Computing Bianchi p-adic L-functions and applications in Iwasawa theory

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Abstract

Using overconvergent cohomology of Bianchi groups, we present an algorithm that explicitly computes the 2-variable p-adic L-function associated to a weight two cuspidal eigenform with small slope over an imaginary quadratic field with class number one. The algorithm uses earlier work of the third author establishing a constructive proof of a control theorem for overconvergent lifts of Bianchi modular forms. Our implementation of this algorithm builds on efficient code built by the first author, Guitart and Sengun that computes 1-variable overconvergent lifts. Our implementation allows us to compute numerical examples of interest in the topic of higher codimension Iwasawa theory, supporting a pseudo-nullity conjecture of Coates–Sujatha.

Contents

1. Introduction

The *Bloch–Kato conjectures* predict a sweeping and very deep relationship between special values of L-functions and arithmetic Selmer groups. Some of the most striking successes in studying this connection have come from Iwasawa theory, and in particular via the study of the *Iwasawa–Greenberg main conjecture.* This is a p-adic reformulation of Bloch–Kato in which we replace the classical L-function with a *p-adic L-function.* The Iwasawa–Greenberg main conjecture is then phrased as an equality of codimension one cycles, predicting that the p-adic L-function generates the characteristic ideal of the Pontryagin dual of a Selmer group. In view of this connection, it is desirable to explicitly compute p-adic L-functions. Remarkably, these explicit numerical computations have found spectacular applications in the study of the Birch and Swinnerton-Dyer conjecture and, in particular, in the construction of explicit points on rational elliptic curves.

While the Iwasawa–Greenberg main conjecture predicts an equality of codimension one cycles, there has recently been much interest in studying *higher codimension* Iwasawa theory, and it is in this framework that our manuscript sits. Several of the main results ([BCG$^+$19, Theorem 5.2.5], [LP19a, Theorem 1], [LP19b, Theorem 1]) in higher codimension Iwasawa theory seek to generalize the Iwasawa–Greenberg main conjectures and are phrased as an equality of codimension two cycles. These aforementioned results are established under the validity of certain pseudo-nullity (a specific structural property) conjectures. We consider applications of our numerical computations of 2-variable Bianchi p-adic L-functions to support a pseudo-nullity conjecture of Coates-Sujatha [CS05, Conjecture B]. It involves the *fine Selmer group*, which is a subgroup of the usual Selmer group that appears in the formulation of the Iwasawa-Greenberg main conjectures. The origins of the idea behind this conjecture date back to pseudo-nullity conjectures of Greenberg [Gre76], [Gre01, Conjecture 3.5] involving classical Iwasawa modules. The definition of pseudo-nullity and the statement of the pseudo-nullity conjecture of Coates–Sujatha (Conjecture 1.2) are recalled in Section 1.3. To support Conjecture 1.2 using our numerical computations, we exploit a method appearing in work of Lei and the second author [LP19a]. In the settings of our supporting numerical examples, there is naturally more than one distinct p-adic L-function to consider and we show that these p-adic L-functions have no common irreducible factor in a 2-variable Iwasawa algebra. In fact, it is under the validity of this criterion in our method on no common irreducible factors that the aforementioned results in higher codimension Iwasawa theory are established.

Although there has been almost no theoretical progress towards a general proof of pseudo-nullity conjectures (both of Greenberg and Coates–Sujatha), even a general methodology towards obtaining numerical evidence is lacking. The only exception that we know of is work of McCallum–Sharifi [MS03] towards Greenberg's pseudo-nullity conjecture specialized to the number field $\mathbb{Q}(\mu_p)$. Their methodology in turn is based on a deep conjecture about a cup-product pairing introduced by them. In [LP19a], to produce numerical examples using the criterion of no common irreducible factors, the authors had to restrict to the case when the Mordell-Weil rank of the elliptic curve over the base field (say r) was zero. Along with an additional hypothesis, this allowed them to reduce to the "1-*dimensional case*", where they could use an earlier construction of 1-variable cyclotomic p-adic L-functions [PS11]. Such a reduction would not work if $r \geq 1$. In the formulation of [CS05, Conjecture B], the Galois extension underlying the Iwasawa algebra is an *admissible p-adic Lie extension* with p-adic Lie dimension $d \geq 2$. In fact, Coates–Sujatha discuss a closely related phenomenon with $d = 1$ when pseudo-nullity fails. With this point in mind, the phenomenon of pseudo-nullity seems to be an artefact of working over admissible p-adic Lie extensions with $d \geq 2$ and the reduction to the "1-dimensional case" might not always succeed. This is an important motivation behind our emphasis on computing 2-variable p-adic L-functions. Our numerical examples are chosen (so that $r \geq 1$) to illustrate our methodology with precisely the closely-related exceptional cases in mind.

Notation. We introduce the notations required to describe our algorithm that computes 2-variable Bianchi p-adic L-functions. Let p be an odd prime number. Let K be an imaginary quadratic field[1] with class number one, where the prime p splits as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Let $\mathcal{O}_K$ denote the ring of integers of K. Let $K_{\mathfrak{p}}$, $K_{\overline{\mathfrak{p}}}$ denote the unique $\mathbb{Z}_p$ extensions of K unramified outside $\mathfrak{p}$, $\overline{\mathfrak{p}}$ respectively. Let $G_{\mathfrak{p}}$, $G_{\overline{\mathfrak{p}}}$ denote $\mathrm{Gal}(K_{\mathfrak{p}}/K)$, $\mathrm{Gal}(K_{\overline{\mathfrak{p}}}/K)$ respectively. Let $\widetilde{K}$ denote the maximal abelian extension of K, unramified outside p. Let K_∞ denote the compositum of all the $\mathbb{Z}_p$-extensions of K. Let $\widetilde{G}$, G_∞ denote $\mathrm{Gal}(\widetilde{K}/K)$, $\mathrm{Gal}(K_\infty/K)$ respectively. Let Δ denote $\ker(\widetilde{G} \twoheadrightarrow G_\infty)$. Since the class number of K equals one, the natural restriction maps $G_\infty \twoheadrightarrow G_{\mathfrak{p}}$ and $G_\infty \twoheadrightarrow G_{\overline{\mathfrak{p}}}$ allow us to identify G_∞ with $G_{\mathfrak{p}} \times G_{\overline{\mathfrak{p}}}$. Recall that we have the following isomorphisms of topological groups:

$$\widetilde{G} \cong \Delta \times G_\infty, \qquad G_\infty \cong \mathbb{Z}_p^2, \qquad G_{\mathfrak{p}} \cong \mathbb{Z}_p, \qquad G_{\overline{\mathfrak{p}}} \cong \mathbb{Z}_p.$$

The abelian group Δ is finite, of exponent dividing $p-1$. For each character ψ in $\mathrm{Hom}(\Delta, \mathbb{Z}_p^\times)$, we have its corresponding idempotent e_ψ in the group ring $\mathbb{Z}_p[\Delta]$. These characters are often called *branch characters*. The topological $\mathbb{Z}_p$-algebra $e_\psi\mathbb{Z}_p[[\widetilde{G}]]$ is isomorphic to $\mathbb{Z}_p[[G_\infty]]$. We have the following natural isomorphism of topological $\mathbb{Z}_p$-algebras:

$$\begin{aligned} \mathbb{Z}_p[[\widetilde{G}]] &\cong \mathbb{Z}_p[[G_\infty]][\Delta] \\ &\cong \prod_{\psi \in \mathrm{Hom}(\Delta, \mathbb{Z}_p^\times)} e_\psi \mathbb{Z}_p[[\widetilde{G}]] \end{aligned} \tag{1.1}$$

Henceforth, we shall fix a branch character $\psi : \Delta \to \mathbb{Z}_p^\times$ and simply denote the corresponding component $e_\psi\mathbb{Z}_p[[\widetilde{G}]]$ by $\mathbb{Z}_p[[G_\infty]]$. Let us choose a topological generator $g_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$. Under the action of complex conjugation on $g_{\mathfrak{p}}$, one obtains a topological generator $g_{\overline{\mathfrak{p}}}$ of $G_{\overline{\mathfrak{p}}}$. This further allows to consider the following isomorphism of topological $\mathbb{Z}_p$-algebras:

$$\begin{aligned} \mathbb{Z}_p[[G_\infty]] &\cong \mathbb{Z}_p[[x,y]], \\ (g_{\mathfrak{p}}, 1) &\to x+1, \\ (1, g_{\overline{\mathfrak{p}}}) &\to y+1 \end{aligned} \tag{1.2}$$

Throughout this paper, we shall fix the isomorphism in (1.2) and denote the 2-variable power series ring $\mathbb{Z}_p[[x,y]]$ appearing in (1.2) by Λ. Let $\chi : \mathrm{Gal}\left(\overline{K}/K\right) \to \mathbb{Z}_p^\times$ be a finite continuous character with conductor prime to p. In the main text, ψ and χ were switched! I've changed the introduction accordingly. This convention matches [PS11] — Chris ♠ Let $\kappa : \mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \widetilde{G}_\infty \to \mathrm{GL}_1(\Lambda)$ denote the tautological character.

Let $\mathfrak{n}$ be an ideal in $\mathcal{O}_K$ satisfying the following condition:

[1] so K equals $\mathbb{Q}(\sqrt{-D})$, where D belongs to $\{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

(p-STAB) — $\mathfrak{n} \subset p\mathcal{O}_K$.

Let Γ denote the congruence subgroup $\Gamma_0(\mathfrak{n})$ of $\mathrm{SL}_2(\mathcal{O}_K)$. Let f be a weight 2 Bianchi cuspidal eigenform of level Γ with Hecke eigenvalues $\{a_I \in \mathbb{Z}_p : I \subset \mathcal{O}_K\}$, satisfying the following condition:

(SMALL SLOPE) — The p-adic valuations of the eigenvalues $a_{\mathfrak{p}}$, $a_{\overline{\mathfrak{p}}}$ for the actions of the $U_{\mathfrak{p}}$, $U_{\overline{\mathfrak{p}}}$ operators on f are strictly less than 1.

Let $\mathbb{D}_2(\mathbb{Q}_p)$ denote the space of $\mathbb{Q}_p$-valued p-adic distributions on $\mathbb{Z}_p^2$, endowed with a *weight two* action of Γ. Let $\mathbb{D}_2^0(\mathbb{Z}_p)$ denote the subspace of $\mathbb{Z}_p$-valued measures (bounded distributions) on $\mathbb{Z}_p^2$, also endowed with a weight two action of Γ. Let $\mathrm{Symb}_\Gamma(\mathbb{D}_2(\mathbb{Q}_p))$ denote the corresponding space of overconvergent modular symbols valued in $\mathbb{D}_2(\mathbb{Q}_p)$. There is a natural action of the Hecke algebra (of weight two and level Γ) on $\mathrm{Symb}_\Gamma(\mathbb{D}_2(\mathbb{Q}_p))$. Since f satisfies (SMALL SLOPE), a control theorem [Wil17, Thm. 6.10] shows that there exists a non-zero element Φ_f in $\mathrm{Symb}_\Gamma(\mathbb{D}_2(\mathbb{Q}_p))$, unique up to a scalar in $\mathbb{Q}_p$, that is stable under the action of the Hecke algebra and whose Hecke eigenvalues coincide with those of f. Associated to the tuple (p, f, ψ, χ), there exists a (locally analytic) distribution $\mu_{f,\psi,\chi}$ in $\mathbb{D}_2(\mathbb{Q}_p)$, built using Φ_f, which interpolates the critical L-values of the modular form f. We elucidate this property in Section 2.1.1.

1.1. Main Aim. The starting point of our algorithm is a precise formula, given in equation (3.3), that computes *moments* of the distribution $\mu_{f,\psi,\chi}$ in terms of the overconvergent modular symbol Φ_f. To illustrate in the introduction the main output of our algorithm and keeping in mind the applications to Iwasawa theory in Section 1.3, we will place ourselves in the following simplifying situation:

(BASE-CHANGE) — f equals the cuspidal Bianchi eigenform corresponding to the base change to K of the weight two p-stabilized cuspidal eigenform over $\mathbb{Q}$ associated to an elliptic curve E, defined over $\mathbb{Q}$.

Let $T_p(E)$ denote the p-adic Tate module associated to $E/\mathbb{Q}$. One can consider a 4-dimensional Galois representation $\tau : G_{\mathbb{Q}} \to \mathrm{GL}_4(\Lambda)$, given by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathfrak{L}_\tau := \mathrm{Ind}_K^{\mathbb{Q}}\left(\mathrm{Res}_K^{\mathbb{Q}} T_p(E) \otimes_{\mathbb{Z}_p} \Lambda(\kappa^{-1})\right)$. Observe that $\mathfrak{L}_\tau$ is a free Λ-module of rank 4. Let $\mathfrak{D}_\tau := \mathfrak{L}_\tau \otimes_\Lambda \mathrm{Hom}_{\mathrm{cont}}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$ denote the associated discrete Λ-module.

(Ord): When E has good ordinary reduction at p. In this case, it is known that $\mu_{f,\psi,\chi}$ belongs to $\mathbb{D}_2^0(\mathbb{Z}_p)$. Recall that the Amice transform provides an isomorphism $\mathbb{D}_2^0(\mathbb{Z}_p) \cong \Lambda$ of topological $\mathbb{Z}_p$-algebras. The purpose of the algorithm in our paper in this case is to compute the Amice transform, denoted $L_{p,f,\psi,\chi}(x,y)$, of $\mu_{f,\psi,\chi}$. To wit, we have

$$\begin{aligned} \mathbb{D}_2^0(\mathbb{Z}_p) &\cong \Lambda, \\ \mu_{f,\psi,\chi} &\leftrightarrow L_{p,f,\psi,\chi}(x,y). \end{aligned}$$

Corresponding to the fact that E has good ordinary reduction at p, one observes that the Galois representation τ satisfies Greenberg's Panchishkin condition[2] and one can formulate an *Iwasawa-Greenberg main conjecture* associated to τ. See [Gre94] for more details. The application to Iwasawa theory stems from the fact that it is $L_{p,f,\psi,\chi}(x,y)$

[2] In fact, τ satisfies another distinct Panchishkin condition since p splits in K.

that appears in the formulation of this conjecture, and is conjectured to generate the characteristic divisor in Λ of the Pontryagin dual of a Selmer group (which we denote $\mathrm{Sel}(\mathbb{Q},\mathfrak{D}_\tau)^\vee$).

(SS): When E has good supersingular reduction at p with $a_p(E) = 0$. In this case, the distribution $\mu_{f,\psi,\chi}$ does not belong to $\mathbb{D}_2^0(\mathbb{Z}_p)$. However, by suitably modifying this distribution (following an approach of Pollack [Pol03] in the 1-variable case over $\mathbb{Q}$), Loeffler [Loe14] has constructed four distributions $\mu^{\pm,\pm}_{f,\psi,\chi}$ in $\mathbb{D}_2^0(\mathbb{Z}_p)$. These distributions are referred to as *doubly-signed* p-adic L-functions. The purpose of the algorithm in our paper in this case is to compute the Amice transform, denoted $L^{\pm,\pm}_{p,f,\psi,\chi}(x,y)$, of $\mu^{\pm,\pm}_{f,\psi,\chi}$. To wit, we have

$$\begin{aligned} \mathbb{D}_2^0(\mathbb{Z}_p) &\cong \Lambda, \\ \mu^{\pm,\pm}_{f,\psi,\chi} &\leftrightarrow L^{\pm,\pm}_{p,f,\psi,\chi}(x,y). \end{aligned}$$

The application to Iwasawa theory in this setting stems from conjectures of Kim [Kim14]. Each of the doubly signed p-adic L-functions $L^{\pm,\pm}_{p,f,\psi,\chi}(x,y)$ is conjectured to generate the characteristic divisor in Λ of the corresponding doubly signed Selmer group (which we denote $\mathrm{Sel}^{\pm,\pm}(\mathbb{Q},\mathfrak{D}_\tau)^\vee$) constructed by Kim [Kim14], which in turn is inspired by work over $\mathbb{Q}$ of Kobayashi [Kob03].

1.2. Description of the algorithm. The principal objects appearing in our algorithm are the following:

— The overconvergent modular symbols valued in $\mathbb{D}_2(\mathbb{Q}_p)$: $\mathrm{Symb}_\Gamma(\mathbb{D}_2(\mathbb{Q}_p))$,

— The parabolic cohomology group valued in distributions: $H^1_{\mathrm{par}}(\Gamma,\mathbb{D}_2(\mathbb{Q}_p))$,

— The parabolic cohomology group with trivial coefficients $H^1_{\mathrm{par}}(\Gamma,\mathbb{Q}_p)$.

Each of these objects are naturally endowed with an action of the Hecke algebra (for weight 2 and level Γ) and are related to each other by Hecke-equivariant maps as illustrated below:

$$\begin{array}{ccccccc} \mathrm{Symb}_\Gamma(\mathbb{D}_2(\mathbb{Q}_p)) & \xrightarrow{\delta} & H^1_{\mathrm{par}}(\Gamma,\mathbb{D}_2(\mathbb{Q}_p)) & \qquad & \Phi_f & \xrightarrow{\delta} & [\Theta_f] \\ & & \downarrow{\scriptstyle\varrho} & & & & \downarrow{\scriptstyle\varrho} \\ & & H^1_{\mathrm{par}}(\Gamma,\mathbb{Q}_p)\,. & & & & [\vartheta_f]. \end{array} \tag{1.3}$$

The objects and the maps appearing in (1.3) are described in Section **??**. We will simply remark here that the map δ is a *connecting homomorphism* in group theory, whereas the map ϱ is obtained from the natural *specialization map* $\mathbb{D}_2(\mathbb{Q}_p) \twoheadrightarrow \mathbb{Q}_p$.

We now outline the main steps involved in our algorithm. the outline is very bare ones now. It will be nice to expand on the steps a little more, especially Step 2 where the Bianchi setting really kicks in and Step 3, where we need to cleverly use code of Aurel Page, solve word presentations etc. Writing the outline perhaps makese sense to do at the end. Perhaps here or at the end, it will also be worthwhile to discuss how computationally intensive each step is — Bharath ♡

(Step 1) — *Constructing a cocycle $[\vartheta_f]$ with trivial coefficients.*

By multiplicity one can we add a suffix, like multiplicity one for blah-blah — Bharath , ♡ there exists a one-dimensional $\mathbb{Q}_p$ vector subspace V_f of $H^1_{\mathrm{par}}(\Gamma, \mathbb{Q}_p)$ that is stable under the action of the Hecke algebra, with Hecke eigenvalues coinciding with those of f. In (STEP 1), we construct a cocycle ϑ in $Z^1(\Gamma, \mathbb{Q}_p)$ such that its corresponding cohomology class $[\vartheta_f]$ in $H^1_{\mathrm{par}}(\Gamma, \mathbb{Q}_p)$ generates V_f.

(STEP 2) — *Constructing an overconvergent lift* $[\Theta_f]$

The control theorem of [Wil17, Thm. 6.10] is also applicable in the setting of group cohomology [Wil18, Thm. 3.7] Reference Wil18 missing — Bharath . There thus ♡ exists a unique cohomology class $[\Theta_f]$ in $H^1_{\mathrm{par}}(\Gamma, \mathbb{D}_2(\mathbb{Q}_p))$, stable under the action of the Hecke algebra, such that $\rho([\Theta_f])$ equals $[\vartheta_f]$. In (STEP 2), we construct a cocycle Θ in $Z^1(\Gamma, \mathbb{D}_2(\mathbb{Q}_p))$ such that its corresponding cohomology class in $H^1_{\mathrm{par}}(\Gamma, \mathbb{D}_2(\mathbb{Q}_p))$ equals $[\Theta_f]$.

(STEP 3) — *Inverting the connecting homomorphism* δ *to obtain* Φ_f

(STEP 4) — *Computing the moments of the distribution* $\mu_{f,\psi,\chi}$.

(STEP 5) — *Computing the* 2*-variable power series*.

1.3. Applications to higher codimension Iwasawa theory. Suppose f satisfies (BASE-CHANGE). Let S denote the set of primes in $\mathbb{Q}$ containing p, ∞ along with all the primes dividing the conductor of E and the discriminant of $K/\mathbb{Q}$. The discrete fine Selmer group associated to τ is defined below:

$$Ш^1(\mathbb{Q}, \mathfrak{D}_\tau) := \ker\left(H^1(\mathbb{Q}_S/\mathbb{Q}, \mathfrak{D}_\tau) \to \bigoplus_{\nu \in S} H^1(\mathbb{Q}_\nu, \mathfrak{D}_\tau)\right).$$

Definition 1.1. A finitely generated torsion Λ-module M is said to be *pseudo-null* if its characteristic divisor $\mathrm{Div}(M)$ in Λ equals zero.

When E has good ordinary reduction at p, one can establish that $Ш^1(\mathbb{Q}, \mathfrak{D}_\tau)^\vee$ is a finitely generated torsion Λ-module (essentially using a result of Rubin in the CM case and of Kato [Kat04] in the non-CM case). When E has good supersingular reduction at p, one can establish that $Ш^1(\mathbb{Q}, \mathfrak{D}_\tau)^\vee$ is a finitely generated torsion Λ-module using a result of Kobayashi [Kob03]. We have the following conjecture, due to Coates-Sujatha ([CS05, Conjecture B]):

Conjecture 1.2.
The Λ*-module* $Ш^1(\mathbb{Q}, \mathfrak{D}_\tau)^\vee$ *is pseudo-null.*

We start with a few observations:

1. When the rank (say $r := \mathrm{Rank}_{\mathbb{Z}} E(K)$) of the Mordell-Weil group of E/K satisfies $r \geq 1$, the cyclotomic specializations of the 2-variable p-adic L-functions that we consider will have a common irreducible factor in the 1-variable Iwasawa algebra corresponding to this Mordell-Weil group.

2. When $r \geq 2$, the Pontryagin dual of the fine Selmer group over every $\mathbb{Z}_p$-extension L/K will have a positive $\mathbb{Z}_p$-rank and consequently will not be pseudo-null over the corresponding 1-variable Iwasawa algebra for $\mathrm{Gal}\, LK$. Verify that $r \geq 2$ is sufficient to demonstrate that these fine Selmer groups aren't pseudo-null and then later add a proof somewhere, perhaps an appendix — Bharath . ♡

To illustrate the method on no common irreducible factors, for the sake of simplicity only a few examples are selected in Tables 1 and 2 Add a reference to an online source where we establish lots more examples, including $r = 0$ — Bharath . Keeping in mind the observations stated earlier, we have chosen the "simplest" examples that satisfy $r \geq 1$. ♡

(Ord): When E has good ordinary reduction at p. The Iwasawa-Greenberg main conjecture amounts to the following:

Conjecture 1.3. *The p-adic L-function $L_{p,f,\psi,\chi}(x,y)$ is non-zero. Furthermore, we have the following equality of characteristic divisors of Λ:*

$$\mathrm{Div}\left(\mathrm{Sel}(\mathbb{Q},\mathfrak{D}_\tau)^\vee\right) = \mathrm{Div}\left(L_{p,f,\psi,\chi}(x,y)\right).$$

♡

What we really only need for the application to pseudo-nullity is the Euler-system inequality $\mathrm{Div}\left(\mathrm{Sel}(\mathbb{Q},\mathfrak{D}_\tau)^\vee\right) \leq \mathrm{Div}\left(L_{p,f,\psi,\chi}(x,y)\right)$. Is this known due to Skinner-Urban? Need to find more info on the current status of this. — Bharath

We will focus on elliptic curves $E/\mathbb{Q}$ with complex multiplication, whose endomorphism algebra (an imaginary quadratic field say L) is chosen not to equal K. Let δ denote the non-trivial element in $\mathrm{Gal}\, L\mathbb{Q}$. We have the following natural surjections of Λ-modules:

$$\mathrm{Sel}(\mathbb{Q},\mathfrak{D}_\tau)^\vee \twoheadrightarrow Ш^1(\mathbb{Q},\mathfrak{D}_\tau)^\vee, \qquad \delta(\mathrm{Sel}(\mathbb{Q},\mathfrak{D}_\tau)^\vee) \twoheadrightarrow Ш^1(\mathbb{Q},\mathfrak{D}_\tau)^\vee \tag{1.4}$$

Assume Conjecture 1.3 holds. Equation (1.4) lets us deduce that the pseudo-nullity conjecture (Conjecture 1.2) holds if we can establish that following hypothesis:

(ORD:GCD) — There exists no irreducible element in Λ that divides both $L_{p,f,\psi,\chi}(x,y)$ and $\delta(L_{p,f,\psi,\chi}(x,y))$.

♡

As stated above, it's not entirely accurate since $\delta(\mathrm{Sel}(\mathbb{Q},\mathfrak{D}_\tau)^\vee)$ and $\delta(L_{p,f,\psi,\chi}(x,y))$ may not make sense. Instead, I really have to look at local and global class field theory to get the other p-adic L-function from L_p, something like a p-adic functional equation. Argh! annoying but should be doable! — Bharath

In this manuscript, we simply focus on the examples given in Table 1, where we establish **??**, motivated by its connection to Conjecture 1.2. See Section on examples — Bharath . ♡

Cremona's Label	K	p	$\mathrm{Rank}_{\mathbb{Z}}(E(\mathbb{Q}))$	$\mathrm{Rank}_{\mathbb{Z}}(E(K))$	End. algebra
$27a1$	$\mathbb{Q}(\sqrt{-2})$	7	0	1	$\mathbb{Q}(\sqrt{-3})$
$32a1$	$\mathbb{Q}(\sqrt{-7})$	5	0	1	$\mathbb{Q}(\sqrt{-1})$
$108a1$	$\mathbb{Q}(\sqrt{-7})$	7	0	2	$\mathbb{Q}(\sqrt{-3})$
$121b1$	$\mathbb{Q}(\sqrt{-19})$	3	1	2	$\mathbb{Q}(\sqrt{-11})$

Table 1: *Examples of CM $E/\mathbb{Q}$ with good ordinary reduction at p*

(SS): When E has good supersingular reduction at p with $a_p(E) = 0$. We have the following conjecture due to Kim ([Kim14, Conjecture 3.1], [Wan16, Conjecture 6.7]):

Conjecture 1.4. *Let $\bullet, \circ \in \{+,-\}$. The p-adic L-function $L^{\bullet,\circ}_{p,f,\psi,\chi}(x,y)$ is non-zero. Furthermore, we have the following equality of characteristic divisors of Λ:*

$$\mathrm{Div}\left(\mathrm{Sel}^{\bullet,\circ}(\mathbb{Q},\mathfrak{D}_\tau)^\vee\right) = \mathrm{Div}\left(L^{\bullet,\circ}_{p,f,\psi,\chi}(x,y)\right).$$

For each $\bullet, \circ \in \{+,-\}$, we have the following natural surjections of Λ-modules:

$$\mathrm{Sel}^{\bullet,\circ}(\mathbb{Q},\mathfrak{D}_\tau)^\vee \twoheadrightarrow Ш^1(\mathbb{Q},\mathfrak{D}_\tau)^\vee \tag{1.5}$$

Assume Conjecture 1.4 holds. Equation (1.5) lets us deduce that the pseudo-nullity conjecture (Conjecture 1.2) holds if we can establish that following hypothesis:

(SS:GCD) — There exists no irreducible element in Λ that divides all four of the p-adic L-functions $L^{\pm,\pm}_{p,f,\psi,\chi}(x,y)$.

In this manuscript, we simply focus on the examples given in Table 2, where we establish (SS:GCD), motivated by its connection to Conjecture 1.2. See Section on examples — Bharath . ♡

A result of Kobayashi ([Kob03]) allows us to deduce that the equal doubly-signed p-adic L-functions $L^{+,+}_{p,f,\psi,\chi}(x,y)$ and $L^{-,-}_{p,f,\psi,\chi}(x,y)$ are non-zero (see [CÇSS17, Proposition 3.5], [LP19a, Remark 8.5]). In general however, it has not been currently established that the mixed doubly-signed p-adic L-functions are non-zero. Consequently, we are also able to provide numerical examples when the mixed doubly-signed p-adic L-functions are non-zero, establishing part of Conjecture 1.4 in this case as well.

Cremona's Label	K	p	$\mathrm{Rank}_{\mathbb{Z}}(E(\mathbb{Q}))$	$\mathrm{Rank}_{\mathbb{Z}}(E(K))$	End. algebra
$14a1$	$\mathbb{Q}(\sqrt{-11})$	5	0	1	$\mathbb{Q}$
$37a1$	$\mathbb{Q}(\sqrt{-2})$	3	1	1	$\mathbb{Q}$
$121b1$	$\mathbb{Q}(\sqrt{-19})$	7	1	2	$\mathbb{Q}(\sqrt{-11})$

Table 2: *Examples of $E/\mathbb{Q}$ with good supersingular reduction at p*

1.3.1. Relationship to the Rankin-Selberg p-adic L-function. Suppose that f satisfies (Base-Change). Observe that as a $\Lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module, $\mathfrak{L}_\tau$ is isomorphic to

$$\left(T_p(E) \otimes_{\mathbb{Z}_p} \mathrm{Ind}_K^{\mathbb{Q}} \Lambda(\kappa_{\mathfrak{p}}^{-1})\right)(\kappa_{\mathrm{cyc}}^{-1}).$$

Here, $\kappa_{\mathfrak{p}} : \mathrm{Gal}(\overline{K}/K) \twoheadrightarrow G_{\mathfrak{p}} \hookrightarrow \Lambda^\times$ is the tautological character and one can view the induction of the character $\kappa_{\mathfrak{p}}^{-1}$ as being associated to a *CM Hida family*. The character κ_{cyc} is the tautological cyclotomic character for the cyclotomic $\mathbb{Z}_p$-extension. This perspective allows us to view the Galois representation τ as the cyclotomic deformation of the tensor product of E and this CM-Hida family. Suppose now that E has good ordinary reduction at p. Hida has constructed a 2-variable Rankin-Selberg p-adic L-function in the fraction field of Λ, where one can vary the cyclotomic variable and the weight-variable of the CM-Hida family. We consider the Rankin-Selberg p-adic L-function that interpolates critical L-values as we fix the cyclotomic variable to equal 1 and vary over the classical weight one specializations of the CM Hida family (so the elliptic curve E has the dominant weight). We refer the interested reader to Hida's works [Hid93, Hid85, Hid88, Hid96] for the precise interpolation properties. It is possible to formulate two Iwasawa-Greenberg main conjectures over the ring Λ — the first corresponding to the 2-variable Bianchi p-adic L-function and the second corresponding to Hida's 2-variable Rankin-Selberg p-adic L-function. These conjectures assert that each of these p-adic L-functions generate the characteristic ideal of the Λ-module $\mathrm{Sel}(\mathbb{Q}, \mathfrak{D}_\tau)^\vee$. As a result, we expect the ratio of these p-adic L-functions to be a unit in the ring Λ. Indeed, this expectation is confirmed in a preprint of Skinner-Zhang [SZ14, Lemma 9.5], which asserts that the ratio of the complex periods appearing in the constructions of the p-adic L-functions is in fact a unit in $\mathbb{Z}_p$. Up to this unit in $\mathbb{Z}_p$, our algorithm would thus compute in this setting Hida's 2-variable Rankin-Selberg p-adic L-function as well.

1.4. Concluding Remarks. In the computations, we restrict to imaginary quadratic fields with class number one for simplicity. In this case, the space of modular symbols is rich enough to capture all of the arithmetic of the form f. In the general case when the class number of K equals h, we would instead need to consider a direct sum decomposition of h copies of modular symbols, with Hecke operators at non-principal primes permuting the factors in these copies; see, for example, [Wil17] for a detailed description of the general case. Such a decomposition arises since the Bianchi three-fold in this situation decomposes into a disjoint union of h connected components (via strong approximation for GL_2/K).

For several practical reasons in our implementation, we choose to first work with arithmetic cohomology (the parabolic cohomology groups) instead of directly working with modular symbols (as in [PS11, Gre07]). This is mainly since in the setting of Bianchi groups, extensive implementation to work with arithmetic cohomology groups already exists; for example, such computations form the basis of [GMŞ15]. Also, this approach might generalize more naturally to other settings, for example, pursuing the constructions of Barrera Salazar [BS15] for p-adic L-functions over *real* quadratic fields, where modular symbols themselves do not exist and one is forced to work with higher degree cohomology groups.

We refer the interested reader to [Bha07, Och09, Jha12, Lim15, She18] where there are examples of specific elliptic curves when the pseudo-nullity conjecture of Coates–Sujatha has been verified (in a setup different to ours).

1. Analyse the run time of our algorithm. What are our practical limitations in computing things.

2. Explain how we verify our code by existing theoretical results, relating it to 1-variable p-adic L-functions of Pollack. And in the root number = -1 case, showing that the appropriate anti-cyclotomic specializations are zero.

3. Also add a line saying that we target a very small sample size (6) just to simplify things and to illustrate our methods better. Mention that we have a larger list of numerical computations online somewhere.

4. Explain how our code compares to existing code of Pollack - Harron and co. [Include remarks on Rob Pollack and co, and in particular emphasise the difficulty of

computing 2-variable things: they restrict to index in the 100s, it takes them many hours and they compute to precision 7. We should do better?]

5. Write down all the attributions here — Stevens, Pollack, Greenberg etc.

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2. Bianchi modular forms and p-adic L-functions

We recap the underlying theory of Bianchi modular forms, which are a natural generalisation of classical modular forms – that is, automorphic forms for $\mathrm{GL}_2/\mathbb{Q}$ – to GL_2/K, where K is an imaginary quadratic field. Throughout this paper, we take K to have class number 1 for simplicity (in line with our later computations). Let $\mathcal{O}_K$ be the ring of integers.

2.1. Background. For each congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$, there is a finite-dimensional $\mathbb{C}$-vector space $S_2^K(\Gamma)$ of 'weight 2 Bianchi cusp forms', defined as harmonic vector-valued functions

$$\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2) : \mathcal{H}_3 \longrightarrow \mathbb{C}^3$$

on the upper half-space $\mathcal{H}_3 := \mathbb{C} \times \mathbb{R}_{>0}$ satisfying a suitable transformation property under an action of Γ. See [Wil17, §1] for a more detailed exposition of all of this (standard) theory in our conventions (noting that $S_2^K(\Gamma)$ for us is denoted $S_{0,0}(\Gamma)$ *op. cit.*).

There is a Hecke action on $S_2^K(\Gamma)$, given by double coset operators and indexed by ideals of $\mathcal{O}_K$.

2.1.1. L-functions and p-adic L-functions. If $\mathcal{F} \in S_2^K(\Gamma)$ is an eigenform with Hecke eigenvalues $\{a_I : I \subset \mathcal{O}_K\}$, and χ is a Hecke character of K, one defines the L-function

$$L(\mathcal{F}, s) = \sum_{I \subset \mathcal{O}_K} a_I \chi(I) N(I)^{-s}$$

as the corresponding Dirichlet series (see [Wil17, §1.2]). This sum converges absolutely for $\mathrm{Re}(s) >> 0$ and admits analytic continuation to arbitrary $s \in \mathbb{C}$. Morever, there exists a complex period $\Omega_{\mathcal{F}} \in \mathbb{C}^\times$ such that

$$\frac{L(\mathcal{F}, \chi, 1)}{(2\pi i)^2 \Omega_{\mathcal{F}}} \in \overline{\mathbb{Q}} \tag{2.1}$$

for all finite order Hecke characters χ (see [Hid94, §8]). The conjecture of Coates–Perrin-Riou then predicts the existence of a *p-adic L-function* interpolating these algebraic numbers. More precisely, this should be a p-adic distribution $\mu_{\mathcal{F}}$ on

$$\mathrm{Gal}(K^{\mathrm{ab},p}/K) \cong \mathrm{Cl}_K(p^\infty) := K^\times \mathbb{A}_K^\times / \mathbb{C}^\times \widehat{\mathcal{O}}_K^{\times,(p)} \cong (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \mathcal{O}_K^\times,$$

where $K^{\mathrm{ab},p}$ is the maximal abelian extension of K unramified outside p, the first isomorphism is given by class field theory, and the second follows from our assumption of class number 1. The required interpolation property is that for all finite order Hecke characters of p-power conductor – which are naturally characters of $\mathrm{Cl}_K(p^\infty)$ – we have

$$\mu_f(\chi) := \int_{\mathrm{Cl}_K(p^\infty)} \chi(z) d\mu_{\mathcal{F}}(z) = (*)\frac{L(\mathcal{F},\chi,1)}{(2\pi i)^2\Omega_f}, \tag{2.2}$$

where $(*)$ is a precise interpolation factor (described explicitly in this case in [BSW19, Thm. 3.12] with $r = 0$). Moreover, $\mu_{\mathcal{F}}$ should satisfy a precise growth property.

2.1.2. Base-change. Of particular importance to us is the existence of a *base-change* map. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N corresponding to a newform $f \in S_2(\Gamma_0(N))$ under modularity. We can consider E instead to have coefficients over K. As predicted by Langlands, there exists a Bianchi newform f/K, the *base-change of f to K*, whose L-function is equal to $L(E/K, s)$. We have $f/K \in S_2^K(\Gamma_0(\mathfrak{n}))$, where $\Gamma_0(\mathfrak{n}) \subset \mathrm{SL}_2(\mathcal{O}_K)$ is the subgroup of matrices that are upper-triangular mod $\mathfrak{n}$, and where $\mathfrak{n}|N\mathcal{O}_K$ is the conductor of E/K. (Note that if N is coprime to the discriminant of $K/\mathbb{Q}$, we have $\mathfrak{n} = N\mathcal{O}_K$). The Hecke eigenvalues of f/K can be described simply in terms of those of f (see e.g. [BSW18, §7.2]). Note that

$$L(f/K, s) = L(E/K, s) = L(E/\mathbb{Q}, s)L(E/\mathbb{Q}, \chi_K, s) = L(f, s)L(f, \chi_K, s),$$

where χ_K is the quadratic Hecke character whose kernel cuts out $K/\mathbb{Q}$.

2.2. Bianchi modular symbols. We are interested in computational aspects of Bianchi modular forms, and particularly their L-functions, but the analytic definition is impractical for such purposes. Fortunately, the space $S_2^K(\Gamma)$ admits a computational avatar via the space of *modular symbols.* Let

$$\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(K))$$

denote the space of 'paths between cusps' in $\mathcal{H}_3$, and let V be any right $\mathrm{SL}_2(K)$-module. Fix an ideal $\mathfrak{n}$ and let $\Gamma := \Gamma_0(\mathfrak{n}) \subset \mathrm{SL}_2(K)$. We define the space of *V-valued modular symbols for* Γ to be the space

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Hom}_\Gamma(\Delta_0, V)$$

of functions satisfying the Γ-invariance property that

$$(\phi|\gamma)(\delta) := \phi(\gamma\delta)|\gamma = \phi(\delta) \quad \forall \delta \in \Delta_0, \gamma \in \Gamma,$$

where Γ acts on the cusps by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot r = (ar + b)/(cr + d)$. Note that this space is entirely algebraic in nature, and far more amenable to computations.

Now let $\mathcal{F} \in S_2^K(\Gamma)$. To $\mathcal{F}$ one may attach an explicit differential form $\delta_{\mathcal{F}}$ on $\mathcal{H}_3$ as follows: let (z, t) be a co-ordinate on $\mathcal{H}_3$, and note that $dz, d\overline{z}, dt$ span the $\mathbb{C}$-valued 1-forms on $\mathcal{H}_3$. Then define

$$\delta_{\mathcal{F}} := \mathcal{F}_0(z, t)dz - \mathcal{F}_1(z, t)dt - \mathcal{F}_2(z, t)d\overline{z} \in \Omega^1(\mathcal{H}_3)$$

(see e.g. [CW94]). For $r, s \in \mathbb{P}^1(K)$, the map

$$\begin{aligned}\phi_{\mathcal{F}} : \Delta_0 &\longrightarrow \mathbb{C},\\ \{r \to s\} &\longmapsto \int_r^s \delta_f,\end{aligned}$$

is well-defined and the transformation property satisfied by $\mathcal{F}$ ensures it is Γ-invariant, thus giving an element $\phi_f \in \mathrm{Symb}_\Gamma(\mathbb{C})$.

The space $\mathrm{Symb}_\Gamma(\mathbb{C})$ admits an action of the Hecke operators, indexed by ideals of $\mathcal{O}_F$ and generated by the operators $T_{\mathfrak{l}}$ for $\mathfrak{l} \nmid \mathfrak{n}$ prime and $U_{\mathfrak{l}}$ for $\mathfrak{l}|\mathfrak{n}$. Of particular importance are the $U_{\mathfrak{p}}$-operators for $\mathfrak{p}|p$, defined as

$$\phi|U_{\mathfrak{p}}\{r \to s\} = \sum_{a=0}^{p-1} \phi\big|\left(\begin{smallmatrix} 1 & a \\ 0 & \pi \end{smallmatrix}\right)\{r - s\} = \sum_{a=0}^{p-1} \phi\left\{\tfrac{r+a}{\pi} \to \tfrac{s+a}{\pi}\right\},$$

where $\mathfrak{p} = (\pi)$. More generally, if the action of Γ on V extends to an action of the semigroup

$$\Sigma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) : ad - bc \neq 0, p \nmid a, p|c\},$$

then the formula $\phi|U_{\mathfrak{p}} = \sum_{a=0}^{p-1} \phi|\left(\begin{smallmatrix} 1 & a \\ 0 & \pi \end{smallmatrix}\right)$ defines an action of $U_{\mathfrak{p}}$ on $\mathrm{Symb}_\Gamma(V)$.

Proposition 2.1. *The resulting map*

$$\iota : S_2^K(\Gamma) \longrightarrow \mathrm{Symb}_\Gamma(\mathbb{C})$$

is injective and induces a splitting

$$\mathrm{Symb}_\Gamma(\mathbb{C}) \cong S_2^K(\Gamma) \oplus \mathrm{Eis}_2^K(\Gamma).$$

For a prime $\mathfrak{l} \nmid \mathfrak{n}$ of norm ℓ, the Hecke operator $T_{\mathfrak{l}}$ acts on $\mathrm{Eis}_2^K(\Gamma)$ as multiplication by $\ell + 1$.

Proof. See, for example, [Wil17] for the injection. For part (ii), observe that by [BSW19, Lemma 8.2] we have a Hecke-equivariant isomorphism

$$\mathrm{Symb}_\Gamma(\mathbb{C}) \cong \mathrm{H}^1_{\mathrm{c}}(Y_\Gamma, \mathbb{C}),$$

where $Y_\Gamma := \Gamma\backslash\mathcal{H}_3$, and the compactly supported cohomology admits a well-understood splitting $\mathrm{H}^1_{\mathrm{cusp}}(Y_\Gamma, \mathbb{C}) \oplus \mathrm{Eis}_\Gamma(\mathbb{C})$ (see [Har87, §3.2.5]). Moreover, under the injection ι, the space $S_2^K(\Gamma)$ is mapped isomorphically onto $\mathrm{H}^1_{\mathrm{cusp}}(Y_\Gamma, \mathbb{C})$. The other direct summand corresponds to Bianchi Eisenstein series, giving the claimed action of Hecke operators. □

The passage from an eigenform $\mathcal{F}$ to $\phi_{\mathcal{F}}$ thus encodes much of the interesting algebraic data attacehed to $\mathcal{F}$. In particular, it retains the Hecke action, the Hecke eigenvalues and thus also sees the L-function of $\mathcal{F}$. In particular, in [CW94] an explicit formula is given relating $\phi_{\mathcal{F}}\{0 \to \infty\}$ to the critical value $L(\mathcal{F}, 1)$. A twisted version, computing $L(\mathcal{F}, \chi, 1)$ for finite order Hecke characters of K, is contained in [Wil17, Prop. 2.8].

2.3. Overconvergent modular symbols. We briefly recap the main ideas in the construction of [Wil17]. The idea behind the overconvergent modular symbol construction is that to p-adically interpolate the (algebraic parts of the) values $L(\mathcal{F},\chi,1)$, we can use the connection to modular symbols, and p-adically interpolate these instead. To do this, we require algebraic coefficients. By [Hid94, §8], there exists a finite extension $L/\mathbb{Q}_p$ such that we may consider $\phi_{\mathcal{F}}/\Omega_{\mathcal{F}} \in \mathrm{Symb}_\Gamma(L)$, where Ω_f is the same complex period from (2.1), which can then be interpolated p-adically via *overconvergent modular symbols*.

For such an interpolation to exist, it is crucial to pass to a level subgroup $\Gamma \subset \Gamma_0(p\mathcal{O}_F)$. If $(p)|\mathfrak{n}$, then $\Gamma_0(\mathfrak{n}) \subset \Gamma_0(p\mathcal{O}_F)$ already; if not, then it is necessary to 'p-stabilise' to ensure this condition holds (see for example [BSW18, §2.4]). We now assume $\Gamma \subset \Gamma_0(p\mathcal{O}_F)$ without further comment.

We pass to an (infinite-dimensional) coefficient module; namely, let $L/\mathbb{Q}_p$ be a finite extension, let $\mathbb{A}_2(L)$ denote the space of convergent power series on $\mathbb{Z}_p^2$, and let

$$\mathbb{D}_2(L) := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}_2(L), L),$$

the space of *p-adic analytic distributions on $\mathbb{Z}_p^2$*. As $\Gamma \subset \Gamma_0(p)$, the left action of Γ on $\mathbb{A}_2(L)$ by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot g(x,y) = g\left(\frac{b+dx}{a+cx}, \frac{\bar{b}+\bar{d}y}{\bar{a}+\bar{c}y}\right),$$

is well-defined and induces dually a right action on $\mathbb{D}_2(L)$. The space of *overconvergent modular symbols* is $\mathrm{Symb}_\Gamma(\mathbb{D}_2(L))$.

Dualising the inclusion of L into $\mathbb{A}_2(L)$ gives a surjection $\mathbb{D}_2(L) \to L$ of Γ-modules, and hence a (Hecke-equivariant) map

$$\rho : \mathrm{Symb}_\Gamma(\mathbb{D}_2) \to \mathrm{Symb}_\Gamma(L).$$

Whilst this map will have a huge kernel, crucially, the Hecke action allows us to control it.

Theorem 2.2. [Wil17, Thm. 6.10] *Let $\mathcal{F} \in S_2^K(\Gamma)$ be an eigenform, with $U_{\mathfrak{p}}\mathcal{F} = \alpha_{\mathfrak{p}}\mathcal{F}$ for each $\mathfrak{p}|p$. If $v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) < 1$ for all $\mathfrak{p}$, then the restriction of ρ to the $\mathcal{F}$-eigenspaces of the Hecke operators is an isomorphism.*

If $\mathcal{F}$ satisfies this condition, we say it has *small slope*. If $\mathcal{F}$ is such a form, then the theorem says that there is a *unique* $\Phi_{\mathcal{F}} \in \mathrm{Symb}_\Gamma(\mathbb{D}_2(L))$ lifting $\phi_{\mathcal{F}}/\Omega_{\mathcal{F}}$.

2.4. The p-adic L-function of a Bianchi modular form. Let $\mathcal{F}$ be a small slope Bianchi eigenform with associated overconvergent modular symbol $\Phi_{\mathcal{F}}$. Recall from §2.1.1 that the p-adic L-function should be a p-adic distribution on $\mathrm{Cl}_K(p^\infty) = (\mathbb{Z}_p^\times)^2/\mathcal{O}_K^\times$. Note that there is a natural inclusion $\mathbb{A}(\mathrm{Cl}_K(p^\infty), L) \subset \mathbb{A}_2(L)$ of the subspace of analytic functions on $\mathrm{Cl}_K(p^\infty)$, and thus a natural restriction map $\mathbb{D}_2(L) \to \mathbb{D}(\mathrm{Cl}_K(p^\infty))$.

Definition 2.3. Define the *p-adic L-function of $\mathcal{F}$* to be

$$\mu_{\mathcal{F}} := \Phi_{\mathcal{F}}\{0 \to \infty\}\big|_{\mathrm{Cl}_K(p^\infty)}.$$

By [Wil17, Thm. 7.4], this is actually in the much smaller subspace of *locally analytic distributions*, and can thus be evaluated at finite order Hecke characters. It then satisfies the expected interpolation property (2.2) and has the expected growth condition. In the small slope case, this defines $\mu_{\mathcal{F}}$ uniquely.

3. Computing p-adic L-series from modular symbols

In this section, we show how to compute the p-adic L-series of a Bianchi modular form from its p-adic L-function (as a distribution on $\mathrm{Cl}_K(p^\infty)$). Since we are working in the case where p is split, this essentially just the product of two copies of the theory for classical modular forms, as found in [PS11, §9], and which we first briefly recall.

3.1. The classical case.

3.1.1. Distributions on $\mathbb{Z}_p^\times$. For simplicity, we assume that $p \geq 3$, though the case $p = 2$ can be obtained with very little (and completely standard) modification. In the classical case, the p-adic L-function is naturally a distribution on the Galois group $\mathrm{Gal}_p := \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab},p\infty}/\mathbb{Q})$, which by class field theory is isomorphic to the narrow ray class group

$$\mathrm{Cl}_{\mathbb{Q}}^{+}(p^\infty) = \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}_{>0}\widehat{\mathbb{Z}}^{\times,(p)} \cong \mathbb{Z}_p^\times.$$

Let now $f \in S_2(\Gamma)$ be a classical eigenform (for $\mathrm{GL}_2/\mathbb{Q}$) with $U_pf = \alpha_p f$ with $v_p(\alpha_p) < 1$. By [PS11], there is a unique overconvergent modular symbol $\Phi_f \in \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), \mathbb{D})$ attached to f, where $\mathbb{D}$ is the space of distributions on $\mathbb{Z}_p$. Then $\Phi_f\{0 \to \infty\} \in \mathbb{D}$ is by definition a distribution on $\mathbb{Z}_p$, and we obtain a distribution μ_f on $\mathbb{Z}_p^\times$ by restriction. This distribution is, by [PS11], the p-adic L-function attached to f, and is entirely encoded in its moments

$$\left\{\mu_f(z^j) := \int_{\mathbb{Z}_p^\times} z^j d\mu_f(z) : j \geq 0\right\}.$$

We have a simple description of the moments of μ_f in terms of Φ_f, and thus, in particular, we can compute them by knowing Φ_f.

Proposition 3.1. *If Φ_f is the overconvergent modular symbol attached to a small slope classical cuspidal eigenform, then*

$$\Phi\{0 \to \infty\}(z^j\mathbf{1}_{a+p\mathbb{Z}_p}(z)) = \alpha_p^{-1}\Phi\left\{\tfrac{a}{p} \to \infty\right\}\left((a+pz)^j\right),$$

where α_p is the U_p-eigenvalue of f. In particular, we recover

$$\mu_f(z^j) = \alpha_p^{-1}\sum_{a=1}^{p-1}\Phi\left\{\tfrac{a}{p} \to \infty\right\}\left((a+pz)^j\right).$$

Proof. Apply $\alpha_p^{-1}U_p$, which acts as 1 on Φ_f, and note that if $a = b$, then

$$\Phi\{b/p \to \infty\}((b+pz)^j\mathbf{1}_{a+p\mathbb{Z}_p}(b+pz)) = 0. \qquad \square$$

The final statement follows since $\mathbf{1}_{\mathbb{Z}_p^\times} = \sum_{a=1}^{p-1}\mathbf{1}_{a+p\mathbb{Z}_p}$.

3.1.2. Passing to p-adic L-series. The distribution μ_f is completely canonical. Even in higher weight situations, it is canonical up to a fixed choice of periods. For practical purposes, however, to compute p-adic L-series from μ_f we wish to break this into $p-1$ pieces via the decomposition

$$\mathbb{Z}_p^\times = (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p), \tag{3.1}$$

and then (non-canonically) identify each one with a distribution on $\mathbb{Z}_p$ (depending on a choice of topological generator for $1+p\mathbb{Z}_p$). On the algebraic side, this corresponds to considering the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, identifying its Galois group $\mathcal{G}$ with $\mathbb{Z}_p$ via the choice of a topological generator, and studying Selmer groups over $\mathcal{G}$.

Notation 3.2: In (3.1), denote projection to the first and second factors by $z \mapsto \{z\}$ and $z \mapsto \langle z\rangle$ respectively.

We can identify distributions μ on $\mathbb{Z}_p^\times$ with analytic functions on *weight space*

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times),$$

which – as a rigid space – is $p-1$ copies of the open unit disc. Given an element ϕ in weight space, we can write ϕ as $\{\phi\}\langle\phi\rangle$, where $\{\phi\}$ is a homomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}_p^\times$, and the disc in $\mathcal{W}$ in which ϕ lives is completely determined by $\{\phi\}$. From μ_f we then obtain $p-1$ analytic functions on the open unit disc, one for each character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Each has a power series representation, which is what we compute.

Fix now a character ψ on $(\mathbb{Z}/p\mathbb{Z})^\times$, corresponding to some fixed disc in $\mathcal{W}$. Let T be a parameter on this open unit disc. The part of the (analytic) function attached to μ_f defined over this disc is defined as

$$L_p(\mu,\psi,T) := \int_{\mathbb{Z}_p^\times} \psi(z)(T+1)^{\log_\gamma(\langle z\rangle)} d\mu_f(z),$$

where $\gamma = p+1$, our fixed choice of topological generator for $1+p\mathbb{Z}_p$. (Of course, for $z \in 1+p\mathbb{Z}_p$, we have $\log_\gamma(z) = \log_p(z)/\log_p(\gamma)$). Writing $z = \{z\}\langle z\rangle$ and expanding the log, we obtain the power series representation $L_p(\mu,\psi,T) = \sum_{n\geq 0} d_n(\psi)T^n$, where $d_n(\psi)$ is defined as

$$d_n(\psi) = \sum_{a=0}^{p-1} \psi(a) \int_{a+p\mathbb{Z}_p} \left[\sum_{j\geq 0} c_j^{(n)} \left(\frac{z}{\{a\}} - 1\right)^j\right] d\mu(z).$$

Here, $c_j^{(n)}$ is defined by from the equation

$$\binom{\log_\gamma(z+1)}{n} = \sum_{j\geq 0} c_j^{(n)} z^j, \tag{3.2}$$

Note also that for a prime to p, we have $\int_{a+p\mathbb{Z}_p} h(z)d\mu(z) = \int_{a+p\mathbb{Z}_p} h(z)d\Phi\{0\to\infty\}(z)$, that is, restriction to $\mathbb{Z}_p^\times$ is already built in. For further details on all of the above, see [PS11, §9].

3.2. The Bianchi case. Now we turn to the Bianchi case. Let f be a small slope classical cuspidal Bianchi eigenform, and $\Phi_f \in \mathrm{Symb}_\Gamma(\mathbb{D}_2)$ the attached overconvergent modular symbol. As outlined in §2.1.1, the p-adic L-function of f is most naturally a distribution μ on $\mathrm{Cl}_K(p^\infty) \cong (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times/\mathcal{O}_K^\times$, where in the last assertion we are assuming class number one for simplicity. Since p is split, we identify this with $(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)/\mathcal{O}_K^\times$. Evaluating Φ_f at $\{0\to\infty\}$, we obtain a distribution on all of $\mathbb{Z}_p^2$. Inside $\mathbb{A}_2$ is the subspace of functions with support in $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ that are invariant under $\mathcal{O}_K^\times$, and we pass to the p-adic L-function μ_f by restricting $\Phi_f\{0\to\infty\}$ to this subspace.

One can consider the direct analogue of the weight space above, that is, considering continuous homomorphisms $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, and show that this decomposes as the disjoint union of $(p-1)^2$ products of open unit discs, parametrised by characters of $(\mathcal{O}_K/p)^\times = (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^\times$. By directly generalising the approach above, given a measure μ on $\mathrm{Cl}_K(p^\infty)$ and such a character ψ, one can define an associated (two-variable) analytic function $L_p(\mu,\psi,T_1,T_2)$ on the corresponding product of open discs. Because the p-adic L-function actually lives on $\mathrm{Cl}_K(p^\infty)$ rather than $(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times$, such a function can only be defined when ψ descends to the quotient $(\mathcal{O}_K/p)^\times/\mathcal{O}_K^\times$.

Fix such a character ψ. Writing $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, we see that $\psi = \psi_{\mathfrak{p}}\psi_{\overline{\mathfrak{p}}}$, where $\psi_{\mathfrak{p}}$ is the restriction to $(\mathcal{O}_K/\mathfrak{p})^\times$. We also have parameters $z_{\mathfrak{p}}, z_{\overline{\mathfrak{p}}}$ on $\mathrm{Cl}_K(p^\infty)$, and $T_{\mathfrak{p}}, T_{\overline{\mathfrak{p}}}$ on the product of open discs. Using the same arguments as above, we find the following.

Proposition 3.3. *The p-adic L-series attached to ψ and μ is*

$$L_p(\mu,\psi,T_{\mathfrak{p}},T_{\overline{\mathfrak{p}}}) = \sum_{m\geq 0}\sum_{n\geq 0} d_{m,n}(\psi)T_{\mathfrak{p}}^m T_{\overline{\mathfrak{p}}}^n,$$

where – for $c_i^{(m)}$ as in (3.2) – we define

$$d_{m,n}(\psi) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\psi_{\mathfrak{p}}(a)\psi_{\overline{\mathfrak{p}}}(b) \times \int_{(a+p\mathbb{Z}_p)\times(b+p\mathbb{Z}_p)} \left[\sum_{i\geq 1}\sum_{j\geq 1} c_i^{(m)}c_j^{(n)}\left(\frac{z_{\mathfrak{p}}}{\{a\}}-1\right)^i\left(\frac{z_{\overline{\mathfrak{p}}}}{\{b\}}-1\right)^j\right]d\mu(\mathbf{z}).$$

3.2.1. Obtaining power series from the moments of μ. As before, it is simple to obtain the moments of μ_f by applying the U_p operator. Write $V_{a,b}$ for the open compact set $(a+p\mathbb{Z}_p)\times(b+p\mathbb{Z}_p)\subset(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times$. We compute that

$$\begin{aligned}\int_{V_{a,b}} f(z_{\mathfrak{p}},z_{\overline{\mathfrak{p}}})d\mu(\mathbf{z}) &:= \Phi\{0\to\infty\}(f(z_{\mathfrak{p}},z_{\overline{\mathfrak{p}}})\mathbb{1}_{V_{a,b}}) \\ &= \alpha_p^{-1}\Phi\{c/p\to\infty\}\big(f(c+pz_{\mathfrak{p}},\overline{c}+pz_{\overline{\mathfrak{p}}})\big),\end{aligned} \tag{3.3}$$

where $c = c_{a,b}\in\mathcal{O}_K$ is such that

$$c\equiv a \pmod{\mathfrak{p}},\quad \overline{c}\equiv b \pmod{\overline{\mathfrak{p}}}.$$

To see this, one applies the operator $\alpha_p^{-1}U_{p\mathcal{O}_K} = \alpha_p^{-1}U_{\mathfrak{p}}U_{\overline{\mathfrak{p}}}$, which acts as the identity on Φ, and note that the indicator function kills all but the a term of $U_{\mathfrak{p}}$ and the b term of $U_{\overline{\mathfrak{p}}}$, corresponding to the c term of U_p; see [Wil17, §7.1] for more details. Since for us f is a polynomial function, it is simple to compute this value by taking a linear combination of the moments of $\Phi\{c/p\to\infty\}$.

3.2.2. Twists. Twisting by characters of p-power conductor is built into the definitions above, but we can also twist by finite order characters of prime-to-p conductor. Consider a finite order character χ of conductor $(\mathfrak{d}) = \mathfrak{D}\subset\mathcal{O}_K$ prime to p. From Φ, one defines a twisted symbol

$$\Phi_\chi := \sum_{b \ (\mathrm{mod}\ \mathfrak{D})}\chi(b)\left[\Phi\{b/\mathfrak{d}\to\infty\}\middle| \begin{pmatrix}1 & b\\ 0 & \mathfrak{d}\end{pmatrix}\right],$$

then computes $L_p(\mu,\psi\chi,T) := L_p(\mu_\psi,\chi,T)$, where $\mu_\chi := \Phi_\chi\{0\to\infty\}|_{\mathrm{Cl}_K(p^\infty)}$. This case is treated in [BSW19, §3.4].

4. Rephrasing via arithmetic cohomology

The above gives a complete algorithm for constructing p-adic L-series from Bianchi modular symbols. For practical reasons, however, this space would be hard to compute in, since this requires presenting $\mathrm{Div}^0(\mathbb{P}^1(K))$ as a $\mathbb{Z}[\Gamma]$-module (and, in particular, solving the word problem in such a presentation). Instead, we work with the arithmetic cohomology groups $\mathrm{H}^1(\Gamma, \mathbb{D}_2)$, for which extensive implementation already exists. This approach might also generalise more naturally to different settings, for example, pursuing the constructions of [BS15] over *real* quadratic fields, where modular symbols themselves do not exist and one is forced to work with higher degree cohomology groups.

4.1. Definition and basic properties. See Marc's papers (which one explains it best, Marc?), or [PP09]; include things such as U_p operators.

A downside of computing with arithmetic cohomology over modular symbols is that we are not free to evaluate at the same range of divisors. In particular, we have a map

$$\begin{aligned} \delta : \mathrm{Symb}_\Gamma(\mathbb{D}_2) &\longrightarrow \mathrm{H}^1(\Gamma, \mathbb{D}_2) \\ \Phi &\longmapsto \big(\varphi : \gamma \mapsto \Phi\{\gamma\cdot\infty \to \infty\}\big). \end{aligned}$$

Given φ, we can thus read off the values $\Phi\{r \to s\}$ *only* for r, s equivalent to the cusp ∞. This poses a problem for our algorithm, since we need to evaluate at pairs $\{a/p \to \infty\}$, where a is coprime to p, and in general a/p will *not* give the same cusp as ∞. To obtain the information we need from φ, then, requires a careful study of the map δ.

From the general theory, the kernel and cokernel of δ are Eisenstein; in particular, it is an isomorphism on the cuspidal part. It follows that there is a *unique* cuspidal lift Φ of φ under δ. In the next section, we show how to explicitly invert δ to obtain this class Φ from φ, and thus how to obtain the p-adic L-function from φ.

4.2. Explicit inversion of δ. We now give an explicit and computable recipe for inverting δ on cuspidal arithmetic cohomology classes, taking care to be precise at every step.

4.2.1. Motivation: a snake diagram. The map δ can be realised in cohomology as

$$\delta : \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta_0, \mathbb{D}_2) \longrightarrow \mathrm{H}^1(\Gamma, \mathbb{D}_2),$$

and is the connecting map in a long exact sequence given by the snake lemma. In particular, for a right Γ-module $\mathcal{M}$, let:

- $C^i(\mathcal{M}) \coloneqq C^i(\Gamma, M) = \mathcal{M}$-valued i-cochains for Γ,
- $Z^i(\mathcal{M}) \coloneqq \mathcal{M}$-valued i-cocycles for Γ,
- and $B^i(\mathcal{M}) \coloneqq \mathcal{M}$-valued i-coboundaries for Γ.

Explicitly, as we are using right modules, a 1-cocycle is a map $z : \Gamma \to \mathcal{M}$ such that $z(\gamma_1\gamma_2) = z(\gamma_1) + z(\gamma_2)|\gamma_1^{-1}$. Also write $(C^i/B^i)(\mathcal{M})$ for the i-cochains modulo the i-coboundaries.

Recall that $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(K))$, and let $\Delta = \mathrm{Div}(\mathbb{P}^1(K)) = \mathbb{Z}[\mathbb{P}^1(K)]$. Now, the degree map gives a short exact sequence $0 \to \Delta_0 \to \Delta \to \mathbb{Z} \to 0$, and hence – for any right Γ-module M – a short exact sequence

$$0 \to M \longrightarrow \mathrm{Hom}(\Delta, M) \longrightarrow \mathrm{Hom}(\Delta_0, M) \to 0,$$

of Γ-modules, identifying $M \cong \mathrm{Hom}(\mathbb{Z}, M')$, and where Γ act on $\mathrm{Hom}(\Delta, M')$ by $(\phi|\gamma)(D) = \phi(\gamma D)|\gamma$. This gives rise to a diagram

$$\begin{tikzcd}
 & & & \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta_0, M)) \arrow[d] & \\
 & \frac{C^0}{B^0}(\Gamma, M) \arrow[r] \arrow[d] & \frac{C^0}{B^0}(\Gamma, \mathrm{Hom}(\Delta, M)) \arrow[r] \arrow[d] & \frac{C^0}{B^0}(\Gamma, \mathrm{Hom}(\Delta_0, M)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^1(\Gamma, M) \arrow[r] \arrow[d] & Z^1(\Gamma, \mathrm{Hom}(\Delta, M)) \arrow[r] & Z^1(\Gamma, \mathrm{Hom}(\Delta_0, M)) & \\
 & \mathrm{H}^1(\Gamma, M) & & &
\end{tikzcd}$$

from which we obtain the snake exact sequence

$$\cdots \to \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta, M)) \xrightarrow{\alpha} \mathrm{H}^0(\gamma, \mathrm{Hom}(\Delta_0, M)) \xrightarrow{\delta} \mathrm{H}^1(\Gamma, M) \xrightarrow{\beta} \mathrm{H}^1(\Gamma, \mathrm{Hom}(\Delta, M)) \to \cdots .$$

In proving that this is exact, one takes an element in $\ker(\beta)$ – such as our class φ – and explicitly realises it in the image of δ. Thus we can lift δ by pursuing this diagram chase.

4.2.2. An explicit formula for inverting δ. In practice, we will work at the level of cocycles, and this will depend on choices made throughout the algorithm (whilst the corresponding cohomology classes, and the p-adic L-functions, will not). Suppose we are given a cohomology class $\varphi \in H^1(M)$ represented by a cocycle φ_0. If $M = \mathbb{D}_2$ and φ arises as the lift of a cuspidal classical eigenclass, then it is in $\ker(\beta)$. Then:

Proposition 4.1. *If $\varphi \in \ker(\beta)$, then for any $c_i \in \mathbb{P}^1(K)$, the restriction of φ_0 to $\mathrm{Stab}_\Gamma(c_i)$ is a coboundary in $Z^1(\mathrm{Stab}_\Gamma(c_i), M)$; explictly, there exists $v_i \in M$ such that*

$$\varphi_0(\gamma) = v_i|\gamma^{-1} - v_i$$

for all $\gamma \in \mathrm{Stab}_\Gamma(c_i)$.

Proof. This is essentially a consequence of Shapiro's lemma, but with slightly more explicit control on the coboundaries. The map β is induced by the map of cocycles that sends the cocycle $\varphi_0 : \Gamma \to M$ to

$$\varphi_0 \longmapsto \Big(\beta\,(\varphi_0) : \gamma \mapsto [r \mapsto \varphi_0(\gamma)]\Big),$$

where r is any element of $\mathbb{P}^1(K)$ (recalling that $\Delta = \mathbb{Z}[\mathbb{P}^1(K)]$ is freely generated such r). In particular, each $\gamma \in \Gamma$ is just sent to a constant function in $\mathrm{Hom}(\Delta, M)$. For $\varphi \in \ker(\beta)$, we must have $\beta(\varphi_0)$ is a coboundary, and hence that there exists $v \in \mathrm{Hom}(\Delta, M)$ such that

$$\beta(\varphi_0)(\gamma) = v|\gamma^{-1} - v.$$

In particular, we have

$$\beta(\varphi_0)(\gamma)(r) = v(\gamma^{-1}r)|\gamma^{-1} - v(r) = \varphi_0(\gamma)$$

for all $c \in \mathbb{P}^1(K)$, by definition of $\beta(\varphi_0)$. For c_i as above and $\gamma \in \mathrm{Stab}_\Gamma(c_i)$, we have

$$\varphi_0(\gamma) = v(c_i)|\gamma^{-1} - v(c_i) = v_i|\gamma^{-1} - v_i,$$

for $v_i := v(c_i) \in M$, as required. □

Let now $c_1, ..., c_t \in \mathbb{P}^1(K)$ be a complete set of representatives for the cusps, and for each $r \in \mathbb{P}^1(K)$, let $g_r \in \Gamma$ and $i(r) \in \{1, ..., t\}$ be such that

$$g_r \cdot r = c_{i(r)}.$$

Assuming $\varphi \in \ker(\beta)$, let $v_1, ..., v_t \in \mathbb{D}_2$ be the distributions arising from Proposition 4.1. We can compute the v_i by computing the action of g as a linear operator on $\mathbb{D}_2$ (up to some precision), and then solving the resulting linear system for a sufficiently large set of elements of the stabiliser[3].

Again motivated by the snake lemma, we now define $\widetilde{\Phi} \in \mathrm{Hom}_\Gamma(\Delta, \mathbb{D}_2) = \mathrm{H}^0(\Gamma, \mathrm{Hom}(\Delta, \mathbb{D}_2))$ by setting

$$\begin{aligned} \widetilde{\Phi} : \mathbb{P}^1(K) &\longrightarrow \mathbb{D}_2, \\ r = g_r^{-1}c_{i(r)} &\longmapsto \varphi(g_r)\big|g_r + v_{i(r)}\big|g_r \end{aligned}$$

and extending linearly, and $\Phi = \alpha(\widetilde{\Phi}) : \Delta_0 \longrightarrow \mathbb{D}_2$ defined by

$$\Phi\{r \to s\} := \widetilde{\Phi}(s) - \widetilde{\Phi}(r).$$

Proposition 4.2. *The map Φ gives a well-defined element of* $\mathrm{Symb}_\Gamma(\mathbb{D}_2)$ *with* $\delta(\Phi) = \varphi$.

Proof. The map Φ is linear in $r \to s$, since it is defined as the difference $\widetilde{\Phi}(s) - \widetilde{\Phi}(r)$. It is mapped to φ under δ; setting $r = \gamma \cdot \infty$ and $s = \infty$, we have $g_r = \gamma^{-1}$, $g_s = 1$, and $c_{i(r)} = c_{i(s)} = c_1$. Then by definition,

$$\begin{aligned} \delta(\Phi)(\gamma) := \Phi\{\gamma \cdot \infty \to \infty\} &= \varphi(1)|1 + v_1 - \varphi(\gamma^{-1})|\gamma^{-1} - v_1|\gamma^{-1} \\ &= \varphi(\gamma) + \left[v_1 - v_1|\gamma^{-1}\right], \end{aligned}$$

using that $\varphi(1) = 0 = \varphi(\gamma\gamma^{-1}) = \varphi(\gamma) + \varphi(\gamma^{-1})|\gamma^{-1}$. The term in the square brackets is a coboundary; thus the cocycle $\delta(\Phi)$ represents the same cohomology class as φ.

It remains to show that Φ is Γ-equivariant. Let $\gamma \in \Gamma$. Note that if $g_r \cdot r = c_{i(r)}$, then $g_r\gamma^{-1} \cdot \gamma r = c_{i(r)}$, so that $c_{i(\gamma r)} = c_{i(r)}$ and $g_{\gamma r} = g_r\gamma^{-1}$. Then

$$\begin{aligned} \widetilde{\Phi}(\gamma r) &= \varphi(g_{\gamma r})|g_{\gamma r} + v_{i(\gamma r)}|g_{\gamma r} \\ &= \varphi(g_r\gamma^{-1})|g_r\gamma^{-1} + v_{i(r)}|g_r\gamma^{-1} \\ &= \varphi(g_r)|g_r\gamma^{-1} + \varphi(\gamma^{-1})|g_r^{-1}g_r\gamma^{-1} + v_{i(r)}|g_r\gamma^{-1} \\ &= \widetilde{\Phi}(r)|\gamma^{-1} + \varphi(\gamma^{-1})|\gamma^{-1}. \end{aligned}$$

[3] In practice, this is actually more subtle than it appears: in the linear system, the variables will appear with varying degrees of precision, due to the filtration appearing in the explicit lifting theorem.

The second term is independent of r, so cancels in the difference $\widetilde{\Phi}(s) - \widetilde{\Phi}(r)$. It follows that

$$\Phi\{\gamma r \to \gamma s\}|\gamma = \widetilde{\Phi}(s)|\gamma - \widetilde{\Phi}(r)|\gamma = \Phi\{r \to s\},$$

as required. □

In general, the map Φ thus defined is not an eigensymbol. In particular, whilst there is a unique cuspidal lift Φ_{cusp} of φ under δ, the map Φ can be any element of $\Phi_{\mathrm{cusp}} + \ker(\delta)$. However, we have:

Proposition 4.3. *For all sufficiently large primes $\mathfrak{l}$ of K, coprime to $\mathfrak{p}\mathfrak{n}$, the symbol*

$$\Phi_{\mathcal{F}} := \frac{T_{\mathfrak{l}} - \ell - 1}{a_{\mathfrak{l}} - \ell + 1}\Phi \in \mathrm{Symb}_{\Gamma}(\mathbb{D}_2)$$

is the uniquely determined overconvergent (cuspidal) eigensymbol mapped to φ under δ, where ℓ is the norm of $\mathfrak{l}$ and $a_{\mathfrak{l}}$ is the $T_{\mathfrak{l}}$-eigenvalue of the Bianchi modular form $\mathcal{F}$.

Proof. By the long exact sequence given by the snake lemma, the kernel of δ is given by the image of $\mathrm{Hom}_{\Gamma}(\Delta, \mathbb{D}_2)$ in $\mathrm{Symb}_{\Gamma}(\mathbb{D}_2)$, or, more precisely, the Eisenstein subspace. We know that for prime $\mathfrak{l} \nmid p\mathfrak{n}$ of norm ℓ, the Hecke operator $T_{\mathfrak{l}}$ acts on the Eisenstein subspace by $\ell + 1$ (see, for example, [PS11, Rem. 5.2] for this in the rational case; more generally, it can be obtained by studying $\mathrm{Hom}(\Delta, \mathbb{D}_2)$ as a Hecke-module). For sufficiently large $\mathfrak{l}$, the Hasse bound implies that $a_{\mathfrak{l}} \neq \ell + 1$. By the remarks above, the operator $T_{\mathfrak{l}} - \ell - 1$ kills any Eisenstein contribution, and thus acts as a projector onto the cuspidal subspace. Now renormalising by $a_{\mathfrak{l}} - \ell - 1$ gives a cuspidal eigensymbol in $\delta^{-1}(\varphi)$. Such a symbol is unique by strong multiplicity one combined with Theorem 2.2. □

5. The algorithms

5.1. Computing with the classical cohomology. There are well-established routines for computing classical cohomology classes attached to elliptic curves over number fields, as used by the first author and his collaborators in in [CITE MARC'S PAPERS]. These rely on computing a presentation of the group Γ, using existing Magma code of Aurel Page to compute a fundamental domain for the action of Γ on the upper half-space $\mathcal{H}_3$. Given this, $\mathrm{H}^1(\Gamma, \mathbb{Z})$ is the abelianisation of Γ, and distinguished classes are constructed by computing the kernels of sufficiently many operators $T_{\mathfrak{q}} - \alpha_{\mathfrak{q}}$.

In practice, working with Bianchi groups is a difficult computational problem, and one that greatly restricts the scope of computations. This is particularly the case in the base-change setting. Consider, for example, the smallest elliptic curve of rank 1, which has conductor 37, and say we want to compute the 3-adic L-function; then we must work with the group $\Gamma_0(3 \cdot 37\mathcal{O}_K)$, which has index $\sim 12,000$ in $\mathrm{SL}_2(\mathcal{O}_K)$. The first rank 2 example has conductor 389, leading to index $\sim 1,300,000$, far beyond current state-of-the-art (for example, in [?], the highest conductors appearing for Bianchi groups are of index 500,000).

To counter this computational difficulty, we used ...[DESCRIBE? OR LEAVE OUT?]

5.2. Computing with distributions. Two variable distributions with coefficients in an extension $L/\mathbb{Q}_p$ are naturally in bijection with doubly-indexed bounded sequences in L (see [Wil17, Proposition 3.6]), the map being given by

$$\mu \mapsto \{\mu(x_i y_j) : i, j \geq 0\}.$$

We can compute with these distributions using the *finite approximation modules op. cit.*.

5.2.1. Computing the action of $\Sigma_0(p)^2$ *on distributions.* It is important to explicitly understand the action of $\Sigma_0(p)^2$ on these distributions. In particular, there are an increasing number of conventions regarding actions (both left and right) on distributions, and we wish to be as clear as possible about the ones we adopt.

Definition 5.1. Consider the usual basis $\{x^i y^j : i, j \geq 0\}$ of $\mathbb{A}(L)$, and order it first by total degree $i + j$ and then lexicographically in j and then i. To make this explicit, we label these basis monomials by defining

$$v_{n,i} := x^{n-i} y^i,$$

which are then ordered $v_{0,0}, v_{1,0}, v_{1,1}, v_{2,0}, v_{2,1}, \ldots$, corresponding to $1, x, y, x^2, xy, \ldots$.

Recall that $\Sigma_0(p)^2$ acts on the *left* of $\mathbb{A}(L)$ by

$$g \cdot f(x, y) = f\left(\frac{b + dx}{a + cx}, \frac{\overline{b} + \overline{d}x}{\overline{a} + \overline{c}x}\right), \quad g = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}\right].$$

Denote by $\psi_{\mathbb{A}}(g)$ the matrix of g acting on $\mathbb{A}(L)$ in the basis $\{v_{n,i}\}$.

We want to compute the dual action on distributions. The natural dual action is a *right* action. However, since we will ultimately compute with the group cohomology – which prefers *left* actions – it is convenient here to pass to the right action on $\mathbb{A}_2(L)$ defined by $f|g = g^{-1} \cdot f$, inducing a left dual action.

Now write

$$v_{n,i}^* := \mathcal{X}^{n-i} \mathcal{Y}^i$$

for the dual basis, where $v_{n,i}^*(v_{m,j}) = \delta_{mn}\delta_{ij}$. For , then the matrix of g acting on $\mathbb{D}(L)$, on the *left*, in this dual basis, is given by

$$\psi_{\mathbb{D}}(g) := \psi_{\mathbb{A}}(g^{-1})^T.$$

We are reduced, then, to computing the action on $\mathbb{A}$.

In practice, we use the following algorithm. Define

$$r = \frac{b + dx}{a + cx}, \quad s = \frac{\overline{b} + \overline{d}y}{\overline{a} + \overline{c}x}.$$

We then start by considering $r^0 s^0 = 1$. Suppose we have computed $r^{n-i} s^i$ for all $i \in \{0, ..., n\}$. The coefficients of $r^{n-i} s^i$ give the column (at $v_{n,i}$) of the matrix of g acting on $\mathbb{A}$. We then compute at step $n + 1$ by multiplying by r or s as necessary, computing up to the desired precision.

5.3. An explicit lifting theorem. Including commutative diagram relating arithmetic cohomology and modular symbols with the action of the $U_{\mathfrak{p}}, U_{\overline{\mathfrak{p}}}$ operators. (Not well-defined on cocycles in general).

Complexity in computing this: lots of Up operators. Fox derivative and gradient.

6. Data

[Include our data, in some form or other] ♡

Bookkeeping:

1. I chose f to represent the Bianchi modular form instead of the calligraphic $\mathcal{F}$. Makes sense. — Chris ♠

2. I chose to represent the modular symbols by Φ_f and φ_f. I chose to represent the objects in parabolic cohomology by $[\Theta_f]$ and $[\vartheta_f]$. Any cocycle corresponding to these cohomology classes are represented without the brackets by Θ and ϑ. In particular, also notice that the overconvergent objects are represented by an uppercase Greek alphabet Φ and $[\Theta]$, whereas the objects with trivial coefficients are lowercase "var" Greek φ and $[\vartheta]$. I like these conventions! — Chris ♠

3. I am starting to suspect that the the constant term of the 2-variable power series is zero, if the Elliptic curve has rank 2 over K — can this be seen from the analytic side? The analytic p-adic L-function will certainly be zero as a distribution. This probably translates to the power series being zero when evaluated at the character as well, but I'm not so confident in that statement: and this would equally apply in rank 1. — Chris ♠

4. If I wanted to remove some text from the pdf, I placed stuff under the comment environment in the tex file.

5. My 2 cents — for the purposes of narrative flow, I think it may be better to have in an appendix rather than the main text — the theory of modular symbols over $\mathbb{Q}$, along with description of the main conjectures and explanations of why the examples we work out are interesting. I personally think the descriptions of the conjectures/why the examples are interesting should come either in the introduction or in one of the opening sections, before we do any algorithmic stuff. The modular symbols over $\mathbb{Q}$ would be suited to an appendix, though I don't think we should move the material from section 3.1, as that motivates/explains 3.2 very concretely. — Chris ♠ Can we then label modular forms over Q by g instead of f? — Bharath ♡

♠

6. I generally like to start papers with some extended fluff explaining what we're doing: sort of like a second abstract, but detailing the context in which the paper sits. In particular it is nice for the reader to understand what we've done and why we've done it within the first 5 minutes of picking up the paper. I've written a sketch, and you two can veto/reduce if you object! — Chris ♡

   Actually what you said makes a lot of sense, seems better to start the paper with some motivation than just writing down a bunch of notations. I expanded on what you wrote and added some context for higher codimension Iwasawa theory. I chose to focus almost entirely on the application to pseudo-nullity in the second abstract since the bit about the algorithm and modular symbols appears in the actual abstract and immeditely afterwards — Bharath

♠

7. Actually, the p-adic L-function is in $\mathbb{D}_2$! This is the space of distributions, not measures. It is not, however, a measure, that is a bounded distribution/distribution of growth 0. We should perhaps denote the subspace of measures by $\mathbb{D}_2^0$? — Chris
   Okay, how about now? — Bharath ♡

♠

8. It's really worth checking whether such examples exist. It will be a big boost for our paper to be able to say this — Chris

   So what I did was go to Mathscinet and check all those papers that refer the Math Annalen paper of Coates-Sujatha. I feel this is a more or less exhaustive list with relevant numerical examples – [Bha07, Och09, Jha12, She18, Lim15, LP19a]. This drove me crazy — to go and check each of the numerical examples there. While I did find some examples where the MW rank was 1, most of the data however seem to target specific curves. I tried to phrase the introduction to still emphasize why computing 2-variable objects is important. The writing there is the best motivation I could come up with.

9. By the way, I also commented out the trick which allows us to compute all the moments except the zeroth since we found a way to get around that. If you still feel this is important, could we add this in an appendix instead?

— Bharath

To do:

1. Sketch of sharp/flat construction
2. description of how to get the classical class/citation of Marc's papers
3. More detail on the pseudo-nullity conjecture
4. description of resultant calculations to show coprimality
5. Evidence!!!
6. Fox derivatives and Fox gradients (RH Fox, free differential calculus)
7. Tietze algorithm

References

[BCG+19] FM Bleher, T Chinburg, R Greenberg, M Kakde, G Pappas, R Sharifi, and MJ Taylor. Higher Chern classes in Iwasawa theory. *to appear in Amer. J. Math., available at arXiv:1512.00273*, 2019. (↑ 2.)

[Bha07] Amala Bhave. Analogue of Kida's formula for certain strongly admissible extensions. *J. Number Theory*, 122(1):100–120, 2007. (↑ 8 and 19.)

[BS15] Daniel Barrera Salazar. Overconvergent cohomology of Hilbert modular varieties and p-adic L-functions. *Ann. I. Fourier*, 2015. To appear. (↑ 8 and 14.)

[BSW18] Daniel Barrera Salazar and Chris Williams. Families of Bianchi modular symbols: critical base-change p-adic L-functions and p-adic Artin formalism. 2018. Submitted. (↑ 9 and 11.)

[BSW19] Daniel Barrera Salazar and Chris Williams. Exceptional zeros and $\mathcal{L}$-invariants of Bianchi modular forms. *Trans. Amer. Math. Soc.*, 372(1):1–34, 2019. (↑ 9, 10, and 14.)

[CÇSS17] Francesc Castella, Mirela Çiperiani, Christopher Skinner, and Florian Sprung. On the Iwasawa main conjectures for modular forms at non-ordinary primes. 2017. Preprint. (↑ 7.)

[CS05] J. Coates and R. Sujatha. Fine Selmer groups of elliptic curves over p-adic Lie extensions. *Math. Ann.*, 331(4):809–839, 2005. (↑ 2 and 5.)

[CW94] John Cremona and Elise Whitley. Periods of cusp forms and elliptic curves over imaginary quadratic fields. *Math. Comp.*, 62(205):407–429, 1994. (↑ 10.)

[GMŞ15] Xavier Guitart, Marc Masdeu, and Mehmet Haluk Şengün. Darmon points on elliptic curves over number fields of arbitrary signature. *Proc. Lond. Math. Soc.*, 111(2):484 – 518, 2015. (↑ 8.)

[Gre76] Ralph Greenberg. On the Iwasawa invariants of totally realy number fields. *Amer. J. Math.*, 98(1):263–284, 1976. (↑ 2.)

[Gre94] Ralph Greenberg. Iwasawa theory and p-adic deformations of motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 193–223. Amer. Math. Soc., Providence, RI, 1994. (↑ 4.)

[Gre01] Ralph Greenberg. Iwasawa theory—past and present. In *Class field theory—its centenary and prospect (Tokyo, 1998)*, volume 30 of *Adv. Stud. Pure Math.*, pages 335–385. Math. Soc. Japan, Tokyo, 2001. (↑ 2.)

[Gre07] Matthew Greenberg. Lifting modular symbols of non-critical slope. *Israel J. Math.*, 161:141–155, 2007. (↑ 8.)

[Har87] Günter Harder. Eisenstein cohomology of arithmetic groups: The case GL_2. *Invent. Math.*, 89:37–118, 1987. (↑ 10.)

[Hid85] Haruzo Hida. A p-adic measure attached to the zeta functions associated with two elliptic modular forms. I. *Invent. Math.*, 79(1):159–195, 1985. (↑ 7.)

[Hid88] Haruzo Hida. A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II. *Ann. Inst. Fourier (Grenoble)*, 38(3):1–83, 1988. (↑ 7.)

[Hid93] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *LMS Student Texts*. Cambridge University Press, 1993. (↑ 7.)

[Hid94] Haruzo Hida. On the critical values of L-functions of GL(2) and GL(2)×GL(2). *Duke Math.*, 74:432–528, 1994. (↑ 9 and 11.)

[Hid96] Haruzo Hida. On the search of genuine p-adic modular L-functions for GL(n). *Mém. Soc. Math. Fr. (N.S)*, no. 67:vi+110, 1996. (↑ 7.)

[Jha12] Somnath Jha. Fine Selmer group of Hida deformations over non-commutative p-adic Lie extensions. *Asian J. Math.*, 16(2):353–365, 2012. (↑ 8 and 19.)

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, 295:117–290, 2004. (↑ 5.)

[Kim14] Byoung Du Kim. Signed-Selmer groups over the $\mathbb{Z}_p^2$-extension of an imaginary quadratic field. *Canad. J. Math.*, 66(4):826–843, 2014. (↑ 4 and 6.)

[Kob03] Shin-ichi Kobayashi. Iwasawa theory for elliptic curves at supersingular primes. *Invent. Math.*, 152(1):1–36, 2003. (↑ 4, 5, and 7.)

[Lim15] Meng Fai Lim. On the pseudo-nullity of the dual fine Selmer groups. *Int. J. Number Theory*, 11(7):2055–2063, 2015. (↑ 8 and 19.)

[Loe14] David Loeffler. P-adic integration on ray class groups and non-ordinary p-adic L-functions. In T. Bouganis and O. Venjakob, editors, *Iwasawa 2012: State of the art and recent advances*, volume 7 of *Contributions in Mathematical and Computational Sciences*, pages 357 – 378. Springer, 2014. (↑ 4.)

[LP19a] Antonio Lei and Bharathwaj Palvannan. Codimension two cycles in Iwasawa theory and elliptic curves with supersingular reduction. *Forum Math. Sigma*, 7:e25, 81 pages, 2019. (↑ 2, 7, and 19.)

[LP19b] Antonio Lei and Bharathwaj Palvannan. Codimension two cycles in Iwasawa theory and tensor product of Hida families. *arXiv preprint arXiv:1901.09301*, 2019. (↑ 2.)

[MS03] William G. McCallum and Romyar T. Sharifi. A cup product in the Galois cohomology of number fields. *Duke Math. J.*, 120(2):269–310, 2003. (↑ 2.)

[Och09] Yoshihiro Ochi. A remark on the pseudo-nullity conjecture for fine Selmer groups of elliptic curves. *Comment. Math. Univ. St. Pauli*, 58(1):1–7, 2009. (↑ 8 and 19.)

[Pol03] Robert Pollack. On the p-adic L-function of a modular form at a supersingular prime. *Duke Math. J.*, 118(3):523–558, 2003. (↑ 4.)

[PP09] David Pollack and Robert Pollack. A construction of rigid analytic cohomology classes for congruence subgroups of $\mathrm{SL}_3(\mathbb{Z})$. *Canad. J. Math.*, 61:674–690, 2009. (↑ 14.)

[PS11] Robert Pollack and Glenn Stevens. Overconvergent modular symbols and p-adic L-functions. *Annales Scientifique de l'Ecole Normale Superieure*, 2011. (↑ 2, 3, 8, 11, 12, 13, and 17.)

[She18] Sudhanshu Shekhar. Comparing the corank of fine Selmer group and Selmer group of elliptic curves. *J. Ramanujan Math. Soc.*, 33(2):205–217, 2018. (↑ 8 and 19.)

[SZ14] Christopher Skinner and Wei Zhang. Indivisibility of Heegner points in the multiplicative case. 2014. preprint, arXiv:1407.1099. (↑ 7.)

[Wan16] Xin Wan. Iwasawa main conjecture for supersingular elliptic curves. 2016. preprint, arXiv:1411.6352v4. (↑ 6.)

[Wil17] Chris Williams. *P*-adic *L*-functions of Bianchi modular forms. *Proc. Lond. Math. Soc.*, 114 (4):614 – 656, 2017. (↑ 3, 5, 8, 9, 10, 11, 14, and 17.)

[Wil18] Chris Williams. Lifting non-ordinary cohomology classes for SL_3. *Publ. Mat.*, 62(2):651–675, 2018. (↑ 5.)

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