ST2132 Cheatsheet
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Chapter 02 - Random Variables

More Disributions

Beta Distribution Beta Function

$X \sim B(a, b)$ where $a > 0, b > 0$ has support $[0, 1]$ and

$$f(X) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, 0 \le x \le 1$$

Note that $\text{Unif}(0,1) \sim B(1,1)$ is a special case.
Beta function $B(u, v)$ is an integral that

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Chi-Square Distribution

If $X = Z^2$ where $Z \sim \mathcal{N}(0,1)$, then

$$f_X(x) = \frac{x^{-1/2}}{\sqrt{2\pi}} e^{-x/2},\ x \ge 0$$

In fact, X follows χ^2 distribution with df 1, and $X \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$.

Functions of a Random Variable

Properties of CDF

(*Ch 2.3 Prop. C*) Let $Z = F(X)$, then $Z \sim \text{Unif}(0,1)$
(*Ch 2.3 Prop D*) Let $U \sim \text{Unif}(0,1)$, and let $X = F^{-1}(U)$, then the CDF of X is F.

Inverse CDF Method

For a r.v. X with CDF F to be generated, let $U = F(X)$ and write it as $X = F^{-1}(U)$, then generate with following steps:

1. Generate u from a $\text{Unif}(0,1)$.
2. Deliver $x = F^{-1}(u)$.

Chapter 03 - Joint Distributions

Joint Distributions

Copula

A **copula**, $C(u, v)$, is a joint CDF where the marginal distributions are standard uniform. It has properties as shown below:

- $C(u, v)$ is defined over $[0, 1] \times [0, 1]$ and is non-decreasing
- $P(U \le u) = C(u, 1)$ and $P(V \le v) = C(1, v)$
- joint density function $c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v,) \ge 0$

Construct joint distributions from marginal distributions given using copula: For any two CRVs X and Y and a copula $C(u, v)$ given,

$$F_{XY}(x, y) = C(F_X(x), F_Y(y))$$

is a joint distribution that has marginal distributions $F_X(x)$ and $F_Y(y)$. Correspondingly, the joint density is

$$f_{XY}(x, y) = c(F_X(x), F_Y(y))\, f_X(x) f_Y(y)$$

Farlie Morgenstern Family

For any two CRVs X and Y with their CDFs $F(x)$ and $G(y)$ given, it is shown that for any constant $|\alpha| \le 1$,

$$H(x, y) = F(x)\, G(y) \left[1 + \alpha(1 - F(x))(1 - G(y))\right]$$

is a bivariate joint CDF of X and Y, with its marginal CDFs equal to $F(x)$ and $G(y)$.
Farlie Morgenstern copula: $C(u, v) = uv(1 + \alpha(1-u)(1-v))$ is the copula used in the Farlie Morgenstern Family.

Bivariate Normal Distribution

If X and Y are jointly distributed with bivariate normal,

$$f(X, Y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]}$$

where $-1 < \rho < 1$ is the correlation coefficient and the other 4 parameters are reflected in marginal distributions,

$$X \sim \mathcal{N}(\mu_x, \sigma_x^2),\ Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$$

For a joint distribution to be considered bivariate normal, it must satisfy both:

1. Its two marginal distributions are normal
2. The contours for its joint density function are elliptical

Conditional Joint Distributions

Rejection Method

For a r.v. X with density function $f(x)$ to be generated, if $f(x) > 0$ for $a \le x \le b$, then

1. Let $M = M(x)$ s.t. $M(x) \ge f(x)$ for $a \le x \le b$
2. Let $m = m(x) = \frac{M(x)}{\int_a^b M(t)dt}$ (i.e. m is a pdf of support $[a, b]$)
3. Generate T with density m.
4. Generate U which follows $\text{Unif}(0,1)$ independent of T.
5. If $M(T) \times U \le f(T)$ then deliver T; otherwise, go base to Step 1 and repeat.

Bayesian Inference

Unknown parameter θ is treated as a distribution, not a value. **Prior distribution** $f_\Theta(\theta)$ represents our knowledge (assumption) about θ before observing data X. After observation, we have a better estimation using the **posterior distribution** $f_{\Theta|X}(\theta|x)$, where

$$f_{\Theta|X} = \frac{f_{X\Theta}(x, \theta)}{f_X(x)} = \frac{f_{X|\Theta}(x|\theta) f_\Theta(\theta)}{\int f_{X|\Theta}(x|t) f_\Theta(t)dt}$$

In short, it means

$$\text{Posterior density} \propto \text{likelihood} \times \text{prior density}$$

Functions of Joint Distributions

(*Ch 3.6.2 Prop. A*) Suppose X and Y are jointly distributed and $u = g_1(x, y),\ v = g_2(x, y)$ can be inverted as $x = h_1(u, v),\ y = h_2(u, v)$ then

$$f_{UV}(u, v) = f_{XY}(h_1(u, v), h_2(u, v))\ |J^{-1}(h_1, h_2)|$$

Sum/Quotient of Random Variables

Suppose X and Y have JDF f. Then for $U = X + Y$,

$$f_U(u) = \int_{-\infty}^{\infty} f(x, u - x)\,dx,$$

and for $V = X/Y$,

$$f_V(v) = \int_{-\infty}^{\infty} |x| f(x, xv)\,dx.$$

Order Statistics

(*Ch 3.7 Thm. A*) Density function of $X_{(k)}$, the k-th order statistics,

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x)[1 - F(x)]^{n-k}$$

Chapter 04 - Expected Values

Model for Measurement Error

Let x_0 denotes the true value of a quantity being measured. Then the measurement, X, can be modeled as:

$$X = x_0 + \beta + \epsilon$$

where β is **bias**, a constant and ϵ is the random component of error. $E(\epsilon) = 0$ and $\text{var}(\epsilon) = \sigma^2$.
Mean Squared Error (**MSE**) is a measure of the overall measurement error,

$$\text{MSE} = E[(X - x_0)^2]\ \text{(Definition)} = \sigma^2 + \beta^2$$

Conditional Expectation & Prediction

Find Expectation & Variance by Conditioning

$$E(Y) = E[E(Y|X)],\ \text{var}(Y) = \text{var}[E(Y|X)] + E[\text{var}(Y|X)]$$

Random Sum

$$E(T) = E(N)E(X),\ \text{var}(T) = [E(X)]^2\text{var} + E(N)\text{var}(X)$$

Predictions

Suppose X and Y are jointly distributed. If X is observed, the predictor of Y that minimises MSE would be

$$h(Y) = E(Y|X)$$