[redacted]. *Explain why any function $F(x)$ is a sum of an even function and an odd function in just one way. Hint:* $F_+(x) = \dfrac{F(x)+F(-x)}{2}$ *is even. What is the even part of e^x? What is the odd part?*

First, $F_+(x)$ is even: $F_+(-x) = \dfrac{F(-x)+F(x)}{2} = F_+(x)$.

Now notice that $F(x) = F_+(x) + F_-(x)$ where $F_-(x) = \dfrac{F(x)-F(-x)}{2}$. I claim that $F_-(x)$ is odd: $F_-(-x) = \dfrac{F(-x)-F(x)}{2} = -F_-(x)$.

So $F(x)$ *is* the sum of an even function and an odd function.

To show that this decomposition is unique, we suppose we have another decomposition $\tilde{F}_+(x) + \tilde{F}_-(x) = F(x)$, where $\tilde{F}_+(x)$ is even and $\tilde{F}_-(x)$ is odd.

Then $F_+(x) + F_-(x) = F(x) = \tilde{F}_+(x) + \tilde{F}_-(x)$, so $F_+(x) - \tilde{F}_+(x) = \tilde{F}_-(x) - F_=(x)$. But the left hand side is even and the right hand side is odd, so they both must be zero, which says that $F_+(x) = \tilde{F}_+(x)$ and $F_-(x) = \tilde{F}_-(x)$.

This decomposition might seem familiar from hyperbolic trig function formulas: The even part of e^x is $\dfrac{e^x + e^{-x}}{2} = \cosh x$, and the odd part of e^x is $\dfrac{e^x - e^{-x}}{2} = \sinh x$.

2

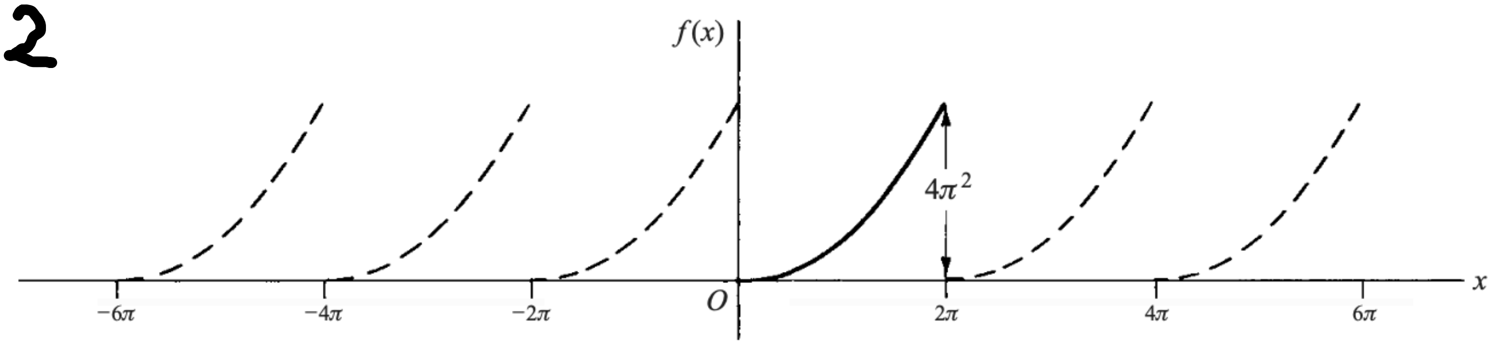

Fig. 13-7

Period $= 2L = 2\pi$ and $L = \pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$
$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2}, \qquad n \neq 0$$

If $n = 0$, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x^2\,dx = \dfrac{8\pi^2}{3}$.

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$
$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

Then $f(x) = x^2 = \dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$.

This is valid for $0 < x < 2\pi$. At $x = 0$ and $x = 2\pi$ the series converges to $2\pi^2$.

(*b*) If the period is not specified, the Fourier series cannot be determined uniquely in general.

3 Using the results of Problem 13.6, prove that $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$.

At $x = 0$ the Fourier series of Problem 13.6 reduces to $\dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2}$.

By the Dirichlet conditions, the series converges at $x = 0$ to $\frac{1}{2}(0 + 4\pi^2) = 2\pi^2$.

Then $\dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2} = 2\pi^2$, and so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

4

Expand $f(x) = x, 0 < x < 2$, in a half range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 13-12 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4$, $L = 2$.

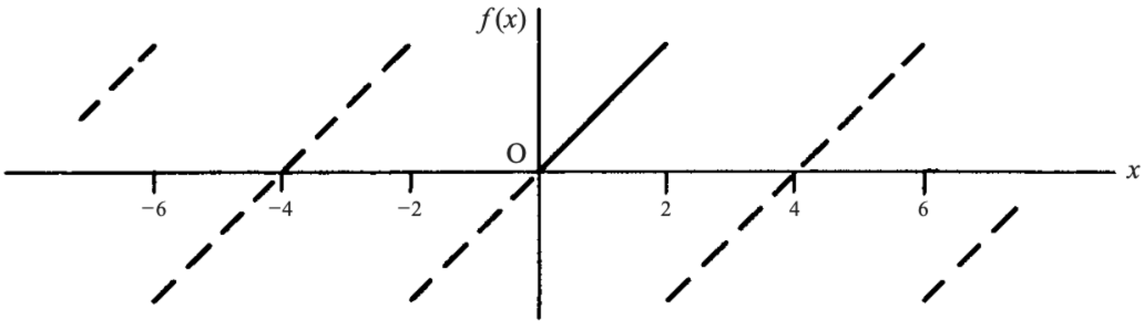

Fig. 13-12

Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$
$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then
$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$
$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(*b*) Extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 13-13 below. This is the *even extension* of $f(x)$. Then $2L = 4$, $L = 2$.

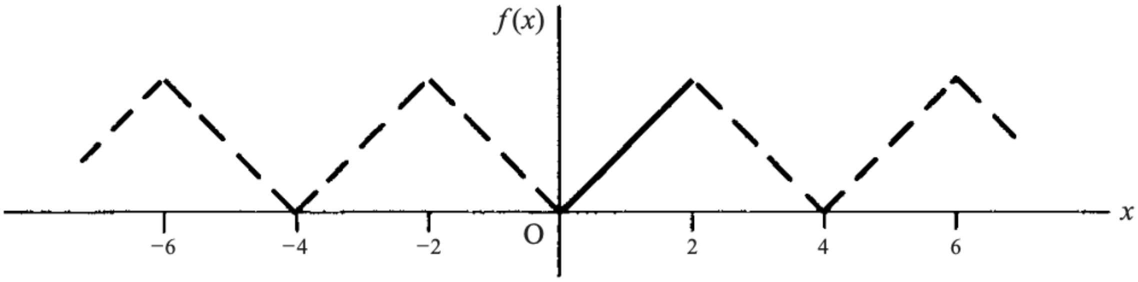

Fig. 13-13

Thus $b_n = 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx \\ &= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 \\ &= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{If } n \neq 0 \end{aligned}$$

If $n = 0$, $a_0 = \int_0^2 x\,dx = 2$.

Then
$$\begin{aligned} f(x) &= 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2} \\ &= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right) \end{aligned}$$

It should be noted that the given function $f(x) = x$, $0 < x < 2$, is represented *equally well* by the two *different* series in (*a*) and (*b*).