

1. (2 points) Consider the following problem:

INPUT: A set  $S = \{(x_i, y_i) | 1 \leq i \leq n\}$  of intervals over the real line.

OUTPUT: A maximum cardinality subset  $S'$  of  $S$  such that no pair of intervals in  $S'$  overlap.

Consider the following algorithm:

Repeat until  $S$  is empty

1. Select the interval  $I$  that overlaps the least number of other intervals.
2. Add  $I$  to final solution set  $S'$ .
3. Remove all intervals from  $S$  that overlap with  $I$ .

Prove or disprove that this algorithm solves the problem.

2. (2 points) Consider the following Interval Coloring Problem.

INPUT: A set  $S = \{(x_i, y_i) | 1 \leq i \leq n\}$  of intervals over the real line. Think of interval  $(x_i, y_i)$  as being a request for a room for a class that meets from time  $x_i$  to time  $y_i$ .

OUTPUT: Find an assignment of classes to rooms that uses the fewest number of rooms.

Note that every room request must be honored and that no two classes can use a room at the same time.

- (a) Consider the following iterative algorithm. Assign as many classes as possible to the first room (we can do this using the greedy algorithm discussed in class, and in the class notes), then assign as many classes as possible to the second room, then assign as many classes as possible to the third room, etc. Does this algorithm solve the Interval Coloring Problem? Justify your answer.
- (b) Consider the following algorithm. Process the classes in increasing order of start times. Assume that you are processing class  $C$ . If there is a room  $R$  such that  $R$  has been assigned to an earlier class, and  $C$  can be assigned to  $R$  without overlapping previously assigned classes, then assign  $C$  to  $R$ . Otherwise, put  $C$  in a new room. Does this algorithm solve the Interval Coloring Problem? Justify your answer.

HINT: Let  $s$  be the maximum number of intervals that overlap at one particular point in time. Obviously, you need at least  $s$  rooms. Therefore any algorithm that uses only  $s$  rooms is obviously optimal. This lower bound on the number of rooms required allows you to prove optimality without using an exchange argument.

3. (2 points) We consider two change making problems:

- (a) Consider the Change Problem in pre-Euro Austria. The input to this problem is an integer  $L$ . The output should be the minimum cardinality collection of coins required to make  $L$  shillings of change (that is, you want to use as few coins as possible). In Austria the coins are worth 1, 5, 10, 20, 25, 50 Shillings. Assume that you have an unlimited number of coins of each type. Formally prove or disprove that the greedy algorithm (that takes as many coins as possible from the highest denominations) correctly solves the Change Problem. So for example, to make change for 234 Shillings the greedy algorithms would take four 50 shilling coins, one 25 shilling coin, one 5 shilling coin, and four 1 shilling coins.

- (b) Consider the Change Problem in Binaryland. The input to this problem is an integer  $L$ . The output should be the minimum cardinality collection of coins required to make  $L$  nibbles of change (that is, you want to use as few coins as possible). In Binaryland the coins are worth  $1, 2, 2^2, 2^3, \dots, 2^{1000}$  nibbles. Assume that you have an unlimited number of coins of each type. Prove or disprove that the greedy algorithm (that takes as many coins of the highest value as possible) solves the change problem in Binaryland.

HINT: The greedy algorithm is correct for one of the above two subproblems and is incorrect for the other. For the problem where greedy is correct, use the following proof strategy: Assume to reach a contradiction that there is an input  $I$  on which greedy is not correct. Let  $OPT(I)$  be a solution for input  $I$  that is better than the greedy output  $G(I)$ . Show that the existence of such an optimal solution  $OPT(I)$  that is different than greedy is a contradiction. So what you can conclude from this is that for every input, the output of the greedy algorithm is the unique optimal/correct solution.

4. (2 points) You wish to drive from point  $A$  to point  $B$  along a highway minimizing the time that you are stopped for gas. You are told beforehand the capacity  $C$  of your gas tank in liters, your rate  $F$  of fuel consumption in liters/kilometer, the rate  $r$  in liters/minute at which you can fill your tank at a gas station, and the locations  $A = x_1, \dots, B = x_n$  of the gas stations along the highway. So if you stop to fill your tank from 2 liters to 8 liters, you would have to stop for  $6/r$  minutes. Consider the following two algorithms:
- (a) Stop at every gas station, and fill the tank with just enough gas to make it to the next gas station.
  - (b) Stop if and only if you don't have enough gas to make it to the next gas station, and if you stop, fill the tank up all the way.

For each algorithm either prove or disprove that this algorithm correctly solves the problem. Your proof of correctness must use an exchange argument.

5. (2 points) Consider the following problem. The input is a collection  $A = \{a_1, \dots, a_n\}$  of  $n$  points on the real line. The problem is to find a minimum cardinality collection  $S$  of unit intervals that cover every point in  $A$ . Another way to think about this same problem is the following. You know a collection of times ( $A$ ) that trains will arrive at a station. When a train arrives there must be someone manning the station. Due to union rules, each employee can work at most one hour at the station. The problem is to find a scheduling of employees that covers all the times in  $A$  and uses the fewest number of employees.
- (a) Prove or disprove that the following algorithm correctly solves this problem. Let  $I$  be the interval that covers the most number of points in  $A$ . Add  $I$  to the solution set  $S$ . Then recursively continue on the points in  $A$  not covered by  $I$ .
  - (b) Prove or disprove that the following algorithm correctly solves this problem. Let  $a_j$  be the smallest (leftmost) point in  $A$ . Add the interval  $I = (a_j, a_j + 1)$  to the solution set  $S$ . Then recursively continue on the points in  $A$  not covered by  $I$ .

HINT: One of the above greedy algorithms is correct and one is incorrect for the other. The proof of correctness must use an exchange argument.

6. (2 points) We consider a greedy algorithm for two related problems
- (a) The input to this problem consists of an ordered list of  $n$  words. The length of the  $i$ th word is  $w_i$ , that is the  $i$ th word takes up  $w_i$  spaces. (For simplicity assume that there are no spaces between words.) The goal is to break this ordered list of words into lines, this is called a layout. Note that you can not reorder the words. The length of a line is the sum of the lengths of the words on that line. The ideal line length is  $L$ . No line may be longer than  $L$ , although it may be shorter. The penalty for having a line of length  $K$  is  $L - K$ . *The total penalty is the sum of the line penalties.* The problem is to find a layout that minimizes the total penalty.

Prove or disprove that the following greedy algorithm correctly solves this problem.

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For i= 1 to n
    Place the ith word on the current line if it fits
    else place the ith word on a new line

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- (b) The input to this problem consists of an ordered list of  $n$  words. The length of the  $i$ th word is  $w_i$ , that is the  $i$ th word takes up  $w_i$  spaces. (For simplicity assume that there are no spaces between words.) The goal is to break this ordered list of words into lines, this is called a layout. Note that you can not reorder the words. The length of a line is the sum of the lengths of the words on that line. The ideal line length is  $L$ . No line may be longer than  $L$ , although it may be shorter. The penalty for having a line of length  $K$  is  $L - K$ . *The total penalty is the **maximum** of the line penalties.* The problem is to find a layout that minimizes the total penalty.

Prove or disprove that the following greedy algorithm correctly solves this problem.

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For i= 1 to n
    Place the ith word on the current line if it fits
    else place the ith word on a new line

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HINT: The greedy algorithm is correct for one of the above two problems and is incorrect for the other. The proof of correctness must be done using an exchange argument.

7. (4 points) The setting for this problem is storage system with a fast memory consisting of  $k$  pages and a slow memory consisting of  $n$  pages. At any time, the fast memory can hold copies of up to  $k$  of the pages in slow memory. The input consists of a sequence of pages from slow memory, think of these as being accesses to memory. If an accessed page is not in fast memory, then it must be swapped into fast memory, and if the fast memory was full, some page must be selected to be evicted from fast memory. The goal is to determine the pages to evict so as to minimize the total number of evictions. Consider for example that  $k = 2$ ,  $n = 4$  pages are named A, B, C and D, and the access sequence is A B C A. Then after the first two pages, the fast memory contains A and B. When C is accessed then either A or B must be evicted. If B is evicted then no further evictions are necessary, and the total number of evictions is 1. If A was evicted, then either B or C must be evicted when A is accessed again, and the total number of evictions would be 2. Give a greedy for this problem and prove that it is correct using an exchange argument.
8. (4 points) Consider the following problem. The input consists of the lengths  $\ell_1, \dots, \ell_n$ , and access probabilities  $p_1, \dots, p_n$ , for  $n$  files  $F_1, \dots, F_n$ . The problem is to order these files on a tape so as to minimize the expected access time. If the files are placed in the order  $F_{s(1)}, \dots, F_{s(n)}$  then the expected access time is

$$\sum_{i=1}^n p_{s(i)} \sum_{j=1}^i \ell_{s(j)}$$

Don't let this formula throw you. The term  $p_{s(i)} \sum_{j=1}^i \ell_{s(j)}$  is the probability that you access the  $i$ th file times the length of the first  $i$  files.

For each of the below algorithms, either give a proof that the algorithm is correct using an exchange argument, or a proof that the algorithm is incorrect.

- Order the files from shortest to longest on the tape. That is,  $\ell_i < \ell_j$  implies that  $s(i) < s(j)$ .
  - Order the files from most likely to be accessed to least likely to be accessed. That is,  $p_i < p_j$  implies that  $s(i) > s(j)$ .
  - Order the files from smallest ratio of length over access probability to largest ratio of length over access probability. That is,  $\frac{\ell_i}{p_i} < \frac{\ell_j}{p_j}$  implies that  $s(i) < s(j)$ .
9. (4 points) Consider the following problem. The input consists of  $n$  skiers with heights  $p_1, \dots, p_n$ , and  $n$  skis with heights  $s_1, \dots, s_n$ . The problem is to assign each skier a ski to minimize the average

difference between the height of a skier and his/her assigned ski. That is, if the  $i$ th skier is given the  $\alpha(i)$ th ski, then you want to minimize:

$$\frac{1}{n} \sum_{i=1}^n |p_i - s_{\alpha(i)}|$$

- (a) Consider the following greedy algorithm. Find the skier and ski whose height difference is minimized. Assign this skier this ski. Repeat the process until every skier has a ski. Prove or disprove that this algorithm is correct.
- (b) Consider the following greedy algorithm. Give the shortest skier the shortest ski, give the second shortest skier the second shortest ski, give the third shortest skier the third shortest ski, etc. Prove or disprove that this algorithm is correct.

HINT: One of the above greedy algorithms is correct and one is incorrect for the other. The proof of correctness must be done using an exchange argument.

10. (4 points) We consider the following scheduling problem:

INPUT: A collection of jobs  $J_1, \dots, J_n$ , where the  $i$ th job is a tuple  $(r_i, x_i)$  of non-negative integers specifying the release time and size of the job.

OUTPUT: A preemptive feasible schedule for these jobs on one processor that minimizes the total completion time  $\sum_{i=1}^n C_i$ .

A *schedule* specifies for each unit time interval, the unique job that is run during that time interval. In a *feasible* schedule, every job  $J_i$  has to be run for exactly  $x_i$  time units after time  $r_i$ . The *completion time*  $C_i$  for job  $J_i$  is the earliest time when  $J_i$  has been run for  $x_i$  time units. Examples of these basic definitions can be found below.

We consider two greedy algorithms for solving this problem that schedule times in an online fashion, that is the algorithms are of the following form:

$t = 0$

while there are jobs left not completely scheduled

Among those jobs  $J_i$  such that  $r_i \leq t$ , and that have previously been scheduled for less than  $x_i$  time units, pick a job  $J_m$  to schedule at time  $t$  according to some rule;  
increment  $t$

One can get different greedy algorithms depending on the rule for selecting  $J_m$ . For each of the following greedy algorithms, prove or disprove that the algorithm is correct. Proofs of correctness must use an exchange argument. Hint: The most obvious exchange argument does not work. If you think that the first thing that you tried worked, you might want to reevaluate.

**SJF:** Pick  $J_m$  to be the job with minimal size  $x_i$ . Ties may be broken arbitrarily.

**SRPT:** Let  $y_{i,t}$  be the total time that job  $J_i$  has been run before time  $t$ . Pick  $J_m$  to be a job that has minimal remaining processing time, that is, that has minimal  $x_i - y_{i,t}$ . Ties may be broken arbitrarily.

As an example of SJF and SRPT consider the following instance:  $J_1 = (0, 100)$ ,  $J_2 = (10, 10)$  and  $J_3 = (1, 4)$ . Both SJF and SRPT schedule job  $J_1$  between time 0 and time 1, and job  $J_3$  between time 1 and time 5, when job  $J_3$  completes, and job  $J_1$  again between time 5 and time 10. At time 10, SJF schedules job  $J_2$  because its original size 10 is less than job  $J_1$ 's original size 100. At time 10, SRPT schedules job  $J_2$  because its remaining processing time 10 is less than job  $J_1$ 's remaining processing time 94. Both SJF and SRPT schedule job  $J_2$  between time 10 and 20, when  $J_2$  completes, and then job  $J_1$  from time 20 until time 114, which job  $J_1$  completes. Thus for both SJF and SRPT on this instance  $C_1 = 114$ ,  $C_2 = 20$  and  $C_3 = 5$  and thus both SJF and SRPT have total completion time 139.

11. (4 points) Consider the following problem.

INPUT: Positive integers  $r_1, \dots, r_n$  and  $c_1, \dots, c_n$ .

OUTPUT: An  $n$  by  $n$  matrix  $A$  with 0/1 entries such that for all  $i$  the sum of the  $i$ th row in  $A$  is  $r_i$  and the sum of the  $i$ th column in  $A$  is  $c_i$ , if such a matrix exists.

Think of the problem this way. You want to put pawns on an  $n$  by  $n$  chessboard so that the  $i$ th row has  $r_i$  pawns and the  $i$ th column has  $c_i$  pawns.

Consider the following greedy algorithm that constructs  $A$  row by row. Assume that the first  $i - 1$  rows have been constructed. Let  $a_j$  be the number of 1's in the  $j$ th column in the first  $i - 1$  rows. Now the  $r_i$  columns with maximum  $c_j - a_j$  are assigned 1's in row  $i$ , and the rest of the columns are assigned 0's. That is, the columns that still needs the most 1's are given 1's. Formally prove that this algorithm is correct using an exchange argument.

12. (8 points) You have  $n$  heterosexual men and  $n$  heterosexual women. Each man ranks the women in order of preference. Each woman ranks the men in order of preference. Consider the following incredibly stereotypical courting algorithm. On stage  $i$ , each man goes to pitch woo on the porch of the woman that he prefers most among all women that have not rejected him yet. At the end of the stage the woman rejects all the men on her porch but the one that she favors most. Note that a women may not reject a man in some stage, but later end up rejecting that man if a better prospect arrives on her porch. If it should ever happen that there is exactly one man on each porch, the algorithm terminates, and each woman marries the man on her porch. (You may be interested to know that medical schools really use this algorithm to fill intern positions.)
- Give an upper bound as a function of  $n$  of the number of stages in this algorithm.
  - A marriage assignment is stable if there does not exist a man  $x$  and a woman  $y$  such that  $x$  prefers  $y$  to his assigned mate, and  $y$  prefers  $x$  to her assigned mate. Clearly adultery is a risk if a marriage assignment is not stable. Prove that this algorithm leads to a stable marriage.
  - A stable marriage  $M$  is man optimal if for every man  $x$ ,  $M$  is the best possible stable marriage. That is, in every stable marriage other than  $M$ ,  $x$  ends up with a woman no more preferable to him than the woman he is married to in  $M$ . Prove or disprove the above algorithm produces a man optimal stable marriage.
  - A stable marriage  $M$  is woman optimal if for every woman  $y$ ,  $M$  is the best possible stable marriage. That is, in every stable marriage other than  $M$ ,  $y$  ends up with a man no more preferable to her than the man she is married to in  $M$ . Prove or disprove the above algorithm produces a woman optimal stable marriage.
  - A stable marriage  $M$  is man pessimal if for every man  $x$ ,  $M$  is the worst possible stable marriage. That is, in every stable marriage other than  $M$ ,  $x$  ends up with a woman no less preferable to her than the woman he is married to in  $M$ . Prove or disprove the above algorithm produces a man pessimal stable marriage.
  - A stable marriage  $M$  is woman pessimal if for every woman  $y$ ,  $M$  is the worst possible stable marriage. That is, in every stable marriage other than  $M$ ,  $y$  ends up with a man no less preferable to her list than the man she is married to in  $M$ . Prove or disprove the above algorithm produces a woman pessimal stable marriage.
13. (8 points) The input to this problem is a tree. Some of the vertices of the tree are designated as infected. Other of the vertices are designated as cities (cities are not infected). And some of the vertices are neither infected or are cities. Each city  $c$  has a population  $P_c$ . Each edge  $e$  has a time  $t_e$  required to cut the edge. The output is an ordering  $e_1, e_2, \dots, e_k$  of some subset  $C$  of the edges. The edges  $C$  must have the property that their removal means that no city is reachable from an infected vertex in the resulting forest. The objective is to minimize the average time that a person has to wait until infection can no longer reach them; so this is  $\sum_c P_c \sum_{k=1}^{w(c)} t_{e_k}$ , where  $w(c)$  is the smallest number such that the removal of edges  $e_1, \dots, e_{w(c)}$  disconnects city  $c$  from every infected vertex. Consider the following greedy algorithm: The next edge  $e$  cut should always be the one that minimizes the ratio of  $t_e$  over the number of people that would be disconnected by this cut.

- (a) Show that the above greedy algorithm is not correct even if there is only 1 infected vertex.
- (b) Show that the above greedy algorithm (for an arbitrary number of infected vertices) is not correct even if each edge  $e$  takes unit time to cut, that is if  $t_e = 1$ .
- (c) Show that the above greedy algorithm is correct if there is only 1 infected vertex and if each edge  $e$  takes unit time to cut, that is if  $t_e = 1$ .
- (d) Give a different algorithm (for an arbitrary number of infected vertices) that is correct if each edge  $e$  takes unit time to cut, that is if  $t_e = 1$ . There is an algorithm that might fairly be called greedy. Prove the correctness of your algorithm.

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14. (8 points) The setting for this problem is a line network model by a line graph. There are  $n$  nodes in this graph, and each node is connected to a left neighbor and a right neighbor (except the leftmost node has no left neighbor and the rightmost node has no right neighbor). The input consists of  $k$  packets where packet  $p$  consists of an integer release time  $r_p$  when the packet  $p$  arrives in the system, a source node  $s_p$  at which the packet  $p$  arrives, and a destination node  $t_p$  (which must be to the right of  $s_p$ ) to which the packet  $p$  must reach. Between consecutive integer times, one packet may be forwarded between each pair of adjacent routers. Once a packet  $p$  reaches its destination  $t_p$ , it leaves the system. We want all packets to reach their destinations. For example if the input consisted of the following triples of the form  $(name, r_p, s_p, t_p)$ : (A, 0, 1, 3) (B, 0, 2, 4) (C 1, 2, 3), then one way to have each packet reach its destination is:

- between time 0 and 1 pack A is forwarded from node 1 to node 2, and packet B is forwarded from node 2 to node 3
- between time 1 and time 2 packet A is forwarded from node 2 to node 3 (leaving the system at time 3) and packet B is forwarded from node 3 to node 4 (leaving the system at time 3). We could have forward packet C instead of packet A, but it is not possible to forward both at this time
- between time 2 and time 3, packet C is forward from node 2 to node 3 (leaving the system at time 4)

- (a) Assume that the objective is to minimize the maximum time that a packet leaves the network. That is, we want to clear the network of packets as quickly as possible. Give a greedy algorithm for this problem and prove that it is correct.
  - (b) Assume that  $n = 2$ , so that the network consists of two nodes and one edge. Further assume that the objective is to minimize the maximum waiting time for any packet. The waiting time for a packet is difference between the time that the packet reaches its destination and the release time for that packet. That is, we want the packet that waits the longest to wait as little time as possible. Give a greedy algorithm for this problem and prove that it is correct.
  - (c) Assume all the release times are zero. Further assume that the objective is to minimize the sum of the waiting times of the packets. This is equivalent to minimizing the average waiting times of the packets. Give a greedy algorithm for this problem and prove that it is correct.
15. (8 points) Consider the following bridge crossing problem where  $n$  people with speeds  $s_1, \dots, s_n$  wish to cross the bridge as quickly as possible. The rules remain:
- It is nighttime and you only have one flashlight.
  - A maximum of two people can cross at any one time
  - Any party who crosses, either 1 or 2 people must have the flashlight with them.
  - The flashlight must be walked back and forth, it cannot be thrown, etc.
  - A pair must walk together at the rate of the slower person's pace.

Give an efficient algorithm to find the fastest way to get a group of people across the bridge. You **must** have a proof of correctness for your method.

16. Divorce