SPUM 102 Lecture 2. Γ function

Remark: Proposition 命题
Theorem 定理
Lemma 引理
Corollary 推论

Γ (Gamma) ζ (Zeta) $\zeta(s) = \sum_n \frac{1}{n^s}$

4.1. Definition

Def. $\Gamma(z) = \int_0^\infty z^{t-1} e^{-t} dt$, $\mathrm{Re}\, z > 0$ (Integrating w.r.t. t)

Proposition. $\mathrm{Re}\, z > 0 \Rightarrow \int_0^1 t^{z-1} e^{-t} dt$ & $\int_1^\infty z^{t-1} e^{-t}$ converge $\Leftrightarrow \Gamma(z)$ analytic.

Ex1. Maxwell speed distribution: $\boxed{dN = 4\pi N \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-mv^2/2kT} v^2 dv}$, find $\langle v^n \rangle$.

$$\langle v^n \rangle = \frac{1}{N} \int_0^\infty v^n dN$$

Remark: $/$, —

$$= \frac{1}{N} \int_0^\infty v^n 4\pi N \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-mv^2/2kT} v^2 dv$$

Remark: $\equiv$, $:=$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^\infty e^{-mv^2/2kT} v^{n+2} dv \qquad \left(\text{Let } \frac{mv^2}{2kT} \equiv x\right)$$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \frac{1}{2} \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \int_0^\infty e^{-x} x^{(n+1)/2} dx$$

$$= \frac{2}{\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{\frac{n}{2}} \Gamma\left(\frac{n+3}{2}\right)$$

4.2. Properties.

1) $\Gamma(1) = 1$ Proof. trivial.

2) $\Gamma(z+1) = z\Gamma(z)$ Proof. Integral by part

Corollary: $\Gamma(n+1) = n!$, $n \in \mathbb{N}$.

3) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

Corollary: $\Gamma(z) \neq 0$ on $\mathbb{C}$.

4) $\Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma(z+\frac{1}{2})$.

Ex2. Find $\Gamma(\frac{1}{2})$.

Let $z = \frac{1}{2}$ and insert it into 3).

Remark. Some special values:

$\Gamma(-\frac{3}{2}) = \frac{4\sqrt{\pi}}{3}$

$\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(1) = 1$

$\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$

$\Gamma(2) = 2$

$\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$

$\Gamma(3) = 2$

$\Gamma(\frac{7}{2}) = \frac{15\sqrt{\pi}}{8}$

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Stirling's formula: $\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z$ when $|z| \to \infty$

$$\Rightarrow \boxed{\ln n! = n\ln n - n}$$

Ex 3. N bosons, 2 states, find entropy $S = k_B \ln \overset{\text{number of state}}{\Omega}$, where $\Omega = \frac{N!}{(xN)!\,[(1-x)N]!}$

$$S = k_B \ln \Omega = k_B \ln\left[\frac{N!}{(xN)!\,[(1-x)N]!}\right] = k_B\{\ln N! - \ln (xN)! - \ln [(1-x)N]!\}$$
$$= k_B\{N\ln N - \cancel{N} - xN\ln(xN) + \cancel{xN} - (1-x)N\ln[(1-x)N] + \cancel{(1-x)N}\}$$
$$= Nk_B\{\ln N - x\ln xN - (1-x)\ln[(1-x)N]\}$$
$$= Nk_B\{\cancel{\ln N} - x\ln x - \cancel{x\ln N} - (1-x)\ln(1-x) - \cancel{(1-x)\ln N}\}$$
$$= Nk_B[-x\ln x - (1-x)\ln(1-x)]$$

4.3. ψ function.

Def. $\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$

Remark. Special values:

$\psi(1) = -\gamma$

$\psi(\frac{1}{2}) = -\gamma - 2\ln 2$

Properties: (induced from $\Gamma(z)$):

1) $\psi(z+1) = \psi(z) + \frac{1}{z}$

   Corollary: $\boxed{\psi(z+n) = \psi(z) + \sum_{n=0}^{n-1}\frac{1}{z+n}}$

2) $\psi(1-z) = \psi(z) + \pi\cot\pi z$

3) $\psi(2z) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi(z+\frac{1}{2}) + \ln 2$

4) $\boxed{\lim_{n\to\infty}[\psi(z+n) - \ln n] = 0}$

5) $\boxed{\psi(n) = H_n - \gamma}$, where $H_n = \sum_{k=1}^{n}\frac{1}{k}$

Euler constant: $\gamma = \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{n}\right) - \ln n\right] = \int_0^\infty\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx \approx 0.5772156649015 32\cdots$

$$\boxed{\psi(n+\tfrac{1}{2}) = -\gamma - 2\ln 2 + \sum_{k=1}^{n}\frac{2}{2k-1}} \quad \text{for } n \in \mathbb{Z}$$

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