CE2001/ CZ2001: Algorithms

Mathematical Background (LAMS)

Dr. Loke Yuan Ren

These slides are used for revision only. It will not covered in the lecture

Learning Objectives

At the end of this topic, students should be able to:

- Review basic mathematical concepts useful for algorithm analysis such as:
    1. Sets and Functions
    2. Floor and Ceiling Functions
    3. Power and Logarithm Functions

Learning Objectives

At the end of this topic, students should be able to:

- Review basic mathematical concepts useful for algorithm analysis such as:
    4. Summations and Series
    5. Limits
    6. Differentiation of Functions
    7. Proof by Induction

Sets

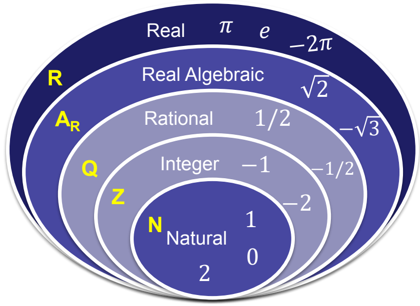
Sets

A **set** is a collection of well defined and distinct objects.

References:
Students Working Together Clipart | Clipart Panda - Free Clipart ... (n.d.). Retrieved May 16, 2016, from http://cliparts.co/clipart/9853.
Furniture Clipart | Clipart Panda - Free Clipart ... (n.d.). Retrieved May 16, 2016, from http://cliparts.co/clipart/9853.

Sets

Example: Number sets N, Z, Q, R, C, etc.



A **relation (x,y)** in a set A is a collection of ordered pairs of elements in A.

Functions

Functions

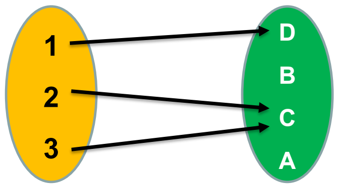
- A **function** is a relation between a set of inputs and a set of potential outputs with the property that each input is related to exactly one output.

Example:

$f(1) = D$

$f(2) = C$

$f(3) = C$



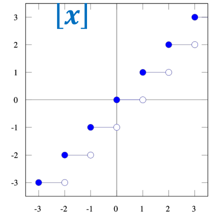
Formal description of a function typically involves the function's name, its domain, its codomain, and a rule of correspondence.

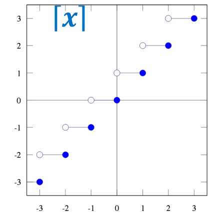
$f : R \to R$ $\qquad f(x) = x^2$

Ceiling and Floor Functions

Ceiling and Floor Functions

- The floor and ceiling functions map a real number to the largest previous or the smallest following integer, respectively.
- $\lfloor x \rfloor$(floor of x) = the largest integer not greater than x
- $\lceil x \rceil$(ceiling of x) = the smallest integer not less than x





References:
U. (n.d.). File:Floor function.svg. Retrieved May 16, 2016, from https://commons.wikimedia.org/w/index.php?curid=670036.
U. (n.d.). File:Ceiling function.svg. Retrieved May 16, 2016, from https://commons.wikimedia.org/w/index.php?curid=696232.

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Examples:

$$\lfloor 5.5 \rfloor = 5, \quad \lfloor 5 \rfloor = 5$$

$$\lceil 5.5 \rceil = 6, \quad \lceil 6 \rceil = 6$$

Facts and Formulas:

$$\lfloor x \rfloor \le x \le \lceil x \rceil$$

Power Function (Exponentiation)

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Power Function (Exponentiation)

- **Exponentiation** is a mathematical operation, written as b^n, involving two numbers, the base b and the exponent (a.k.a. index or power) n.
- When n is a positive integer, exponentiation corresponds to repeated multiplication.

Facts and Formulas:

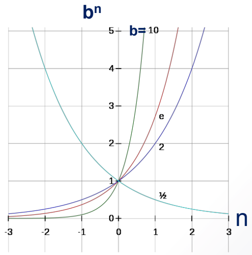
$b^0 = 1$

$b^1 = b$

$b^2 = b \times b$

…

$b^n = b^{n-1} \times b$



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Logarithm Function

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Logarithm Function

- The **logarithm** of a number to the base **b** is the exponent by which the base b has to be raised to produce that number.

$$y = \log_b(x) \quad \Leftrightarrow \quad x = b^y$$

E.g., $\log_{10}(1000)$ (reads as log of 1000 to base 10) is 3, as $1000 = 10^3$

Facts and Formulas:

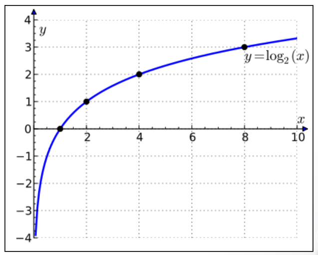
$\log_b(1) = 0$ (since $b^0 = 1$)

$\log_b(0)$ undefined (since $b^? = 0$)

$\log_b(x)^n = n \log_b(x)$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b(x) = \log_k(x)/\log_k(b)$



Reference: U. (n.d.). File:Binary logarithm plot with ticks.svg. Retrieved May 16, 2016, from https://commons.wikimedia.org/w/index.php?curid=15408195

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Summations and Series

Summations and Series

- A **series** is, informally speaking, the sum of the terms of a sequence.

Example:

$$S = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2} \quad \longleftarrow \text{finite series}$$

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n \to \infty} \frac{1}{2^n} \quad \longleftarrow \text{infinite series}$$

Arithmetic Series (AS)

- Each successive term is produced by adding a constant number to the previous term.

$$\sum_{i=0}^{n-1} [a + id] = a + [a+d] + [a+2d] + \ldots + [a+(n-1)d]$$

$$= \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[a + L]$$

a: first term
d: difference
n: number of terms
L: last term

Example: $\sum_{i=0}^{n-1} 1 + i \; = 1 + 2 + 3 + \ldots + n \; = \frac{n}{2}[1+n]$

Geometric Series (GS)

- Each successive term is produced by multiplying the previous term by a constant number.

$$\sum_{i=0}^{n-1} ar^i = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

$$= a\frac{(1-r^n)}{1-r}$$

$$= a\frac{(r^n - 1)}{r-1}$$

a: first term
r: ratio
n: number of terms

Example: $\sum_{i=0}^{n-1} 2^i = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = \frac{1-2^n}{1-2} = 2^n - 1$

Limits

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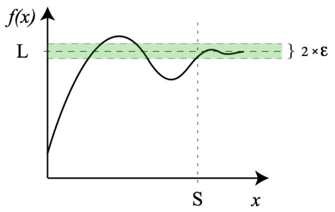
Limits

- A **limit** is the value that a function or sequence "approaches" as the input or index approaches some value.

$$\lim_{x \to c} f(x) = L$$ means

The limit of **f(x)**, as **x** approaches **c**, is **L**.



Example:

$$\lim_{x \to 0} \log(x) = -\infty$$

$$\lim_{n \to \infty} \frac{4n}{e^n} = 0$$

Reference: U. (n.d.). File:Limit-at-infinity-graph.png. Retrieved May 16, 2016, from https://commons.wikimedia.org/w/index.php?curid=644080 English Wikipedia - Transferred from en.wikipedia to Commons by Maksim.

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Differentiation of Functions

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Differentiation of Functions

- Differentiating a function is to find the derivative or rate of change of the function

Example:

$$\frac{d}{dx} c = 0 \qquad \frac{d}{dx} x = 1$$

$$\frac{d}{dx} x^2 = 2x \qquad \frac{d}{dx} cx^n = cnx^{n-1}$$

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Some Useful Formula

Terminologies:

log(x) means $\log_{10}(x)$

lg(x) means $\log_2(x)$

ln(x) means $\log_e(x)$

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} \ln(x) = \frac{1}{x}$$

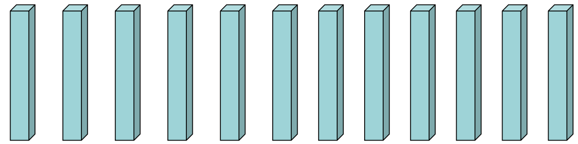
$$\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x) \qquad \frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} f'(x)$$

$$\frac{d}{dx} 2^{f(x)} = 2^{f(x)} \ln 2\, f'(x) \qquad \frac{d}{dx} \log_b(x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx} a^{f(x)} = a^{f(x)} \ln a\, f'(x) \qquad \frac{d}{dx} \log_b f(x) = \frac{1}{f(x) \ln b} f'(x)$$

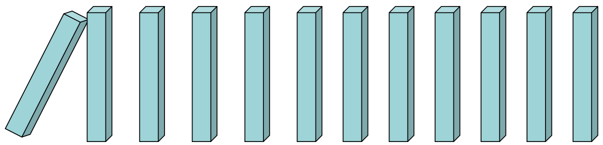
Proof by Mathematical Induction

Proof by Mathematical Induction



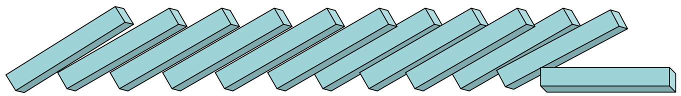
Mathematical induction -- Domino effect:

Proof by Mathematical Induction



Mathematical induction -- Domino effect:
(i) We prove that the first bar will fall
(ii) We prove that if a bar will fall, the next bar will fall

Proof by Mathematical Induction



Mathematical induction -- Domino effect:
(i) We prove that the first bar will fall
(ii) We prove that if a bar will fall, the next bar will fall

➡ every bar will fall.

Mathematical Induction

Basic Principle:

- Let T be a theorem to be proved;
- Express T in terms of positive integer **n**;
- T is true for any of n (for $n \geq c$ where c is a small constant) if the following are true:

  (i) Base Case: Prove that T holds for n = each of a small set of basic values.

  (ii) Induction Step: Prove that if T holds for any value k, then T holds for k+1. OR

  (ii-1) Strong induction: Prove that if T holds for all k, $c \leq k < n$, then T holds for n.

  e.g. T(n) = an algorithm gives correct results for an array of size n.

Mathematical Induction: Prove Theorem

Prove Theorem: There are at most 2^d nodes at depth **d** of a binary tree.

Proof:

- By definition of a binary tree, each node has at most 2 children. Let **d** denote the depth of the tree.
- We will prove the result by induction on **d.**

  i. **Base case: At d = 0**, there is at most 1 root node, i.e. 2^0 node.

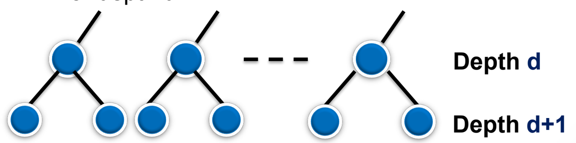
Mathematical Induction: Prove Theorem

Proof:

ii. **Induction Step:** We assume that the tree has, for any depth d, at most 2^d nodes at that depth.

Prove that at depth d+1, there are at most $2^{(d+1)}$ nodes.

- By assumption, at depth d, there are at most 2^d nodes.
- Each of the node at depth d can have at most 2 children, hence there are at most $2*2^d$ nodes. Thus the result is true for depth d+1.



Combining Base Case and Induction Step, the result is true for all depths of a binary tree.

Summary

Summary

- Sets, Relations and Functions
- Floor and Ceiling Functions
- Power and Logarithm Functions
- Summations and Series
- Limits
- Differentiation of Functions
- Proof by Induction