MA2104 Cheatsheet
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Chapter 01 - Vectors in 3D Space

Vectors

Vector projection of a onto b: $\text{proj}_b a = \frac{a \cdot b}{b \cdot b} b$
Scalar projection of a onto b: $\text{comp}_b a = \frac{a \cdot b}{\|b\|}$

Dot & Cross Product

$$a \cdot b = \|a\|\|b\| \cos \theta, \quad \|a \times b\| = \|a\|\|b\| \sin \theta$$

where θ is the angle between vectors a and b.

Prop Ch01.3.5 - Scalar Triple Product

$$|a \cdot (b \times c)| = \left| \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|$$

is the volume of the parallelepiped determined by vectors a, b, c.

Chapter 02 - Curves and Surfaces

Curve

Tangent Vector

Tangent vector to a curve C paramaterised by $R(t) = \langle f(t), g(t), h(t) \rangle$ at $R(a)$ on the curve is given by

$$R'(a) = \langle f'(a), g'(a), h'(a) \rangle.$$

Arc Length Formula

The length of curve $C : R(t) = \langle f(t), g(t), h(t) \rangle$ between $R(a)$ and $R(b)$ is

$$\int_a^b \|R'(t)\| \, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} \, dt.$$

provided the first derivatives are continuous.

Surfaces

Cylinder

A surface is a cylinder if there is a plane P such that all the planes parallel to P intersect the surface in the same curve.

Quadric Surfaces

- Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
- Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Elliptic Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
- Hyperboloid of 1 Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of 2 Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

Chapter 03 - Multivariable Functions

Limit, Continuity & Differentiability

Limit for 2D Functions

For function f with domain $D \subset \mathbb{R}^2$ that contains points arbitrarily close to (a, b), then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.
The limit exists iff. the limit **exists and is the same for all continuous paths** to (a, b).

Clairaut's Theorem

For function f defined on $D \subset \mathbb{R}^2$ that contains (a, b), if the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Differentiability for 2D Functions

For function f defined on $D \subset \mathbb{R}^2$ and differentiable at (a, b) within the interior of D,

$$\lim_{(h,k)->(0,0)} \frac{f(a+h, b+k) - f(a, b) - L(h, k)}{\sqrt{h^2 + k^2}} = 0,$$

where $L : \mathbb{R}^2 \to \mathbb{R}$ is a linear map defined as the total derivative of f at (a, b):

$$L(h, k) = D_{f(a,b)}(h, k) = f_x(a, b)h + f_y(a, b)k.$$

Notes about Differentiability

Consider f at a point (a, b):

- f_x and f_y exist $\not\Longrightarrow$ f differentiable
- f_x and f_y exist & continuous $\Longrightarrow$ f differentiable (*Differentiability Theorem*)
- f differentiable $\not\Longrightarrow$ f_x and f_y continuous

Linear Approximation

$$f(a+h, b+k) \approx f(a, b) + f_x(a, b)h + f_y(a, b)k$$

Gradient Vector

Gradient Vector

The gradient vector of f defined on $D \subset \mathbb{R}^2$ at $(a, b) \in D$ is defined as:

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

Directional Directive

The directional directive of f defined on $D \subset \mathbb{R}^2$ in the direction of the unit vector $u = \langle u_1, u_2 \rangle$ is

$$D_{f(a,b)}(u) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = \nabla f(a, b) \cdot u.$$

Perpendicular Vector of Level Sets

$\nabla f(a, b)$ is orthogonal to the $f(a, b)$-level curve of f at (a, b).

Chapter 04 - Calculus on Surfaces

Implicit Differentiation

Prop Ch04.1.4

For F defined on $D \subset \mathbb{R}^3$ where $F(a, b, c) = k$ defines z as a differentiable function of x and y near (a, b, c), and $F_z(a, b, c) \neq 0$,

$$\frac{\partial z}{\partial x}(a, b, c) = -\frac{F_x(a, b, c)}{F_z(a, b, c)}, \quad \frac{\partial z}{\partial y}(a, b, c) = -\frac{F_y(a, b, c)}{F_z(a, b, c)}$$

Extrema

Extreme Value Theorem

If $f : D \to \mathbb{R}$ is continuous on a **closed and bounded** set $D \subset \mathbb{R}^2$, then f has at least one global maximum and one global minimum.

Steps for Finding Global Extrema

For $f : D \to \mathbb{R}$ where D is closed and bounded,

1. Find all critical points of f and their corresponding f-values.
2. Find the extreme values of f on boundary of D.
3. Compare.

Method of Lagrange Multiplier

To find the extrema of differentiable $f : D \to \mathbb{R}$ subject to curve $C : g(x, y) = k$ for some $k \in \mathbb{R}$,

1. Find all points (a, b) for $\nabla g(a, b) \neq 0$ and values λ s.t.
$$\nabla f(a, b) = \lambda \nabla g(a, b), \ g(a, b) = k,$$
and evaluate f at all these points.
2. Find the extreme values of f on the boundary of C.
3. Compare.

Chapter 05 & 06 - Integration

Fubini's Theorem

If f is continuous on the rectangle $D = [a, b] \times [c, d]$, then,

$$\iint_D f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Its equivalence in $\mathbb{R}^3$ for triple integral also holds.

Change of Coordinates

Double Integral in Polar Coordinates

Transform between (x, y) and (r, θ):

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \end{cases}$$

In addition, $dA = dxdy = rdrd\theta$.

Triple Integral in Cylindrical Coordinates

Transform between (x, y, z) and (r, θ, z):

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

In addition, $dV = dxdydz = rdrd\theta dz$.

Triple Integral in Spherical Coordinates

Transform between (x, y, z) and (ρ, θ, ϕ):

$$\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(y/x) \\ \phi = \cos^{-1}(z/\rho) \le \pi \end{cases}$$

In addition, $dV = dxdydz = \rho^2\sin\phi d\rho d\theta d\phi$.

Application of integration

For a given region D in $\mathbb{R}^2$, the area of the region can be calculated as $\mathrm{Area}(D) = \iint_D 1dA$.
For a given solid E in $\mathbb{R}^3$, the volume of the solid can be calculated as $\mathrm{Volume}(E) = \iiint_E 1dV$.

Chapter 07 - Change of Coordinates

Planar Transformation

A map $T : S \to R$ is a planar transformation if it is a differentiable map whose inverse is differentiable.
Therefore, to show $T : S \to R$ is a planar transformation, it must satisfy:

1. T is differentiable
2. The inverse T^{-1} exists
3. The inverse T^{-1} is differentiable

Change of Coordinates

2D Jacobian Determinant

The *Jacobian* of the transformation $T(u, v) = (x(u, v), y(u, v))$ is defined

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of Variable in Double Integral

Let $T : S \to R$ be a planar transformation, where S lies in the uv-plane and R lies in the xy-plane. Let A and A' denote the area in the xy- and uv-plane respectively. For a two-var. function from xy-plane to $\mathbb{R}$,

$$\iint_R f(x,y)dA = \iint_R f \circ T(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dA'$$

Equivalently,

$$\iint_R f(x,y)dxdy = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

The 3D Jacobian has similar expressions and properties.

Inverse Function Theorem

Chapter 08 - Line Integrals

Line Integral

Line Integral of Functions

For curve C parameterised by $R(t) = (x(t), y(t), z(t), \ a \le t \le b$, and a 3-var function $f(x, y, z)$, the line integral

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\|R'(t)\|dt$$

Line Integral of Vector Fields

Let $\mathbf{C} = (C, o)$ be a smooth oriented curve in $\mathbb{R}^3$ parameterised by $R(t) = (x(t), y(t), z(t)), \ a \le t \le b$, and let $\mathbf{F} = \mathbf{F}(x, y, z)$ be a continuous vector field along C. Then, the line integral of $\mathbf{F}$ along $\mathbf{C}$

$$\int_{\mathbf{C}} \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t),y(t),z(t))\cdot R'(t)dt$$

For $\mathbf{F} = \langle X, Y, Z\rangle$ in its component form,

$$\int_{\mathbf{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathbf{C}} Xdx + Ydy + Zdz$$

In addition, if $-\mathbf{C}$ is the curve $\mathbf{C}$ with opposite orientation,

$$\int_{\mathbf{C}} \mathbf{F}\cdot d\mathbf{r} = -\int_{-\mathbf{C}} \mathbf{F}\cdot d\mathbf{r}$$

Conservative Vector Fields

Whenever a vector field $\mathbf{F}$ on some open domain stratifies $\mathbf{F} = \nabla f$ for some differentiable function f, then we call $\mathbf{F}$ a *conservative vector field* and f the *potential function* of $\mathbf{F}$.

Tests for Conservativity

- To show conservative: find a f s.t. $\nabla f = \mathbf{F}$ (by definition).
- To show conservative: if $\mathbf{F}(x, y) = \langle X, Y\rangle$ is defined over an **open and simply-connected** region $D \subset \mathbb{R}^2$, then need to show
$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$$
- To show non-conservative: find two oriented curves, $\mathbf{C_1}$ and $\mathbf{C_2}$, with same starting and ending points, s.t.
$$\int_{\mathbf{C_1}} \mathbf{F}\cdot d\mathbf{r} \ne \int_{\mathbf{C_2}} \mathbf{F}\cdot d\mathbf{r}$$

Gradient Theorem

For a 3-var function f whose gradient vector ∇f is continuous along $\mathbf{C} = (C, 0)$ parameterised by $R(t) = (x(t), y(t), z(t)), \ a \le t \le b$,

$$\int_{\mathbf{C}} \nabla f\cdot d\mathbf{r} = f(R(b)) - f(R(a))$$

Green's Theorem, Version I

Let $\mathbf{C} = (C, o)$ be a **positively oriented**, piecewise differentiable loop in $\mathbb{R}^2$, and let D be the region bounded by C. Then for vector field $\mathbf{F} = \langle X, Y\rangle$,

$$\int_{\mathbf{C}} \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)dA$$

Green's Theorem, Version II

Let $\mathbf{C} = (C, o)$ be a **positively oriented**, piecewise differentiable loop in $\mathbb{R}^2$, and let D be the region bounded by C. Then for vector field $\mathbf{F}$, let $\mathbf{n}(x, y)$ denote the **outward pointing** unit normal vector to S,

$$\int_{\mathbf{C}} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA$$

Algebraically, the outward pointing unit normal vector is

$$\mathbf{n}(t) = \frac{\langle y'(t), -x'(t)\rangle}{\|R'(t)\|} = \frac{\langle y'(t), -x'(t)\rangle}{\sqrt{x'(t)^2+y'(t)^2}}$$

Note, the integral is called the outward flux of $\mathbf{F}$ across C.

Chapter 09 - Surface Integrals

Surface Integral

Surface Integral of Functions

Let $R : D \to S$ be a (differentiable) parameterisation of surface S, then

$$\iint_S f(x,y,z)dS = \iint_D f(x(u,v),y(u,v),z(u,v))\|R_u \times R_v\|dA$$

Special case: when S is the graph of a 2-var function $g(x, y)$ for $(x, y) \in D$ for some domain D,

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\left(\sqrt{g_x^2+g_y^2+1}\right)dA$$

Orientation on Surface

A (differentiable) surface $S \in \mathbb{R}^3$ is *orientable* if it is possible to define for every $(x, y, z) \in S$, a unit normal vector $\mathbf{n}(x, y, z)$ to S with initial point (x, y, z) such that $\mathbf{n}$ varies continuously.
An orientable surface has two orientations:

$$\mathbf{n} = \pm\frac{R_u \times R_v}{\|R_u \times R_v\|}$$

Surface Integral of Vector Fields

Let $\mathbf{S} = (S, \mathbf{n})$ be an oriented surface where $R : D \to S$ is the parameterisation for S, and let $\mathbf{F}$ be a vector field along the surface,

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{D} \mathbf{F} \cdot (R_u \times R_v)\, dA$$

Special case: when S is the graph of a 2-var function $g(x, y)$, let $\mathbf{S}$ denote S with **upward orientation**, and let $\mathbf{F} = \langle X, Y, Z \rangle$,

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \Big(-\mathbf{X}\frac{\partial g}{\partial x} - \mathbf{Y}\frac{\partial g}{\partial Y} + \mathbf{Z} \Big)\, dA$$

Gauss' Theorem

Divergence

For a vector field $\mathbf{F} = \langle X, Y, Z \rangle$, the *divergence* of $\mathbf{F}$ is defined as

$$\text{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial X}{\partial x}(x, y, z) + \frac{\partial Y}{\partial y}(x, y, z) + \frac{\partial Z}{\partial z}(x, y, z)$$

Gauss' Theorem

Let E be a solid region where the boundary surface S is piece-wise smooth, and let $\mathbf{S}$ denote S with **outward orientation**. Then for a vector field $\mathbf{F}$ defined over S,

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{E} \text{div}\, \mathbf{F}\, dV$$

Stokes' Theorem

Curl

For a vector field $\mathbf{F} = \langle X, Y, Z \rangle$, the *curl* of $\mathbf{F}$ is defined as

$$\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right\rangle$$

Induced Orientation

For oriented surface $\mathbf{S} = (S, \mathbf{n})$ with boundary C being a simple loop, the *induced orientation*, $\mathbf{o}$ of $\mathbf{n}$ is one such that if you want along C in the orientation $\mathbf{o}$ with your head pointing in the direction of $\mathbf{n}$, then S will always be on your left.

Stokes' Theorem

For oriented surface $\mathbf{S} = (S, \mathbf{n})$ bounded by a simple curve C, and let $\mathbf{C} = (C, \mathbf{o})$ be the oriented loop with induced orientation, then for a vector field $\mathbf{F}$,

$$\iint_{\mathbf{S}} \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$$

Properties of Gradient, Divergence & Curl

For any function $f = f(x, y, z)$,

$$\nabla \times (\nabla f) = \text{curl}\, (\nabla f) = \mathbf{0}$$

For any vector fields $\mathbf{F} = \mathbf{F}(x, y, z)$,

$$\nabla \cdot (\nabla \times \mathbf{F}) = \text{div}\, (\text{curl}\, \mathbf{F}) = 0$$