# SPUM 102. Lecture 5. Fourier transform & Laplace transform

Supplementary: Analytic continuation 解析延拓

Considering $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, $|z|<1$ because LHS diverge at $|z|\geq 1$.

However, domain of RHS: $z \in \mathbb{C} \setminus 1$

Why don't we let $\sum_{n=0}^{\infty} z^n := \frac{1}{1-z}$ when $|z|\geq 1$?

Def. Analytic continuation:

$f(z_1)$ $z_1 \in U$, $F(z_2)$ $z_2 \in V$, $U \subset V$, $F(z) = f(z)$ $z \in U \Rightarrow F$ is the A.C. of $f$. $F$ unique.

For the example above, we pick $z=-1$: $1-1+1-1\cdots := \frac{1}{2}$. $(\neq (1-1)+(1-1)+\cdots)$

$z=2$: $1+2+4+8+\cdots := -1$

## 5.1. Fourier series F.S.

We know the power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Any other ways? Fourier series.

Advantages: ① easy to differentiate & integrate;

② each term contains one characteristic frequency;

③ can be used to solve ODE/PDE as numerical method;

④ can be used to discribe non-differentiable functions

Dirichlet condition: ① periodic function, $f(x+p) = f(x)$;

② single-valued, continuous except for finite points;

③ have a limit number of maximum & minimum;

④ $\int_0^p |f(x)|\, dx$ converge.

Ex1. 随便出几个判断题

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Then, we will only consider the functions obeys the dilichlet condition.

Generally, a function is a composition of an odd function and an even function:

$$f(x) = \underbrace{\frac{1}{2}\left[f(x)+f(-x)\right]}_{\text{even}} + \underbrace{\frac{1}{2}\left[f(x)-f(-x)\right]}_{\text{odd}}$$

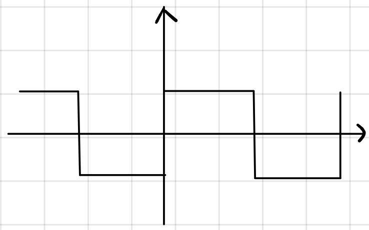
Def. Fourier series: $\boxed{f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left[\underbrace{a_r \cos\frac{2\pi r x}{L}}_{\text{even}} + \underbrace{b_r \sin\frac{2\pi r x}{L}}_{\text{odd}}\right]}$ orthogonality

Fourier coefficients:

$$\boxed{\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ a_r &= \frac{2}{L}\int_0^L f(x)\cos\frac{2\pi r x}{L}\,dx \\ b_r &= \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi r x}{L}\,dx \end{aligned}}$$

Ex2. F. S. of $\sin(x + \frac{\pi}{4})$.

Ex3. Square wave function $f(t) = \begin{cases} -1, & -\frac{1}{2}T \leq t < 0 \\ 1, & 0 \leq t < \frac{1}{2}T \end{cases}$

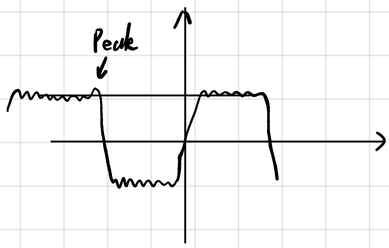


$$f(t) = \sum_{r=1}^{\infty} b_r \sin\frac{2\pi r t}{T} \qquad (\text{odd}, \ a_0 = a_r = 0)$$

$$b_r = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\frac{2\pi r t}{T}\,dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\frac{2\pi r t}{T}\,dt = \frac{4}{T}\int_0^{\frac{T}{2}} \sin\frac{2\pi r t}{T}\,dt$$

$$= \frac{4}{T}\cdot\frac{T}{2\pi r}\cos\frac{2\pi r t}{T}\Big|_0^{\frac{T}{2}} = \frac{2}{\pi r}\left[1-(-1)^r\right]$$

$$\Rightarrow f(t) = \frac{4}{\pi}\left[\sin\frac{2\pi t}{T} + \frac{1}{3}\sin\frac{6\pi t}{T} + \frac{1}{5}\sin\frac{10\pi t}{T} + \cdots\right]$$




Gibbs phenomenon: When doing F.S. on the discontinuous function, F.S. will always overshoot its value, although more terms will reduce it.

For non-periodic function: expand its value.

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Integration & differentiation.

Ex4. Find $f(x)=x^3$, $0<x<2$, knowing that $x^2=\underbrace{\frac{4}{3}+\sum_{r=1}^{\infty}\frac{(-1)^n}{\pi^2r^2}\cos\frac{\pi r x}{2}}_{G.}$

$$f(x)=\int 3x^2=\int(4+3G)\,dx=\underbrace{4x}_{\text{not a F.S.}}+3\int G\,dx+C$$

$$4x=2\frac{d}{dx}x^2=2G'$$

$$\Rightarrow f(x)=2G'+3\int G\,dx+C \quad\Rightarrow \text{a F.S.}$$

5.2. Complex Fourier Series.

We know that $e^{irx}=\cos rx+i\sin rx$

Therefore, we have

$$\boxed{\begin{gathered} f(x)=\sum_{r=-\infty}^{\infty}c_r e^{-\frac{2\pi i r x}{L}} \\ c_r=\frac{1}{L}\int_0^L f(x)e^{-\frac{2\pi i r x}{L}}\,dx. \end{gathered}}$$

where the coefficients

Relation of $c_r$ and $a_r, b_r$: $c_r=\frac{1}{2}(a_r+ib_r)$, $c_{-r}=\frac{1}{2}(a_r-ib_r)$.

Ex5. $f(x)=x$, $-2<x<2$.

$$c_0=0$$

$$c_r=\frac{1}{L}\int_{-2}^{2}xe^{-\frac{2\pi i r x}{4}}\,dx=\frac{2i}{\pi r}(-1)^r$$

$$f(x)=\sum_{r=-\infty}^{\infty}\frac{2i}{\pi r}(-1)^r e^{-\frac{\pi i r x}{2}}$$

Parseval's theorem: $\frac{1}{L}\int_0^L|f(x)|^2=\sum_{r=-\infty}^{\infty}|c_r|^2=\frac{a_0^2}{4}+\frac{1}{2}\sum_{r=1}^{\infty}(a_r^2+b_r^2)$

5.3. Fourier transform F.T.

Requirement: Dirichlet condition, but can become non-periodic

$\mathcal{F}$ 1 2 3

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It is basically the change of domain, and it is inversible (inverse F.T., I.F.T).

Def. F.T.

$$\boxed{\begin{aligned} \mathcal{F}[f(x)] &= \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\underbrace{e^{-i\omega t}}_{\text{kernel}}\,dt \\ \mathcal{F}^{-1}[\tilde{f}(\omega)] &= f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)\underbrace{e^{i\omega t}}_{\text{kernel}}\,dt. \end{aligned}}$$

In general:
$$F(k) = \int_a^b f(x)\,K(k,x)\,dx$$

Remark: In different definition, $\frac{1}{\sqrt{2\pi}}$ may change to another side.

It is so basical that it has not been used in the transformation of signal.

Ex6. $f(x) = \begin{cases} 0, & t<0 \\ Ae^{-\lambda t}, & t\geq 0,\ \lambda>0 \end{cases}$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} Ae^{-\lambda t}e^{-i\omega t}dt = \frac{A}{\sqrt{2\pi}}\int_0^{\infty} e^{(-\lambda-i\omega)t}dt = \frac{A}{\sqrt{2\pi}(\lambda+i\omega)}$$

F.T. of Gauss distribution: $\mathcal{F}\left[\frac{1}{\sigma_t\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma_t^2}}\right] = \frac{1}{\sqrt{2\pi}}e^{-\frac{\sigma_t^2\omega^2}{2}}$

It shows that, F.T. of a Gauss distribution is another Gauss distribution, and $\sigma_t\sigma_\omega = \sigma_t\frac{1}{\sigma_t} = 1 \Rightarrow \Delta p\Delta x = \frac{\hbar}{2}$.

F.T. of the delta function: $\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}}$, $\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{\sqrt{2\pi}}$

## 5.4. Properties of F.T.

① Differentiation: $\mathcal{F}[f'(t)] = i\omega\tilde{f}(\omega)$

② Integration: $\mathcal{F}\left[\int f(t)\,dt\right] = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi C\overset{\text{"+C" part}}{\delta(\omega)}$

③ Scaling: $\mathcal{F}[f(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right)$

④ Translation: $\mathcal{F}[f(t+a)] = e^{ia\omega}\tilde{f}(\omega)$

⑤ Exp. multi.: $\mathcal{F}[e^{\alpha t}f(t)] = \tilde{f}(\omega+i\alpha)$