R: Noetherian ring.

A f.g. torsion R-module M is said to be pseudo-null if

$M_P = 0$ for all height one primes P in R.

e.g. if $R = \mathbb{Z}_p[[x_1, x_2]]$, then

$\frac{\mathbb{Z}_p[[x_1, x_2]]}{(p, x_1, x_2)}$ is P.N. (supported only at the max ideal which has height 3)

$\frac{\mathbb{Z}_p[[x_1, x_2]]}{(x_1, x_2)} \cong \mathbb{Z}_p$ ← Not finite. (supported only at the height two prime (x_1, x_2))

Let F : # field

S : finite set of primes containing the primes above p and ∞.

F_S : maximal S-ramified extension of F.

Given

$\rho : Gal(F_S/F) \longrightarrow GL_d(R)$ say some finite ext of $\mathbb{Z}_p[[x_1,..,x_n]]$

a Galois representation

L : an algebraic extension of F.

T_ρ : free R-module of rank d, that gives us ρ

D_ρ : $T_\rho \otimes_R Hom_{cont}(R, \mathbb{Q}_p/\mathbb{Z}_p)$.

$H^1(F_S/L, D_\rho)$: Global cohomology group

$Ш^1(F_S/L, D_\rho)$: fine Selmer group

③

defined to be the subgroup of
$H^1(F_S/L, D_\rho)$ that is
"locally trivial" at all primes above L.

Question: When is (under what conditions on L, ρ etc?)
the $R[[Gal(L/F)]]$-module $Ш^1(F_S/L, D_\rho)^\vee$ a
pseudo-null module?

say $Gal(L/F)$ is a p-adic Lie group

Remark 1. If F : # field
L : compositum of all $\mathbb{Z}_p$-extensions
ρ : trivial character values in $\mathbb{Z}_p$.
S : contains all primes above p and ∞ only.

then Greenberg has conjectured that
$Ш^1(F_S/L, D_\rho)^\vee$ is a p.n. $\mathbb{Z}_p[[Gal(L/F)]]$-module

Leopoldt $\Rightarrow \cong \mathbb{Z}_p[[X_1, \dots, X_{r_2+1}]]$

Remark 2: The Iwasawa algebra $\mathbb{Z}_p[[Gal(L/F)]]$ can be non-commutative. There is notion of pseudo-null modules for non-commutative rings due to Venjakob.

Remark 3: If F : # field

$E_{/F}$: elliptic curve

S : finite set of primes, containing p, ∞, bad primes of E

L : algebraic extension of F s.t.

- $F_{cyc} \subseteq L \subseteq S$
- $Gal(L/F)$ is a p-adic Lie group with dimension ≥ 2,

$\rho : Gal(F_S/F) \to GL_2(\mathbb{Z}_p)$: p-adic Tate module of E

then Coates-Sujatha have conjectured that $Ш^1(F_S/L)^\vee$ is a p.n. $\mathbb{Z}_p[[Gal(L/F)]]$-module.

Remark. By Shapiro's lemma, we can express all of this for Galois representations over $Gal(F_S/F)$, but with values in $GL_d(\mathbb{Z}_p[[Gal(L/F)]])$.

Motivation for

Our project: Provide evidence when

$F = K$: imaginary quadratic

p splits.

$E_{/K}$: elliptic curve

$\tilde{K}_\infty = L$: compositum of $\mathbb{Z}_p^2$-extension of K.

Conjecture: $Ш^1(K_S/L, E[p^\infty])^\vee$ is a pseudo-null $\mathbb{Z}_p[[Gal(L/K)]]$-module

$$\mathbb{Z}_p[[Gal(L/K)]] \cong \mathbb{Z}_p[[x_1, x_2]].$$

<u>Strategy</u>: In this setup.

There are

$$\mathrm{Sel}^{\pm\pm}(K_S/K_\infty^{n}, E[p^\infty]) \subseteq H^1(K_S/K_\infty^{n}, E[p^\infty])$$

$$\mathrm{Sel}^{Gr}(K_S/K_\infty^{n}, E[p^\infty])$$

such that

- for all primes $v \in S$, not lying above p, the cocycles are locally trivial
- some condition (Gr or $\pm,\pm$) at primes above p.

So, we have inclusions

$$Ш^1(K_S/K_\infty^{n}, E[p^\infty]) \subseteq \begin{matrix} \mathrm{Sel}^{Gr}(K_S/K_\infty^{n}, E[p^\infty]) \\ \mathrm{Sel}^{\pm\pm}(K_S/K_\infty^{n}, E[p^\infty]) \end{matrix}$$

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On the dual side, we have surjections

$$\mathrm{Sel}^{Gr}(K_S/K_\infty^{n}, E[p^\infty])^\vee \twoheadrightarrow Ш^1(K_S/K_\infty^{n}, E[p^\infty])^\vee.$$

$$\mathrm{Sel}^{\pm\pm}(K_S/K_\infty^{n}, E[p^\infty])^\vee \twoheadrightarrow$$

So,

$$\mathrm{Supp}_{ht=1}\left(\mathrm{Sel}^{\pm\pm}(K_S/K_\infty^{n}, E[p^\infty])^\vee\right) \supseteq \mathrm{Supp}_{ht=1}\left(Ш^1(K_S/K_\infty^{n}, E[p^\infty])^\vee\right)$$

$$\mathrm{Supp}_{ht=1}\left(\mathrm{Sel}^{Gr}(K_S/K_\infty^{n}, E[p^\infty])^\vee\right)$$

To prove, $Ш^1(K_S/K_\infty^{n}, E[p^\infty])^\vee$ is p.n.

$$\Updownarrow$$

$$\mathrm{Supp}_{ht=1}\left(Ш^1(K_S/K_\infty^{n}, E[p^\infty])^\vee\right) = \phi$$

it suffices to show (say) (not necessary)

$$(**)\quad \mathrm{Supp}_{ht=1}\left(\mathrm{Sel}^{++}(K_S/K_\infty^{n}, E[p^\infty])^\vee\right) \cap \mathrm{Supp}_{ht=1}\left(\mathrm{Sel}^{+-}(K_S/K_\infty^{n}, E[p^\infty])^\vee\right) = \phi$$

(Any pair, other than $(++, +-)$ would also do).

Suppose the Iwasawa Main Conjectures.

Let $? \in \{(\pm\pm), G_n\}$.

Then

$$\mathrm{Char}\left(\mathrm{Sel}^{?}(K_S/K_\infty^{n}, E[p^\infty])^\vee\right) = (\Theta^{?}).$$

So, to prove $(**)$,

assuming the main conjectures,

Suffices to show

Θ^{++} and Θ^{+-} have no common irreducible factor.

(Since, assuming the M.C.

$\mathrm{Supp}_{\mathrm{ht}=1}\left(\mathrm{Sel}^{?}(K_S/K_\infty^{n}, E[p^\infty])^\vee\right)$

equals all the prime ideals generated by irreducible dividing $\Theta^{?}$)