Computing Bianchi p-adic L-functions and applications in Iwasawa theory

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Abstract

Let $\mathcal{F}$ be a cuspidal eigenform of weight 2 over an imaginary quadratic field K. In this paper, we use the overconvergent cohomology of Bianchi groups to explicitly compute p-adic L-functions attached to $\mathcal{F}$. We give several applications to Iwasawa theory: in particular, we provide new examples where Coates and Sujatha's psuedo-nullity conjecture holds for rational elliptic curves.

To do:

1. Sketch of sharp/flat construction
2. description of how to get the classical class/citation of Marc's papers
3. More detail on the pseudo-nullity conjecture
4. description of resultant calculations to show coprimality
5. Evidence!!!
6. Fox derivatives and Fox gradients (RH Fox, free differential calculus)
7. Tietze algorithm

1. Introduction

Many standard conjectures in Iwasawa theory predict that "pseudo-null" modules are ubiquitous. However, the evidence for these conjectures, computational or otherwise, is limited. In this paper, we provide some of the first computational evidence for a pseudo-nullity conjecture of John Coates and Sujatha Ramdorai. Our main tool for doing this is an explicit algorithm for constructing p-adic L-functions attached to modular forms over imaginary quadratic fields, which – combined with a particular case of the Iwasawa main conjectures – allows further study of the structure of the Selmer group of the conjecture.

1.1. The pseudo-nullity conjecture

We state the conjecture in question. Let K denote an imaginary quadratic field, and let K_∞ denote the compositum of the $\mathbb{Z}_p$-extensions of K. Let p be a rational prime that splits in K, and let $E/\mathbb{Q}$ be an elliptic curve with good supersingular reduction at p. The conjecture relates a Selmer group attached to the base-change E/K. Let S denote the set of primes of K dividing the conductor of E/K and the discriminant of the imaginary quadratic field K. Let K_S denote the maximal extension of K unramified outside S. The fine Selmer group, denoted $Ш^1(E[p^\infty], K_\infty)$, is defined by

$$Ш^1(E[p^\infty], K_\infty) := \ker\left(H^1\left(\mathrm{Gal}\, K_S K_\infty, E[p^\infty]\right) \longrightarrow \prod_{\nu \in S}\prod_{\eta|\nu} H^1\left(G_\eta, E[p^\infty]\right)\right).$$

Here, G_η denotes the decomposition group for the prime η inside the Galois group $\mathrm{Gal}\, K_S K_\infty$. ♠

INSERT SOME ARITHMETIC SIGNIFICANCE OF THIS SELMER GROUP — Chris

The Pontryagin dual of the fine Selmer group, denoted $Ш^1(E[p^\infty], K_\infty)^\vee$, turns out to be a finitely generated module over the Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}\, K_\infty K]]$, which is (non-canonically) isomorphic to the unique factorization domain $\mathbb{Z}_p[[x_1, x_2]]$.

This project is motivated by the following conjecture of John Coates and Sujatha Ramdorai, which is named “Conjecture B” in [?].

Conjecture. *The $\mathbb{Z}_p[[x_1, x_2]]$-module $Ш^1(E[p^\infty], K_\infty)^\vee$ is pseudo-null. That is, there exist two elements*

$$\theta_1, \theta_2 \in \mathrm{Ann}_{\mathbb{Z}_p[[x_1,x_2]]}\left(Ш^1(E[p^\infty], K_\infty)^\vee\right)$$

such that they have no irreducible factor in common.

We study this conjecture through the prism of p-adic L-functions. In [Loe14], Loeffler attached four p-adic L-functions $L_p^{\pm,\pm}$ to E/K, which can be viewed Sharp/Flat construction — Chris as elements in the ring $\mathbb{Z}_p[[x_1, x_2]]$. It is expected, if one believes one of the divisibilities of an appropriate Iwasawa Main Conjecture, that $L_p^{\pm,\pm}$ belongs to $\mathrm{Ann}_{\mathbb{Z}_p[[x_1,x_2]]}\left(Ш^1(E[p^\infty], K_\infty)^\vee\right)$; assuming this, it would thus be possible to verify the conjecture by numerically computing two of these p-adic L-functions and checking that they are coprime in the this annihilator. ♠

1.2. Explicit Bianchi p-adic L-functions

To compute the p-adic L-functions of [Loe14], there is an algorithm of the third author. The main result of [Wil17] was a construction of a p-adic L-function attached to very general classes of Bianchi modular forms – that is, modular forms over K – using a generalisation of Stevens' theory of *overconvergent modular symbols*. A particular feature of this approach, as explored for classical modular forms in [PS11], is its amenability to computation.

To $E/\mathbb{Q}$, one can attach a classical modular form f under modularity, and through base-change this corresponds to a Bianchi modular form $\mathcal{F}$. By assumption, this will have level $N\mathcal{O}_F$ prime to p, but for each prime $\mathfrak{p}|p$, there are two $\mathfrak{p}$-stabilisations to level $\mathfrak{p}N\mathcal{O}_F$, and hence four p-stabilisations to level $pN\mathcal{O}_F$. The four p-adic L-functions attached to these stabilisations in [Wil17] are precisely those of [Loe14]. We give a sketch of the explicit construction in §2.

We actually use a modified version of the construction of [Wil17]. Existing code is much more developed for computations with arithmetic group cohomology, rather than modular symbols, so we develop a cohomological version of the construction. This approach, however, introduces new theoretical complications, which we explain and treat in the main text. In particular, it requires explicitly inverting the natural map from modular symbols to group cohomology.

[Include remarks on Rob Pollack and co, and in particular emphasise the difficulty of computing 2-variable things: they restrict to index in the 100s, it takes them many hours and they compute to precision 7. We should do better?]

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2. Bianchi modular forms and p-adic L-functions

We recap the underlying theory of Bianchi modular forms, which are a natural generalisation of classical modular forms – that is, automorphic forms for $\mathrm{GL}_2/\mathbb{Q}$ – to GL_2/K, where K is an imaginary quadratic field. Throughout this paper, we take K to have class number 1 for simplicity (in line with our later computations). Let $\mathcal{O}_K$ be the ring of integers.

2.1. Background

For each congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$, there is a finite-dimensional $\mathbb{C}$-vector space $S_2^K(\Gamma)$ of 'weight 2 Bianchi cusp forms', defined as harmonic vector-valued functions

$$\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2) : \mathcal{H}_3 \longrightarrow \mathbb{C}^3$$

on the upper half-space $\mathcal{H}_3 := \mathbb{C}\times\mathbb{R}_{>0}$ satisfying a suitable transformation property under an action of Γ. See [Wil17, §1] for a more detailed exposition of all of this (standard) theory in our conventions (noting that $S_2^K(\Gamma)$ for us is denoted $S_{0,0}(\Gamma)$ *op. cit.*).

There is a Hecke action on $S_2^K(\Gamma)$, given by double coset operators and indexed by ideals of $\mathcal{O}_K$.

2.1.1. L-functions and p-adic L-functions. If $\mathcal{F} \in S_2^K(\Gamma)$ is an eigenform with Hecke eigenvalues $\{a_I : I \subset \mathcal{O}_K\}$, and χ is a Hecke character of K, one defines the L-function

$$L(\mathcal{F}, s) = \sum_{I \subset \mathcal{O}_K} a_I \chi(I) N(I)^{-s}$$

as the corresponding Dirichlet series (see [Wil17, §1.2]). This sum converges absolutely for $\mathrm{Re}(s) >> 0$ and admits analytic continuation to arbitrary $s \in \mathbb{C}$. Morever, there exists a complex period $\Omega_{\mathcal{F}} \in \mathbb{C}^\times$ such that

$$\frac{L(\mathcal{F}, \chi, 1)}{(2\pi i)^2 \Omega_{\mathcal{F}}} \in \overline{\mathbb{Q}} \tag{2.1}$$

for all finite order Hecke characters χ (see [Hid94, §8]). The conjecture of Coates–Perrin-Riou then predicts the existence of a *p-adic L-function* interpolating these algebraic numbers. More precisely, this should be a p-adic distribution $\mu_{\mathcal{F}}$ on

$$\mathrm{Gal}(K^{\mathrm{ab},p}/K) \cong \mathrm{Cl}_K(p^\infty) := K^\times \mathbb{A}_K^\times / \mathbb{C}^\times \widehat{\mathcal{O}}_K^{\times,(p)} \cong (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \mathcal{O}_K^\times,$$

where $K^{\mathrm{ab},p}$ is the maximal abelian extension of K unramified outside p, the first isomorphism is given by class field theory, and the second follows from our assumption of class number 1. The required interpolation property is that for all finite order Hecke characters of p-power conductor – which are naturally characters of $\mathrm{Cl}_K(p^\infty)$ – we have

$$\mu_f(\chi) := \int_{\mathrm{Cl}_K(p^\infty)} \chi(z) d\mu_{\mathcal{F}}(z) = (*) \frac{L(\mathcal{F},\chi,1)}{(2\pi i)^2 \Omega_f}, \tag{2.2}$$

where $(*)$ is a precise interpolation factor (described explicitly in this case in [BSW19, Thm. 3.12] with $r = 0$). Moreover, $\mu_{\mathcal{F}}$ should satisfy a precise growth property.

2.1.2. Base-change. Of particular importance to us is the existence of a *base-change* map. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N corresponding to a newform $f \in S_2(\Gamma_0(N))$ under modularity. We can consider E instead to have coefficients over K. As predicted by Langlands, there exists a Bianchi newform f/K, the *base-change of f to K*, whose L-function is equal to $L(E/K,s)$. We have $f/K \in S_2^K(\Gamma_0(\mathfrak{n}))$, where $\Gamma_0(\mathfrak{n}) \subset \mathrm{SL}_2(\mathcal{O}_K)$ is the subgroup of matrices that are upper-triangular mod $\mathfrak{n}$, and where $\mathfrak{n}|N\mathcal{O}_K$ is the conductor of E/K. (Note that if N is coprime to the discriminant of $K/\mathbb{Q}$, we have $\mathfrak{n} = N\mathcal{O}_K$). The Hecke eigenvalues of f/K can be described simply in terms of those of f (see e.g. [BSW18, §7.2]). Note that

$$L(f/K,s) = L(E/K,s) = L(E/\mathbb{Q},s)L(E/\mathbb{Q},\chi_K,s) = L(f,s)L(f,\chi_K,s),$$

where χ_K is the quadratic Hecke character whose kernel cuts out $K/\mathbb{Q}$.

2.2. Bianchi modular symbols

We are interested in computational aspects of Bianchi modular forms, and particularly their L-functions, but the analytic definition is impractical for such purposes. Fortunately, the space $S_2^K(\Gamma)$ admits a computational avatar via the space of *modular symbols*. Let

$$\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(K))$$

denote the space of 'paths between cusps' in $\mathcal{H}_3$, and let V be any right $\mathrm{SL}_2(K)$-module. Fix an ideal $\mathfrak{n}$ and let $\Gamma := \Gamma_0(\mathfrak{n}) \subset \mathrm{SL}_2(K)$. We define the space of *V-valued modular symbols for* Γ to be the space

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Hom}_\Gamma(\Delta_0, V)$$

of functions satisfying the Γ-invariance property that

$$(\phi|\gamma)(\delta) := \phi(\gamma\delta)|\gamma = \phi(\delta) \quad \forall \delta \in \Delta_0, \gamma \in \Gamma,$$

where Γ acts on the cusps by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot r = (ar+b)/(cr+d)$. Note that this space is entirely algebraic in nature, and far more amenable to computations.

Now let $\mathcal{F} \in S_2^K(\Gamma)$. To $\mathcal{F}$ one may attach an explicit differential form $\delta_{\mathcal{F}}$ on $\mathcal{H}_3$ as follows: let (z,t) be a co-ordinate on $\mathcal{H}_3$, and note that $dz, d\overline{z}, dt$ span the $\mathbb{C}$-valued 1-forms on $\mathcal{H}_3$. Then define

$$\delta_{\mathcal{F}} := \mathcal{F}_0(z,t)dz - \mathcal{F}_1(z,t)dt - \mathcal{F}_2(z,t)d\overline{z} \in \Omega^1(\mathcal{H}_3)$$

(see e.g. [CW94]). For $r,s \in \mathbb{P}^1(K)$, the map

$$\begin{aligned} \phi_{\mathcal{F}} : \Delta_0 &\longrightarrow \mathbb{C}, \\ \{r \to s\} &\longmapsto \int_r^s \delta_f, \end{aligned}$$

is well-defined and the transformation property satisfied by $\mathcal{F}$ ensures it is Γ-invariant, thus giving an element $\phi_f \in \mathrm{Symb}_\Gamma(\mathbb{C})$.

The space $\mathrm{Symb}_\Gamma(\mathbb{C})$ admits an action of the Hecke operators, indexed by ideals of $\mathcal{O}_F$ and generated by the operators $T_{\mathfrak{l}}$ for $\mathfrak{l} \nmid \mathfrak{n}$ prime and $U_{\mathfrak{l}}$ for $\mathfrak{l}|\mathfrak{n}$. Of particular importance are the $U_{\mathfrak{p}}$-operators for $\mathfrak{p}|p$, defined as

$$\phi|U_{\mathfrak{p}}\{r \to s\} = \sum_{a=0}^{p-1} \phi\big| \left(\begin{smallmatrix} 1 & a \\ 0 & \pi \end{smallmatrix}\right) \{r - s\} = \sum_{a=0}^{p-1} \phi\left\{\tfrac{r+a}{\pi} \to \tfrac{s+a}{\pi}\right\},$$

where $\mathfrak{p} = (\pi)$. More generally, if the action of Γ on V extends to an action of the semigroup

$$\Sigma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) : ad - bc \neq 0, p \nmid a, p|c\},$$

then the formula $\phi|U_{\mathfrak{p}} = \sum_{a=0}^{p-1} \phi| \left(\begin{smallmatrix} 1 & a \\ 0 & \pi \end{smallmatrix}\right)$ defines an action of $U_{\mathfrak{p}}$ on $\mathrm{Symb}_\Gamma(V)$.

Proposition 2.1. *The resulting map*

$$\iota : S_2^K(\Gamma) \longrightarrow \mathrm{Symb}_\Gamma(\mathbb{C})$$

is injective and induces a splitting

$$\mathrm{Symb}_\Gamma(\mathbb{C}) \cong S_2^K(\Gamma) \oplus \mathrm{Eis}_2^K(\Gamma).$$

For a prime $\mathfrak{l} \nmid \mathfrak{n}$ of norm ℓ, the Hecke operator $T_{\mathfrak{l}}$ acts on $\mathrm{Eis}_2^K(\Gamma)$ as multiplication by $\ell + 1$.

Proof. See, for example, [Wil17] for the injection. For part (ii), observe that by [BSW19, Lemma 8.2] we have a Hecke-equivariant isomorphism

$$\mathrm{Symb}_\Gamma(\mathbb{C}) \cong \mathrm{H}^1_{\mathrm{c}}(Y_\Gamma, \mathbb{C}),$$

where $Y_\Gamma := \Gamma\backslash\mathcal{H}_3$, and the compactly supported cohomology admits a well-understood splitting $\mathrm{H}^1_{\mathrm{cusp}}(Y_\Gamma, \mathbb{C}) \oplus \mathrm{Eis}_\Gamma(\mathbb{C})$ (see [Har87, §3.2.5]). Moreover, under the injection ι, the space $S_2^K(\Gamma)$ is mapped isomorphically onto $\mathrm{H}^1_{\mathrm{cusp}}(Y_\Gamma, \mathbb{C})$. The other direct summand corresponds to Bianchi Eisenstein series, giving the claimed action of Hecke operators. □

The passage from an eigenform $\mathcal{F}$ to $\phi_{\mathcal{F}}$ thus encodes much of the interesting algebraic data attacehed to $\mathcal{F}$. In particular, it retains the Hecke action, the Hecke eigenvalues and thus also sees the L-function of $\mathcal{F}$. In particular, in [CW94] an explicit formula is given relating $\phi_{\mathcal{F}}\{0 \to \infty\}$ to the critical value $L(\mathcal{F}, 1)$. A twisted version, computing $L(\mathcal{F}, \chi, 1)$ for finite order Hecke characters of K, is contained in [Wil17, Prop. 2.8].

2.3. Overconvergent modular symbols

We briefly recap the main ideas in the construction of [Wil17]. The idea behind the overconvergent modular symbol construction is that to p-adically interpolate the (algebraic parts of the) values $L(\mathcal{F},\chi,1)$, we can use the connection to modular symbols, and p-adically interpolate these instead. To do this, we require algebraic coefficients. By [Hid94, §8], there exists a finite extension $L/\mathbb{Q}_p$ such that we may consider $\phi_{\mathcal{F}}/\Omega_{\mathcal{F}} \in \mathrm{Symb}_\Gamma(L)$, where Ω_f is the same complex period from (2.1), which can then be interpolated p-adically via *overconvergent modular symbols*.

For such an interpolation to exist, it is crucial to pass to a level subgroup $\Gamma \subset \Gamma_0(p\mathcal{O}_F)$. If $(p)|\mathfrak{n}$, then $\Gamma_0(\mathfrak{n}) \subset \Gamma_0(p\mathcal{O}_F)$ already; if not, then it is necessary to 'p-stabilise' to ensure this condition holds (see for example [BSW18, §2.4]). We now assume $\Gamma \subset \Gamma_0(p\mathcal{O}_F)$ without further comment.

We pass to an (infinite-dimensional) coefficient module; namely, let $L/\mathbb{Q}_p$ be a finite extension, let $\mathbb{A}_2(L)$ denote the space of convergent power series on $\mathbb{Z}_p^2$, and let

$$\mathbb{D}_2(L) := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}_2(L), L),$$

the space of *p-adic analytic distributions on $\mathbb{Z}_p^2$*. As $\Gamma \subset \Gamma_0(p)$, the left action of Γ on $\mathbb{A}_2(L)$ by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot g(x,y) = g\left(\frac{b+dx}{a+cx}, \frac{\bar{b}+\bar{d}y}{\bar{a}+\bar{c}y}\right),$$

is well-defined and induces dually a right action on $\mathbb{D}_2(L)$. The space of *overconvergent modular symbols* is $\mathrm{Symb}_\Gamma(\mathbb{D}_2(L))$.

Dualising the inclusion of L into $\mathbb{A}_2(L)$ gives a surjection $\mathbb{D}_2(L) \to L$ of Γ-modules, and hence a (Hecke-equivariant) map

$$\rho : \mathrm{Symb}_\Gamma(\mathbb{D}_2) \to \mathrm{Symb}_\Gamma(L).$$

Whilst this map will have a huge kernel, crucially, the Hecke action allows us to control it.

Theorem 2.2. [Wil17, Thm. 6.10] *Let $\mathcal{F} \in S_2^K(\Gamma)$ be an eigenform, with $U_{\mathfrak{p}}\mathcal{F} = \alpha_{\mathfrak{p}}\mathcal{F}$ for each $\mathfrak{p}|p$. If $v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) < 1$ for all $\mathfrak{p}$, then the restriction of ρ to the $\mathcal{F}$-eigenspaces of the Hecke operators is an isomorphism.*

If $\mathcal{F}$ satisfies this condition, we say it has *small slope*. If $\mathcal{F}$ is such a form, then the theorem says that there is a *unique* $\Phi_{\mathcal{F}} \in \mathrm{Symb}_\Gamma(\mathbb{D}_2(L))$ lifting $\phi_{\mathcal{F}}/\Omega_{\mathcal{F}}$.

2.4. The p-adic L-function of a Bianchi modular form

Let $\mathcal{F}$ be a small slope Bianchi eigenform with associated overconvergent modular symbol $\Phi_{\mathcal{F}}$. Recall from §2.1.1 that the p-adic L-function should be a p-adic distribution on $\mathrm{Cl}_K(p^\infty) = (\mathbb{Z}_p^\times)^2/\mathcal{O}_K^\times$. Note that there is a natural inclusion $\mathbb{A}(\mathrm{Cl}_K(p^\infty), L) \subset \mathbb{A}_2(L)$ of the subspace of analytic functions on $\mathrm{Cl}_K(p^\infty)$, and thus a natural restriction map $\mathbb{D}_2(L) \to \mathbb{D}(\mathrm{Cl}_K(p^\infty))$.

Definition 2.3. Define the *p-adic L-function of $\mathcal{F}$* to be

$$\mu_{\mathcal{F}} := \Phi_{\mathcal{F}}\{0 \to \infty\}\big|_{\mathrm{Cl}_K(p^\infty)}.$$

By [Wil17, Thm. 7.4], this is actually in the much smaller subspace of *locally analytic distributions*, and can thus be evaluated at finite order Hecke characters. It then satisfies the expected interpolation property (2.2) and has the expected growth condition. In the small slope case, this defines $\mu_{\mathcal{F}}$ uniquely.

3. Computing p-adic L-series from modular symbols

In this section, we show how to compute the p-adic L-series of a Bianchi modular form from its p-adic L-function (as a distribution on $\mathrm{Cl}_K(p^\infty)$). Since we are working in the case where p is split, this essentially just the product of two copies of the theory for classical modular forms, as found in [PS11, §9], and which we first briefly recall.

3.1. The classical case

3.1.1. Distributions on $\mathbb{Z}_p^\times$. For simplicity, we assume that $p \geq 3$, though the case $p = 2$ can be obtained with very little (and completely standard) modification. In the classical case, the p-adic L-function is naturally a distribution on the Galois group $\mathrm{Gal}_p \coloneqq \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab},p\infty}/\mathbb{Q})$, which by class field theory is isomorphic to the narrow ray class group

$$\mathrm{Cl}_{\mathbb{Q}}^+(p^\infty) = \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}_{>0}\widehat{\mathbb{Z}}^{\times,(p)} \cong \mathbb{Z}_p^\times.$$

Let now $f \in S_2(\Gamma)$ be a classical eigenform (for $\mathrm{GL}_2/\mathbb{Q}$) with $U_pf = \alpha_p f$ with $v_p(\alpha_p) < 1$. By [PS11], there is a unique overconvergent modular symbol $\Phi_f \in \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), \mathbb{D})$ attached to f, where $\mathbb{D}$ is the space of distributions on $\mathbb{Z}_p$. Then $\Phi_f\{0 \to \infty\} \in \mathbb{D}$ is by definition a distribution on $\mathbb{Z}_p$, and we obtain a distribution μ_f on $\mathbb{Z}_p^\times$ by restriction. This distribution is, by [PS11], the p-adic L-function attached to f, and is entirely encoded in its moments

$$\left\{\mu_f(z^j) \coloneqq \int_{\mathbb{Z}_p^\times} z^j d\mu_f(z) : j \geq 0\right\}.$$

We have a simple description of the moments of μ_f in terms of Φ_f, and thus, in particular, we can compute them by knowing Φ_f.

Proposition 3.1. *If Φ_f is the overconvergent modular symbol attached to a small slope classical cuspidal eigenform, then*

$$\Phi\{0 \to \infty\}(z^j\mathbf{1}_{a+p\mathbb{Z}_p}(z)) = \alpha_p^{-1}\Phi\left\{\tfrac{a}{p} \to \infty\right\}\left((a+pz)^j\right),$$

where α_p is the U_p-eigenvalue of f. In particular, we recover

$$\mu_f(z^j) = \alpha_p^{-1}\sum_{a=1}^{p-1}\Phi\left\{\tfrac{a}{p} \to \infty\right\}\left((a+pz)^j\right).$$

Proof. Apply $\alpha_p^{-1}U_p$, which acts as 1 on Φ_f, and note that if $a = b$, then

$$\Phi\{b/p \to \infty\}((b+pz)^j\mathbf{1}_{a+p\mathbb{Z}_p}(b+pz)) = 0. \qquad \square$$

The final statement follows since $\mathbf{1}_{\mathbb{Z}_p^\times} = \sum_{a=1}^{p-1}\mathbf{1}_{a+p\mathbb{Z}_p}$.

3.1.2. Passing to p-adic L-series. The distribution μ_f is completely canonical. Even in higher weight situations, it is canonical up to a fixed choice of periods. For practical purposes, however, to compute p-adic L-series from μ_f we wish to break this into $p-1$ pieces via the decomposition

$$\mathbb{Z}_p^\times = (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p), \tag{3.1}$$

and then (non-canonically) identify each one with a distribution on $\mathbb{Z}_p$ (depending on a choice of topological generator for $1+p\mathbb{Z}_p$). On the algebraic side, this corresponds to considering the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, identifying its Galois group $\mathcal{G}$ with $\mathbb{Z}_p$ via the choice of a topological generator, and studying Selmer groups over $\mathcal{G}$.

Notation 3.2: In (3.1), denote projection to the first and second factors by $z \mapsto \{z\}$ and $z \mapsto \langle z\rangle$ respectively.

We can identify distributions μ on $\mathbb{Z}_p^\times$ with analytic functions on *weight space*

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times),$$

which – as a rigid space – is $p-1$ copies of the open unit disc. Given an element ϕ in weight space, we can write ϕ as $\{\phi\}\langle\phi\rangle$, where $\{\phi\}$ is a homomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}_p^\times$, and the disc in $\mathcal{W}$ in which ϕ lives is completely determined by $\{\phi\}$. From μ_f we then obtain $p-1$ analytic functions on the open unit disc, one for each character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Each has a power series representation, which is what we compute.

Fix now a character ψ on $(\mathbb{Z}/p\mathbb{Z})^\times$, corresponding to some fixed disc in $\mathcal{W}$. Let T be a parameter on this open unit disc. The part of the (analytic) function attached to μ_f defined over this disc is defined as

$$L_p(\mu,\psi,T) := \int_{\mathbb{Z}_p^\times} \psi(z)(T+1)^{\log_\gamma(\langle z\rangle)} d\mu_f(z),$$

where $\gamma = p+1$, our fixed choice of topological generator for $1+p\mathbb{Z}_p$. (Of course, for $z \in 1+p\mathbb{Z}_p$, we have $\log_\gamma(z) = \log_p(z)/\log_p(\gamma)$). Writing $z = \{z\}\langle z\rangle$ and expanding the log, we obtain the power series representation $L_p(\mu,\psi,T) = \sum_{n\geq 0} d_n(\psi)T^n$, where $d_n(\psi)$ is defined as

$$d_n(\psi) = \sum_{a=0}^{p-1} \psi(a) \int_{a+p\mathbb{Z}_p} \left[\sum_{j\geq 0} c_j^{(n)} \left(\frac{z}{\{a\}} - 1\right)^j\right] d\mu(z).$$

Here, $c_j^{(n)}$ is defined by from the equation

$$\binom{\log_\gamma(z+1)}{n} = \sum_{j\geq 0} c_j^{(n)} z^j, \tag{3.2}$$

Note also that for a prime to p, we have $\int_{a+p\mathbb{Z}_p} h(z)d\mu(z) = \int_{a+p\mathbb{Z}_p} h(z)d\Phi\{0\to\infty\}(z)$, that is, restriction to $\mathbb{Z}_p^\times$ is already built in. For further details on all of the above, see [PS11, §9].

3.2. The Bianchi case

Now we turn to the Bianchi case. Let f be a small slope classical cuspidal Bianchi eigenform, and $\Phi_f \in \mathrm{Symb}_\Gamma(\mathbb{D}_2)$ the attached overconvergent modular symbol. As outlined in §2.1.1, the p-adic L-function of f is most naturally a distribution μ on $\mathrm{Cl}_K(p^\infty) \cong (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times/\mathcal{O}_K^\times$, where in the last assertion we are assuming class number one for simplicity. Since p is split, we identify this with $(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/\mathcal{O}_K^\times$. Evaluating Φ_f at $\{0 \to \infty\}$, we obtain a distribution on all of $\mathbb{Z}_p^2$. Inside $\mathbb{A}_2$ is the subspace of functions with support in $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ that are invariant under $\mathcal{O}_K^\times$, and we pass to the p-adic L-function μ_f by restricting $\Phi_f\{0 \to \infty\}$ to this subspace.

One can consider the direct analogue of the weight space above, that is, considering continuous homomorphisms $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, and show that this decomposes as the disjoint union of $(p-1)^2$ products of open unit discs, parametrised by characters of $(\mathcal{O}_K/p)^\times = (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^\times$. By directly generalising the approach above, given a measure μ on $\mathrm{Cl}_K(p^\infty)$ and such a character ψ, one can define an associated (two-variable) analytic function $L_p(\mu, \psi, T_1, T_2)$ on the corresponding product of open discs. Because the p-adic L-function actually lives on $\mathrm{Cl}_K(p^\infty)$ rather than $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$, such a function can only be defined when ψ descends to the quotient $(\mathcal{O}_K/p)^\times/\mathcal{O}_K^\times$.

Fix such a character ψ. Writing $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, we see that $\psi = \psi_{\mathfrak{p}}\psi_{\overline{\mathfrak{p}}}$, where $\psi_{\mathfrak{p}}$ is the restriction to $(\mathcal{O}_K/\mathfrak{p})^\times$. We also have parameters $z_{\mathfrak{p}}, z_{\overline{\mathfrak{p}}}$ on $\mathrm{Cl}_K(p^\infty)$, and $T_{\mathfrak{p}}, T_{\overline{\mathfrak{p}}}$ on the product of open discs. Using the same arguments as above, we find the following.

Proposition 3.3. *The p-adic L-series attached to ψ and μ is*

$$L_p(\mu, \psi, T_{\mathfrak{p}}, T_{\overline{\mathfrak{p}}}) = \sum_{m \geq 0} \sum_{n \geq 0} d_{m,n}(\psi) T_{\mathfrak{p}}^m T_{\overline{\mathfrak{p}}}^n,$$

where – for $c_i^{(m)}$ as in (3.2) *– we define*

$$d_{m,n}(\psi) = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \psi_{\mathfrak{p}}(a)\psi_{\overline{\mathfrak{p}}}(b)$$
$$\times \int_{(a+p\mathbb{Z}_p)\times(b+p\mathbb{Z}_p)} \left[\sum_{i \geq 1} \sum_{j \geq 1} c_i^{(m)} c_j^{(n)} \left(\frac{z_{\mathfrak{p}}}{\{a\}} - 1\right)^i \left(\frac{z_{\overline{\mathfrak{p}}}}{\{b\}} - 1\right)^j\right] d\mu(\mathbf{z}).$$

3.2.1. Obtaining power series from the moments of μ. As before, it is simple to obtain the moments of μ_f by applying the U_p operator. Write $V_{a,b}$ for the open compact set $(a + p\mathbb{Z}_p) \times (b + p\mathbb{Z}_p) \subset (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$. We compute that

$$\begin{aligned}\int_{V_{a,b}} f(z_{\mathfrak{p}}, z_{\overline{\mathfrak{p}}}) d\mu(\mathbf{z}) &:= \Phi\{0 \to \infty\}(f(z_{\mathfrak{p}}, z_{\overline{\mathfrak{p}}})\mathbb{1}_{V_{a,b}}) \\ &= \alpha_p^{-1}\Phi\{c/p \to \infty\}\big(f(c + pz_{\mathfrak{p}}, \overline{c} + pz_{\overline{\mathfrak{p}}})\big),\end{aligned}$$

where $c = c_{a,b} \in \mathcal{O}_K$ is such that

$$c \equiv a \pmod{\mathfrak{p}}, \quad \overline{c} \equiv b \pmod{\overline{\mathfrak{p}}}.$$

To see this, one applies the operator $\alpha_p^{-1}U_{p\mathcal{O}_K} = \alpha_p^{-1}U_{\mathfrak{p}}U_{\overline{\mathfrak{p}}}$, which acts as the identity on Φ, and note that the indicator function kills all but the a term of $U_{\mathfrak{p}}$ and the b term of $U_{\overline{\mathfrak{p}}}$, corresponding to the c term of U_p; see [Wil17, §7.1] for more details. Since for us f is a polynomial function, it is simple to compute this value by taking a linear combination of the moments of $\Phi\{c/p \to \infty\}$.

3.2.2. Twists. Twisting by characters of p-power conductor is built into the definitions above, but we can also twist by finite order characters of prime-to-p conductor. Consider a finite order character χ of conductor $(\mathfrak{d}) = \mathfrak{D} \subset \mathcal{O}_K$ prime to p. From Φ, one defines a twisted symbol

$$\Phi_\chi := \sum_{b \pmod{\mathfrak{D}}} \chi(b)\left[\Phi\{b/\mathfrak{d} \to \infty\}\middle| \begin{pmatrix} 1 & b \\ 0 & \mathfrak{d}\end{pmatrix}\right],$$

then computes $L_p(\mu, \psi\chi, T) := L_p(\mu_\chi, \psi, T)$, where $\mu_\chi := \Phi_\chi\{0 \to \infty\}|_{\mathrm{Cl}_K(p^\infty)}$. This case is treated in [BSW19, §3.4].

4. Rephrasing via arithmetic cohomology

The above gives a complete algorithm for constructing p-adic L-series from Bianchi modular symbols. For practical reasons, however, this space would be hard to compute in, since this requires presenting $\mathrm{Div}^0(\mathbb{P}^1(K))$ as a $\mathbb{Z}[\Gamma]$-module (and, in particular, solving the word problem in such a presentation). Instead, we work with the arithmetic cohomology groups $\mathrm{H}^1(\Gamma, \mathbb{D}_2)$, for which extensive implementation already exists. This approach might also generalise more naturally to different settings, for example, pursuing the constructions of [BS15] over *real* quadratic fields, where modular symbols themselves do not exist and one is forced to work with higher degree cohomology groups.

4.1. Definition and basic properties

See Marc's papers (which one explains it best, Marc?), or [PP09]; include things such as U_p operators.

A downside of computing with arithmetic cohomology over modular symbols is that we are not free to evaluate at the same range of divisors. In particular, we have a map

$$\begin{aligned} \delta : \mathrm{Symb}_\Gamma(\mathbb{D}_2) &\longrightarrow \mathrm{H}^1(\Gamma, \mathbb{D}_2) \\ \Phi &\longmapsto \big(\varphi : \gamma \mapsto \Phi\{\gamma\cdot\infty \to \infty\}\big). \end{aligned}$$

Given φ, we can thus read off the values $\Phi\{r \to s\}$ *only* for r, s equivalent to the cusp ∞. This poses a problem for our algorithm, since we need to evaluate at pairs $\{a/p \to \infty\}$, where a is coprime to p, and in general a/p will *not* give the same cusp as ∞. To obtain the information we need from φ, then, requires a careful study of the map δ.

From the general theory, the kernel and cokernel of δ are Eisenstein; in particular, it is an isomorphism on the cuspidal part. It follows that there is a *unique* cuspidal lift Φ of φ under δ. In the next section, we show how to explicitly invert δ to obtain this class Φ from φ, and thus how to obtain the p-adic L-function from φ.

4.2. Explicit inversion of δ

We now give an explicit and computable recipe for inverting δ on cuspidal arithmetic cohomology classes, taking care to be precise at every step.

4.2.1. Motivation: a snake diagram. The map δ can be realised in cohomology as

$$\delta : \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta_0, \mathbb{D}_2) \longrightarrow \mathrm{H}^1(\Gamma, \mathbb{D}_2),$$

and is the connecting map in a long exact sequence given by the snake lemma. In particular, for a right Γ-module $\mathcal{M}$, let:

- $C^i(\mathcal{M}) \coloneqq C^i(\Gamma, M) = \mathcal{M}$-valued i-cochains for Γ,
- $Z^i(\mathcal{M}) \coloneqq \mathcal{M}$-valued i-cocycles for Γ,
- and $B^i(\mathcal{M}) \coloneqq \mathcal{M}$-valued i-coboundaries for Γ.

Explicitly, as we are using right modules, a 1-cocycle is a map $z : \Gamma \to \mathcal{M}$ such that $z(\gamma_1\gamma_2) = z(\gamma_1) + z(\gamma_2)|\gamma_1^{-1}$. Also write $(C^i/B^i)(\mathcal{M})$ for the i-cochains modulo the i-coboundaries.

Recall that $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(K))$, and let $\Delta = \mathrm{Div}(\mathbb{P}^1(K)) = \mathbb{Z}[\mathbb{P}^1(K)]$. Now, the degree map gives a short exact sequence $0 \to \Delta_0 \to \Delta \to \mathbb{Z} \to 0$, and hence – for any right Γ-module M – a short exact sequence

$$0 \to M \longrightarrow \mathrm{Hom}(\Delta, M) \longrightarrow \mathrm{Hom}(\Delta_0, M) \to 0,$$

of Γ-modules, identifying $M \cong \mathrm{Hom}(\mathbb{Z}, M')$, and where Γ act on $\mathrm{Hom}(\Delta, M')$ by $(\phi|\gamma)(D) = \phi(\gamma D)|\gamma$. This gives rise to a diagram

$$\begin{tikzcd}
& & & \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta_0, M)) \arrow[d] & \\
& \frac{C^0}{B^0}\big(\Gamma, M\big) \arrow[r] \arrow[d] & \frac{C^0}{B^0}\big(\Gamma, \mathrm{Hom}(\Delta, M)\big) \arrow[r] \arrow[d] & \frac{C^0}{B^0}\big(\Gamma, \mathrm{Hom}(\Delta_0, M)\big) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^1(\Gamma, M) \arrow[r] \arrow[d] & Z^1(\Gamma, \mathrm{Hom}(\Delta, M)) \arrow[r] & Z^1(\Gamma, \mathrm{Hom}(\Delta_0, M)) & \\
& \mathrm{H}^1(\Gamma, M) & & &
\end{tikzcd}$$

from which we obtain the snake exact sequence

$$\cdots \to \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta, M)) \xrightarrow{\alpha} \mathrm{H}^0(\gamma, \mathrm{Hom}(\Delta_0, M)) \xrightarrow{\delta} \mathrm{H}^1(\Gamma, M) \xrightarrow{\beta} \mathrm{H}^1(\Gamma, \mathrm{Hom}(\Delta, M)) \to \cdots .$$

In proving that this is exact, one takes an element in $\ker(\beta)$ – such as our class φ – and explicitly realises it in the image of δ. Thus we can lift δ by pursuing this diagram chase.

4.2.2. An explicit formula for inverting δ. In practice, we will work at the level of cocycles, and this will depend on choices made throughout the algorithm (whilst the corresponding cohomology classes, and the p-adic L-functions, will not). Suppose we are given a cohomology class $\varphi \in H^1(M)$ represented by a cocycle φ_0. If $M = \mathbb{D}_2$ and φ arises as the lift of a cuspidal classical eigenclass, then it is in $\ker(\beta)$. Then:

Proposition 4.1. *If $\varphi \in \ker(\beta)$, then for any $c_i \in \mathbb{P}^1(K)$, the restriction of φ_0 to $\mathrm{Stab}_\Gamma(c_i)$ is a coboundary in $Z^1(\mathrm{Stab}_\Gamma(c_i), M)$; explictly, there exists $v_i \in M$ such that*

$$\varphi_0(\gamma) = v_i|\gamma^{-1} - v_i$$

for all $\gamma \in \mathrm{Stab}_\Gamma(c_i)$.

Proof. This is essentially a consequence of Shapiro's lemma, but with slightly more explicit control on the coboundaries. The map β is induced by the map of cocycles that sends the cocycle $\varphi_0 : \Gamma \to M$ to

$$\varphi_0 \longmapsto \Big(\beta\left(\varphi_0\right) : \gamma \mapsto [r \mapsto \varphi_0(\gamma)]\Big),$$

where r is any element of $\mathbb{P}^1(K)$ (recalling that $\Delta = \mathbb{Z}[\mathbb{P}^1(K)]$ is freely generated such r). In particular, each $\gamma \in \Gamma$ is just sent to a constant function in $\mathrm{Hom}(\Delta, M)$. For $\varphi \in \ker(\beta)$, we must have $\beta(\varphi_0)$ is a coboundary, and hence that there exists $v \in \mathrm{Hom}(\Delta, M)$ such that

$$\beta(\varphi_0)(\gamma) = v|\gamma^{-1} - v.$$

In particular, we have

$$\beta(\varphi_0)(\gamma)(r) = v(\gamma^{-1}r)|\gamma^{-1} - v(r) = \varphi_0(\gamma)$$

for all $c \in \mathbb{P}^1(K)$, by definition of $\beta(\varphi_0)$. For c_i as above and $\gamma \in \mathrm{Stab}_\Gamma(c_i)$, we have

$$\varphi_0(\gamma) = v(c_i)|\gamma^{-1} - v(c_i) = v_i|\gamma^{-1} - v_i,$$

for $v_i := v(c_i) \in M$, as required. $\square$

Let now $c_1, ..., c_t \in \mathbb{P}^1(K)$ be a complete set of representatives for the cusps, and for each $r \in \mathbb{P}^1(K)$, let $g_r \in \Gamma$ and $i(r) \in \{1, ..., t\}$ be such that

$$g_r \cdot r = c_{i(r)}.$$

Assuming $\varphi \in \ker(\beta)$, let $v_1, ..., v_t \in \mathbb{D}_2$ be the distributions arising from Proposition 4.1. We can compute the v_i by computing the action of g as a linear operator on $\mathbb{D}_2$ (up to some precision), and then solving the resulting linear system for a sufficiently large set of elements of the stabiliser[1].

Again motivated by the snake lemma, we now define $\widetilde{\Phi} \in \mathrm{Hom}_\Gamma(\Delta, \mathbb{D}_2) = \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta, \mathbb{D}_2))$ by setting

$$\begin{aligned} \widetilde{\Phi} : \mathbb{P}^1(K) &\longrightarrow \mathbb{D}_2, \\ r = g_r^{-1}c_{i(r)} &\longmapsto \varphi(g_r)\big|g_r + v_{i(r)}\big|g_r \end{aligned}$$

and extending linearly, and $\Phi = \alpha(\widetilde{\Phi}) : \Delta_0 \longrightarrow \mathbb{D}_2$ defined by

$$\Phi\{r \to s\} := \widetilde{\Phi}(s) - \widetilde{\Phi}(r).$$

Proposition 4.2. *The map Φ gives a well-defined element of* $\mathrm{Symb}_\Gamma(\mathbb{D}_2)$ *with $\delta(\Phi) = \varphi$.*

Proof. The map Φ is linear in $r \to s$, since it is defined as the difference $\widetilde{\Phi}(s) - \widetilde{\Phi}(r)$. It is mapped to φ under δ; setting $r = \gamma \cdot \infty$ and $s = \infty$, we have $g_r = \gamma^{-1}$, $g_s = 1$, and $c_{i(r)} = c_{i(s)} = c_1$. Then by definition,

$$\begin{aligned} \delta(\Phi)(\gamma) := \Phi\{\gamma \cdot \infty \to \infty\} &= \varphi(1)|1 + v_1 - \varphi(\gamma^{-1})|\gamma^{-1} - v_1|\gamma^{-1} \\ &= \varphi(\gamma) + \left[v_1 - v_1|\gamma^{-1}\right], \end{aligned}$$

using that $\varphi(1) = 0 = \varphi(\gamma\gamma^{-1}) = \varphi(\gamma) + \varphi(\gamma^{-1})|\gamma^{-1}$. The term in the square brackets is a coboundary; thus the cocycle $\delta(\Phi)$ represents the same cohomology class as φ.

[1]In practice, this is actually more subtle than it appears: in the linear system, the variables will appear with varying degrees of precision, due to the filtration appearing in the explicit lifting theorem.

It remains to show that Φ is Γ-equivariant. Let $\gamma \in \Gamma$. Note that if $g_r \cdot r = c_{i(r)}$, then $g_r\gamma^{-1} \cdot \gamma r = c_{i(r)}$, so that $c_{i(\gamma r)} = c_{i(r)}$ and $g_{\gamma r} = g_r\gamma^{-1}$. Then

$$\begin{aligned}\widetilde{\Phi}(\gamma r) &= \varphi(g_{\gamma r})|g_{\gamma r} + v_{i(\gamma r)}|g_{\gamma r}\\ &= \varphi(g_r\gamma^{-1})|g_r\gamma^{-1} + v_{i(r)}|g_r\gamma^{-1}\\ &= \varphi(g_r)|g_r\gamma^{-1} + \varphi(\gamma^{-1})|g_r^{-1}g_r\gamma^{-1} + v_{i(r)}|g_r\gamma^{-1}\\ &= \widetilde{\Phi}(r)|\gamma^{-1} + \varphi(\gamma^{-1})|\gamma^{-1}.\end{aligned}$$

The second term is independent of r, so cancels in the difference $\widetilde{\Phi}(s) - \widetilde{\Phi}(r)$. It follows that

$$\Phi\{\gamma r \to \gamma s\}|\gamma = \widetilde{\Phi}(s)|\gamma - \widetilde{\Phi}(r)|\gamma = \Phi\{r \to s\},$$

as required. □

In general, the map Φ thus defined is not an eigensymbol. In particular, whilst there is a unique cuspidal lift Φ_{cusp} of φ under δ, the map Φ can be any element of $\Phi_{\mathrm{cusp}} + \ker(\delta)$. However, we have:

Proposition 4.3. *For all sufficiently large primes $\mathfrak{l}$ of K, coprime to $p\mathfrak{n}$, the symbol*

$$\Phi_{\mathcal{F}} := \frac{T_{\mathfrak{l}} - \ell - 1}{a_{\mathfrak{l}} - \ell + 1}\Phi \in \mathrm{Symb}_\Gamma(\mathbb{D}_2)$$

is the uniquely determined overconvergent (cuspidal) eigensymbol mapped to φ under δ, where ℓ is the norm of $\mathfrak{l}$ and $a_{\mathfrak{l}}$ is the $T_{\mathfrak{l}}$-eigenvalue of the Bianchi modular form $\mathcal{F}$.

Proof. By the long exact sequence given by the snake lemma, the kernel of δ is given by the image of $\mathrm{Hom}_\Gamma(\Delta, \mathbb{D}_2)$ in $\mathrm{Symb}_\Gamma(\mathbb{D}_2)$, or, more precisely, the Eisenstein subspace. We know that for prime $\mathfrak{l} \nmid p\mathfrak{n}$ of norm ℓ, the Hecke operator $T_{\mathfrak{l}}$ acts on the Eisenstein subspace by $\ell + 1$ (see, for example, [PS11, Rem. 5.2] for this in the rational case; more generally, it can be obtained by studying $\mathrm{Hom}(\Delta, \mathbb{D}_2)$ as a Hecke-module). For sufficiently large $\mathfrak{l}$, the Hasse bound implies that $a_{\mathfrak{l}} \neq \ell + 1$. By the remarks above, the operator $T_{\mathfrak{l}} - \ell - 1$ kills any Eisenstein contribution, and thus acts as a projector onto the cuspidal subspace. Now renormalising by $a_{\mathfrak{l}} - \ell - 1$ gives a cuspidal eigensymbol in $\delta^{-1}(\varphi)$. Such a symbol is unique by strong multiplicity one combined with Theorem 2.2. □

4.2.3. Remark: an alternative formula using U_p. In certain cases, there are even more explicit formula – inspired by Prop. 3.1 – for computing p-adic L-functions from the knowledge only of φ (without inverting δ to compute the whole modular symbol Φ). We highlight this only in the rational case for simplicity, though the same ideas go through for Bianchi modular forms too.

Let $\varphi \in \mathrm{H}^1(\Gamma, \mathbb{D}_2)$ be an overconvergent eigenclass. Recall if $\delta(\Phi) = \varphi$, then $\varphi(\gamma) = \Phi\{\gamma\infty \to \infty\}$, and in particular we can directly evaluate Φ at any pair of cusps that are both equivalent to ∞. The idea is then to use the Hecke operators to rewrite our target values – namely, the moments $\Phi\{a/p \to \infty\}((a+pz)^j)$ – in terms of values of $\Phi\{r \to \infty\}$, where each r is Γ-equivalent to ∞, and which we can then read off from φ. For $\Gamma = \Gamma_0(N)$, this is true if and only if $r = a/N$, where a is prime to N. We must thus express the $\Phi\{a/p \to \infty\}$ in terms of values $\Phi\{\beta/N \to \infty\} = \varphi(\gamma)$, where $\gamma = \left(\begin{smallmatrix}\beta & *\\ N & *\end{smallmatrix}\right) \in \Gamma_0(N)$.

Suppose for simplicity that $N = q \neq p$ is prime. Then we have:

Proposition 4.4. *Let d_q denote the order of q in $(\mathbb{Z}/p\mathbb{Z})^\times$, and write a_q for the U_q-eigenvalue of Φ. Then we have*

$$\big(1 - a_q^{-d_q} q^{jd_q}\big)\Phi\left\{\tfrac{a}{p} \to \infty\right\}((a+pz)^j) = \\ a_p^{-1}\sum_{m=0}^{d_q-1} a_q^{-m} q^{jm} \sum_{\substack{\beta\in(\mathbb{Z}/Np\mathbb{Z})^\times \\ \beta\equiv a/q^m \pmod p}} \Phi\left\{\tfrac{\beta}{qp} \to \infty\right\}\big((\beta+qpz)^j\big).$$

We will not give the full proof, but instead observe that one obtains this by applying the operator $\alpha_q^{-1}U_q$, which acts as the identity, to force q to be in the denominator. At each iteration, there is one problem term where the q in the denominator cancels, and we end up with only p (rather than pq) in the denominator. Iterating this process at the problem term, after d_q iterations one arrives at the original expression $\Phi\{a/p \to \infty\}((a+pz)^j)$ multiplied by $a_q^{-d_q}q^{jd_q}$, giving the claimed formula.

We can now find an element $\gamma = \left(\begin{smallmatrix}\beta & * \\ pq & *\end{smallmatrix}\right) \in \Gamma_0(N)$, and $\gamma\infty = \beta/pq$; thus $\Phi\{\beta/pq - \infty\} = \varphi(\gamma)$ and we can compute this directly, without inverting δ. Similar formula hold for N a product of primes, and also in the Bianchi setting, where now we consider moments attached to a pair of integers (j,k).

Remark 4.5: Note that if $j \neq 0$ (or, in the Bianchi setting, when either $j \neq 0$ or $k \neq 0$) the scalar $1 - a_q^{-d_q}q^{jd_q}$ is always non-zero, and thus we can always recover the *higher* moments required to define the p-adic L-series without inverting δ. However, when $j = 0$ (or $j = k = 0$), it is possible for the scalar to vanish. In particular, if f is attached to an elliptic curve E of rank 1 with good reduction at p, then the sign of the functional equation forces the existence of some prime q such that $a_q(E) = 1$, and then the scalar vanishes. This leads to the curious phenomenon where we can compute all but the first moment; and yet the first moment is precisely the value of the *classical* modular symbol we started with. Thus via this method, in this case we can compute all of the overconvergent moments, but not the classical starting moment!

5. The algorithms

5.1. Computing with the classical cohomology

There are well-established routines for computing classical cohomology classes attached to elliptic curves over number fields, as used by the first author and his collaborators in in [CITE MARC'S PAPERS]. These rely on computing a presentation of the group Γ, using existing Magma code of Aurel Page to compute a fundamental domain for the action of Γ on the upper half-space $\mathcal{H}_3$. Given this, $\mathrm{H}^1(\Gamma,\mathbb{Z})$ is the abelianisation of Γ, and distinguished classes are constructed by computing the kernels of sufficiently many operators $T_{\mathfrak{q}} - \alpha_{\mathfrak{q}}$.

In practice, working with Bianchi groups is a difficult computational problem, and one that greatly restricts the scope of computations. This is particularly the case in the base-change setting. Consider, for example, the smallest elliptic curve of rank 1, which has conductor 37, and say we want to compute the 3-adic L-function; then we must work with the group $\Gamma_0(3\cdot 37\mathcal{O}_K)$, which has index $\sim 12{,}000$ in $\mathrm{SL}_2(\mathcal{O}_K)$. The first rank 2 example has conductor 389, leading to index $\sim 1,300,000$, far beyond current state-of-the-art (for example, in [?], the highest conductors appearing for Bianchi groups are of index 500,000).

To counter this computational difficulty, we used ...[DESCRIBE? OR LEAVE OUT?]

5.2. Computing with distributions

Two variable distributions with coefficients in an extension $L/\mathbb{Q}_p$ are naturally in bijection with doubly-indexed bounded sequences in L (see [Wil17, Proposition 3.6]), the map being given by

$$\mu \mapsto \{\mu(x_i y_j) : i, j \geq 0\}.$$

We can compute with these distributions using the *finite approximation modules op. cit.*.

5.2.1. Computing the action of $\Sigma_0(p)^2$ *on distributions.* It is important to explicitly understand the action of $\Sigma_0(p)^2$ on these distributions. In particular, there are an increasing number of conventions regarding actions (both left and right) on distributions, and we wish to be as clear as possible about the ones we adopt.

Definition 5.1. Consider the usual basis $\{x^i y^j : i, j \geq 0\}$ of $\mathbb{A}(L)$, and order it first by total degree $i + j$ and then lexicographically in j and then i. To make this explicit, we label these basis monomials by defining

$$v_{n,i} := x^{n-i} y^i,$$

which are then ordered $v_{0,0}, v_{1,0}, v_{1,1}, v_{2,0}, v_{2,1}, \ldots$, corresponding to $1, x, y, x^2, xy, \ldots$.

Recall that $\Sigma_0(p)^2$ acts on the *left* of $\mathbb{A}(L)$ by

$$g \cdot f(x,y) = f\left(\frac{b+dx}{a+cx}, \frac{\overline{b}+\overline{d}x}{\overline{a}+\overline{c}x}\right), \quad g = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}\right].$$

Denote by $\psi_{\mathbb{A}}(g)$ the matrix of g acting on $\mathbb{A}(L)$ in the basis $\{v_{n,i}\}$.

We want to compute the dual action on distributions. The natural dual action is a *right* action. However, since we will ultimately compute with the group cohomology – which prefers *left* actions – it is convenient here to pass to the right action on $\mathbb{A}_2(L)$ defined by $f|g = g^{-1} \cdot f$, inducing a left dual action.

Now write

$$v_{n,i}^* := \mathcal{X}^{n-i} \mathcal{Y}^i$$

for the dual basis, where $v_{n,i}^*(v_{m,j}) = \delta_{mn}\delta_{ij}$. For , then the matrix of g acting on $\mathbb{D}(L)$, on the *left*, in this dual basis, is given by

$$\psi_{\mathbb{D}}(g) := \psi_{\mathbb{A}}(g^{-1})^T.$$

We are reduced, then, to computing the action on $\mathbb{A}$.

In practice, we use the following algorithm. Define

$$r = \frac{b+dx}{a+cx}, \quad s = \frac{\overline{b}+\overline{d}y}{\overline{a}+\overline{c}x}.$$

We then start by considering $r^0 s^0 = 1$. Suppose we have computed $r^{n-i}s^i$ for all $i \in \{0, ..., n\}$. The coefficients of $r^{n-i}s^i$ give the column (at $v_{n,i}$) of the matrix of g acting on $\mathbb{A}$. We then compute at step $n+1$ by multiplying by r or s as necessary, computing up to the desired precision.

5.3. An explicit lifting theorem

Including commutative diagram relating arithmetic cohomology and modular symbols with the action of the $U_{\mathfrak{p}}, U_{\overline{\mathfrak{p}}}$ operators. (Not well-defined on cocycles in general).

Complexity in computing this: lots of Up operators. Fox derivative and gradient.

6. Data

[Include our data, in some form or other]

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