BC2410: Prescriptive Analytics 6 Feb. 2022

Linear Optimization Problems

1. A company manufactures 3 products a, b and c, which sells \$ 14, \$ 15 and \$ 22 per unit respectively. These prices are constant and independent of the market state they are addressed to, and it is also supposed that any produced quantity can be sold. For the manufacturing of these products four types of raw materials are required. The prices of raw materials, the raw material units needed for each product type and the corresponding available quantities within a certain time period are included in the following table.

Raw material	Unit price	Products a	b	c	Available raw material units
1	3	0	2	3	50
2	2	3	2	1	200
3	0. 5	4	4	6	200
4	1	0	0	2	100

   The company's goal is to determine the quantities of each product which should be produced in order to achieve the highest profit.

   Define in detail the decision variables and form the objective function and all constraints of the problem.

2. A company which manufactures canoes employs 120 employees, each of whom working 30 hours per week. Half of them work in the carpenter department, 20 persons in the plastics department, and the rest of them at the completion department. The company manufactures the simple canoes with net unit profit \$7 and the luxury canoes with corresponding profit \$10. A simple canoe requires 4.5 hours in the carpenter department and two hours in each of the other two departments. The working hours for each luxury canoe are 5, 1 and 4 at the carpenter department, plastics department and completion department respectively. Marketing calculations have shown that not less than 1/3 and not more than 2/3 of the total number of the canoes should be luxurious.

   How will the company maximize its overall net profit?

3. Solar Oil Company is a gasoline refiner and wholesaler. It sells two products to gas stations: regular and premium gasoline. It makes these two final products by blending together four raw gasolines and some chemical additives (the amount and cost of the additives per barrel are assumed to be independent of the mixture). Each gasoline has an octane rating that reflects its energy content. Table in the following lists the octane, purchase price per barrel, and availability at that price per day. This table also gives the required minimum octane for each final gasoline, the net selling price per barrel (removing the cost of the additives), and the expected daily demand for gas at that price. Solar Oil can sell all the gas it produces up to that amount. The blending of gasoline is approximately a linear operation in terms of volume and octane. If x barrels of 80 octane gasoline are blended with y barrels of 90 octane gasoline, this produces $x + y$ barrels of gasoline with an octane of $(80x + 90y)/(x + y)$. There is no significant volume gain or loss, and octane of the mixture is a weighted average of the octanes of the inputs.

		Octane	**Cost (\$/b)**	**Available daily**
Raw gasolines	1	86	17	20,000
	2	88	18	15,000
	3	92	20.5	15,000
	4	96	23	10,000
		Octane	**Price (\$/b)**	**Maximum daily demand**
Products	Regular	89	19.5	35,000
	Premium	93	22	23,000

4. Solve the the following problems graphically:

(a)

$$\begin{aligned} \max\ & x_1 + x_2 \\ \text{s.t. } & x_1 + 2x_2 \leq 10 \\ & 2x_1 + x_2 \leq 16 \\ & -x_1 + 2x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \max\ & x_1 + 3x_2 \\ \text{s.t. } & x_1 + 2x_2 \geq 6 \\ & x_1 - x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

(c) $\min 2x_1 - x_2$ subject to the constraints in (b)

Solutions:

1. Let x_i be the quantity and p_i be the price of product i, b_j the available units and c_j the unit price of raw material j, a_{ij} units of raw material j for producing product i, where $i \in \{a, b, c\}$ and $j \in \{1, 2, 3, 4\}$, we then have

$$\begin{aligned} \max\ & (\boldsymbol{p} - \boldsymbol{c})^\top \boldsymbol{x} \\ \text{s.t. } & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}; \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

2. Denote x and y, respectively, as the simple and luxury canoes to produce. We can formulate the problem as follows:

$$\begin{aligned} \max\ & 7x + 10y \\ \text{s.t. } & 4.5x + 5y \leq 60 \times 30 \\ & 2x + y \leq 20 \times 30 \\ & 2x + 4y \leq 40 \times 30 \\ & \frac{1}{3} \leq \frac{y}{x+y} \leq \frac{2}{3} \\ & x, y \geq 0; x, y \in \mathbb{Z} \end{aligned}$$

Note that the fourth constraint is not a linear equality or linear inequality. We can linearize it as:

$$x \leq 2y; y \leq 2x.$$

3. The manager of Solar Oil's operation wants to maximize the company's profit. The first question is: What quantities does the manager control? What can the manager manipulate to influence profit? It is incomplete simply to say that the manager controls the amount of each final product to make. The manager controls,

and must determine, how to make each final product and how much to make. This can be expressed by letting x_{ij} = the number of barrels of raw gas $i(= 1, 2, 3, 4)$ used per day to make final product $j(= R, P)$ be the decision variables. Each barrel of raw gas i that is blended in final product j and then sold generates a profit equal to its selling price minus its cost. The objective function can be formulated as:

$\max 2.5x_{1R} + 1.5x_{2R} - x_{3R} - 3.5x_{4R} + 5.0x_{1P} + 4.0x_{2P} + 1.5x_{3P} - x_{4P}$

Note that the coefficients for some variables are negative. For example, Solar loses $1.00 on each barrel of raw gas 4 that is blended into premium. Does this imply that the optimal value for these variables must be zero and that they can be dropped from the problem? No! In blending operations, it is common for some low-cost materials to be combined with high-cost materials. Although it appears that we are losing money on the high-cost materials, they make the low-cost materials more valuable, and often the final product cannot be made without them. For example, tungsten steel combines low-cost iron ore or scrap (worth $100/ton) with tungsten (costing thousands of dollars per ton) to make steel that might sell for $500 per ton. The manufacturer loses money on the tungsten (on a per ton basis) but is more than compensated by the enhanced value of the iron ore. Therefore, we do not omit variables from the problem unless we can prove that their optimum value is zero.

The next step is to identify the constraints. The availability constraint for each raw gasoline is

barrels of raw gas i used per day $\leq$ barrels of gas i available per day

The number of barrels of raw gas i used each day is the amount used to make regular gasoline x_{iR} plus the amount used each day to make premium gasoline x_{iP}. The availability constraints can be written as

$$
\begin{aligned}
x_{1R} + x_{1P} &\leq 20,000 \\
x_{2R} + x_{2P} &\leq 15,000 \\
x_{3R} + x_{3P} &\leq 15,000 \\
x_{4R} + x_{4P} &\leq 10,000
\end{aligned}
$$

The demand constraints put an upper limit on how much regular and premium gasoline can be sold. The total amount made of each is the sum of the raw gasolines allocated to making each gasoline each day. In other words,

$$
\begin{aligned}
x_{1R} + x_{2R} + x_{3R} + x_{4R} &\leq 35,000 \\
x_{1P} + x_{2P} + x_{3P} + x_{4P} &\leq 23,000
\end{aligned}
$$

If the model formulation is left at this stage, the optimal solution is to mix the lowest cost gasolines into the final products, regardless of octane. Therefore, we need to include constraints that guarantee the variables will take on values that produce final gasolines with at least the minimum specified octane ratings. The octane rating of the regular gasoline that is produced will be a weighted average of the octanes of the raw gasolines used; that is,

octane of regular = [86 (barrels of raw gas 1 used/day to make regular) + 88 (barrels of raw gas 2 used/day to make regular) + $\cdots$ + 96 (barrels of raw gas 4 used/day to make regular)]/[total barrels of raw gases blended into regular gasoline]

which should be at least 89. Substituting the appropriate variable names for these quantities produces the constraint

$$(86x_{1R} + 88x_{2R} + 92x_{3R} + 96x_{4R})/(x_{1R} + x_{2R} + x_{3R} + x_{4R}) \geq 89,$$

which can be linearized as

$$-3x_{1R} - x_{2R} + x_{3R} + 7x_{4R} \geq 0$$

Using the same approach to guarantee an octane of 93 for premium gas produces the constraint

$$-7x_{1R} - 5x_{2R} - x_{3R} + 3x_{4R} \geq 0$$

Finally, all variables should be nonnegative in value.

4. Omitted.