Investigation 5: Lorenz Equations

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Initial Value Problem

The Lorenz Equations (1) are an initial value problem that are chaotic in nature. They are useful as a model equation because there is no special domain to be descretized; allowing the primary focus to be on the unsteady equation solver.

$$\dot{\vec{x}} = \begin{bmatrix} \sigma(\vec{x}_2 - \vec{x}_1) \\ \vec{x}_1(\rho - \vec{x}_3) - \vec{x}_2 \\ \vec{x}_1\vec{x}_2 - \beta\vec{x}_3 \end{bmatrix} \tag{1}$$

Along with the statement of the derivative, an initial value and values of the constants must be supplied.

$$\vec{x}_0 = \lfloor -10 \quad -10 \quad 30 \rfloor \tag{2}$$

The equation can then be solved by integrating over t.

$$\vec{x} = \int \frac{d\vec{x}}{dt} dt \tag{3}$$

Transient Solvers

A transient solver integrates the equation in time where the value in the future is unknown.

Euler's Method

Euler's Method approximates the function in the future with its derivative and its value at some point (although not necessarily the same point). The method is $O(h)$ since it only approximates the integrated function at a single point.

Explicit Euler

The explicit Euler's Method is the simplest to evaluate.

$$\vec{x}(t+h) = \vec{x}(t) + \dot{\vec{x}}(t)h \tag{4}$$

While the explicit method is easily implemented, in practice, h must be so small as to outweigh any benefits gained with computation speed and ease of programming.

Implicit Euler

The implicit Euler's Method involves taking the derivative in the future.

$$\vec{x}(t+h) = \vec{x}(t) + \dot{\vec{x}}(t+h)h \tag{5}$$

This is much more stable than taking the derivative in the present, however since $\dot{\vec{x}}$ is also a function of $\vec{x}$, evaluating the derivative becomes somewhat more difficult.

Point Iteration

Since the future value is unknown, it must be iterated for. An easy option is to use point iteration.

$$\vec{x}_{k+1}(t+h) = \vec{x}(t) + \dot{\vec{x}}(\vec{x}_k, t+h)h \tag{6}$$

A challenge is approximating a good initial guess. One option is to use Explicit Euler

$$\vec{x_1} = \vec{x}(t) + \dot{\vec{x}}(\vec{x}(t), t)h \tag{7}$$

However, since Explicit Euler is so unstable for large h, a good option is to under relax to slow down convergence.

$$\vec{x}_{k+1} = \vec{x}(t) + \omega\dot{\vec{x}}(\vec{x}_k, t+h)h; \quad \omega \in (0,1] \tag{8}$$

Newton's Method

Newton's method can also be applied to determine $\vec{x}(t+h)$

$$R = \vec{x} - \vec{x_0} - \dot{\vec{x}}(t+h, \vec{x})h \tag{9}$$

$$DR = J_R = \begin{bmatrix} \frac{1}{h} - \sigma & \sigma & 0 \\ \rho - x_3 & \frac{1}{h} - 1 & -x_1 \\ x_2 & x_1 & \frac{1}{h} - \beta \end{bmatrix} h \tag{10}$$

$$J_R(\vec{x}_k)\Delta\vec{x} = -R(\vec{x}_k) \tag{11}$$

Notably, the amount of iterations required went up with Newton Iteration using Crank-Nicolson. Since this is opposite of the expected trend, it could be an implementation error.

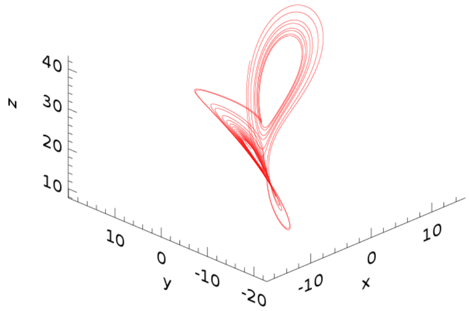

Figure 1: Solution

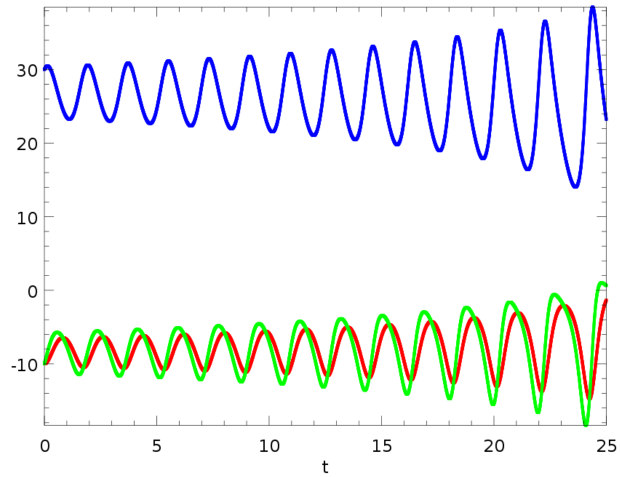

Figure 2: Solution vector

Crank-Nicolson

By evaluating $\dot{\vec{x}}$ at more than one point, a higher order scheme can be developed. One such scheme is Crank-Nicolson.

$$\vec{x}(t+h) = \vec{x}(t) + (\dot{\vec{x}}(t) + \dot{\vec{x}}(t+h))\frac{h}{2} \qquad (12)$$

This is equivalent to the trapezoid rule for numerical integration and is second order. A weighting factor $\omega \in [0,2]$ can be applied although the method is only second order for $\omega = 1$. Notably, $\omega = 0$ is Explicit Euler's Method and $\omega = 1$ is Implicit Euler's Method.

$$\vec{x}(t+h) = \vec{x}(t) + ((2-\omega)\dot{\vec{x}}(t) + \omega\dot{\vec{x}}(t+h))\frac{h}{2} \qquad (13)$$

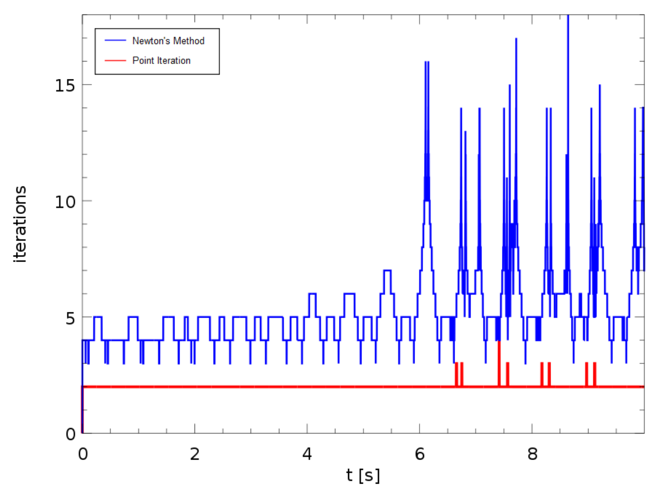

Figure 3: Iteration Comparison

Solution

The Lorenz Equations are highly dependent on the initial conditions: A small change in the initial conditions may produce a large change in the output. As a result of this unpredictability, integration error can significantly affect the resulting solution. When the timestep is the same, Crank-Nicolson produces a very different solution than that of Implicit Euler.