

Note8 Bessel Equations

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1 Bessel Equations of Order ν

1. 1 Find the Indicial and Recurrence Equations
1. 2 Find the First Independent Solution
 1. 2. 1 Find the First Independent Solution with the Larger r_1
 1. 2. 2 The Bessel Function of the First Kind
1. 3 Find the Second Independent Solution ($\nu \notin \mathbb{N}$)
1. 4 Find the Second Independent Solution ($\nu \in \mathbb{N}$)
 1. 4. 1 Reduction of Order
 1. 4. 2 The Other Method
 1. 4. 3 The Bessel Function of the Second Kind

2 Reduce Differential Equations to Bessel Equation

2. 1 $x^2 y'' + xy' - (x^2 + \nu^2) y = 0$
2. 2 $x^2 y'' + xy' + (a^2 x^2 - \nu^2) y = 0$
2. 3 $x^2 y'' + axy' + (x^2 - \nu^2) y = 0$
2. 4 * For thoughts: $y'' - xy = 0$

Let's apply the Method of Frobenius to solve Bessel equations.

And analyze the solutions (Bessel functions).

1 Bessel Equations of Order ν

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$$

Having a regular singular point at 0.

The Method of Frobenius can be applied.

1.1 Find the Indicial and Recurrence Equations

Choose the **Frobenius ansatz**

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k \quad a_0 \neq 0$$

Besides,

$$xp(x) = 1, \quad p_0 = 1$$

$$x^2 q(x) = x^2 - \nu^2, \quad q_0 = -\nu^2, \quad q_2 = 1$$

Setting

$$F(x) := x(x-1) + p_0x + q_0 = x^2 - v^2$$

We get the **indicial equation** and **recurrence equations**

$$F(r) = r^2 - v^2 = 0$$
$$a_m F(r+m) = - \sum_{k=0}^{m-1} (q_{m-k} + (r+k)p_{m-k}) a_k, \quad m \geq 1$$

Which gives us

$$r^2 - v^2 = 0$$
$$a_1((r+1)^2 - v^2) = 0$$
$$a_m = - \frac{a_{m-2}}{(m+r+v)(m+r-v)}, \quad m \geq 2$$

It obviously turns out $r_1 = v$ and $r_2 = -v$.

From the result in class we know if $r_1 - r_2 = 2v \notin \mathbb{N}$, two independent solutions would be found easily.

And if $r_1 - r_2 = 2v \in \mathbb{N}$, we may use the special technique.

However, we will see actually for **Bessel Equations**, the condition is slightly **less strict**:

If $v \notin \mathbb{N}$, then r_1 and r_2 give two independent solutions.

1.2 Find the First Independent Solution

1.2.1 Find the First Independent Solution with the Larger r_1

With the **LARGER** $r_1 = v$, we have

$$a_1((v+1)^2 - v^2) = 0$$
$$a_m = - \frac{a_{m-2}}{(m+2v)m}, \quad m \geq 2$$

So $a_1 = a_3 = a_5 = \dots = 0$ and

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (1+v)(2+v) \cdots (k+v)}$$

Question:

Notice v may not be an integer. Don't write as factories.

Then how do you simplify this solution?

1.2.2 The Bessel Function of the First Kind

Recall **Euler Gamma function's** property:

$$\Gamma(s+1) = s\Gamma(s)$$

So it gives

$$(1+v)(2+v)\cdots(k+v) = \frac{\Gamma(k+1+v)}{\Gamma(1+v)}$$

And by setting $a_0 = \frac{2^{-v}}{\Gamma(1+v)}$, we will have the first independent solution be **the Bessel function of the first kind of order v**

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+v)} \left(\frac{x}{2}\right)^{2k}$$

Question:

Which region of x does $J_v(x)$ defined?

Take $v = 1$ as example, we have

$$J_1(x) = \frac{x}{2} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k+1)! k!}$$

1.3 Find the Second Independent Solution ($v \notin \mathbb{N}$)

Starting from if $2v$ **is not an integer**, with the **SMALLER** $r_2 = -v$, we have

$$\begin{aligned} a_1((v-1)^2 - v^2) &= 0, & a_1(2v-1) &= 0 \\ a_m &= -\frac{a_{m-2}}{(m-2v)m}, & m &\geq 2 \end{aligned}$$

We have $a_1 = a_3 = a_5 = \cdots = 0$ and

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (1-v)(2-v)\cdots(n-v)}$$

Similarly,

$$(1-v)(2-v)\cdots(k-v) = \frac{\Gamma(k+1-v)}{\Gamma(1-v)}$$

And by setting $a_0 = \frac{2^{-v}}{\Gamma(1-v)}$, the second independent solution will be **the Bessel function of the first kind of negative order $-v$**

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1-v)} \left(\frac{x}{2}\right)^{2k}$$

Then the **general solution** is

$$y(x) = C_1 J_v(x) + C_2 J_{-v}(x)$$

But actually, If 2ν **is an odd integer**, which means ν **is not an integer**, the above results also holds.

And the combined conclusion is if ν is not an integer, the above results will hold.

1.4 Find the Second Independent Solution ($\nu \in \mathbb{N}$)

1.4.1 Reduction of Order

Set $y_2(x) = c(x) \cdot J_\nu(x)$, then

$$\begin{aligned} x^2 y_2'' + x y_2' + (x^2 - \nu^2) y_2 &= 0 \\ \Rightarrow x^2 (c''(x) J_\nu(x) + 2c'(x) J_\nu'(x) + c(x) J_\nu''(x)) \\ &+ x (c'(x) J_\nu(x) + c(x) J_\nu'(x)) + (x^2 - \nu^2) c(x) \cdot J_\nu(x) = 0 \\ \Rightarrow x^2 J_\nu(x) c''(x) + (2x^2 J_\nu'(x) + x J_\nu(x)) c'(x) &= 0 \\ \Rightarrow \ln|c'(x)| = (-2 \ln|J_\nu(x)| - \ln|x|) \\ \Rightarrow c'(x) &= \frac{1}{x \cdot J_\nu^2(x)} \\ \Rightarrow c(x) &= \int \frac{dx}{x \cdot J_\nu^2(x)} \end{aligned}$$

So a second independent solution is given as

$$y_2(x) = J_\nu(x) \int \frac{dx}{x \cdot J_\nu^2(x)}$$

1.4.2 The Other Method

$$\begin{aligned} x_2(t) &= \frac{\partial}{\partial r} \left(t^r \sum_{k=0}^{\infty} a_k(r) t^k \right) \Big|_{r=r_2} = c \cdot x_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} a'_k(r_2) t^k \\ \frac{a'_{2k}(r)}{a_{2k}(r)} &= \frac{d}{dr} \ln|a_{2k}(r)| \end{aligned}$$

Practice:

Using 5 minutes to try solving out the second solution by yourself.

Do you find any problems?

Instead of computing $a'_{2k}(r_2)$, let's find these new constants in another way. Assume

$$y_2(x) = a J_\nu(x) \ln x + x^{-1} \left[\sum_{k=0}^{\infty} c_k x^k \right], \quad x > 0$$

Computing y_2' , $y_2''(x)$, substituting in the original Bessel Equation, and make use of $J_v(x)$ is a solution(as we have done by reduction of order), we can obtain all the constants a, c_0, c_1, \dots

Let's try with the Bessel Equation of order 1. Notice $a_0(r) = 1$, so we can set $c_0 = 1$

$$y_2(x) = aJ_1(x) \ln x + x^{-1} \left[1 + \sum_{k=1}^{\infty} c_k x^k \right], \quad x > 0$$

Substituting back and we get

$$2axJ_1'(x) + \sum_{k=0}^{\infty} [(k-1)(k-2)c_k + (k-1)c_k - c_k] x^{k-1} + \sum_{k=0}^{\infty} c_k x^{k+1} = 0$$

Substituting for $J_1(x)$ then

$$-c_1 + [0 \cdot c_2 + c_0] x + \sum_{k=2}^{\infty} [(k^2 - 1) c_{k+1} + c_{k-1}] x^k = -a \left[x + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1) x^{2k+1}}{2^{2k} (k+1)! k!} \right]$$

This first gives us $c_1 = 0, c_0 = -a = 1$.

Further even powers on the left must vanish, so $(k^2 - 1) c_{k+1} + c_{k-1}$ must vanish for even k , and then $c_1 = c_3 = \dots = 0$.

And from odd powers on the left we have

$$[(2m+1)^2 - 1] c_{2m+2} + c_{2m} = \frac{(-1)^m (2m+1)}{2^{2m} (m+1)! m!}, \quad m = 1, 2, 3, \dots$$

When we set $m = 1$, we get

$$(3^2 - 1) c_4 + c_2 = (-1)3 / (2^2 \cdot 2!)$$

Hence, c_2 can be selected in arbitrary, and then we just gain the second independent solution.

In practice, we always choose $c_2 = \frac{1}{2^2}$, and then we would be possible to simplify:

$$c_{2m} = \frac{(-1)^{m+1} (H_m + H_{m-1})}{2^{2m} m! (m-1)!}$$

Where H_m is the **Hermite polynomials**, $H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$, $H_0 = 0$. So

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x} \left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} m! (m-1)!} x^{2m} \right], \quad x > 0$$

1.4.3 The Bessel Function of the Second Kind

Actually the second independent solution of Bessel Equations are written as **the Bessel function of the second kind of order v** , which can be some linear combination of $J_v(x)$ and the second independent solution $y_2(x)$. In our specific case here of order 1, we set **the Bessel function of the second kind of order 1** as

$$Y_1(x) = \frac{2}{\pi} [-y_2(x) + (\gamma - \ln 2) J_1(x)]$$

But, in practice, **the Bessel function of the second kind of order v** can be found from $J_v(x)$ and $J_{-v}(x)$:

$$Y_v(x) = \frac{J_v(x) \cos \pi v - J_{-v}(x)}{\sin \pi v}$$

And then the **general solution** can be written as

$$y(x) = C_1 J_v(x) + C_2 Y_v(x)$$

2 Reduce Differential Equations to Bessel Equation

2.1 $x^2 y'' + xy' - (x^2 + v^2) y = 0$

Exercise:

Show that the general solution of this equation can be expressed as

$$y(x) = C_1 J_v(-ix) + C_2 Y_v(-ix)$$

2.2 $x^2 y'' + xy' + (a^2 x^2 - v^2) y = 0$

Exercise:

Show that the general solution of this equation can be expressed as

$$y(x) = C_1 J_v(ax) + C_2 Y_v(ax)$$

2.3 $x^2 y'' + axy' + (x^2 - v^2) y = 0$

Exercise:

Show that the general solution of this equation can be expressed as

$$y(x) = x^{\frac{1-a}{2}} [C_1 J_n(x) + C_2 Y_n(x)]$$

Hint:

using the substitution $y(x) = x^{\frac{1-a}{2}} z(x)$

2.4 * For thoughts: $y'' - xy = 0$

*Exercise:

Show that the general solution of this equation can be expressed as

$$y(x) = C_1 \sqrt{x} J_{\frac{1}{3}} \left(\frac{2}{3} i x^{\frac{3}{2}} \right) + C_2 \sqrt{x} J_{-\frac{1}{3}} \left(\frac{2}{3} i x^{\frac{3}{2}} \right)$$