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Supplementary Material

1 EQUILIBRIA AND STABILITY ANALYSIS

1.1 Equilibria

Setting all the derivatives in the delay differential system (9) to zero results in $E^c = I_a^c = I_s^c = I_d^c = 0$ hence all equilibria of the system are disease-free. We are then left with the reduced system

$$\begin{cases} \dot{S}_{-1} = 0, \\ \dot{S}_1 = 0, \\ \dot{R} = 0. \end{cases} \tag{1.1}$$

It follows that any fixed point $X = (S_{-1}, S_1, E, I_a, I_s, I_d, R)^\top$ of Eq (9) is element of the solution sub-space

$$\mathcal{D}_0 = \{S_{-1} \geq 0, S_1 \geq 0, I = 0, R \geq 0 | S_{-1} + S_1 + R = N_0\} \tag{1.2}$$

where $I(t) = E(t) + I_a(t) + I_s(t) + I_d(t)$. In other words, any equilibrium point is of the form $X^c = (S_{-1}^c, S_1^c, 0, 0, 0, 0, R^c)^\top$ where $0 \leq S_{-1}^c \leq N_0$, $0 \leq S_1^c \leq N_0 - S_{-1}^c$ and $R^c = N_0 - S_{-1}^c - S_1^c$.

1.2 Stability of equilibria

From Eq (10a), it appears that starting from an equilibrium point satisfying $R_0 > 0$ always corresponds to a lower reproductive number $\mathcal{R}(0)$ as compared to $R_0 = 0$. The reason is that $R_0 > 0$ reduces the probability of contacts between susceptible individuals and any introduced infectious individual. Given that the compartment R is a dead end of the system, it immediately follows that an equilibrium X_0 satisfying $R_0 = 0$ is not globally stable (since after disease introduction, we necessarily end up with $R^c > 0$). It further follows that any equilibrium X_0 satisfying $R_0 < N_0$ is not globally stable because after disease introduction, we end up with $R^c > R_0$. In accordance, the following Theorem states that any equilibrium point X^c of system (9) is globally asymptotically unstable, unless $R^c = N_0$ (i.e. we have $S_{-1}^c = S_1^c = 0$, and a zero reproductive number: $\mathcal{R}(\infty) = 0$).

DRAFT

THEOREM 1 (Global Asymptotic Stability of Equilibria). *Any equilibrium point* $X^c = (S^c_{-1}, S^c_1, 0, 0, 0, 0, R^c)^\top$ *of the delay differential system* (9) *is globally asymptotically unstable, provided that* $R^c < N_0$. *Accordingly, only the equilibrium point* $X^c = (0, 0, 0, 0, 0, 0, N_0)^\top$ *is globally asymptotically stable.*

After ruling out global stability in any situation of interest, we next investigate the local stability of equilibria. Without loss of generality, we restrict attention to $R^c < N_0$ since there is no possibility of an outbreak when $R^c = N_0$ ($\mathcal{R}(\infty) = 0$).

THEOREM 2 (Local Asymptotic Stability of Equilibria). *Any equilibrium point* $X^c = (S^c_{-1}, S^c_1, 0, 0, 0, 0, R^c)^\top$ *of the delay differential system* (9) *such that* $R^c < N_0$ *is locally asymptotically stable if the basic reproductive number satisfies* $\mathcal{R}(0) < 1$, *and unstable when* $\mathcal{R}(0) \geq 1$.

PROOF OF THEOREM 2. The differential system (9) is locally asymptotically stable (l.a.s.) at an equilibrium point X^c if all eigenvalues of the Jacobian matrix of the system evaluated at X^c have negative real parts (Martcheva, 2015). The Jacobian matrix of model (9) at any equilibrium point has the block structure

$$\boldsymbol{J}^c = \begin{pmatrix} \boldsymbol{J}_S & \boldsymbol{J}^c_{SI} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{J}^c_I & \mathbf{0} \\ \mathbf{0} & \boldsymbol{J}^c_{RI} & \boldsymbol{J}^c_R \end{pmatrix} \tag{1.3a}$$

where $\boldsymbol{J}_S = \mathbf{0}$ is the 2×2 matrix of all zeros,

$$\boldsymbol{J}^c_{SI} = -\begin{pmatrix} 0 & \beta_{-1a}\bar{S}^c_{-1} & \beta_{-1s}\bar{S}^c_{-1} \\ 0 & \beta_{1a}\bar{S}^c_1 & \beta_{1s}\bar{S}^c_1 \end{pmatrix}, \tag{1.3b}$$

$$\boldsymbol{J}^c_I = \begin{pmatrix} -\theta & \sum_i \beta_{ia}\bar{S}^c_i & \sum_i \beta_{is}\bar{S}^c_i \\ (1-\sigma)(1-\pi)\theta & -g_a & 0 \\ \sigma(1-\pi)\theta & 0 & -g_s \end{pmatrix}, \tag{1.3c}$$

$$\boldsymbol{J}^c_{RI} = \begin{pmatrix} \pi\theta & \gamma_a & \gamma_s \\ 0 & \rho_a & \rho_s \end{pmatrix}, \quad \text{and} \quad \boldsymbol{J}^c_R = \begin{pmatrix} -\rho_d & 0 \\ \rho_d & 0 \end{pmatrix} \tag{1.3d}$$

with $\beta_{ij} = \beta_{ij}(0)$, $\bar{S}^c_i = \frac{S_i}{N_0}$, $g_a = \gamma_a + \rho_a$ and $g_s = \gamma_s + \rho_s$. From this block structure, the eigenvalues of $\boldsymbol{J}^c$ are those of $\boldsymbol{J}^c_S$, $\boldsymbol{J}^c_I$ and $\boldsymbol{J}^c_R$ (using Schur complements). Obviously, $\boldsymbol{J}^c_S$ has two zero eigenvalues ($d_1 = d_2 = 0$) and $\boldsymbol{J}^c_R$ has one zero eigenvalue ($d_3 = 0$ in addition to $d_4 = -\rho_d$), hence all eigenvalues of $\boldsymbol{J}^c$ do not have negative real parts. We thus have a critical case of stability of equilibrium states (Barsuk and Paladi, 2021).

DRAFT

Note that from the block structure of $\boldsymbol{J}^c$, we can separately study the diagonal blocks $\boldsymbol{J}_S^c$, $\boldsymbol{J}_I^c$ and $\boldsymbol{J}_R^c$. Since the order-2 matrix $\boldsymbol{J}_S^c$ is null, it has rank zero, and thus satisfies $rank\,(\boldsymbol{J}_S^c) = 2 - k$ where $k = 2$ is the multiplicity of its zero eigenvalues. Likely, $rank\,(\boldsymbol{J}_R^c) = 1 = 2 - k$ where $k = 1$ is the multiplicity of its zero eigenvalue. Then, by Eq (9) of Barsuk and Paladi (2021), the stability of the system only depends on d_4 and the eigenvalues of $\boldsymbol{J}_I^c$ which must all have negative real parts to ensure l.a.s. First note that $d_4 = -\rho_d < 0$. The characteristic polynomial P_c of $\boldsymbol{J}_I^c$ is given by

$$P_c(d) = d^3 + K_2 d^2 + K_1 d + \theta g_a g_s \left[1 - \mathcal{R}(0)\right]$$

on setting $K_1 = \theta(g_a + g_s)\left[1 - \frac{(1-\sigma)(1-\pi)}{g_a+g_s}\sum_i \beta_{ia}\bar{S}_i - \frac{\sigma(1-\pi)}{g_a+g_s}\sum_i \beta_{is}\bar{S}_i\right] + g_a g_s$, and $K_2 = \theta + g_a + g_s$. The Routh-Hurwitz stability conditions (see Eq (A.22) in (May, 1973, page 196)) corresponding to the polynomial P_c are:

$$K_2 > 0, \quad K_0 > 0, \quad \text{and} \quad K_1 > \frac{K_0}{K_2}$$

where $K_0 = \theta g_a g_s \left[1 - \mathcal{R}(0)\right]$.

Since $\theta > 0$ and $g_j > 0$ for any $j \in \{a, s\}$, we have (i): $K_2 > 0$ by definition. If $\mathcal{R}(0) \geq 1$, then $K_0 \leq 0$. It follows that at least one eigenvalue of $\boldsymbol{J}_I^c$ has a positive real part when $\mathcal{R}(0) \geq 1$, and instability is established. When $\mathcal{R}(0) < 1$ on the contrary, we have (ii): $K_0 > 0$. Next, notice that $\mathcal{R}(0) < 1$ is equivalent to $1 > \frac{(1-\sigma)(1-\pi)}{g_a}\sum_i \beta_{ia}\bar{S}_i + \frac{\sigma(1-\pi)}{g_s}\sum_i \beta_{is}\bar{S}_i$. This implies that $1 > \frac{(1-\sigma)(1-\pi)}{g_a+g_s}\sum_i \beta_{ia}\bar{S}_i + \frac{\sigma(1-\pi)}{g_a+g_s}\sum_i \beta_{is}\bar{S}_i$ since $g_a > 0$ and $g_s > 0$. This in turn implies that $\left[1 - \frac{(1-\sigma)(1-\pi)}{g_a+g_s}\sum_i \beta_{ia}\bar{S}_i - \frac{\sigma(1-\pi)}{g_a+g_s}\sum_i \beta_{is}\bar{S}_i\right] > 0$ hence $K_1 > g_a g_s$ on the one hand. On the other hand, $\mathcal{R}(0) < 1$ implies that $\frac{K_0}{K_1} < g_a g_s$ since $0 < \frac{\theta}{\theta+g_a+g_s} < 1$ and $0 < 1 - \mathcal{R}(0) < 1$. It follows that $\frac{K_0}{K_1} < g_a g_s < K_1$, hence (iii): $K_1 > \frac{K_0}{K_2}$ if $\mathcal{R}(0) < 1$. Statements (i), (ii) and (iii) ensure that all the three eigenvalues of $\boldsymbol{J}_I^c$ have negative real parts when $\mathcal{R}(0) < 1$, and l.a.s. is established. □

2 PERSISTENCE OF THE DISEASE

The global instability of equilibria of system (9) does not imply that the disease will uniformly persist, even when $\mathcal{R}(0) > 1$. Indeed, since $\partial R(t)/\partial t > 0$ when $E(t) + I_a(t) + I_s(t) + I_d(t) > 0$ and the population is closed ($N(t) = N_0$), the removed compartment R can only increase in size, and it always does when there is

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an exposed or an infectious individual in the population. As a result, the disease-free region $\mathcal{D}_0$ is globally asymptotically stable, i.e.

$$\lim_{t\to\infty} X(t) \in \mathcal{D}_0, \tag{2.1}$$

for any value of $\mathcal{R}(0)$. Accordingly, the disease always dies out, i.e.

$$\lim_{t\to\infty} I(t) = 0, \tag{2.2}$$

although this might not occur quickly after an outbreak, possibly occurring at $R(\infty) = N_0$ (i.e. $S_{-1}(\infty) = S_1(\infty) = 0$, with final size $F_\infty = 100\%$).

3 CRITICAL CONTROL PARAMETERS

Let $p(t) = \frac{S(t)}{N_0 - I_d(t)}$ denote the proportion of susceptible individuals in the mixing population. We derive here some consequences of THEOREM 2 for policy design.

3.1 Critical early detection probability

COROLLARY 2.1 (Critical Early Detection Probability). *Suppose* $\mathcal{R}(0) > 1$ *when* $\pi = 0$. *Then, the epidemic can be controlled through contact tracing and early isolation of exposed individuals. In this case,* ceteris paribus*, the critical (minimal) early detection probability is*

$$\pi^* = 1 - \left[\beta_0 \left(\phi_a \frac{1-\sigma}{\gamma_a + \rho_a} + \phi_s \frac{\sigma}{\gamma_s + \rho_s}\right) \sum_i \frac{S_{i0}}{N_0} (1 - \kappa m_{i0})\right]^{-1}. \tag{3.1}$$

That is, $\pi > \pi^*$ *is required to sufficiently lower* $\mathcal{R}(0)$ *and expect the disease dying out quickly.*

3.2 Critical level of protection

COROLLARY 2.2 (Critical Level of Protection). *Suppose that* $\mathcal{R}(0) < 1$ *when* $\kappa = 0$ *and* $\mathcal{R}(0) > 1$ *when* $\kappa = 1$. *Then,* ceteris paribus*, the critical (minimal) level of protection (by prophylactic behavior) is:*

$$\kappa^* = \frac{(1 - \frac{R_0}{N_0})\mathcal{R}_o - 1}{\mathcal{R}_o \sum_i m_{i0} \frac{S_{i0}}{N_0}}, \tag{3.2}$$

DRAFT

that is, $\kappa > \kappa^*$ is required to sufficiently lower $\mathcal{R}(0)$ and expect the disease dying out quickly.

3.3 Critical prophylactic proportion

COROLLARY 2.3 (Critical Prophylactic Proportion). *Suppose $\mathcal{R}(t) > 1$ when $m_i(t) = 0$ and $\mathcal{R}(t) < 1$ when $m_i(t) = 1$ for all i. Then the (withing risk tolerance group) critical (minimal) prophylactic proportion required to ensure disease eradication given $p(t)$ is:*

$$m^*(t) = \frac{1}{\kappa}\left[1 - \frac{1}{p(t)\mathcal{R}_o}\right], \tag{3.3}$$

that is, $m_i(t) > m^*(t)$ is required withing each group i to sufficiently lower $\mathcal{R}(t)$ and expect the disease dying out quickly. In particular, if $m_{i0} = m_0$, $\mathcal{R}(0) > 1$ when $m_0 = 0$ and $\mathcal{R}(0) < 1$ when $m_0 = 1$, then, ceteris paribus*, the critical (minimal) prophylactic proportion is:*

$$m_0^* = \frac{1}{\kappa}\left[1 - \frac{1}{(1 - \frac{R_0}{N_0})\mathcal{R}_o}\right], \tag{3.4}$$

that is, $m_0 > m_0^*$ is required to sufficiently lower $\mathcal{R}(0)$ and expect that an introduction of an infectious individual does not kick off an epidemic.

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