Introduction to Computer Graphics and Foundation Mathematics

Module 1. Lecture 2

Learning Objectives

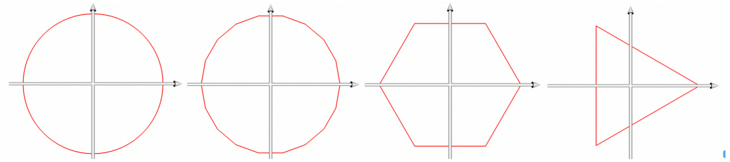
- To understand that definition of geometry with a computer requires using its digital representation in a form of coordinates as well as various mathematical functions defining the coordinates in given coordinate domains with selected sampling resolutions.
- To develop the skills of making images with raw mathematics and from first principles.
- The idea of “reasoning from first principles” is to break down complicated problems into basic elements and then reassemble them from the ground up.

Foundation Mathematics

- How to draw with a computer
- Coordinate systems
- Analytic functions
- Pythagorean theorem
- Trigonometry
- Matrices
- Vectors

How to draw with a computer

- Computers use digits, therefore we have to digitize anything we want to draw, and this will be a discrete representation sampling objects to be displayed within the computer precision limits.
- The digits can be obtained using various coordinate systems as a frame of reference and based on some mathematical models (formulas, equations, procedures, algorithms) which then have to be digitally sampled limited by the precision of the computer.
- It is always a task of interpolation or approximation based on a digital model used in the computer.



Foundation Mathematics

- How to draw with a computer
- Coordinate systems
- Functions
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Invention of Coordinates

- **Rene Descartes**, "La Geometrie" (1637). Started with geometric curves and produces their equations.
- Translated into Latin in 1649 by the name Cartesius.
- Revolutionized mathematics by providing the first systematic link between Euclidean geometry ("Elements" 300 BC) and algebra.
- Discovered an ability to define geometry by algebraic formulas.
- Invented system of measuring locations of points in space by numbers (coordinates) – signed numbers used to **uniquely** determine the position of a point or other geometric elements in the modeling space
- **Pierre de Fermat** started with an algebraic equation and then described the geometric curve which satisfied it.
- Cartesian coordinate system, polar coordinates, etc.
- Bridge between algebra and geometry was built
- Revolutionized geometry
- Analytic geometry, linear algebra, complex analysis, differential geometry, calculus, … and eventually computer graphics

Computer Graphics

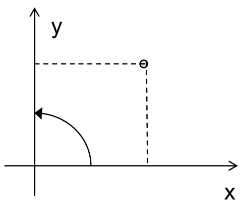
- Making images with a computer
- Printing 2D images and 3D models with a computer
- Visualization: from digital model of geometry to colors visible on the monitor or hardcopy
- Direct implementation of analytic geometry in computers
- Digital computers approximate since limited by precision of data representation

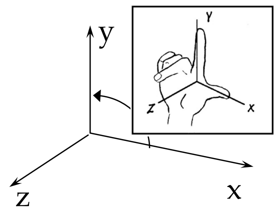
Coordinate Systems (T1 and all other)

- Coordinates are signed numbers used to **uniquely** determine the position of a point or other geometric elements in the modeling space
- Number of coordinates = dimension of space, i.e. plane – 2 coordinates, 3D space – 3 coordinates
- Coordinate systems:
  - Cartesian Coordinate System (2D and 3D).
  - Polar
  - Cylindrical
  - Spherical
- Time will be considered as yet another dimension (coordinate)

Cartesian Coordinate System

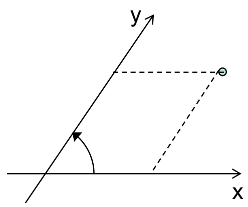
- Cartesian Coordinate System (2D and 3D). Cartesius is a Latinized name of Descartes.
- Coordinate axes are located at 90 degrees to each other
- Location of point P is defined by coordinates which are distances from the point to the axes measured along the lines orthogonal to the axes
- Cartesian coordinates are the foundation of analytic geometry linear algebra, complex analysis, differential geometry, multivariate calculus, group theory, and more.
- *Right-handed* 2D coordinate system: if rotation of the first axis towards the second axis is counter-clockwise.
- *Right-handed 3D coordinate system:* if the three vectors are situated like the thumb, index finger and middle finger pointing straight up from your palm. Also, rotation of the first axis towards the second axis is counter-clockwise as seen from the third axis. Also, while curling fingers from the first to the second axis, the extended thumb will show the direction of the third axis.

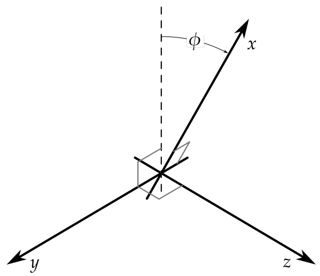




Non-orthogonal (Skewed) Coordinate Systems

- Coordinate axes are placed at an angle to each other
- Location of point P is defined by coordinates which are distances from the point to the axes measured along the lines parallel to the other axis

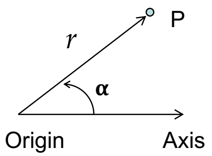


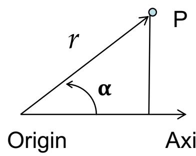


Not used in this course however very much used in different applications, e.g., crystallography

Polar Coordinate System

- Defined by an axis and an origin point on it.
- Location of a 2D point P is defined as a distance *r* from the origin to the point and an angle **α** between the axis and the vector cast towards the point
- Usually, distance *r* is positive, however it depends on the problem
- Usually, angle **α** is from 0 to 2π, however it depends on the problem
- Usually, positive **α** is measured in a counter-clockwise way



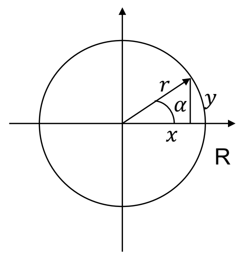


$x = r\cos\alpha$

$y = r\sin\alpha$

$x^2 + y^2 = r^2$

Polar Coordinate System



$r = f(\alpha) \quad \alpha \in [*,*]$???

$r = R$

$\alpha \in [0, 2\pi]$

$x = r\cos(u2\pi)$

$y = r\sin(u2\pi)$

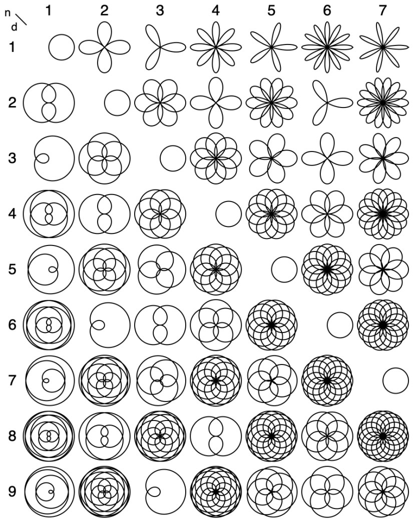
$u \in [0, 1]$

$x^2 + y^2 = r^2 = R^2$

$(r\cos(u2\pi))^2 + (r\sin(u2\pi))^2 = r^2$

$r = R$

Polar Coordinate System



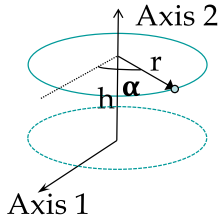
Rose curves defined by
$r = \cos(k\alpha),\ \alpha \in [0, 2\pi]$
for various values of $k=n/d$.

$x = r\cos(\alpha) = \cos(k\alpha)\cos(\alpha)$
$y = r\sin(\alpha) = \cos(k\alpha)\sin(\alpha)$
$\alpha \in [0, 2\pi]$

$x = \cos(ku2\pi)\cos(u2\pi)$
$y = \cos(ku2\pi)\sin(u2\pi)$
$u \in [0, 1]$

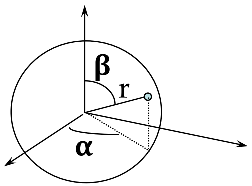
Cylindrical Coordinate System

- Extension of polar system to 3D
- Defined by two orthogonal axes and an origin which is their intersection
- Location of a 3D point **P** is defined as a displacement *h* along the second axis towards the plane orthogonal to the two axes and containing the point **P**, a distance *r* from the second axis to the point, an angle **α** between the first axis and the vector cast towards the point
- Distance *h* can be positive and negative
- Distance *r* is positive
- Angle **α** is from 0 to 2π
- Positive **α** is measured counter-clockwise as seen towards the origin



Spherical Coordinate System 3’55”

- Extension of polar system to 3D
- Defined by three orthogonal axes and an origin which is their intersection
- Location of a 3D point P is defined by a distance *r* from the origin to the point, an two angles **α** (azimuth) and **β** (zenith) between the first axis and the vector cast towards the point
- Distance *r* is positive
- Angle **α** is from 0 τo 2π
- Angle **β** is from 0 τo 1π



 https://youtu.be/FDyenWWlPdU 3’55”

Foundation Mathematics

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Mathematical Functions (T2 and all)

- In mathematics, a **function** associates one quantity, the *argument* of the function, also known as the *input*, with another quantity, the *value* of the function, also known as the *output*. A function assigns exactly one output to each input. Values from the *input domain* map to the values in the *function range*.
- Coordinate functions:
  - Explicit way of function definition
    - $x=f(y)$, $y=f(x)$, $z=f(x,y)$, $g=f(x,y,z)$
  - Implicit way of function definition
    - $f(x)=0$, $f(x,y)=0$, $f(x,y,z)=0$
  - Parametric way of function definition (pa'ram'etric, pa'rameter)
    - $x=f_x(\tau)$ $y=f_y(\tau)$ $z=f_z(\tau)$ $\tau=[\tau_1, \tau_2]$
    - $x=f_x(u,v)$ $y=f_y(u,v)$ $z=f_z(u,v)$ $u=[u_1, u_2]$ $v=[v_1,v_2]$
    - $x=f_x(u,v,w)$ $y=f_y(u,v,w)$ $z=f_z(u,v,w)$ $u=[u_1,u_2]$ $v=[v_1,v_2]$ $w=[w_1,w_2]$

Mathematical Functions

- Conversion from explicit $y=f(x)$ to implicit $f(x,y)=0$:
  - by making an equation $y\text{-}f(x)=0$
- Conversion from explicit $y=f(x)$ to parametric $x=f_x(u)$ $y=f_y(u)$:
  - by introducing parameter(s). The simplest, but not always the best way, is to assign $x=u$, which yields $y=f(u)$
- Conversion from parametric $x=f_x(u)$ $y=f_y(u)$ to explicit $y=f(x)$ or implicit $f(x,y)=0$
  - by expressing parameter u as a function of x from the first equation and then by substituting it into the second equation
  - by eliminating parameter u while doing algebraic manipulations with the two equations (raising to power, multiplications, additions, subtractions, divisions, etc.).
- Watch video in the course site in the "Lecture Supplement" about conversion between parametric and explicit/implicit functions

Foundation Mathematics

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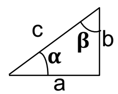
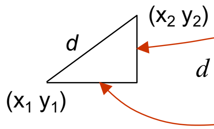
Pythagorean Theorem

- $c^2=a^2+b^2$



$a=c\cdot\cos\alpha$ $b=c\cdot\sin\alpha$
$b=c\cdot\cos\beta$ $a=c\cdot\sin\beta$
$b=a\cdot\tan\alpha$ $a=b\cdot\tan\beta$

- Distance d between two points with coordinates $(x_1\ y_1)$ and $(x_2\ y_2)$ as a consequence of the theorem



$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

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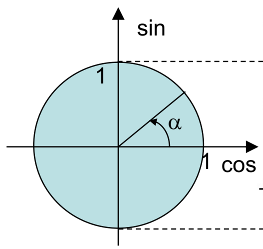
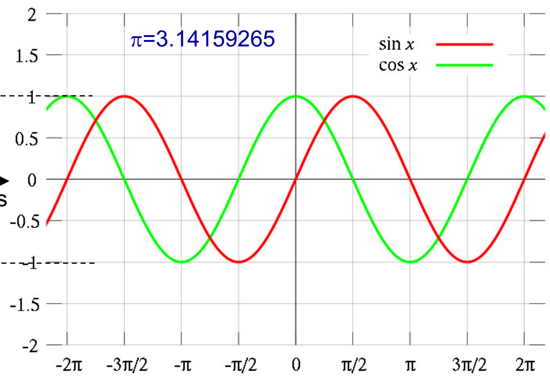
Trigonometry





- $\sin^2\alpha + \cos^2\alpha = 1$
- $\sin(-\alpha) = -\sin\alpha$
- $\cos(-\alpha) = \cos\alpha$
- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Any oscillation like in bouncing motion, surface height, change of color, etc.

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Matrices (M4, T8-10)

- A **matrix** (plural **matrices**) is a rectangular array of numbers denoted as:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

- Basic matrix operations:
  - Addition $$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix} = \begin{bmatrix} a+g & b+h & c+i \\ d+j & e+k & f+l \end{bmatrix}$$
  - Scalar multiplication $$r \cdot \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} r\cdot a & r\cdot b & r\cdot c \\ r\cdot d & r\cdot e & r\cdot f \end{bmatrix}$$
  - Transpose $$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Matrices

- Matrix multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}=\begin{bmatrix} ag+bi+ck & ah+bj+cl \\ dg+ei+fk & dh+ej+fl \end{bmatrix}$$

Multiplication of two matrices is defined only if the number of columns of the left matrix is the same as the number of rows of the right matrix.

Matrices

- Determinant

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}=ad-cb$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}=a\det\begin{bmatrix} e & f \\ h & k \end{bmatrix}-b\det\begin{bmatrix} d & f \\ g & k \end{bmatrix}+c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Sarrus' rule

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Vectors (M5, T11)

Vector is a geometric object that has both a magnitude (or length) and direction. A vector is visually represented by an arrow, connecting an **initial point** *A* with a **terminal point** *B*, and denoted by $\overrightarrow{AB}$ or **AB.**

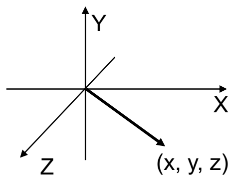
A vector is what is needed to "carry" the point *A* to the point *B*.

A vector can be represented by identifying the coordinates of its initial and terminal point. For instance, the points A = (a,b,c) and B = (d,e,f).

Vector Coordinates

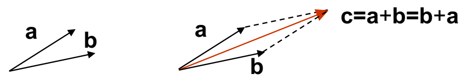
- To calculate with vectors, they are defined by the coordinates of their endpoints assuming that the tail of the vector coincides with the origin. The endpoint coordinates are arranged into column or raw vectors, particularly when dealing with matrices

$$\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} x & y & z \end{bmatrix}$$

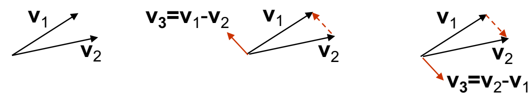


Calculations with Vectors

- Two vectors are said to be equal if they have the same magnitude and direction. Equivalently they will be equal if their coordinates are equal.
- Sum of vectors: **a**=[a_1 a_2 a_3], **b**=[b_1 b_2 b_3]
  c=**a**+**b**=[a_1+b_1 a_2+b_2 a_3+b_3] **c**=**b**+**a**=[a_1+b_1 a_2+b_2 a_3+b_3]

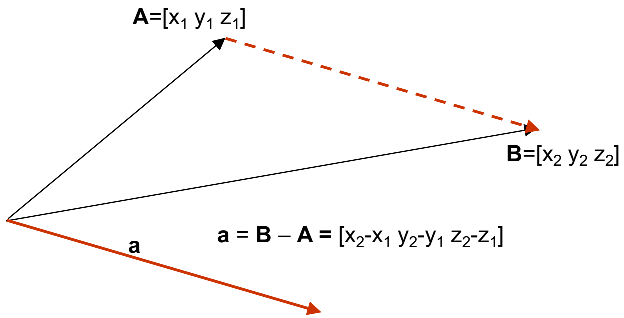


- Subtraction of vectors **a**=[a_1 a_2 a_3], **b**=[b_1 b_2 b_3]
  c=**a**-**b**=[a_1-b_1 a_2-b_2 a_3-b_3] **c**=**b**-**a**=[b_1-a_1 b_2-a_2 b_3-a_3]



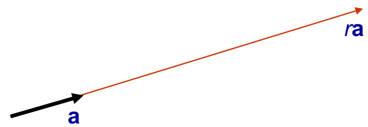
Calculations with Vectors

- Definition of a vector **a** from point A with coordinates (x_1, y_1, z_1) to point B with coordinates (x_2, y_2, z_2)



Calculations with Vectors

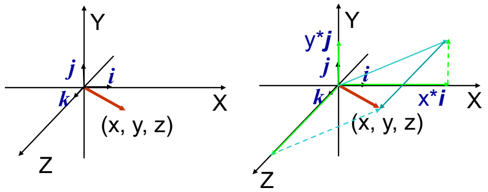
- A vector may also be multiplied, or re-*scaled*, by a real number *r*
  a=[a_1 a_2 a_3] ***r*a** = [ra_1 ra_2 ra_3]



Calculations with Vectors

- Another way to represent a vector is using unit length vectors defining X, Y, and Z coordinate axes and coordinates x, y, z

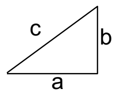
  $\mathbf{a} = x^*\boldsymbol{i} + y^*\boldsymbol{j} + z^*\boldsymbol{k}$



- Vector magnitude calculation:

  $\mathbf{a}=[x\ y\ z]$ $\quad ||a||= \sqrt{x^2+y^2+z^2}$

  which is a consequence of the by Pythagorean formula $c^2=a^2+b^2$



Calculations with Vectors

- A **unit vector** is any vector with a length of one; normally unit vectors are used simply to indicate direction. A vector of arbitrary length can be divided by its length to create a unit vector. This is known as **normalizing** a vector.

  $\mathbf{a}=[x\ y\ z]$ $\quad ||a||= \sqrt{x^2+y^2+z^2}$

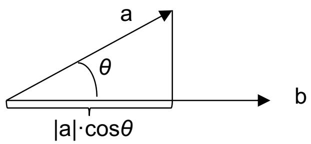
  normalized $\mathbf{a}_n$= [x/||a|| y/||a|| z/||a||]

Calculations with Vectors (M5, T11)

- *Dot product* or *scalar product* of two vectors:

  $\mathbf{a}=[a_1\ a_2\ a_3]$, $\mathbf{b}=[b_1\ b_2\ b_3]$ $\quad \mathbf{a}\cdot\mathbf{b} = |a|\cdot|b|\cdot\cos\theta$ where θ is the measure of the angle between **a** and **b**

  Projection of one vector onto another followed by scaling



  The dot product can also be defined as the sum of the products of the components of each vector as

  $\mathbf{a}\cdot\mathbf{b} = a_1\cdot b_1 + a_2\cdot b_2 + a_3\cdot b_3$

  The result of dot product is a **number** or scalar.

  For unit vectors, the dot product is a *cos* of the angle between them

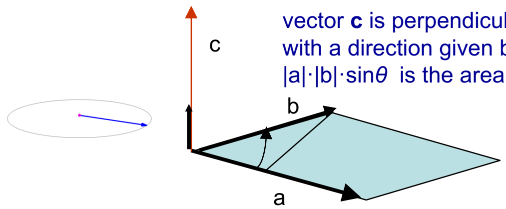
Calculations with Vectors (T1, M5)

- *Vector product* or *cross product* of 2 vectors

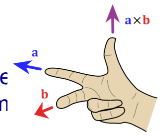
  $\mathbf{a}=[a_1\ a_2\ a_3]$, $\mathbf{b}=[b_1\ b_2\ b_3]$ $\quad \mathbf{a} \times \mathbf{b} = (|a|\cdot|b|\cdot\sin\theta)\ \mathbf{n}$

  $c=\mathbf{a}\times\mathbf{b}$ = [$a_2b_3-a_3b_2$ $\quad a_3b_1-a_1b_3$ $\quad a_1b_2-a_2b_1$] which is actually by Sarrus' rule:

$$c=\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \boldsymbol{i}\det\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \boldsymbol{j}\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \boldsymbol{k}\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$



vector **c** is perpendicular to both **a** and **b** with a direction given by the right-hand rule $|a|\cdot|b|\cdot\sin\theta$ is the area of the parallelogram



Counter-clockwise rotation from **a** to **b**

Summary

- Visualization requires to define some mathematical model to be rendered into images
- Shapes consist of geometry, colors, image textures, and geometrical textures
- All shape components can be defined in their own coordinate systems and merged together into one object
- Shapes can be further transformed and eventually grouped into one application coordinate system
- Viewer and light sources have to be defined to render the scene
- Vector and matrix algebra is used intensively in computer graphics to make it dimension independent and computationally efficient
- Computer graphics is fun!