ST3236 Cheatsheet
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Probability Theory, Review

Boole's Inequality

For any events $A_1, A_2, \ldots$,

$$\mathbb{P}\Big(\bigcup_{n\geq 1} A_n\Big) \leq \sum_{n\geq 1} \mathbb{P}(A_n)$$

Conditional Expectation, Properties

- Linearity: $\mathbb{E}(aX_1 + bX_2|Y) = a\mathbb{E}(X_1|Y) + b\mathbb{E}(X_2|Y)$
- Law of Iterated Expectation: $\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X)$
- Tower Property: $\mathbb{E}(\mathbb{E}(X|Y,Z)|Y) = \mathbb{E}(X|Y)$
- Independence: $\mathbb{E}(X|Y) = \mathbb{E}(X)$, for X and Y independent.

Markov Chain (MC) Basics

A (discrete-time) **Markov Chain** (MC) is a stochastic process $\{X_n\}_{n=0}^{\infty}$ that satisfies,

$$\begin{aligned}&\mathbb{P}(X_{n+1} = t_{n+1}|X_n = t_n, X_{n-1} = t_{n-1}, \ldots, X_1 = t_1)\\ =&\mathbb{P}(X_{n+1} = t_{n+1}|X_n = t_n)\end{aligned}$$

,whenever $\mathbb{P}(X_n = t_n, X_{n-1} = t_{n-1}, \ldots, X_1 = t_1) > 0$. This property is also called the **Markovian Property**.

A **Time-homogeneous** Markov Chain is one where the conditional probability $\mathbb{P}(X_{n+1} = j|X_n = i)$ does not depend on n. Equivalently, it means for all $n \geq 0$ and $i, j \in S$,

$$\mathbb{P}(X_{n+1} = j|X_n = i) = \mathbb{P}(X_1 = j|X_0 = i)$$

For time-homogeneous MCs, $p_{ij} = \mathbb{P}(X_1 = j|X_0 = i)$ is the **1-step transition probability**, and $P = ((p_{ij}))_{i,j\in S}$ is the **transition matrix**. Similarly, $p_{ij}^{(k)} = \mathbb{P}(X_k = j|X_0 = i)$ is the **k-Step Transition Probability**.

Chapman-Komogorov Equation

$$p_{ij}^{(k)} = \sum_{i_1,\ldots,i_{k-1}\in S} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{k-1}j}$$

Equivalently, it means $P^{(k)} = P^k$.

Accessing States, MC

Accessible States

For states i, j of a MC, j is **accesible** from i, $i \to j$, if there exists $k \geq 0$ such that,

$$p_{ij} = \mathbb{P}(X_k = j|X_0 = i) > 0$$

States i and j **communicate**, i.e. $i \leftrightarrow j$ iff. $i \to j \wedge j \to i$.

$\leftrightarrow$ is a **Equivalence Relation**. The equivalent classes of S partitioned by it are called **Communicating classes**.

Notes:

- $i \leftrightarrow i$ as $p_{ii}^{(0)} = 0$ for all $i \in S$
- For $i \in S_i$ and $j \in S_j$, it is possible that $i \to j$.
- For $i \in S_i$ and $j \in S_j$, $i \to j \implies j \not\to i$.

A MC is **irreducible** if there is only one communicating class for the state space. Otherwise, the MC is **reducible**.

Essential States

A state i is **essential** if for every state j it satisfies

$$i \to j \implies j \to i$$

A state that is not essential is called **inessential**.

Properties of Essential States

- If i essential and $i \to j$ then j is essential.
- All states in one communicating class are either all essential or all inessential.
- A **finite state** MC must have at least one essential state.
- An absorbing state is essential.