Seminar 102. Lecture 6. Laplace transform.

[Ref: 吴崇试 C8,18; Riley C13.2]

Notations: | Important formul
Laplace transfo

Recall: Integral transform: "change of domain" $F(k) \Leftrightarrow f(x)$

$$\underbrace{F(k)}_{\text{image}} = \int_a^b \underbrace{K(k,x)}_{\text{kernel}} \underbrace{f(x)}_{\text{pre-image}} dx$$

Fourier transform	$K = e^{-ikx}$	$-\infty < x < \infty$	
Laplace transform	$K = e^{-kx}$	$0 \le x < \infty$	
Sine transform	$K = \sin kx$	$0 \le x < \infty$	
Cosine transform	$K = \cos kx$	$0 \le x < \infty$	
Hankel transform	$K = x J_n(kx)$	$0 \le x < \infty$	Bessel function
Mellin transform	$K = x^{k-1}$	$0 \le x < \infty$	

6.1. Def. of L.T.

$$\boxed{F(p) = \int_0^\infty e^{-pt} f(t)\, dt}$$

use "t, p" in usual

Notations: $\boxed{\begin{array}{lll} F(p) = \mathcal{L}\{f(t)\} & \text{or} & F(p) \doteqdot f(t) \\ f(t) = \mathcal{L}^{-1}\{F(p)\} & \text{or} & f(t) \doteqdot F(p) \end{array}}$

Remark: Let $f(t)|_{t<0} = 0$, i.e. $f(t-t') \equiv f(t-t')\eta(t-t')$, where

$$\eta(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$$

is the Heaviside unit step function.

EX1. Find the L.T. of $f(t) = 1$.

~~6.2. Properties.~~

$$1 \doteqdot \int_0^\infty 1 \cdot e^{-pt} dt = -\frac{1}{p} e^{-pt}\Big|_0^\infty = \frac{1}{p}, \; \mathrm{Re}\, p > 0.$$

$\square$

EX2. Find the L.T. of $f(t)=e^{\alpha t}$.

$$e^{\alpha t} \doteqdot \int_0^\infty e^{\alpha t}\cdot e^{-pt}\,dt = \frac{1}{p-\alpha},\quad Re(p-\alpha)>0. \qquad \square$$

6.2. Properties of L.T.

① L.T. is a linear transform. [Proof: by definition $\square$]

~~② F(p) is analytic at Re p > s₀, where s₀ is called~~

Remark:

$$\sin\omega t = \frac{1}{2i}(e^{i\omega t}-e^{-i\omega t}) \doteqdot \frac{1}{2i}\left(\frac{1}{p-i\omega}-\frac{1}{p+i\omega}\right) = \frac{\omega}{p^2+\omega^2}$$

$$\cos\omega t = \frac{1}{2}(e^{i\omega t}+e^{-i\omega t}) \doteqdot \frac{1}{2}\left(\frac{1}{p-i\omega}+\frac{1}{p+i\omega}\right) = \frac{p}{p^2+\omega^2}$$

② $F(p)$ is analytic at $Re\,p > s_0$, where s_0 is called abscissa of convergence

③ ~~When Re(p)~~ $\lim_{Re\,p\to+\infty} F(p)=0$,

$$\lim_{Im\,p\to\pm\infty} F(p)=0.$$

④. The derivative of L.T.:

$$\boxed{f'(t) \doteqdot pF(p)-f(0)}$$

Proof: $\int_0^\infty f(t)e^{-pt}dt = f(t)e^{-pt}\Big|_0^\infty + p\int_0^\infty f(t)e^{-pt}dt$ $\square$

Corollary: $\boxed{f^{(n)}(t) \doteqdot p^n F(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}$

⑤ The integration of L.T.

$$\boxed{\int_0^t f(t')dt' \doteqdot \frac{1}{p}F(p)}$$

Proof. Def. $\int dt$. $\Rightarrow$ $\left(\int_0^t f(t')dt'\right) \doteqdot \mathcal{L}\left\{\int_0^t f(t')dt'\right\}$

④ $\frac{d}{dt}$ $\Rightarrow$

$$f(t) \doteqdot p\mathcal{L}\left\{\int_0^t f(t')dt'\right\} - 0$$

$$F(p) = p\mathcal{L}\left\{\int_0^t f(t')dt'\right\}$$

$$\frac{1}{p}F(p) \doteqdot \int_0^t f(t')dt'$$

6-3. Application of L.T. on solving ODEs.

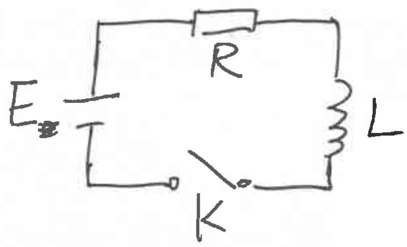
$$\frac{d}{dt} \Rightarrow p \qquad \int_0^t \Rightarrow \frac{1}{p}$$

So we can use L.T. to transform calculus into algebras.

In the circuit analysis, we have [Ref: E~~CEN~~1005]

$$\boxed{\begin{array}{ll} \text{Resistor:} & u = iR \\ \text{Inductor:} & u = L\dfrac{di}{dt} \\ \text{Capacitor:} & i = C\dfrac{du}{dt} \end{array}} \quad \text{and} \quad \boxed{\begin{array}{l} \text{Kirchhoff laws:} \\ \text{KVL: } \sum_{\text{node}} i = 0 \\ \text{KCL: } \sum_{\text{loop}} u = 0 \end{array}}$$

Ex3. Consider a RL circuit, find $i(t)$ after the K is closed



From the KVL, we have an ODE

$$L\frac{di}{dt} + Ri = E,$$

and BCs (Boundary condition): $\quad i(0) = 0.$

Apply L.T. $\quad i(t) \doteqdot I(p)$, then $\quad L\frac{di}{dt} \doteqdot LpI - i(0) = LpI$,

$$LpI + RI = \frac{1}{p}E$$

$$\Rightarrow I(p) = \frac{E}{p}\cdot\frac{1}{Lp+R} = \frac{E}{R}\left(\frac{1}{p} - \frac{1}{Lp+R}\right).$$

$$\Rightarrow i(t) = \mathcal{L}^{-1}\{I(p)\} = \frac{E}{R}\left(1 - e^{-Rt/L}\right). \qquad \square$$

Ex4. Solve the ODE:

$$y''(t) - y'(t) - 2y(t) = 0$$

and BCs: $\quad y(0) = 1, \; y(t)|_{t\to\infty}$ converge. $\qquad$ Remark: "定解问题" (Definite solution problem)

Assuming we have L.T.: $y(t) \doteqdot Y(p)$:

$$y'(t) \doteqdot pY(p) - 1, \quad y''(t) \doteqdot p^2Y(p) - p - \underbrace{y'(0)}_{\text{unknown}}$$

Thefore, the original ODE becomes

$$[p^2Y(p)-p-y'(0)]-[pY(p)-1]-2Y(p)=0$$

$$\Rightarrow Y(p)=\frac{p-1+y'(0)}{p^2-p-2}=\frac{1}{3}\left(\frac{1}{p-2}+\frac{2}{p+1}\right)+\frac{1}{3}y'(0)\left(\frac{1}{p-2}-\frac{1}{p+1}\right)$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}\{Y(p)\}=\frac{1}{3}(2-y'(0))e^{-t}+\frac{1}{3}(1+y'(0))e^{2t}.$$

From $y(t)|_{t\to\infty}$ converge, we have $y'(0)=-1$.

Therefore, we have

$$y(t)=e^{-t}.$$

$\square$

Ex5. Consider a LC circuit, find $i(t)$.

From $u=\frac{q}{C}$, $u=L\frac{di}{dt}$, KVL, we have

$$\frac{q(t)}{C}=L\frac{di}{dt}, \quad q(t)=-\int_0^t i(t')\,dt'+q_0$$

$$\Rightarrow L\frac{di}{dt}+\frac{1}{C}\int_0^t i(t')\,dt'=\frac{q_0}{C}$$ (this is an integro-differential funct

Apply L.T. $i(t)\doteqdot I(p)$:

$$LpI(p)+\frac{1}{C}\cdot\frac{1}{p}I(p)=\frac{1}{p}\cdot\frac{q_0}{C}$$

$$\Rightarrow I(p)=\frac{q_0}{LCp^2+1}$$

Recall that $\sin wt \doteqdot \frac{1}{p^2+w^2}$, we have $i(t)=\mathcal{L}^{-1}\{I(p)\}=\frac{q_0}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}$

That indicates that LC circuit ~~with~~ will produce a wave with $w=\frac{1}{\sqrt{LC}}$. $\square$

6.4. Inverse of L.T., special case

The uniqueness of the inverse of L.T.:

I.L.T. is unique if and only if the $f(t)$ is ~~unique~~ continuous.

We thereby only consider the $f(t)$ that is continuous.

① The inverse of derivatives: (if we know $F(p) \doteq f(t)$)

$$\boxed{F^{(n)}(p) \doteq (-t)^n f(t)}$$

Proof: $F^{(n)}(p) = \left(\frac{d}{dp}\right)^n \int_0^\infty f(t) e^{-pt} dt = \int_0^\infty f(t) \cdot (-t)^n e^{-pt} dt.$ $\square$

· Corollary: From $\frac{1}{p} \doteq 1$, we have

$$\frac{1}{p^2} \doteq -\frac{d}{dp}\frac{1}{p} \doteq t$$

$$\frac{1}{p^3} = \frac{1}{2}\left(\frac{d}{dp}\right)^2 \frac{1}{p} \doteq \frac{1}{2} t^2 \quad \cdots$$

$$\Rightarrow \quad \frac{1}{p^n} \doteq t^{n-1} \cdot \frac{1}{(n-1)^n}$$

Ex6. Find the I.L.T. of $\frac{1}{p^3(p+\alpha)}$.

$$\frac{1}{p^3(p+\alpha)} = \frac{1}{\alpha}\frac{1}{p^3} - \frac{1}{\alpha^2}\frac{1}{p^2} + \frac{1}{\alpha^3}\frac{1}{p} - \frac{1}{\alpha^3}\frac{1}{p+\alpha}$$

$$\doteq \frac{1}{2\alpha} t^2 - \frac{1}{\alpha^2} t + \frac{1}{\alpha^3} - \frac{1}{\alpha^3} e^{-\alpha t}.$$ $\square$

② The inverse of the integrations

$$\boxed{\int_p^\infty F(p')\,dp' \doteq \frac{1}{t} f(t)}$$

Proof. $\frac{d}{dp}\left(\int_p^\infty F(p')dp'\right) \doteq (-t)\left(\frac{1}{t} f(t)\right)$

$$-F(p) = -f(t)$$ $\square$

Corollary: $\frac{\sin \omega t}{t} \doteq \int_p^\infty \frac{\omega}{p'^2+\omega^2}\,dp' = \frac{\pi}{2} - \arctan\frac{p}{\omega}$.

Corollary: If the integration exists at $p \to 0$, we have

$$\boxed{\int_0^\infty F(p)\,dp = \int_0^\infty \frac{f(t)}{t}\,dt.}$$

Ex7. Solve $\int_0^\infty \frac{f(t)}{t}\,dt$ where $f(t)=\sin t$.

$$\int_0^\infty \frac{\sin t}{t} = \int_0^\infty \frac{1}{p^2+1}\,dp = \frac{\pi}{2}.$$

③ $F(p)$ is analytic at $p=\infty$:

$$\boxed{F(p)=\sum_{n=1}^{\infty} c_n p^{-n} \Rightarrow f(t)=\sum_{n=0}^{\infty}\frac{c_{n+1}}{n!}t^n}$$

④ The convolution theorem of L.T.: [Remark: for F.T., it is $\mathcal{F}(f)\mathcal{F}(g)=\mathcal{F}\{f*g\}$]

If ~~$F_1(p)F_2(p)$~~ $F_*(p)\doteq f_*(t)$, $G(p)\doteq g(t)$, then

$$\boxed{F_*(p)G(p)\doteq\int_0^t f(t')g(t-t')\,dt'.}$$

Proof: $F_*(p)\,G(p)=\int_0^\infty f(t')e^{-pt'}dt'\int_0^\infty g(t'')e^{-pt''}dt''$

$$=\int_0^\infty f(t')\,dt'\int_0^\infty g(t'')e^{-p(t'+t'')}dt''$$

$$\begin{pmatrix} t\equiv t'+t'' \\ t''=t-t' \end{pmatrix} \quad =\int_0^\infty f(t')\,dt'\int_{t'}^\infty g(t-t')e^{-pt}dt$$

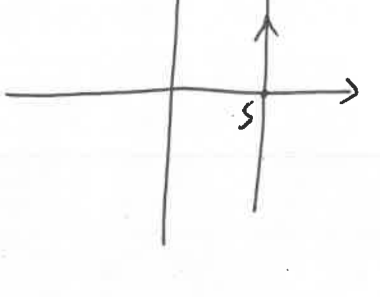
$$=\int_0^\infty e^{-pt}dt\int_0^t f(t')g(t-t')\,dt'.$$

Remark: $\int_0^\infty dt'\int_{t'}^\infty dt \equiv \int_0^\infty dt\int_0^t dt'$ from multivariable calculus. □

6.5. Inverse of L.T., General case.

Theorem. If $\int_{s-i\infty}^{s+i\infty}|F(p)|\,dp$ $(s>s_0)$ converge, $\forall s$ then for $Re(p)>s_0$, we have

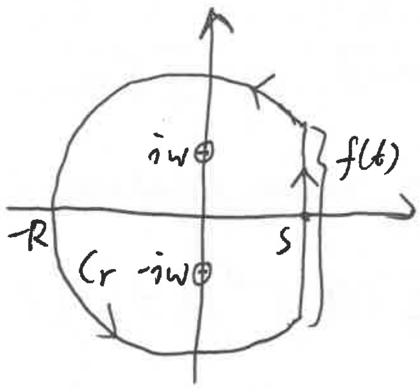
$$\boxed{F(p)\doteq\frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty}F(p)e^{pt}\,dp}.$$



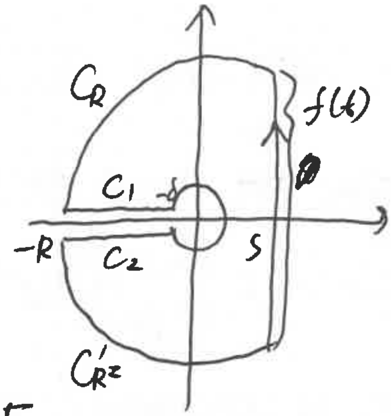
~~Proof:~~ i.e. a complex integral.

Proof: Refer to 吴崇试 P.128. □

Practically, we usually refer to the contour (围道) below:



For singlevalue function



For multivariable function

EX8. Find the I.L.T. of $F(p)=\frac{1}{(p^2+w^2)^2}$, $w>0$.

Using the contour above, we have

$$f(t)=\frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty}\frac{1}{(p^2+w^2)^2}e^{pt}dp, \quad (t>0),$$

and the singularities are shown above: $p=\pm iw$.

According to the Jordon's lemma, we have

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{(p^2+w^2)^2}e^{pt}dp=0.$$

Therefore, according to the Residue theorem, we have

$$f(t)=\sum \mathrm{Res}\left\{\frac{1}{(p^2+w^2)^2}e^{pt}\right\}-0=\mathrm{Res}\left\{\frac{e^{pt}}{(p^2+w^2)^2}\right\}\Bigg|_{p=iw}+\mathrm{Res}\left\{\frac{e^{pt}}{(p^2+w^2)^2}\right\}\Bigg|_{p=-iw}$$

$$=\left\{\left[\frac{t}{(p+iw)^2}-\frac{2}{(p+iw)^3}\right]e^{pt}\right\}_{p=iw}+\left\{\left[\frac{t}{(p-iw)^2}-\frac{2}{(p-iw)^3}\right]e^{pt}\right\}_{p=-iw}$$

$$=\frac{1}{2w^3}(\sin wt-wt\cos wt).$$

6.6. Using L.T. ~~ta~~ to find the ~~*~~ series.

If ~~an~~ a ~~serie~~ series can be represented by $\sum F(n)$, where $F(p)$ is ~~the fun~~ an image function, then it can be represented by

$$\sum_n F(n) = \sum_n \int_0^\infty f(t) e^{-nt}\,dt = \int_0^\infty \sum_n f(t) e^{-nt}\,dt = \int_0^\infty f(t)\left(\sum_n e^{-nt}\right)dt$$

Over there, we exchanged the sequence of the integration and sum. It requires the ~~fu~~ function $(f(t)e^{-nt})$ to be differentiable, integradable and continuous. (uniform convergence).

EX 9. Find $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

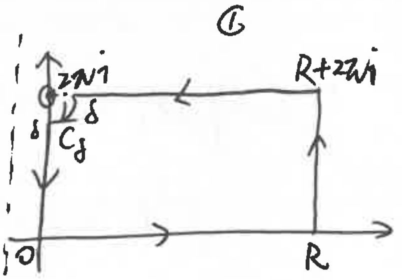
Consider a L.T. $F(n) \doteq f(x)$, we ~~hope~~ have $\frac{1}{n^2} \doteq x$, then

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \int_0^\infty x e^{-nx}\,dx = \int_0^\infty x \sum_{n=1}^{\infty} e^{-nx}\,dx = \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,dx = \int_0^\infty \frac{x}{e^x - 1}\,dx.$$

Therefore, we can use the complex integral to find S:

$$\text{Consider } \oint_C \frac{z^2}{e^z-1}\,dz = \underbrace{\int_0^R \frac{x^2}{e^x-1}\,dx}_{A} + \underbrace{\int_0^{2\pi} \frac{(R+iy)^2}{e^{R+iy}-1}\,i\,dy}_{B}$$

$$+ \underbrace{\int_R^\delta \frac{(x+2\pi i)^2}{e^x-1}\,dx}_{C} + \underbrace{\int_{C_\delta} \frac{z^2}{e^z-1}\,dz}_{D} + \underbrace{\int_{2\pi-\delta}^{0} \frac{(iy)^2}{e^{iy}-1}\,dz}_{E} = 0$$



where ~~only B, C and D contains imaginary part~~:

$$\lim_{R\to\infty} B = 0,\quad \lim_{\delta\to 0} D = -\frac{1}{4}\cdot 2\pi i \operatorname{Res}\left(\frac{z^2}{e^z-1}\right)_{z=2\pi i} = \cancel{2\pi^3 i,\ C = \int_0^{\pi}} -\frac{\pi i}{2}\cdot(2\pi i)^2 = 2\pi^3 i$$

$$A + C = -\int_0^\infty \frac{4\pi i x - 4\pi^2}{e^x - 1}\,dx,\quad E = \int_0^{2\pi} \frac{y^2 e^{-iy/2}}{2\sin(y/2)}\,dy$$

$$\Rightarrow -\int_0^\infty \frac{4\pi i x - 4\pi^2}{e^x-1} + \int_0^{2\pi} \frac{y^2 e^{iy/2}}{2\sin(y/2)}\,dy + 2\pi^3 i = 0$$

The imaginary part: $4\pi \underbrace{\int_0^\infty \frac{x}{e^x-1}\,dx}_{S} + \frac{1}{2}\int_0^{2\pi} y^2\,dy = 2\pi^3 \Rightarrow S = -\frac{1}{3}\pi^2 + \frac{1}{2}\pi^2 = \frac{\pi^2}{6}$.