MA2108 Cheatsheet
by Yiyang, AY23/34

1. Pre-requisites

Well-Ordering Principle of $\mathbb{N}$ Every non-empty subset $S \in \mathbb{N}$ has a least (smallest) element.

2. The Real Numbers

Algebraic Properties, ~

Different types of means

- **Arithmetic Means** $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k$
- **Geometric Means** $G_n = \left(\prod_{k=1}^{n} a_k\right)^{1/n}$
- **Harmonic Means** $H_n = n\left(\sum_{k=1}^{n} a_k^{-1}\right)^{-1}$

, for $n \in \mathbb{N}_{\geq 2}$ and $a_1, a_2, ..., a_n \in \mathbb{R}$ are positive. For the means, we have the **AM-GM-HM Inequality** :

$$H_n \leq G_n \leq A_n$$

, taking "=" iff. $a_1 = ... = a_n$.

Bernoulli's Inequality For $x > -1$, we have $(1+x)^n \geq 1 + nx$, for any $n \in \mathbb{N}$.

Triangle Inequity $|a+b| \leq |a| + |b|$, for all $a, b \in \mathbb{R}$.
Derived: [1] $||a| - |b|| \leq |a - b|$, [2] $|a-b| \leq |a| + |b|$.

Neighbourhood

For any $a \in \mathbb{R}$ and $\epsilon > 0$, the **ϵ-neighbourhood of a** is the set:

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

Theorem 2.2.8 For $a \in \mathbb{R}$, if $x \in V_\epsilon(a)$ for every $\epsilon > 0$, then $x = a$.

Completeness Properties, ~

For a non-empty $S \subseteq \mathbb{R}$, it is **Bounded Above** (**Bounded Below**) if S has an upper bound (a lower bound). S is **Bounded** if it is bounded above and below, and is **Unbounded**, otherwise.

For a non-empty $S \subseteq \mathbb{R}$, u is the **Supremum** of S if the following conditions are met, and we denote it as $\sup S$:

1. u is an upper bound of S.
2. $\forall v \in \mathbb{R}$, if v is an upper bound of S, then $v \geq u$.

For a non-empty $S \subseteq \mathbb{R}$, w is the **Infinum** of S if the following conditions are met, and we denote it as $\inf S$:

1. w is a lower bound of S.
2. $\forall v \in \mathbb{R}$, if v is a lower bound of S, then $v \leq w$.

Note: Sup. and Inf. are **uniquely determined**, if they exist.
Alternative Definition (Similarly for Infinum):
Lemma 2.3.4 For u an upper bound of $S \subseteq \mathbb{R}$, $u = \sup S$ iff.

$$\forall \epsilon > 0, \exists s_\epsilon \in S,\ u - \epsilon < s_\epsilon$$

For a non-empty $S \subseteq \mathbb{R}$, u is the **Maximum** (**Minimum**) of S, if $u = \sup S$ ($u = \inf S$) and $u \in S$.
Note: Sup. and Inf. are not necessarily elements in S (if they exist), but maximum and minimum are.

Supremum Property of $\mathbb{R}$ Every non-empty subset of $\mathbb{R}$ that has an upper bound has a supremum.

The Archimedeam Property If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ s.t. $x < n_x$.

Corollary 2.4.6 If $x > 0$, then $\exists n \in \mathbb{N}$ such that $n - 1 \leq x < n$.

Density Theorems For $x, y \in \mathbb{R}$ with $x < y$, tehre exists $r \in \mathbb{Q}$ ($z \in \mathbb{R}\backslash\mathbb{Q}$) s.t. $x < r < y$ ($x < z < y$).

Intervals

A sequence of intervals $I_n, n \in \mathbb{N}$ is **Nested** if

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_n \supseteq I_{n+1} \supseteq ...$$

Properties: [1] If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested seq. of closed bounded intervals, then $\exists \xi \in \mathbb{R}$ s.t. $\xi \in I_n, \forall n \in \mathbb{N}$. [2] If $I_n = [a_n, b_n], n \in \mathbb{N}$ satisfying $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then ξ contained in all I_n is unique.

3. Sequences & Series

Sequence & Convergence

Sequence in $\mathbb{R}$: a real-valued function $X : \mathbb{R} \to \mathbb{R}$. We write $x_n = X(n)$ for the n-th term of the sequence, and denote the sequence as $(x_n, : n \in \mathbb{N})$.

A sequence $X = (x_n)$ in $\mathbb{R}$ is **Convergent** to $x \in \mathbb{R}$ iff. for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ s.t.

$$n \geq K(\epsilon) \implies |x_n - x| < \epsilon$$

, and we call x the **Limit** of (x_n), denoted as $\lim_{n\to\infty} x_n = x$. A sequence is **Divergent** if it is not convergent.
Technique for proving convergence:

1. Express $|x_n - x|$ in terms of n and find a simpler upper bound $L = L(n)$, i.e. $|x_n - x| < L$.
2. Let $\epsilon > 0$ be arbitrary, find $K \in \mathbb{N}$ s.t. for all $n \geq K$, $L = L(n) < \epsilon$, then

$$n \geq K \implies |x_n - x| < L < \epsilon$$

Squeeze Theorem If $x_n \leq y_n \leq z_n$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then

$$\lim_{n\to\infty} y_n = a$$

A sequence $X = (x_n)$ is **Bounded** if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Monotone Convergence Theorem Let (x_n) be a monotone sequence of real numbers, then (x_n) is convergent iff. (x_n) is bounded. If it is bounded and increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. (Similarly for decreasing.)

For a sequence (x_n), it **tends to** $+\infty$, i.e. $\lim_{n\to\infty} x_n = +\infty$ if for all $\alpha \in \mathbb{R}$, there exists $K = K(\alpha) \in \mathbb{N}$ such that if $n \geq K(\alpha)$, then $x_n > \alpha$. (Similarly for $\lim_{n\to\infty} x_n = -\infty$.)
A sequence (x_n) is **Properly Divergent** if $\lim_{n\to\infty} x_n = \pm\infty$.

Subsequences

A **Subsequence** of $X = (x_n)$ is $X' = (x_{n_k})$:

$$X' = (x_{n_1}, x_{n_2}, ..., x_{n_3})$$

, where $n_1 < n_2 < ... < n_k < ...$ is a strictly increasing sequence in $\mathbb{N}$. Note: $n_k \geq n, \forall k$.

Theorem 3.4.2 If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x,

$$\lim_{n_k\to\infty} x_{n_k} = \lim_{k\to\infty} x_{n_k} = x$$

Theorem 3.4.5 If (x_n) has either of the following properties, it is divergent: [1] (x_n) has two convergent subsequences with different limits. [2] (x_n) is unbounded.

Theorem 3.4.7 Every sequence has a monotone subsequence.

Bolzano-Weierstrass Theorem Every bounded sequence has a convergent subsequence.

Cauchy Sequences

A **Cauchy Sequence** (x_n) is a sequence where for all $\epsilon > 0$, there exists $H = H(\epsilon) \in \mathbb{N}$ such that

$$\forall n, m \in \mathbb{N}, n, m \geq H \implies |x_n - x_m| < \epsilon$$

Cauchy Criterion A sequence is convergent iff. it is Cauchy.

A **Contractive Sequence** (x_n) is a sequence where there exists $C \in (0,1)$ s.t.

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|,\ \forall n \in \mathbb{N}$$

Theorem 3.5.8 Every contractive sequence is Cauchy.

Infinite Series

For (x_n), its **(Infinite) Series** is sequence (s_n), where $s_n = \sum k = 1^n x_k$ is called a **Partial Sum** of the series, and x_k is a **Term**.
Tests for infinite series' convergence:

- **n-th Term Test** - If $\sum x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.
- Cauchy Criterion Test
- **Partial Sum Bounded Test**, for series w. non-negative terms - Suppose $x_n \geq 0, \forall n \in \mathbb{N}$, then $\sum_{x_n}$ converges iff. (s_n) is bounded.
- **Comparison Test** - For $(x_n), (y_n)$ with some $K \in \mathbb{N}$, s.t. $n \geq K \implies 0 \leq x_n \leq y_n$. Then [1] $\sum y_n$ converges $\implies \sum x_n$ converges, and [2] $\sum x_n$ diverges $\implies \sum y_n$ diverges.
- **Limit Comparison Test** - For **strictly positive** $(x_n), (y_n)$ with limit $r = \lim_{n\to\infty}(\frac{x_n}{y_n})$. Then [1] if $r = 0$, $\sum y_n$ converges $\implies \sum x_n$ converges. [2] if $r > 0$, $\sum y_n$ converges iff $\sum x_n$ converges.

Intermediate Results & Lemmas