MA1101R Exercise Question Cheatsheet
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Chapter 2 - Matrices

Ex2Qn9

Suppose the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ has non-trivial solution. Then the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has either no solution or infinitely many solution.

Ex2Qn11(e)

There are no square matrices A and B of same order such that $AB - BA = I$.

Ex2Qn23 (Block matrix multiplication)

Let A be an $m \times n$ matrix,

- For matrices B_1 and B_2 of size $n \times p$ and $n \times q$ respectively,
$$A(B_1\ B_2) = (AB_1\ AB_2)$$
- For matrices D_1 and D_2 of size $p \times m$ and $q \times m$ respectively,
$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} A = \begin{pmatrix} D_1 A \\ D_2 A \end{pmatrix}$$

Ex2Qn60

Suppose A is an invertible matrix, then $adj(A)$ is invertible.

Chapter 3 - Vector Space

Ex3Qn30

Let $u_1, u_2, ..., u_k$ be vectors in $\mathbb{R}^n$ and P a square matrix of order n,

- If $Pu_1, Pu_2, ..., Pu_k$ are linearly independent, then $u_1, u_2, ..., u_k$ are linearly independent.
- If $u_1, u_2, ..., u_k$ are linearly independent, *and P is invertible*, then $Pu_1, Pu_2, ..., Pu_k$ are linearly independent.

Ex3Qn41

Let V be a vector space,

- suppose S is a finite subset of V such that $span(S) = V$, then there exists a subset S' such that S' is a basis for V.
- suppose T is a finite subset of V such that T is linearly independent, then there exists a basis T^* for V such that $T \subseteq T^*$

Ex3Qn43

Let V, W be two subspaces of $\mathbb{R}^n$, then
$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

Ex3Qn45

Let V, W be two subspaces of a given vector space,

- there exists a basis S_1 for V and a basis S_2 for W, such that $S_1 \cap S_2$ is a basis for $V \cap W$.
- there exists a basis T_1 for V and a basis T_2 for W, such that $T_1 \cup T_2$ is a basis for $V + W$.

Chapter 4 - Rank & Nullity

Ex4Qn20

Suppose A and B are two matrices such that $AB = \boldsymbol{0}$, then column space of B is contained in the nullspace of A.

Ex4Qn21

There is no matrix whose row space and nullspace both contain the same nonzero vector.

Ex4Qn22

Let A be an $m \times n$ matrix and P an $m \times m$ matrix. If P is invertible, then $\text{rank}(PA) = \text{rank}(A)$. (The inverse is not true)

Ex4Qn23

For two matrices A, B of the same size,
$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$$

Ex4Qn24

Let A be an $m \times n$ matrix. Suppose the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^n$, then the linear system $A^T\boldsymbol{y} = \boldsymbol{0}$ has only the trivial solution.

Ex4Qn25

For a matrix A of size $m \times n$,

- nullspace of A is equal to nullspace of $A^T A$
- $\text{nullity}(A) = \text{nullity}(A^T A)$
- $\text{rank}(A) = \text{rank}(A^T A)$
- $\text{rank}(A) = \text{rank}(AA^T)$

(However, $\text{nullity}(A) \neq \text{nullity}(AA^T)$)

Chapter 5 - Orthogonality

Ex5Qn9

Let $\{u_1, u_2, ..., u_n\}$ be an orthogonal set of vectors in a vector space, then
$$\|u_1 + u_2 + ... + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + ... + \|u_n\|^2$$

Ex5Qn18 Uniqueness of (Orthogonal) Projection

Let V be a subspace of $\mathbb{R}^n$ and $\boldsymbol{u}$ a vector in $\mathbb{R}^n$. $\boldsymbol{u}$ can written uniquely as $\boldsymbol{u} = \boldsymbol{n} + \boldsymbol{p}$ such that $\boldsymbol{n}$ is a vector orthogonal to V and $\boldsymbol{p}$ a vector in V.

Ex5Qn19

Let A be a square matrix of order n such that $A^2 = A^T = A$,

- for any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $(A\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (A\boldsymbol{v})$
- for any vector $\boldsymbol{w} \in \mathbb{R}^n$, $A\boldsymbol{w}$ is the projection of $\boldsymbol{w}$ onto the subspace $V = \{\boldsymbol{u} \in \mathbb{R}^n | A\boldsymbol{u} = \boldsymbol{u}\}$ of $\mathbb{R}^n$

Ex5Qn32

Let A be an orthogonal matrix of order n and let $\boldsymbol{u}, \boldsymbol{v}$ be any two vectors in $\mathbb{R}^n$,

- $\|\boldsymbol{u}\| = \|A\boldsymbol{u}\|$
- $d(\boldsymbol{u}, \boldsymbol{v}) = d(A\boldsymbol{u}, A\boldsymbol{v})$
- the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to the angle between $A\boldsymbol{u}$ and $A\boldsymbol{v}$

Ex5Qn33

Let A be an orthogonal matrix of order n and $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$

- $T = \{A\boldsymbol{u}_1, A\boldsymbol{u}_2, ..., A\boldsymbol{u}_n\}$ is a basis for $\mathbb{R}^n$
- S is orthogonal $\rightarrow$ T is orthogonal
- S is orthonormal $\rightarrow$ T is orthonormal

Chapter 6 - Diagonalisation

Ex6Qn3

Let λ be an eigenvalue of a square matrix A,

- λ^n is an eigenvalue of A^n for any $n \in \mathbb{Z}_{\geq 1}$
- $\frac{1}{\lambda}$ is an eigenvalue for A^{-1} if A is invertible
- λ is an eigenvalue for A^T

Ex6Qn4

Let A be a square matrix such that $A^2 = A$. If λ is an eigenvalue of A, then $\lambda = 0$ or 1.

Ex6Qn16

Let A be a stochastic matrix,

- 1 is an eigenvalue of A,
- if λ is an eigenvalue of A, then $|\lambda| \leq 1$.

(A stochastic matrix $(a_{i_j})_{m \times n}$ is one where all entries are non-negative and sum of entries of each column is 1, i.e. $a_{1i} + a_{2i} + ... + a_{ni} = 1$, for all $i = 1, 2, ..., n$)

Ex6Qn25

Let $\boldsymbol{u}$ be a column matrix, then $I - \boldsymbol{u}\boldsymbol{u}^T$ is orthogonally diagonablisable.

Ex6Qn26

Let A be a symmetry matrix. If $\boldsymbol{u}, \boldsymbol{v}$ are two eigenvalues of A associated with different eigenvalues, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

Ex6Qn30

For two orthogonally diagonalisable matrices of same order, A, B, $A + B$ is orthogonally diagonalisable . (However, AB might not be.)

Chapter 7 - Linear Transformations

Ex7Qn8

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T \circ T = T$,

- if T is not the zero transformation, then there exits a nonzero vector $u \in \mathbb{R}^n$ such that $T(\boldsymbol{u}) = \boldsymbol{u}$
- if T is no the identity transformation, then there exists a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $T(\boldsymbol{v}) = 0$

Ex7Qn10

A linear operator T on $\mathbb{R}^n$ is called isometry if $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

- if T is an isometry on $\mathbb{R}^n$, then $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$
- let A be the standard matrix for a linear operator T. T is isometry iff. A is an orthogonal matrix.

Ex7Qn16

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. $\mathrm{Ker}(T) = \{\mathbf{0}\}$ iff T is one-to-one (i.e. $\forall\ \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n, \boldsymbol{u} \neq \boldsymbol{v} \to T(\boldsymbol{u}) \neq T(\boldsymbol{v})$)

Ex7Qn17

Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations,

- $\mathrm{Ker}(S) \subseteq \mathrm{Ker}(T \circ S)$
- $\mathrm{R}(T \circ S) \subseteq \mathrm{R}(T)$