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C1 $z = x + iy = ae^{i\theta}$

z^* , $zz^* = |z|^2$

多值性 $z = ae^{i(\theta+2k\pi)}$ $f(ae^{i\theta}) \neq f(ae^{i\theta+2\pi})$

multivalued function $\mathrm{LN}z = \mathrm{LN}ae^{i\theta+2k\pi} = i(\theta+2k\pi)+\ln a$

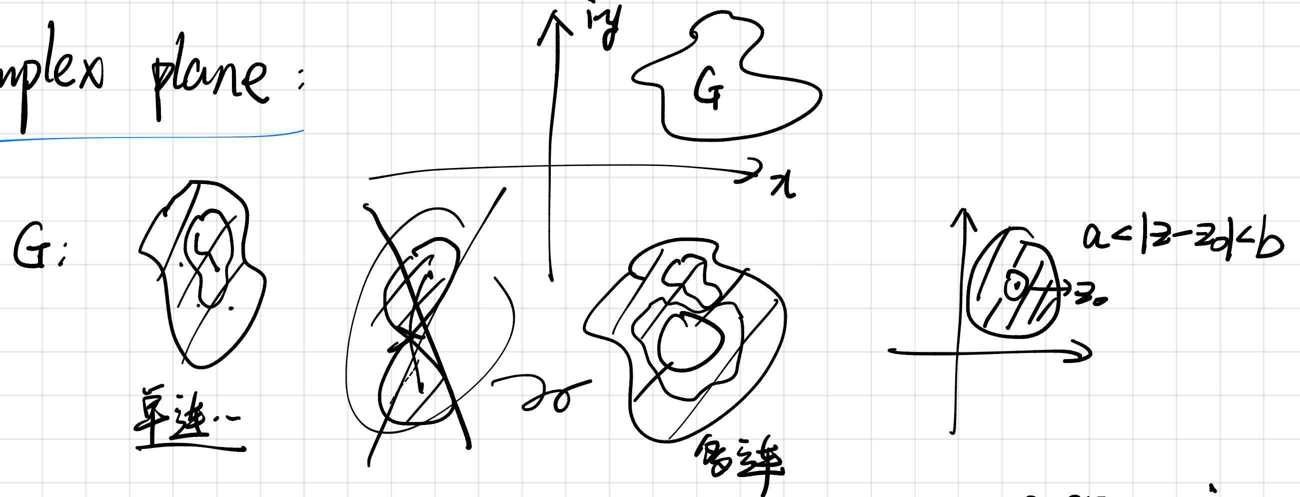
$\ln z = i\theta + \ln a$ $(\theta \in [0, 2\pi))$

Complex series: $z_n = x_n + iy_n$ 性质完全等同 …

Bolzano-Weierstrass thm. 有界序列必有1聚点

聚点 $\forall z, \forall \varepsilon > 0$ s.t. 无穷个 n, $|z_n - z| < \varepsilon$, z 聚点, 必极限

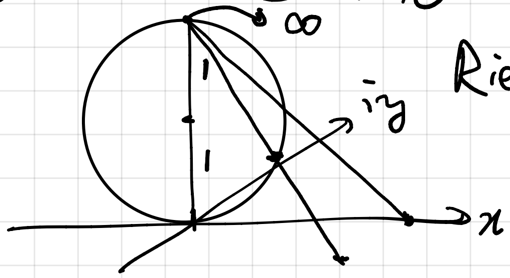
Complex plane:



open region
closed

区域/Region: G {① Interior point ② 连通性

无穷远点: $\mathbb{C}$ 扩充



Riemann or $b = \frac{1}{z}$

$z \to \infty, b \to 0$

laurent series

Complex variable: $f(z) = u + iv$, $u(x,y), v(x,y)$

image: $z \longrightarrow f(z)$

e.g. $x+iy \xrightarrow{f(z)=\frac{1}{z}} u, v$ $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

C2 Analytic function

$f(z) = u(x,y) + iv(x,y)$ 可导

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ ... Green theorem

$\Rightarrow$ Cauchy-Riemann equ.

$z = z_0$

$\left\{ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right.$ → 可导 z_0 → 解析 z_0 邻域

$f(x+iy)$ 而不是 $f(x,y)$

Conjugated harmonic function

$\nabla^2 u = 0$，一定找到唯一 v, s.t. $f(z) = u + iv$ 解析

u, v 互为 CHF

Singularity

meromorphic 亚纯 f $\mathbb{C} \setminus$ 一些点解析

holomorphic 全纯 f $\mathbb{C}$ 解析

$f(z) = \frac{1}{z}$, $z = 0$ 不解析

Singularity point / 奇点

多值函数 multivalued function

① $\sqrt{z-a} = f$

def: $w^2 = z - a$, $w = \sqrt{z-a}$ 宗量: $z-a$ (= 自变量)

给定 z, 2个 w 对应

$w = \sqrt{-1}$, $w = i, -i$

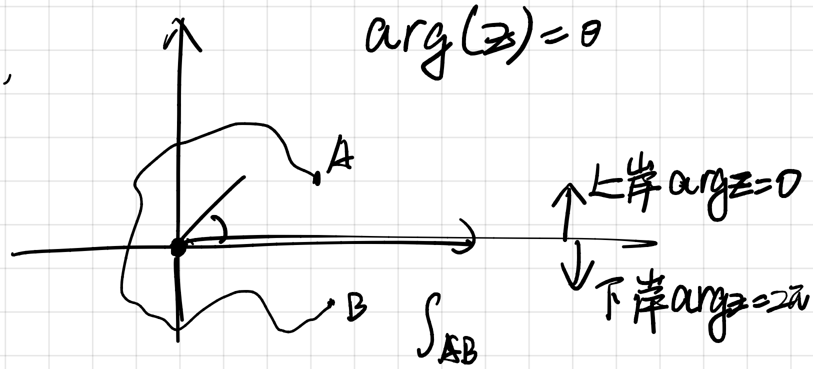
② $\ln a$

Method to make 多值变单值

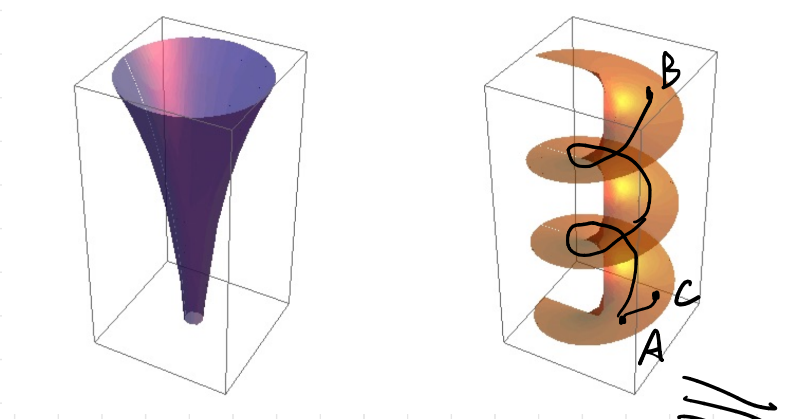
① 割线 连接2个 sing.

$\ln z$

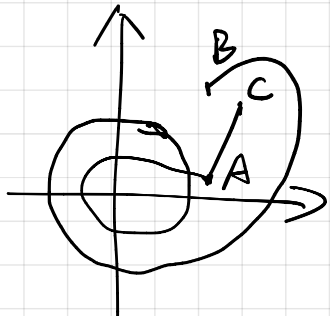
scant line
principle value



② Riemann 面



≡ of $\ln x$



$\operatorname{Arg}(B)=a+4\pi$

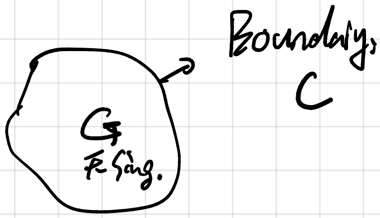
$\operatorname{Arg}(A)=b$

C3. Complex integral

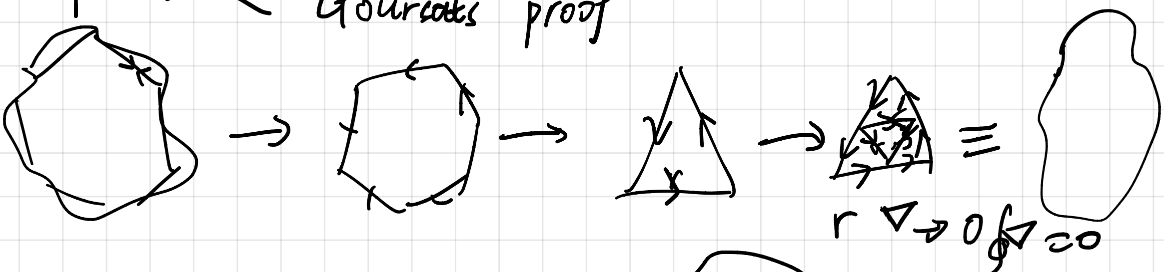
Mainly focus on Analytic function

meromorphic

对于 f, $\oint_C f(z)dz=0$

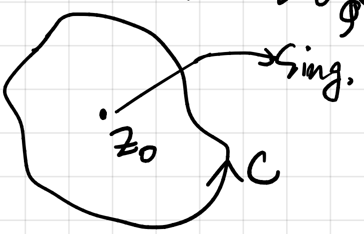


proof < stoke's thm. / Goursat's proof



Cauchy Integral Formula

1) $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$



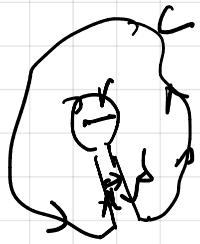
e.g.① $\oint_C \frac{e^z}{z-1}dz$

$= 2\pi i e^{z_0}\Big|_{z_0=1}$

$= 2\pi i e$

$f(z)=e^z$, $z_0=1$, C包含 $z=1$

Proof

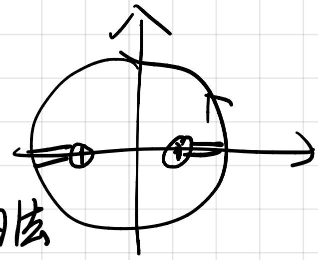


$\delta, r \to 0$

key hole

② $\oint_C \frac{\cos\pi z}{(z-1)(z+1)}dz$ C: $0<|z|<2$

$= 2\pi i \frac{\cos\pi z}{(z+1)}\Big|_{z=1} + 2\pi i \frac{\cos\pi z}{(z-1)}\Big|_{z=-1}$



2) $\oint_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)$

归纳法

C4 无穷级数 Infinite Series

Complex series: $S_n = \sum u_n$, $u_n = x_n + i y_n$

Function series: $S_n = \sum u_n(z)$ at region G

Def. 绝对收敛: $\sum |u_n|$ conv., $\sum u_n$ 绝 conv.

一致收敛: $\forall \varepsilon > 0, \exists N(\varepsilon)$, N与z无关

s.t. $\forall n > N(\varepsilon)$, $\forall z \in G$, 有 $\left|S(z) - \sum_{k=1}^{n} u_k(z)\right| < \varepsilon$

$S(z)$: 和函数

⇐ Weierstrass's M判别法:

if we can find a_k conv. s.t. $\forall u_k(z)$, $|u_k(z)| < a_k$

Properties:

① continuous

若 $u_k(z)$ 在G cont. 则 $S(z)$ cont.

② $\int \sum_k u_k(z)\,dz = \sum_k \int u_k(z)\,dz$

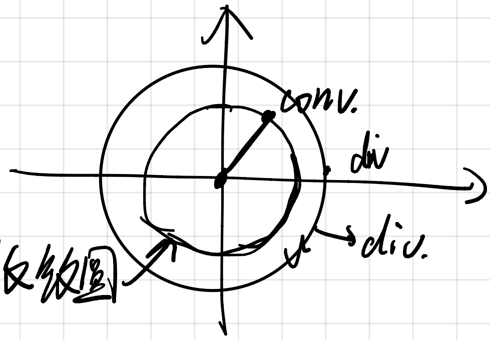
③ $\frac{d^n}{dz^n} \sum_k u_k(z)\,dz = \sum_k \frac{d^n}{dz^n} u_k(z)\,dz$

(Weierstrass thm.)

Power series: $\underbrace{\sum_{n=0}^{\infty} c_n (z-z_0)^n}_{=u(z)}$

Abel 定理：收敛圆内，绝对收敛
一致

收敛圆



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C5 Power series

Taylor series $\overset{def:}{:}$ $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$

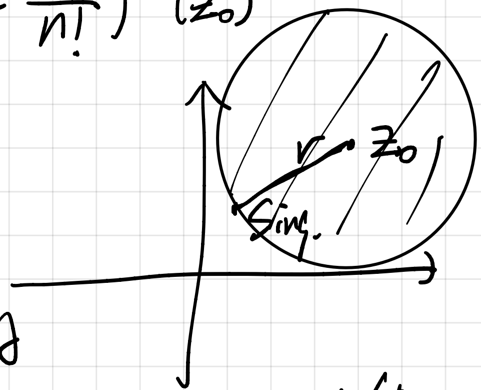
if $f(z)$ Analytic

$\zeta \equiv \xi$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi = \frac{1}{n!} f^{(n)}(z_0)$$

收敛半径 r：z_0 到奇点的距离

只有收敛圆内有效



唯一性. given f, z_0, a_n 完全确定的

Proof. 有 a_n, b_n

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

when, $z \to z_0$, ∵ 一致收敛

$\therefore a_0 = b_0$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} b_n (z-z_0)^n\right)$$

$a_1 = b_1$

......

求法：公式法、级数乘法、待定系数法

级数乘法：

e.g. $\frac{1}{1-3z+2z^2} = \frac{1}{(1-z)} \cdot \frac{1}{(1-2z)} = -\frac{1}{1-z} + \frac{2}{1-2z}$

$\sum \times \sum = \sum$

$\sum a_m \sum b_n$: $a_1 b_1 + a_1 b_2 + \cdots$
$+ a_2 b_1 + a_2 b_2 + \cdots$
$+ \cdots$

$\frac{1}{x+a}$, $\frac{1}{x^2+1}$, $\frac{x}{x^2+1}$ dx dx^2

$\ln|x+a|$ arctan$x$ $\frac{1}{2}\ln(1+x^2)$

待定系数法：$\tan z = \frac{\sin z}{\cos z} = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}$

where $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}$

$\cos z = \sum_{l=0}^{\infty} (-1)^l \cdot \frac{1}{(2l)!} z^{2l}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+1} = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1} \sum_{l=0}^{\infty} (-1)^l \cdot \frac{1}{(2l)!} z^{2l}$$

$=$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{2k+1} \cdot \frac{(-1)^{n-k}}{(2n-2k)!}\right) z^{2n+1}$$

$\sum_{k=0}^{n} (-1)^k \cdot \frac{1}{(2n-2k)!} a_{2k+1} = \frac{1}{(2n+1)!}$ 递推公式

Substitution:

$2n+1 = 2k+1+2l$

l 用 n、和 k 表示

$l = n-k$

$\sum_{l=0}^{\infty} = \sum_{k=n}^{0}$

$k=n$

多值函数：割线

$\ln z =$

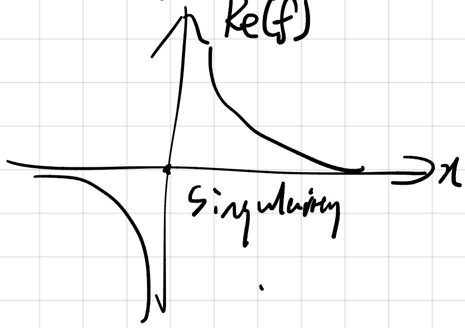
$\frac{1}{z}$ $-\frac{1}{z^2}$ $2\frac{1}{z^3}$ $-3!\frac{1}{z^4}$ ...

无穷远点：令 $b=\frac{1}{z}$ → $f(z)=g(\frac{1}{b})$ ... b^n → z^{-n}

∞点 ≡ 0点

要是孤立的才这样

Laurent series: $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$

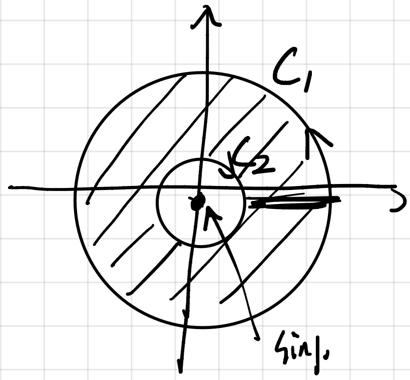


$f(z)=\frac{1}{z}$ ，$z_0=0$ 处展开

和区域密切相关

$n\geq 0: a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta$

$n\leq 0: a_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta$



唯一性

求法：

① 唯一性

e.g. $\frac{1}{z^3}\sin z\Big|_{z_0=0}=\sum_{n=-3}^{\infty}(-1)^n\frac{1}{(2n+1)!}z^{2n+1}$

提出 $\frac{1}{z^n}$

② 待定系数法 $\sin\frac{1}{z}=\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^n}{(2n+1)}z^{-2n-1}$

奇点的分类

孤立奇点

Removable

① 可去奇点：$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ 不含负幂项

e.g. $f(z)=\frac{\sin z}{z}\Big|_{z=0}$

② 极点 / Pole $f(z)=\sum_{n=m}^{\infty}a_n(z-z_0)^n$ 有限负幂项

e.g. $f(z)=\frac{1}{z^3}$，乘 z^m，可去

③ 本性奇点 $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ 无限负幂项

e.g. $f(z)=\sin(\frac{1}{z})$