MA2108 Cheatsheet by Yiyang, AY23/34

1. Pre-requisites

Well-Ordering Principle of $\mathbb{N}$ Every non-empty subset $S \in \mathbb{N}$ has a least (smallest) element.

2. The Real Numbers

Algebraic Properties, ~

Different types of means

- **Arithmetic Means** $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k$
- **Geometric Means** $G_n = \left(\prod_{k=1}^{n} a_k\right)^{1/n}$
- **Harmonic Means** $H_n = n\left(\sum_{k=1}^{n} a_k^{-1}\right)^{-1}$

, for $n \in \mathbb{N}_{\geq 2}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ are positive. For the means, we have the **AM-GM-HM Inequality** :

$$H_n \leq G_n \leq A_n$$

, taking "=" iff. $a_1 = \ldots = a_n$.

Bernoulli's Inequality For $x > -1$, we have $(1+x)^n \geq 1 + nx$, for any $n \in \mathbb{N}$.

Triangle Inequity $|a+b| \leq |a| + |b|$, for all $a, b \in \mathbb{R}$.
Derived: [1] $||a| - |b|| \leq |a - b|$, [2] $|a - b| \leq |a| + |b|$.

Neighbourhood

For any $a \in \mathbb{R}$ and $\epsilon > 0$, the **ϵ-neighbourhood of a** is the set:

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

Theorem 2.2.8 For $a \in \mathbb{R}$, if $x \in V_\epsilon(a)$ for every $\epsilon > 0$, then $x = a$.

Completeness Properties, ~

For a non-empty $S \subseteq \mathbb{R}$, it is **Bounded Above** (**Bounded Below**) if S has an upper bound (a lower bound). S is **Bounded** if it is bounded above and below, and is **Unbounded**, otherwise.

For a non-empty $S \subseteq \mathbb{R}$, u is the **Supremum** of S if the following conditions are met, and we denote it as $\sup S$:

1. u is an upper bound of S.
2. $\forall v \in \mathbb{R}$, if v is an upper bound of S, then $v \geq u$.

For a non-empty $S \subseteq \mathbb{R}$, w is the **Infinum** of S if the following conditions are met, and we denote it as $\inf S$:

1. w is a lower bound of S.
2. $\forall v \in \mathbb{R}$, if v is a lower bound of S, then $v \leq w$.

Note: Sup. and Inf. are **uniquely determined**, if they exist.
Alternative Definition (Similarly for Infinum):
Lemma 2.3.4 For u an upper bound of $S \subseteq \mathbb{R}$, $u = \sup S$ iff.

$$\forall \epsilon > 0, \exists s_\epsilon \in S,\ u - \epsilon < s_\epsilon$$

For a non-empty $S \subseteq \mathbb{R}$, u is the **Maximum** (**Minimum**) of S, if $u = \sup S$ ($u = \inf S$) and $u \in S$.
Note: Sup. and Inf. are not necessarily elements in S (if they exist), but maximum and minimum are.

Supremum Property of $\mathbb{R}$ Every non-empty subset of $\mathbb{R}$ that has an upper bound has a supremum.

The Archimedeam Property If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ s.t. $x < n_x$.

Corollary 2.4.6 If $x > 0$, then $\exists n \in \mathbb{N}$ such that $n - 1 \leq x < n$.

Density Theorems For $x, y \in \mathbb{R}$ with $x < y$, tehre exists $r \in \mathbb{Q}$ ($z \in \mathbb{R}\backslash\mathbb{Q}$) s.t. $x < r < y$ ($x < z < y$).

Intervals

A sequence of intervals $I_n, n \in \mathbb{N}$ is **Nested** if

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

Properties: [1] If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested seq. of closed bounded intervals, then $\exists \xi \in \mathbb{R}$ s.t. $\xi \in I_n, \forall n \in \mathbb{N}$. [2] If $I_n = [a_n, b_n], n \in \mathbb{N}$ satisfying $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then ξ contained in all I_n is unique.

3. Sequences & Series

Sequence & Convergence

Sequence in $\mathbb{R}$: a real-valued function $X : \mathbb{R} \to \mathbb{R}$. We write $x_n = X(n)$ for the n-th term of the sequence, and denote the sequence as $(x_n : n \in \mathbb{N})$.

A sequence $X = (x_n)$ in $\mathbb{R}$ is **Convergent** to $x \in \mathbb{R}$ iff. for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ s.t.

$$n \geq K(\epsilon) \implies |x_n - x| < \epsilon$$

, and we call x the **Limit** of (x_n), denoted as $\lim_{n\to\infty} x_n = x$. A sequence is **Divergent** if it is not convergent.
Technique for proving convergence:

1. Express $|x_n - x|$ in terms of n and find a simpler upper bound $L = L(n)$, i.e. $|x_n - x| < L$.
2. Let $\epsilon > 0$ be arbitrary, find $K \in \mathbb{N}$ s.t. for all $n \geq K$, $L = L(n) < \epsilon$, then
$$n \geq K \implies |x_n - x| < L < \epsilon$$

Squeeze Theorem If $x_n \leq y_n \leq z_n$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then

$$\lim_{n\to\infty} y_n = a$$

A sequence $X = (x_n)$ is **Bounded** if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Monotone Convergence Theorem Let (x_n) be a monotone sequence of real numbers, then (x_n) is convergent iff. (x_n) is bounded. If it is bounded and increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. (Similarly for decreasing.)

For a sequence (x_n), it **tends to** $+\infty$, i.e. $\lim_{n\to\infty} x_n = +\infty$ if for all $\alpha \in \mathbb{R}$, there exists $K = K(\alpha) \in \mathbb{N}$ such that if $n \geq K(\alpha)$, then $x_n > \alpha$. (Similarly for $\lim_{n\to\infty} x_n = -\infty$.)
A sequence (x_n) is **Properly Divergent** if $\lim_{n\to\infty} x_n = \pm\infty$.

Subsequences

A **Subsequence** of $X = (x_n)$ is $X' = (x_{n_k})$:

$$X' = (x_{n_1}, x_{n_2}, \ldots, x_{n_3})$$

, where $n_1 < n_2 < \ldots < n_k < \ldots$ is a strictly increasing sequence in $\mathbb{N}$. Note: $n_k \geq n, \forall k$.

Theorem 3.4.2 If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x,

$$\lim_{n_k\to\infty} x_{n_k} = \lim_{k\to\infty} x_{n_k} = x$$

Theorem 3.4.5 If (x_n) has either of the following properties, it is divergent: [1] (x_n) has two convergent subsequences with different limits. [2] (x_n) is unbounded.

Theorem 3.4.7 Every sequence has a monotone subsequence.

Bolzano-Weierstrass Theorem Every bounded sequence has a convergent subsequence.

Cauchy Sequences

A **Cauchy Sequence** (x_n) is a sequence where for all $\epsilon > 0$, there exists $H = H(\epsilon) \in \mathbb{N}$ such that

$$\forall n, m \in \mathbb{N}, n, m \geq H \implies |x_n - x_m| < \epsilon$$

Cauchy Criterion A sequence is convergent iff. it is Cauchy.

A **Contractive Sequence** (x_n) is a sequence where there exists $C \in (0, 1)$ s.t.

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|,\ \forall n \in \mathbb{N}$$

Theorem 3.5.8 Every contractive sequence is Cauchy.

Infinite Series

For (x_n), its **(Infinite) Series** is sequence (s_n), where $s_n = \sum k = 1^n x_k$ is called a **Partial Sum** of the series, and x_k is a **Term**.
Tests for infinite series' convergence:

- **n-th Term Test** - If $\sum x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.
- Cauchy Criterion Test
- **Partial Sum Bounded Test**, for series w. non-negative terms - Suppose $x_n \geq 0, \forall n \in \mathbb{N}$, then $\sum_{x_n}$ converges iff. (s_n) is bounded.
- **Comparison Test** - For $(x_n), (y_n)$ with some $K \in \mathbb{N}$, s.t. $n \geq K \implies 0 \leq x_n \leq y_n$. Then [1] $\sum y_n$ converges $\implies$ $\sum x_n$ converges, and [2] $\sum x_n$ diverges $\implies$ $\sum y_n$ diverges.
- **Limit Comparison Test** - For **strictly positive** $(x_n), (y_n)$ with limit $r = \lim_{n\to\infty}(\frac{x_n}{y_n})$. Then [1] if $r = 0$, $\sum y_n$ converges $\implies$ $\sum x_n$ converges. [2] if $r > 0$, $\sum y_n$ converges iff $\sum x_n$ converges.

Absolute Convergence

Series $\sum x_n$ is **Absolutely Convergent** if series $\sum |x_n|$ is convergent. A series is **Conditionally Convergent** if it is convergent but not absolutely convergent.
Tests for absolutely convergence:

- Limit Comparison Test - Consider convergence of positive sequences $|x_n|$ and $|y_n|$ if $(x_n), (y_n)$ non-negative.
- **Root Test** - For (x_n), [1] if $\exists r \in \mathbb{R}, 0 < r < 1$ and $K \in \mathbb{N}$ s.t. $|x_n|^{1/n} \leq r,\ \forall n \geq K$, then $\sum x_n$ is abs. convergent. [2] If $\exists r \in \mathbb{R}, r > 1$ and $K \in \mathbb{N}$ s.t. $|x_n|^{1/n} \geq r > 1,\ \forall n \geq K$, then $\sum x_n$ is **divergent**.
- **Ratio Test** - For (x_n) nonzero, [1] if $\exists r \in \mathbb{R}, 0 < r < 1$ and $K \in \mathbb{N}$ s.t. $|\frac{x_{n+1}}{x_n}| \leq r,\ \forall n \geq K$, then $\sum x_n$ is abs. convergent. [2] If $\exists K \in \mathbb{N}$ s.t. $|\frac{x_{n+1}}{x_n}| \geq 1,\ \forall n \geq K$, then $\sum x_n$ is **divergent**.

4. Limits

For $A \subseteq \mathbb{R}$, c is the **Cluster Point** of A iff. $\forall \delta > 0$, there exists $x \in A$ s.t. $0 < |x - c| < \delta$.
Theorem 4.1.2 (Sequential Criterion) $c \in \mathbb{R}$ is a cluster point of A iff. there exists a sequence (a_n) in A s.t. $\lim a_n = c$ and $a_n \neq c, \forall n \in \mathbb{N}$.
Limit of a function $f : A \to \mathbb{R}$ at $c \in A$, $L = \lim_{x\to c} f(x)$ iff. $\forall \epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t. $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$.
Theorem 4.1.8 (Sequential Criterion) $\lim_{x\to c} f(x) = L$ iff. for every seq. (x_n) in A w. $lim_{n\to\infty} x_n = c$ and $x_n \neq c, \forall n \in \mathbb{N}$, $lim_{n\to\infty} f(x_n) = L$.

For $f : A \to \mathbb{R}$ and c a cluster point of A, f is **Bounded** on a neighbourhood of c if $\exists V_\delta(c)$ and constant $M > 0$ s.t. $|f(x)| < M, \forall x \in A \cap V_\delta(c)$.
Theorem 4.2.2 If $f : A \to \mathbb{R}$has a limit at cluster point c, then f is bounded on some neighbourhood of c.

Theorem 4.2.9 If $\lim_{x\to c} f(x) > 0$, then $\exists V_\delta(c)$ s.t. $f(x) > 0, \forall x \in A \cap V_\delta(c), x \neq c$.
Similar statements for $f(x) < 0$.

For $A \subseteq \mathbb{R}$, function $f : A \to \mathbb{R}$ and a cluster point c of A, **Right Hand Limit** $L_+ = \lim_{x\to c^+} f(x)$ iff. $\forall \epsilon > 0, \exists \delta > 0, x \in V_\delta(c)\ \{c\} \implies f(x) \in V_\epsilon(L_+)$.
Similar definition for **Left-Hand Limit** $L_- = \lim_{x\to c^-} f(x)$.
Sequential Criteria for One-sided Limits exist.
Theorem 4.3.3 $\lim_{x\to c} f(x) = L$ iff. both $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist and

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

5. Continuous Functions

Continuity

For $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$, f is **Continuous** at $c \in A$ iff $\forall \epsilon > 0, \exists \delta > 0$ s.t. $x \in V_\delta(c) \implies f(x) \in V_\epsilon(f(c))$.
f is continuous at $c \in A$ iff. $\lim_{x\to c} f(x) = f(c)$.
(Sequential Criterion) $f : A \to \mathbb{R}$ is continuous at $x = c$ iff. for every sequence (x_n) in A s.t. $x_n \to c$, we have $f(x_n) \to f(c)$.

Continuous Function on Intervals

Boundedness Theorem If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.
Note: It only applies to **closed bounded** intervals.
Max-Min Theorem If f is continuous on $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.
Location of Roots Theorem If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists a point c in (a, b) s.t. $f(c) = 0$.
Bolzano's Intermediate Value Theorem For interval I and function f continuous on I, and $a, b \in I$ with $f(a) \leq f(b)$, then for any $k \in [f(a), f(b)]$, $\exists c \in I$ s.t. $f(c) = k$.
Preservation of Closed Intervals Theorem For f continuous on $[a, b]$,

$$f([a, b]) := \{f(x) : x \in [a, b]\} = [m, M]$$

, with $m = \inf f([a, b])$ and $M = \sup f([a, b])$.

Monotonicity & Bijectivity

A function $f : A \to \mathbb{R}$ is **Increasing** (**Decreasing**) on A if $\forall x_1, x_2 \in A, x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$ $(x_1 \leq x_2 \implies f(x_1) \geq f(x_2))$.
f is **Monotone** if it is increasing or decreasing.
Strictly ~: $x_1 < x_2 \implies f(x_1) < f(x_2)$ and so on.
For a function $f : A \to B$, it is

- **Injective** (**One-One**), iff $\forall x_1 \neq x_2 \in A, f(x_1) \neq f(x_2)$.
- **Surjective**, iff $f(A) = B$.
- **Bijective**, iff it is injective and surjective.

Uniform Continuity

For $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$, f is **Uniformly Continuous** on A if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t.

$$\forall x, y \in A, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

i.e. $\delta = \delta(\epsilon)$ is independent of $x, y \in A$.
Note: f is **not uniformly continuous** on A iff. $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0, \exists x_\delta, y_\delta \in A$ with $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon$.
Sequential Criterion

- Uniformly continuous - For any $(x_n), (y_n)$ in A with $\lim_{n\to\infty} x_n - y_n = 0$, we have $\lim_{n\to\infty} f(x_n) - f(y_n) = 0$.
- Not uniformly continuous - There exists $\epsilon_0 > 0$ and $(x_n), (y_n)$ in A, $\lim_{n\to\infty} x_n - y_n = 0$ and $\lim_{n\to\infty} f(x_n) - f(y_n) \geq \epsilon_0$.

Uniform Continuity Theorem If f is continuous on a **closed bounded** interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

A function $f : A \to \mathbb{R}$ is a **Lipschitz Function** on A iff. there exists $K > 0$ s.t.

$$|f(x) - f(y)| \leq K|x - y|, \forall x, y \in A$$

Theorem 5.4.5 If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.
Theorem 5.4.7 If $f : A \to \mathbb{R}$ is uniformly continuous on A and (x_n) a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.
i.e. Uniformly continuous functions preserve Cauchy sequences.
Continuous Extension Theorem f is uniformly continuous on interval (a, b) iff. it can be defined at the endpoints a and b s.t. the extended function is continuous on $[a, b]$.
Note: Define $f(a) = \lim_{x\to a^+} f(x)$ and $f(b) = \lim_{x\to b^-} f(x)$ provided both limits exist.

Jumps

Theorem 5.6.1 For interval $I \subseteq \mathbb{R}$ and increasing function $f : I \to \mathbb{R}$, $c \in I$ not an endpoint, then

- $\lim_{x\to c^-} f(x) = \sup\{f(x) : x \in I, x < c\}$
- $\lim_{x\to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}$

Jump of f at c is defined as

$$j_f(c) = \lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x)$$

, and at endpoints, $j_f(a) = \lim_{x\to a^+} f(x) - f(a)$ and so on for b.
Theorem 5.6.4 For interval $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ monotone on I, the set of points $D \subseteq I$ at which f is discontinuous is a countable set.
Continuous Inverse Theorem For interval $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ strictly monotone and continuous, the inverse f^{-1} exists and is also strictly monotone and continuous on $J = f(I)$.

11. Topology Introduction

Open & Closed Sets

A set V is the **Neighbourhood** of a point $x \in \mathbb{R}$ iff there exists $\epsilon > 0$ s.t. $V_\epsilon(x) \subseteq V$.
A subset $G \subseteq \mathbb{R}$ is **Open** in $\mathbb{R}$ iff. for each $x \in G$, there exists $\epsilon_x > 0$ s.t. $V_{\epsilon_x}(x) \subseteq G$.
A subset $F \subseteq \mathbb{R}$ is **Closed** in $\mathbb{R}$ if the complement $C(F) = \mathbb{R}\ \ F$ is open in $\mathbb{R}$.
Note: [1] $\mathbb{R}$ and $\emptyset$ are both open and closed. [2] $\mathbb{Z}$ is closed but not open. [3] $\mathbb{Q}$ is neither open nor closed.
Open & Closed Set Properties

- Open: [1] Union of any collection of open subsets is open. [2] Intersection of finitely many open subsets is open.
- Closed: [1] Intersection of any collection of closed subsets is closed. [2] Union of finitely many closed subsets is closed.

Characterisation of Closed Sets Theorem A subset $F \subseteq \mathbb{R}$ is closed iff. any convergence sequence (x_n) in F has $\lim_{n\to\infty} x_n \in F$.
Theorem 11.1.8 A subset $F \subseteq \mathbb{R}$ is closed iff. it contains all its cluster points.
Theorem 11.1.9 A subset $G \subseteq \mathbb{R}$ is open iff it is the union of countably many disjoint open intervals in $\mathbb{R}$.

Global Continuity Theorem A function $f : A \to \mathbb{R}$ is continuous on A iff. for every open set $G \subseteq \mathbb{R}$, there exists open set $H \subseteq \mathbb{R}$ such that $H \cap A = f^{-1}(G)$ where $f^{-1}(G) = \{x \in A : f(x) \in G\}$.
Corollary 11.3.3 Function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff. $f^{-1}(G)$ is open in $\mathbb{R}$ for every open G.

Metric Space

A **Metric** on a set S is a function $d : S \times S \to \mathbb{R}$ that satisfies

- **Positivity** $d(x, y) \geq 0, \forall x, y \in S$
- **Definiteness** $d(x, y) = 0 \iff x = y$
- **Symmetry** $d(x, y) = d(y, x), \forall x, y \in S$
- **Triangle Inequality** $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in S$

A **Metric Space** (S, d) is a set S with a metric d on S.
Generalised definition for a metric space (S, d)

- **Neighbourhood**: $V_\epsilon(x_0) = \{x \in S : d(x, x_0) < \epsilon\}$ for $\epsilon > 0$ and $x_0 \in S$

- **Boundedness** of $K \subseteq S$: $\exists M > 0, x_0 \in S, d(x, x_0) \leq M, \forall x \in K$.
- **Convergence** to $x \in S$ of sequence (x_n): $\forall \epsilon > 0, \exists K = K(\epsilon) \in \mathbb{N}, n \geq K \implies x_n \in V_\epsilon(x)$
- **Continuity** of $f : S_1 \to S_2$ at $c \in S_1$: $\forall \epsilon > 0, \exists \delta > 0, d_1(x, c) < \delta \implies d_2(f(x), f(c)) < \epsilon$.
- **Open & Closed Set**

Compact Set

For a metric space S, an **Open Cover** of a subset $A \subseteq S$ is a collection $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ of open subsets of S satisfying

$$A \subseteq \cup_{\lambda \in \Lambda} G_\lambda$$

If $\mathcal{G}' \subseteq \mathcal{G}$ whose union also contains A, then $\mathcal{G}'$ is a **Subcover** of $\mathcal{G}$. If $\mathcal{G}'$ is finite, it is a **Finite Subcover** of $\mathcal{G}$.
For a metric space S, a subset $K \subseteq S$ is **Compact** iff. for every open cover of K there is a finite subcover.
Heine-Borel Theorem For a metric space (S, d), a subset $K \subseteq S$ is compact iff. it is closed and bounded.
Bolzano-Weierstrass Theorem A bounded sequence in (S, d) has a convergent subsequence.
Theorem 11.2.6 $K \subseteq S$ is compact iff. every sequence in K has a subsequence that converges to a point in K.
Preservation of Compactness Theorem If (S, d) is compact and $f : S \to \mathbb{R}$ is continuous, then $f(S)$ is compact in $\mathbb{R}$.

A subset $U \subseteq S$ is **Disconnected** iff. U has an open cover $\{A, B\}$ s.t. $A \cap B \cap U = \emptyset$ and $A \cap U = \emptyset, B \cap U = \emptyset$. Otherwise it is **Connected**.
Note: $E \subseteq \mathbb{R}$ is connected iff E is an interval, i.e. $x, y \in E, x < y \implies [x, y] \in E$.
Intermediate Value Theorem For $f : S \to \mathbb{R}$ continuous, if E is connected then $f(E)$ is connected.

Intermediate Results & Lemmas

(Tut10Qn4) For any two functions $f, g : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R}$, if $f(x) = g(x), \forall x \in \mathbb{Q}$, then $f(x) = g(x), \forall x \in \mathbb{R}$.
Useful statements

- $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.