```
In[•]:= Id = {{1, 0}, {0, 1}};
        sigX = {{0, 1}, {1, 0}};
        sigY = {{0, -I}, {I, 0}};
        sigZ = {{1, 0}, {0, -1}};
        II = KroneckerProduct[Id, Id];
        XX = KroneckerProduct[sigX, sigX];
        IZ = KroneckerProduct[Id, sigZ];
        IY = KroneckerProduct[Id, sigY];
        IX = KroneckerProduct[Id, sigX];
        XZ = KroneckerProduct[sigX, sigZ];
        YY = KroneckerProduct[sigY, sigY];
        ZZ = KroneckerProduct[sigZ, sigZ];
        Phi00 = 1 / 2 {{1, 0, 0, 1}, {0, 0, 0, 0}, {0, 0, 0, 0}, {1, 0, 0, 1}};
        Phi01 = 1 / 2 {{0, 0, 0, 0}, {0, 1, 1, 0}, {0, 1, 1, 0}, {0, 0, 0, 0}};
        Phi10 = 1 / 2 {{1, 0, 0, -1}, {0, 0, 0, 0}, {0, 0, 0, 0}, {-1, 0, 0, 1}};
        Phi11 = 1 / 2 {{0, 0, 0, 0}, {0, 1, -1, 0}, {0, -1, 1, 0}, {0, 0, 0, 0}};
        H = 1 / Sqrt[2] {{1, 1}, {1, -1}};
        Hy = 1 / Sqrt[2] {{1, 1}, {-I, I}};

In[•]:= Clear[x, t]

In[•]:= v1 = 1 / 2 {{Cos[x]}, {Cos[x]}, {Cos[x]}, {Cos[x]}} +
           1 / Sqrt[2] {{0}, {Sin[x]}, {-Sin[x]}, {0}};
        V1 = KroneckerProduct[v1, Transpose[v1]];
        v2 = -1 / 2 {{Sin[x]}, {Sin[x]}, {Sin[x]}, {Sin[x]}} +
           1 / Sqrt[2] {{0}, {Cos[x]}, {-Cos[x]}, {0}};
        V2 = KroneckerProduct[v2, Transpose[v2]];
        Simplify[Tr[V1]]
        Simplify[V2.V1]
        v = {{Cos[x]}, {Sin[x]}, {Sin[x]}, {Cos[x]}} / Sqrt[2];
        V = KroneckerProduct[v, Transpose[v]];

In[•]:= x = Pi / 4;

In[•]:= Xp = 1 / 2 {{1, 1}, {1, 1}};
        Xm = 1 / 2 {{1, -1}, {-1, 1}};
        Yp = 1 / 2 {{1, I}, {-I, 1}};
        Ym = 1 / 2 {{1, -I}, {I, 1}};

In[•]:= M1 = {{1, 0}, {0, Sqrt[1 - t]}};
        M2 = Sqrt[t] {{0, 1}, {0, 0}};
        N1 = {{Sqrt[1 - t], 0}, {0, 1}};
        N2 = Sqrt[t] {{0, 0}, {1, 0}};
        ChA[X_] := Transpose[N1].X.N1 + Transpose[N2].X.N2
```

```
In[•]:= Simplify[Eigenvalues[ChA[Xp]]]
```

$$\text{Out[•]}= \left\{\frac{1}{2}\left(1-\sqrt{1-t}\right),\ \frac{1}{2}\left(1+\sqrt{1-t}\right)\right\}$$

```
In[•]:= Simplify[ChAAD[II]]
```

$$\text{Out[•]}= \left\{\left\{\sqrt{1-t}\ \text{Conjugate}\left[\sqrt{1-t}\right]+\sqrt{t}\ \text{Conjugate}\left[\sqrt{t}\right],\ 0,\ 0,\ 0\right\},\ \{0,\ 1,\ 0,\ 0\},\right.$$
$$\left\{0,\ 0,\ 1-t+(-1+2t)\ \text{Conjugate}[t]+2\sqrt{-(-1+t)\ t}\ \text{Conjugate}\left[\sqrt{-(-1+t)\ t}\right],\ 0\right\},$$
$$\left.\left\{0,\ 0,\ 0,\ \sqrt{1-t}\ \text{Conjugate}\left[\sqrt{1-t}\right]+\sqrt{t}\ \text{Conjugate}\left[\sqrt{t}\right]\right\}\right\}$$

```
In[•]:= ChAAFD[X_] := Transpose[KroneckerProduct[M1, N1]].X.KroneckerProduct[M1, N1] +
          Transpose[KroneckerProduct[M1, N2]].X.KroneckerProduct[M1, N2] +
          Transpose[KroneckerProduct[M2, N1]].X.KroneckerProduct[M2, N1] +
          Transpose[KroneckerProduct[M2, N2]].X.KroneckerProduct[M2, N2]
```

```
In[•]:= Simplify[Eigenvalues[ChAAD[KroneckerProduct[Xp, Xm]]], t ≥ 0]
```

$$\text{Out[•]}= \left\{\frac{1}{4}\left(2-2\sqrt{1-t}-t\right),\ \frac{1}{4}\left(2+2\sqrt{1-t}-t\right),\ \frac{t}{4},\ \frac{t}{4}\right\}$$

```
In[•]:= Simplify[Eigenvalues[ChAAD[KroneckerProduct[Xp, Xm]]]]
```

$$\text{Out[•]}= \left\{\frac{1}{4}\left(2-2\sqrt{1-t}-t\right),\ \frac{1}{4}\left(2+2\sqrt{1-t}-t\right),\ \frac{t}{4},\ \frac{t}{4}\right\}$$

```
In[•]:= Simplify[Eigenvalues[ChAAD[KroneckerProduct[Xm, Xm]]]]
```

$$\text{Out[•]}= \left\{\frac{1}{4}\left(2-2\sqrt{1-t}-t\right),\ \frac{1}{4}\left(2+2\sqrt{1-t}-t\right),\ \frac{t}{4},\ \frac{t}{4}\right\}$$

```
In[•]:= Simplify[Eigenvalues[ChAAFD[Phi10]]]
```

$$\text{Out[•]}= \{1-t,\ 0,\ 0,\ -(-1+t)\ t\}$$

```
In[•]:= Simplify[Eigenvalues[ChAAFD[Phi11]], 1 >= t ≥ 0]
```

$$\text{Out[•]}= \left\{\frac{t}{2},\ \frac{t}{2},\ \frac{1}{2}\left(1-t+t^2+(-1+t)\ \sqrt{1+t^2}\right),\ \frac{1}{2}\left(1-t+t^2-(-1+t)\ \sqrt{1+t^2}\right)\right\}$$

```
In[•]:= Clear[t]
```

```
In[•]:= x = 0.1;
```

```
In[•]:= t = .9
```

$$\text{Out[•]}= 0.9$$

```
In[•]:= NMaximize[ 1/2 Root[
          -10 t^3 + 15 t^3 Cos[2 x] - 6 t^3 Cos[4 x] + t^3 Cos[6 x] + (64 t - 76 t^2 + 48 t^3 + 32 t Cos[2 x] -
                112 t^2 Cos[2 x] + 32 t^3 Cos[2 x] + 32 t Cos[4 x] - 68 t^2 Cos[4 x] + 48 t^3 Cos[4 x]) #1 +
            (-64 + 16 t - 16 t Cos[2 x] + 64 t^2 Cos[2 x]) #1^2 + 32 #1^3 &, 2], {x}]
```

$$\text{Out[•]}= \{0.45,\ \{x \to 1.5708\}\}$$

In[•]:=
```
Eigenvalues[ChAAFD[V]]
```

Out[•]=
$$\left\{\frac{1}{4}\left(t-t\cos[2x]\right),\ \frac{1}{2}\,\text{Root}\left[-10t^3+15t^3\cos[2x]-6t^3\cos[4x]+t^3\cos[6x]+\left(64t-76t^2+48t^3+32t\cos[2x]-112t^2\cos[2x]+32t^3\cos[2x]+32t\cos[4x]-68t^2\cos[4x]+48t^3\cos[4x]\right)\#1+\left(-64+16t-16t\cos[2x]+64t^2\cos[2x]\right)\#1^2+32\#1^3\ \&,\ 1\right],\ \frac{1}{2}\,\text{Root}\left[-10t^3+15t^3\cos[2x]-6t^3\cos[4x]+t^3\cos[6x]+\left(64t-76t^2+48t^3+32t\cos[2x]-112t^2\cos[2x]+32t^3\cos[2x]+32t\cos[4x]-68t^2\cos[4x]+48t^3\cos[4x]\right)\#1+\left(-64+16t-16t\cos[2x]+64t^2\cos[2x]\right)\#1^2+32\#1^3\ \&,\ 2\right],\ \frac{1}{2}\,\text{Root}\left[-10t^3+15t^3\cos[2x]-6t^3\cos[4x]+t^3\cos[6x]+\left(64t-76t^2+48t^3+32t\cos[2x]-112t^2\cos[2x]+32t^3\cos[2x]+32t\cos[4x]-68t^2\cos[4x]+48t^3\cos[4x]\right)\#1+\left(-64+16t-16t\cos[2x]+64t^2\cos[2x]\right)\#1^2+32\#1^3\ \&,\ 3\right]\right\}$$

In[•]:=
```
NMaximize[Eigenvalues[ChAAFD[V], {x, t}]]
```

••• Eigenvalues : Sequence specification (+n, −n, {+n}, {−n}, {m, n}, or {m, n, s }) expected at position 2 in ≪1≫.

••• NMaximize : NMaximize called with 1 argument; between 2 and 4 arguments are expected.

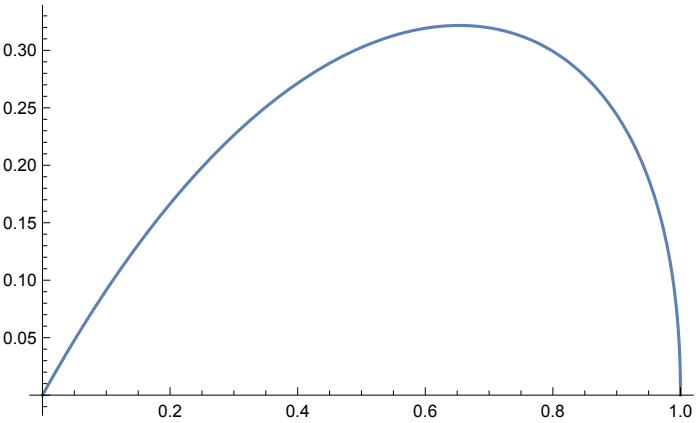
Out[•]=
$$\text{NMaximize}\Big[\text{Eigenvalues}\Big[\Big\{\Big\{\frac{1}{2}(1-t)\cos[x]^2+\frac{1}{2}t\sin[x]^2,\ \frac{1}{2}\sqrt{1-t}\cos[x]\sin[x],\ \frac{1}{2}(1-t)^{3/2}\cos[x]\sin[x]+\frac{1}{2}\sqrt{1-t}\,t\cos[x]\sin[x],\ \frac{1}{2}(1-t)\cos[x]^2\Big\},\ \Big\{\frac{1}{2}\sqrt{1-t}\cos[x]\sin[x],\ \frac{\sin[x]^2}{2},\ \frac{1}{2}(1-t)\sin[x]^2,\ \frac{1}{2}\sqrt{1-t}\cos[x]\sin[x]\Big\},\ \Big\{\frac{1}{2}(1-t)^{3/2}\cos[x]\sin[x]+\frac{1}{2}\sqrt{1-t}\,t\cos[x]\sin[x],\ \frac{1}{2}(1-t)\sin[x]^2,\ (1-t)\,t\cos[x]^2+\frac{1}{2}(1-t)^2\sin[x]^2+\frac{1}{2}t^2\sin[x]^2,\ \frac{1}{2}(1-t)^{3/2}\cos[x]\sin[x]+\frac{1}{2}\sqrt{1-t}\,t\cos[x]\sin[x]\Big\},\ \Big\{\frac{1}{2}(1-t)\cos[x]^2,\ \frac{1}{2}\sqrt{1-t}\cos[x]\sin[x],\ \frac{1}{2}(1-t)^{3/2}\cos[x]\sin[x]+\frac{1}{2}\sqrt{1-t}\,t\cos[x]\sin[x],\ \frac{1}{2}(1-t)\cos[x]^2+\frac{1}{2}t\sin[x]^2\Big\}\Big\},\ \{x,t\}\Big]\Big]$$

In[•]:=
```
Simplify[(-t-Sqrt[1+t^2])^2+1]
```

Out[•]=
$$2\left(1+t^2+t\sqrt{1+t^2}\right)$$

In[]:=
```
Plot[{1 / 2 (2 + 2 Sqrt[1 - t] - t) - 1 / 2 (1 - t + t^2 + Sqrt[(-1 + t)^2 (1 + t^2)]) - (1 - t)}, {t, 0, 1}]
```

Out[]=




```
Clear[x, t]
```

In[]:=
```
t = .95
```

Out[]= 0.95

In[]:=
```
x = 0;
Eigenvalues[ChAAFD[V1]]
```

Out[]= {0.05, 0.0475, 0., 0.}

In[]:=
```
x = 0;
Eigenvalues[ChAAFD[V2]]
```

Out[]= {0.510733, 0.475, 0.475, 0.441767}

In[]:=
```
x = Pi / 4;
Eigenvalues[ChAAFD[V1]]
```

Out[]= {0.374303, 0.2375, 0.2375, 0.150697}

In[]:=
```
x = Pi / 4;
Eigenvalues[ChAAFD[V2]]
```

Out[]= {0.374303, 0.2375, 0.2375, 0.150697}

```
Max[Eigenvalues[ChAAFD[V1]]] + Max[Eigenvalues[ChAAFD[V2]]]
```

In[]:=
```
x = Pi / 4;
t = .995;
Max[Eigenvalues[ChAAFD[V1]]] + Max[Eigenvalues[ChAAFD[V2]]]
```

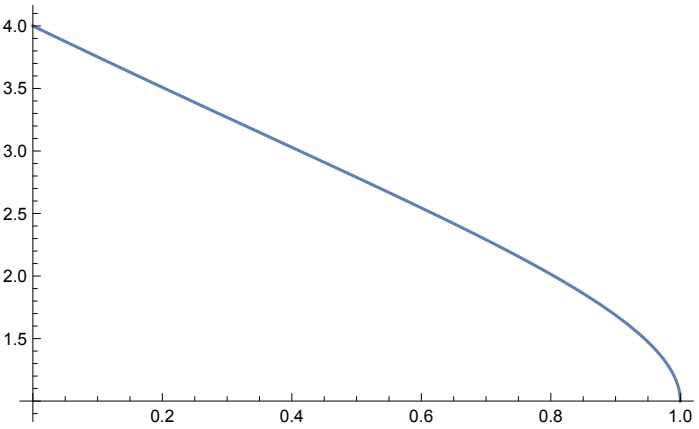
Out[]= 0.573211

In[]:=
```
x = Pi / 4 + .25;
t = .995;
Max[Eigenvalues[ChAAFD[V1]]] + Max[Eigenvalues[ChAAFD[V2]]]
```

Out[]= 0.564609

In[]:= `Plot[(1 + Sqrt[1 - t])^2 - 1/2 ((1 - t) t), {t, 0, 1}]`

Out[]=



In[]:= `Simplify[Tr[ChA[Xp].Xp]]`

Out[]= $\frac{1}{2}\left(1+\sqrt{1-t}\right)$

In[]:= `Simplify[Tr[ChA[Xp].Xp]]`

Out[]= $\frac{1}{2}\left(1+\sqrt{1-t}\right)$

In[]:= `Simplify[1 / 2 Tr[sigX.ChA[sigX]]]`

Out[]= $\sqrt{1-t}$

In[]:= `Simplify[1 / 2 Tr[sigX.ChA[sigZ]]]`

Out[]= 0

In[]:= `Simplify[1 / 2 Tr[sigX.ChA[sigZ]]]`

Out[]= $-\frac{t}{2}$

In[]:= `A = {{Sqrt[1 - t], 0, 0}, {0, Sqrt[1 - t], 0}, {-t / 2, 0, 1 - t / 2}};`

In[]:= `MatrixForm[A]`

Out[]//MatrixForm=

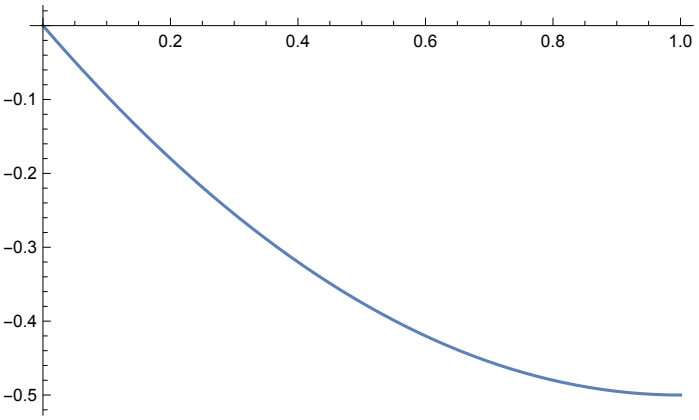
$$\begin{pmatrix} \sqrt{1-t} & 0 & 0 \\ 0 & \sqrt{1-t} & 0 \\ -\frac{t}{2} & 0 & 1-\frac{t}{2} \end{pmatrix}$$

In[]:= `Eigenvalues[A.Transpose[A]]`

Out[]= $\left\{\frac{2-t}{2}, 1-t, \frac{1}{2}\left(2-3t+t^2\right)\right\}$

In[•]:=
```
Plot[{1/2 (2 - 3 t + t^2) - (1 - t / 2)}, {t, 0, 1}]
```

Out[•]=



In[•]:=
```
sx = a * Sqrt[1 / 2];
sy = 0;
sz = a * Sqrt[1 / 2];
tx = a;
ty = 0;
tz = a;
```

In[•]:=
```
FullSimplify[Eigenvalues[II + sx * IX + sy * IY + sz * IZ + tx * XX + ty * YY + tz * ZZ], 0 ≤ a ≤ 1]
```

Out[•]= $\{1 + a - \sqrt{2}\ a,\ 1 + \left(-1 + \sqrt{2}\right) a,\ 1 - \left(1 + \sqrt{2}\right) a,\ 1 + a + \sqrt{2}\ a\}$

In[•]:=
```
Solve[1 + a - Sqrt[2] a == 0]
```

Out[•]= $\left\{\left\{a \to \frac{1}{-1 + \sqrt{2}}\right\}\right\}$

In[•]:=
```
Solve[1 - (1 + Sqrt[2]) a == 0]
```

Out[•]= $\left\{\left\{a \to \frac{1}{1 + \sqrt{2}}\right\}\right\}$

In[•]:=
```
N[1 / (1 + Sqrt[2])]
```

Out[•]= 0.414214

In[•]:=
```
sx = a / Sqrt[3];
sy = a / Sqrt[3];
sz = a / Sqrt[3];
tx = a / 2;
ty = a / 2;
tz = a / 2;
```

In[]:=
```
FullSimplify[Eigenvalues[II + sx * IX + sy * IY + sz * IZ + tx * XX + ty * YY + tz * ZZ], 0 ≤ a ≤ 1]
```

Out[]=
$$\left\{1-\frac{a}{2},\ 1+\frac{3\,a}{2},\ 1+\left(-\frac{1}{2}+\sqrt{2}\right)\,a,\ 1-\frac{1}{2}\left(1+2\,\sqrt{2}\right)\,a\right\}$$

In[]:=
```
Solve[1 - 1/2 (1 + 2 Sqrt[2]) a == 0]
```

Out[]=
$$\left\{\left\{a\to\frac{2}{1+2\,\sqrt{2}}\right\}\right\}$$

In[]:=
```
sx = a / Sqrt[3];
sy = a / Sqrt[3];
sz = a / Sqrt[3];
tx = a / 2;
ty = -a / 2;
tz = a / 2;
```

In[]:=
```
a = 2 / 3;
```

In[]:=
```
FullSimplify[Eigensystem[II + sx * IX + sy * IY + sz * IZ + tx * XX + ty * YY + tz * ZZ], 0 ≤ a ≤ 1]
```

Out[]=
$$\left\{\left\{\frac{2}{3}\left(2+\sqrt{2}\right),\ \frac{4}{3},\ -\frac{2}{3}\left(-2+\sqrt{2}\right),\ 0\right\},\right.$$
$$\left\{\left\{-1+\frac{6}{3+\sqrt{3}-\sqrt{6}},\ \frac{(1+i)\,\sqrt{3}}{3+\sqrt{21-6\,\sqrt{6}}},\ -\frac{(1-i)\,\sqrt{3}}{-3-3\,\sqrt{2}+\sqrt{3}},\ 1\right\},\right.$$
$$\left\{-1,\ \frac{(1+i)\left(-3+2\,\sqrt{3}\right)}{-3+\sqrt{3}},\ -\frac{(1-i)\,\sqrt{3}}{-3+\sqrt{3}},\ 1\right\},$$
$$\left\{-1+\frac{-3+\sqrt{3}+\sqrt{6}}{\sqrt{2}},\ -\frac{(1+i)\,\sqrt{3}}{-3+3\,\sqrt{2}+\sqrt{3}},\ -\frac{(1-i)\,\sqrt{3}}{-3+3\,\sqrt{2}+\sqrt{3}},\ 1\right\},$$
$$\left.\left.\left\{-1,\ \frac{(1+i)\left(3+2\,\sqrt{3}\right)}{3+\sqrt{3}},\ -\frac{(1-i)\,\sqrt{3}}{3+\sqrt{3}},\ 1\right\}\right\}\right\}$$

In[]:=
```
sx = Sqrt[(1 + 3 b) (1 - b)] / Sqrt[3];
sy = Sqrt[(1 + 3 b) (1 - b)] / Sqrt[3];
sz = Sqrt[(1 + 3 b) (1 - b)] / Sqrt[3];
tx = b;
ty = -b;
tz = b;
```

In[]:=
```
Clear[a, b]
```

In[]:=
```
FullSimplify[Eigenvalues[II + sx * IX + sy * IY + sz * IZ + tx * XX + ty * YY + tz * ZZ], 0 ≤ a ≤ 1]
```

Out[]=
$$\left\{0,\ 2\,(1+b),\ 1-b-\sqrt{1+(2-3\,b)\,b},\ 1-b+\sqrt{1+(2-3\,b)\,b}\right\}$$

In[]:=
```
sx = a;
sy = a;
sz = a;
tx = b;
ty = -b;
tz = b;
```

In[]:=
```
FullSimplify[Eigenvalues[II + sx * IX + sy * IY + sz * IZ + tx * XX + ty * YY + tz * ZZ], 0 ≤ a ≤ 1]
```

Out[]= $\left\{1-\sqrt{3}\ a-b,\ 1+\sqrt{3}\ a-b,\ 1+b-\sqrt{3\,a^2+4\,b^2},\ 1+b+\sqrt{3\,a^2+4\,b^2}\right\}$

In[]:=
```
Solve[1 - b - Sqrt[1 + (2 - 3 b) b] == 0]
```

Out[]= $\{\{b\to 0\},\{b\to 1\}\}$

In[]:=
```
Solve[1/2 (2 + a - 2 Sqrt[2] a) == 0]
```

Out[]= $\left\{\left\{a\to \frac{2}{-1+2\sqrt{2}}\right\}\right\}$

In[]:=
```
N[2/(-1 + 2 Sqrt[2])]
```

Out[]= 1.09384

In[]:=
```
N[2/(1 + 2 Sqrt[2])]
```

Out[]= 0.522408

In[]:=
```
Solve[1 - b - Sqrt[a^2 + 4 b^2] == 0]
```

Out[]= $\left\{\left\{a\to -\sqrt{1-2\,b-3\,b^2}\right\},\left\{a\to \sqrt{1-2\,b-3\,b^2}\right\}\right\}$

In[]:=
```
N[1 / (1 + Sqrt[2])]
```

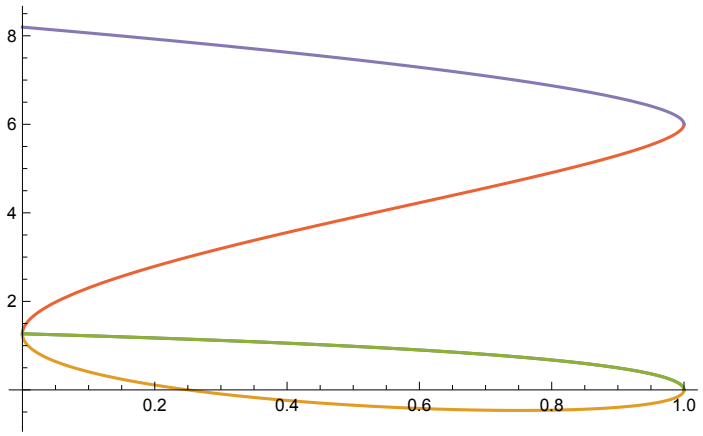
Out[]= 0.414214

In[]:=
```
N[2 / Sqrt[5]]
```

Out[]= 0.894427

In[ ]:=
```
Plot[{3 + Sqrt[3 - 3 a] - Sqrt[3] Sqrt[4 - a], 3 - Sqrt[3 - 3 a] - 3 Sqrt[a], (3 + Sqrt[3 - 3 a] - Sqrt[3] Sqrt[4 - a]),
  (3 - Sqrt[3 - 3 a] + 3 Sqrt[a]), (3 + Sqrt[3 - 3 a] + Sqrt[3] Sqrt[4 - a])}, {a, 0, 1}]
```

Out[ ]=



In[ ]:=
```
Solve[3 - Sqrt[3 - 3 a] - 3 Sqrt[a] == 0]
```

Out[ ]= $\left\{\left\{a \to \frac{1}{4}\right\}, \{a \to 1\}\right\}$