Delaunay Triangulations
Height Interpolation

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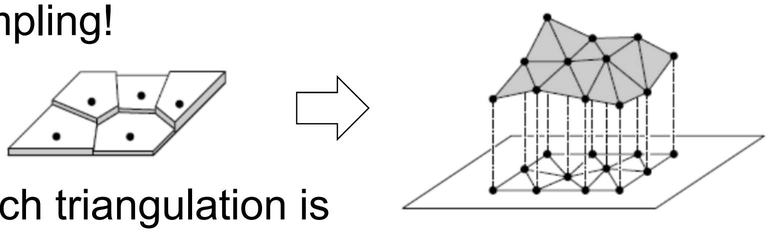
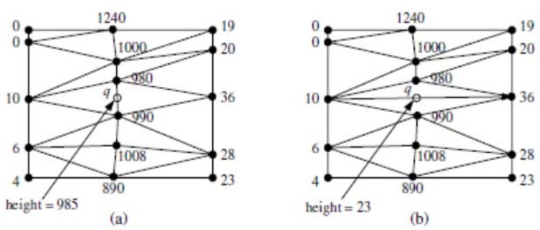
How to Represent a Terrain?

- Sampling!



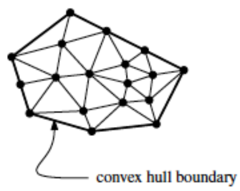
- Which triangulation is better?



Planar Point Sets

- A *maximal planar subdivision* is a subdivision S such that no edge connecting two vertices can be added to S without destroying its planarity.
- Let P := {p_1, p_2, . . . , p_n} be a set of points in the plane, the *triangulation* of *P* is now defined as a maximal planar subdivision whose vertex set is *P*.

Convex Hull vs Triangulation of Planar Point Set



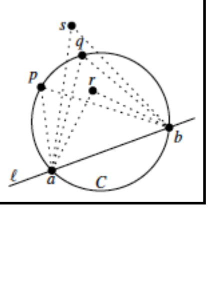
- The convex hull boundary union of the bounded faces of T is always the convex hull of *P*.
- *Let P be a set of n points in the plane, not all collinear, and let k denote the number of points in P that lie on the boundary of the convex hull of P. Then any triangulation of P has* $2n-2-k$ *triangles and* $3n-3-k$ *edges.*

Angle-Vector

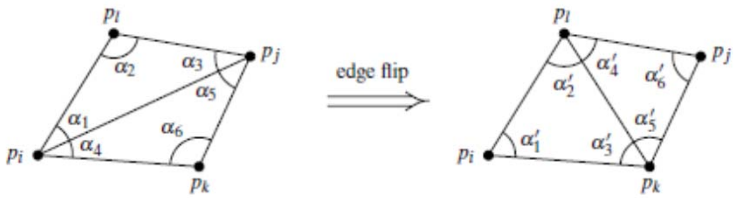
- Consider the $3m$ angles of the triangles of T, sorted by increasing value. Let $\alpha_1, \alpha_2, \ldots, \alpha_{3m}$ be the resulting sequence of angles; hence, $\alpha_i \le \alpha_j$, for $i < j$. We call $A(\mathrm{T}) := (\alpha_1, \alpha_2, \ldots, \alpha_{3m})$ the *angle-vector* of T.
- We say that the angle-vector of T is larger than the angle-vector of T' if A(T) is lexicographically larger than A(T'), or, in other words, if there exists an index i with $1 \le i \le 3m$ such that $\alpha_j = \alpha_j'$ for all $j < i$, and $\alpha_i > \alpha_i'$. => *We denote this as A(T) >= A(T')*

Angle-Optimal Triangulations

- A triangulation T is called *angle-optimal* if A(T) >= A(T') for all triangulations T' of P.
- Thales's Theorem: *Let C be a circle, l a line intersecting C in points a and b, and p, q, r, and s points lying on the same side of l. Suppose that p and q lie on C, that r lies inside C, and that s lies outside C. Then \angle{arb} > \angle{apb}* = \angle{*aqb*} > *\angle{asb}.*



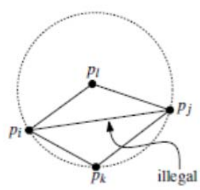
Edge Flip



- We call the edge *e* = \overline{p_ip_j} an *illegal edge* if min{α_i} < min{α_i'}.
- *Let* T *be a triangulation with an illegal edge e. Let* T' *be the triangulation obtained from* T *by flipping e. Then A*(T') > *A*(T).

How to Find Illegal Edge?

- *Let edge \overline{p_ip_j} be incident to triangles $p_ip_jp_k$ and $p_ip_jp_l$, and let C be the circle through p_i, p_j, and p_k. The edge \overline{p_ip_j} is illegal if and only if the point p_l lies in the interior of C.*
- *If the points p_i, p_j, p_k, p_l form a convex quadrilateral and do not lie on a common circle, then exactly one of \overline{p_ip_j} and \overline{p_kp_l} is an illegal edge.*



- A *legal triangulation* is a triangulation that does not contain any illegal edge.

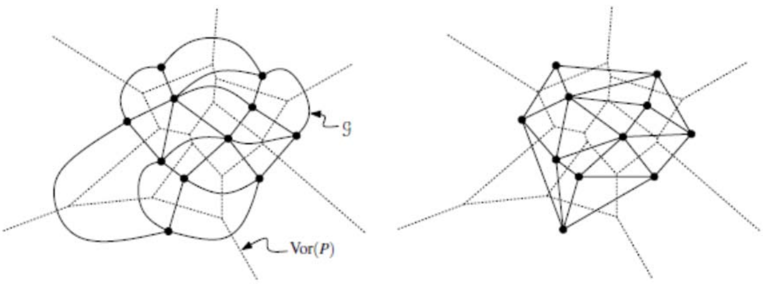
Legalize Triangulation

Algorithm LEGALTRIANGULATION($\mathcal{T}$)
Input. Some triangulation $\mathcal{T}$ of a point set P.
Output. A legal triangulation of P.

while $\mathcal{T}$ contains an illegal edge $\overline{p_i p_j}$
    do (* Flip $\overline{p_i p_j}$ *)
        Let $p_i p_j p_k$ and $p_i p_j p_l$ be the two triangles adjacent to $\overline{p_i p_j}$.
        Remove $\overline{p_i p_j}$ from $\mathcal{T}$, and add $\overline{p_k p_l}$ instead.
return $\mathcal{T}$

Delaunay Triangulation

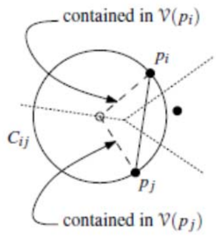
- Dual graph of Voronoi diagram

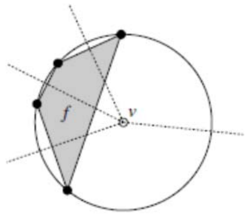


- *Delaunay graph* of *P* is denoted by DG(*P*).

Plane Graph

- The Delaunay graph of a planar point set is a plane graph.





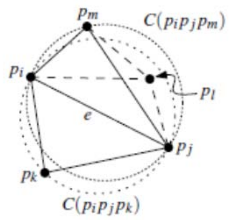
- We assume points are in “general position”, i.e., no 4 points on a circle

Rephrase Theorem for VD in terms of Delaunay Graphs

- *Let P be a set of points in the plane.*
  - *Three points p_i, p_j, $p_k \in P$ are vertices of the same face of the Delaunay graph of P if and only if the circle through p_i, p_j, p_k contains no point of P in its interior.*
  - *Two points p_i, $p_j \in P$ form an edge of the Delaunay graph of P if and only if there is a closed disc C that contains p_i and p_j on its boundary and does not contain any other point of P.*

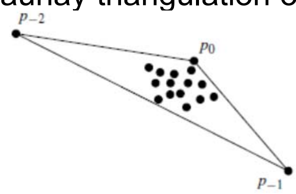
Additional Theorems

- *Let P be a set of points in the plane, and let* T *be a triangulation of P. Then* T *is a Delaunay triangulation of P if and only if the circumcircle of any triangle of* T *does not contain a point of P in its interior.*
- *Let P be a set of points in the plane. A triangulation* T *of P is legal if and only if* T *is a Delaunay triangulation of P.*
- *Let P be a set of points in the plane. Any angle-optimal triangulation of P is a Delaunay triangulation of P.*
- *Any Delaunay triangulation of P max the min angle over all triangulations of P.*

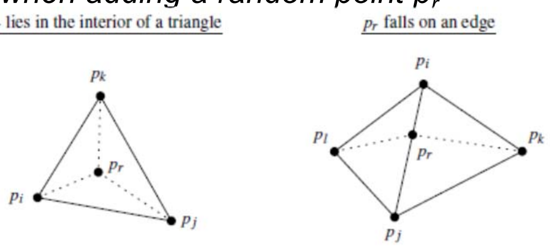


Computing the Delaunay Triangulation

- Computing a Delaunay triangulation of $P \cup \{p_{-1}, p_{-2}\}$



- *Two cases when adding a random point p_r*



Delaunay Triangulation Algorithm

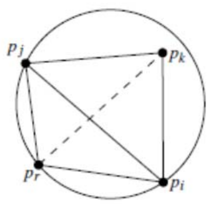
Algorithm DELAUNAYTRIANGULATION(P)
Input. A set P of $n+1$ points in the plane.
Output. A Delaunay triangulation of P.

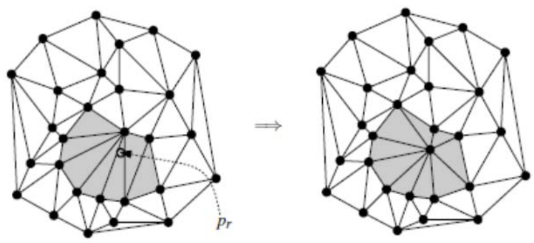
1. Let p_0 be the lexicographically highest point of P, that is, the rightmost among the points with largest y-coordinate.
2. Let p_{-1} and p_{-2} be two points in $\mathbb{R}^2$ sufficiently far away and such that P is contained in the triangle $p_0 p_{-1} p_{-2}$.
3. Initialize $\mathcal{T}$ as the triangulation consisting of the single triangle $p_0 p_{-1} p_{-2}$.
4. Compute a random permutation $p_1, p_2, \ldots, p_n$ of $P \setminus \{p_0\}$.
5. **for** $r \leftarrow 1$ **to** n
6.   **do** (* Insert p_r into $\mathcal{T}$: *)
7.   Find a triangle $p_i p_j p_k \in \mathcal{T}$ containing p_r.
8.   **if** p_r lies in the interior of the triangle $p_i p_j p_k$
9.   **then** Add edges from p_r to the three vertices of $p_i p_j p_k$, thereby splitting $p_i p_j p_k$ into three triangles.
10.   LEGALIZEEDGE($p_r, \overline{p_i p_j}, \mathcal{T}$)
11.   LEGALIZEEDGE($p_r, \overline{p_j p_k}, \mathcal{T}$)
12.   LEGALIZEEDGE($p_r, \overline{p_k p_i}, \mathcal{T}$)
13.   **else** (* p_r lies on an edge of $p_i p_j p_k$, say the edge $\overline{p_i p_j}$ *)
14.   Add edges from p_r to p_k and to the third vertex p_l of the other triangle that is incident to $\overline{p_i p_j}$, thereby splitting the two triangles incident to $\overline{p_i p_j}$ into four triangles.
15.   LEGALIZEEDGE($p_r, \overline{p_i p_l}, \mathcal{T}$)
16.   LEGALIZEEDGE($p_r, \overline{p_l p_j}, \mathcal{T}$)
17.   LEGALIZEEDGE($p_r, \overline{p_j p_k}, \mathcal{T}$)
18.   LEGALIZEEDGE($p_r, \overline{p_k p_i}, \mathcal{T}$)
19. Discard p_{-1} and p_{-2} with all their incident edges from $\mathcal{T}$.
20. **return** $\mathcal{T}$

Legalize Edge

LEGALIZEEDGE($p_r, \overline{p_i p_j}, \mathcal{T}$)

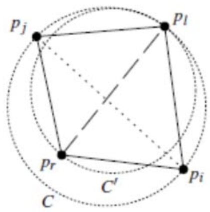
1. (* The point being inserted is p_r, and $\overline{p_i p_j}$ is the edge of $\mathcal{T}$ that may need to be flipped. *)
2. **if** $\overline{p_i p_j}$ is illegal
3.   **then** Let $p_i p_j p_k$ be the triangle adjacent to $p_r p_i p_j$ along $\overline{p_i p_j}$.
4.   (* Flip $\overline{p_i p_j}$: *) Replace $\overline{p_i p_j}$ with $\overline{p_r p_k}$.
5.   LEGALIZEEDGE($p_r, \overline{p_i p_k}, \mathcal{T}$)
6.   LEGALIZEEDGE($p_r, \overline{p_k p_j}, \mathcal{T}$)



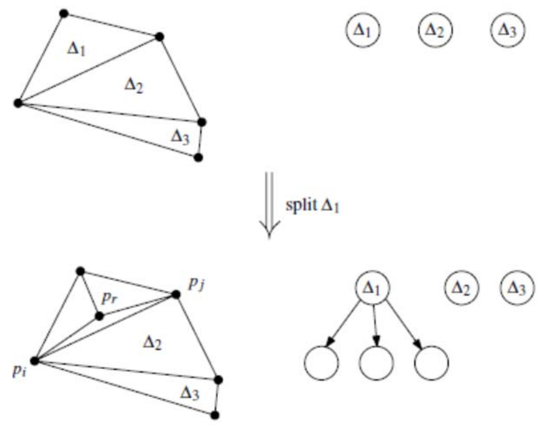


Correctness

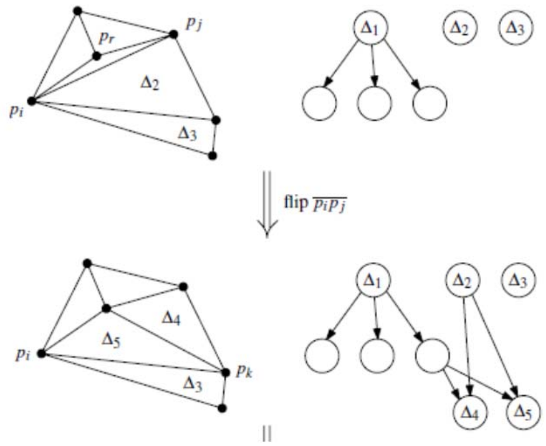
- *Every new edge created in* DELAUNAYTRIANGULATION *or in* LEGALIZEEDGE *during the insertion of p_r is an edge of the Delaunay graph of $\{p_{-2}, p_{-1}, p_0, \dots, p_r\}$.*



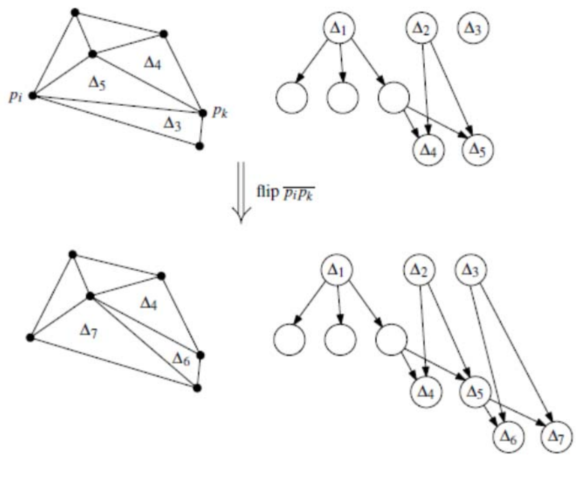
Process of DG Creation (1/3)



Process of DG Creation (2/3)



Process of DG Creation (3/3)



How to Pick P_{-1} and P_{-2}

- Treat them "symbolically"
- $p = (x_p, y_p)$ is higher than $q = (x_q, y_q)$ if
  - $y_p > y_q$ or
  - $y_p = y_q$ and $x_q > x_p$.
  - The lexicographic ordering of P is defined by this relation.
- Let l_{-1} be a horizontal line lying below P, and l_{-2} be a horizontal line lying above P
  - We choose p_{-1} on l_{-1} sufficiently far to the right and p_{-2} on l_{-2} sufficiently far to the left

Determine whether or not \line{$p_i p_j$} is illegal

- Equivalent statements
  - p_j lies to the left of the line from p_i to p_{-1}
  - p_j lies to the left of the line from p_{-2} to p_i
  - p_j is lexicographically larger than p_i
- Let \line{$p_i p_j$} be the edge to be tested, and let p_k and p_l be the other vertices of the triangles incident to \line{$p_i p_j$}
  - \line{$p_i p_j$} is an edge of \triangle{$p_0 p_{-1} p_{-2}$}. These edges are always legal.
  - The indices i, j, k, l are all non-negative. This is the normal case; none of the points involved in the test is treated symbolically.
  - All other cases. In this case, \line{$p_i p_j$} is legal if and only if min(k, l) < min(i, j).

of Triangles Created in Step r (1/2)

- $P_r := \{p_1, \ldots, p_r\}$
- $DG_r := DG(\{p_{-2}, p_{-1}, p_0\} \cup P_r)$
- *The expected number of triangles created by* DELAUNAYTRIANGULATION *is at most* $9n+1$*.*
  - Initial p_r insertion creates 3 triangles and each subsequent flip creates 2 triangles
  - If after the insertion of p_r there are k edges of DG_r incident to p_r, then we have created at most $2(k-3)+3 = 2k-3$ new triangles.

of Triangles Created in Step r (2/2)

- What is the average of k ($\deg(p_r, DG_r)$)?
  - DGr has at most 3(r+3) – 6 edges
  - The total degree is less than 2[3(r+3) – 9]=6r.
  - Backward analysis tells us it's 6!
- The # of triangles created in step r:

$$\begin{aligned}\mathrm{E}[\text{number of triangles created in step } r] &\leqslant \mathrm{E}[2\deg(p_r, \mathcal{DG}_r) - 3] \\ &= 2\mathrm{E}[\deg(p_r, \mathcal{DG}_r)] - 3 \\ &\leqslant 2\cdot 6 - 3 = 9\end{aligned}$$

Time Complexity

- *The Delaunay triangulation of a set P of n points in the plane can be computed in O(nlogn) expected time, using O(n) expected storage.*
  - The time spent by the algorithm (not counting the time to find the point location) is proportional to the number of created triangles, which is O(n).
  - The storage space is obviously O(n).

Point Location Steps (1/2)

- Locating p_r is O(1) + linear time in the number of triangles that were present at some earlier stage, but have been destroyed, and contain p_r.
- Two cases $p_i p_j p_k$ can be destroyed:
  1. A new point p_l has been inserted inside (or on the boundary of) $p_i p_j p_k$, and $p_i p_j p_k$ was split into three (or two) subtriangles.
  2. An edge flip has replaced $p_i p_j p_k$ and an adjacent triangle $p_i p_j p_l$ by the pair $p_k p_i p_l$ and $p_k p_j p_l$.
- In all cases we can say the circumcircle of a Delaunay triangle $p_i p_j p_k$ contains p_r.

Point Location Steps (2/2)

- Denote the subset of points in *P* that lie in the circumcircle of a given triangle Δ by *K*(Δ). the total time for the point location steps is $O(n + \sum_{\Delta} \mathrm{card}(K(\Delta)))$
- What is $\mathrm{card}(K(\Delta))$?
  - When r = 1 it will be n
  - When r = n it will be 0
  - Randomization allows us to "interpolate" => O(n/r)

Formal Proof (1/5)

- *If P is a point set in general position, then* $\sum_{\Delta} \mathrm{card}(K(\Delta)) = O(n \log n)$, *where the summation is over all Delaunay triangles Δ created by the algorithm.*
- We denote the set of triangles of DG_r by T_r. Now the set of Delaunay triangles created in stage *r* equals $T_r \setminus T_{r-1}$
- The above becomes: $\sum_{r=1}^{n} \left(\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta)) \right)$

Formal Proof (2/5)

- For a point q, let $k(P_r, q)$ denote the number of triangles $\Delta \in T_r$ such that $q \in K(\Delta)$, and let $k(P_r, q, p_r)$ be the number of triangles $\Delta \in T_r$ such that not only $q \in K(\Delta)$ but for which we also have that p_r is incident to Δ.

$$\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \text{card}(K(\Delta)) = \sum_{q \in P \setminus P_r} k(P_r, q, p_r)$$

Formal Proof (3/5)

- Fix p_r, consider all possible permutations of P where $P_r = P^*_r$, the value $k(P_r, q, p_r)$ then depends only on the choice of p_r. Since a triangle $\Delta \in T_r$ is incident to a random point $p \in P^*_r$ with probability at most $3/r$, we get $\mathrm{E}[k(P_r, q, p_r)] \leqslant \frac{3k(P_r, q)}{r}$
- Sum this over all $q \in P \setminus P_r$

$$\mathrm{E}[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \text{card}(K(\Delta))] \leqslant \frac{3}{r} \sum_{q \in P \setminus P_r} k(P_r, q)$$

Formal Proof (4/5)

- Every $q \in P \backslash P_r$ is equally likely to appear as p_{r+1} => $\mathrm{E}[k(P_r, p_{r+1})] = \frac{1}{n-r} \sum_{q \in P \setminus P_r} k(P_r, q)$

  => $\mathrm{E}[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))] \leqslant 3\left(\frac{n-r}{r}\right) \mathrm{E}[k(P_r, p_{r+1})]$
- $k(P_r, p_{r+1})$ is the number of triangles Δ of T_r that have $p_{r+1} \in K(\Delta)$.
  - These triangles are exactly the triangles of T_r that will be destroyed by the insertion of p_{r+1}.

$$\mathrm{E}[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))] \leqslant 3\left(\frac{n-r}{r}\right) \mathrm{E}[\mathrm{card}(\mathcal{T}_r \setminus \mathcal{T}_{r+1})]$$

Formal Proof (5/5)

- The number of triangles in T_m is precisely $2(m+3)-2-3 = 2m+1$. Therefore, the number of triangles *destroyed* by the insertion of point p_{r+1} is exactly two less than the number of triangles *created* by the insertion of p_{r+1}

$$\mathrm{E}[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))] \leqslant 3\left(\frac{n-r}{r}\right)\left(\mathrm{E}[\mathrm{card}(\mathcal{T}_{r+1} \setminus \mathcal{T}_r)] - 2\right)$$

- The number of triangles created by the insertion of p_{r+1} is identical to the number of edges incident to p_{r+1} in T_{r+1}, and that the expected number of these edges is at most 6.

$$\mathrm{E}[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))] \leqslant 12\left(\frac{n-r}{r}\right)$$

Homework Assignment 9

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Bonus

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