@Chen Siyi

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Final Part1

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> For **homogeneous linear ODEs** with **variable coefficients**, sometimes finding an explicit solution is difficult, then we use the method of **power series ansatz** to solve/approximate solutions.
>
> Recall: **homogeneous**, **linear**, **ordinary**, **variable coefficients**.

1 Summary of Power Series Ansatz

1. Analyze the equation, decide whether we can use power series ansatz around some point
2. Choose which form of ansatz to use
3. Plug into the ansatz, get recurrence relationship of the coefficients

4. Set initial value of coeffiencients. solve for coefficients to get one or more independent solutions
5. If not enough independent solutions are found, using reduction of order to find more solutions
6. Obtain the general solution

2 Ansatz1: ODE with Analytic Coefficients

$$x\prime\prime + P(t)x\prime + Q(t)x = 0$$

Where $P(t)$ and $Q(t)$ are **analytic in a neiborhood of** t_0.

> " a neighborhood of t_0" contains t_0

Then we can choose the ***ansatz***

$$x(t) = \sum_0^\infty a_k(t - t_0)^k$$

Accordingly,

$x\prime(t) = \sum_0^\infty ka_k(t - t_0)^{k-1}$

$x\prime\prime(t) = \sum_0^\infty k(k - 1)a_k(t - t_0)^{k-2}$

Plug the three equations back, we can obtain the relationship of the coefficients {a_0, a_1, a_2, ...}.

Depending on the situation, after setting values for first n terms (always 2), we can solve 1 to n (expected) independent solutions.

If not enough indepedent solutions are found, sometimes we can use reduction of order to find more.

> **Comments**:
>
> - The solutions found should be valid within its radius of convergence

> **Radius of Convergence of a Power Series**:
>
> - $\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$
> - $\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n}$

3 ODE Having Singular Points

The general form of a **homogeneous linear second-order ODE** with **variable coefficients**:

$$P(t)x\prime\prime + Q(t)x\prime + R(t)x = 0$$

It is said to have a **singular point** at t_0 if $P(t_0) = 0$.

Generally around singular points, it's hard to decide or find continuous solutions. But there're some specific cases we can deal with.

3.1 Regular Singular Points

$$x\prime\prime + p(t)x\prime + q(t)x = 0$$

is said to have a ***regular singular point*** at t_0 if the functions $(t - t_0)p(t)$ and $(t - t_0)^2q(t)$ are ***analytic in a neighborhood of*** t_0. A singular point which is not regular is said to be ***irregular***.

The general claim is: if an equation has a regular sigular point at t_0, then we can assume $p(t) = \frac{p_{-1}}{t-t_0} + \sum_{j=0}^{\infty} p_j(t-t_0)^j$, $q(t) = \frac{q_{-2}}{(t-t_0)^2} + \frac{q_{-1}}{t-t_0} + \sum_{j=0}^{\infty} q_j(t-t_0)^j$ and use the ansatz $x(t) = (t-t_0)^r \sum_{k=0}^{\infty} a_k(t-t_0)^k$ to find solutions.

4 Ansatz2: Euler's Equation

$$t^2x'' + \alpha t x' + \beta x = 0, \quad \alpha, \beta \in \mathbb{R}$$

> **Analysis:**
>
> This is exactly the case where the equation $x'' + \alpha\frac{1}{t}x' + \beta\frac{1}{t^2}x = 0, \quad \alpha, \beta \in \mathbb{R}$ is having a regular singular point at $t = 0$.
>
> But for this specific case of the Euler's Equation, we can choose an easier ansatz.

We can choose the ***ansatz***

$$x(t) = t^r$$

Inserting back and solve for r we get

$$r = -\frac{\alpha - 1}{2} \pm \frac{1}{2}\sqrt{(\alpha-1)^2 - 4\beta}$$

- $(\alpha - 1)^2 - 4\beta > 0$

  $$x\left(t; c_1, c_2\right) = c_1t^{r_1} + c_2t^{r_2}, \quad c_1, c_2 \in \mathbb{R}$$

- $(\alpha - 1)^2 - 4\beta = 0$, $r_1 = r_2 = \frac{1-\alpha}{2}$, need to use reduction of order

  $$x\left(t; c_1, c_2\right) = c_1t^{r_1} + c_2t^{r_1}\ln t, \quad c_1, c_2 \in \mathbb{R}$$

> **Reduction of order:**
>
> For equation $y\prime\prime + p(t)y\prime + q(t)y = 0$, and a known solution $y_1(x)$, let $y_2(x) = v(x)y_1(x)$, then you can solve for $v(x)$ using
>
> $$y_1(t)v\prime\prime + (2y_1\prime(t) + p(t)y_1(t))v\prime = 0$$

- $(\alpha - 1)^2 - 4\beta < 0$

  After getting $x_1(t) = t^{r_1} = t^{\lambda}(\cos(\mu \ln t) + i \sin(\mu \ln t))$.
  $x_2(t) = t^{r_1} = t^{\lambda}(\cos(\mu \ln t) - i \sin(\mu \ln t))$, further have

  $$x\left(t; c_1, c_2\right) = c_1t^{\lambda}\cos(\mu \ln t) + c_2t^{\lambda}\sin(\mu \ln t), \quad c_1, c_2 \in \mathbb{R}$$

5 Ansatz3: The Method of Frobenius

5.1 Basic Method

$$x'' + p(t)x' + q(t)x = 0$$

$$t^2 x'' + t(tp(t))x' + t^2 q(t)x = 0$$

If it has a ***regular singular point*** at $t = 0$, then we can write out

$$\begin{aligned} tp(t) &= \sum_{j=0}^{\infty} p_j t^j \\ t^2 q(t) &= \sum_{j=0}^{\infty} q_j t^j \end{aligned}$$

> p_j and q_j are known constants for us

We choose the ***Frobenius ansatz***

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k \qquad a_0 \neq 0$$

Accordingly,

$x'(t) = \sum_{k=0}^{\infty} (r+k) a_k t^{r+k-1}$

$x''(t) = \sum_{k=0}^{\infty} (r+k)(r+k-1) a_k t^{r+k-2}$

Plug back into the equations we then get

$$\left(r(r-1) + p_0 r + q_0\right) a_0 = 0$$

$$\left((r+m)(r+m-1) + q_0 + (r+m)p_0\right) a_m + + \sum_{k=0}^{m-1} \left(q_{m-k} + (r+k)p_{m-k}\right) a_k = 0 \qquad m \geq 1$$

Setting

$$F(x) := x(x-1) + p_0 x + q_0$$

We get the ***indicial equation*** and ***recurrence equations*** to solve for a_k

$$\begin{aligned} F(r) &= 0 \\ a_m F(r+m) &= -\sum_{k=0}^{m-1} \left(q_{m-k} + (r+k)p_{m-k}\right) a_k, \quad m \geq 1 \end{aligned}$$

With the recurrence equations, you can usually generate out a easier recurrence equation.

For good and different r_i solved by the indical equation, llus some assumed initial values for a_0, a_1, ..., we are possible to solve for all a_k.

If everything goes fine, with $r_1 \neq r_2$ are two GOOD solutions, you get two INDEPENDENT solutions.

> **Question**
>
> Find the series solution to the below equation in the vicinity of $x_0 = 0$

> $$2x^2y'' + 7x(x+1)y' - 3y = 0$$

> **Answer**

5.2 Find a Second Independent Solution

5.2.1 Problem

But things can go wrong if $r_1 = r_2 + N$, $N \in \mathbb{N}$

- $r_1 = r_2$: then need further work to obtain another solution
- $r_1 = r_2 + N$, $N \in \mathbb{N}^+$: then though r_1 gives a solution, for r_2, due to $F(r_2 + N) = F(r_1) = 0$,
  - if the right-side of the recurrence equation vanishes for $F(r_2 + m) = F(r_2 + N)$, then a_N is arbitrary, by setting a_N as zero when dealing with r_1(but you may not be able to do this), and as an arbitrary non-zero number when dealing with r_2, we may further find a second independent solution. Though we can also use another general method
  - if the right-side of the recurrence equation doesn't vanish, need further work to obtain another solution

5.2.2 One Possible Solution

The recurrence equations can **give a relationship** $a_k(r)$, where you can view a_k as a function of r. Then we have

$$x_2(t) = \frac{\partial}{\partial r}\left(t^r \sum_{k=0}^{\infty} a_k(r) t^k\right)\Bigg|_{r=r_2} = c \cdot x_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} a_k'(r_2)\, t^k$$

where the constant $c \in R$ may vanish. If $r_1 = r_2$, then $c = 1$.

And a tricky way to find $a_{2k}'(r_2)$ is to use

$$\frac{a_{2k}'(r)}{a_{2k}(r)} = \frac{d}{dr} \ln|a_{2k}(r)|$$

6 Bessel Equations of Order v

$$x^2 y'' + xy' + \left(x^2 - v^2\right) y = 0$$

> Having a regular singular point at 0.
>
> The Method of Frobenius can be applied.

6.1 Find the Indical and Recurrence Equations

Choose the ***Frobenius ansatz***

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k \qquad a_0 \neq 0$$

Besides,

$$xp(x) = 1, \qquad p_0 = 1$$

$$x^2 q(x) = x^2 - v^2, \qquad q_0 = -v^2, \quad q_2 = 1$$

Setting

$$F(x) := x(x-1) + p_0 x + q_0 = x^2 - v^2$$

We get the ***indicial equation*** and ***recurrence equations***

$$\begin{aligned} F(r) &= r^2 - v^2 = 0 \\ a_m F(r+m) &= -\sum_{k=0}^{m-1} \left(q_{m-k} + (r+k) p_{m-k}\right) a_k, \quad m \geq 1 \end{aligned}$$

Which gives us

$$\begin{aligned} & r^2 - v^2 = 0 \\ & a_1\left((r+1)^2 - v^2\right) = 0 \\ & a_m = -\frac{a_{m-2}}{(m+r+v)(m+r-v)}, \quad m \geq 2 \end{aligned}$$

It obviously turns out $r_1 = v$ and $r_2 = -v$.

> If $r_1 - r_2 = 2v \notin \mathbb{N}$, then r_1 and r_2 give two independent solutions.
>
> But for ***Bessel Equations***, the condition is slightly ***less strict***:
>
> ***If $v \notin \mathbb{N}$, then r_1 and r_2 give two independent solutions.***

6.2 Find the First Independent Solution

6.2.1 Find the First Independent Solution with the Larger r_1

With the ***LARGER*** $r_1 = v$, we have

$$\begin{aligned} & a_1\left((v+1)^2 - v^2\right) = 0 \\ & a_m = -\frac{a_{m-2}}{(m+2v)m}, \quad m \geq 2 \end{aligned}$$

So $a_1 = a_3 = a_5 = \cdots = 0$ and

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1+v)(2+v)\cdots(k+v)}$$

6.2.2 The Bessel Function of the First Kind

Recall ***Euler Gamma function***'s property:

$$\Gamma(s+1) = s\Gamma(s)$$

So it gives

$$(1+v)(2+v)\cdots(k+v) = \frac{\Gamma(k+1+v)}{\Gamma(1+v)}$$

And by setting $a_0 = \frac{2^{-v}}{\Gamma(1+v)}$, we will have the first independent solution be ***the Bessel function of the first kind of order*** v

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+v)} \left(\frac{x}{2}\right)^{2k}$$

Take $v = 1$ as example, we have

$$J_1(x) = \frac{x}{2} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k+1)!k!}$$

6.3 Find the Second Independent Solution ($v \notin \mathbb{N}$)

<u>Starting from if $2v$ ***is not an integer***</u>, with the ***SMAllER*** $r_2 = -v$, we have

$$\begin{aligned} &a_1((v-1)^2 - v^2) = 0, \quad a_1(2v-1) = 0 \\ &a_m = -\frac{a_{m-2}}{(m-2v)m}, \quad m \geq 2 \end{aligned}$$

We have $a_1 = a_3 = a_5 = \cdots = 0$ and

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1-v)(2-v)\cdots(n-v)}$$

Similarly,

$$(1-v)(2-v)\cdots(k-v) = \frac{\Gamma(k+1-v)}{\Gamma(1-v)}$$

And by setting $a_0 = \frac{2^{-v}}{\Gamma(1+v)}$, the second independent solution will be ***the Bessel function of the first kind of negative order*** $-v$

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1-v)} \left(\frac{x}{2}\right)^{2k}$$

Then the <u>***general solution***</u> is

$$y(x) = C_1 J_v(x) + C_2 J_{-v}(x)$$

But actually, If $2v$ ***is an odd integer***, which means v ***is not an integer***, the above results also holds.

And the combined conclusion is ***if v is not an integer, the above results will hold***.

6.3.1 Another Example: $v = \frac{1}{2}$

Recall what you have seen in class with $v = \frac{1}{2}$, you are "lucky" enough to find a second independent solution directly with $r_2 = -\frac{1}{2}$. (Exactly the case where $2v \in \mathbb{N}$ but $v \notin \mathbb{N}$!)

Which is in slide 533, and there actually exsits a small typo.

You use $r_1 = \frac{1}{2}$ to get the Bessel function of the first kind of order 1/2 $J_{1/2} = \sqrt{\frac{2}{\pi t}} \sin t$ and use $r_2 = -\frac{1}{2}$ to get the Bessel function of the second kind of order 1/2 $Y_{1/2}(t) = \sqrt{\frac{2}{\pi t}} \cos t$ (Notice the minus sign!). Actually,

$$J_{\frac{1}{2}}(x) = Y_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

$$J_{-\frac{1}{2}}(x) = -Y_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

6.4 Find the Second Independent Solution ($v \in \mathbb{N}$)

6.4.1 Reduction of Order

Set $y_2(x) = c(x) \cdot J_\nu(x)$, then

$$\begin{aligned}
&x^2 y_2'' + x y_2' + \left(x^2 - \nu^2\right) y_2 = 0 \\
\Rightarrow &x^2 \left(c''(x) J_\nu(x) + 2c'(x) J_\nu'(x) + c(x) J_\nu''(x)\right) \\
&+ x\left(c'(x) J_\nu(x) + c(x) J_\nu(x)\right) + \left(x^2 - \nu^2\right) c(x) \cdot J_\nu(x) = 0 \\
\Rightarrow &x^2 J_\nu(x) c''(x) + \left(2x^2 J_\nu'(x) + x J_\nu(x)\right) c'(x) = 0 \\
\Rightarrow &\ln|c'(x)| = \left(-2\ln|J_\nu(x)| - \ln|x|\right) \\
\Rightarrow &c'(x) = \frac{1}{x \cdot J_\nu^2(x)} \\
\Rightarrow &c(x) = \int \frac{dx}{x \cdot J_\nu^2(x)}
\end{aligned}$$

So a second independent solution is given as

$$y_2(x) = J_\nu(x) \int \frac{dx}{x \cdot J_\nu^2(x)}$$

6.4.2 The Second Method only for $v = 0$

$$x_2(t) = \frac{\partial}{\partial r}\left(t^r \sum_{k=0}^{\infty} a_k(r) t^k\right)\Bigg|_{r=r_2} = c \cdot x_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} a_k'(r_2) t^k$$

$$\frac{a_{2k}'(r)}{a_{2k}(r)} = \frac{d}{dr}\ln|a_{2k}(r)|$$

> Will fail except for $v = 0$, because $\frac{\partial}{\partial r}\left(t^r \sum_{k=0}^{\infty} a_k(r)t^k\right)$ has no definition at $r = r_2$

6.4.3 The Third Method

Let's find these new constants in another way. Using the "ansatz"

$$y_2(x) = aJ_v(x)\ln x + x^{-v}\left[\sum_{k=0}^{\infty} c_k x^k\right], \quad x > 0$$

Computing $y_2\prime$, $y_2\prime\prime(x)$, substituting in the original Bessel Equation, and make use of $J_v(x)$ is a solution(as we have done by reduction of order), we can obtain all the constants $a, c_0, c_1, \ldots$

For example, if you try with order 1, where you also choose $c_2 = \frac{1}{2^2}$, you would get $c_1 = c_3 = \cdots = 0$ and:

$$c_{2m} = \frac{(-1)^{m+1}\left(H_m + H_{m-1}\right)}{2^{2m} m!(m-1)!}$$

Where $H_m(x) := \sum_{i=1}^{m} \frac{1}{i}$, $H_0 = 0$, is the Harmonic Numbers. In conclusion:

$$y_2(x) = -J_1(x)\ln x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m\left(H_m + H_{m-1}\right)}{2^{2m} m!(m-1)!} x^{2m}\right], \quad x > 0$$

6.4.4 The Bessel Function of the Second Kind

Actually the second independent solution of Bessel Equations can be a more beautiful form: ***the Bessel function of the second kind of order*** v, which is some linear combinition of $J_v(x)$ and a second independent solution $y_2(x)$ we find. In our specific case for $y_2(x)$ of order 1, we set ***the Bessel function of the second kind of order 1*** as

$$Y_1(x) = \frac{2}{\pi}[-y_2(x) + (\gamma - \ln 2)J_1(x)]$$

But, in practice, ***the Bessel function of the second kind of order*** v can be found from $J_v(x)$ and $J_{-v}(x)$:

$$Y_v(x) = \frac{J_v(x)\cos\pi v - J_{-v}(x)}{\sin\pi v}$$

And then the ***general solution*** can be written as

$$y(x) = C_1 J_v(x) + C_2 Y_v(x)$$

7 Transform Differential Equations to Bessel Equation

> **Key Take-away:**
>
> - $u = u(x) = \frac{y}{f(x)}$, f is a known function

$$\frac{d^2y}{dx^2} = \frac{d^2(f(x)u(x))}{dx^2} = \frac{d(f'(x)u(x) + f(x)u'(x))}{dx}$$

- $z = z(x)$, z is a known function

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{d^2z}{dx^2}\right)$$

7.1 $x^2y'' + xy' + \left(a^2x^2 - v^2\right)y = 0$

Exercise:

Transform this equation to a Bessel equation of order v

7.2 $x^2y'' + axy' + \left(x^2 - v^2\right)y = 0$

Exercise:

Transform this equation to a Bessel equation using the substitution $y(x) = x^{\frac{1-a}{2}}z(x)$. What's the order?

7.3 $y'' - xy = 0$

Exercise:

Show that the general solution of this equation can be expressed as

$y(x) = C_1\sqrt{x}J_{\frac{1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right) + C_2\sqrt{x}J_{-\frac{1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right)$

Hint:

be careful with $\frac{d^2y}{dx^2}$