Geometric Shapes: Curves

Module 3

Lecture 2

Lecture 2: Learning Objectives

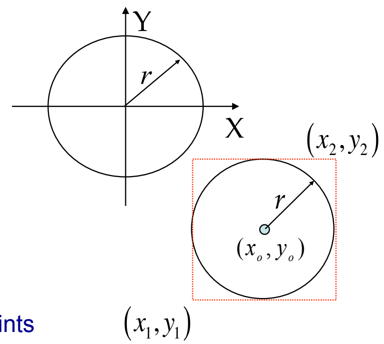
- To understand how curves can be used in solving data visualization problems
- To understand curves as objects with 1 degree of freedom
- To understand what mathematical representation is the most efficient for defining and displaying curves
- To understand how different coordinate systems can be used together for deriving mathematical representations of curves

Circle.
Implicit Representation

$$r^2 - x^2 - y^2 = 0$$

$$r^2 - (x - x_o)^2 - (y - y_o)^2 = 0$$

$$x \in [x_1, x_2], y \in [y_1, y_2]$$

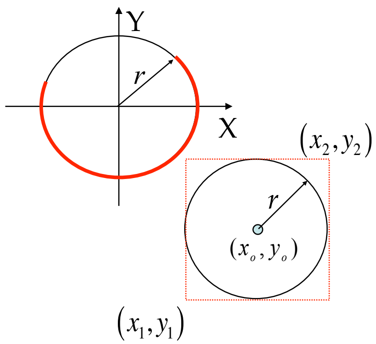


- Drawing is done by sampling points within the x and y domains. Slow.

Circle.
Implicit Representation

$$r^2 - x^2 - y^2 = 0$$

$$r^2 - (x - x_o)^2 - (y - y_o)^2 = 0$$

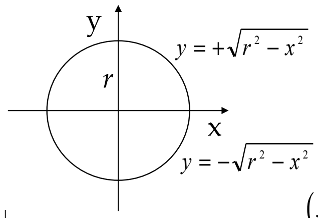


- **Arc** domain in x and y?
- Impossible to do it using only $x \in [x_1, x_2], y \in [y_1, y_2]$ Requires angular values as in polar coordinates

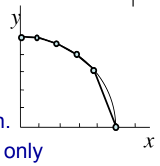
Circle.
Explicit Representation

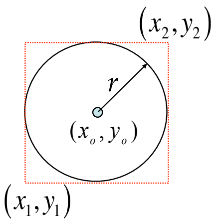
$$y = \pm\sqrt{r^2 - x^2}$$

$$y = \pm\sqrt{r^2 - (x - x_o)^2} + y_o$$



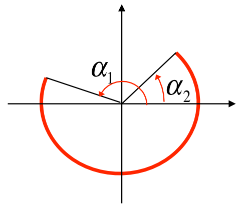
- Axes dependency: 2 formulas for the upper and lower semicircles
- Drawing is done by incrementing x or y and obtaining y and x, respectively. It is fast but with irregular segment length interpolation.
- Impossible to define arc domain with only $x \in [x_1, x_2], y \in [y_1, y_2]$
  Requires angular values as in polar coordinates





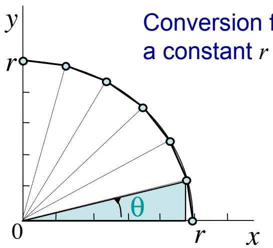
Circle.
Explicit Representation in Polar Cordinates

- In polar coordinates: $r = r(\alpha)$
- Origin-centred circle: *r=constant_radius*
  $\alpha \in [0, 2\pi]$
- Arc is defined by the domain of $\alpha \in [\alpha_1, \alpha_2]$



- Fast drawing is done by incrementing angle α and obtaining radius r
- Other (not origin centred) circle-arc locations are problematic to define in polar coordinates

Circle.
Parametric Representation



Conversion from polar coordinates $r(\alpha)$ to Cartesian with a constant r

$$x = r\cos(\theta)$$

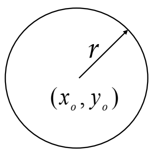
$$y = r\sin(\theta)$$

$$\theta = \alpha$$

One parameter!

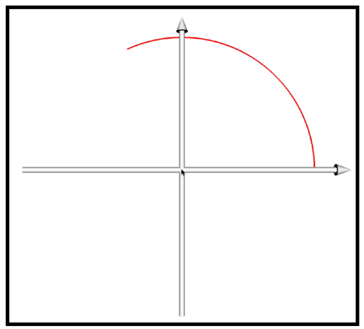
$$x = r\cos(\theta) + x_o \qquad 0 \le \theta \le 2\pi \qquad \text{for a circle}$$

$$y = r\sin(\theta) + y_o \qquad \theta_1 \le \theta \le \theta_2 \qquad \text{for an arc}$$



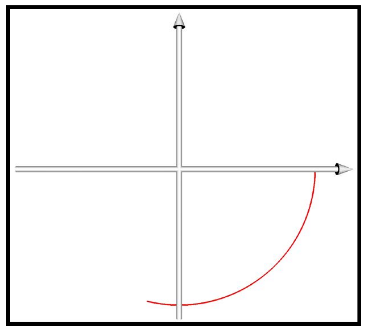
Drawing is done by incrementing parameter θ and obtaining x and y.
It is axes independent, fast and with a uniform length of the segments interpolating the circle.

Circle.
Parametric Representation



$x = r\cos(\theta) \qquad y = r\sin(\theta)$

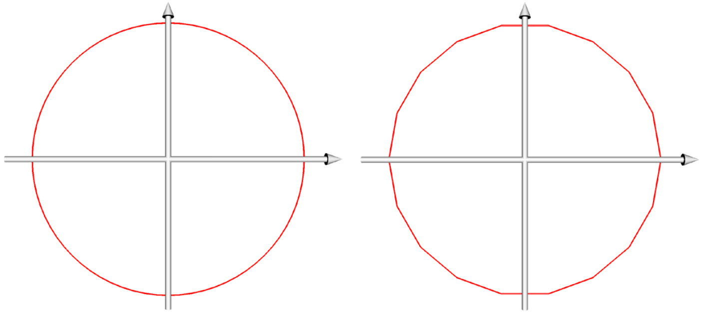
$\theta \in [0, 2\pi]$



$x = r\cos(\theta) \qquad y = r\sin(\theta)$

$\theta \in [0, \underline{-2\pi}]$

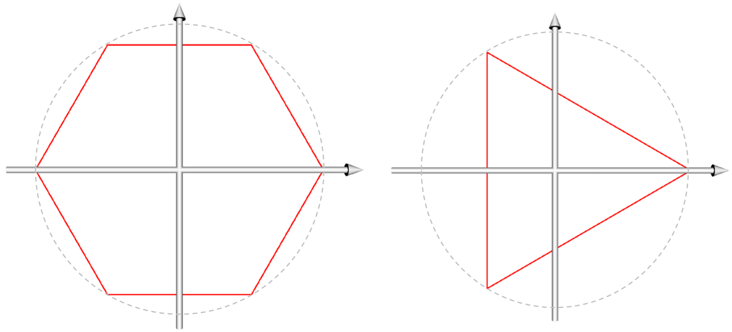
Circle and Beyond.
Parametric Representation



72 sampling points (segments) 18 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta) \quad \theta \in [0, 2\pi]$$

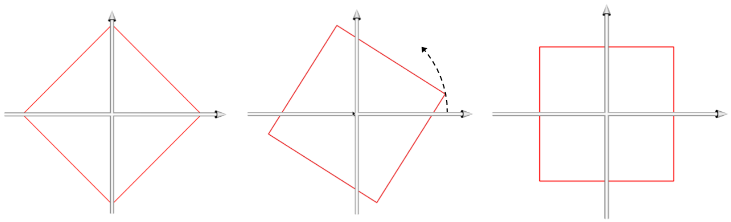
Circle and Beyond.
Parametric Representation



6 sampling points 3 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta) \quad \theta \in [0, 2\pi]$$

Circle and Beyond.
Parametric Representation



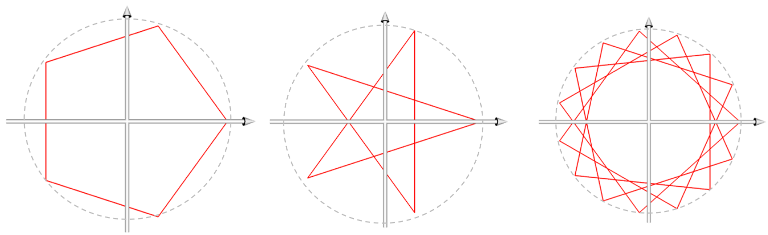
4 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 2\pi]$$

4 sampling points and offset

$$x = r\cos(\theta + \frac{\pi}{4}) \quad y = r\sin(\theta + \frac{\pi}{4})$$
$$\theta \in [0, 2\pi]$$

Circle and Beyond.
Parametric Representation



5 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 2\pi]$$

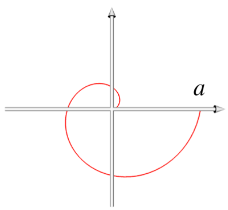
5 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 4\pi]$$

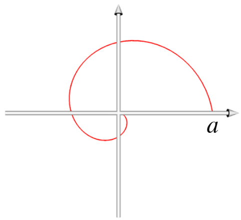
15 sampling points

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$
$$\theta \in [0, 8\pi]$$

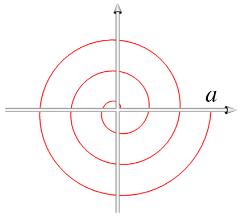
Circle and Beyond.
Parametric Representation



$$x = a \cdot u \cdot \cos(2\pi u)$$
$$y = a \cdot u \cdot \sin(2\pi u)$$
$$u \in [0,1]$$



$$x = a \cdot u \cdot \cos(-2\pi u)$$
$$y = a \cdot u \cdot \sin(-2\pi u)$$
$$u \in [0,1]$$

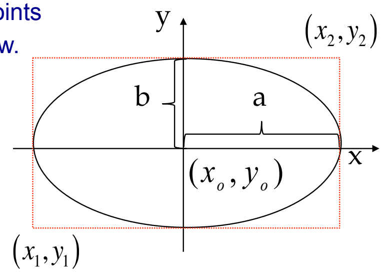


$$x = a \cdot u \cdot \cos(6\pi u)$$
$$y = a \cdot u \cdot \sin(6\pi u)$$
$$u \in [0,1]$$

Ellipse.
Implicit Representation

$$1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0 \qquad 1 - \left(\frac{x - x_0}{a}\right)^2 - \left(\frac{y - y_0}{b}\right)^2 = 0$$

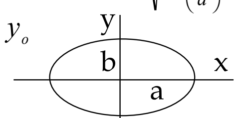
- Drawing is done by sampling points within the x and y domains. Slow.
- **Arc** domain in x and y?
- Impossible to do it using only
  $x \in [x_1, x_2], y \in [y_1, y_2]$
- Requires angular values as in polar coordinates



Ellipse.
Explicit Representation

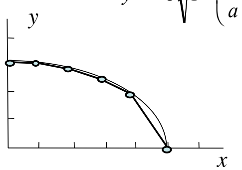
$$y = \pm b\sqrt{1 - \left(\frac{x}{a}\right)^2} \qquad y = \pm b\sqrt{1 - \left(\frac{x - x_o}{a}\right)^2} + y_o$$

$$y = +b\sqrt{1 - \left(\frac{x}{a}\right)^2}$$



$$y = -b\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

- Axes dependency: 2 formulas for the upper and lower semi-ellipses
- Drawing is done by incrementing x or y and obtaining y and x, respectively. It is fast but with irregular segment length interpolation.
- Impossible to define arc domain with only
  $x \in [x_1, x_2], y \in [y_1, y_2]$



Ellipse.
Explicit Representation in Polar Coordinates

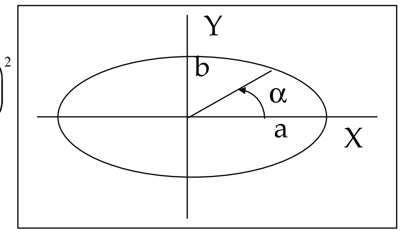
Origin-centred Ellipse in Polar Coordinates

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{r\cos\alpha}{a}\right)^2 + \left(\frac{r\sin\alpha}{b}\right)^2$$

$$r^2\left(\left(\frac{\cos\alpha}{a}\right)^2 + \left(\frac{\sin\alpha}{b}\right)^2\right) = 1$$

$$r = \frac{1}{\sqrt{\left(\frac{\cos\alpha}{a}\right)^2 + \left(\frac{\sin\alpha}{b}\right)^2}}$$

$$0 \le \alpha < 2\pi$$



Ellipse.
Parametric Representation

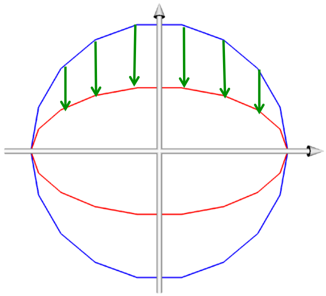
$$x = 1 \cdot \cos\theta + x_0$$
$$y = 1 \cdot \sin\theta + y_0$$

By scaling of a circle with radius 1 by *a* and *b* along X and Y

$$x = \underline{a}\cos\theta + x_0$$
$$y = \underline{b}\sin\theta + y_0$$

$0 \le \theta \le 2\pi$ for an ellipse

$\theta_1 \le \theta \le \theta_2$ for an arc

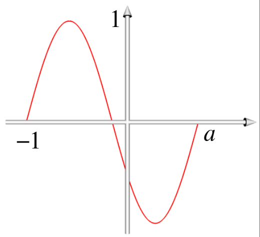


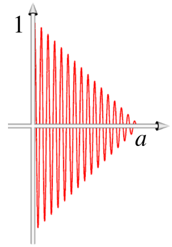
Parameter θ is not a polar angle α!

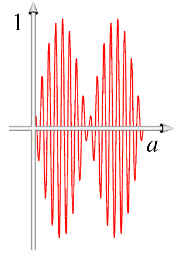
Curves.
Parameterisation, Modulation

$y = \sin(x)$, $x \in [0, 2\pi]$ $\Rightarrow$ $x = u \quad y = \sin(u)$, $u \in [0, 2\pi]$

$x = -1 + u \cdot (a - (-1))$ $y = \sin(u2\pi)$ $u \in [0,1]$	$x = u \cdot a$ $y = (1-u) \cdot \sin(16 \cdot 2\pi \cdot u)$ $u \in [0,1]$	$x = u \cdot a$ $y = \lvert\sin(2\pi \cdot u)\rvert \cdot \sin(16 \cdot 2\pi \cdot u)$ $u \in [0,1]$







Summary

2D curves can be defined analytically by

- **Implicit functions**
  $f(x,y)=0$ – Slow for rendering
- **Explicit functions**
  $y=f(x)$ or $x=f(y)$ – Fast but axes dependent
- **Parametric functions**
  One parameter only
  $x=x(t), \quad y=y(t) \quad t=[t_1, t_2]$ – Fast and axes independent