CS 1510: Algorithm Design
Homework 3 - Greedy Problems 7
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September 3, 2013

Problem 7

Our greedy algorithm goes through each page j in the fast memory and locates its first reapparance a_j in the slow memory , then evicts the page which reappears the furthest away in the slow memory.
We will prove that the algorithm G solves the problem.
Suppose towards a contradiction that there is an input I on which the algorithm G produces unacceptable output. Let $Opt(I)$ be an optimal output which agrees with $G(I)$ for the maximum number of steps of all such optimal outputs.
Since G produces unacceptable output and Opt produces the correct solution, they must disagree on at least one interval. Let k be the first such interval where they disagree, so that $G_k \neq O_k$.
Construct $Opt'(I) = Opt(I) - O_k + G_k$, by swapping the positions of G_k and O_k in Opt. Then clearly $Opt'(I)$ agrees with G for one additional term than Opt. Additionally, since G evicts the page from the fast memory which reappears furthest in time in the slow memory, then the next appearance of the page G_k accesses must reappear after Opt_k. Because of this, in Opt' the page O_k represents would need to be reloaded before the page that G_k represents, thus replacing O_k with G_k in Opt' will not force an additional eviction of the page that G_k represents.
Also note that since we're only adjusting in memory (fast and slow) the pages represented by O_k and G_k, the remaining pages will be unaffected by this swap from Opt to Opt'. Since no aditional evictions are necessary, Opt' will have at most the same number of evictions as Opt. So if Opt was optimal the Opt' is optimal as well.
Thus we have shown that there exists $Opt'(I)$ which is an optimal output which agrees with $G(I)$ for one step further than $Opt(I)$, and so we have a contradiction with the fact that $Opt(I)$ agrees with $G(I)$ for the maximum number of steps. So our original assumption that there was an input on which G produced unacceptable output was incorrect, and we are done.