Geometric Shapes: Surfaces

Module 3
Lecture 4

Geometric Shapes

- Geometry has no color and texture
- Points
- Curves
- **Surfaces**
- Solid objects

Learning objectives

- To understand how surfaces can be used in solving data visualization problems
- To understand surfaces as objects with 2 degree of freedom
- To understand what mathematical representations are the most efficient for defining and displaying surfaces
- To understand how different coordinate systems can be used together for deriving mathematical representations of surfaces
- To understand surfaces as objects created by moving curves

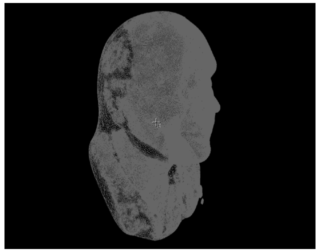
Surfaces

- Polygonal representation - polygon meshes
- Analytic representations
  - Explicit representation
  - Implicit representation
  - Parametric representation

Polygonal Representation

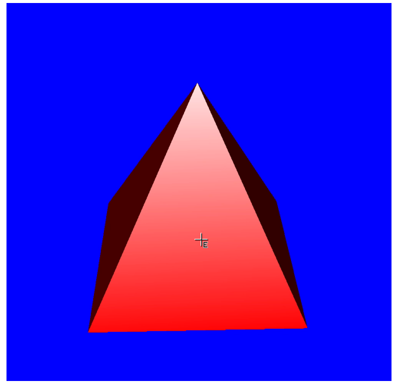
- List of vertices
- List of polygons formed by the vertices
- List of normal vectors built at the vertices (optional)

- Order of vertices is important
- Usually only one visible side where the normal is pointing out from
- Examples of common polygon mesh data formats:.OBJ (Wavefront), .STL (STereo Lithography), Indexed FaceSet in VRML and X3D



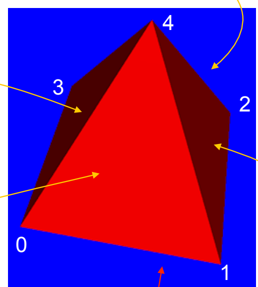
VRML Polygon Mesh

```
coord Coordinate { point [
 0,000000  0.000000   0.000000,
-0.005000 -0.505000 -1.619200,
-0.105000 -0.485000 -1.619200,
. . . . . . . . . . . . . . . . . . . . . . . . . ]  }
normal Normal { vector [
 1.000000  0.000000  0.000000
 0.176000 -0.601000 -0.780000
-0.306000 -0.340000 -0.889000
. . . . . . . . . . . . . . . . . . . . . . . . . ]  }
coordIndex [
70 65 62 -1
23 63 6 -1
18 19 17 -1
. . . . . . . . . . . . . . . . . . . . . . . . . ] }
```



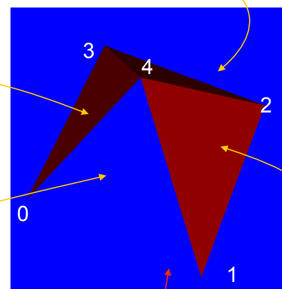
VRML Polygon Mesh

```
geometry IndexedFaceSet {
coord Coordinate { point [
-1.0  -1.0   1.0,     #vertex 0
 1.0   -1.0   1.0,    #vertex 1
 1.0   -1.0  -1.0,    #vertex 2
-1.0  -1.0  -1.0,     #vertex 3
 0.0    1.0   0.0     #vertex 4
 ]  }
coordIndex [
0,  3,  2,  1,  -1,
0,  1,  4, -1,
1,  2,  4, -1,
2,  3,  4, -1,
3,  0,  4, -1,
                  ]
    }
```



VRML Polygon Mesh

```
geometry IndexedFaceSet {
coord Coordinate { point [
-1.0  -1.0   1.0,     #vertex 0
 1.0   -1.0   1.0,    #vertex 1
 1.0   -1.0  -1.0,    #vertex 2
-1.0  -1.0  -1.0,     #vertex 3
 0.0    1.0   0.0     #vertex 4
 ]  }
coordIndex [
0,  3,  2,  1,  -1,
0,  4,  1, -1,
1,  2,  4, -1,
2,  3,  4, -1,
3,  0,  4, -1,
                  ]
    }
```

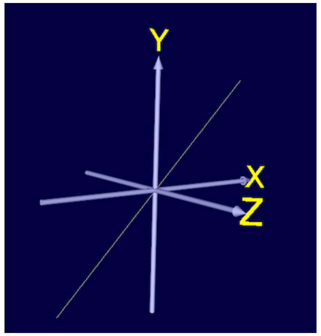


We have learnt that

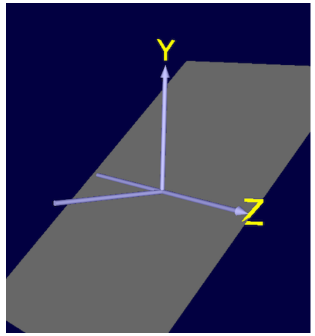
- Surfaces can be defined (interpolated) by polygon meshes
- Polygon meshes are defined by set of vertices, set of normals at the vertices, and polygons formed by the vertices
- Order how the vertices are listed to form a polygon define the direction of the normal to the polygon. Right-hand rule is used to define the normal direction
- Examples of common polygon mesh data formats: .OBJ (Wavefront), .STL (STereo Lithography), Indexed FaceSet in VRML and X3D
- Explicit, implicit and parametric functions can be used for defining surfaces analytically for any precision and compactness of the model

Let’s Increase the Dimension

2D: $y - x = 0$

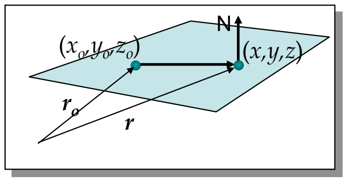


3D: $y - x = 0$



Plane Surface

$$Ax + By + Cz + D = 0$$



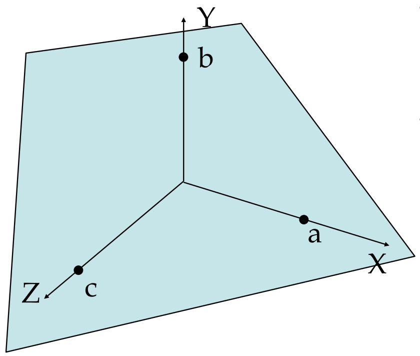
- Implicit function
- Explicit functions can be derived from the implicit but are seldom used
- The values of the coefficients A,B,C, and D can be obtained by solving a set of three plane equations using the coordinates for three non-collinear points in the plane. $\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + 1 = 0$
- N=[A B C] – normal vector to the plane: cross product of two vectors
- For any point $\boldsymbol{r}_o = (x_o, y_o, z_o)$: $\mathbf{N} \bullet (\boldsymbol{r} - \boldsymbol{r}_o) = 0$ i.e., 90° angle

$$A(x - x_o) + B(y - y_0) + C(z - z_o) = 0$$

See the detailed video exercise in Lecture Supplement

Plane Surface

- Implicit equation in intercepts



$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$$

$$Ax + By + Cz + D = 0$$

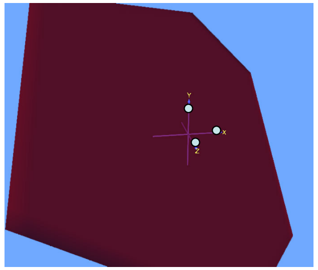
$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + 1 = 0$$

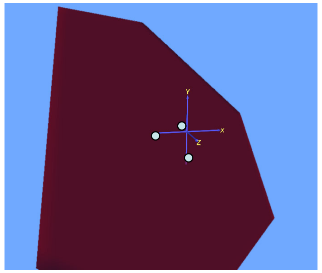
$$\frac{A}{-D}x + \frac{B}{-D}y + \frac{C}{-D}z = 1$$

$$a = -\frac{D}{A}, \quad b = -\frac{D}{B}, \quad c = -\frac{D}{C}$$

Plane Surface Implicitly

$x/1.2+y/1+z/1-1 = 0$ $x/1.2+y/1+z/1+1 = 0$



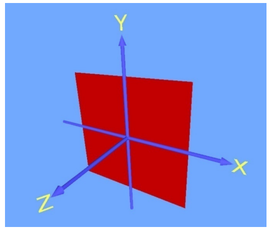


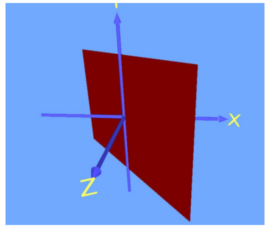
The displayed size of the plane
Is defined by the XYZ domain

Plane Parametrically

$x=u$ $y=v$ $z=0$
$u=[-1, 1]$ $v=[-1, 1]$ – rectangle
$u,v=[-\infty, \infty]$ - plane

$x=u$ $y=v$ $z=u$



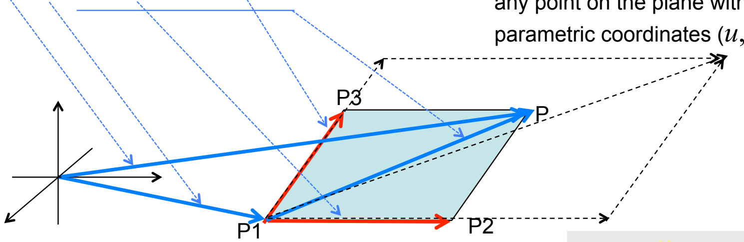


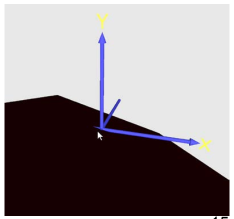
Two parameters !
Parametrically defined rectangle

Any Plane Parametricaly

By the sum of two vectors:
$P=P1+u*(P2-P1)+v*(P3-P1)$, $u,v=[-\infty,\infty]$

any point on the plane with parametric coordinates (u,v)



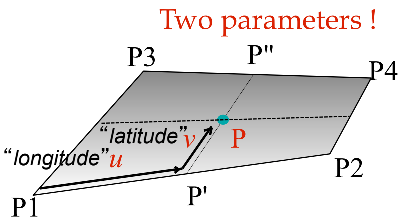


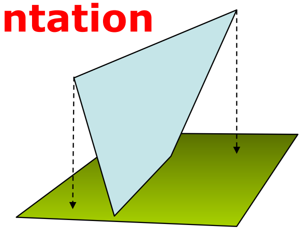
Two parameters !
Parametrically defined plane

Bilinear Surface Parametric Representation

- Line equation parametrically: $P' = P_1 + u(P_2 - P_1)$
- One parametric coordinate: u
- Surface has 2 parametric coordinates (e.g. u, v)
- Let's use crossing lines to define points on the surface

Bilinear Surface
Parametric Representation





$$P' = P_1 + u(P_2 - P_1)$$
$$P'' = P_3 + u(P_4 - P_3)$$
$$P = P' + v(P'' - P')$$

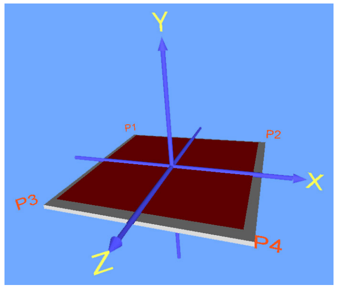
$$P = P_1 + u(P_2 - P_1) + v(P_3 - P_1 + u(P_4 - P_3 - (P_2 - P_1)))$$

Point P_1, P_2, P_3 and P_4 may be any points in a 3D space so that even "twisted" surfaces may result

Bilinear Surface
Parametric Representation

P_1	P_2	P_3	P_4
$x_1=-1$	$x_2=1$	$x_3=-1$	$x_4=1$
$y_1=0$	$y_2=0$	$y_3=0$	$y_4=0$
$z_1=-1$	$z_2=-1$	$z_3=1$	$z_4=1$

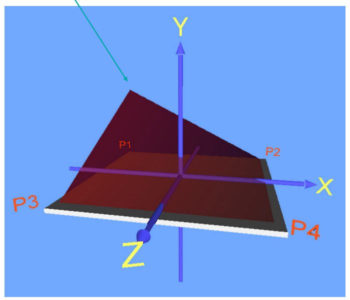
$$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$$
$$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$$
$$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$$



Bilinear Surface
Parametric Representation

P_1	P_2	P_3	P_4
$x_1=-1$	$x_2=1$	$x_3=-1$	$x_4=1$
$y_1=\mathbf{1}$	$y_2=0$	$y_3=0$	$y_4=0$
$z_1=-1$	$z_2=-1$	$z_3=1$	$z_4=1$

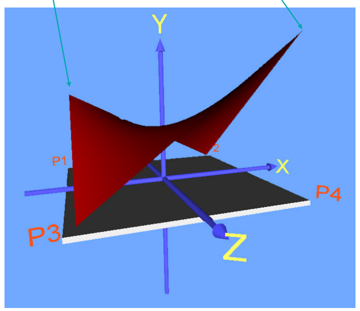
$$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$$
$$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$$
$$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$$



Bilinear Surface
Parametric Representation

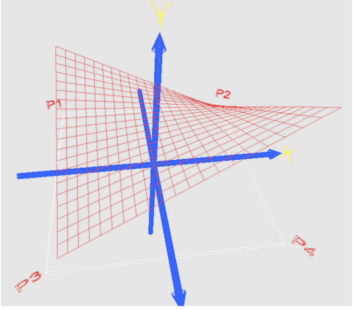
P_1	P_2	P_3	P_4
$x_1=-1$	$x_2=1$	$x_3=-1$	$x_4=1$
$y_1=\mathbf{1}$	$y_2=0$	$y_3=0$	$y_4=\mathbf{1.5}$
$z_1=-1$	$z_2=-1$	$z_3=1$	$z_4=1$

$$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$$
$$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$$
$$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$$



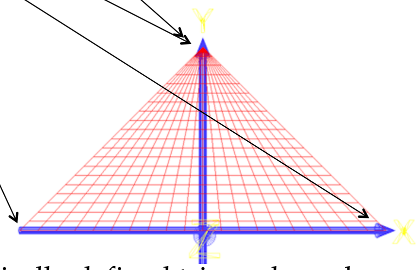
Bilinear Surface Parametric Representation

P_1	P_2	P_3	P_4	
$x_1=-1$	$x_2=1$	$x_3=-1$	$x_4=1$	$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$
$y_1=1$	$y_2=0$	$y_3=0$	$y_4=1.5$	$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$
$z_1=-1$	$z_2=-1$	$z_3=1$	$z_4=1$	$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$



Bilinear Surface Parametric Representation

P_1	P_2	P_3 =	P_4	
$x_1=-1$	$x_2=1$	$x_3=0$	$x_4=0$	$x(u,v)=x_1+u\cdot(x_2-x_1)+v\cdot(x_3-x_1+u\cdot(x_4-x_3-x_2+x_1))$
$y_1=0$	$y_2=0$	$y_3=1$	$y_4=1$	$y(u,v)=y_1+u\cdot(y_2-y_1)+v\cdot(y_3-y_1+u\cdot(y_4-y_3-y_2+y_1))$
$z_1=0$	$z_2=0$	$z_3=0$	$z_4=0$	$z(u,v)=z_1+u\cdot(z_2-z_1)+v\cdot(z_3-z_1+u\cdot(z_4-z_3-z_2+z_1))$



Parametrically defined triangular polygon

Summary

- Plane surfaces can be defined by explicit, implicit and parametric functions
- In implicit (linear) equation Ax+By+Cz+D=0, **N**=[A B C] while D defines displacement from the origin
- To get the plane equation, derive **N** first (e.g., by cross product of two vectors), then substitute any *x,y,z* on the plane to derive D.
- Implicit equation in intercepts can be easily written from x/a+y/b+z/c=1
- Parametric definition of plane is based on linear equations of two parameters
- Bilinear interpolation is an extension of linear interpolation for interpolating functions of two variables (e.g., *u, v*). It performs linear interpolation first in one direction, and then again in the other direction. Although each step is linear, the interpolation as a whole is not linear but rather quadratic in the sample location
- Bilinear surface defines a 4-sided polygon, including non-planar surfaces.
- Bilinear surface can be used for writing parametric functions of a triangle as well: two of the vertices are simply merged together which will also simplify the defining formulas.