CZ2003 Tutorial 9 (2020/2021, Semester 1)

3D Transformations

1. The following matrix defines a 3D rotation transformation

$$\begin{bmatrix} \cos 30^\circ & 0 & -\sin 30^\circ & 0 \\ 0 & 1 & 0 & 0 \\ \sin 30^\circ & 0 & \cos 30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Identify the rotation axis, rotation direction and rotation angle of this transformation.

2. The following VRML code defines a Transform node:

Transform {
rotation 1 0 1 3.1415926
scale 1 1 3
translation 2 0 1
children [...]
}

Assuming a column represented position vector, write in a proper order the individual matrices composing this transformation. The final matrix is not required.

3. Assuming a column represented position vector, write in a proper order individual matrices implementing the transformation of reflection about a straight line defined parametrically by $x = 1 - t, y = 0, z = 2t, \quad t \in (-\infty, \infty)$. The final matrix is not required.

4. A semi-circle on the ZX plane is shown in Fig.4 (left). It undergoes a sweeping by a full rotation about the Y-axis and a translation along the Y-axis by 2 units simultaneously, which produces a surface as shown in Fig.4 (right). By utilizing transformation matrices, derive a parametric representation of the surface.

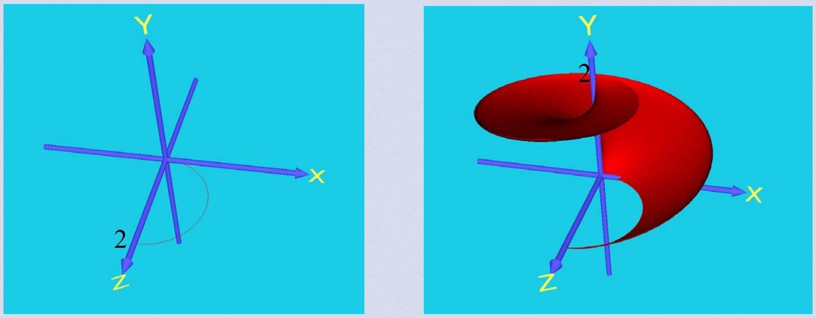


Fig.4

1. The following matrix defines a 3D rotation transformation

$$\begin{bmatrix} \cos 30^\circ & 0 & -\sin 30^\circ & 0 \\ 0 & 1 & 0 & 0 \\ \sin 30^\circ & 0 & \cos 30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-30) & 0 & \sin(-30) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-30) & 0 & \cos(-30) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identify the rotation axis, rotation direction and rotation angle of this transformation.

Rotation axis: Y axis

Rotation angle: 30°

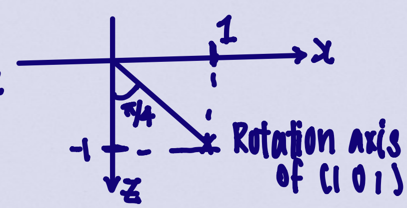
Rotation direction: clockwise.

2. The following VRML code defines a Transform node:

```
Transform {
        rotation     1 0 1   3.1415926
        scale        1 1 3
        translation  2 0 1
children [...]
}
```

$a_x\ a_y\ a_z$ θ

↳ positive so rotate z axis



Assuming a column represented position vector, write in a proper order the individual matrices composing this transformation. The final matrix is not required.

order is always scale first, then rotation the finally translation.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{translation by } \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate about axis y. to set back the rotation axis}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate about axis z}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate about Y axis to set rotation axis to Z axis}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{scale x, y, z axes.}} \vdots$$

3. Assuming a column represented position vector, write in a proper order individual matrices implementing the transformation of reflection about a straight line defined parametrically by $x = 1 - t, y = 0, z = 2t, \quad t \in (-\infty, \infty)$. The final matrix is not required.

sample 2 points on the line:

when $t=0$

$x=1$
$y=0$
$z=0$

when $t=1$:

$x=0$
$y=0$
$z=2$.

① the direction of the line.

$$\begin{pmatrix}0\\0\\2\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

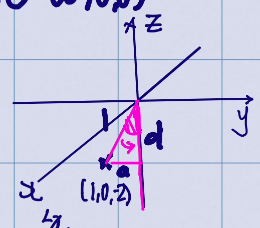
② Perform a translation to move point $(1,0,0)$ to make the line aligned with Z axis

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After translation, points are $(0,0,0)$ and $(1,0,-2)$

$l = \sqrt{(1)+0+(-2)^2} = \sqrt{5}$

$d=-2$
$a=1$



③ Rotation (about Y axis)

$$R_1 = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

④ Reflection about Z axis

$$R_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⑤ Rotation back (about Y axis)

$$R_3 = \begin{bmatrix} 2/\sqrt{5} & 0 & -1/\sqrt{5} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⑥ Translate back

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{individual matrix}}$

$T_2 R_3 R_2 R_1 T_1$ //

4. A semi-circle on the ZX plane is shown in Fig.4 (left). It undergoes a sweeping by a full rotation about the Y-axis and a translation along the Y-axis by 2 units simultaneously, which produces a surface as shown in Fig.4 (right). By utilizing transformation matrices, derive a parametric representation of the surface.

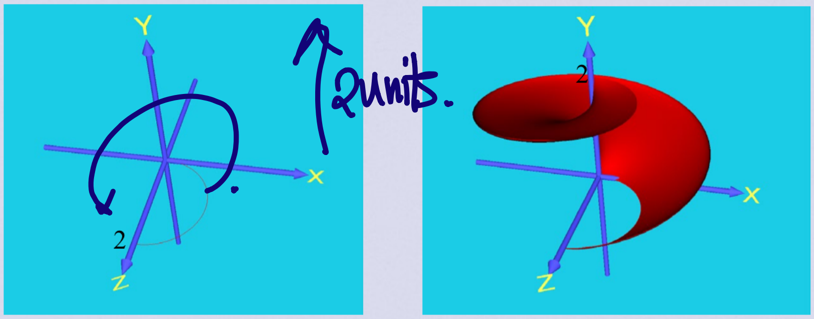


Fig.4

1. parametric equation of semi-circle.

$x = 2\sin(\pi u)$
$y = 0$
$z = 2\cos(\pi u) + 2$

2. construct the rotation matrix. Rotation about the y axis, counter clockwise.

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \theta \in [0, 2\pi]$$

3. deal with the translation.

let $f(\theta)$ be a displacement func

$f(\theta) = A\theta + B$
$f(0) = 0.$ $\quad A \times 0 + B = 0 \Rightarrow B = 0$
$f(2\pi) = 2$ $\quad A \times 2\pi + B = 2$
$A = 2/2\pi$
$= 1/\pi$ $\quad f(\theta) = \theta/\pi$

4. Apply transformations to line segment.

$$\begin{bmatrix} x(u,\theta) \\ y(u,\theta) \\ z(u,\theta) \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta/\pi \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{translation}} \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \end{bmatrix}}_{\text{rotation}} \begin{bmatrix} x_0(u) \\ y_0(u) \\ z_0(u) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 1 & \sin\theta & 0 \\ 0 & 1 & 0 & \theta/\pi \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0(u) \\ y_0(u) \\ z_0(u) \\ 1 \end{bmatrix} \qquad 4\times 4 \;\times\; 4\times 1$$

$$= \begin{bmatrix} x_0(u)\cos\theta + y_0(u) + z_0(u)\sin\theta \\ y_0(u) + \theta/\pi \\ -x_0(u)\sin\theta + z_0(u)\cos\theta \\ 1 \end{bmatrix} \qquad \text{since } y_0(u)=0$$

$$= \begin{bmatrix} x_0(u)\cos\theta + z_0(u)\sin\theta \\ \theta/\pi \\ -x_0(u)\sin\theta + z_0(u)\cos\theta \end{bmatrix}$$

5. Reparameterize

let $\theta = 2\pi v$, $v \in [0,1]$

final eqn are

$$x(u,v) = 2\sin(\pi u)\cos(2\pi v) + (2\cos(\pi u)+2)\sin(2\pi v)$$

$$y(u,v) = \frac{2\pi v}{\pi} = 2v$$

$$z(u,v) = -2\sin(\pi u)\sin(2\pi v) + (2\cos(\pi u)+2)\cos(2\pi v)$$