SPUM Lecture 3 Residue theorem

3.1. Intro to Residues

Def. $\mathrm{Res}\, f(z_k)$ $\quad f(z)=\sum_{l=-\infty}^{\infty} a_l^{(k)}(z-z_k)^l$, $\mathrm{Res}\, f(z_k)=a_{-1}^{(k)}$

Residue formula: $\boxed{\oint_C f(z)\,dz = 2\pi i \sum_k \mathrm{res}\, f(z_k)}$

Proof: Recall Cauchy Integral formula: $\oint_C \frac{f(z)}{z-z_k}\,dz = 2\pi i f(z_k)$

Let $g(z)=\frac{f(z)}{z-z_k}$, then at $z=z_k$, $f(z)=f(z_k)+f'(z_k)(z-z_k)+O(z^2)$

$\Rightarrow g(z)=\frac{f(z_k)}{z-z_k}+f'(z_k)+O(z)$

(consider 1 singularity only)

$\Rightarrow$ for $g(z)$, $a_{-1}^{(k)}=f(z_k)$

Then, $\oint_C g(z)\,dz = 2\pi i f(z_k) = 2\pi i\, a_{-1}^{k} = 2\pi i\, \mathrm{Res}\, g(z_k)$

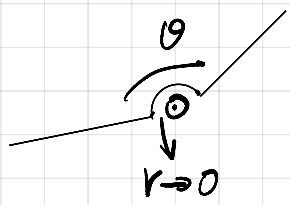
Recall how to obtain a_{-1}: $\quad a_{-1}=\frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1}\left[(z-z_k)^m f(z)\right]_{z=z_k}$

For Poles: $\quad a_{-1}=(z-z_k)f(z)\big|_{z=z_k}$

Therefore, we have a full understanding of $\oint_C f(z)\,dz$. By

① No singularity: 0 ② have singularity: $\sum 2\pi i\, \mathrm{Res}\, f(z_k)$

③ Go through singularity: $i\theta\, \mathrm{Res}\, f(z_k)$



Ex 0. $f(z)=\frac{1}{z}$, $\mathrm{Res}\, f(0)=1$ Ex 1. 随便编一个

Ex 2. $f(z)=\frac{1}{(z^2+1)^3}$, $\mathrm{Res}\, f(\pm i)=\frac{1}{2!}\frac{d^2}{dz^2}\left[(z\mp i)^3\cdot\frac{1}{(z^2+1)^3}\right]_{z=\pm i}=\mp\frac{3}{16}i$

Remark: If $f(z)=\frac{P(z)}{Q(z)}$, $Q(z_0)=0$, then $\mathrm{Res}\, f(z_0)=\frac{P(z_0)}{Q'(z_0)}$

Residue at ∞: $\quad \mathrm{Res}\, f(\infty)=\frac{1}{2\pi i}\oint_{C'} f(z)\,dz$

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3.2. Application 1. Trigonal integration : $I=\int_0^{2\pi} f(\sin\theta,\cos\theta)d\theta$

By sustitution : $\boxed{z=e^{i\theta}}$, then $\boxed{\sin\theta=\frac{z^2-1}{2iz}\qquad \cos\theta=\frac{z^2+1}{2z}\ ,\quad d\theta=\frac{1}{iz}dz}$

Therefore, we have $I=\oint_{|z|=1} f\left(\frac{z^2-1}{2iz},\frac{z^2+1}{2z}\right)\frac{1}{iz}\,dz$

Ex 3. $\int_0^{\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta=\frac{1}{2}\oint_{|z|=1}\frac{\left(\frac{z^2-1}{2iz}\right)^2}{a+b\,\frac{z^2+1}{2z}}\cdot\frac{1}{iz}dz$ $(a>b>0)$

$$=\oint_{|z|=1}\frac{(z^2-1)^2}{4z^2(2az+b(z^2+1))}dz$$

3.3. Application 2. Integration consisting triogonal function: $\begin{cases}\int_{-\infty}^{\infty} f(x)\sin px\,dx\\ \int_{-\infty}^{\infty} f(x)\cos px\,dx\end{cases}$

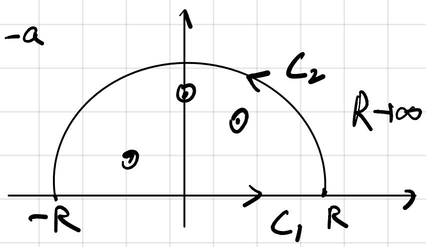
Consider $f(z)\,e^{ipz}$,

then, at C_1, we have $f(z)e^{ipz}=f(x)e^{ipx}=f(x)\cos px+f(x)\,i\sin px$

at C_2, we have $\boxed{\text{Jordan Lemma: } \lim_{R\to\infty}\int_{C_2} f(z)e^{ipz}dz=0 \quad \text{if} \quad f(z)\to 0}$

Ex 3. $\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx \ (=I)$

Consider $\oint_C\frac{ze^{iz}}{z^2+a^2}dz=2\pi i\,\mathrm{Res}\left.\frac{ze^{iz}}{z^2+a^2}\right|_{z=ai}=\pi i e^{-a}$



$$=\int_{-R}^{R}\frac{xe^{ix}}{x^2+a^2}dx+\int_{C_2}\frac{ze^{iz}}{z^2+a^2}dz$$

$$\Rightarrow \int_{-R}^{R}\frac{xe^{ix}}{x^2+a^2}dx=\pi i e^{-a}$$

$$\Rightarrow \int_{-R}^{R}\frac{x\sin x}{x^2+a^2}dx=\pi e^{-a}$$

$$\Rightarrow I=\frac{\pi}{2}e^{-a}$$

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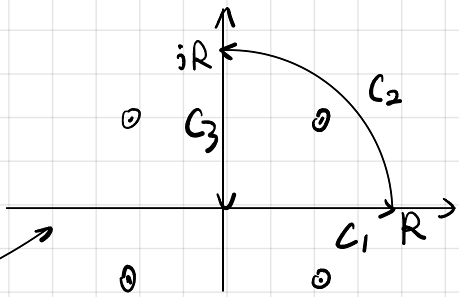
3.4. Application 3. Infinite integral : $I=\int_{-\infty}^{\infty} f(x)dx$

Use the path above again. Things will become easy if we find $\lim_{R\to\infty}\int_{C_2} f(z)dz = 0$.

Ex4. $I=\int_0^{\infty}\frac{1}{1+x^4}dx$



Consider $\oint \frac{1}{1+z^4}dz$, singularities: $z=e^{i(\frac{\pi}{4}+\frac{k\pi}{2})}$

Therefore, we construct the path

$$\oint_C \frac{1}{1+z^4}dz = \underbrace{\int_{C_1}\frac{1}{1+x^4}dx}_{I} + \underbrace{\int_{C_3}\frac{i}{1+(iy)^4}dy}_{iJ} + \underbrace{\int_{C_2}\frac{1}{1+z^4}dz}_{0} = I+iJ$$

$$\boxed{\begin{aligned} 1 &= e^{2\pi i} \\ i &= e^{\frac{\pi i}{2}} \end{aligned}}$$

$$\mathrm{Res}\,\frac{1}{1+z^4}\Big|_{z=e^{\frac{i\pi}{4}}} = (z-e^{\frac{i\pi}{4}})\cdot\frac{1}{(1+iz^2)(1-iz^2)}\Big|_{z=e^{\frac{i\pi}{4}}}$$

$$=\frac{(z-e^{\frac{i\pi}{4}})}{(1+e^{\frac{i\pi}{4}}z)(1-e^{\frac{5i\pi}{4}}z)(1+e^{\frac{3i\pi}{4}}z)(1-e^{\frac{3i\pi}{4}}z)}\Big|_{z=e^{\frac{i\pi}{4}}}$$

$$=\frac{-e^{\frac{i\pi}{4}}\cancel{(1-e^{-\frac{i\pi}{4}}z)}}{(1-e^{\frac{i\pi}{4}}z)(1-e^{\frac{3i\pi}{4}}z)(1-e^{\frac{5i\pi}{4}}z)\cancel{(1-e^{\frac{7i\pi}{4}}z)}}\Big|_{z=e^{\frac{i\pi}{4}}}$$

$$=\frac{-e^{-\frac{i\pi}{4}}}{(1-e^{\frac{i\pi}{4}}z)(1-e^{\frac{3i\pi}{4}}z)(1-e^{\frac{5i\pi}{4}}z)}\Big|_{z=e^{\frac{i\pi}{4}}}$$

$$=\frac{-e^{-\frac{i\pi}{4}}}{(1-e^{\frac{i\pi}{2}})(1-e^{i\pi})(1-e^{\frac{3}{2}i\pi})}$$

$$=\frac{\frac{\sqrt{2}}{2}(i-1)}{\underbrace{(1-i)(1-(-1))(1+i)}_{2}}$$

$$=\frac{\sqrt{2}}{2}(i-1)\cdot\frac{1}{4}$$

$$\Rightarrow 2\pi i\cdot\frac{1}{4}\cdot\frac{\sqrt{2}}{2}(i-1) = \frac{\sqrt{2}}{4}\pi(1-i) = I+iJ$$

$$I=\frac{\sqrt{2}}{4}\pi$$

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3.5. Application 5. Series $\sum_n f(n)$ $\quad a_n \Rightarrow f(n)$

We know that $\frac{1}{2\pi i}\oint F(z)dz = \underline{\sum_{k=1}^{K}} \underline{\mathrm{res}F(z_k)}$,

what if $z_k = n = 0,1,\cdots,N$, and $\mathrm{Res}F(z_k)=f(n)$? Note: $K \geq N$

Recall for poles: $\mathrm{Res} f(z_k) = (z-z_k)f(z)\big|_{z_k}$

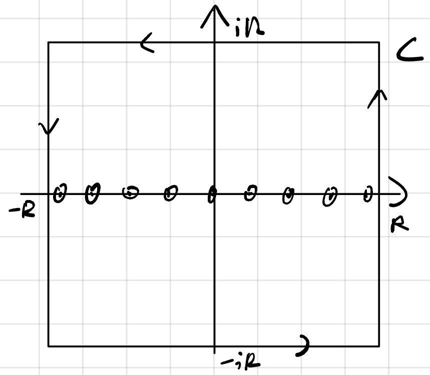
If we have $g(z)$ s.t. $g(z)$ have pole at $0,1,\cdots$, with residue 1.

Let $F=fg$, we have $\mathrm{Res}\,F(n) = \mathrm{Res} f(n)g(n) = f(n)\,\cancel{\mathrm{Res}\,g(n)} = f(n)$.

One possible $\boxed{g(z)=\pi\cot\pi z}$. (check)

$$\boxed{\textbf{Theorem. } \sum_{-\infty}^{\infty} f(n) = \lim_{N\to\infty}\sum_{n=-N}^{N} f(n) = -\sum_{k=1}^{K}\mathrm{Res}\left[\pi\cot\pi z\, f(z)\right]}$$

A typical path: (RHS). By choosing different path, different N can be solved.



Ex5. $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$, find $S=\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Consider $f(z)=\frac{1}{z^2}$. $\begin{cases} z=0: \text{3rd order singularity,} \\ z\in\mathbb{N}\setminus 0: \text{pole.}\end{cases}$

We have $\overset{(\lim R\to\infty)}{\oint_C} f(z)\pi\cot\pi z\,dz = 0$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\mathrm{Res} f(n)\cot n\pi = 2S + \mathrm{Res} f(n)\cot n\pi\big|_{n=0} = \frac{0}{2\pi i} = 0$$

$$\Rightarrow S = -\frac{1}{2}\,\mathrm{Res}\left[\frac{\cot n\pi}{n^2}\right]_{n=0} = -\frac{1}{2}\cdot\left(-\frac{\pi^2}{3}\right) = \frac{\pi^2}{6}$$