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Note7 Power Series Solutions

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> For **homogeneous linear ODEs** with **variable coefficients**, sometimes finding an explicit solution is difficult, then we use the method of **power series ansatz** to solve/approximate solutions.
>
> Recall: **homogeneous**, **linear**, **ordinary**, **variable coefficients**.

1 Summary of Power Series Ansatz

1. Analyze the equation, decide whether we can use power series ansatz around some point
2. Choose which form of ansatz to use
3. Plug into the ansatz, get recurrence relationship of the coefficients
4. Set initial value of coeffiencients. solve for coefficients to get one or more independent solutions
5. If not enough independent solutions are found, using reduction of order to find more solutions
6. Obtain the general solution

2 ODE with Analytic Coefficients

$$x\prime\prime + p(t)x\prime + q(t)x = 0$$

Where $P(t)$ and $Q(t)$ are **analytic in a neiborhood of** t_0.

> " a neighborhood of t_0" contains t_0

Then we can choose the ansatz

$$x(t) = \sum_{0}^{\infty} a_k (t - t_0)^k$$

Accordingly,

$x\prime(t) = \sum_0^\infty ka_k(t-t_0)^{k-1}$

$x\prime\prime(t) = \sum_0^\infty k(k-1)a_k(t-t_0)^{k-2}$

Plug the three equations back, we can obtain the relationship of the coefficients $\{a_0, a_1, a_2, ...\}$.

Depending on the situation, after setting values for first n terms (always 2), we can solve 1 to n (expected) independent solutions.

If not enough indepedent solutions are found, sometimes we can use reduction of order to find more.

> **Comments**:
>
> - The solutions found should be valid within its radius of convergence

> **Radius of Convergence of a Power Series**:
>
> - $\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$
> - $\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n}$

3 ODE with Coefficents having Singular Points

The general form of a **homogeneous linear second-order ODE** with **variable coefficients**:

$$P(t)x\prime\prime + Q(t)x\prime + R(t)x = 0$$

It is said to have a **singular point** at t_0 if $P(t_0) = 0$.

Generally around singular points, it's hard to decide or find continuous solutions. But there're some specific cases we can deal with.

3.1 Regular Singular Points

$$x\prime\prime + p(t)x\prime + q(t)x = 0$$

is said to have a ***regular singular point*** at t_0 if the functions $(t-t_0)p(t)$ and $(t-t_0)^2q(t)$ are analytic in a neighborhood of t_0. A singular point which is not regular is said to be ***irregular***.

The general claim is: if an equation has a regular sigular point at t_0, then we can assume $p(t) = \frac{p_{-1}}{t-t_0} + \sum_{j=0}^\infty p_j(t-t_0)^j$, $q(t) = \frac{q_{-2}}{(t-t_0)^2} + \frac{q_{-1}}{t-t_0} + \sum_{j=0}^\infty q_j(t-t_0)^j$ and use the ansatz $x(t) = (t-t_0)^r \sum_{k=0}^\infty a_k(t-t_0)^k$ to find solutions.

4 Euler's Equation

$$t^2x'' + \alpha tx' + \beta x = 0, \quad \alpha, \beta \in \mathbb{R}$$

> **Analysis:**
>
> This is exactly the case where the equation $x'' + \alpha\frac{1}{t}x' + \beta\frac{1}{t^2}x = 0, \quad \alpha, \beta \in \mathbb{R}$ is having a regular singular point at $t = 0$.

Then we can choose the ansatz

$$x(t) = t^r$$

Inserting back and solve for r we get

$$r = -\frac{\alpha - 1}{2} \pm \frac{1}{2}\sqrt{(\alpha - 1)^2 - 4\beta}$$

- $(\alpha - 1)^2 - 4\beta > 0$

$$x(t; c_1, c_2) = c_1 t^{r_1} + c_2 t^{r_2}, \quad c_1, c_2 \in \mathbb{R}$$

- $(\alpha - 1)^2 - 4\beta = 0$, $r_1 = r_2 = \frac{1-\alpha}{2}$, need to use reduction of order

$$x(t; c_1, c_2) = c_1 t^{r_1} + c_2 t^{r_1} \ln t, \quad c_1, c_2 \in \mathbb{R}$$

> **Reduction of order:**
>
> For equation $y\prime\prime + p(t)y\prime + q(t)y = 0$, and a known solution $y_1(x)$, let $y_2(x) = v(x)y_1(x)$, then you can solve for $v(x)$ using
>
> $$y_1(t)v\prime\prime + (2y_1\prime(t) + p(t)y_1(t))v\prime = 0$$

- $(\alpha - 1)^2 - 4\beta < 0$
  After getting $x_1(t) = t^{r_1} = t^{\lambda}(\cos(\mu \ln t) + i \sin(\mu \ln t))$.
  $x_2(t) = t^{r_1} = t^{\lambda}(\cos(\mu \ln t) - i \sin(\mu \ln t))$, further have

$$x(t; c_1, c_2) = c_1 t^{\lambda} \cos(\mu \ln t) + c_2 t^{\lambda} \sin(\mu \ln t), \quad c_1, c_2 \in \mathbb{R}$$

5 The Method of Frobenius

5.1 Basic Method

$$x\prime\prime + p(t)x\prime + q(t)x = 0$$

$$t^2 x\prime\prime + t(tp(t))x\prime + t^2 q(t)x = 0$$

If it has a ***regular singular point*** at $t = 0$, then we can write out

$$tp(t) = \sum_{j=0}^{\infty} p_j t^j$$
$$t^2 q(t) = \sum_{j=0}^{\infty} q_j t^j$$

> p_j and q_j are known constants for us

We choose the ***Frobenius ansatz***

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k \qquad\qquad a_0 \neq 0$$

Accordingly,

$x'(t) = \sum_{k=0}^{\infty} (r+k) a_k t^{r+k-1}$

$x''(t) = \sum_{k=0}^{\infty} (r+k)(r+k-1) a_k t^{r+k-2}$

Plug back into the equations we then get

$$\left(r(r-1) + p_0 r + q_0\right) a_0 = 0$$

$$\left((r+m)(r+m-1) + q_0 + (r+m)p_0\right) a_m + + \sum_{k=0}^{m-1} \left(q_{m-k} + (r+k)p_{m-k}\right) a_k = 0 \qquad\qquad m \geq 1$$

Setting

$$F(x) := x(x-1) + p_0 x + q_0$$

We get the ***indicial equation*** and ***recurrence equations*** to solve for a_k

$$\begin{aligned} F(r) &= 0 \\ a_m F(r+m) &= -\sum_{k=0}^{m-1} \left(q_{m-k} + (r+k)p_{m-k}\right) a_k, \quad m \geq 1 \end{aligned}$$

With the recurrence equations, you can usually generate out a easier recurrence equation.

For good and different r_i solved by the indical equation, llus some assumed initial values for a_0, a_1, ..., we are possible to solve for all a_k.

If everything goes fine, with $r_1 \neq r_2$ are two GOOD solutions, you get two INDEPENDENT solutions.

> **Question**
>
> Find the series solution to the below equation in the vicinity of $x_0 = 0$
>
> $$2x^2 y'' + 7x(x+1)y' - 3y = 0$$

> **Answer**

5.2 Find a Second Independent Solution

But things can go wrong if

- $r_1 = r_2$: then need further work to obtain another solution (how? Later.)
- $r_1 = r_2 + N$: then though r_1 gives a solution, for r_2, due to $F(r_2 + N) = F(r_1) = 0$,
  - if the right-side of the recurrence equation vanishes for $F(r_2 + m) = F(r_2 + N)$, then a_N is arbitrary, by setting a_N as zero when dealing with r_1, and as an arbitrary non-zero number when dealing with r_2, we can further find a second independent solution. Though we can also use another general method (how? Later.)
  - if the right-side of the recurrence equation doesn't vanish, need further work to obtain another solution (how? Later.)

> Noticing the above 3 cases have one thing in common: $r_1 = r_2 + N$, $N \in \mathbb{N}$ including 0. There's a general method for the above cases.

The recurrence equations can give a relationship $a_k(r)$. Then we have

$$x_2(t) = \frac{\partial}{\partial r}\left(t^r \sum_{k=0}^{\infty} a_k(r) t^k\right)\Bigg|_{r=r_2} = c \cdot x_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} a_k'(r_2)\, t^k$$

where the constant $c \in R$ may vanish. If $r_1 = r_2$, then $c = 1$.

And a tricky way to find $a_{2k}'(r_2)$ is to use

$$\frac{a_{2k}'(r)}{a_{2k}(r)} = \frac{d}{dr} \ln |a_{2k}(r)|$$