MA2104 Cheatsheet
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Chapter 01 - Vectors in 3D Space

Vectors

Vector projection of a onto b: $\text{proj}_b a = \frac{a \cdot b}{b \cdot b} b$
Scalar projection of a onto b: $\text{comp}_b a = \frac{a \cdot b}{\|b\|}$

Dot & Cross Product

$$a \cdot b = \|a\|\|b\| \cos\theta, \quad \|a \times b\| = \|a\|\|b\| \sin\theta$$

where θ is the angle between vectors a and b.

Prop Ch01.3.5 - Scalar Triple Product

$$|a \cdot (b \times c)| = \left| \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|$$

is the volume of the parallelepiped determined by vectors a, b, c.

Chapter 02 - Curves and Surfaces

Curve

Tangent Vector

Tangent vector to a curve C paramaterised by $R(t) = \langle f(t), g(t), h(t) \rangle$ at $R(a)$ on the curve is given by

$$R'(a) = \langle f'(a), g'(a), h'(a) \rangle.$$

Arc Length Formula

The length of curve $C : R(t) = \langle f(t), g(t), h(t) \rangle$ between $R(a)$ and $R(b)$ is

$$\int_a^b \|R'(t)\| \, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} \, dt.$$

provided the first derivatives are continuous.

Surfaces

Cylinder

A surface is a cylinder if there is a plane P such that all the planes parallel to P intersect the surface in the same curve.

Quadric Surfaces

- Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
- Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Elliptic Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
- Hyperboloid of 1 Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of 2 Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

Chapter 03 - Multivariable Functions

Limit, Continuity & Differentiability

Limit for 2D Functions

For function f with domain $D \subset \mathbb{R}^2$ that contains points arbitrarily close to (a, b), then

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.
The limit exists iff. the limit **exists and is the same for all continuous paths** to (a, b).

Clairaut's Theorem

For function f defined on $D \subset \mathbb{R}^2$ that contains (a, b), if the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Differentiability for 2D Functions

For function f defined on $D \subset \mathbb{R}^2$ and differentiable at (a, b) within the interior of D,

$$\lim_{(h,k) -> (0,0)} \frac{f(a+h, b+k) - f(a, b) - L(h, k)}{\sqrt{h^2 + k^2}} = 0,$$

where $L : \mathbb{R}^2 \to \mathbb{R}$ is a linear map defined as the total derivative of f at (a, b):

$$L(h, k) = D_{f(a,b)}(h, k) = f_x(a, b)h + f_y(a, b)k.$$

Notes about Differentiability

Consider f at a point (a, b):

- f_x and f_y exist $\not\Longrightarrow$ f differentiable
- f_x and f_y exist & continuous $\Longrightarrow$ f differentiable (*Differentiability Theorem*)
- f differentiable $\not\Longrightarrow$ f_x and f_y continuous

Linear Approximation

$$f(a+h, b+k) \approx f(a, b) + f_x(a, b)h + f_y(a, b)k$$

Gradient Vector

Gradient Vector

The gradient vector of f defined on $D \subset \mathbb{R}^2$ at $(a, b) \in D$ is defined as:

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

Directional Directive

The directional directive of f defined on $D \subset \mathbb{R}^2$ in the direction of the unit vector $u = \langle u_1, u_2 \rangle$ is

$$D_{f(a,b)}(u) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = \nabla f(a, b) \cdot u.$$

Perpendicular Vector of Level Sets

$\nabla f(a, b)$ is orthogonal to the $f(a, b)$-level curve of f at (a, b).

Chapter 04 - Calculus on Surfaces

Implicit Differentiation

Prop Ch04.1.4

For F defined on $D \subset \mathbb{R}^3$ where $F(a, b, c) = k$ defines z as a differentiable function of x and y near (a, b, c), and $F_z(a, b, c) \neq 0$,

$$\frac{\partial z}{\partial x}(a, b, c) = -\frac{F_x(a, b, c)}{F_z(a, b, c)}, \quad \frac{\partial z}{\partial y}(a, b, c) = -\frac{F_y(a, b, c)}{F_z(a, b, c)}$$

Extrema

Extreme Value Theorem

If $f : D \to \mathbb{R}$ is continuous on a **closed and bounded** set $D \subset \mathbb{R}^2$, then f has at least one global maximum and one global minimum.

Steps for Finding Gloabl Extrema

For $f : D \to \mathbb{R}$ where D is closed and bounded,

1. Find all critical points of f and their corresponding f-values.
2. Find the extreme values of f on boundary of D.
3. Compare.

Method of Lagrange Multiplier

To find the extrema of differentiable $f : D \to \mathbb{R}$ subject to curve $C : g(x, y) = k$ for some $k \in \mathbb{R}$,

1. Find all points (a, b) and **non-zero** value λ s.t.

   $$\nabla f(a, b) = \lambda \nabla g(a, b), \; g(a, b) = k,$$

   and evaluate f at all these points.
2. Find the extreme values of f on the boundary of C.
3. Compare.