Transient Diffusion

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Introduction

Derivation

The transient conduction equation (1) was modeling using the Finite Volume Method.

$$\nabla \cdot k\nabla T + q = \rho C_P \dot{T} \tag{1}$$

In addition to integrating over the control volume, the timestep is also integrated over.

$$\int_{\Delta t}\int_V (\nabla \cdot k\nabla T + q - \rho C_P \dot{T}) dV dt \tag{2}$$

If T_P and q_P are presumed to predominate the control volume and ρC_P is constant, by the divergence theorem equation (3) is obtained.

$$\sum_{\partial V} k\nabla T \cdot \hat{n}_i + q\Delta x - \rho C_P \dot{T}\Delta x \tag{3}$$

$$\left(\sum_{\partial V} k\nabla T \cdot \hat{n}_i + q\Delta x\right)^{k+m} - \rho C_P (T_{k+1} - T_k)\frac{\Delta x}{\Delta t} \tag{4}$$

Depending on the selection of m the result is Implicit Euler (1), Explicit Euler (0), or Crank-Nicolson ($\frac{1}{2}$).

Transient Solver Performance

Implicit Euler being unconditionally stable performed extremely well although there was an associated computational cost with solving a linear system.

Explicit Euler can be put into the form of equation (5).

$$\{T\}^{k+1} = ([A]\{T\}^k + \{b\})\frac{\Delta t}{\rho C_P \Delta x} \tag{5}$$

This allows for extremely quick timestepping since only matrix multiplication needs to be carried out. However, the stable timestep was extremely low.

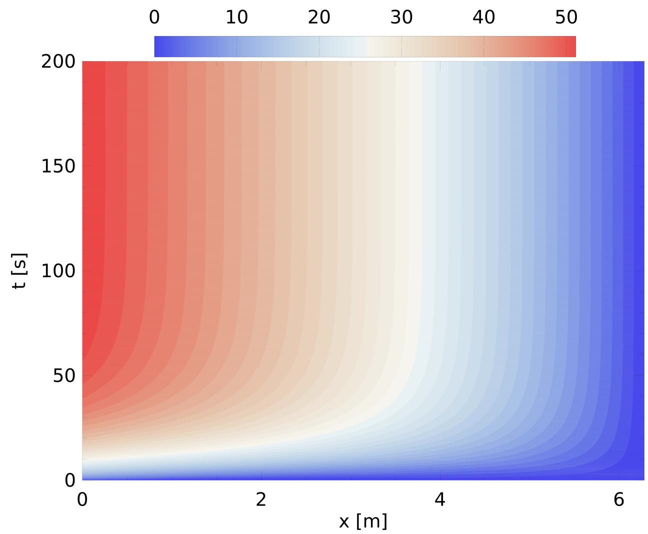

Figure 1: Constant heat flux solution

Since Crank Nicolson is second order, as the time step decreases, Crank Nicolson becomes much more accurate. Notably, the instability of Explicit Euler means that the stable timestep is somehwat low

Manufactured Solutions

The Method of Manufactured Solutions can be applied. With $T = \sin x \cos t$ the source term in equation (6) is obtained.

$$S = (k\cos t - \sin t)\sin x - q \tag{6}$$

Error can be quantified by equation (7)

$$\epsilon \propto \sqrt{\frac{1}{N}\sum_{i,j}^{N}(\hat{T}(x_i, t_j) - \tilde{T}_{i,j})^2} \tag{7}$$

$$\epsilon(h, \Delta t) = O(h^2, \Delta t^n) \tag{8}$$

Error is some function of both grid spacing h and timestep Δt. Since the special discretization is second order, the error should go down with the square of h if $\Delta t = 0$.

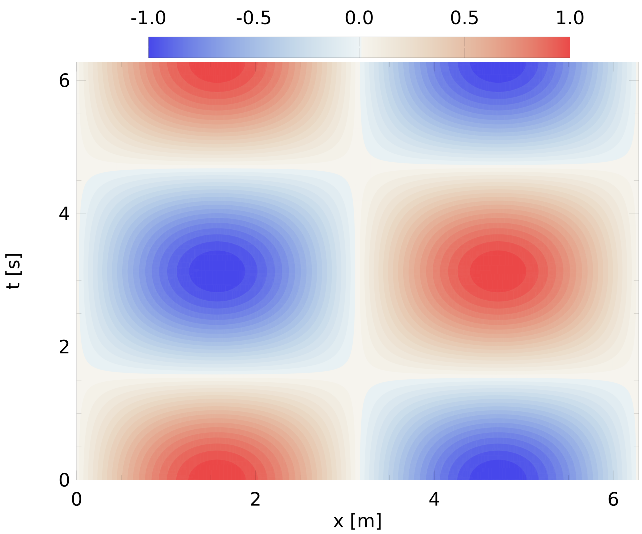

Figure 2: Manufactured solution

If Implicit or Explicit Euler is used, then $n = 1$ and the error should be proportional to Δt if $h = 0$.

Notably, error seems to be converging to a value above zero. This could be due to spacial error since it is non-zero.

Improvement

Since error is cummulative in the time domain and dependant on the length of time, a better characterization of error is needed. One option could be to normalize the error by the simulation time.

$$\epsilon = \frac{O(h^2, \Delta t)}{t} \tag{9}$$

This would give an error that is less dependant on the simulation time. Which may make error somewhat less dependant on time.

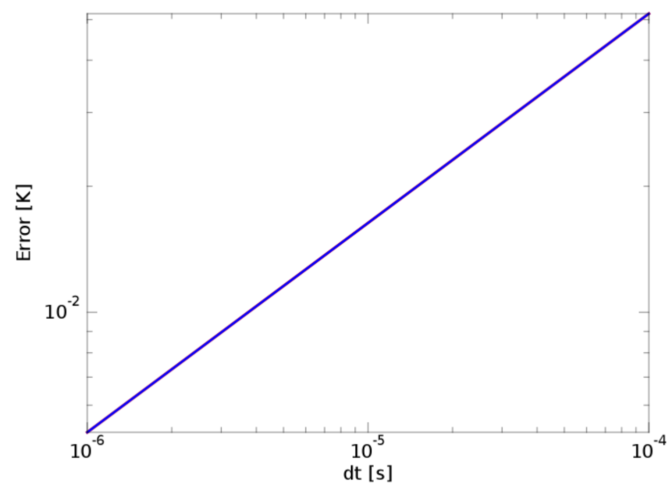

Figure 3: Crank-Nicolson and Euler Implicit Global Error

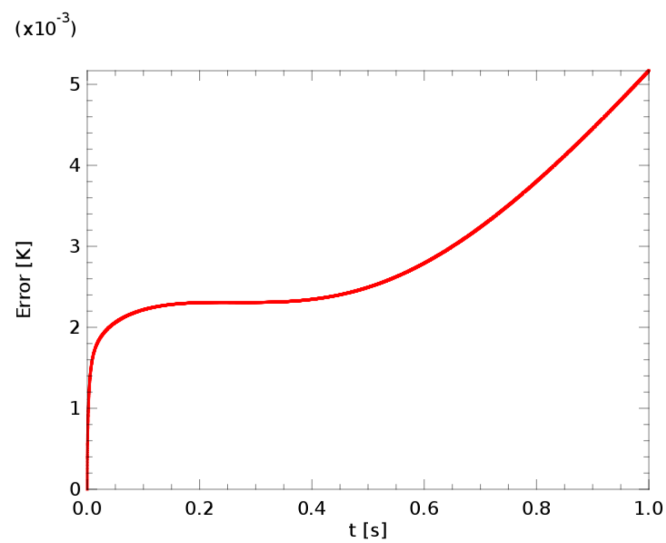

Figure 4: Cumulative Error