Setup:

K: imaginary quadratic field, where p splits into $\mathfrak{P}$ and $\mathfrak{q}$.

Let $f_{/\mathbb{Q}}$ be a C.M. modular form, with C.M. by K.

So, the 2-dimensional Galois rep $\rho_f : G_{\mathbb{Q}} \to GL_2(R)$ is such that

(R is a D.V.R., finite over $\mathbb{Z}_p$)

$$\rho_f \simeq \mathrm{Ind}_K^{\mathbb{Q}} (\psi_{\mathfrak{P}} \, \varepsilon)$$

where

$$\psi_{\mathfrak{P}} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow 1 + m_R$$

is an infinite character ramified only at $\mathfrak{P}$

$\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to R^{\times}$ is a finite character.

g : Eigenform over $\mathbb{Q}$.

$\rho_g : G_{\mathbb{Q}} \to GL_2(R)$ (extend R to contain eigenvalues of both f and g)

is the 2-dim rep associated to g.

V_f : 2-dimensional $\mathrm{Frac}(R)$ vector space (giving ρ_f)

V_g : 2-dimensional $\mathrm{Frac}(R)$ vector space (giving ρ_g).

$G_{\mathbb{Q}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$G_K : \mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

(say $\mathcal{L} = \mathrm{Frac}(R)$)

<u>Proposition</u>: We have the following isomorphism of 4-dimensional $G_{\mathbb{Q}}$ representations

$$\boxed{\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\left[\mathrm{Res}_{G_K}^{G_{\mathbb{Q}}}(V_g)\otimes_L L(\psi_p\varepsilon)\right] \cong V_g \otimes_L V_f}$$

Fact: Since V_g, V_f are absolutely irreducible $G_{\mathbb{Q}}$-representations (in particular, semi-simple) in char 0, $V_g \otimes_L V_f$ is semi-simple $G_{\mathbb{Q}}$-representation.

This is due to a theorem of Chevalley

[page 88 in Chevalley 1955
Théorie des groupe de Lie, Tome III.
Théorèmes généraux sur les algèbres de Lie]

This book is difficult to find.

Check

i) Marco Maculan: Tensor prod and semi-stability

OR

ii) Richard Molnar: Tensor prod and semi-simple modular rep of finite groups and restricted lie algebras

OR

iii) Tensor product of simple representations

<u>Lemma 1</u>: Suppose a 1-dimensional representation $L(\chi)$ is a semi-simple component of $V_f \otimes V_g$ as a $G_{\wp}$-representation.

Then, V_g is an induced representation from G_K. $\left[\text{Set } V_g^* := \mathrm{Hom}_L(V_g, L)\right]$

Proof of Lemma 1:

There exists a non-zero map in

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(V_f \otimes_L V_g,\ L(\chi)\right)$$

$$\cong \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(V_f,\ V_g^* \otimes_L L(\chi)\right)$$ Hom-Tensor adjunction

$$\cong \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}_p}}(L(\psi,\varepsilon)),\ V_g^*(\chi)\right)$$ f has c.m. by K

irreducible

Since $V_g^*(\chi)$ is irreducible and this Hom space is non-zero, by Schur's lemma

$$\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}_p}}\left(L(\psi_p\varepsilon)\right) \cong V_g^*(\chi).$$

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$\Rightarrow g$ has c.m. by K.

$$\Rightarrow V_g \simeq \mathrm{Hom}_L\Big(\underbrace{\mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\big(L(\psi_p\varepsilon)\big)}_{V_f},\ L(\chi)\Big).$$

In this case, to prove.

$$\boxed{\begin{aligned}&\mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\left[\mathrm{Res}_{G_K}^{G_\mathbb{Q}}(V_g)\otimes_L L(\psi_p\varepsilon)\right].\\ &\quad\cong V_g\otimes_L V_f\end{aligned}}$$

$$\text{R.H.S.} \simeq \mathrm{Hom}_L(V_f, V_f)(\chi) \quad \text{.....} \quad 1$$

$$\text{L.H.S.} \simeq \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\left[\mathrm{Res}_{G_K}^{G_\mathbb{Q}}\mathrm{Hom}_L(V_f, L)\otimes L(\psi_p\varepsilon)\right](\chi)$$

$$\simeq \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\left[\mathrm{Hom}_L\big(L_p(\psi\varepsilon)\oplus L_p(\psi^c\varepsilon^c), L(\psi_p\varepsilon)\big)\right](\chi)$$

$$\cong \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\left(\mathrm{Hom}_L\left(L_p(\psi\varepsilon), L_p(\psi\varepsilon)\right)\right) \oplus \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\left(\mathrm{Hom}_L\left(L(\psi^c\varepsilon^c), L(\psi\varepsilon)\right)\right)(x) \quad (7)$$

$$\cong \left[\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\mathbb{1}) \oplus \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\left(L\left(\underline{\underline{(\psi_p^c\varepsilon^c)^{-1}\psi_p\varepsilon}}\right)\right)\right](x)$$

$$\cong \left[\mathbb{1} \oplus \varepsilon_K \oplus \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\left(L\left(\frac{\psi_p\varepsilon}{\psi_p^c\varepsilon^c}\right)\right)\right](x). \quad \ldots\ldots (2)$$

(1) = (2) since, for any character β

$$\mathrm{Hom}_K^{\mathbb{Q}}\left(\mathrm{Ind}_K^{\mathbb{Q}}(\beta), \mathrm{Ind}_K^{\mathbb{Q}}(\beta)\right)$$

$$\cong \mathbb{1} \oplus \underbrace{\varepsilon_K \oplus \mathrm{Ind}_K^{\mathbb{Q}}\left(\frac{\beta}{\beta^c}\right)}$$

This is classical. See for e.g. Page 258 of Hida's Mod forms and Galois cohomology.

Quadratic character associated to K

$\varepsilon_K \oplus \mathrm{Ind}_K^{\mathbb{Q}}(\beta/\beta^c)$ turns out to equal trace zero adjoint $Ad^0(\beta)$.

$\beta^c(x) = \beta(cxc^{-1})$ for $x \in Gal(\overline{\mathbb{Q}}/K)$

c is a lift of complex conjugation from $Gal(K/\mathbb{Q})$ to $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Lemma 2: Suppose there is no 1-dimensional $G_{\mathfrak{p}}$-component to $V_f \otimes V_g$. Then, $V_f \otimes V_g$ is an irreducible $G_{\mathfrak{p}}$-rep.

Proof: Since

$$
\begin{aligned}
4 &= 4+0 \\
&= 2+1+1 \\
&= 3+1 \\
&= 2+2
\end{aligned}
$$

We just have to show there is no 2-dimensional irreducible

By Lemma 1, g does not have C.M. by K.

Proof by contradiction.

Suppose W is a 2-dimensional irreducible $G_{\mathfrak{p}}$-representation, which is a Jordan-Holder component of $V_f \otimes V_g$.

So,

$$0 \neq \mathrm{Hom}_{G_{\mathbb{Q}}}\left(V_f \otimes V_g, W\right)$$

$$\cong \mathrm{Hom}_{G_{\mathbb{Q}}}\left(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(L(\psi_p \varepsilon)), V_g^* \otimes W\right)$$

Hom - Tensor adjunction

$$\cong \mathrm{Hom}_{G_K}\left(L(\psi_p \varepsilon), \mathrm{Res}(V_g^*) \otimes \mathrm{Res}(W)\right)$$

(Since $[G_{\mathbb{Q}} : G_K] < \infty$ Ind - Res are adjoints to each other (both left, right))

$$\cong \mathrm{Hom}_{G_K}\left(L(\psi_p \varepsilon) \otimes \mathrm{Res}(V_g^*), \mathrm{Res}(W)\right)$$

Since g does not have CM by K

$\mathrm{Res}_{G_K}^{G_{\mathbb{Q}}}(V_g^*)$ is irreducible.

Since the Hom set is not zero, then

$$\mathrm{Res}(V_g^*)(\Psi_p \varepsilon) \simeq \mathrm{Res}(W).$$

Take determinants, we get equality of characters

$$\mathrm{Res}\left(\det(V_g^*)\right)\cdot(\Psi_p \varepsilon)^2 = \mathrm{Res}\left(\det(W)\right)$$

$$\Rightarrow (\Psi_p \varepsilon)^2 = \mathrm{Res}\left(\boxed{\frac{\det(W)}{\det(V_g^*)}}\right)$$

This is impossible since

$(\Psi_p \varepsilon)^2$ is an infinite character ramified at P and unramified at q.

However

$\mathrm{Res}\left(\boxed{}\right)$ must be ramified at P and q

OR

unramified at P and q

since it is the restriction of a character of $G_{\mathbb{Q}}$

(ramification behavior of primes above p must be identical).

So, if g does not have C.M. by K, then $V_g \otimes V_f$ is irreducible.

So, in this case to prove that

$$\boxed{\begin{aligned} &\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\left[\mathrm{Res}_{G_K}^{G_{\mathbb{Q}}}(V_g) \otimes_L L(\psi_p \varepsilon)\right]. \\ &\cong V_g \otimes_L V_f \end{aligned}}$$

Suffices to show that the following Hom set is non-zero.

$$\mathrm{Hom}_{G_{\mathbb{Q}}}\left(V_g \otimes V_f \,,\, \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \left(\mathrm{Res}_{G_K}^{G_{\mathbb{Q}}}(V_g) \otimes_L L(\psi_p \varepsilon) \right) \right)$$

$$\cong \mathrm{Hom}_{G_K}\left(\mathrm{Res}(V_g) \otimes \mathrm{Res}(V_f) \,,\, \mathrm{Res}_{G_K}^{G_{\mathbb{Q}}}(V_g) \otimes_L L(\psi_p \varepsilon) \right).$$

Since

$$\mathrm{Res}(V_f) \cong L(\psi_p \varepsilon) \oplus L(\psi_p^c \varepsilon^c)$$

$\Rightarrow$

$$\mathrm{Res}(V_g) \otimes \mathrm{Res}(V_f) \cong \underline{\mathrm{Res}(V_g) \otimes L(\psi_p \varepsilon)} \oplus \mathrm{Res}(V_g) \otimes L(\psi_p^c \varepsilon^c)$$

So this Hom set is non-zero.

This proves the proposition.