# 3D Transformations

## Lesson objectives

By the end of the module, you should be able to:

- Identify and explain basic 3D transformations
- Understand and explain 3D affine transformations
- Construct and represent 3D affine transformations using 4×4 matrix or matrices
- Perform computations of 3D affine transformations
- Apply 3D affine transformations



## Homogeneous coordinates example

- Each point in 3D space has Cartesian coordinate representation (x, y, z).
- The representation can be expanded to a four-element representation (xw, yw, zw, w) with w ≠ 0, which is called homogeneous coordinates of point (x, y, z).
- Cartesian → Homogeneous: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\text{homogenizing}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}, w \neq 0$
- Homogeneous → Cartesian: $\begin{bmatrix} x_h \\ y_h \\ z_h \\ h \end{bmatrix} = \begin{bmatrix} \frac{x_h}{h} \\ \frac{y_h}{h} \\ \frac{z_h}{h} \\ 1 \end{bmatrix} \xrightarrow{\text{inhomogenizing}} \begin{bmatrix} \frac{x_h}{h} \\ \frac{y_h}{h} \\ \frac{z_h}{h} \end{bmatrix}$



## Problems to be addressed

- What are basic 3D transformations?
- What are 3D affine transformations?
- How to represent 3D transformations using matrix/ matrices?
- How to perform 3D transformation?

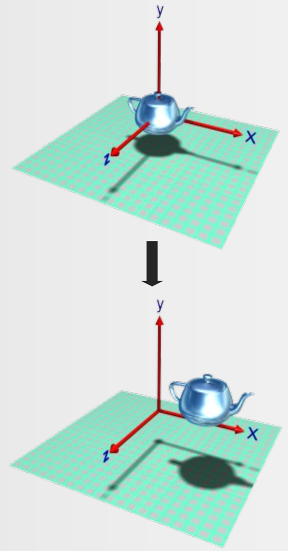
## 2. Basic 3D transformations

- Basic set:
  - Translation
  - Scaling
  - Rotation
- Other:
  - Reflection





## 3D translation



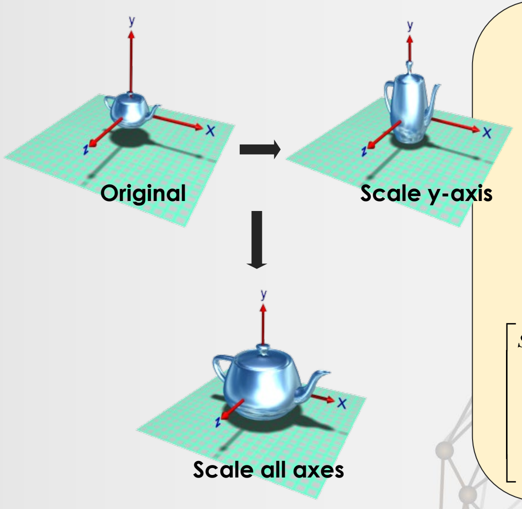

$$x' = x + d_x$$
$$y' = y + d_y$$
$$z' = z + d_z$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T(d_x,d_y,d_z)} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} x+d_x \\ y+d_y \\ z+d_z \\ 1 \end{bmatrix}}_{P'}$$





## 3D scaling




$$x' = s_x \times x$$
$$y' = s_y \times y$$
$$z' = s_z \times z$$

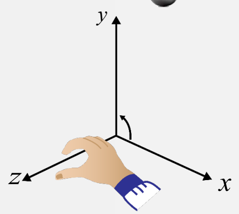
$$\underbrace{\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{S(s_x,s_y,s_z)} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} xs_x \\ ys_y \\ zs_z \\ 1 \end{bmatrix}}_{P'}$$



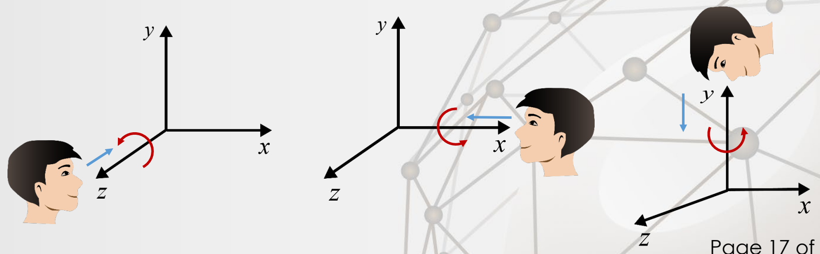


## 3D rotation

- **Right-handed** systems:

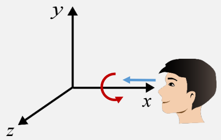



- Positive rotation angles for right-handed systems: (counter-clockwise rotations)

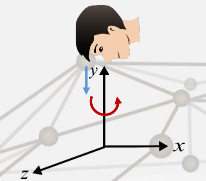


## Recap

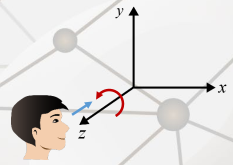
- About x-axis: $\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$




- About y-axis: $\mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$




- About z-axis: $\mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$






## 3D reflection

- In 3D, there are three situations for reflection:
  - Reflection about a point;
  - Reflection about a line; and
  - Reflection about a plane.
- Reflection about the origin: (x,y,z) becomes (-x,-y,-z).

$$\mathrm{Ref}_o = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



## Reflection about coordinate axes

- About x-axis: x remains unchanged. $\mathrm{Ref}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- About y-axis: y remains unchanged. $\mathrm{Ref}_y = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- About z-axis: z remains unchanged. $\mathrm{Ref}_z = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



## Reflection about coordinate planes

- About xy plane:
  x, y remain unchanged. $\mathrm{Ref}_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- About yz plane:
  y, z remain unchanged. $\mathrm{Ref}_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- About zx plane:
  z, x remain unchanged. $\mathrm{Ref}_{zx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## 3. Affine transformations

- Affine transformations map straight lines to straight lines and preserve ratios of distances along straight lines.
- Basic 3D transformations and reflections are all affine transformations.



## 3D affine transformations

- 3D affine transformations can always be represented by:

$$\begin{aligned} x' &= ax + by + cz + m \\ y' &= dx + ey + fz + n \\ z' &= gx + hy + pz + l \end{aligned}$$

where

- a, b, c, d, e, f, g, h, p, m, n, l are constants.
- (x, y, z) are the coordinates of the point to be transformed.
- (x', y', z') are the coordinates of the transformed point.

- The general matrix form of affine transformations is:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & m \\ d & e & f & n \\ g & h & p & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



## Example 1

**Q:** An affine transformation defined by the matrix $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

which applies to point P = (-1, -1, 2). Compute the coordinates of the transformed point P'.

**Ans:**
First, construct the homogeneous coordinates of point P = (-1, -1, 2, 1).

Then, $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \\ 1 \end{bmatrix}$

Therefore, P' = (-2, -2, 3).



## Example 2

**Q:** A 3D object is defined by the vertices with following coordinates:

(0, 0, 0), (1, 0, 0), (1, 0, -1), (0, 0, -1)
(0, 1, 0), (1, 1, 0), (1, 1, -1), (0, 1, -1)

The object is transformed into another object with the following coordinates for the transformed vertices by an affine transformation:

(5, 0, 2), (5, 6, 6), (-1, -3, -3), (-1, -9, -7)
(6, 7, 4), (6, 13, 8), (0, 4, -1), (0, -2, -5)

Derive the transformation matrix.

## Use of affine transformations in VRML

- In VRML, the Transform node contains several fields that define a transformation: translation, rotation, and scaling.

```
Transform {
        translation    dx  dy  dz
         rotation       ax  ay  az theta
         scale          sx  sy  sz
         children [ …]
  }
```

  - dx, dy, dz are the translation amounts along x-, y-, z-axes.
  - Here, the rotation axis is from the origin to point (ax, ay, az), and theta (in radian) is the rotation angle value.
  - sx, sy, sz are the three scaling factors along x-, y-, z-axes.
  - The order is always scale first, then rotation, finally translation.
- One can use nested transforms to change the order.





## 4. Examples

We will study three examples with complicated affine transformations.





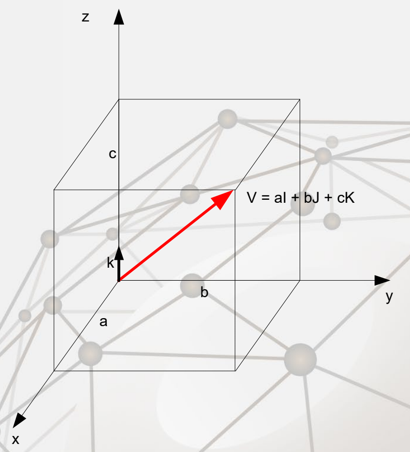
## 4.1 Align a vector to the z-axis

**Problem:** How to align vector V = (a, b, c) to z-axis?

**Basic idea:**

– Rotate V about the x-axis to bring it to the zx plane

– Then, rotate it around the y-axis to align it to the z-axis



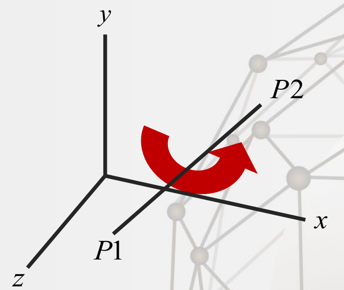





## 4.2 Rotation about an arbitrary axis

**Problem:** Derive the matrices for a rotation about an arbitrary axis $P_1P_2$ in 3D by an angle of a, where $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

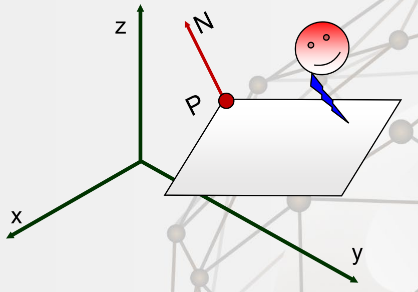
**Basic idea:** Do some pre-processes to turn the problem to a rotation about the coordinate axes.



## 4.3 Reflection about an arbitrary plane

**Problem:** Derive matrices for a reflection about a plane defined by the plane normal N and a point P on the plane.

**Basic idea:** Do some pre-processes to turn the problem to a reflection about the xy plane.



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## 5. Applications in sweeping

- Translational sweeping
- Rotational sweeping



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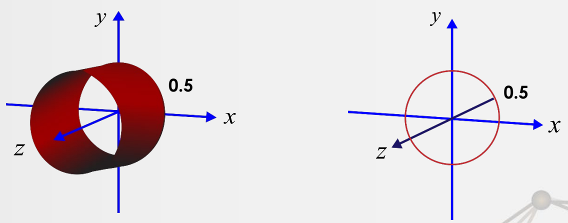
## Translational sweeping revisited




- A cylinder of height 0.8 can be generated by translating the back circle along the z-axis.
  - The back circle: $x = 0.5\cos\varphi,\ y = 0.5\sin\varphi,\ z = 0,\ 0 \le \varphi \le 2\pi$
  - Any point on the back circle has coordinates:

$$\begin{bmatrix} 0.5\cos\varphi \\ 0.5\sin\varphi \\ 0 \end{bmatrix}$$



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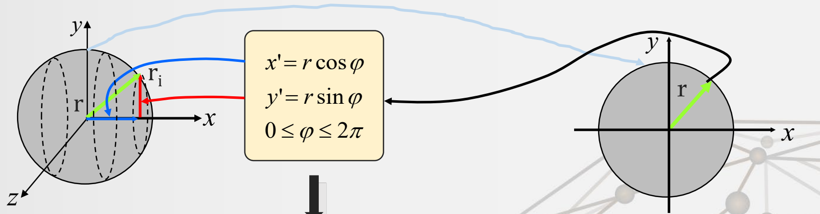
## Rotational sweeping revisited

- The sphere is generated by rotating a circle on the xy plane about the x-axis by 180 degrees.




$$x' = r\cos\varphi$$
$$y' = r\sin\varphi$$
$$0 \le \varphi \le 2\pi$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \\ 1 \end{bmatrix}$$

$\theta = [0, \pi]$

$$x = r\cos\varphi$$
$$y = r\sin\varphi\cos\theta$$
$$z = r\sin\varphi\sin\theta$$
$$0 \le \varphi \le 2\pi$$
$$0 \le \theta \le \pi$$

# 6. Summary

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- 3D point and homogeneous coordinates
- Basic 3D transformations and their 4x4 matrix representation
- Composition of transformations
- 3D affine transformations and their 4x4 matrix representations
- Applications