ST4238 Cheatsheet
by Yiyang, AY22/23

1. Poisson Processes

Overview

Definition 1

A **Poisson Process** with rate $\lambda > 0$ is an integer-valued stochastic process $\{X(t), t \geq 0\}$ for which

- for any time points $t_0 = 0 < t_1 < t_2 < ... < t_n$, increments
$$X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$$
are independent r.v.
- for $s \geq 0$ and $t > 0$, the r.v. $X(s+t) - X(s) \sim Pois(\lambda t)$
- $X(0) = 0$

Note: The process above is **homogeneous**. If $\lambda = \lambda(t)$ varies with time, then the process is **non-homogeneous**, and replace the second property above with

$$X(s+t) - X(s) \sim Pois\Big(\int_s^{s+t} \lambda(u)du\Big)$$

Note: A **Cox Process** is where $\lambda(t)$ is a stochastic process itself.

To **homogeniz a non-homogeneous Poisson Process**:

1. Define $\Lambda(t) = \int_0^t \lambda(u)du$
2. Define a new process $Y(s) = X(t)$ where $s = \Lambda(t)$
3. Now $Y(s)$ is a homogeneous Poisson Process with rate 1.

Definition 2 - LRE

Let $N((s,t])$ be a r.v. counting number of occurrences in the interval $(s,t]$. Then $N((s,t])$ is a **Poisson Point Process** of intensity $\lambda > 0$ if

- for any time points $t_0 = 0 < t_1 < ... < t_n$, increments
$$N((t_0,t_1]), N((t_1,t_2]), ..., N((t_{n-1},t_n])$$
are independent r.v.
- $\exists \lambda > 0$, s.t. as $h \to 0$, $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$.
- as $h \to 0$, $P(N((t,t+h]) \geq 2) = \lambda + o(h)$.

Note: For non-homogeneous, $P(N((t,t+h]) \geq 1) = \lambda(t)h + o(h)$.

Law of Rare Events

Let $\epsilon_1, \epsilon_2, ...$ be independent Ber. r.v.'s with $P(\epsilon_i = 1) = p_i$ and let $S_n = \sum i = 1^n \epsilon_i$. The exact probability for S_n and Poisson probability with $\lambda = \sum_{i=1}^n p_i$ differ by at most

$$|P(S_n = k) - e^{-\lambda}\frac{\lambda^k}{k!}| \leq \sum i = 1^n p_i^2$$

Note: In the case of $p_1 = ... = \lambda/n$, RHS becomes λ^2/n.

Definition 3 - Sojourn Time

Consider a sequence $\{S_n, n \geq 0\}$ of i.i.d. $Exp(\lambda)$. Define a counting process by specifying the occurrence time of n-th event

$$W_n = S_0 + S_1 + ... + S_{n-1}$$

The new counting process will be a Poisson Process with rate λ.

Waiting & Sojourn Time

The **Waiting Time**, W_n of a Poisson Process $X(t)$ is the time of n-th occurrence, for $n \in \mathbb{N}$. We set $W_0 = 0$.

The **Sojourn Time**, $S_n = W_{n+1} - W_n$ is the time where the process sojourns in state n, for $n \in \mathbb{Z}_{\geq 0}$.

For homogeneous Poisson Processes,

$$S_n \sim Exp(\lambda)$$
$$W_n \sim Gamma(n, \lambda)$$

Properties

Arrival Time

Given $X(t) = n$, the joint distribution of waiting time $W_1, ..., W_n$ is

$$f(w_1, w_2, ..., w_n | X(t) = n) = \frac{n!}{t^n}, 0 \leq w_1 \leq w_2 \leq ... \leq 2_n \leq t$$

It is the joint distribution of n **ranked** independent $Unif(0,t)$ r.v.'s.

Given $X(t) = n$, the distribution of the k-th waiting time has the same distribution as that of the k-th order statistic of n independent $Unif(0,t)$ r.v.'s.

$$f_k(x) = \frac{n!}{(n-k)!(k-1)!}\frac{1}{t}(\frac{x}{t})^{k-1}(1-\frac{x}{t})^{n-k}, 0 \leq k \leq n$$

Merging & Splitting Processes

For $\{N_1(t), t \geq 0\}, \{N_2(t), t \geq 0\}, ..., \{N_m(t), t \geq 0\}$ independent Poisson Processes with rates $\lambda_1, \lambda_2, ..., \lambda_m$, let $N(t) = \sum_{i=1}^m N_i(t), t \geq 0$ be the **Merging Process**. Then $\{N(t), t \geq 0\}$ is also a Poisson Process with rate $\lambda = \lambda_1 + \lambda_2 + ... + \lambda_m$.

For $\{N(t), t \geq 0\}$ a Poisson Process with rate λ, if each event occurred can be of type A and B with probability p and $1-p$ independently, then let $X(t)$ and $Y(t)$ be the **Splitting Processes** counting number of type A and B occurrences. Then $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are Poisson Processes with rates λp and $\lambda(1-p)$, and they are independent.

Comparision of Two Processes

Consider $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ two independent Poisson Processes with rates λ_1 and λ_2. Define W_n^X and W_m^Y as the waiting time of the n-th and m-th waiting time of $X(t)$ and $Y(t)$ respectively.

$$P(W_n^X < W_m^Y) = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} (\frac{\lambda_1}{\lambda_1+\lambda_2})^k (\frac{\lambda_2}{\lambda_1+\lambda_2})^{n+m-1-k}$$

Analysis: It is equivalent as getting n or more heads in $n+m-1$ tosses where getting a head has probability $\lambda_1/(\lambda_1+\lambda_2)$.

Variants

Compound Poisson Process

A stochastic process $\{X(t), t \geq 0\}$ is a **Compound Poisson Process** if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \ t \geq 0$$

, where $\{N(t), t \geq 0\}$ is a Poisson Process with rate λ and $Y_i \sim F$ is a family of i.i.d. r.v.'s independent of $\{N(t), t \geq 0\}$.
Note: We need [1] rate λ, and [2] distribution F to specify a compound Poisson Process.

$E[X(t)] = \lambda t E[Y_i]$
$Var[X(t)] = \lambda t\big(E[Y_i]^2 + Var(Y_i)\big)$

To merge two compound Poisson Processes $X(t)$ and $Y(t)$ with parameters (λ_1, F_1) and (λ_2, F_2) as $N(t) = X(t) + Y(t)$, the result is a compound Poisson Process with parameters

$$\lambda = \lambda_1 + \lambda_2$$
$$F(x) = \frac{\lambda_1}{\lambda_1+\lambda_2}F_1(x) + \frac{\lambda_2}{\lambda_1+\lambda_2}F_2(x)$$

Conditional Poisson Process

A stochastic process $\{N(t), t \geq 0\}$ is a **Conditional Poisson Process** if there is a positive r.v. L such that $\{N(t)|L = \lambda, t \geq 0\}$ is a Poisson Process with rate λ.
If L has pdf $g(\cdot)$, then pdf for increment of $N(t)$ is

$$P(N(t+s) - N(s) = n) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!} g(\lambda) d\lambda$$

$E[N(t)] = E(L)t$
$Var(N(t)) = tE(L) + t^2 Var(L)$
Conditional probability of L given $N(t) = n$ (posterior),

$$P(L \leq x | N(t) = n) = \frac{\int_0^x e^{-\lambda t}(\lambda t)^n g(\lambda) d\lambda}{\int_0^\infty e^{-\lambda t}(\lambda t)^n g(\lambda) d\lambda}$$

Multi-Dimensional Poisson Process

Let S be a subset of $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$. Let $\mathcal{A}$ be the set of subsets of S and for any set $A \in \mathcal{A}$, define $|A|$ as the size of A. Then $\{N(A) : A \in \mathcal{A}\}$ is a homogeneous Poisson process with $\lambda > 0$ if,

- for each $A \in \mathcal{A}$, $N(A) \sim Pois(\lambda|A|)$
- for every finite collection $\{A_1, ..., A_n\}$ of disjoint subsets of S, r.v.'s $N(A_1), ..., N(A_n)$ are independent.

2. Continusous Time Markov Chains

Overview

Definition

For a stochastic process $\{X(t), t \geq 0\}$, if for all $s > u \geq 0, t > 0$,

$$P(X(s+t) = j|X(s) = i, X(u) = k) = P(X(s+t) = j|X(s) = i)$$

, then we call $\{X(t), t \geq 0\}$ a **Continuous-Time Markov Chain**, and the property **Markovian Property**.

A CTMC $\{X(t), t \geq 0\}$ has **Stationary / Homogeneous Transition Probabilities** if for all $s \geq 0, t > 0$ and states i, j,

$$P(X(s+t) = j|X(s) = i) \text{ is independent of } s$$

Parameterisation 1

A CTMC $\{X(t), t \geq 0\}$ can be specified with

- **State Space** S
- **Waiting Time Rate Vector** $\vec{\nu}$, where the time $X(t)$ stays in state $i \in S$ follows $Exp(\nu_i)$
- **Jump Probabilities** P_{ij}, the probability of $X(t)$ currently in state $i \in S$ and moves to $j \in S$ at first transition.

Note: By definition, $P_{ii} = 0$ and $\sum_{j \neq i} P_{ij} = 1$ for all $i \in S$.
Note: For an absorbing state i (e.g. $i = 0$ in Birth & Death Process), we may set $\nu_i = 0$.

Transition Probability Function

Define the **Transition Probabilities** of a CTMC $X(t)$ as

$$P_{ij}(t) := P(X(t+s) = j|X(s) = i)$$

Note: A transition probability matrix can uniquely specify a CTMC.
Note: $P_{ij} \neq P_{ij}(t)$ since there might not be exactly one transition.

Chapman-Kolmogorov Equation

$$P_{ij}(t+s) = \sum_{k \in S} P_{ik}(t) P_{kj}(s)$$

Poisson Process as CTMC

A Poisson Process $\{X(t), t \geq 0\}$ with rate λ can be modelled as a CTMC with state space $S = \{0, 1, 2, ...\}$, rates $\nu_i = \lambda, \forall i \in S$, and jump matrix P where $P_{i,i+1} = 1$ and $P_{ij} = 0, \forall j \neq i+1$.
Note: CTMCs are not necessarily Poisson Processes.

Discretisation of CTMC

For a CTMC $\{X(t), t \geq 0\}$, $\{Y_1(n)\}_{n \geq 0}$ discretises it at equal intervals if for some constant $l > 0$,

$$Y_1(n) = X(nl), n = 0, 1, 2, ...$$

Analysis: $Y_1(n)$ has state space S and transition matrix $P(l)$.

For a CTMC $\{X(t), t \geq 0\}$, $\{Y_2(n)\}_{n \geq 0}$ is the **Embedded Chain** if it only considers the states visited by $X(t)$.
Analysis: $Y_2(n)$ has state space S and transition matrix P.

Infinitesimal Generator

Instantaneous Transition Rates

Lemma: Transition Rates, for a CTMC,

- $\lim_{h \to 0} \frac{P_{ii}(h) - P_{ii}(0)}{h} = -\nu_i$
- $\lim_{h \to 0} \frac{P_{ij}(h) - P_{ij}(0}{)} h = \nu_i P_{ij}$, for all $i \neq j$

For any pair of states $i \neq j \in S$, define **Instantaneous Transition Rates** as

$$q_{ij} := \nu_i P_{ij}$$

The **Infinitesimal Generator** G of a CTMC is defined as

$$G_{ii} = -\nu_i, \; G_{ij} = q_{ij}, \; i \neq j$$

Note: $P'(0) = G$.

Kolmogorov's Forward Equations

For all states i, j, and times $t \geq 0$,

$$P'_{ij}(t) = \sum_{k \neq i} P_{ik}(t) q_{kj} - \nu_j P_{ij}(t)$$
$$\equiv P'(t) = P(t)G$$

Kolmogorov's Backward Equations

For all states i, j, and times $t \geq 0$,

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$
$$\equiv P'(t) = GP(t)$$

Note: G uniquely decides $P(t)$.

Appendix: Probability Theory

Gamma

$X \sim Gamma(\alpha, \lambda)$ for shape α, and rate $\lambda > 0$. ($1/\lambda$ is scale parameter)

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0$$

Statistics: $E(X) = \frac{\alpha}{\lambda}$, $\text{var}(X) = \frac{\alpha}{\lambda^2}$
MGF: $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}, \quad t < \beta$
Special case: $Exp(\lambda) = Gamma(1, \lambda)$, $\chi^2_n = Gamma(\frac{n}{2}, \frac{1}{2})$
Properties: $Gamma(a, \lambda) + Gamma(b, \lambda) = Gamma(a+b, \lambda)$, and $cX \sim Gamma(\alpha, \frac{\lambda}{c})$
Gamma function $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$
$\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)!$, $n \in \mathbb{Z}^+$

Beta

$X \sim B(a, b)$ where $a > 0, b > 0$ has support $[0, 1]$

$$f(X) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, 0 \leq x \leq 1$$

Statistics: $E(X) = \frac{1}{1+\beta/\alpha}$, $\text{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ Special case: $\text{Unif}(0,1) = B(1,1)$

Beta function $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$