NANYANG TECHNOLOGICAL UNIVERSITY

CE2001/ CZ2001: Algorithms

Mergesort

Ke Yiping, Kelly

NANYANG TECHNOLOGICAL UNIVERSITY

Learning Objectives

At the end of this lecture, students should be able to:

- Explain the approach of Divide and Conquer
- Describe how Mergesort works by:
  - Recalling the pseudo code
  - Manually executing the algorithm on a toy input array
- Analyse the time complexity of Mergesort, by using:
  - Recurrence equation
  - Recursion tree

NANYANG TECHNOLOGICAL UNIVERSITY

Mergesort
(Divide and Conquer Approach)

NANYANG TECHNOLOGICAL UNIVERSITY

Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```

Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```


Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```


Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```


Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```

Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```


Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```

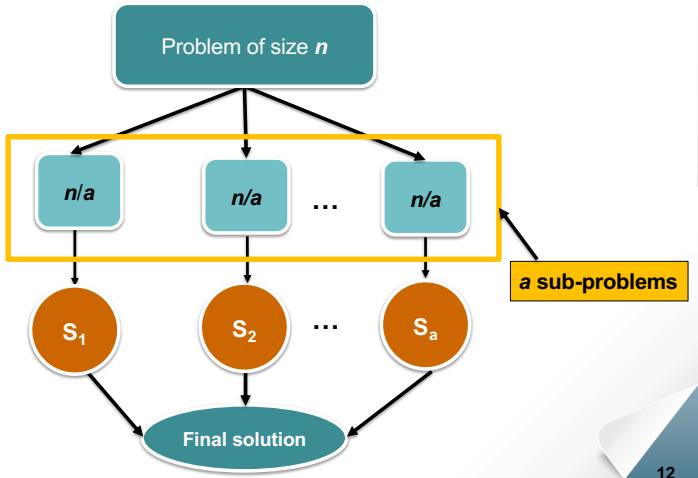

Mergesort

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
        divide the problem into p1, p2, … , pk;
        for each sub-problem ps
            solutions = solve (ps);
        combine all solutions;
    }
}
```


Mergesort



NANYANG TECHNOLOGICAL UNIVERSITY

Mergesort (Algorithm)

NANYANG TECHNOLOGICAL UNIVERSITY

Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```


NANYANG TECHNOLOGICAL UNIVERSITY

Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```


NANYANG TECHNOLOGICAL UNIVERSITY

Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```

Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```


Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```


Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```


Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```

Mergesort (Algorithm)

```
mergeSort(list) {
        if (length of list > 1) {
            Partition list into two (approx.) equal sized
                lists, L1 & L2;
            mergeSort (L1);
            mergeSort (L2);
            merge the sorted L1 & L2;
        }
}
```


Mergesort
(Overview of Pseudo Code)

Mergesort

```
void mergesort(int n, int m)
{   int mid = (n+m)/2;
    if (m-n <= 0)
        return;
    else if (m-n > 1) {
        mergesort(n, mid);
        mergesort(mid+1, m);
    }
    merge(n, m);
}
```

5	4	3	7	6

Mergesort

```
void mergesort(int n, int m)
{   int mid = (n+m)/2;
    if (m-n <= 0)
        return;
    else if (m-n > 1) {
        mergesort(n, mid);
        mergesort(mid+1, m);
    }
    merge(n, m);
}
```

5	4	3	7	6

n m

Mergesort

```
void mergesort(int n, int m)
{  int mid = (n+m)/2;
   if (m-n <= 0)
        return;
   else if (m-n > 1) {
        mergesort(n, mid);
        mergesort(mid+1, m);
   }
   merge(n, m);
}
```

Mergesort

```
void mergesort(int n, int m)
{  int mid = (n+m)/2;
   if (m-n <= 0)
        return;
   else if (m-n > 1) {
        mergesort(n, mid);
        mergesort(mid+1, m);
   }
   merge(n, m);
}
```

if m-n = 0,

m = n

if m-n < 0, Empty array

Mergesort

```
void mergesort(int n, int m)
{  int mid = (n+m)/2;
   if (m-n <= 0)
        return;
   else if (m-n > 1) {
        mergesort(n, mid);
        mergesort(mid+1, m);
   }
   merge(n, m);
}
```

Mergesort (Example)

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
90	25	10	71	94	22	59	74

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
90	25	10	71	94	22	59	74

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
90	25	10	71	94	22	59	74

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
90	25	10	71	94	22	59	74

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
25	90	10	71	94	22	59	74

1 key comparison in merging

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
25	90	10	71	94	22	59	74

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
25	90	10	71	94	22	59	74

1 key comparison in merging

Mergesort

Sort in ascending order

0	1	2	3	4	5	6	7
10	25	71	90	94	22	59	74

3 key comparison in merging

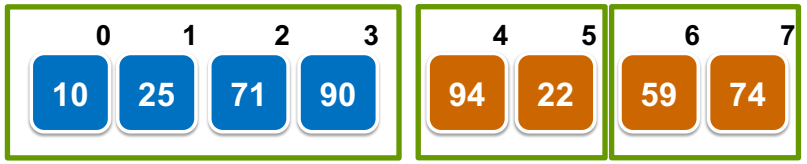
Mergesort

Sort in ascending order

3 key comparison in merging

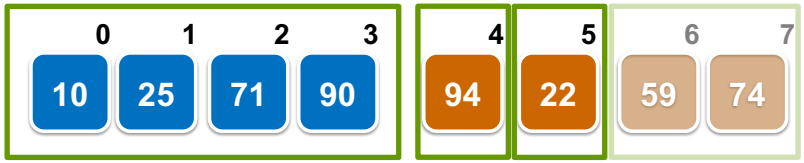
Mergesort

Sort in ascending order



Mergesort

Sort in ascending order



Mergesort

Sort in ascending order

1 key comparison in merging

Mergesort

Sort in ascending order

Mergesort

Sort in ascending order

1 key comparison in merging

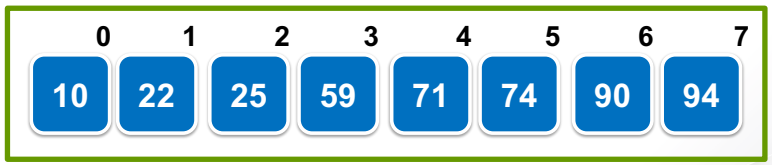
Mergesort

Sort in ascending order

3 key comparison in merging

Mergesort

Sorted in ascending order



7 key comparisons in merging

NANYANG TECHNOLOGICAL UNIVERSITY

Merge (Pseudo Code)

NANYANG TECHNOLOGICAL UNIVERSITY

Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```


NANYANG TECHNOLOGICAL UNIVERSITY

Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```


NANYANG TECHNOLOGICAL UNIVERSITY

Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```

Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```


Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```


Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```


Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```

Merge (Pseudo Code)

```
void merge(int n, int m) {
        if (m-n <= 0) return;
        divide the list into 2 halves;  // both halves are sorted
        while (both halves are not empty) {
                compare the 1st elements of the 2 halves;  // 1 comparison
                if (1st element of 1st half is smaller)
                        1st element of 1st half joins the end of the merged list;
                else if (1st element of 2nd half is smaller)
                        move the 1st element of 2nd half to the end of the
                        merged list;
```


Merge (Pseudo Code)

```
else {   // the 1st elements of the 2 halves are equal
                if (they are the last elements)   break;
                1st element of 1st half joins end of the merged list;
                move the 1st element of 2nd half to the end of the
    merged list;
        }
    } // end of while loop;
} // end of merge
```


Merge (Pseudo Code)

```
else {   // the 1st elements of the 2 halves are equal
                if (they are the last elements)   break;
                1st element of 1st half joins end of the merged list;
                move the 1st element of 2nd half to the end of the
    merged list;
        }
    } // end of while loop;
} // end of merge
```


Merge (Pseudo Code)

```
else {   // the 1st elements of the 2 halves are equal
                if (they are the last elements)   break;
                1st element of 1st half joins end of the merged list;
                move the 1st element of 2nd half to the end of the
    merged list;
        }
    } // end of while loop;
} // end of merge
```

Challenge:
How to do it without auxiliary storage for the merged list?

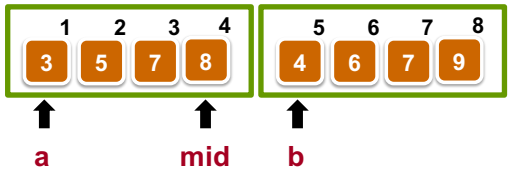
NANYANG TECHNOLOGICAL UNIVERSITY

Merge (Case Scenarios)

NANYANG TECHNOLOGICAL UNIVERSITY

Merge (Case Scenarios)

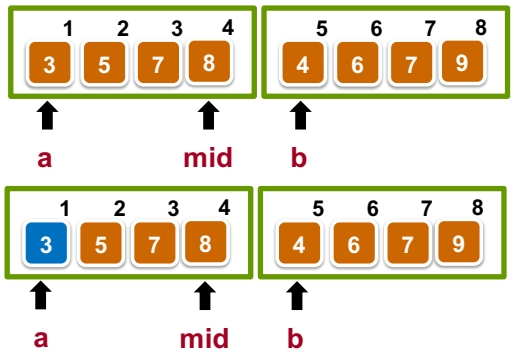
Case 1: 1st element of 1st half is smaller



NANYANG TECHNOLOGICAL UNIVERSITY

Merge (Case Scenarios)

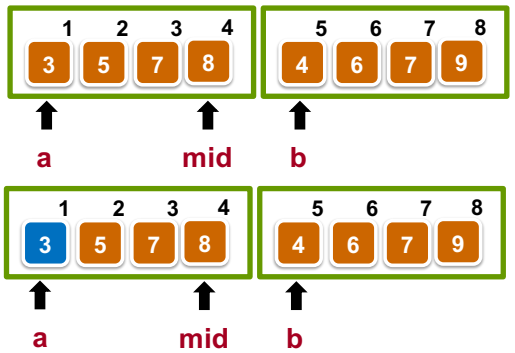
Case 1: 1st element of 1st half is smaller



NANYANG TECHNOLOGICAL UNIVERSITY

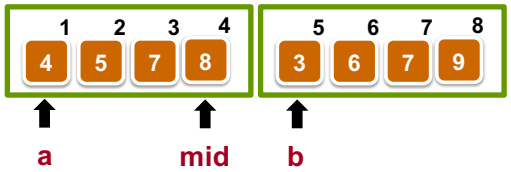
Merge (Case Scenarios)

Case 1: 1st element of 1st half is smaller



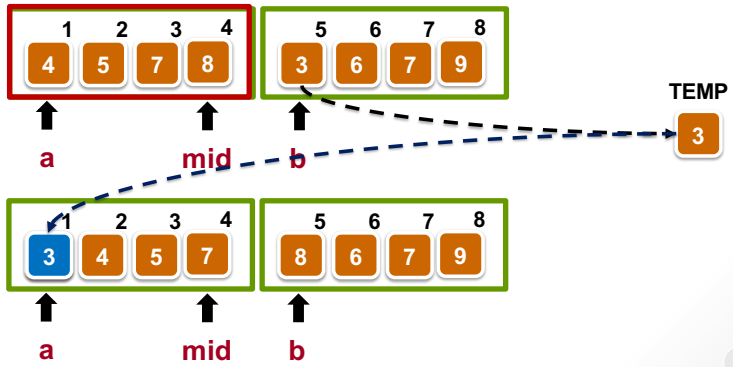
Merge (Case Scenarios)

Case 2: 1st element of 2nd half is smaller



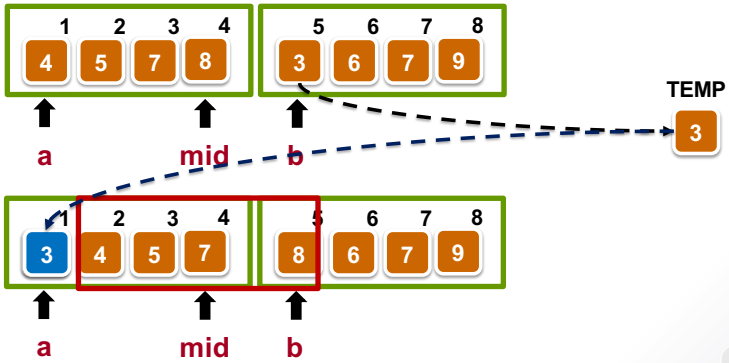
Merge (Case Scenarios)

Case 2: 1st element of 2nd half is smaller



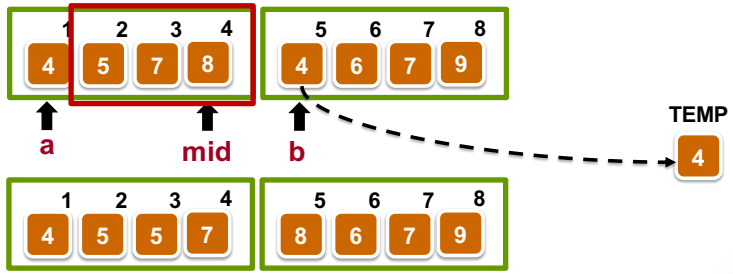
Merge (Case Scenarios)

Case 2: 1st element of 2nd half is smaller



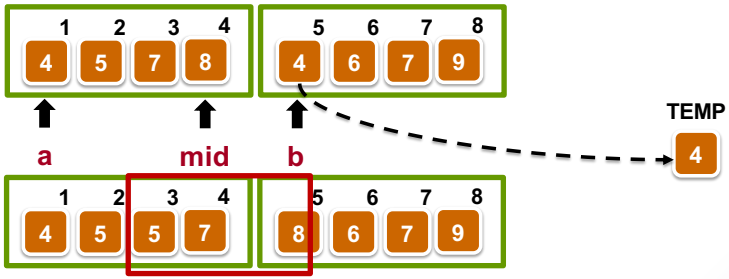
Merge (Case Scenarios)

Case 3: 1st element of 2nd half is equal



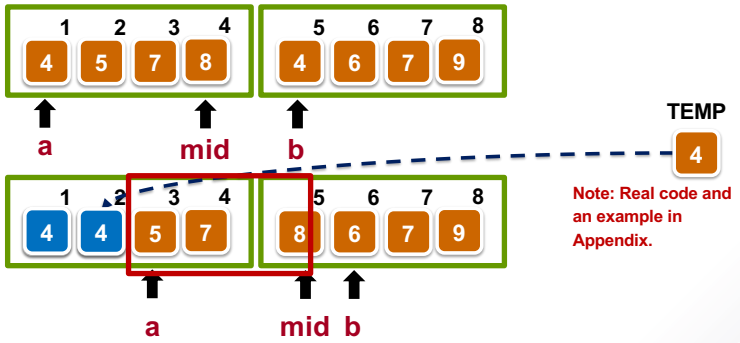
Merge (Case Scenarios)

Case 3: 1st element of 2nd half is equal



Merge (Case Scenarios)

Case 3: 1st element of 2nd half is equal



Note: Real code and an example in Appendix.

Mergesort Algorithm (Recap)

Mergesort Algorithm (Recap)

- Since merging is performed directly on the original array, swapping and shifting are needed
- mergesort() partitions a contiguous array of elements between index n and m into two subarrays

```
void mergesort(int n, int m)
{   int mid = (n+m)/2;
   if (m-n <= 0)
      return;
   else if (m-n > 1) {
      mergesort(n, mid);
      mergesort(mid+1, m);
   }
   merge(n, m);
}
```

Mergesort Algorithm (Recap)

- Since merging is performed directly on the original array, swapping and shifting are needed
- mergesort() partitions a contiguous array of elements between index n and m into two subarrays
- Recursively partitions until m-n<=0, then merge the resulting two subarrays

```
void mergesort(int n, int m)
{   int mid = (n+m)/2;
    if (m-n <= 0)
        return;
    else if (m-n > 1)
        …..
}
```


Mergesort Algorithm (Recap)

- Since merging is performed directly on the original array, swapping and shifting are needed
- mergesort() partitions a contiguous array of elements between index n and m into two subarrays
- Recursively partitions until m-n<=0, then merge the resulting two subarrays
- merge() function merges two sub-arrays of elements between index n and 'mid', and between 'mid+1' and m

```
void mergesort(int n, int m)
{   int mid = (n+m)/2;
    if (m-n <= 0)
    ……..
    merge(n, m);
}
```

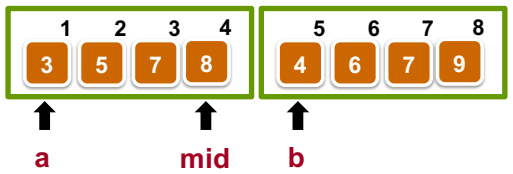

Mergesort Algorithm (Recap)

- Since merging is performed directly on the original array, swapping and shifting are needed
- mergesort() partitions a contiguous array of elements between index n and m into two subarrays
- Recursively partitions until m-n<=0, then merge the resulting two subarrays
- merge() function merges two sub-arrays of elements between index n and 'mid', and between 'mid+1' and m
- During merging, one element from each subarray is compared and the smaller one is inserted into new list

```
void mergesort(int n, int m)
{  …..
   merge(n, m);
}
```

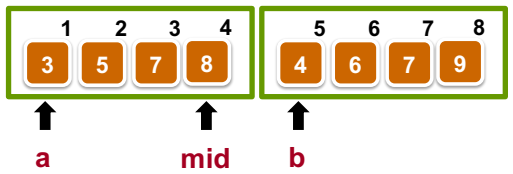

Mergesort Algorithm (Recap)

- Left subarray runs from n to 'mid' with a as running index; right subarray runs from mid+1 to m with b as running index



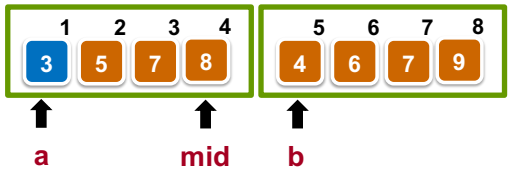
Mergesort Algorithm (Recap)

- Left subarray runs from n to 'mid' with a as running index; right subarray runs from mid+1 to m with b as running index
- slot[a] is the head element of left subarray, slot[b] is the head element of right subarray



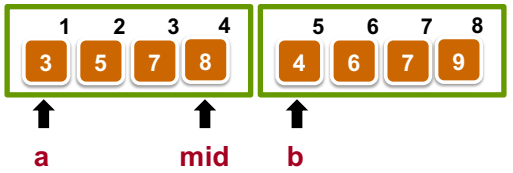
Mergesort Algorithm (Recap)

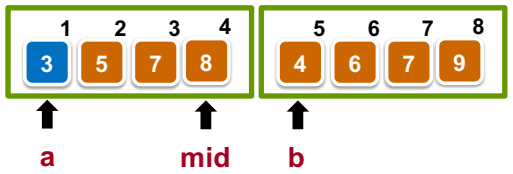
- Left subarray runs from n to 'mid' with a as running index; right subarray runs from mid+1 to m with b as running index
- slot[a] is the head element of left subarray, slot[b] is the head element of right subarray
- During merging, both left and right subarrays shrink towards the right to make space for the newly merged array



Mergesort Algorithm (Recap)

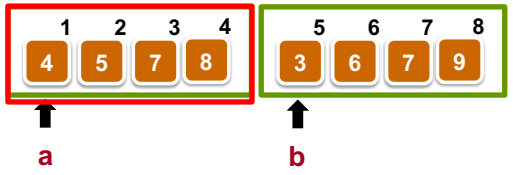
Case 1: if slot[a] < slot[b], there is nothing much to do since smaller element already in correct position (with regard to the merged array)

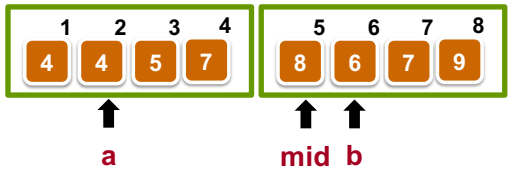




Mergesort Algorithm (Recap)

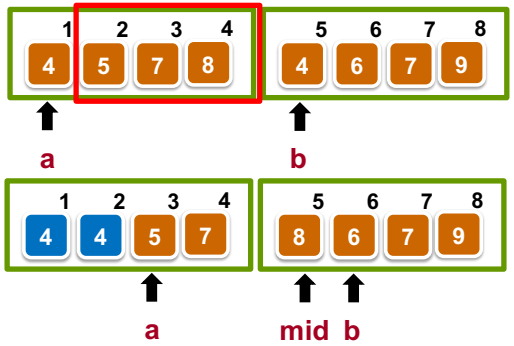
Case 2: if slot[a] > slot[b], then Right-shift (by one) elements of left subarray from index a to 'mid' and insert element at slot[b] into slot[a]





Mergesort Algorithm (Recap)

Case 3: if slot[a] == slot[b], then slot[a] is in the correct position. So, move slot[b] next to beside slot[a], by Right-shifting and swapping



Complexity of Mergesort

Complexity of Mergesort

- After each comparison of keys from the two sub-lists, at least one element is moved to the new merged list and never compared again
- After the last key comparison, at least two elements will be moved into the merged list
- Thus, to merge two sub-lists of n elements in total, the number of key comparisons needed is at most n − 1

Complexity of Mergesort

```
void mergesort(int s, int e) // s=start, e=end
{   int mid = (s+e)/2;
    if (e-s <= 0) return;                  → W(1) = 0
    else if (e-s > 1) {
        mergesort(s, mid);                 → W(n/2)
        mergesort(mid+1, e);               → W(n/2)
    }
    merge(s, e);                           → Worst case: n-1
}
```

W(n)

Complexity of Mergesort

Mergesort performance (assume $n = 2^k$)

$k = \lg n$

Worst case :

$W(1) = 0,$

$W(n) = W(n/2) + W(n/2) + n-1$ Or

$W(2^k) = 2W(2^{k-1}) + 2^k - 1$

$= 2(2W(2^{k-2}) + 2^{k-1} - 1) + 2^k - 1$

$= 2^2W(2^{k-2}) + 2^k - 2 + 2^k - 1$

$= 2^2(2W(2^{k-3}) + 2^{k-2} - 1) + 2^k - 2 + 2^k - 1$

$= 2^3W(2^{k-3}) + 2^k - 2^2 + 2^k - 2 + 2^k - 1$

…

$= 2^kW(2^{k-k}) + k2^k - (1 + 2 + 4 + \ldots + 2^{k-1})$

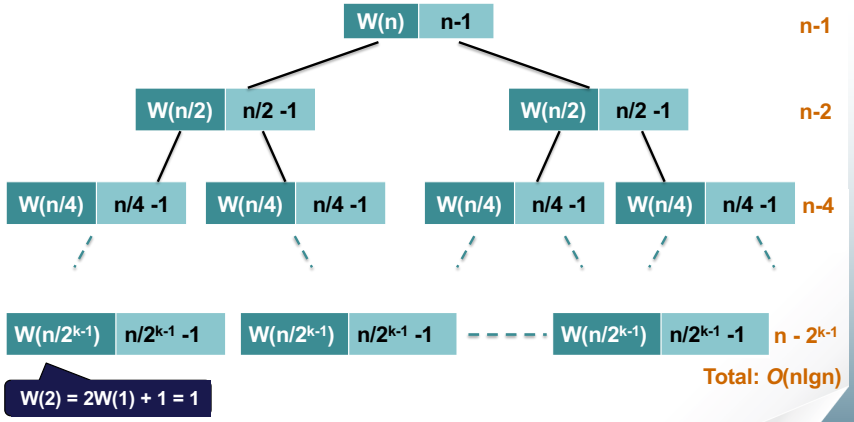
$= k2^k - (2^k - 1)$

$= n \lg n - (n - 1)$

$= O(n \lg n)$

Geometric series

Visually :Recursion Tree



Height of tree is $k = O(\lg n)$

Evaluation of Mergesort

☺ **Strengths:**

- Simple and good runtime behavior
- Easy to implement when using linked list

☹ **Weaknesses:**

- Difficult to implement for contiguous data storage such as array without auxiliary storage (requires data movements during merging)

Summary

- Mergesort uses the Divide and Conquer approach.
- It recursively divide a list into two halves of approximately equal sizes, until the sub-list is too small (no more than two elements).
- Then, it recursively merges two sorted sub-lists into one sorted list.
- The worst-case running time for merging two sorted lists of total size n is $n - 1$ key comparisons.
- The running time of Mergesort is $O(n \lg n)$.