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R: Any complete local ring

$R[[y]]$.

Given: $f \in R[[y]]$, a Weierstrass power series,

λ: λ-invariant of f

$$\left(f = \sum_{n=0}^{\infty} a_n y^n , \quad \lambda = \min \{ n , a_n \in R^{\times} \} \right).$$

$$S := S_{\lambda} \left(\sum_{n=0}^{\infty} b_n y^n \right) = \sum_{n=0}^{\infty} b_{n+\lambda} y^n$$

Y_{prec}: desired precision of y, (y-adic precision)

P_{prec}: desired precision of P (p-adic precision).

Set $M = Y_{prec} + P_{prec}$

Algorithm :

- Set $f_0 = f$

- Step i : $f_i = \dfrac{f_{i-1}}{S(f_{i-1})}$

- Repeat till $i = M$

- Truncate the power series f_M upto $\deg(y) = \lambda$.
(i.e. look at the first $\lambda+1$ terms).

f_M is the desired

③

Why does this work?

Since S_n is an R-linear map

$$S_n : R[[y]] \to R[[y]],$$

we have the following observation:

i) For any $g \in R[[y]]$ a Weierstrass power series, we have

$$\frac{g}{S_{\lambda_g}(g)} = \frac{\sum_{n=0}^{\infty} b_n y^n}{\underbrace{b_{\lambda_g} + \sum_{n=1}^{\infty} b_{n+\lambda_g} y^n}_{\text{unit}}}$$

y^{λ_g} term is $1 \bmod m_{R[[y]]}$

$$\Rightarrow S\left(\frac{g}{S_{\lambda_g}(g)}\right) \equiv 1 \pmod{m_{R[[y]]}}$$

$\|$

As a result, $\Downarrow$

$$\lambda\left(\frac{g}{S_{\lambda_g}(g)}\right) = \lambda_g .$$

(for our algorithm, this means that

$$\lambda_{f_i} = \lambda_f$$

So $S_{\lambda_{f_i}} = S_{\lambda_f}$

)

<u>Proposition</u> : $S_{f_i} \equiv 1 \mod m^{i}_{R[i,j]}$

Assume this proposition.

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Corollary 1:

$\{f_i\}$ form a Cauchy sequence.

Pf:

$$f_{i+1} - f_i = \frac{f_i}{S(f_i)} - f_i = \frac{f_i}{S(f_i)}(1 - S(f_i))$$

$$\in m^i_{R[[y]]}$$

So, if $n_1, n_2 \geq j$

$$(f_{n_1} - f_{n_2}) \in m^j_{R[[y]]}.$$

Let $f_\infty = \lim_{i\to\infty} f_i$.

$$S(f_\infty) = \lim_{i\to\infty} S(f_i) = 1.$$

$\Rightarrow$

⑥

Corollary 2: f_∞ is a polynomial of degree λ_g.

Corollary 3: $f = f_\infty$ (unit).

Observe: $\dfrac{f_{i-1}}{f_i} = S(f_{i-1}) \in 1+m_{R[[y]]}$

$\Rightarrow f = f_1$ (unit) $\in 1+m_{R[[y]]}$

$f_1 = f_2$ (unit) $1+m_{R[[y]]}$ $\Rightarrow f = f_2$ (unit) $1+m_{R[[y]]}$

$\Rightarrow f = f_i$ (unit).

$\Rightarrow \lim\limits_{i\to\infty} \dfrac{f}{f_i} \in 1+m_{R[[y]]}$ ($1+m_{R[[y]]}$ is complete).

$\Rightarrow f = f_\infty$ (unit)

is $1+m_{R[[y]]}$

So, f_∞ is the desired polynomial. (7)

Proof of Proposition:

By induction, (case $i=1$, was shown earlier when we showed that the λ-invariants don't change).

Suppose $S(f_i) = 1 + m_i$, (for some m_i in $m^i_{R[[y]]}$)

then

$$f_{i+1} = \frac{f_i}{S(f_i)} = \frac{h_{n-1} + y^\lambda S(f_i)}{S(f_i)}$$

$$= y^\lambda + h_{n-1}(1 + r_i)$$

belong to $m^i_{R[[y]]}$

$1 + r_i = (1 + m_i)^{-1}$

(polynomial of degree $\leq n-1$,
all of its coefficients (by def'n of λ-invariant)

belong to max. ideal of $\mathcal{O}$.

$$= y^{\lambda} + h_{n-1} + h_{n-1} r_i$$

$$\left(S(h_{n-1} r_i) \in m^{i+1} \right).$$

So,

$$S(f_{i+1}) = S(y^{\lambda}) + S(h_{n-1} r_i)$$

$$= 1 + S(h_{n-1} r_i) \in 1 + m^{i+1}.$$

$\square$.