Follow the notations of previous scans.

Start with the isomorphism of Galois representations:

$$\boxed{\begin{aligned} &\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\Big[\underbrace{\mathrm{Res}^{G_{\mathbb{Q}}}_{G_K}(V_g)\otimes_L L(\psi_p\varepsilon)}_{\text{Bianchi}}\Big]. \\ &\cong \underbrace{V_g\otimes_L V_f}_{\text{Rankin-Selberg}} \end{aligned}}$$

(Both) This Galois representation(s) satisfies the Panchishkin condition of Greenberg.

$$\left(\quad\begin{tikzcd}[column sep=small] & \hat{K}_\infty \arrow[dl, dash] \arrow[dr, dash] & \\ K_{cyc} \arrow[dr, dash] & K_p \arrow[d, dash] & K_{ac} \arrow[dl, dash] \\ & K & \end{tikzcd} \qquad \begin{aligned} \hat{\Gamma} &= \mathrm{Gal}(\hat{K}_\infty/K) \\ \Gamma_{cyc} &= \mathrm{Gal}(K_{cyc}/K) \\ \Gamma_{ac} &= \mathrm{Gal}(K_{ac}/K) \end{aligned}\quad\right)$$

Bianchi-perspective: For the Galois rep over G_K

$\mathrm{Res}^{G_{\mathbb{Q}}}_{G_K}(V_g) \otimes L(\mathcal{E})$ (without the Φ_p twist)

$\mathcal{L}_{\mathrm{Bianchi}}$ contains the homomorphisms:

$$\phi : \tilde{\Gamma} \longrightarrow \overline{\mathbb{Q}}_p^{\times}$$
$$\tilde{\Gamma} \cong \Gamma_{cyc} \times \Gamma_{\mathfrak{p}}$$

(via restriction maps $\tilde{\Gamma} \twoheadrightarrow \Gamma_{cyc}$, $\tilde{\Gamma} \twoheadrightarrow \Gamma_{\mathfrak{p}}$)

given by

$$\varepsilon_{cyc} : \Gamma_{cyc} \to \overline{\mathbb{Q}}_p^{\times} \qquad \varepsilon_{\mathfrak{p}} : \Gamma_{\mathfrak{p}} \to \overline{\mathbb{Q}}_p^{\times}$$

finite characters

Denote this 2-variable p-adic L-function by $\Theta_{\mathrm{Bianchi}}$.

in fraction field of $\mathcal{O}_g[[\tilde{\Gamma}]]$.

(will discuss incorporating twists of $\Gamma_{\mathfrak{p}}$ at the end).

3

Some generalities for Rankin Selberg p-adic L-functions.

Suppose F, G are two Hida families

R_F, R_G are the normalizations of the irreducible components of Hida's ordinary Hecke algebra through F, G resp.

The p-adic L-functions

$L_p(\underline{F}, G, s)$ and $L_p(F, \underline{G}, s)$

are elements of $R_F \hat{\otimes} R_G [[\Gamma_{cyc}]]$.

For the p-adic L-function

$$L_p(\underline{F}, G, s)$$

where F is the dominant Hida family, the critical set

④

$C^{(F)}_{R.S.}$ is the set of homomorphisms

$$\phi : R_F \hat{\otimes} R_G [[\Gamma_{cyc}]] \longrightarrow \overline{\mathbb{Q}}_p$$

classical specializations

given by

$$\phi_{k,F} : R_F \longrightarrow \overline{\mathbb{Q}}_p, \quad \phi_{l,G} : R_G \to \overline{\mathbb{Q}}_p, \quad \phi_{cyc} : \Gamma_{cyc} \to \overline{\mathbb{Q}}_p^{\times}$$

with $\phi_{cyc} = \left(\chi_{p,cyc}\, \omega^{-1}\right)^{-s}$

p-adic cyclotomic character

Teichmüller character

with the condition

$$wt(\phi_{k,F}) > s+1 \geq wt(\phi_{l,G})$$

Similarly, define $C^{(G)}_{R.S.}$ for $L_p(F, \underline{G}, s)$.

And similarly if you specialize for eigenforms

f_k and / or g_k

to get two variable / one-variable p-adic L-functions.

Remark:

i) Suppose f is any eigenform (ordinary or no), then

$L_p(\underline{\underline{f}}, G, s)$ is defined!

ii) If f has weight two, for $L_p(\underline{\underline{f}}, G, s)$ to be defined, G needs to have infinitely many weight one specializations.

This is only possible if G is a C.M. Hida family.

Now consider the case when

$\underline{E}$: elliptic curve (so wt = 2).

G_K : C.M. Hida family corresponding to the Galois representation

$$\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi_p)$$

where $\chi_p : G_K \twoheadrightarrow \Gamma_p \hookrightarrow GL_1(\mathbb{Z}_p[[\Gamma_p]])$ is the tautological character

And the critical set for the

$$L_p(\underline{\underline{E}}, G_K, s)$$ → Rankin - Selberg

matches with the critical set for the corresponding Bianchi p-adic L-function. $\chi_p(\Theta_{Bianchi})$.

(I haven't worked out the exact twist here. Not sure sorry.)

There must be a relation (conjecturally, at least) between these p-adic L-functions, since they correspond to the same Galois representation, (=> the complex L-functions are the same)

they interpolate at the same critical set

(=> At least conjecturally, the corresponding complex and p-adic periods must be related.)