# CS3241 Cheatsheet
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## Overview

### Graphics Basics

**Addictive Colour** - Form a colour by adding amounts of three primaries (RGB).
**Subtractive Colour** - Form Form a colour by filtering white with Cyan (C), Magenta (M), and Yellow (Y). Note: For subtractive colours, M absorbs G and allows R & B to pass through.

### OpenGL Basics

Polygons in OpenGL need to be **Simple** (edges cannot cross), **Convex**, and **Flat** (all vertices in the same plane), or they will not be displayed correctly.

Colour info is stored in each vertex of a polygon, and the shading mode determines how the polygon is coloured:
**Smooth ~** - Interpolation of vertex colours across polygon. The default setting. `glShadeModel(GL_SMOOTH)`.
**Flat ~** - Fill colour is colour of first vertex. `glShadeModel(GL_FLAT)`.

**Hidden Surface Removal** - It deals with 3D objects overlapping over one another from our perspective, by using an extra **z-buffer** that saves depth information.

## Interaction

### Overview

Input devices generate **Triggers**, namely signals, and return **Measures**, which are trigger with other meta-info from the input devices, to the OS. Each trigger generates an **Event** whose measure is put in an **Event Queue** to be examined by the user program.
The user program defines a **Callback** for each type of event to GLUT and it is executed when the event occurs.

### Positioning in Callbacks

Window systems (such as mouse & motion callbacks) measure positions with origin at **top-left corner**. OpenGL measures positions with origin at **bottom-left corner**. Conversion:

$$x_{\text{opengl}} = x_{\text{win}}, \quad y_{\text{opengl}} = h - 1 - y_{\text{win}}$$

### Animation

**Double Buffering** - The usage of two **colour buffers** for display where the **Front Buffer** displays and the **Back Buffer** is written to. It minimises flickering in animation.
Note: Two identical buffers, switched once writing to back is done.

## Geometry

### Representation

**Affine Space** - A frame formed by an origin point and basis vectors.

**Homogeneous Coordinates** ~A 4D system for 3D points & vectors.

- Vector $v = [\alpha_1, \alpha_2, \alpha_3, 0]^T = (\alpha_1, \alpha_2, \alpha_3)$
- Point $p = [\beta_1, \beta_2, \beta_3, w]^T = (\frac{\beta_1}{w}, \frac{\beta_2}{w}, \frac{\beta_3}{w})$

Note: [1] All Homogeneous Coordinate elements are denoted as **Column Vectors**. [2] Point + Vector = Point. [3] Representation for each point is non-unique. We typically use $w = 1$. **Perspective Division** is the process of dividing $x, y, z$ by $w$.

### Transformation

All transformations can be expressed as a $4 \times 4$ matrix, **pre-multiplied** to the Homogeneous Coordinates.
Note: $p' = ABp$ is to apply transformation $B$ then $A$ to element $p$.

*Translation*

$$T(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*Rotation*

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*Scaling*

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = diag(s_x, s_y, s_z, 1)$$

*Shear*

Shearing along $x$-axis maps $(x, y, z)$ to $(x + y \cot\theta, y, z)$:

$$H(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: There is no direct matrix for shearing in OpenGL.

*Inverse of Transformation Matrices*

- $T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z)$
- $R^{-1} = R^T$, as all $R$ orthogonal
- $S^{-1}(s_x, s_y, s_z) = S(1/s_x, 1/s_y, 1/s_z)$

*Combinations of Transformation*

**General Rotation about Origin**: $R(\theta) = R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$, where $\theta_x, \theta_y, \theta_z$ are the Euler angles.
**Rotation about Arbitrary Point**: $M = T(p_f)R(\theta)T(-p_f)$, where $p_f$ is the point and $\theta$ rotational angle.
Note: Matrix ordering is the reverse of the actual transformation's.

### OpenGL Transformation

For each matrix mode, **Current Transformation Matrix** (CTM) stores the $4 \times 4$ homogeneous coordinate matrix, as part of the state and it is applied to all vertices in the pipeline. Some matrix modes include Model-View (`GL_MODEVIEW`) and Projection (`GL_PROJECTION`). OpenGL also maintains a stack for each matrix mode.

## Camera & Viewing

OpenGL Spaces: **Object Space**, for modelling each object where the object is centred in the frame. **World Space**, a common frame for all objects, and for defining lightning and camera pose. **Camera Space**, the local frame for camera where it is at the origin and looking into the **Negative z-direction**, initially the same as the world frame.

An OpenGL **Viewport** defines a rectangular region of the window in which OpenGL can draw.
Note: [1] A window can have multiple viewports. [2] A viewport can be larger than the window, and whatever inside the viewport but outside the window will not be displayed.

### Transformations

**View Transformation** - Transform points from World Frame to Camera Frame. The transformation matrix

$$M_{\text{view}} = RT = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $T$ moves camera position back to world origin, and $R$ rotates axes of camera to coincide with world frame's.
View transformation for normal vectors:

$$M_n = (M_t^{-1})^T$$

, where $M_t$ is the upper left $3 \times 3$ sub-matrix of $M_{\text{view}}$.

**Projection Transformation** - Specify a **View / Clipping Volume** in the camera frame, and project the volume to a $2 \times 2 \times 2$ cube centred at origin, known as the **Normalised Device Coordinate** (NDC) / **Canonical View Volume**.
Note: It preserves depth order and lines.

**Viewport transformation** The canonical view volume is then mapped to the viewport (from NDC to window coordinates)

$$\frac{x_{NDC} - (-1)}{2} = \frac{x_{win} - x_{vp}}{w} \Rightarrow x_{win} = x_{vp} + \frac{w(x_{NDC}+1)}{2}$$
$$\frac{y_{NDC} - (-1)}{2} = \frac{y_{win} - y_{vp}}{h} \Rightarrow y_{win} = y_{vp} + \frac{h(y_{NDC}+1)}{2}$$
$$z_{win} = \frac{z_{NDC}+1}{2}$$

### Projections

Types of projection

- **Perspective projection** - Projectors converge at centre of projection. Objects further from viewer are projected smaller.
- **Parallel Projection** - Projectors are parallel. One special case **Orthographic Projection** is where projectors are orthogonal to projection surface.

Perspective Transformation for Homogeneous Coordinates Any point $q = [x, y, z, 1]^T$ projected perspectively onto vertical plane at $z = d < 0$ will be at $(xd/z, yd/z, d)$, equivalent to $p = [x, y, z, z/d]^T = Mq$ for

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix}$$

*OpenGL Projection Matrix Equivalences*

`glOrtho(L, R, B, T, N, F)`:

$$M_{\text{ortho}} = \begin{bmatrix} \frac{2}{R-L} & 0 & 0 & -\frac{R+L}{R-L} \\ 0 & \frac{2}{T-B} & 0 & -\frac{T+B}{T-B} \\ 0 & 0 & \frac{-2}{F-N} & -\frac{F+N}{F-N} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= S\left(\frac{2}{R-L}, \frac{2}{T-B}, \frac{2}{N-F}\right)T\left(\frac{-(R+L)}{2}, \frac{-(T+B)}{2}, \frac{F+N}{2}\right)$$

`glFrustum(L, R, B, T, N, F)`:

$$M_{\text{persp}} = \begin{bmatrix} \frac{2N}{R-L} & 0 & \frac{R+L}{R-L} & 0 \\ 0 & \frac{2N}{T-B} & \frac{T+B}{T-B} & 0 \\ 0 & 0 & -\frac{F+N}{F-N} & -\frac{2FN}{F-N} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Note: The default projection matrix in OpenGL is an identity and orthographic, equivalent to `glOrtho(-1, 1, -1, 1, -1, 1)`.

# Rendering

## Overview

*Pipeline Stages*

**Vertex Process** - The stage where Model-View and Projection transformations, and lighting and textures are applied. The result is in Clip Space.
**Primitive Assembly** - The stage where primitive types are recalled and their processed vertices are collected for polygons, so that we can redefine the primitive for clipping and rasterisation.
**Perspective Division** - The stage where points with $w \neq 1$ are converted into $w = 1$, from Clip Space into NDC Space.
**Viewport Transformation** - Transformation into Window Space, with depth range linearly scaled from $[-1, 1]$ in NDC to $[0, 1]$ in Window Space.
**Back-Face Culling** - Eliminate polygon if it is **back-facing** and invisible, i.e. when $N_p \cdot N < 0$. Note: Need to specify polygon vertices in Anti-Clockwise order to ensure correct front-face (RHGR).

*Visibility & Occlusion Tests*

**Visibility Tests**: Back-Face Culling, Screen Clipping.
**Occlusion Tests**: Painter's Algorithm, z-Buffer Algorithm.
Note: Perform Visibility before Occlusion Tests.

## Clipping

*Brute Force Approach*

Compute intersections with all sides of the clipping window.
Analysis: Inefficient, require one division per intersection.

*2D Cohen-Sutherland Algorithm*

Define **Outcode** $b_0b_1b_2b_3$ where bit is $1$ iff it is outside $y_{\max}$, $y_{\min}$, $x_{\max}$, $x_{\min}$ respectively (top, btn, right, left) for each endpoint.
For a line with 2 endpoints $A$ and $B$, four cases to consider:

- `Out(A) == Out(B) == 0` - Both points inside all 4 boundaries. Accept the entire line.
- `Out(A) == 0 && Out(B) != 0` - One point inside all and one outside some. Num of intersections == num of 1 bit in B.
- `Out(A) & Out(B) != 0` - Both points outside the same boundary. Discard the entire line.
- `Out(A) & Out(B) == 0 && Out(A) | Out(B) != 0` - May or may not be intersection. Most complicated. Need to compute intersections for all 4 sides.

Analysis: [1] Efficient, as it filters out simple cases. [2] Yet, still need computation for complicated cases. [3] Can extend to 3D using 6-digit outcodes.

*Polygon Clipping*

Clipping a polygon may yield multiple polygons, but clipping a **convex polygon** can yield at most one other.
**Tessellation** - replace concave polygons with a set of smaller simple polygons, typically triangles. Tessellation enables simpler clipping.

*AABB Clipping Algorithm*

**Axis-Aligned Bounding Box** (AABB) - 1) Draw a bounding box for the polygon. 2) Consider whether and whereh we need to perform detailed clipping.
Analysis: Similar idea of "easy filtering first".

## Rasterisation

**Rasterisation** - The process of determining which pixels are inside primitive specified by a set of vertices. It produces a set of fragments.
**Fragments** ~"Potential pixels", with a pixel location and attributes such as colour and depth.

*DDA Algorithm*

Simple iterative method that moves along one axis one pixel at a time. To draw a line $y = mx + b$ from $(x_0, y_0)$ to $(x_e, y_e)$:

```
for (x=x0, y=y0; x <= xe; x++) {
    write_pixel(x, round(y), color);
    y += m;
}
```

Analysis: [1] Simple, and applicable to curves as well. [2] Inefficient due to many floating point calculations. [3] Lines rasterised are discontinuous when $|m| > 1$. Solution is to flip roles of $x$ and $y$.

*Bresenham's Algorithm*

Use a **Decision Variable**, $p_k$ to determine which $y$-value to rasterise for every $x$.

1. Input line and store left end in $(x_0, y_0)$. Assume $|m| <= 1$.
2. Calculate constants $\triangle x, \triangle y, 2\triangle y, 2\triangle y - 2\triangle x$.
3. Fill $(x_0, y_0)$, and obtain $p_0 = 2\triangle y - \triangle x$.
4. At each $x_k$ along the line, starting at $k = 0$, perform the following test: [1] If $p_k < 0$, then next point is $(x_k + 1, y_k)$, and $p_{k+1} = p_k + 2\triangle y$. [2] Otherwise, next point $(x_k + 1, y_k + 1)$ and $p_{k+1} = p_k + 2\triangle y - 2\triangle x$.
5. Repeat Step 4 for $\triangle x - 1$ more times.

Analysis: [1] Only applicable to line (and circle) rasterisation. [2] Efficient, due to no floating points.

*Polygon Rasterisation Algorithms*

**Scan-Line Fill Algorithm** - [1] First tessellate if concave. [2] Compute colours & depths for vertices. [3] For each horizontal line, interpolate colours and z-values.
**Floor Fill Algorithm** - Start with one point inside the polygon, expand to 4 directions using BFS recursively.

## Hidden Surface Removal

**Painter's Algorithm** - Render polygons in back-to-front order so that polygons behind others are simply painted over.
Analysis: It involves **Depth Sorting**, with time complexity $O(n \log n)$ for $n$ polygons.

**Image-Space Approaches** - Look at each pixel / projector and find closest of all polygons along that pixel.
Example: Ray Tracing, z-Buffering (using a z-buffer / depth buffer).
Analysis: Complexity $O(nmk)$ for $k$ polygons and frame buffer sized $n \times m$.

# Others

## From Tutorials & Assignments

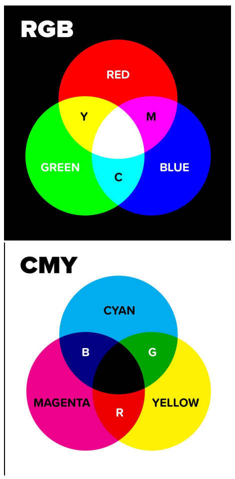
*(Tut1 Qn2)* Human eyes' sensitivity to change in colour: G > R > B.
*(Tut5 Qn4)* **z-Fighting** ~when different fragments at the same pixel have the same or very similar z-buffer values. Solution: [1] Minimise distance `far-near`. [2] Use z-buffers with higher precision.

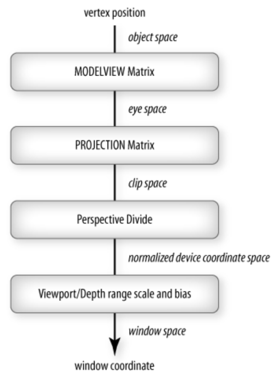
### *From Past Year Papers*

*(AY19/20 Sem1 Midterm Qn7&8)* Vertices for Primitive Assembly are in Clip Space. Vertices for Rasterisation are in Window Space.
*(AY20/21 Sem1 Midterm Qn22)* z-fighting is more serious for polygons further away in a **perspective** projection; z-fighting is unaffected by how far a polygon is in a **orthographic** projection.
*(AY20/21 Sem1 Midterm Qn24)* Back-Face Culling can be performed in Window Sapce or Camera Space, but cannot be performed in the Vertex Processing stage.

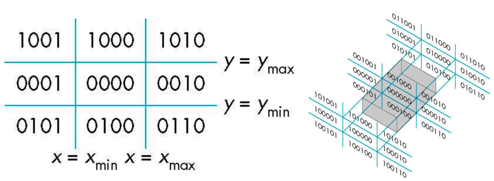
### *Diagrams*

Colour Theory, Transformation Pipeline

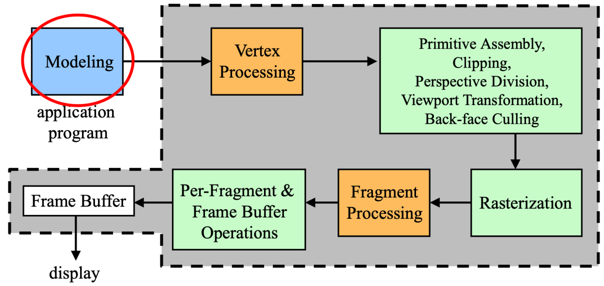



Outcode

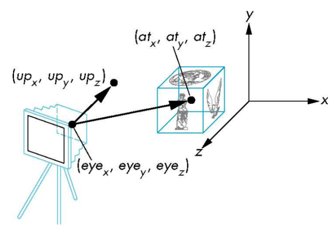



Rendering Pipeline




gluLookAt Coordinate Specification
($u, v, n$ for $x, y, z$ of the Camera)

- **n** = normalize( **eye** – **at** )
- **u** = normalize( **up** ) × **n**
- **v** = **n** × **u**

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# OpenGL Reference

## *Miscellaneous*

*Settings*

`glutInitDisplayMode(...|GLUT_DOUBLE)` uses Double Buffering, as compared to `GLUT_SINGLE` for Single Buffering.

`glViewport(u, v, w, h)` - Define the rectangular viewport.

## *Callback-Related*

`glutDisplayFunc(void (*func)(void))` - when GLUT determines the window to be refreshed (e.g. opened, reshaped, exposed etc.)
`glutIdleFunc` - when there is no trigger (e.g. for animation).
`glutMotionFunc` - when mouse on hold and moving.
`glutPassiveMotionFunc` - when mouse moving but not on hold.
`glutTimerFunc(unsigned int msecs, void (*func)(int value), value)` - Set a timer and trigger the callback when timer elapses.

Other common trigger callbacks: `glutMouseFunc`, `glutReshapeFunc`, `glutKeyboardFunc`.

`glutPostRedisplay()` - Set a flag for posting redisplay, which avoids frequent repeated execution.

## *OpenGL Matrices & Transformation*

`glMatrixMode(GLenum mode)` - Specify the current matrix mode. Modes are `GL_MODELVIEW`, `GL_PROJECTION`, `GL_TEXTURE` or `GL_COLOR`.

*Transformations*

`glLoadIdentity()` - Override CTM with $I_4$.
`glTranslatef(dx, dy, dz)` - Post-mul. translation matrix to CTM.
`glRotatef(theta, vx, vy, yz)` - Post-mul. rotation matrix to CTM.
`glScalef(sx, sy, sz)` - Post-mul. scaling matrix to CTM.
`glLoadMatrixf(m)` - Override CTM with matrix $m_{4\times4}$.
`glMultMatrixf(m)` - Post-mul. $m_{4\times4}$ to CTM.
Note: [1] Each has a Double(`gl...d`) and a Float (`gl...f`) format. [2] $m$ is a 1D array of length 16, ordered by column.

*Matrix Stacks*

`glPushMatrix()` - Push CTM to the stack, and duplicate the CTM.
`glPopMatrix()` - Pop top matrix from the stack and replace CTM.
Note: Typically an indentation is applied for every Push-Pop pair.

## *Camera & Viewing*

`gluLookAt(eyex, eyey,eyez, atx, aty, atz, upx, upy, upz)` - Create a view transformation matrix for the camera specified. `eye*` specify the camera location, `at*` specify where to look at, and `up*` is **Up-Vector** for orientation of camera.
Note: [1] The up vector does not need to be perpendicular to Eye-At. [2] The up is usually $(0, 1, 0)$.

`glOrtho(left, right, bottom, top, near, far)` - Post-mul. to CTM the orthographic projection matrix.
Note: It will map $(-\text{near}, -\text{far})$ to $(-1, 1)$.
`glFrustum(left, right, bottom, top, near, far)` - Post-mul. to CTM the perspective projection matrix.
Note: [1] LRBT all refer to the locations on the near plane. [2] It will map $(-\text{near}, -\text{far})$ to $(-1, 1)$.
`gluPerspective(fovy, aspect, near, far)` - Post-mul. to CTM the "central & symmetric" perspective projection matrix.
Note: `fovy` vertical field of view angle in degrees. `aspect` $w/h$ ratio.
`glViewport(x, y, width, height` - Specify transformation matrix for xy from NDC to window coordinates.