CS4261 Cheatsheet
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Nash Equilibrium

A **Game** is a general abstract framework for strategic interactions, with usually 1) A set of **Players** $N = \{1, ..., n\}$, and subsequently for each player $i \in N$, 2) A set of possible **Actions** $A_i = \{a_{i1}, a_{i2} ...\}$, and 3) a **Utility Function** $u_i : A \mapsto \mathbb{R}$, which indicates the utility Player i can get from an action profile, then lastly 4) a **(Pure) Action Profile** that denotes the actions taken by all the players $\vec{a} \in A_1 \times A_2 \times ... \times A_n = A$.

A **Normal Form Game** is a Matrix Representation of the player utilities for a 2-Player game. Conventionally, each element in the Normal Form Game is a pair of real values $C_{ij} = (u, v) \in \mathbb{R}^2$ where u and v are the utility of the row and column player given an action profile $\vec{a} = (a_{1,i}, a_{2,j})$.

Pure Nash Equilibrium

Given actions taken by everyone else $\vec{a}_{-i}$, the **Best Response** set of Player i is defined

$$BR_i(\vec{a}_{-i}) = \{b \in A_i | b \in \text{argmax } u_i(\vec{a}_{-i}, b)\}$$

An action profile is a **Pure Nash Equilibrium** if it is one best response for everyone given what others have chosen, i.e.

$$\forall i \in N, a_i \in BR_i(\vec{a}_{-i})$$

Analysis: Not all games have Pure Nash Equilibria.

Mixed Nash Equilibrium

Let $\vec{p} \in \Delta(A_i)$ be the probability distribution over Player i's actions. A **Mixed / Randomised Strategy Profile** is given by $\vec{p} = (\vec{p}_1, ..., \vec{p}_n) \in \Delta(A_1) \times ... \times \Delta(A_n)$. Player utility is $u_i(\vec{p}) = \sum_{\vec{a} \in A} u_i(\vec{a}) P(\vec{a}) = \mathbb{E}_{\vec{a} \sim \vec{p}}[u_i(\vec{a})]$.

A mixed profile is a **Mixed Nash Equilibrium** if

$$\forall i \in N, \vec{q}_i \in \Delta(A_i),\ u_i(\vec{p}) \geq u_i(\vec{p}_{-1}, \vec{q}_i)$$

Analysis: All games have Mixed Nash Equilibria.

In a Mixed Nash Equilibrium, a player is **Indifferent** if he gets the same **expected utility** from choosing any action (as a result of the other player playing a mixed strategy).

Compute Nash Equilibria in 2 Player Games

- Compute all NE in which at least one player plays a pure strategy.
- Compute all NE in which both players play mixed strategies. In this case, each player must be indifferent between the two strategies.

Dominant Strategies

A strategy $\vec{p} \in \Delta(A_i)$ **Dominates** $\vec{q} \in \Delta(A_i)$ if

$$\forall \vec{p}_{-i} \in \Delta(A_{-i}),\ u_i(\vec{p}_{-i}, \vec{p}) \geq u_i(\vec{p}_{-i}, \vec{q})$$

There are similar definitions for **Strictly Domination**.

Intuitively, it means no mather what others do, playing $\vec{p}$ is always better than $\vec{q}$.

Dominant Strategy Theorem

If an action $a \in A_i$ is **strictly dominated** by some strategy $\vec{p} \in \Delta(A_i)$, then action a is never played with any positive probability in any Nash Equilibrium.

Note: The theorem enables us to prune actions that will not occur in any Nash Equilibria.

Auction

Single-Item Auction

Types of Single-Item Auction

- **English Auction** - Auctioneer sets a starting price. Bidders take turn raising their bids. The bidder makes the last bid wins and pay his bid.
- **Japanese Auction** - Auctioneer sets a starting price and raises it. A bidder can drop out and not return once dropped. The last standing bidder gets the item and pays the current price.
- **Vickrey / Second-Price Auction** - All bidders submit bids simultaneously. The highest bidder wins and pays the second highest price.

Vickrey Auction Problem Specification

There are n players $N = \{1, 2, ..., n\}$, each with a valuation of the item v_i. The actions are to place a bid at different prices. The payoff for a player is $v - p$ if getting the item, and 0 otherwise.

Analysis

Vickrey Auctions are **Truthful**, i.e. bidding according to one's true valuation is a dominant strategy.
First-Price Auctions are **Not Truthful**.
Note: Dominant strategies are Nash Equilibria in Auction games, but they are not necessarily the only Nash Equilibria.

Multi-Unit Auction

Game Specification

There are n players $N = \{1, 2, ..., n\}$, each with a valuation of the item v_i. There are $k \leq n$ identical copies of the item.

The objective is to design a mechanism where

- Truthful bidding is a dominant strategy
- Items are allocated to the k highest bidders

Vickrey Clarke Groves (VCG) Mechanism

Procedures:

1. Choose some outcome o^* that maximises social welfare $\sum_i v_i(o^*)$
2. Calculate the payment that Player j must take with $p_j = \sum_{i \neq j} v_i(o^*_{-j}) - \sum_{i \neq j} v_i(o^*)$, where o^*_{-j} is the outcome that maximises $\sum_{i \neq j} v_i(o_{-j})$.

Note: The payment for each Player is essentially the **Externality** that he imposes on other players, which is the difference in the max welfare of others between if he is absent and if present.
Analysis: VCG is truthful. Vickrey Auction is a special case of VCG.

Combinatorial Auction

Problem Specification

There are n Players and m possibly distinct items for sale. Each player has a valuation for each subset of the m objects.

VCG is truthful, but it can be computationally intensive, and suffers from **Revenue Non-Monotonicty**, a paradox where adding more players in the bidding game may lead to a decrease in the **Revenue**, i.e. sum of all players' payment $R = \sum_{i=1}^{n} p_i$.
Note: Single-Item Auctions have no revenue non-monotonicity.

Facility Location

Overview

Problem Specification

There are n players $N = \{1, ..., n\}$, each with a location $x_i \in \mathbb{R}$, assuming $x_1 \leq x_2 \leq ... \leq x_n$ for convenience.

The objective is to design $f : \mathbb{R}^n \mapsto \mathbb{R}$ that minimises either of

- **Total Cost** - $\sum_{i \in N} |f(\vec{x}) - x_i|$
- **Max Cost** - $\max_{i \in N} |f(\vec{x}) - x_i|$

Analysis: OPT for Total Cost is **Not Truthful**. OPT for Max Cost is **Truthful** if it always "snaps" to a median player.

Max Cost Approximation Theorems

Deterministic Case

Any deterministic truthful mechanism for facility location has a worst-case approximation ratio ≤ 2 to the maximum cost.

Randomised Case

Any randomised truthful mechanism for facility location has a worst-case approximation ratio $\leq \frac{3}{2}$ to the maximum cost.

Routing Games

In a traffic network, players are drivers trying to find a route that minimises their total traffic time.

- **Proportion Version** There is 1 unit of traffic to allocate in total. Drivers are considered proportion of the total traffic.
- **Atomic Version** The traffic consists of $k \in \mathbb{N}$ drivers, each being an atomic entity.

Price of Anarchy (PoA) is the ratio of the social cost under the worst case Nash Equilibrium and under socially optimal solution

$$PoA = \frac{\text{WorstNash}(G)}{OPT(G)}$$

Analysis: 1) $PoA \geq 1$ with the smaller being the better. 2) All Nash Equilibria in a Routing Game have the same social cost.

Atomic Version

Atomic Routing Game Theorem

In an atomic routing game, a pure NE flow always exists.

Higher Level Idea

Every atomic routing game is a potential game, where all players are inadvertently and collectively optimising a potential function, $\Phi(f)$,

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

Analysis: When a player deviates (changes path), change in the deviator's individual cost is equal to $\triangle\Phi$. "Alignment in individual and social objective".

Cooperative Games

Overview

Problem Specifications

A **Cooperative Game** $\mathcal{G}(N, v)$ consists of a set of players $N = \{1, \ldots, n\}$, and a valuation function $v : 2^n \to \mathbb{R}_{\geq 0}$ for each player. A **Coalition Structure** (CS) is a partition of N while $\mathrm{OPT}(G) = \max_{\mathrm{CS}} \sum_{S \in \mathrm{CS}} v(S)$ is the optimal.

Properties, Cooperative Games

Monotone - For all $S \subseteq T \subseteq N$, $v(S) \leq v(T)$.
Simple - Monotone and for all $S \subseteq N$, $v(S) \in \{0, 1\}$.
Super-additive - For any disjoint $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T)$.
Convex - For $S \subseteq T \subseteq N$ and any $i \in N \backslash T$, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

Properties, Game Payoff

Imputation - An efficient & individual rational payoff allocation $\vec{x}$.
$\vec{x}$ is **Efficient** if satisfying $\sum_{i \in N} x_i = v(N)$.
$\vec{x}$ is **Individual Rational** if $x_i \geq v(i)$ for all $i \in N$.

The Core

An imputation $\vec{x}$ is in the **Core** if it satisfies for all $S \subseteq N$,

$$\sum_{i \in S} x_i = x(s) \geq v(S)$$

Note: A core is a set of vectors, not a set of players.

Properties, the Core

Assume the game $\mathcal{G} = (N, v)$ is simple:
Winning / **Losing Coalition** - A coalition with value 1 / 0.
Veto Player - a player that is in every winning coalition.
Analysis: For p veto, any coalition without p cannot win, and any wit p does not necessarily win.

Veto Player Theorem

For a simple game, $\mathrm{Core}(\mathcal{G}) \neq \emptyset$ iff. $\mathcal{G}$ has veto players.
The core only distribute payoffs among the veto players.

Shapley Value

Definitions

For Player i and $S \subseteq N$, the **Marginal Contribution** of i to S is

$$m_i(S) = v(S \cup \{i\}) - v(S)$$

Given a permutation $\sigma \in \Pi(N)$, the **Predecessors** of i in σ are

$$P_i(\sigma) = \{j \in N | \sigma(j) < \sigma(i)\}$$

We can write $m_i(\sigma) = m_i(P_i(\sigma))$ for marginal contribution.
Shapley Value of a player in a Coalition is his expected marginal contribution.

$$Sh_i = \mathbb{E}\left[m_i(\sigma)\right] = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i(\sigma)$$

Shapley Value Theorem

Shapley Value is the only payoff allocation value satisfying efficiency, linearity, dummy and symmetry.
Note: [1] **Symmetry**: symmetric players are paid equally. [2] **Dummy**: Dummy players are not paid.

Induced Subgraph Games

Problem Specifications

An example of Cooperative Games $\mathcal{G} = (V, E)$ where players are vertices and $e = (u, v) \in E$ with weight w_e describing the utility of u and v in a coalition.

Induced Subgraph Game Core Theorem

The core of an induced subgraph game is non-empty iff. the graph has no negative cut.
Note: A negative cut is a cut (set of edges that parition the graph into two) where the sum of edge weights is negative.

Shapley Value, Induced Subgraph Game

The payoff for each player i will be

$$\phi_i = \frac{1}{2} \sum_{j \in N} w(i, j)$$

Nash Bargaining Solution

Definitions

Problem Specifications

A **Bargaining Game** is a pair $(S, \vec{d})$ for $S \subseteq \mathbb{R}^2$ and $\vec{d} \in \mathbb{R}^2$ with at least one point $(x_1, y_1) \in S$ such that $x_1 \geq d_1$ and $y_1 \geq d_2$ for $\vec{d} = (d_1, d_2)$. **Two players** chooses $x, y \in \mathbb{R}$ respectively. If $(x, y) \in S$, the two players receives x and y respectively. Otherwise, they receive d_1 and d_2. A **solution** is a function $\vec{f} = (f_1, f_2)$ that takes in $(S, \vec{d})$ and outputs a value for the two players each.

Pareto Optimality

An outcome (x_1, x_2) **Pareto Dominates** another outcome (x_2, y_2) if $x_1 \geq x_2$ and $y_1 \geq y_2$ and at least one of these two inequalities is strict. The dominating one is the **Pareto Improvement**. An outcome without a Pareto Improvement is **Pareto Optimal**.
In a Bargaining game, S's top right boundary is the Pareto Frontier.

Properties, Bargaining Game Solutions

Efficiency - No outcome (v_1, v_2) dominates $\left(f_1(S, \vec{d}), f_2(S, \vec{d})\right)$.
Symmetry - Let $S^T = \{(y, x) : (x, y) \in S\}$ and $\vec{d}^T = (d_2, d_1)$, then

$$\left(f_1(S^T, \vec{d}^T), f_2(S^T, \vec{d}^T\right) = \left(f_1(S, \vec{d}), f_2(S, \vec{d}\right)$$

Independence of Irrelevant Alternative (IIA) - Let $S' \subseteq S$ such that $\left(f_1(S, \vec{d}), f_2(S, \vec{d}\right) \in S'$, then,

$$\left(f_1(S', \vec{d}), f_2(S', \vec{d}\right) = \left(f_1(S, \vec{d}), f_2(S, \vec{d}\right)$$

Invariance under Equivalent Representations (IER) - For any $\alpha_1, \alpha_2 \in \mathbb{R}, \vec{\beta} \in \mathbb{R}^2$,

$$f_i((\alpha_1, \alpha_2)S + \vec{\beta},\ (\alpha_1, \alpha_2)\vec{d} + \vec{\beta}) = \alpha_i f_i(S, \vec{d}) + \beta_i,\ i = 1, 2$$

Nash Bargaining Solution

The **Nash Bargaining Solution** for a bargaining game is the solution

$$\underset{(v_1, v_2) \in S}{\operatorname{argmax}} (v_1 - d_1)(v_2 - d_2)$$

The Nash Bargaining Solution is the only solution that satisfies Efficiency, Symmetry, IIA and IER.

Utility Maximises

- **Utilitarian**: $\max \sum_i u_i(A)$.
- **Nash**: $\max \prod_i u_i(A)$.
- **Egalitarian**: $\max \min_i u_i(A)$.

Stable Matching

Definitions

Problem Specifications

In the case of matching medical students S to hospitals H, with $|S| = n, |H| = m$: Each student s has a strict preference order over H, denoted by $\succ_s$. Similarly for $\succ_h$. A matching $M : S \to H$ is a one-to-one mapping.

The goal is to design a market that **Thick**, **Safe** (truthful, fair and encouraging participation), and **Timely**.

Stable Matching

A pair $(s, h) \in S \times H$ **Blocks** matching M, if

$$h \succ_s M(s) \land s \succ_h M^{-1}(h)$$

A matching M is **Stable** if there are no blocking pairs.
Analysis: [1] A stable matching always exists. [2] A stable matching can always be found in polynomial time. e.g. GS Algorithm.

Gale-Shapley Deferred Acceptance Algorithm

Procedures:

1. Start with all students unassigned.
2. While there are unassigned students: Each unassigned student proposes to favourite **not-yet-proposed-to** hospital. Then each hospital looks at **students proposed to it in this round and whoever currently assigned** to it, and picks most preferred; all others remain unassigned.
3. Repeat until all matched.

Analysis: [1] Terminate in $\leq n^2$ iterations with a stable matching.

Fairness for GS Algorithm

Given $s \in S$, a **Valid** hospital $h \in H$ is one that there exists some stable matching M such that $M(s) = h$. $\mathbf{best(s)}$ and $\mathbf{worse(s)}$ are the most and least highly ranked valid hospital for s.

Theorem

The GS Algorithm (with student proposing) assigns each student $s \in S$ to the hospital $\text{best}(s)$, and each hospital $h \in H$ to the student $\text{worst}(h)$.

Fair Allocation of Indivisible Goods

Definitions

Problem Specifications

With players $N = \{1, 2, ..., \}$ and indivisible goods $G = \{g_1, g_2, ..., g_m\}$, Player i has value $v_i(g)$ for good g. An **Allocation**, $A = (A_1, ..., A_n)$ is a partition, with **Bundle** A_i is allocated to Player i.

Valuation is **Additive** if

$$v_i(G') = \sum_{g \in G'} v_i(g), \ \forall G' \subseteq G$$

Fairness

Proportionality - $v_i(A_i) \geq \frac{1}{n} \cdot v_i(G)$ for all $i \in N$.
Envy-freness - $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$.
Envy-freeness Up to One Good (EF1) - For any $i, j \in N$, if $A_j \neq \emptyset$, there exists $g \in A_j$ such that

$$v_i(A_i) \geq v_i(A_j \backslash \{g\})$$

Analysis: EF1 allocations always exist.

Envy-freeness Up to Any Good (EFX) - For any $i, j \in N$ and any $g \in A_j$, we have

$$v_i(A_i) \geq v_i(A_j \backslash \{g\})$$

Analysis: [1] EFX is stronger than EF1. [2] EFX allocations always for $n = 2$ (Cut-and-Choose protocol) and $n = 4$, and its existence for $n \geq 4$ remains an open problem.

Maximum Nash Welfare

An allocation that maximises the Nash welfare, known as **Maximum Nash Welfare** (MNW) allocation, satisfies EF1.
Note: If $MNW = 0$, maximising the number of players with positive utility, then maximise Nash welfare among these players.
Analysis: The allocation is also Pareto Optimal.

Envy-free

Round-Robin

Procedures: Let players take turns choosing their favourite good from the remaining, in the order $1, 2, ..., n, 1, 2, ..., n, 1, 2, ...$ until goods run out.
Note: Require **additive** valuations.

Envy-Cycle Elimination

Procedures: [1] Allocate one good at a time in an arbitrary order. [2] Maintain an **envy graph** with $e : i \to j$ for each i envying j. [3] At each step, the next good is allocated to a player with no incoming edges. [3'] Any cycle that arises is eliminated by giving j's entire bundle to any i for $i \to j$.
Note: Require **Monotone** valuation, not necessarily **additive**.

Proportionality

Maximin Share (MMS) of Player i: Player i divides all goods into n bundles so as to maximise the values of the value of the minimum-value bundle. An relaxation of proportionality:

$$\text{MMS}_i \leq \frac{v_i(G)}{n}$$

Analysis: [1] Solution achieving MMS for every player exists for $n = 2$ (Cut-and-Choose), but might not exist for $n \geq 3$. [2] Solution with at least $\frac{3}{4}\text{MMS}_i$ for each always achievable for all n.

Query Complexity

The Envy-Cycle Elimination Algoritm can be implemented using $O(nm)$ queries, even with monotonic valuations.
For EF1 solution for two agents with monotonic valuations, $O(\log m)$ queries suffice.
Any deterministic EF1 algorithm needs $\Omega(\log m)$ queries.
Any deterministic EFX algorithm needs queries **exponential** in m.

Cake Cutting

Imtermediate Results

From Assignments

Assignment5 Qn3 For a cooperative game with non-empty core, the Shapley vector is not necessarily in the core.