Q1

① $f(z) = \frac{\cos z - 1}{z^2} = \frac{1}{z^2}\left[\sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n}}{(2n)!} - 1\right]$

$= \sum_{n=1}^{+\infty} \frac{(-1)^n z^{2(n-1)}}{(2n)!}$

$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots$

which actually don't contain z^{-n}.

So it's a removable singularity.

(a more intuitive example: $f(z) = \frac{z^2}{z^2}$, let $f(0) = 1$.)

② $\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{-\sin z}{2z} = -\frac{1}{2}$

Q2

Let $\varphi(z) = \frac{1}{f(z)} = 1 - \cos z$

Find all the zeroes : $z_k = 2k\pi \quad (k \in \mathbb{Z})$

Notice $\varphi'(z_k) = \sin z \big|_{z=2k\pi} = 0$

$\varphi''(z_k) = \cos z \big|_{z=2k\pi} = 1 \neq 0$

So actually z_k is a zero of order 2.

So z_k is a pole of order 2.

Q3

① $z=0, \pm i, 2$ are poles

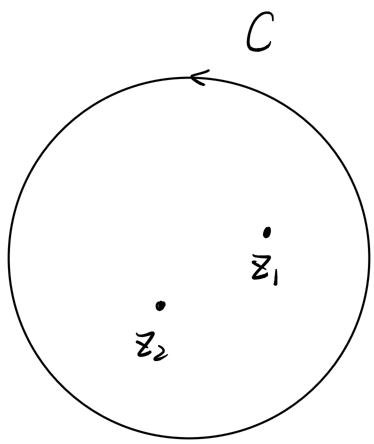
② At $z=0$: $\mathrm{Res}_{z_0} f = \lim_{z\to 0} z f(z) = \left.\dfrac{1}{(z^2+1)(z-2)^2}\right|_{z=0} = \dfrac{1}{4}$

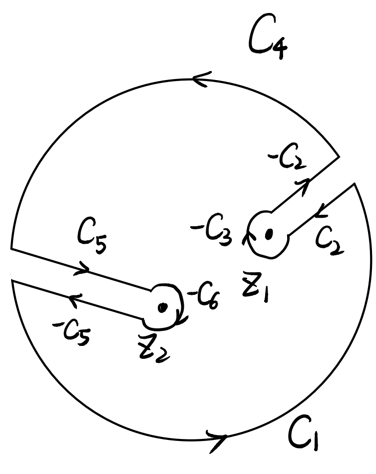
At $z=i$: $\mathrm{Res}_{z_1} f = \lim_{z\to i} (z-i) f(z) = \left.\dfrac{1}{z(z+i)(z-2)^2}\right|_{z=i} = \cdots$

At $z=-i$ $\mathrm{Res}_{z_2} f = \left.\dfrac{1}{z(z-i)(z-2)^2}\right|_{z=-i} =$

At $z=2$: $\mathrm{Res}_{z_3} f = \lim_{z\to 2}\left[\dfrac{1}{z(z^2+1)}\right]' = -\dfrac{13}{100}$

Q4





$$\tilde{C} = C_1 + C_2 - C_3 - C_2 + C_4 + C_5 - C_6 - C_5 = C_1 + C_4 - (C_3 + C_6)$$

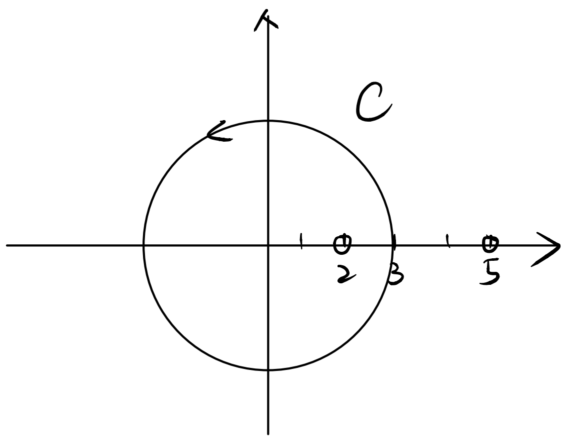
$$\int_{\tilde{C}} f(z)dz = \int_{C_1+C_4} f(z)dz - \int_{C_3} f(z)dz - \int_{C_6} f(z)dz = 0$$

$$\int_C f(z)dz = \int_{C_1+C_4} f(z)dz = \int_{C_3} f(z)dz + \int_{C_6} f(z)dz$$

$$= 2\pi i \mathrm{Res}(f, z_1) + 2\pi i \mathrm{Res}(f, z_2)$$

Q5

Let $f(z) = \frac{1}{(z-2)(z-5)}$, poles $z=2, z=5$



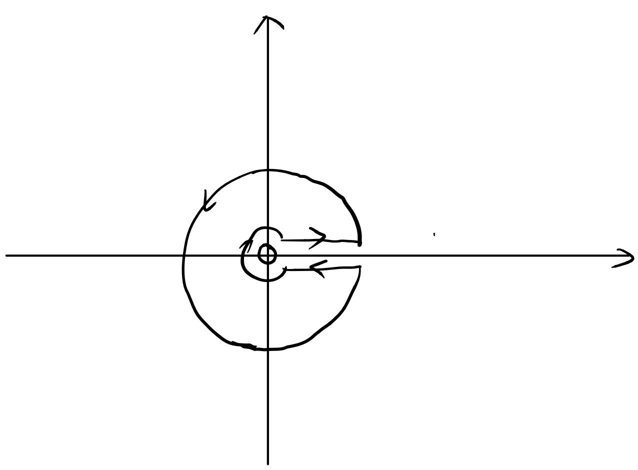
C contains "2".

$$\int_C f(z)\,dz = 2\pi i \sum Res = 2\pi i \cdot Res_2$$

With $Res_2 = \lim_{z \to 2} (z-2)\frac{1}{(z-2)(z-5)} = \left.\frac{1}{z-5}\right|_{z=2} = -\frac{1}{3}$

So $\int_C \frac{1}{(z-2)(z-5)} = -\frac{2\pi i}{3}$.

✱ About logarithm



Q6

Consider the function

$$f(z) = \frac{1}{1+z^2},$$

which is holomorphic in the complex plane except for simple poles at the points i and $-i$. Also, we choose the contour γ_R shown in Figure 1. The contour consists of the segment $[-R, R]$ on the real axis and of a large half-circle centered at the origin in the upper half-plane.
Since we may write

$$f(z) = \frac{1}{(z-i)(z+i)}$$

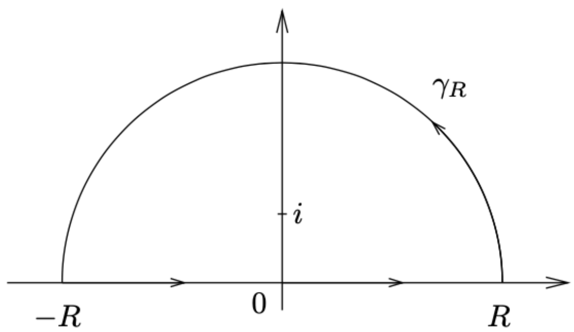


Figure 1. The contour γ_R in Example 1

$$|f(z)| = \left|\frac{1}{1+z^2}\right| = \frac{1}{|1+z^2|} < \frac{1}{|z|^2} = \frac{1}{R^2}$$

we see that the residue of f at i is simply $1/2i$. Therefore, if R is large enough, we have

$$\int_{\gamma_R} f(z)\,dz = \frac{2\pi i}{2i} = \pi.$$

If we denote by C_R^+ the large half-circle of radius R, we see that

$$\left|\int_{C_R^+} f(z)\,dz\right| \le \pi R \frac{B}{R^2} \le \frac{M}{R},$$

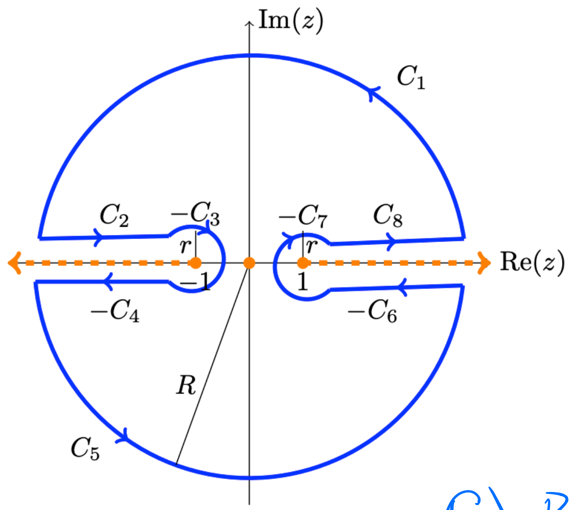
where we have used the fact that $|f(z)| \le B/|z|^2$ when $z \in C_R^+$ and R is large. So this integral goes to 0 as $R \to \infty$. Therefore, in the limit we find that

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi,$$

as desired. We remark that in this example, there is nothing special about our choice of the semicircle in the upper half-plane. One gets the same conclusion if one uses the semicircle in the lower half-plane, with the other pole and the appropriate residue.

Q7

set $f(z) = \frac{1}{z\sqrt{z^2-1}}$



$\mathbb{C} \setminus \mathbb{R}_+$

We use the branch cut for square root that removes the positive real axis. In this branch

$$0 < \arg(z) < 2\pi \quad \text{and} \quad 0 < \arg(\sqrt{w}) < \pi.$$

For $f(z)$, this necessitates the branch cut that removes the rays $[1, \infty)$ and $(-\infty, -1]$ from the complex plane.

The pole at $z = 0$ is the only singularity of $f(z)$ inside the contour. It is easy to compute that

$$\operatorname{Res}(f, 0) = \frac{1}{\sqrt{-1}} = \frac{1}{i} = -i.$$

$\left.\frac{1}{\sqrt{x^2-1}}\right|_{x=0} = \frac{1}{\sqrt{-1}} = \frac{1}{i} = -i$

So, the residue theorem gives us

$$\int_{C_1+C_2-C_3-C_4+C_5-C_6-C_7+C_8} f(z)\, dz = 2\pi i \operatorname{Res}(f, 0) = 2\pi. \tag{2}$$

In a moment we will show the following limits

$$\lim_{R\to\infty} \int_{C_1} f(z)\, dz = \lim_{R\to\infty} \int_{C_5} f(z)\, dz = 0$$
$$\lim_{r\to 0} \int_{C_3} f(z)\, dz = \lim_{r\to 0} \int_{C_7} f(z)\, dz = 0.$$

We will also show

$$\lim_{R\to\infty, r\to 0} \int_{C_2} f(z)\, dz = \lim_{R\to\infty, r\to 0} \int_{-C_4} f(z)\, dz$$
$$= \lim_{R\to\infty, r\to 0} \int_{-C_6} f(z)\, dz = \lim_{R\to\infty, r\to 0} \int_{C_8} f(z)\, dz = I.$$

Using these limits, Equation 2 implies $4I = 2\pi$, i.e.

$$\boxed{I = \pi/2.}$$

All that's left is to prove the limits asserted above.

limits for C_1, C_5 for large z is $\frac{1}{|z|^{\frac{3}{2}}}$.

easy to check.

We get the limit for C_3 as follows. Suppose r is small, say much less than 1. If

$$z = -1 + re^{i\theta}$$

is on C_3 then,

$$|f(z)| = \frac{1}{|z\sqrt{z-1}\sqrt{z+1}|} = \left(\frac{1}{|-1+re^{i\theta}|\sqrt{|-2+re^{i\theta}|}}\right)\frac{1}{\sqrt{r}} \le \frac{M}{\sqrt{r}}.$$

where M is chosen to be bigger than

$$\frac{1}{|-1+re^{i\theta}|\sqrt{|-2+re^{i\theta}|}}$$

for all small r.

Thus,

$$\left|\int_{C_3} f(z)\,dz\right| \le \int_{C_3} \frac{M}{\sqrt{r}}\,|dz| \le \frac{M}{\sqrt{r}} \cdot 2\pi r = 2\pi M\sqrt{r}.$$

This last expression clearly goes to 0 as $r \to 0$.

The limit for the integral over C_7 is similar.

We can parameterize the straight line C_8 by

$$z = x + i\epsilon,$$

where ϵ is a small positive number and x goes from (approximately) 1 to ∞. Thus, on C_8, we have

$$\arg(z^2-1) \approx 0 \quad \text{and} \quad f(z) \approx f(x).$$

All these approximations become exact as $r \to 0$. Thus,

$$\lim_{R\to\infty,\, r\to 0} \int_{C_8} f(z)\,dz = \int_1^\infty f(x)\,dx = I.$$

We can parameterize $-C_6$ by

$$z = x - i\epsilon$$

where x goes from ∞ to 1. Thus, on C_6, we have

$$\arg(z^2-1) \approx 2\pi,$$

so

$$\sqrt{z^2-1} \approx -\sqrt{x^2-1}.$$

This implies

$$f(z) \approx -\frac{1}{x\sqrt{x^2-1}} = -f(x).$$

Thus,

$$\lim_{R\to\infty, r\to 0} \int_{-C_6} f(z)\, dz = \int_{\infty}^{1} -f(x)\, dx = \int_{1}^{\infty} f(x)\, dx = I.$$

We can parameterize C_2 by $z = -x + i\epsilon$ where x goes from ∞ to 1. Thus, on C_2, we have

$$\arg(z^2 - 1) \approx 2\pi,$$

so

$$\sqrt{z^2 - 1} \approx -\sqrt{x^2 - 1}.$$

This implies

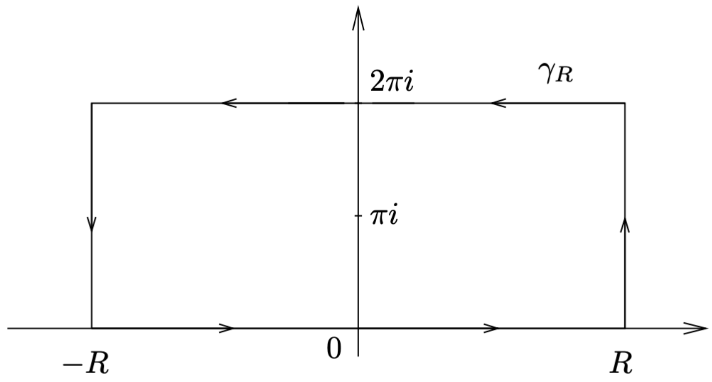
$$f(z) \approx \frac{1}{(-x)(-\sqrt{x^2 - 1})} = f(x).$$

Thus,

$$\lim_{R\to\infty, r\to 0} \int_{C_2} f(z)\, dz = \int_{\infty}^{1} f(x)\, (-dx) = \int_{1}^{\infty} f(x)\, dx = I.$$

The last curve $-C_4$ is handled similarly.

Q8



The only point in the rectangle γ_R where the denominator of f vanishes is $z = \pi i$. To compute the residue of f at that point, we argue as follows: First, note

$$(z - \pi i)f(z) = e^{az}\frac{z - \pi i}{1 + e^z} = e^{az}\frac{z - \pi i}{e^z - e^{\pi i}}.$$

We recognize on the right the inverse of a difference quotient, and in fact

$$\lim_{z \to \pi i} \frac{e^z - e^{\pi i}}{z - \pi i} = e^{\pi i} = -1$$

since e^z is its own derivative. Therefore, the function f has a simple pole at πi with residue

$$\mathrm{res}_{\pi i} f = -e^{a\pi i}.$$

As a consequence, the residue formula says that

$$\int_{\gamma_R} f = -2\pi i e^{a\pi i}. \tag{3}$$

We now investigate the integrals of f over each side of the rectangle. Let I_R denote

$$\int_{-R}^{R} f(x)\,dx$$

and I the integral we wish to compute, so that $I_R \to I$ as $R \to \infty$. Then, it is clear that the integral of f over the top side of the rectangle (with the orientation from right to left) is

$$-e^{2\pi ia}I_R.$$

Finally, if $A_R = \{R + it : \ 0 \le t \le 2\pi\}$ denotes the vertical side on the right, then

$$\left|\int_{A_R} f\right| \le \int_0^{2\pi} \left|\frac{e^{a(R+it)}}{1+e^{R+it}}\right| dt \le Ce^{(a-1)R},$$

and since $a < 1$, this integral tends to 0 as $R \to \infty$. Similarly, the integral over the vertical segment on the left goes to 0, since it can be bounded by Ce^{-aR} and $a > 0$. Therefore, in the limit as R tends to infinity, the identity (3) yields

$$I - e^{2\pi ia}I = -2\pi i e^{a\pi i},$$

from which we deduce

$$\begin{aligned} I &= -2\pi i \frac{e^{a\pi i}}{1 - e^{2\pi ia}} \\ &= \frac{2\pi i}{e^{\pi ia} - e^{-\pi ia}} \\ &= \frac{\pi}{\sin \pi a}, \end{aligned}$$

and the computation is complete.