0 < s < t$$

$\{X(t), t \geq 0\}$ is **Time-Reversible** if $X(t)$ and $Y(t)$ has the same probability structure: [1] Same $\boldsymbol{\nu}$, and [2] Same jump matrix.

Local Balanced Equations

$$\pi_j q_{ji} = \pi_i q_{ij}, \ \forall i, j$$

If it is satisfied, $X(t)$ is time reversible with limiting distribution $\boldsymbol{\pi}$.

Proposition: Time Reversibility Subset
Truncate a time-reversible CTMC $X(t)$ from S to $A \subseteq S$, then it remains time-reversible and has limiting distribution

$$\pi_j^A = \frac{\pi_j}{\sum_{i \in A} \pi_i}, \ \forall j \in A$$

Proposition: Time Reversibility Vectors
For CTMCs $\{X_i(t), t \geq 0\}, i = 1, 2, \ldots, n$ time reversible, the vector process $\{(X_1(t), \ldots, X_n(t)), t \geq 0\}$ is also time reversible.

CTMC Techniques

Uniformization

For a CTMC $\{X(t), t \geq 0\}$, where $\exists \nu \in \mathbb{R}$ s.t. $\nu_i \leq \nu, \forall i \in S$,

$$P_{ij}(t) = \sum_{n=0}^{\infty} (P^*)_{ij} \frac{(\nu t)^n}{n!} e^{-\nu t} \text{ ,where } P^*_{ij} = \begin{cases} 1 - \nu_i/\nu, & i = j \\ (\nu_i/\nu) P_{ij}, & i \neq j \end{cases}$$

Intuition: P^* is the jump matrix after **Uniformisation**. A CTMC with identical ν_i is a Poisson Process with rate ν_i.

CTMC with Absorbing States

For a CTMC $\{X(t), t \geq 0\}$, if there is a state i s.t. $\forall t > 0, s \geq 0$,

$$P(X(t+s) = i|X(s) = i) = 1$$

, (or $P_{ii}(t) = 1, \forall t > 0$,) then we call i an **Absorbing State**.

Assume state 0 is absorbing, **probability of absorbing** $u_i = \lim_{t \to \infty} P(X(t) = 0|X(0) = i)$ from state i by CTMC is **the same as that based on its embedded chain**.
Define **expected time of absorption** w_i for starting at state i. Case 1 When $i = 0$, $w_i = 0$. Case 2a When $u_i < 1$, $w_i = \infty$. Case 2b When $u_i = 1$, $w_i = \mathbb{E}[\text{time till 1st jump}] + \sum_{j \neq i} P_{ij} w_j$.

3. Renewal Process

Overview

A **Renewal Process** is a counting process $\{N(t), t \geq 0\}$ for a sequence of non-negative r.v.s $\{X_1, X_2, \ldots\}$ that are iid. with a distribution F.

- $F(x) = P(X_k \leq x), k = 1, 2, \ldots$, CDF of sojourn time
- $F_k(x) = P(W_k \leq x), k = 1, 2, \ldots$, CDF of waiting time W_k.
- $M(t) = E[N(t)]$, **Renewal Function**, expected # of renewals

Properties

- $N(t) \geq k \iff W_k \leq t$
- $W_{N(t)} \leq t < W_{N(t)+1}$
- $P(N(t) = k) = F_k(t) - F_{k+1}(t)$
- $F_k(t) = \int_0^t F_{k-1}(t-y) dF(y)$, one-step analysis

The Renewal Equation - For a renewal process with sojourn times distributed as F, let $M(t) = E[X(t)]$, then

$$M(t) = F(t) + \int_0^t M(t-x) f(x) dx$$

Waiting time in Renewal Process:

$$\mathbb{E}[W_{N(t)+1}] = \mathbb{E}[X_1](M(t) + 1)$$

Note: Not a Random Sum as $N(t) + 1$ not independent with X_i.

Special Random Variables

- **Excess Time / Residual Time**: $\gamma_t = W_{N(t)+1} - t$
- **Current Life / Age**: $\delta_t = t - W_{N(t)} \geq 0$
- **Total Life**: $\beta_t = \gamma_t + \delta_t$

Special Case: Poission Distribution

- $\gamma_t \sim Exp(\lambda)$
- δ_t follows $Exp(\lambda)$ truncated at t.
- $\mathbb{E}[\beta_t] = 1/\lambda + (1 - \exp(-\lambda t))/\lambda$

Limiting Behaviours

Elementary Renewal Theorem

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\mathbb{E}[X_k]}$$

Central Limit Theorem for Renewal Process
Let $\mu=E(X_k),\sigma^2=\text{Var}(X_k)$, then as $t\to\infty$, $\frac{\text{Var}(N(t))}{t}\to\frac{\sigma^2}{\mu^3}$ and

$$N(t)\sim\mathcal{N}(\frac{t}{\mu},\frac{t\sigma^2}{\mu^3})\text{ approximately}$$

Generalisation

Renewal Reward Process

Given a renewal process $N(t)$ with interarrival times $X_n, n\geq 1$ and suppose there is a reward for each renewal R_n that are i.i.d., the **Renewal Reward Process** $\{R(t),t\geq 0\}$ is

$$R(t)=\sum_{n=1}^{N(t)}R_n$$

Note: [1] R_n can depend on X_n, time. [2] Rewards can occur between/along renewals.

Limiting Theorems for Renewal Reward Process
For $E[R_n]<\infty$ and $E[X_n]<\infty$,

$$\lim_{t\to\infty}\frac{E[R(t)]}{t}=\frac{E[R_n]}{E[X_n]}$$

Example: Avg. Current Life $\lim_{t\to\infty}(\int_0^t\delta(s)ds)/t=E[X^2]/2E[X]$.

Regenerative Process

A stochastic process $\{X(t),t\geq 0\}$ is a **Regenerative Process** if there exists time pts when the process probabilistically restarts itself.
Note: [1] "Restart" includes both same transition & whether current state is same as initial. [2] Neither of MC & RegenProc $\subseteq$ the other.

Delayed Renewal Process

A **Delayed Renewal Process** is one when the component in operation at $t=0$ is not new, but all subsequent ones are.
Analysis: Same set of parameters & limiting behaviours.

4. Brownian Motion

Multi-Normal Distribution

A k-dim random vector $\mathbf{X}=(X_1,...,X_k)'$ with mean vector $\mu\in\mathbb{R}_{k\times 1}$ and covariance matrix $\Sigma\in\mathbb{R}_{k\times k}$ is **multivariate normally distributed** $\mathbf{X}\sim\mathcal{N}(\mu,\Sigma)$ if the joint density function is

$$f(x_1,...,x_k)=\frac{1}{\sqrt{2\pi\|\Sigma\|}}\exp\Big(-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\Big)$$

For any $\mathbf{a}\in\mathbf{R}_{1\times k}$, $aX\sim\mathcal{N}(a\mu,a\Sigma a')$.
For any matrix $\mathbf{Q}\in\mathbf{R}_{m\times k}$ with rank $m\leq k$, $QX\sim\mathcal{N}(Q\mu,Q\Sigma Q')$.
For any parition,

$$\mathbf{X}=\begin{pmatrix}\mathbf{X}_1\\\mathbf{X}_2\end{pmatrix}\sim\mathcal{N}(\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix},\begin{pmatrix}\Sigma_{11}&\Sigma_{12}\\\Sigma_{21}&\Sigma_{22}\end{pmatrix})$$

And the conditional distribution is still normal:

$$\mathbf{X}_1|\mathbf{X}_2\sim\mathcal{N}(\mu_1+\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}(\mathbf{X}_2-\mu_2),\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

Overview

(Standard) Brown Motion

Process $\{X(t),t\geq 0\}$ is a **Brownian Motion** with parameter σ if

- $X(0)=0$,
- $\{X(t),t\geq 0\}$ has stationary & independent increments,
- For every $t>0$, $X(t)\sim\mathcal{N}(0,\sigma^2 t)$.

A **Standard Brownian Motion** $\{B(t),t\geq 0\}$ has $\sigma=0$.
For time $t_1\leq t_2$,

$$\begin{pmatrix}\mathbf{X}_1\\\mathbf{X}_2\end{pmatrix}\sim\mathcal{N}(\mathbf{0},,\begin{pmatrix}\sigma^2t_1&\sigma^2t_1\\\sigma^2t_1&\sigma^2t_2\end{pmatrix})$$

For any time $s,t>0$,

- If $s\geq t$, $X(s)|X(t)\sim\mathcal{N}(X(t),\sigma^2(s-t))$.
- If $s<t$, $X(s)|X(t)\sim\mathcal{N}(\frac{s}{t}X(t),\sigma^2s^2-\sigma^2s^2/t^2)$

Brownian Motion with Drift

A **Brownian Motion with Drift** $\{X(t),t\geq 0\}$ with parameters μ and σ satisfies

- $X(0)=0$,
- $\{X(t),t\geq 0\}$ has stationary & independent increments,
- For every $t>0$, $X(t)\sim\mathcal{N}(\mu t,\sigma^2 t)$.

Note: $X(t)=\sigma B(t)+\mu t$.
For any time $s,t>0$,

$$X(s)|X(t)\sim\mathcal{N}(\mu,\frac{\min(s,t)}{t}[X(t)-\mu t],\sigma^2s-\sigma^2[\min(s,t)]^2/t)$$

Geometric Brownian Motion

A **Geometric Brownian Motion** $Y(t)$ with parameters μ and σ is:

$$Y(t)=e^{\sigma B(t)+\mu t}$$

Note: $Y(0)=1$ and $Y(t)\geq 0,\forall t$.
For any time $s<t$,

$$\mathbb{E}[Y(t)]=M_{X(t)}(1)=e^{\mu t+\sigma^2t/2}$$
$$\text{Var}(Y(t))=M_{X(t)}(2)-(M_{X(t)}(1))^2=e^{2\mu t+\sigma^2t}(e^{\sigma^2t}-1)$$
$$\mathbb{E}[Y(t)|Y(s)]=Y(s)\exp\big(\mu(t-s)+\sigma^2(t-s)/2\big)$$
$$\text{Cov}(Y(s),Y(t))=\exp\big(\mu(t+s)+\sigma^2(t+s)/2\big)(\exp(\sigma^2s)-1)$$

Intermediate Results & Others

From Lecture & Tutorials

For **Birh & Death Process** with +1 λ_i & -1 μ_i, limiting distribution:

$$\pi_n=\pi_0\prod_{i=1}^n\frac{\lambda_{i-1}}{\mu_i},\quad\text{subject to }\sum_{n=0}^\infty\pi_n=1$$

Delay Renewal Example: Consider $Y_1,Y_2,...$ iid. A **Pattern** is a r-dim vector $(y_1,...,y_r)$. Every time $(Y_{n-r+1},...,Y_n)=(y_1,...,y_r)$, a renewal occurs at n, denoted as $I(n)=1$. The counting process $N(n)$ is a Delayed Renewal Process.
Define **Overlapping** $k=\max\{j<r:(y_{r-j+1},...,y_r)=(y_1,...,y_j)\}$ how much two renewals overlap. Let $p=P(I(n)=1)$.
When $k=0$: $E[X_1]=1/p$, $\text{Var}(X_1)=1/p^2-(2r-1)/p$.
When $k>0$: $E[X_1]=E[X_{y_1,...,y_k}]+1/p$, and $\text{Var}(X_1)=\text{Var}(X_{y_1,...,y_k})+p^{-2}-(2r-1)/p+2p^{-3}\sum_{j=r-k}^{r-1}E[I(r)I(r+j)]$.

Appendix: Probability Theory

Gamma Distribution

$X\sim Gamma(\alpha,\lambda)$ for shape α, and rate $\lambda>0$. ($1/\lambda$ scale param.)

$$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x},\quad x\geq 0$$

Statistics: $E(X)=\frac{\alpha}{\lambda}$, $\text{var}(X)=\frac{\alpha}{\lambda^2}$
MGF: $M_X(t)=\left(1-\frac{t}{\lambda}\right)^{-\alpha},\quad t<\beta$
Special case: $Exp(\lambda)=Gamma(1,\lambda),\chi^2_n=Gamma(\frac{n}{2},\frac{1}{2})$
Properties: $Gamma(a,\lambda)+Gamma(b,\lambda)=Gamma(a+b,\lambda)$, and $cX\sim Gamma(\alpha,\frac{\lambda}{c})$
Gamma function $\Gamma(\alpha)=\int_0^\infty e^{-y}y^{\alpha-1}\,dy$
$\Gamma(1)=1,\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ and $\Gamma(n)=(n-1)!,\ n\in\mathbb{Z}^+$

Beta Distribution

$X\sim B(a,b)$ where $a>0,b>0$ has support $[0,1]$

$$f(X)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1},0\leq x\leq 1$$

Statistics: $E(X)=\frac{1}{1+\beta/\alpha}$, $\text{var}(X)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Special case: $\text{Unif}(0,1)=B(1,1)$
Beta function $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Common MGFs

Binomial: $M_X(t)=(1-p+pe^t)^n$
Poisson: $M_X(t)=\exp\big(\lambda(e^t-1)\big)$
Exponential: $M_X(t)=\frac{\lambda}{\lambda-t}$ for $t<\lambda$
Normal: $M_X(t)=\exp\big(\mu t+\sigma^2t^2/2\big)$

Others

$E[(X-\mu)^4]=3\sigma^4$ for $X\sim\mathcal{N}(\mu,\sigma^2)$
$E[(\int_0^T B(s)ds)^2]=\int_0^T\int_0^T E[B(s)B(t)]dtds$