Module 4

2D Transformation

Learning Objectives

- Identify **basic 2D transformations**
- Cartesian coordinates ⇔ **homogeneous coordinates**
- Understand **affine transformations**
- Represent and construct affine transformations by matrix or matrices
- Perform computation with 2D transformations

Sources

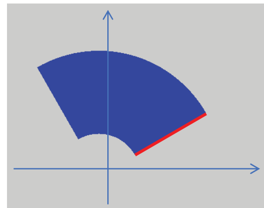
- Textbook (Chapter 4: 2D transformations and viewing)
- Wiki:
  - http://en.wikipedia.org/wiki/Affine_transformation
  - http://en.wikipedia.org/wiki/Transformation_matrix

Outline

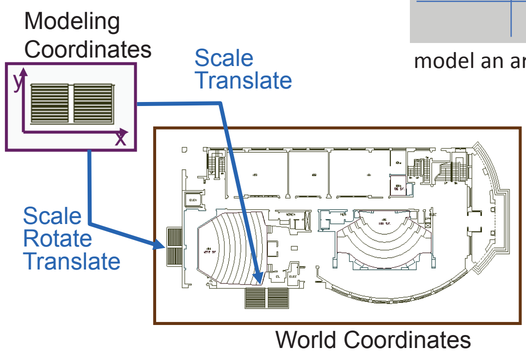
1. Motivation and applications
2. Basic 2D transformations
3. Homogeneous coordinates
4. 2D affine transformations

1. Motivation and applications

- Transformations tell relation between shapes.
- Transformations also provide tools for
  - fast modeling
  - compact representation
  - …

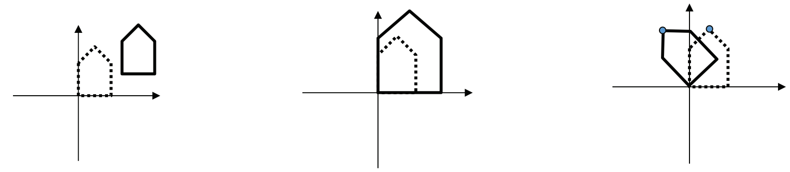


model an area from a line

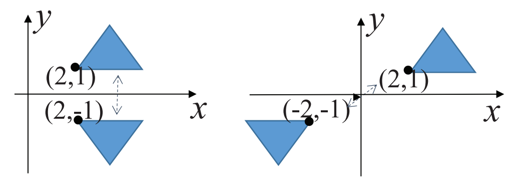


2. Basic 2D transformations

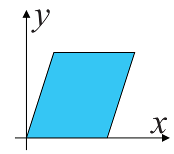
- Translation
- Scaling
- Rotation



- Reflection



- Shear



- Question: What kind of information should be provided to specify these transformations?

Representations

- How to represent these transformations in computer?
- How to apply these transformations?

- Transformation: matrix ***M***
- Point: *column* vector ***p***
- Then the transformed point can be obtained by

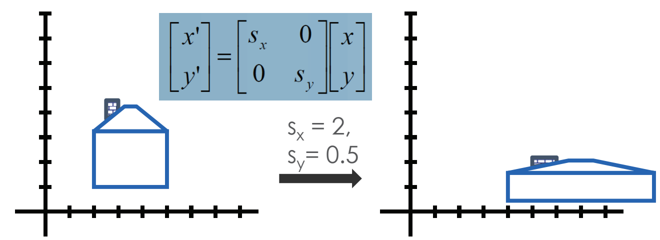
p′ = ***Mp***

for example, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(Note that ***M*** goes on the *left* of ***p***.)

Scaling matrix

- Scaling about the ***origin***, with **scaling factors** (s_x, s_y):
  - The original coordinates are multiplied by the given scale factors
- Uniform scaling: change size, but keep shape
- Non-uniform scaling: change both size & shape

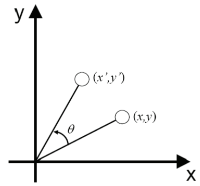


- Question: What if the reference (fixed) point is **not** the origin?

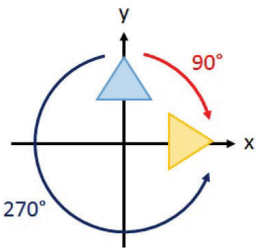
Rotation matrix

- Rotation about the ***origin***, with **an angle** θ:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



- How to deal with clockwise rotation?
  - Clockwise rotation through an angle θ is equivalent to a counterclockwise rotation through angle 2π-θ.
  - Replace θ by 2π-θ or simply use -θ since Sine, Cosine functions are periodic, with period 2π.



- Question: What if the reference (pivot) point is **not** the origin?

Translation: no matrix multiplication!!

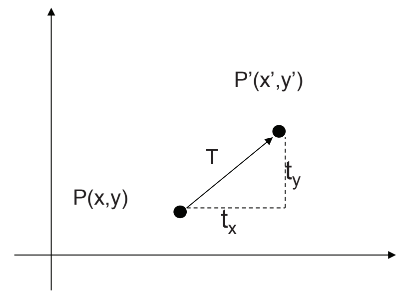
- Translation is represented by the sum of two vectors, instead of matrix product!
  - Moves a point to a new location by adding **translation amounts** to the coordinates of the point.

$x' = x + t_x$

$y' = y + t_y$

i.e.,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



Why matrices?

- Why do we represent transformation in matrix form?
- A typical computer graphics task: given an object consisting of 1000 points, apply 100 transformations to the object.
  - A naïve solution: simply apply 100 transformations to each of 1000 points.

```
for (i = 0; i<1000; i++)
 for (j = 0; j < 100; j++)
  point[i]=apply(transformation[j],point[i]);
```

  - Question: what is the computational complexity of this solution?

    1000*100 iterations.

Why matrices? (cont)

- If each transformation is represented by a matrix, we can
  - first multiply the 100 matrices;
  - then apply the **single** product matrix to each of 1000 points.

```
t = identityMatrix;
for (j=0; j<100; j++)
 t=matrixMatrixMultiply(matrix[j],t);
for (i = 0; i<1000; i++)
 point[i]=matrixVectorMultiply(t,point[i]);
```

  - Question: what is the computational complexity of this approach?

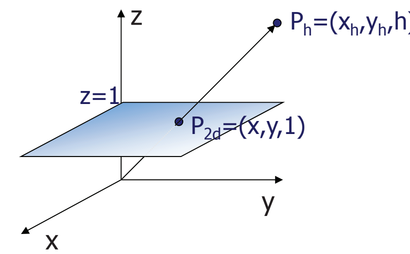
    1000 + 100 iterations.

Why matrices? (cont)

- Furthermore, matrix operations can be highly optimized and carried out efficiently on graphics hardware (GPU).
- Scaling, rotation, reflection and shear transformations can be defined by matrices.
- But, translation is defined by vector addition!
- Is there a way to represent all these transformations in matrices?
  - Yes!! Using homogeneous coordinates for points.

3. Homogeneous coordinates

- Expand 2D Cartesian coordinates (x,y) to a 3-element (x_h,y_h,h) where h is a nonzero value satisfying $x=\frac{x_h}{h},\quad y=\frac{y_h}{h}$
- (x_h,y_h,h) is called **homogeneous coordinates** of point (x,y).
  - In 2D/3D transformation, we simply set $h=1$ in general.
- Each point (x,y) has multiple homogeneous coordinates. For example, $x_h=hx$, $y_h=hy$.
  - $h=1$ $(x,y) \rightarrow (x,y,1)$
  - $h=2$ $(x,y) \rightarrow (2x,2y,2)$



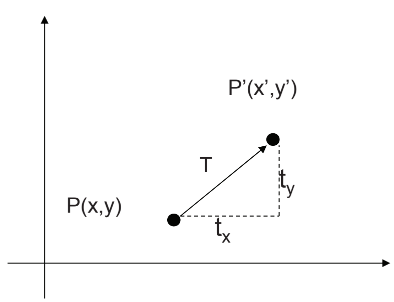
Translation matrix

- Using homogeneous coordinates, translation can be represented by matrix product, with a 3×3 matrix:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$x' = x + t_x$

$y' = y + t_y$



Matrix representation

- For consistency, upgrade the other matrices to 3×3.
  - Scaling:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Leftrightarrow \begin{array}{rcl} x' & = & s_x x \\ y' & = & s_y y \end{array}$$

  - Rotation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

  - Reflections:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Recap

- Homogeneous coordinates expand 2D Cartesian coordinates (x,y) to (x,y,1).
- All basic 2D transformations can be represented by a 3×3 matrix.

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Translation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Scaling about origin

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotation about origin

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Reflection over x-axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

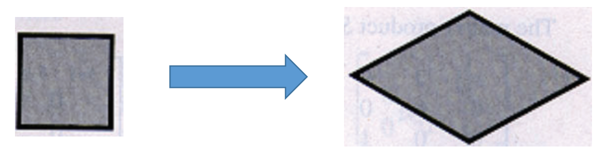
Reflection over y-axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Reflection over origin

4. Affine transformations

- Affine transformations are composites of four transformations: *translation*, *rotation*, *scaling*, and *shear*.
- Affine transformations map straight lines to straight lines and preserve ratios of distances along straight lines.
  - Affine transformations preserve parallelism of lines but not lengths and angles



Affine transformations (cont)

- Affine transformations can always be represented by

$$x' = ax + by + m$$
$$y' = cx + dy + n$$

where
  - a, b, c, d, m, n are constants
  - (x,y) are the coordinates of the point to be transformed
  - (x',y') are the coordinates of the transformed point.

The general matrix form of affine transformations is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & m \\ c & d & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine transformations (cont)

- These three different descriptions of affine transformations actually give methods to answer various questions.
  - In the rest of this lecture, we will see how we apply these principles.

4.1 Question for you

Is the following transformation T an affine transformation?

$$T: \begin{cases} x' &= 2y+1 \\ y' &= x-y+2 \end{cases}$$

Answer:

Question for you

Is the following transformation T an affine transformation?

$$T: \begin{cases} x' &= \sin(x)+2y+1 \\ y' &= x-y+2 \end{cases}$$

Answer:

Question for you

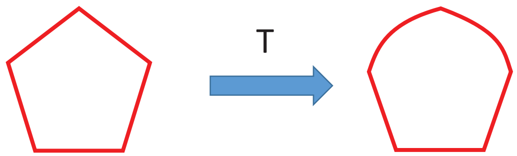
Is the following transformation T an affine transformation?

$$T: \begin{cases} x' &= \sin(3)x+2y+1 \\ y' &= x-y+2 \end{cases}$$

Answer:

Question for you

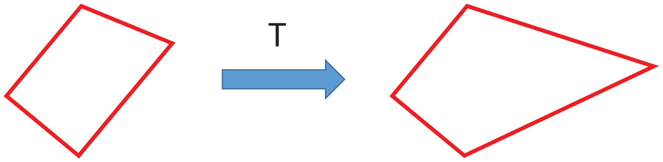
Is the following transformation T an affine transformation?



Answer:

Question for you

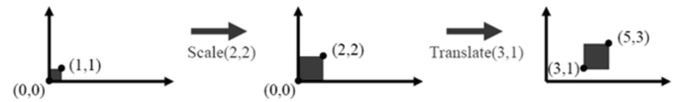
Is the following transformation T an affine transformation?



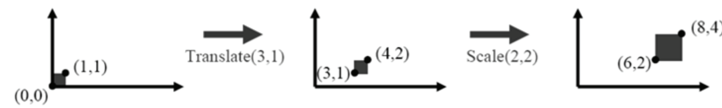
Answer:

4.2 Composition of transformations

- An affine transformation can be defined as a composition of basic transformations, which provides a way to define an affine transformation.
- The order of transformations DOES matter!
  - First scale, then translate



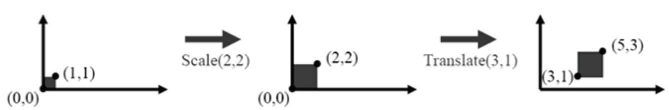
  - First translate, then scale



Composition of transformations

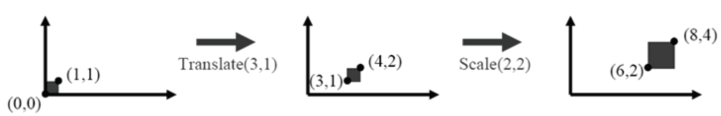
- Implementation
  - First scale, then translate

$$\mathbf{TS} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



  - First translate, then scale

$$\mathbf{ST} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$



Transformation order

- Suppose we are given 3 transformations **A**, **B** and **C**.

  Apply transformation **A** first, followed by **B** and then **C**.
- How to write the matrix product?

$$p_1 = \mathbf{A}p$$

$$p_2 = \mathbf{B}p_1$$

$$p_3 = \mathbf{C}p_2$$

Therefore, $p_3 = \mathbf{C}(\mathbf{B}p_1) = \mathbf{C}(\mathbf{B}(\mathbf{A}p)) = \mathbf{CBA}p$

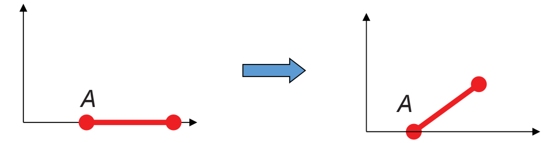
The answer is **CBA**. Since points are *column* vectors, the transformation matrices are pre-multiplied.

Find the matrices by composing basic/ simple transformations

- Problem: How to perform ("non-standard") complicated transformations (which answers the Qs in slides 8 & 9)
- Method:
  - Step 1. Analyze each transformation and do the following:
  - Step 2. If it is not a simple transformation, find some basic/simple transformations and perform them as a pre-process to make it "standard".
  - Step 3. Write in order the matrices for all the transformations performed in the preprocess
  - Step 4. Write the matrix for the required transformation in "standard" form
  - Step 5. Perform the post-process by reversing the transformations performed in the pre-process step and write their matrices in order

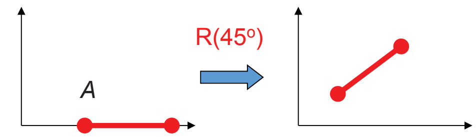
Example 1 (rotation about an arbitrary pivot point)

Q: What is the matrix of the transformation that rotates the red line segment by 45 degrees about its endpoint A=(3,0)?



Hints:

- Wong way: simply rotate the two endpoints of the line segment by 45 degrees.

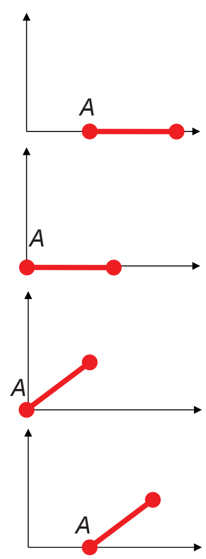


Example 1 (cont)

- Correct way:
  - First translate line so *A* is at origin: Tran(-3,0)
  - Then rotate line 45 degrees: Rot(45º)
  - Then translate back so *A* is where it was: Tran(3,0)
  - As a result, the final matrix should be

    Tran(3,0)Rot(45º)Tran(-3,0) =

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



4.3 Find the matrix using the general matrix form

- Problem: Assuming that object B is obtained from object A by an affine transformation, find the matrix for the transformation.
- Method:
  - Step 1. Assume that the affine transformation is represented by

$$x' = ax + by + m$$
$$y' = cx + dy + n$$

  - Step 2. Choose at least 3 point pairs from objects A and B. Substitute their coordinates as (x,y) and (x',y') into the above two equations. This gives you a set of linear equations.
  - Step 3. Solving the linear equations for a, b, c, d, m, n.
  - Step 4. Using homogeneous coordinates, convert the linear representation of the affine transformation to matrix form.

Example 2

Q: A 2D affine transformation T transforms polygon $A_1A_2A_3A_4A_5$ into polygon $B_1B_2B_3B_4B_5$ where vertices A_1=(0,1), A_2=(1,2), A_3=(3,4), $\mathbf{A_4}$=(6,-2), A_5=(4,0), B_1=(2,3), B_2=(0,6), B_3=(-4,12), B_5=(4,15) and B_k corresponds to A_k for k =1,2,3,4,5. Compute the coordinates of $\mathbf{B_4}$ that corresponds to A_4.

Hints: Let $x' = ax + by + m$

$y' = cx + dy + n$

A_1= (0,1) → B_1= (2,3) gives 2 = b+m, 3 = d +n.

A_5= (4,0) → B_5 = (4,15) gives 4 = 4a+m, 15 = 4c+n

A_2= (1,2) → B_2 = (0,6) gives 0 = a+2b+m, 6 = c+2d+n

Solving the equations gives a=0, b=-2, m=4, c=3, d=0,n = 3.

Example 2 (cont)

Thus the transformation is

x' = -2y+4

y' = 3x +3

For B_4 corresponding to A_4=(6, -2), its coordinates are

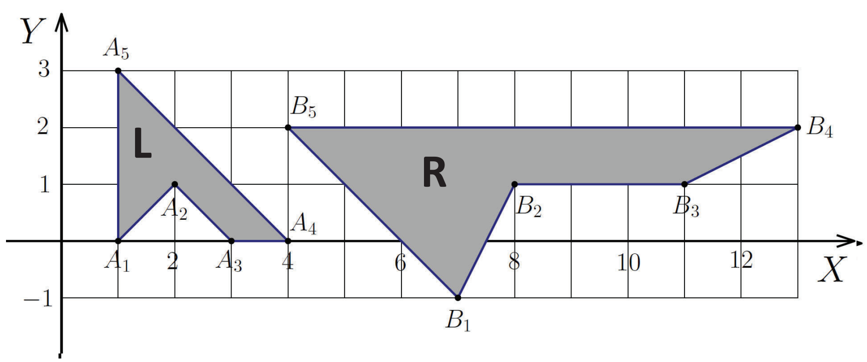
x'= -2*(-2)+4 = 8

y'=3*6+3 = 21

Therefore B_4 = (8,21).

Example 3

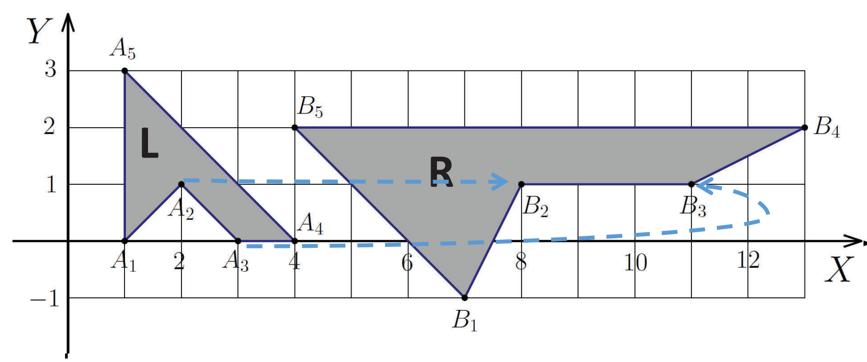
Q: An affine transformations transforms polygon **L** to polygon **R**, as shown in the figure. Derive the 3×3 matrix for this transformation.



Example 3 (cont)

Q: An affine transformations transforms polygon **L** to polygon **R**, as shown in the figure. Derive the 3×3 matrix for this transformation.

Ans: Finding the corresponding points based on the properties of affine transformations. That is, A_2→B_2, A_3→B_3, A_1→B_1, …



Example 3 (cont)

Let $\begin{cases} x' &= ax+by+m \\ y' &= cx+dy+n \end{cases}$

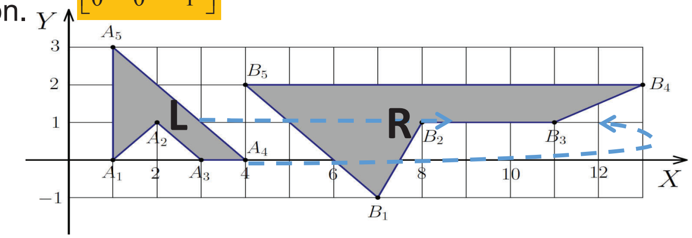
A_1= (1,0) → B_1= (7,-1) gives 7 = a +m, -1 = c +n.

A_3= (3,0) → B_3 = (11,1) gives 11 = 3a+m, 1 = 3c+n

A_2= (2,1) → B_2 = (8,1) gives 8 = 2a+b+m, 1 = 2c+d+n

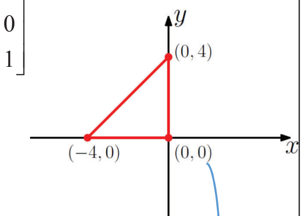
Solving the equations gives a=2, b=-1, m=5, c=1, d=1,n = -2.

Thus the matrix is $\begin{bmatrix} 2 & -1 & 5 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. It is good to apply the matrix to other point pairs to verify the existence of such a transformation.



4.4 Example 4 (applying transformation)

Q: An affine transformation defined by $M = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ applies to a triangle shown in red in the right figure. Find the image of the triangle under this transformation.



Hints: Applying the transformation to points (0,0), (0,4) and (-4,0) gives three new points:

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 1 \end{bmatrix}$$

The image of the triangle is a new triangle with vertices (2,0), (6,8) and (2,-8).

Recap

- Use the geometric definition of affine transformations to analyze the shape deformation.
- Use the general matrix form of affine transformations to find the matrix representation
- Use the combination of basic transformations to define an affine transformation (paying attention to the order!)
- Use matrix-vector multiplication to perform transformation

END ?

Extra example 1: Homogeneous coordinates

Q: Which of the following triples are the homogeneous coordinates of a 2D point with Cartesian coordinates (1,2)?

$$\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}2\\4\\1\end{bmatrix}\begin{bmatrix}2\\4\\2\end{bmatrix}\begin{bmatrix}-1\\-2\\1\end{bmatrix}\begin{bmatrix}-3\\-6\\-3\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix}\begin{bmatrix}1.5\\3\\1.5\end{bmatrix}\begin{bmatrix}-1\\-2\\-1\end{bmatrix}$$

(i) (ii) (iii) (iv) (v) (vi) (vii) (viii) (ix)

Hints: The main idea is to divide each triple by its 3rd component to make the 3rd component be 1, then to extract the first two components and compare to (1,2). For example, consider (iv).

$$\begin{bmatrix}2\\4\\2\end{bmatrix}\rightarrow\begin{bmatrix}2\\4\\2\end{bmatrix}/2=\begin{bmatrix}1\\2\\1\end{bmatrix}\rightarrow\begin{bmatrix}1\\2\end{bmatrix}$$

So (iv) is the homogeneous coordinates of (1,2). Similarly, (i), (vi), (viii), (ix) are also the answers.

Extra example 2

Q: What kind of transformation do the following matrices define?

$$A=\begin{bmatrix}\cos 45^\circ & -\sin 45^\circ & 0\\ \sin 45^\circ & \cos 45^\circ & 0\\ 0 & 0 & 1\end{bmatrix}\quad B=\begin{bmatrix}\cos 45^\circ & \sin 45^\circ & 0\\ -\sin 45^\circ & \cos 45^\circ & 0\\ 0 & 0 & 1\end{bmatrix}\quad C=\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Hints: A defines a rotation about the origin by 45 degrees counter-clockwise.

B defines a rotation about the origin by 45 degrees clockwise. This is because

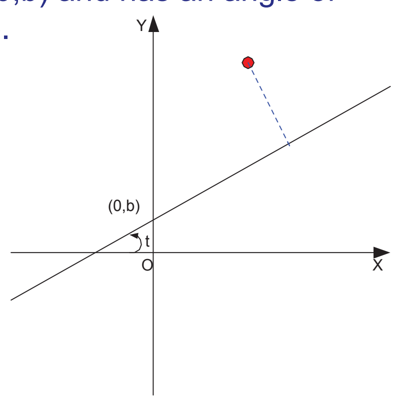
$$B=\begin{bmatrix}\cos 45^\circ & \sin 45^\circ & 0\\ -\sin 45^\circ & \cos 45^\circ & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos(-45^\circ) & -\sin(-45^\circ) & 0\\ \sin(-45^\circ) & \cos(-45^\circ) & 0\\ 0 & 0 & 1\end{bmatrix}$$

C can be considered to be an affine transformation that performs a reflection through the x-axis followed by a uniform scaling with scaling factor 2 with respect to the origin due to

$$C=\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Extra example 3

Q: Derive the transformation matrix for a reflection about a line that passes through point (0,b) and has an angle of t with the x-axis (see the figure).



Ans:

Step 1: Translate (0,b) to origin
Step 2: Rotate -t degrees
Step 3: Mirror reflect about X-axis
Step 4: Rotate t degrees
Step 5: Translate origin to (0,b)

The final matrix is the multiplication of 5 matrices:

M = Tran(0,b) Rot(t) Ref_x Rot(-t) Tran(0,-b)

Extra example 3 (cont)

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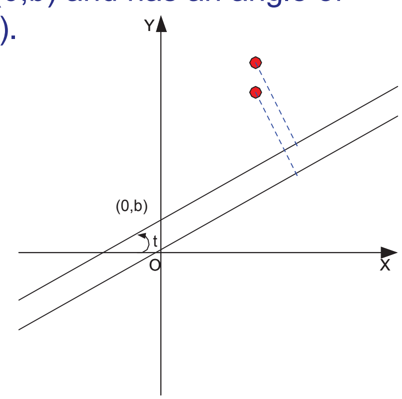

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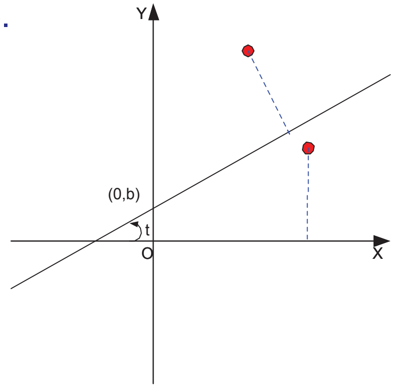

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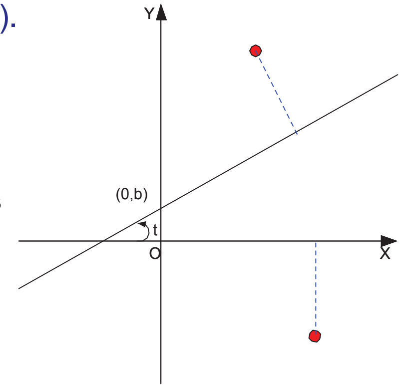

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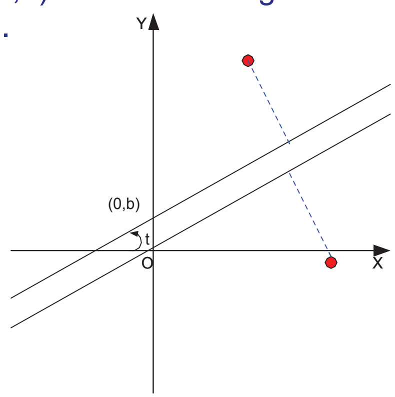

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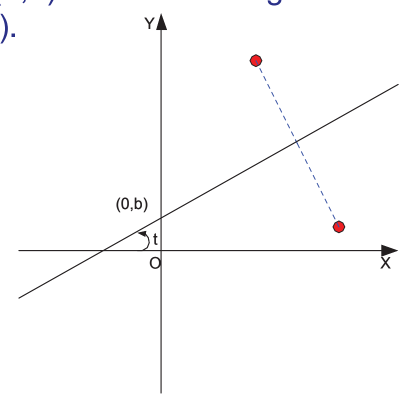

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=

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(-t) & -\sin(-t) & 0 \\ \sin(-t) & \cos(-t) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$