CS 1510: Algorithm Design
Homework 2 - Greedy Problems 1 & 5
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Problem 1

We will prove that the algorithm G solves the problem.
Suppose towards a contradiction that there is an input I on which the algorithm G produces unacceptable output. Let $Opt(I)$ be an optimal output which agrees with $G(I)$ for the maximum number of steps of all such optimal outputs.
Since G produces unacceptable output and Opt produces the correct solution, they must disagree on at least one interval. Let k be the first such interval where they disagree, so that $G_k \neq O_k$.
Construct $Opt'(I) = Opt(I) - O_k + G_k$. Then clearly $Opt'(I)$ agrees with G for one additional term than Opt. Additionally, note that by definition of G, we know G_k overlaps fewer other intervals than O_k. So then if Opt was optimal the Opt' is optimal as well.
Thus we have shown that there exists $Opt'(I)$ which is an optimal output which agrees with $G(I)$ for one step further than $Opt(I)$, and so we have a contradiction with the fact that $Opt(I)$ agrees with $G(I)$ for the maximum number of steps. So our original assumption that there was an input on which G produced unacceptable output was incorrect, and we are done.

Problem 5a

We will disprove that the algorithm correctly solves the problem.
Consider the input
$I = \{.1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0, 1.1$
1.3
$1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5\}$.
The algorithm selects its next interval based on which will cover the most number of points, so there are 2 identical choices - the interval from $[.1, 1.1]$ and from $[1.5, 2.5]$, both of which contain 11 points. The algorithm will select one of these, then the other (it does not matter the order for our purposes), then will be left trying to select an interval that covers the remaining point

(1.3) without overlapping any of the previous intervals. The algorithm will fail, however, because it must select a unit interval and only has the space from $[1.1, 1.5]$ to work with.
Thus with the input I, the algorithm will fail to produce a minimum cardinality set that covers all points with no overlap, and so the algorithm does not correctly solve the problem.

Problem 5b

We will prove the algorithm correctly solves the problem.
Suppose towards a contradiction that there is an input I on which the algorithm G produces unacceptable output. Let $Opt(I)$ be an optimal output which agrees with $G(I)$ for the maximum number of steps of all such optimal outputs.
Since G produces unacceptable output and Opt produces the correct solution, they must disagree on at least one interval. Let k be the first such interval where they disagree, so that $G_k \neq O_k$.
Next we will show that O_k starts before G_k. We know this because G_k starts at the rightmost position it can to still be able not to miss the point a_k. Thus if O_k started after G_k it would miss the point and not be optimal, and if it started at the same position as G_k then it would not be the case that $G_k \neq O_k$.
Thus we know that $Opt'(I) = Opt(I) - O_k + G_k$ is optimal if $Opt(I)$ is optimal, since Opt' will cover at least as many points as Opt, and since G_k ends after O_k, it is possible it covers even more and so could potentially require fewer later intervals.
Thus we have shown that there exists $Opt'(I)$ which is an optimal output which agrees with $G(I)$ for one step further than $Opt(I)$, and so we have a contradiction with the fact that $Opt(I)$ agrees with $G(I)$ for the maximum number of steps. So our original assumption that there was an input on which G produced unacceptable output was incorrect, and we are done.