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CE2001/ CZ2001: Algorithms

Analysis of Algorithms
(Recursive Functions)

Dr. Loke Yuan Ren

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Learning Objectives

At the end of this lecture, students should be able to:

- Express the time complexity of recursive functions in terms of recurrent equations
- Solve the recurrent equations to obtain the time complexity function

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Time Complexity of Recursive Functions

For a simple recursive function, determine:

1. Number of primitive operations for each recursive call
2. Number of recursive calls

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Example 1:

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{       if (n == 1) return 1;
        else return n * factorial(n-1);
}
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For each recursive call when n > 1: c_1
For the last recursive call when n is 1: c_2
Number of recursive calls: n-1 (non-terminating case) + 1
Time complexity: $c_1*(n-1) + c_2$ (this is roughly $c_1 n$)

Time Complexity of Recursive Functions

Example 2:

```
int count (int array[ ], int n, int a)
{ // count how many times item 'a' appears in 'array'
    if  (n == 1)
            if (array[0] == a)
                return 1;
            else return 0;
    if (array[0] == a)
            return 1 + count (&array[1], n-1, a);
    else
            return count (&array[1], n-1, a);
}
```

Time Complexity of Recursive Functions

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    else
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}
```

Address of next array element

Time Complexity of Recursive Functions

Example 2:

```
int count (int array[ ], int n, int a)
{ // count how many times item 'a' appears in 'array'
    if (n == 1)
            if (array[0] == a)
                return 1;
            else return 0;
    if (array[0] == a)
            return 1 + count (&array[1], n-1, a);
    else
            return count (&array[1], n-1, a);
}
```


Time Complexity of Recursive Functions

Consider the number of comparisons between an array element and a:

$W_1 = 1$

$W_n = 1 + W_{n-1}$

$W_n = 1 + W_{n-1}$

$= 1 + 1 + W_{n-2}$

$= 1 + 1 + 1 + W_{n-3}$

....

$= 1 + 1 + \ldots + 1 + W_{n-(n-1)}$

$= n$

W_n is the number of comparisons in count (array[], n, a)

W_{n-1} is the number of comparisons in count (&array[1], n-1, a)

Alternative notation:
W(1) = 1
W(n) = 1 + W(n-1)

n-1 1's

Method of Backward Substitutions

Time Complexity of Recursive Functions

Example 3: Consider the number of multiplications

```
preorder (simple_t * tree)
{
    if (tree != NULL)  {
            tree->item *= 10;
            preorder (tree->left);
            preorder (tree->right);
                }
}
```


Time Complexity of Recursive Functions

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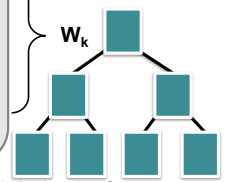
Time Complexity of Recursive Functions

Example 3: Consider the number of multiplications

```
preorder (simple_t * tree)
{
    if (tree != NULL)  {
            tree->item *= 10;      ⟶ 1
            preorder (tree->left);     ⟶ W_L
            preorder (tree->right);    ⟶ W_R
                }
}
```



Let W_k be the no. of multiplications for a tree with **k** depth of the tree.

$W_0 = 0$

$W_k = 1 + W_L + W_R$

Alternative notation:
W(0) = 0
W(n) = 1 + W(L) + W(R)

Time Complexity of Recursive Functions

- Assume that it is a complete binary tree
- The number of nodes is $2^k - 1$ (k is the depth of the tree)

$W_0 = 0$

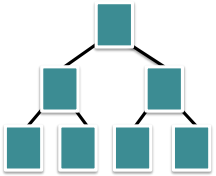
$W_1 = 1$

$W_2 = 1 + W_1 + W_1 = 1 + 2 = 3$

$W_3 = 1 + W_2 + W_2 = 1 + 2\ (1+2) = 1+2+4 = 7$

…

$W_k = 1 + W_{k-1} + W_{k-1} = 1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$



- The # of recursive calls is a geometric series
- Prove the hypothesis can be done by mathematical induction
- Method of Forward Substitutions

Time Complexity of Recursive Functions

```
int fibonacci (int n){
    if( n < 1) return 0;
    if(n == 1 || n ==2) return 1;
    return fibonacci (n-1) + fibonacci (n-2);
}
```

Example 4: Consider the number of function calls

W(n) = W(n-1) + W(n-2)+1

W(1) = W(2) = 1

Neither Forward Substitutions nor Backward Substitutions are able to solve the time complexity.

Approximation Solution

```
int fibonacci (int n){
    if( n < 1) return 0;
    if(n == 1 || n ==2) return 1;
    return fibonacci (n-1) + fibonacci (n-2);
}
```

W(n) = W(n-1) + W(n-2)+1
W(1) = W(2) = 1

Example 4: Consider the number of function calls

$$W(n) = W(n-1) + W(n-2) + 1$$

$$\approx 2W(n-1) + 1$$

$$= 2\ (2\ \text{W(n-2)}+1) + 1$$

$$= 2\ (2\ (\ 2\text{W(n-3)}+1) + 2 + 1$$

…

$$= 2^{n-3}\left(2W\left(n-(n-2)\right)+1\right) + 2^{n-4} + 2^{n-5} + \ldots + 4 + 2 + 1$$

$$= 2^{n-2} + 2^{n-3} + 2^{n-4} + 2^{n-5} + \ldots + 4 + 2 + 1$$

$$= 2^{n-1} - 1$$

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Time Complexity of Recursive Functions

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Can we solve it exactly?

Second-Order Linear Recurrences with Constant Coefficients

$$aW(n) + b\,W(n-1) + c\,W(n-2) = f(n)$$

- Constant coefficients => a, b and c are constant
- Second order => W(n) and W(n-2) are two position apart in the sequence
- f(n)=
  - 0 => homogeneous
  - Otherwise => non-homogeneous

Second-Order Linear Recurrences with Constant Coefficients

$$aW(n) + b\,W(n-1) + c\,W(n-2) = f(n)$$

- Its characteristic equation is

$$ar^2 + br + c = 0$$

- The roots of the characteristic equation are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Two real and distinct root: r_+ and r_- $(b^2 - 4ac > 0)$
2. A real double root: $r_+ == r_-$ $(b^2 - 4ac = 0)$
3. Two distinct root: $b^2 - 4ac < 0$

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A linear combination of two geometric sequence

Homogeneous solution

$W^h(n) = \alpha r_+^n + \beta r_-^n$

$W^h(n) = \alpha r^n + \beta n r^n$

$W^h(n) = \gamma^n[\alpha \cos n\theta + i\beta \sin n\theta]$

Remarks: Case 1 and case 3 are just real and complex roots. The solution is just a trigonometric form of complex number

Second-Order Linear Recurrences with Constant Coefficients

$$aW(n) + b\,W(n-1) + c\,W(n-2) = kr_p^n$$

- The general solution

$$W(n) = W^h(n) + W^p(n)$$

Homogeneous solution

Particular solution

- The particular solution, $W^p(n)$

1. Not equal to any root: $r_p \neq r_+, r_p \neq r_-$
2. Equal to one of the roots: $r_p = r_+$ or $r_p = r_-$
3. Equal to a double root: $r_p = r_+ = r_-$

Particular solution

$W^p(n) = Ar_p^n$

$W^p(n) = Anr_p^n$

$W^p(n) = An^2 r_p^n$

Time Complexity of Recursive Functions

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    if( n < 1) return 0;
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Example 4: Consider the number of function calls

W(n) = W(n-1) + W(n-2)+1

W(1) = W(2) = 1

Neither Forward Substitutions nor Backward Substitutions are able to solve the time complexity.

Can we solve it exactly?

W(n) - W(n-1) - W(n-2) = 1

Time Complexity of Recursive Functions

W(n) - W(n-1) - W(n-2) = 1

$$W(n) = W^h(n) + W^p(n)$$

$W^h(n)$:

The root of the characteristic equation:

$$r = \frac{-1 \pm \sqrt{5}}{2}$$

The homogeneous solution is

$$W^h(n) = \alpha(\frac{1+\sqrt{5}}{2})^n + \beta(\frac{1-\sqrt{5}}{2})^n$$

$W^p(n)$:

$f(n) = 1 \cdot 1^n$

Not equal to any root

The particular solution is

$W^p(n) = A$

$$W(n) = W^h(n) + W^p(n)$$

$$W(n) = \alpha(\frac{1+\sqrt{5}}{2})^n + \beta(\frac{1-\sqrt{5}}{2})^n + A$$

$$W^p(n) - W^p(n-1) - W^p(n-2) = 1$$

$$A - A - A = 1$$

$$A = -1$$

$$W(n) = \alpha(\frac{1+\sqrt{5}}{2})^n + \beta(\frac{1-\sqrt{5}}{2})^n - 1$$

Substitute W(1) = W(2) = 1,

$$W(n) = \frac{2}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{2}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n - 1$$

Time Complexity of Recursive Functions

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}
```

n	W(n)
1	1
2	1
3	3
4	5
5	9
6	15
7	25
8	41
9	67
10	109
11	177
12	287

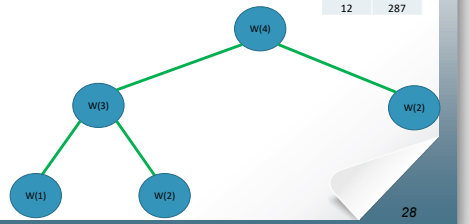
How many function calls are there in this algorithm?

W(n) = W(n-1) + W(n-2) + 1

W(1) = W(2) = 1

$$W(n) = \frac{2}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{2}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n - 1$$

$$\approx 1.618^n$$



Summary

Analysing Recursive Functions by:

- Expressing the time complexity of recursive functions in terms of recurrent equations
- Solving the recurrent equations to obtain the time complexity function
  - Method of forward substitutions
  - Method of backward substitutions
  - 2nd order linear recurrences

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Analysis of Algorithms

(Order of Growth of Time Complexity Functions)

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Learning Objectives

At the end of this lecture, students should be able to:

- Appreciate the order of growth of different time complexity functions
- Analyse the impact of faster algorithms versus faster computers

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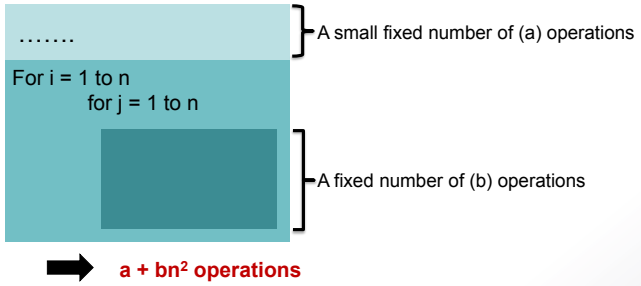
Growth Rate of Time Complexity

We are concerned with the growth rate of resources

1. Constants are not significant when n is large
2. Multipliers are not important for relative growth rate



➡ **$a + bn^2$ operations**

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How Functions Grow?

Algo	1	2	3	4	5	6
Operation (μsec)	$13n$	$13n\log_2 n$	$13n^2$	$130n^2$	$13n^2+10^2$	2^n

How Functions Grow?

Algo	1	2	3	4	5	6
Operation (µsec)	13n	$13n\log_2 n$	$13n^2$	$130n^2$	$13n^2+10^2$	2^n

Input (n) — Time Cost (Second)

10	.00013	.00043	.0013	.013	.0014	.001024
100	.0013	.0086	.13	1.3	.1301	$4*10^{16}$ yrs
10^4	.13	1.73	22 mins	3.61 hrs	22 mins	
10^6	13	259	150 days	1505 days	150 days	

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How Functions Grow?

Algo	1	2	3	4	5	6
Operation (µsec)	13n	13nlogn	$13n^2$	$130n^2$	$13n^2+10^2$	2^n

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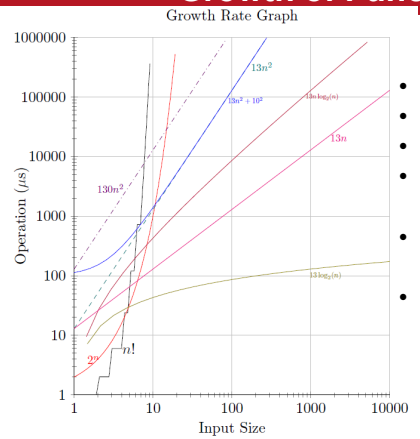
Growth of Functions



- n! is the fastest growth
- 2^n is the second
- 13n is linear
- $13\log_2 n$ is the slowest

- 10^2 can be ignored when n is large
- $13n^2$ and $130n^2$ have similar growth.
  - $130n^2$ slightly faster

Faster Computer Or Faster Algorithm?

- You have devised an algorithm to solve a problem
- Algorithm takes too long to run
- Get a faster computer or improve the algorithm?
- Problem size does not remain the same with faster computer
- People want to do more work with faster computer
- Hence, problem remains

How Would Faster Computer Help?

Scenario

- Suppose old computer executes 10,000 basic operations per hour
- New computer, which is 10 times faster, executes 100,000 basic operations per hour
- If old computer solves a problem of size n in one hour, what is the largest problem the new computer can solve in one hour?

How Would Faster Computer Help?

Findings

- For linear complexity problem, the improvement is 10 times
- For all faster-growing than linear (harder) problems , the improvement is poorer than the linear problems'
- For exponential complexity problem, there is hardly any improvement

How Would Faster Computer Help?

f(n)	n	n'	Change	n'/n
10n	1,000	10,000	n'=10n	10
20n	500	5,000	n'=10n	10
$5n \log_2 n$	250	1,842	3.16n<n'<10n	7.37
$2n^2$	70	223	n'=3.16n	3.16
2^n	13	16	n'=n+3	1.23

How Would Faster Computer Help?

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		5,0		10
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n: Largest problem size solved by old computer, which performs 10,000 operations in 1 hour

n': Largest problem size solved by new computer, which performs 100,000 operations in 1 hour

How Would Faster Computer Help?

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$2n^2$	70	223	n'=3.16n	3.16
2^n	13	16	n'=n+3	1.23

How Would Faster Computer Help?

f(n)	n	n'	Change	n'/n
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20n	500	5,000	n'=10n	10
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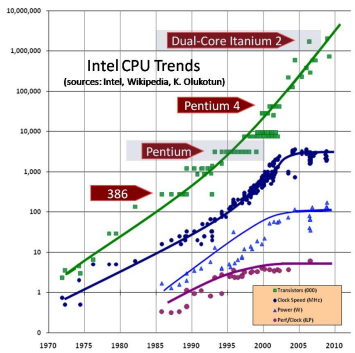
How About Improving The Algorithm?

- **Suppose complexity function is n^2**
  - When n = 1024, it takes 1024×1024 = 1,048,576 primitive steps to run
- **Suppose complexity function is reduced to n $\log_2$ n**
  - Then for n = 1024, it takes 10,240 primitive steps to run
  - More than a factor of 100 times improvement
  - What is the improvement when n = 2048?
  - n^2 = 2048 x 2048 = 4,194,304
  - $n\log_2 n$ = 2048 x $\log_2$ 2048 = 22,528
- **A factor of 200 times improvement!**

Moore's Law



- number of transistors doubles every 2 years
- The cost of computers is halved
- Clock speed is limited at 5Ghz
- It is always good that you have a more efficient algorithm.

Summary

- Order of Growth of Functions
- When a faster computer can help
- Importance of having faster algorithms