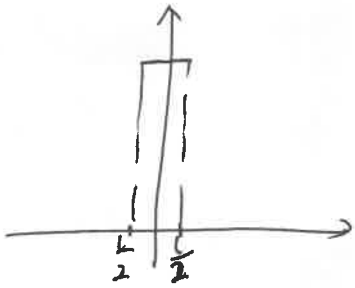
Seminar 102. Lecture 7. Delta function

[Ref. 吴崇试 C10.19, Riley C13.1]

7.1. Definition of Delta function.



Consider a charge distribution:

$$\delta_l(x) = \begin{cases} 0, & x \le -\frac{l}{2} \text{ or } x \ge \frac{l}{2} \\ \frac{1}{l}, & -\frac{l}{2} < x < \frac{l}{2} \end{cases}$$

~~and~~ Apparently, $\int_{-\infty}^{\infty} \delta_l(x)\, dx = 1$. Then, let $l \to 0$, we have

$$\boxed{\delta(x) = \lim_{l \to 0} \delta_l(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0. \end{cases}}$$

It has the following property:

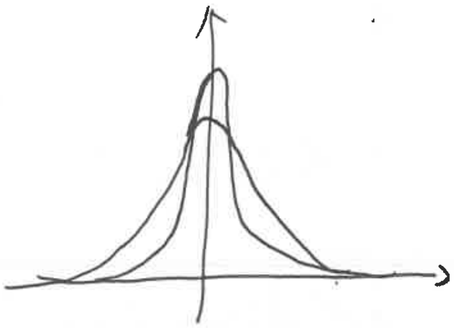
$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0).}$$

where $f(x)$ is called the test function, and it is continuous at $x=0$.

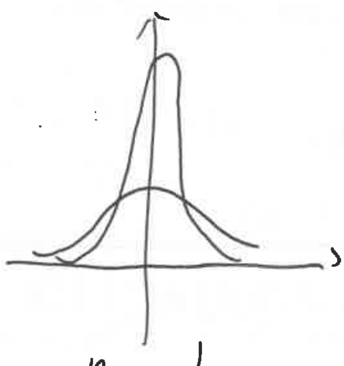
Therefore, we introduce the definition of $\delta(x)$:

$$\boxed{\text{Any limit of sequence } \{\delta_n(x)\} \text{ if } \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\,\delta_n(x)\, dx = f(0).}$$

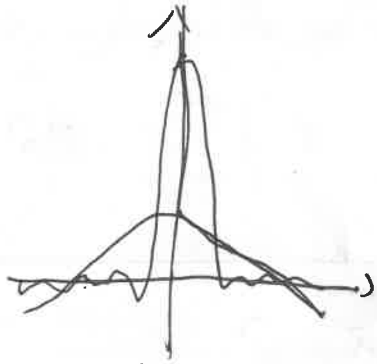
For example, the following sequence is qualified:



$\frac{n}{\sqrt{\pi}} e^{-n^2x^2}$



$\frac{n}{\pi} \frac{1}{1+n^2x^2}$



$\frac{\sin nx}{\pi x}$

7.2. The properties of $\delta(x)$.

It is essential to stress that, all of the ~~fun~~ functions consisting $\delta(x)$ should be understood under an integration:

$$\boxed{f(x)=\delta(x) \Leftrightarrow \int f(x)dx = \begin{cases} 0, & x\neq 0 \\ f(0), & x=0 \end{cases}}$$

~~That,~~ ~~it~~ is because the only non-trivial point of $\delta(x)$ is its singularity. Then, we can look into some properties of $\delta(x)$.

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~~$x\delta(x)=0$~~

$$\boxed{\begin{array}{ll} ① & \delta(-x)=\delta(x) \\ ② & \delta'(-x)=-\delta(x) \\ ③ & \delta(ax)=\frac{1}{|a|}\delta(x) \\ ④ & g(x)\delta(x)=g(0)\delta(x) \end{array}}
\begin{array}{l} \Leftrightarrow \\ \Leftrightarrow \\ \Leftrightarrow \\ \Leftrightarrow \end{array}$$

Using the differentiation of ④, we also have

$$\boxed{⑤\ g(x)\delta'(x)=g(0)\delta'(x)-g'(x)\delta(x).}$$

Proof: $[g(x)\delta(x)]'=[g(0)\delta(x)]'=\begin{cases} g'(x)\delta(x)+g(x)\delta'(x) \\ g(0)\delta'(x) \end{cases}$ □

The differentiation ~~and integration~~ of $\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta'(x)\,dx = f(x)\delta(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty} f'(x)\delta(x)\,dx=-f'(0).$$

The integration of $\delta(x)$:

$$\int_{-\infty}^{x}\delta(x')dx'=\eta(x) \Rightarrow \boxed{\delta(x)=\frac{d\eta(x)}{dx}}.$$

where $\eta(x)$ is the ~~heavistep~~ step function.

The F.T. of $\delta(x)$:

$$\cancel{\delta(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}dx = 1$$

$$\Rightarrow \boxed{\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk}\ \text{, or}\ \boxed{\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos kx\,dk.}$$

The L.T. of $\delta(x)$:

$$\boxed{\delta(t-t_0) \doteq \int_{-\infty}^{\infty}\delta(t-t_0)e^{-pt}dt = e^{-pt_0},\ t_0 > 0.}$$

The 3-D $\delta(x)$:

$$\boxed{\iiint f(\vec{x})\delta(\vec{x})d^3\vec{x} = f(0).} \qquad (x \equiv r)$$

7.3. Using $\delta(x)$ to find the integrations.

~~If we can find the pattern~~

The following representations:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \quad \text{or} \quad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos kx\,dk$$

is useful if we can find such pattern in the integrations.

Ex1. Find $I = \int_{-\infty}^{\infty}\frac{\sin x}{x}dx$.

Consider $F(\lambda) = \int_{-\infty}^{\infty}\frac{\sin\lambda x}{x}dx$

$$\Rightarrow F'(\lambda) = \int_{-\infty}^{\infty}\cos\lambda x\,dx = 2\pi\delta(\lambda).$$

Therefore, $F(\lambda) = \int F'(\lambda)d\lambda = 2\pi\eta(x) + C$

~~where~~ using $F(-\lambda) = F(\lambda)$, we have $C = -\pi$.

Therefore, $I = F(1) = 2\pi - \pi = \pi$.