Module 5

3D Transformation

Learning Objectives

- Identify **basic 3D transformations**
- Understand **3D affine transformations**
- Construct and represent 3D affine transformations using 4×4 matrix or matrices
- Perform computations using 3D affine transformations
- Apply 3D affine transformations

Sources

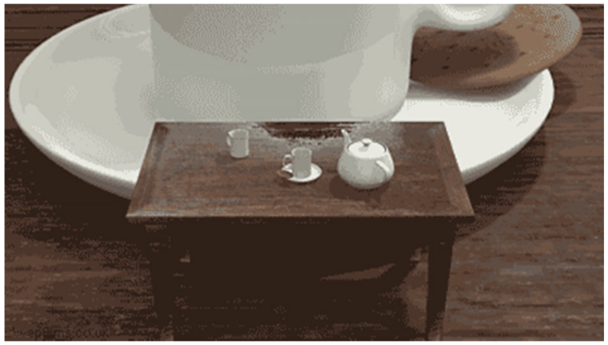
- Textbook (Chapter 5: 3D transformations)
- Wiki:
  - http://en.wikipedia.org/wiki/Affine_transformation
  - http://en.wikipedia.org/wiki/Transformation_matrix

Outline

1. Motivation and applications
2. Basic 3D transformations
3. 3D affine transformations
4. Affine transformations in VRML
5. Applications in sweeping

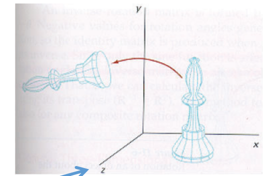
1. Motivation and applications

- Same as the 2D transformation: modeling, motion, representation, etc.
- Development is parallel to that of 2D.
  - Make use of knowledge from 2D
  - Pay attention to differences

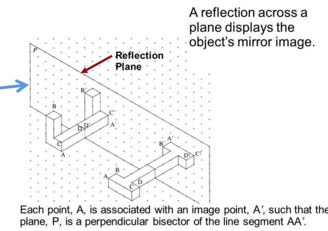


2. Basic 3D transformations

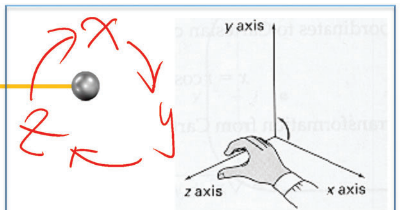
- Translation
- Scaling
- *Rotation*
  - Rotation about an axis (NOT a point)



- *Reflection*
  - over a point
  - over a line
  - over a plane



3D rotation

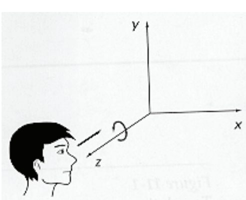


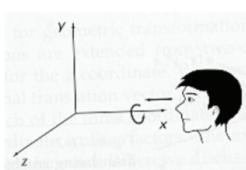
Right-handed systems

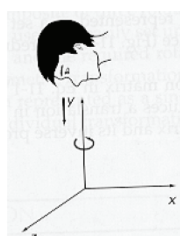
- 2D rotation is about a point.
- 3D rotation is about an axis.
- For right-handed systems:

When looking down a positive axis towards the origin,

- positive rotations are counter-clockwise
- negative rotations are clockwise



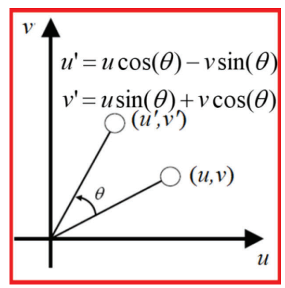


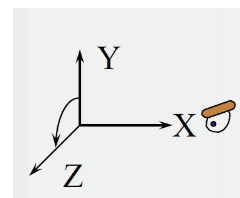


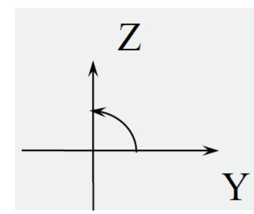
(counter-clockwise rotations)

Rotation about the x-axis

- *x*-coordinate remains unchanged.







Replace u, v by y and z:

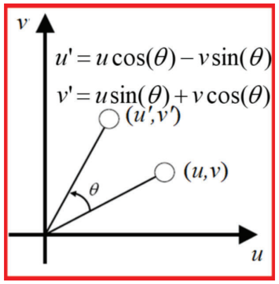
$$\begin{cases} x' &= x \\ y' &= y\cos\theta - z\sin\theta \\ z' &= y\sin\theta + z\cos\theta \end{cases}$$

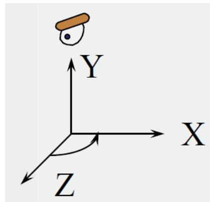
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

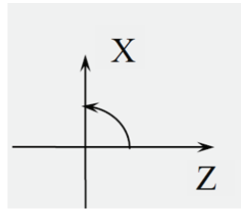
Note that homogeneous coordinates are used here.

Rotation about the y-axis

- *y*-coordinate remains unchanged.







Replace u, v by z and x:

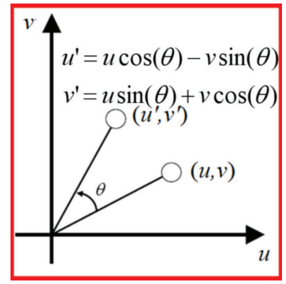
$$\begin{cases} z' = z\cos\theta - x\sin\theta \\ y' = y \\ x' = z\sin\theta + x\cos\theta \end{cases}$$

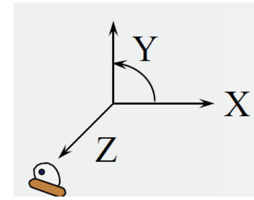
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

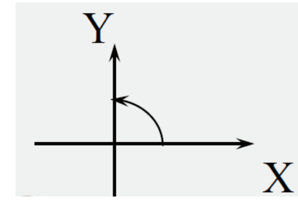
Note that homogeneous coordinates are used here.

Rotation about the z-axis

- *z*-coordinate remains unchanged.







Replace u, v by x and y:

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Note that homogeneous coordinates are used here.

- Question: What if the reference axis is **not** one of the three coordinate axes?

Reflections over a plane

$$\mathrm{Ref}_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{Ref}_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{Ref}_{zx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Over xy plane:
x, y remain unchanged

Over yz plane:
y, z remain unchanged

Over zx plane:
z, x remain unchanged

- Question: What if the reference plane is **not** one of the three coordinate planes?

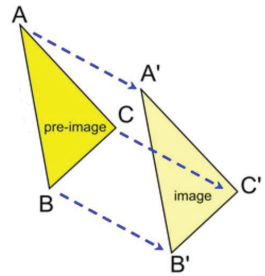
3. 3D affine transformations

- Affine transformations map straight lines to straight lines and preserve ratios of distances along straight lines.
- Basic 3D transformations and reflections are all affine transformations.
- 3D Affine transformations can always be represented by

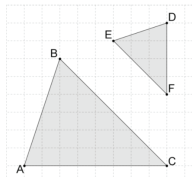
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & m \\ d & e & f & n \\ g & h & p & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Special cases of affine transformations

- **Rigid transformation**: a combination of translations, rotations and reflections.
  - Keep the size and shape
  - Useful in 3D registration



- **Similarity transformation**: a rigid transformation followed by a dilation.
  - The shapes are similar



3.1 Find the matrix using the general matrix form

- Problem: Assuming that object B is obtained from object A by an affine transformation, find the transformation matrix.
- Method (similar to the 2D case):
  - Step 1. Assume that the affine transformation is represented by

$$x' = ax + by + cz + m, \quad y' = dx + ey + fz + n, \quad z' = gx + hy + pz + l.$$

  - Step 2. Choose at least 4 point pairs from objects A and B (note: *make sure that the 4 points from A are not co-planar*). Substitute their coordinates as (x,y,z) and (x',y',z') into the above 3 equations. This gives you 12 linear equations.
  - Step 3. Solving the linear equations for the coefficients *a*, *b*, ….
  - Step 4. Using homogeneous coordinates, convert the linear representation of the affine transformation to matrix form.

3.2 Find the matrices by composing simple transformations

- Problem: How to perform ("non-standard") complicated transformations (which answers the Qs in slides 10 & 11)
- Method (similar to the 2D case):
  - Step 1. Analyze each transformation and do the following:
  - Step 2. If it is not a simple transformation, find some basic/simple transformations and perform them as a pre-process to make it "standard".
  - Step 3. Write in order the matrices for all the transformations performed in the preprocess
  - Step 4. Write the matrix for the required transformation in "standard" form
  - Step 5. Perform the post-process by reversing the transformations performed in the pre-process step and write their matrices in order

Example 1 (from Module 3: Geometric Shapes 4/8)

Example: Scaling by S_x, S_y and S_z with reference to the point (l, m, n)

$$\begin{bmatrix} x'(u) \\ y'(u) \\ z'(u) \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & l(1-s_x) \\ 0 & s_y & 0 & m(1-s_y) \\ 0 & 0 & s_z & n(1-s_z) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(u) \\ y(u) \\ z(u) \\ 1 \end{bmatrix}$$

Prove the above formula.

Proof:

- Step 1. perform a translation to move the reference point $(l.m.n)$ to the origin.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & -l \\ 0 & 1 & 0 & -m \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1 (from Module 3: Geometric Shapes 4/8)

- Step 2. perform the "standard" scaling (w.r.t the origin).

$$S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 3. perform another translation to move the reference point back to its original position ($l.m.n$) .

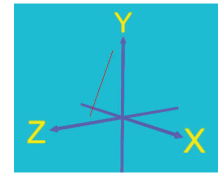
$$T_2 = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 4. Thus the overall matrix is

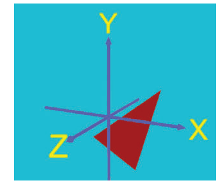
$$M = T_2 S T_1 = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -l \\ 0 & 1 & 0 & -m \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & l(1-s_x) \\ 0 & s_y & 0 & m(1-s_y) \\ 0 & 0 & s_z & n(1-s_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remarks

- We can see from the previous example that if the reference is a ***point***, it is easy to move the point to the origin by a translation.
- However, if the reference is an ***arbitrary line*** in 3D space, it may not be easy to make it a coordinate axis.



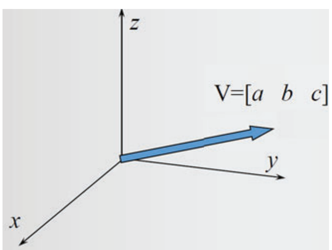
- Similarly, if the reference is an ***arbitrary plane*** in 3D space, it may be nontrivial to make it a coordinate plane.



- The next example gives a *general* solution to these two nontrivial tasks.

Example 2 : Aligning a vector to the z-axis

- Problem: How to align vector V=(a,b,c) to z-axis?

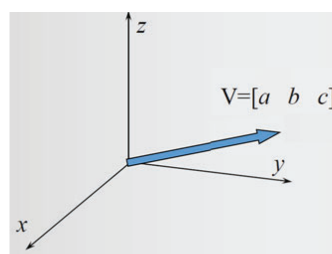


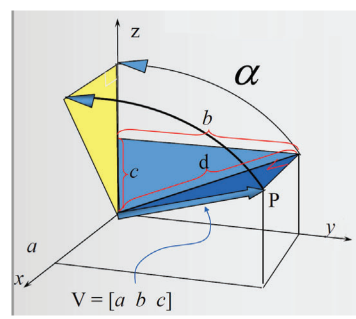
- Basic idea:
  - First, rotate V about the x-axis to bring it to the zx-plane
  - Then, rotate it around the y-axis to align it to the *z*-axis.

Align a vector to the z-axis

Step 1. rotate V about the x-axis by α to bring it to the zx-plane.

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \cos\alpha = \frac{c}{d} \quad \sin\alpha = \frac{b}{d} \quad d = \sqrt{b^2 + c^2}$$

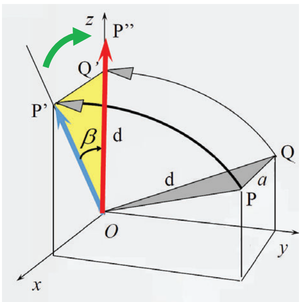


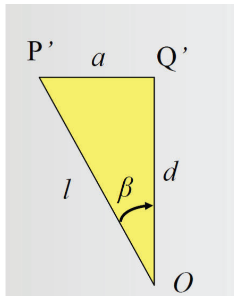


Align a vector to the z-axis

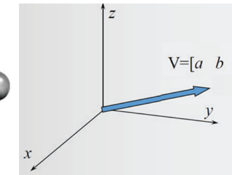
Step 2. rotate V about the y-axis by β clockwise to align it to the z-axis.

$$R_y(-\beta) = \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \cos\beta = \frac{d}{l} \quad \sin\beta = \frac{a}{l} \quad l = \sqrt{a^2+b^2+c^2}$$





Align a vector to the z-axis

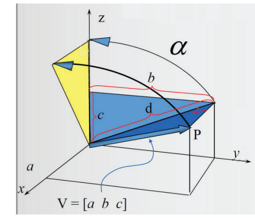


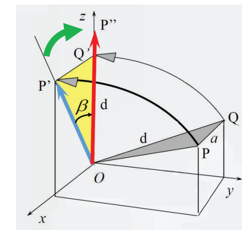
- As a result, vector V=(a,b,c) can be made to align with z-axis by two rotations: $R_y(-\beta)R_x(\alpha)$

  where

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & -\frac{b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, d = \sqrt{b^2+c^2}$$

$$R_y(-\beta) = \begin{bmatrix} \frac{d}{l} & 0 & -\frac{a}{l} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{l} & 0 & \frac{d}{l} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, l = \sqrt{a^2+b^2+c^2}$$





Applications

- Rotation or reflection about a line through the origin with direction [a,b,c]
- Reflection about a plane whose normal is [a,b,c]
- For example, 3D rotation about an arbitrary axis passing through the origin can be achieved by the following steps:
  - Step 1: align the axis with the z-axis via at most 2 simple rotations
  - Step 2: perform the required rotation about the z-axis
  - Step 3: reverse the rotations in Step 1

  The final matrices are

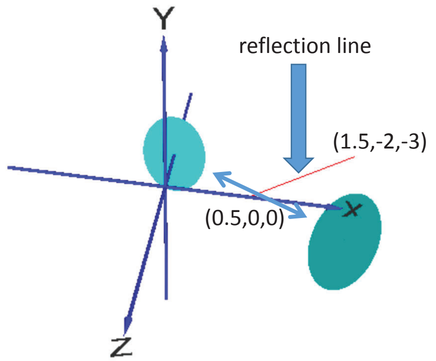
$$\underbrace{R_x(-\alpha)R_y(\beta)}_{\text{Step 3}}\ \underbrace{R_z(\theta)}_{\text{Step 2}}\ \underbrace{R_y(-\beta)R_x(\alpha)}_{\text{Step 1}}$$

Recap

- 3D basic transformations
- 3D affine transformations
  - General matrix form
  - Composition of simple transformations
- Matrix representation & homogeneous coordinates
- Implementation of aligning a vector to z-axis

Example 3

Q: Derive the matrices implementing the reflection about the line from point (0.5,0,0) to point (1.5,-2,-3).



Example 3 (cont)

Answer:

1) the direction of the line is: (2.5,-2,-3)-(0.5,0,0) = [1 -2 -3];

2) we perform a translation to move point (0.5,0,0) to the origin;

3) we apply the method of "aligning a vector to the z-axis";

4) perform the reflection;

5) Reverse steps 2) & 3).

The final matrices are:

$$\begin{bmatrix}1&0&0&0.5\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{-3}{\sqrt{13}}&\frac{-2}{\sqrt{13}}&0\\0&\frac{2}{\sqrt{13}}&\frac{-3}{\sqrt{13}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{13}}{\sqrt{14}}&0&\frac{1}{\sqrt{14}}&0\\0&1&0&0\\-\frac{1}{\sqrt{14}}&0&\frac{\sqrt{13}}{\sqrt{14}}&0\\0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{\text{reflect over z}}$$

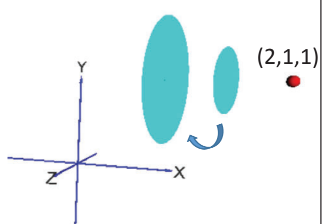
$$\underbrace{\begin{bmatrix}\frac{\sqrt{13}}{\sqrt{14}}&0&-\frac{1}{\sqrt{14}}&0\\0&1&0&0\\\frac{1}{\sqrt{14}}&0&\frac{\sqrt{13}}{\sqrt{14}}&0\\0&0&0&1\end{bmatrix}}_{\text{rotate about y}}\underbrace{\begin{bmatrix}1&0&0&0\\0&\frac{-3}{\sqrt{13}}&\frac{2}{\sqrt{13}}&0\\0&\frac{-2}{\sqrt{13}}&\frac{-3}{\sqrt{13}}&0\\0&0&0&1\end{bmatrix}}_{\text{rotate about x}}\underbrace{\begin{bmatrix}1&0&0&-0.5\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{\text{translate}}$$

Example 4

Q: A 3D object is uniformly scaled 2 times relative to point (2, 1, 1) and then reflected about a plane defined by $-y+z-2=0$. Assuming a column represented position vector, write in a proper order the individual matrices composing this transformation. The final single matrix is not required.

Answer: First, the scaling can be accomplished by the following three matrices:

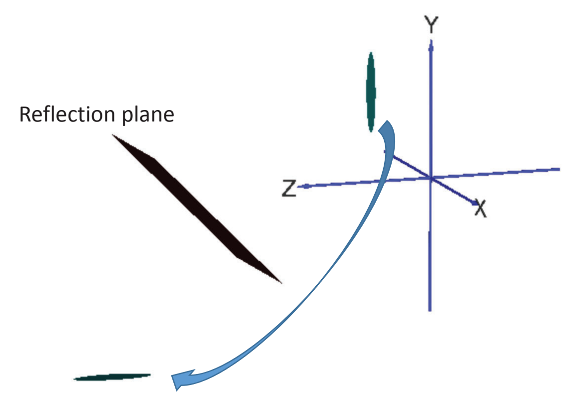
$$\underbrace{\begin{bmatrix}1&0&0&2\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{bmatrix}}_{\text{translation}}\underbrace{\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}}_{\text{scaling}}\underbrace{\begin{bmatrix}1&0&0&-2\\0&1&0&-1\\0&0&1&-1\\0&0&0&1\end{bmatrix}}_{\text{translation}}$$



effect of this scaling

Example 4 (cont)

Second, consider the reflection through the plane:



Example 4 (cont)

Letting y = 0, from equation –y+z-2=0 we obtain z=2. Thus we obtain a point (0,0,2) that is on the plane.

The normal of the plane is [0 -1 1]. If we use the method of "aligning a vector to the z-axis" to perform one translation and two rotations, we can make the plane align with the xy-plane. Hence we obtain the matrices that implement the reflection about the plane:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{\sqrt{2}}&0&\frac{0}{\sqrt{2}}&0\\0&1&0&0\\-\frac{0}{\sqrt{2}}&0&\frac{\sqrt{2}}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{\sqrt{2}}&0&-\frac{0}{\sqrt{2}}&0\\0&1&0&0\\\frac{0}{\sqrt{2}}&0&\frac{\sqrt{2}}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$$

Example 4 (cont)

Finally, combining the matrices of scaling and reflections gives the answer:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-2\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&2\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}6&0&0&0\\0&6&0&0\\0&0&6&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-2\\0&1&0&-1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

4. Affine transformations in VRML

- In VRML, the Transform node contains several fields that define a transformation: translation, rotation, and scaling.

```
Transform {
        translation  dx  dy  dz
        rotation      ax  ay  az  theta
        scale          sx   sy   sz
        children [ …]
}
```

  - Here the rotation axis is from the origin to point (ax,ay,az), and theta (in radian) is the rotation angle value.
  - sx, sy, sz are the 3 scaling factors along x, y, z axes.
  - dx, dy, dz are the translation amounts along x, y, z axes.
  - The order is always scale first, then rotation, finally translation.
- One can use nested transforms to change the order.

Example 4

The following codes show two transformations applied to a box.

```
Transform {  translation  0 0 3 rotation 1 0 0 1.2 scale   2 1 0.3
   children[  Shape {
              appearance Appearance {material Material {diffuseColor 0 1 0}}
              geometry Box{size  1  1  1}
   }]
}

Transform {   rotation 1 0 0 1.2    scale   2 1 0.3
  children[
     Transform{  translation 0 0 3
       children[ Shape {
              appearance Appearance {material Material {diffuseColor 1 0 0}}
              geometry Box{size  1  1  1}
            }]
  }]
}
```

Example 4 (cont)

- Write the matrices in order for the first transformation (i.e., green box).

- Write the matrices in order for the second transformation (i.e., red box).

Example 4 (cont)

- Write the matrices in order for the first transformation (i.e., green box). .

Hint:
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\cos(1.2)&-\sin(1.2)&0\\0&\sin(1.2)&\cos(1.2)&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&0.3&0\\0&0&0&1\end{bmatrix}$$

- Write the matrices in order for the second transformation (i.e., red box)..

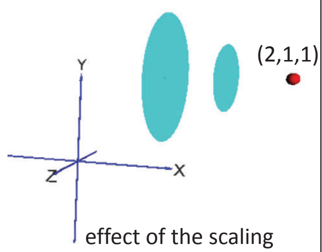
Hint:
$$\begin{bmatrix}1&0&0&0\\0&\cos(1.2)&-\sin(1.2)&0\\0&\sin(1.2)&\cos(1.2)&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&0.3&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&3\\0&0&0&1\end{bmatrix}$$

Question for you

Q: How to use VRML to implement the uniform scaling with scaling factor 2 relative to point (2, 1, 1)?

$$\begin{bmatrix}1&0&0&2\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-2\\0&1&0&-1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

translation   scaling   translation



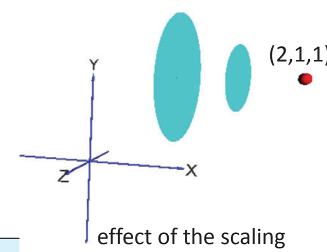
effect of the scaling

Question for you

Q: How to use VRML to implement the uniform scaling with scaling factor 2 relative to point (2, 1, 1)?

$$\begin{bmatrix}1&0&0&2\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-2\\0&1&0&-1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

translation   scaling   translation



effect of the scaling

```
Transform {   translation  2  1 1    scale   2 2 2
   children[
      Transform{  translation -2 -1 -1
          children[  Shape {…} ]
      }
   ]
}
```

5. Applications in sweeping

- Rotational sweeping by rotation transformation(s)
- Translational sweeping by translation transformation(s)

- General approach for deriving the representation of sweeping surfaces:
  - Step 1: write the section/profile curve in parametric functions
  - Step 2: multiply the coordinates by the 3D rotation matrix (pay attention to the range of the rotation angle)
  - Step 3: if translational sweeping is involved, simply add the displacements to the corresponding coordinates (or multiply the coordinates by the translation matrix)
  - Step 4: if needed, reparameterize the representation such that parameters u and v have the domain [0,1].

Example 5

Define by mathematical functions a 3D surface which is obtained by rotational and translational sweeping of the straight line segment S as displayed in Figure Q2.

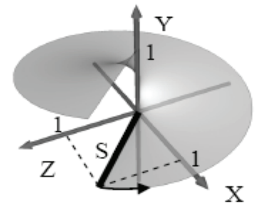
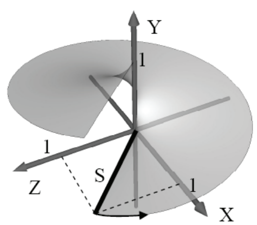


Figure Q2

Answer:

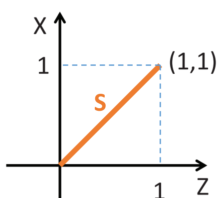
Note that the surface is generated by rotating and translating the line segment S.

Example 5 (cont)



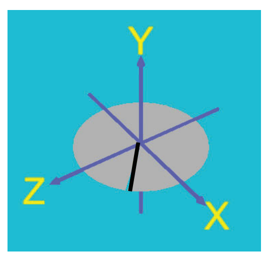
- Step 1: identify the building block (profile curve), which is line segment S. Derive the parametric representation of the line segment.

$$\begin{cases} x_0(u) &= u \\ y_0(u) &= 0 \qquad u \in [0,1] \\ z_0(u) &= u \end{cases}$$

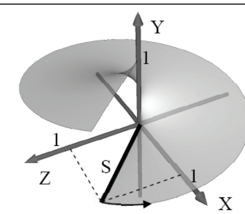


- Step 2: construct the rotation matrix. It is a rotation about y-axis counter-clockwise.

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta \in [0, 2\pi]$$



Example 5 (cont)



- Step 3: deal with the translation. (Note that we should use the same parameter θ for translation. Why?)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & f(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta \in [0, 2\pi]$$

where $f(\theta)$ is a displacement function. By observing the shape, we can let $f(\theta)$ be a linear function: $f(\theta) = A\theta + B$. We compute the two unknown coefficients A and B by forming 2 equations:

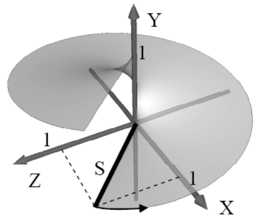
$$\begin{aligned} f(0) &= 0 \\ f(2\pi) &= 1 \end{aligned} \Longrightarrow \begin{aligned} A \times 0 + B &= 0 \\ A \times 2\pi + B &= 1 \end{aligned}$$

Thus $A = \frac{1}{2\pi}, B = 0$ and $f(\theta) = \frac{\theta}{2\pi}$.

- Step 4: apply the transformations to the line segment.

$$\begin{bmatrix} x(u,\theta) \\ y(u,\theta) \\ z(u,\theta) \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\theta}{2\pi} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{translation}} \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{rotation}} \begin{bmatrix} x_0(u) \\ y_0(u) \\ z_0(u) \\ 1 \end{bmatrix} = \begin{bmatrix} u(\cos\theta + \sin\theta) \\ \frac{\theta}{2\pi} \\ u(-\sin\theta + \cos\theta) \\ 1 \end{bmatrix}$$

Example 5 (cont)



- Step 5: Reparameterize θ.
  Let $\theta = 2\pi v$, $v \in [0,1]$.
  Then the final equations are:

$$\begin{cases} x(u,v) &= u\,[\cos(2\pi v) + \sin(2\pi v)] \\ y(u,v) &= v \\ z(u,v) &= u\,[-\sin(2\pi v) + \cos(2\pi v)] \end{cases} \qquad u, v \in [0,1]$$

- Question for you: if we introduce another parameter β (instead of θ) in step 3, what kind of shape do we define?

Recap

- Applications of "aligning a vector to the z-axis" in handling complicated affine transformations
- Recognition and construction of affine transformations in VRML
- Use of rotation and translation transformations in sweeping methods

END ?

Extra example 1

Q: What are the Cartesian coordinates corresponding to homogeneous coordinates (10, -12, 8, 2)?

Hint:

(10, -12, 8, 2) → (10, -12, 8, 2)/2 = (5,-6,4,1) → (5, -6,4)

The Cartesian coordinates are (5, -6, 4).

Extra example 2

Q: Derive the matrix of reflection about the plane specified by three points B=(1,0,0), C=(0,1,0), and D=(0,0,1).

Hint: The plane's equation is x/1 + y/1 + z/1 -1 =0. Thus the plane normal N = [1 1 1]. Moving B to the origin and using the method of "aligning a vector to the z-axis, we can get the reflection by the following matrices:

$$\begin{bmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{\sqrt{3}}&0&\frac{1}{\sqrt{3}}&0\\0&1&0&0\\-\frac{1}{\sqrt{3}}&0&\frac{\sqrt{2}}{\sqrt{3}}&0\\0&0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}}_{\text{reflect across xy}}$$

$$\underbrace{\begin{bmatrix}\frac{\sqrt{2}}{\sqrt{3}}&0&-\frac{1}{\sqrt{3}}&0\\0&1&0&0\\\frac{1}{\sqrt{3}}&0&\frac{\sqrt{2}}{\sqrt{3}}&0\\0&0&0&1\end{bmatrix}}_{\text{rotate about y}}\underbrace{\begin{bmatrix}1&0&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}}_{\text{rotate about x}}\underbrace{\begin{bmatrix}1&0&0&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{\text{translate}}$$