CZ2003 Tutorial 8 (2020/2021, Semester 1)

2D Transformations

1. A quadrilateral on the XY plane is defined by four corner points whose homogeneous coordinates are (2, -2, 1), (-6, -3, -3), (0, 0.5, 0.5) and (0, 4, -4). Analyze whether the quadrilateral is a square, a rectangle, or a trapezium.

2. Find an affine transformation that transforms polygon **A** to polygon **A'** with vertex $\mathbf{v_a}$ corresponding to $\mathbf{v_a}'$, $\mathbf{v_b}$ to $\mathbf{v_b}'$, $\mathbf{v_c}$ to $\mathbf{v_c}'$, etc, as shown in Figure Q2. A point P = (2, 3) lies on polygon **A.** Compute the coordinates of the image of P under this transformation.

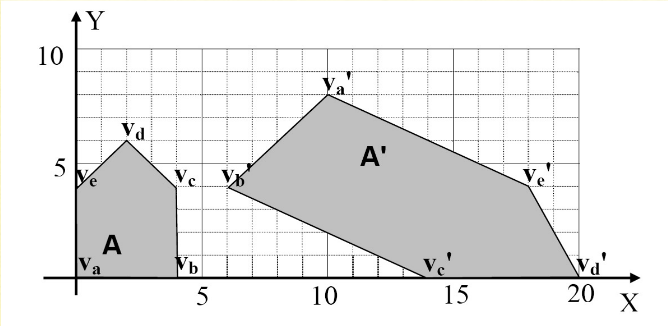


Figure Q2

3. A 2D geometric object is scaled with respect to the point with coordinates (1,1) in the x-coordinate by 5 times and in the y-coordinate by 3 times. Then the object is rotated about the origin by 90° in the clockwise direction. Finally, the object is reflected through the x-axis. Write in a proper order the matrices constituting this transformation.

4. Suppose that **R** and **S** represent a rotation transformation and a scaling transformation. Both transformations are performed with respect to the origin. Discuss the conditions under which **SR** and **RS** define the same transformation.

1. A quadrilateral on the XY plane is defined by four corner points whose homogeneous coordinates are (2, -2, 1), (-6, -3, -3), (0, 0.5, 0.5) and (0, 4, -4). Analyze whether the quadrilateral is a square, a rectangle, or a trapezium.

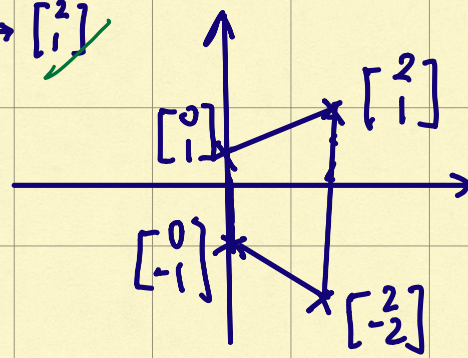
first convert the homogenous coordinates into Cartesian coordinates

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ -2 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} -6 \\ -3 \\ -3 \end{bmatrix} \rightarrow \begin{bmatrix} -6 \\ -3 \\ -3 \end{bmatrix} / -3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix} \checkmark$$



By observation, the quadrilateral is a trapezium.

2. Find an affine transformation that transforms polygon **A** to polygon **A'** with vertex $\mathbf{v_a}$ corresponding to $\mathbf{v_a}$', $\mathbf{v_b}$ to $\mathbf{v_b}$', $\mathbf{v_c}$ to $\mathbf{v_c}$', etc, as shown in Figure Q2. A point P = (2, 3) lies on polygon **A.** Compute the coordinates of the image of P under this transformation.

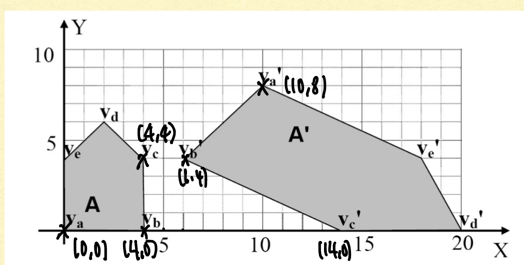


Figure Q2

matrix of affine transformations:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & m \\ c & d & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = ax + by + m$$

$y' = cx + dy + n.$

1. Find affine transformations that transforms A to A', B to B'

$v_A \to v_A'$

$(0,0) \to (10,8)$

$10 = a(0) + b(0) + m$

$m = 10$

$8 = c(0) + d(0) + n$

$n = 8$

$V_B \to V_B'$

$(4,0) \to (6,4)$

$6 = a(4) + b(0) + m$

$4a + 10 = 6$

$a = -1$

$4 = c(4) + d(0) + n$

$4c + 8 = 4$

$c = -1$

$V_C \to V_C'$

$(4,4) \to (14,0)$

$14 = a(4) + b(4) + 10$

$4(-1) + 4b = 4$

$4b = 8$

$b = 2$

$0 = c(4) + d(4) + 8$

$-8 = -4 + d(4)$

$-4 = d(4)$

$d = -1$

2. Compute coordinates of image P.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 10 \\ -1 & -1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$P = (2, 3)$

$x' = ax + by + m$

$= (-1)(2) + (2)(3) + 10$

$= 14$

$y' = cx + dy + n$

$= (-1)(2) + (-1)(3) + 8$

$= 3$

$\therefore$ the coordinates are $(14,3)$

3. A 2D geometric object is scaled with respect to the point with coordinates (1,1) in the x-coordinate by 5 times and in the y-coordinate by 3 times. Then the object is rotated about the origin by 90° in the clockwise direction. Finally, the object is reflected through the x-axis. Write in a proper order the matrices constituting this transformation.

$S(S_x, S_y) = S(5,3)$

Translation back should be directly after scaling as only the scaling is w.r.t to (1,1). The rotation and reflection both w.r.t original origin.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Reflection about x axis}} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rotation } -90^\circ} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{trans back}} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{scale}} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{trans to origin}}$$

4. Suppose that **R** and **S** represent a rotation transformation and a scaling transformation. Both transformations are performed with respect to the origin. Discuss the conditions under which **SR** and **RS** define the same transformation.

Let R be rotation

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let S be the scaling: $\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$SR = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x\cos\theta & -s_x\sin\theta & 0 \\ s_y\sin\theta & s_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RS = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x\cos\theta & -s_y\sin\theta & 0 \\ s_x\sin\theta & s_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$SR = RS$ will only occur if $s_x = s_y$ meaning scaling along x & y by the same scale factor.

So $SR = RS$ iff $s_x \sin\theta = s_y \sin\theta$ which means $s_x = s_y$ or $\sin\theta = 0$

Therefore SR & RS define transformation if the scaling is a uniform scaling or the rotation angle is multiple of π

4. Let **R** be a 2D rotation about the origin and **T** a 2D translation. Do **RT** and **TR** define the same composite transformation? Justify your answer mathematically.