Geometric Shapes: Constructive Solid Geometry

Module 3
Lecture 7

Previously we Learnt

- Implicit
  - $f(x,y)=0$ 2D curve
  - $f(x,y,z)=0$ 3D surface
- Explicit
  - $y=f(x)$ $x=f(y)$ 2D curves *(seldom used)*
  - $z=f(x,y)$ $y=f(x,z)$ $x=f(y,z)$ 3D surfaces *(seldom used)*
- Parametric
  - 2D/3D Curves: 1 parameter
  - Surfaces: 2 parameters
  - 3D Solids: 3 parameters

Solid Objects

- Voxels (volume elements)
- Parametric representation
- Explicit (variant of implicit) representation

Using Explicit Functions for Defining Solid Objects

Let's change in any implicit function "=" to "≤" or "≥"

$f(x,y)=0$ $g=f(x,y)\le 0$ or $g=f(x,y)\ge 0$
$f(x,y,z)=0$ $g=f(x,y,z)\le 0$ or $g=f(x,y,z)\ge 0$

It becomes an explicit function in +1 dimension (scientific name **Frep**), i.e. $g=f(x,y,z)\ge 0$ evaluates some coordinate or value in the dimension other than x, y, z.

In this course, we will ONLY use $\ge \mathbf{0}$:

$\boldsymbol{g=f(x,y)\ge 0}$ and $\boldsymbol{g=f(x,y,z)\ge 0}$

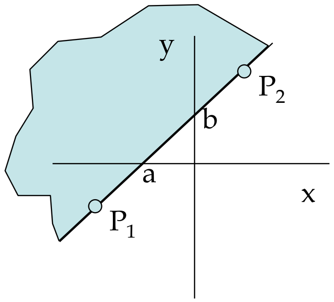
to be consistent with the rendering algorithm and other mathematics used in the remaining part of this module.

Half-plane
Implicit Representation

- Implicit

$$Ax + By + C \geq 0$$

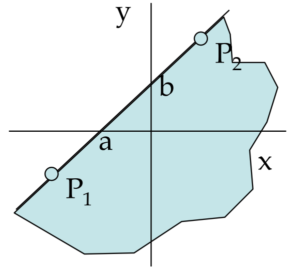
$$\frac{x}{a} + \frac{y}{b} - 1 \geq 0$$



Half-plane
Implicit Representation

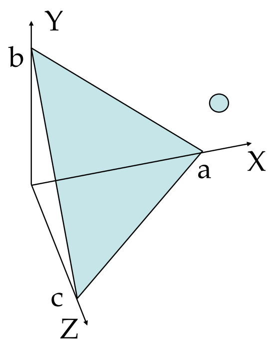
$$-(Ax + By + C) \geq 0$$

$$-\frac{x}{a} - \frac{y}{b} + 1 \geq 0$$



Plane-bounded Half-space

- Above the plane

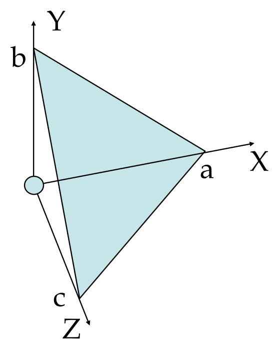


$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \Rightarrow$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 \geq 0$$

Plane-bounded Half-space

- Below the plane

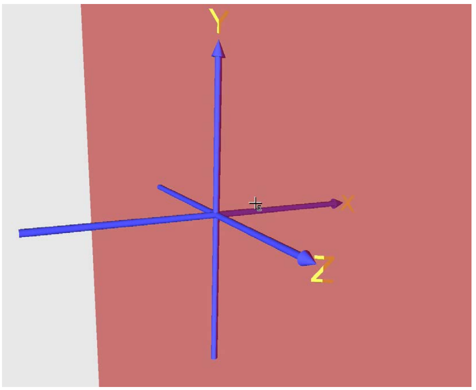


$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \Rightarrow$$

$$1 - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} \geq 0$$

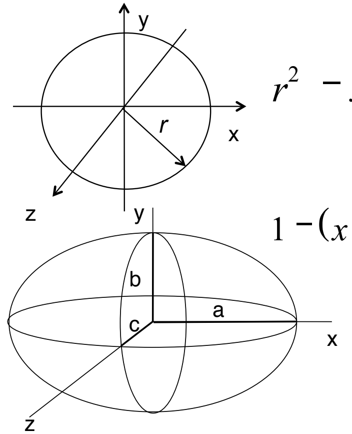
Plane-bounded Half-space

- $x \geq 0$



The displayed size of the half-space surface is defined by the XYZ domain

Solid Sphere and Ellipsoid



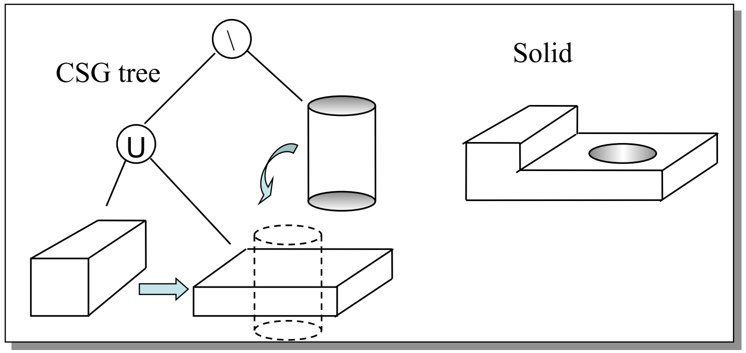
$$r^2 - x^2 - y^2 - z^2 \geq 0$$

$$1 - (x/a)^2 - (y/b)^2 - (z/c)^2 \geq 0$$

Constructive Solid Geometry (CSG)

- CSG is a family of schemes introduced for representing rigid solids as Boolean constructions and combinations of solid components.
- The three basic operators *union* U, *intersection* ∩, and *difference* \ are applied to primitive objects.
- In CSG, objects are represented as binary trees, called *CSG trees*. Each leaf is a primitive object and each non-terminal node is either a Boolean operator or a motion (translation, rotation) which operates on the subnodes.

Constructive Solid Geometry



Implicit Representation of CSG

$G: \quad f(x,y,z) \ge 0$

$G_3 = G_1 \cup G_2: \quad f_3 = f_1 \vee f_2 = \max(f_1, f_2)$ Union

$G_3 = G_1 \cap G_2: \quad f_3 = f_1 \wedge f_2 = \min(f_1, f_2)$ Intersection

$G_3 = -G_1: \quad f_3 = -f_1$ Outer part or Complement

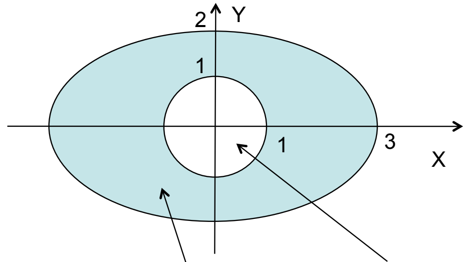
$G_3 = G_1 \setminus G_2: \quad f_3 = f_1 \setminus f_2 = \min(f_1, -f_2)$ Subtraction

Example:

$G_5 = G_1 \cup ((G_2 \cap G_3) \setminus G_4):$

$f_5 = f_1 \vee ((f_2 \wedge f_3) \setminus f_4) = \max(f_1, \min(\min(f_2, f_3), -f_4)) \ge 0$

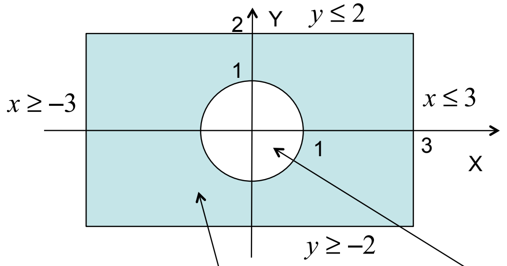
Implicit Representation of Boolean Operations



$$f(x,y) = \min\left(\left(1-\left(\frac{x}{3}\right)^2-\left(\frac{y}{2}\right)^2\right), -\left(1^2-x^2-y^2\right)\right) \ge 0$$

subtraction

Implicit Representation of Boolean Operations

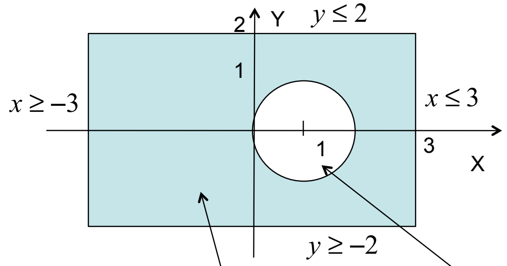


$$f(x,y) = \min\left(\min(3-x, x+3, 2-y, y+2), -\left(1^2-x^2-y^2\right)\right) \ge 0$$

intersection

subtraction

Implicit Representation of Boolean Operations

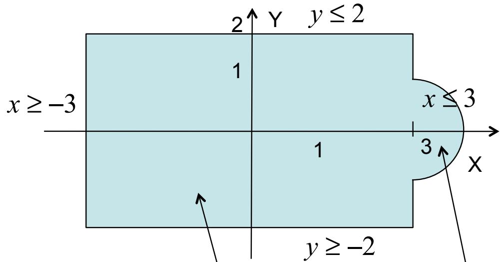


$$f(x,y) = \min\left(\min(3-x, x+3, 2-y, y+2), -\left(1^2-(x-1)^2-y^2\right)\right) \ge 0$$

intersection

subtraction

Implicit Representation of Boolean Operations

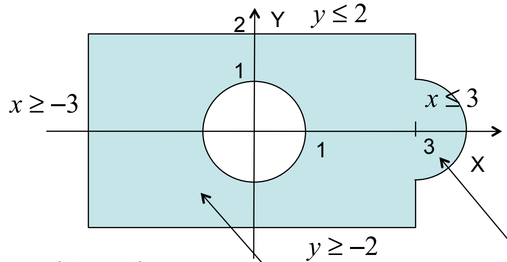


$$f(x,y) = \max\left(\min(3-x, x+3, 2-y, y+2), \left(1^2-(x-3)^2-y^2\right)\right) \ge 0$$

union

intersection

Implicit Representation of Boolean Operations



subtraction

union

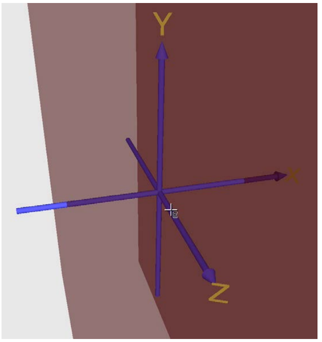
intersection

$$f(x,y) = \min\left(\max\left(\min(3-x, x+3, 2-y, y+2), \left(1^2-(x-3)^2-y^2\right)\right), -\left(1^2-x^2-y^2\right)\right) \ge 0$$

CSG with Implicit Functions

- *min(x+1,1-x)* ≥ 0

intersection

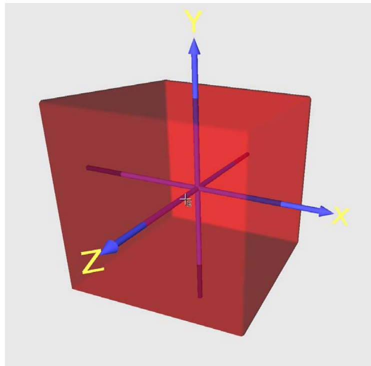


The displayed size of the half-space surface is defined by the XYZ domain

CSG with Implicit Functions

- *min(x+1,1-x,y+1,1-y,z+1,1-z)* ≥ 0

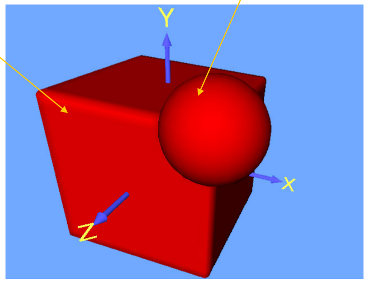
intersection



CSG with Implicit Functions

- $max(min(x+1,1-x,y+1,1-y,z+1,1-z),$
  $0.5^2-(x-1)^2-(y-1)^2-(z-1)^2 \ ) \geq 0$

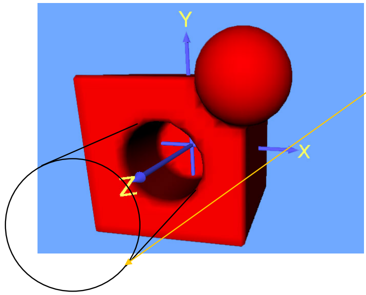
union



CSG with Implicit Functions

- $min(max(min(x+1,1-x,y+1,1-y,z+1,1-z),$
  $0.5^2-(x-1)^2-(y-1)^2-(z-1)^2 \ ), - min(0.5^2-x^2-y^2,z)) \geq 0$

difference



Summary. Constructive Solid Geometry using Function Reprentations (FReps)

$$G: \quad f(x,y,z) \geq 0$$

$$G_3 = G_1 \cup G_2: \quad f_3 = f_1 \vee f_2 = \max(f_1, f_2) \qquad \text{Union}$$

$$G_3 = G_1 \cap G_2: \quad f_3 = f_1 \wedge f_2 = \min(f_1, f_2) \qquad \text{Intersection}$$

$$G_3 = -G_1: \quad f_3 = -f_1 \qquad \text{Outer part or Complement}$$

$$G_3 = G_1 \setminus G_2: \quad f_3 = f_1 \setminus f_2 = \min(f_1, -f_2) \qquad \text{Subtraction}$$

Example:

$$G_5 = G_1 \cup ((G_2 \cap G_3) \setminus G_4):$$

$$f_5 = f_1 \vee ((f_2 \wedge f_3) \setminus f_4) = \max(f_1, \min(\min(f_2, f_3), -f_4)) \geq 0$$

Summary

- Solid objects can be defined by
  - Changing implicit equations of surfaces into inequalities ≤0 or ≥0
  - Parametric equations with 3 parameters (3D solids)
- Sweeping can be easily formulated by using parametric functions for the cases of translational and rotational sweeping
- Boolean or Set-theoretic operations can be defined for implicit functions and implicit inequalities when min/max functions are used; We assume only inequalities ≥0 for the functions-arguments and functions-results of the operations.

Analytic Shape Representations. Summary

- Implicit
  - $f(x,y)= 0$ 2D curve
  - $f(x,y,z)=0$ 3D surface
- Explicit
  - $y=f(x)$ $x=f(y)$ 2D curves
  - $z=f(x,y)$ $y=f(x,z)$ $x=f(y,z)$ 3D surfaces
  - $g=f(x,y,z) \geq 0$ solid objects

    U $\max(f_1, f_2) \geq 0$ $\cap$ $\min(f_1, f_2) \geq 0$ \ $\min(f_1, -f_2) \geq 0$
- Parametric
  - 2D/3D Curves: 1 parameter
  - Surfaces: 2 parameters
  - 3D Solids: 3 parameters