Say in the lab frame $K$ we have a spherical shell radiating energy inwards radially, and the energy density in radiation is $u_{rad}$. A particle is moving radially outwards (say in the $\hat{x}$ direction) with ultra relativistic speed $\beta c$, $\gamma \gg 1$. I want to find the energy density, $u'_{rad}$, in radiation in the reference frame of the particle, $K'$.

In general, we have that $u_{rad} = \frac{1}{c} \int \int I_\nu d\Omega d\nu$, where $I_\nu$ is the specific intensity in radiation. I am instructed to use this to find the radiation energy density in the reference frame.

So,

$$u'_{rad} = \frac{1}{c} \iint I'_{\nu'} d\Omega' d\nu'$$

The trick (I guess) is to move all variables and the specific intensity to the lab frame. From Rybicki and Lightman eq. 4.110 (page 146) we know that $\frac{I_\nu}{\nu^3}$ is a relativistic invariant. Also, from Doppler's effect, we know that $\nu' = \gamma \nu (1 - \beta \cos\theta)$, so we get

$$u'_{rad} = \frac{1}{c} \iint \nu'^3 \frac{I'_{\nu'}}{\nu'^3} d\Omega' d\nu' = \frac{\gamma^4}{c} \iint \frac{I_\nu}{\nu^3} \nu^3 (1 - \beta \cos\theta)^4 d\nu d\Omega'$$
$$= \frac{\gamma^4}{c} \iint I_\nu (1 - \beta \cos\theta)^4 d\nu d\Omega'$$

So we are left to transfering $d\Omega'$. We know from the aberration formulas that

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \Rightarrow d\cos\theta' = \frac{1 - \beta^2}{(1 - \beta\cos\theta)^2} d\cos\theta$$

so we get that

$$d\Omega' = d\cos\theta' d\phi' = \frac{1 - \beta^2}{(1 - \beta\cos\theta)^2} d\cos\theta d\phi = \frac{1 - \beta^2}{(1 - \beta\cos\theta)^2} d\Omega$$

since $\phi$ remains the same (Rybicki and Lightman p. 110). We finally get that

$$u'_{rad} = \frac{1 - \beta^2}{c} \iint I_\nu (1 - \beta\cos\theta)^2 d\nu d\Omega$$

The integral now is in the lab frame, where $I_\nu$ is isotropic. So we need to evaluate the angular integral only:

$$\int_{-1}^{1} (1 - \beta u)^2 du = \frac{2}{3} \left(\beta^2 + 3\right)$$

So we end up with

$$u'_{rad} = \gamma^4 \frac{2\pi}{c}(1-\beta^2)\frac{2}{3}\left(\beta^2+3\right)\int I_\nu d\nu$$

Since $1-\beta^2 = \gamma^{-2}$, we get

$$u'_{rad} = \gamma^2 \frac{4\pi}{c}\int I_\nu d\nu \left(\frac{\beta^2}{3}+1\right) = \gamma^2\left(\frac{\beta^2}{3}+1\right)u_{rad}$$

Unfortunately, my official answer says that the result is $\gamma^2\left(\frac{\beta^2}{3}+\boldsymbol{\beta}+1\right)u_{rad}$
What did I miss?