K : imaginary quadratic field.

K_{cyc} : cyclotomic $\mathbb{Z}_p$-extension of K.

$\tilde{K}_\infty$: Compositum of all $\mathbb{Z}_p^2$-extensions of K.

Suppose p splits in K.

$$p\mathcal{O}_K = \mathfrak{p}\mathfrak{q}.$$

fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Say this chooses $\mathfrak{p}$.

$K_{\mathfrak{p}}$: unique $\mathbb{Z}_p$-extension of K, only ramified at $\mathfrak{p}$.

$K_{\mathfrak{q}}$: unique $\mathbb{Z}_p$-extensions of K, only ramified at $\mathfrak{q}$.

$$\begin{tikzcd}
 & & \tilde{K}_\infty \arrow[dll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
K_{cyc} & K_{\mathfrak{p}} & & K_{\mathfrak{q}} & K_{ac} \\
 & & K \arrow[ull, dash] \arrow[ul, dash] \arrow[ur, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

Suppose f is a C.M. modular form, with C.M. by K.

(since p splits in K, f should have ordinary reduction at p).

Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{Z}_p)$

be the 2-dimensional Galois rep associated to f.

Then $\rho_f \simeq \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} (\psi_p \, \varepsilon)$

where

$\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{Z}_p^\times$ is a finite order character

and $\psi_p : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to 1 + p\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p^\times$

infinite order character satisfying the following properties ③

- $\psi_p : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \twoheadrightarrow \mathrm{Gal}(K_p/K) \to 1+p\mathbb{Z}_p$

(factors through $\mathrm{Gal}(K_p/K)$, so this character is ramified only at $\mathfrak{p}$)

- $\mathbb{C}_p(\psi_p)$ has Hodge-Tate weight $k-1$

(Things to keep in mind, we had identified G_{K_p} with $G_{\mathbb{Q}_p}$ via

$$\mathbb{Q}_p \xrightarrow{\sim} K_p$$

And, if χ_p denotes the p-adic cyclotomic character, the convention is that $\mathbb{C}_p(\chi_p)$ has Hodge Tate weight 1)

4

Let T_f be the free $\mathbb{Z}_p$-module of rank two, on which $G_{\mathbb{Q}}$ acts to give us ρ_f.

On the algebraic side, we can associate the Selmer group to the 2-dimensional Galois representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to GL_2(\mathbb{Z}_p[[\hat{\Gamma}]])$$

given by

$$T_\rho := \underbrace{\mathrm{Res}^{G_{\mathbb{Q}}}_{G_K}(T_f)}_{\mathbb{Z}_p\text{-rank } 2} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\hat{\Gamma}]](\chi^{-1}),$$

free $\mathbb{Z}_p[[\hat{\Gamma}]]$-module of rank two.

Here $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/K) \twoheadrightarrow \hat{\Gamma} \hookrightarrow GL_1(\mathbb{Z}_p[[\hat{\Gamma}]])$ is the tautological character.

Let $\Lambda := \mathbb{Z}_p[[\Gamma]]$

Let $\Lambda^\vee := \mathrm{Hom}_{cont}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$

Pontryagin dual

$D_\rho = T_\rho \otimes_\Lambda \Lambda^\vee$: $\rightsquigarrow$ discrete Λ-module

(The analogy to keep in mind is

for an elliptic curve

$T_p(E)$: p-adic Tate module,

free $\mathbb{Z}_p$-module of rank two

$E[p^\infty] \cong T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$: Discrete $\mathbb{Z}_p$-module

$\cong \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \oplus \mathbb{Z}_p$

Note $\mathbb{Z}_p^\vee = \mathbb{Q}_p/\mathbb{Z}_p$.

)

The Selmer group is a subgroup of the Galois cohomology group

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/K), D_\rho).$$

But by Shapiro's lemma, we can work with the induced representation

$$\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\rho) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_4(\Lambda)$$

given by the action of $G_{\mathbb{Q}}$ on

$$\underbrace{\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(T_\rho) \cong \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\Big(\underbrace{\mathrm{Res}_{G_K}^{G_{\mathbb{Q}}} T_f \otimes_{\mathbb{Z}_p} \Lambda(\kappa^{-1})}_{\Lambda\text{-mod rank } 2}\Big)}_{\Lambda\text{-mod rank } 4.}.$$

Now the Selmer group can be identified inside the Galois cohomology group:

$$\boxed{H^1\left(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\ \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(T_p)\otimes_{\Lambda}\Lambda^{\vee}\right)}$$