Module 5 - Homework 5

Colt Thomas

March 26, 2021

Problem 1

The joint probability density function of X and Y is defined below. Let Z=X+Y, where X and Y are dependent random variables. Find the probability density function of Z.

$$f_{X,Y}(x,y) \begin{cases} 2e^{-(x+y)} & 0 \leq y \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

For this problem it is easier to first find the CDF, and then the PDF since CDFs can be represented as a probability over a range. Let's find $F_X(x) = P(Z \leq z)$. To do this, we will need to find the bounds of X and Y. Note that $P(Z \leq z) = P(X + Y \leq z)$. Let D be an area bound by the range of X and Y. We have $y = z - x$, given by the definition of Z, as our first bound. Since $0 \leq y \leq x \leq \infty$, we also have a bound of $y = 0$ and $y = x$. This forms the plot shown in Figure below.

Now that we know our bounds, we can quantify this into integral bounds such that:

$$F_Z(z) = \int \int_D f_{X,Y}(x,y)dxdy$$

$$F_Z(z) = \int_0^{z/2} \int_0^x f_{X,Y}(x,y)dydx + \int_{z/2}^{z} \int_0^{z-x} f_{X,Y}(x,y)dydx$$

Note that because the area is an equilateral triangle, you can just double half of one of the intervals:

$$F_Z(z) = \int_0^{z/2} \int_0^x 2e^{-(x+y)}dydx + \int_{z/2}^{z} \int_0^{z-x} 2e^{-(x+y)}dydx$$

$$F_Z(z) = 2\int_0^{z/2} e^{-x} \int_0^x e^{-y}dydx + 2\int_{z/2}^{z} e^{-x} \int_0^{z-x} e^{-y}dydx$$

$$F_Z(z) = 2\int_0^{z/2} e^{-x} - 1dx + 2\int_{z/2}^{z} e^{-x} - e^{-z}dx$$

$$F_Z(z) = 1 - e^{-z} - ze^{-z}$$

Take the derivative to find the PDF. Note that the integral from $(0, \infty)$ is 1:

$$\boxed{f_Z(z) = \begin{cases} ze^{-z} & z \geq 0 \\ 0 & z < 0 \end{cases}}$$

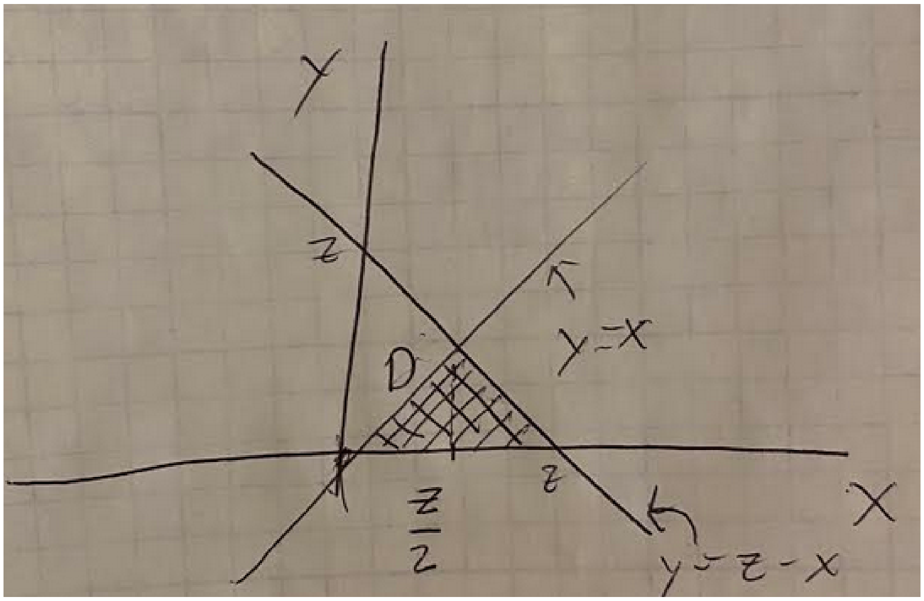

Figure 1: A plot of the area D bound by the joint PDF for problem 1.

Problem 2

Suppose $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ with X and Y independent. Find the joint PDF of Z and W where

$$Z = X^2 + Y^2$$
$$W = X^2 - Y^2$$

For this problem I will use the expected value, variance and moment generating functions to find the joint PDF of Z and W. First, let's find the expected values:

$$E[Z] = E[X^2] + E[Y^2]$$
$$E[W] = E[X^2] - E[Y^2]$$

One of the properties of a gaussian with zero mean is the following:

$$Var(X) = E[X^2] - E[X]^2 = E[X^2] - 0^2 = E[X^2]$$

So we then get:

$$\boxed{E[Z] = \sigma_X^2 + \sigma_Y^2}$$

$$\boxed{E[W] = \sigma_X^2 - \sigma_Y^2}$$

Next we just need the variance:

$$Var(Z) = Var(X^2 + Y^2) = Var(X^2) + Var(Y^2) - 2Cov(X^2, Y^2)$$

$$Var(W) = Var(X^2 - Y^2) = Var(X^2) + Var(Y^2) - 2Cov(X^2, Y^2)$$

Note that X and Y have zero mean and are thus jointly normal, and $Var(Z) = Var(W)$. Page 219 eq 6-199 shows a proof that if two random variables are jointly normal with zero mean:

$$E[X^2Y^2] = E[X^2]E[Y^2] + 2E[XY]^2$$

Also remember that X and Y are independent:

$$E[X^2Y^2] = E[X^2]E[Y^2] + 2E[X]^2E[Y]^2$$

The definition of covariance is:

$$Cov(X^2, Y^2) = E[X^2Y^2] - E[X^2]E[Y^2]$$

Let's plug in these values for covariance into our equation above:

$$Var(Z) = Var(X^2)+Var(Y^2)-2Cov(X^2, Y^2) = Var(X^2)+Var(Y^2)-2(E[X^2Y^2]-E[X^2]E[Y^2])$$

$$Var(Z) = Var(X^2)+Var(Y^2)-2(E[X^2]E[Y^2]+2E[X]^2E[Y]^2-E[X^2]E[Y^2])$$

$$Var(Z) = Var(X^2) + Var(Y^2) - 2(E[X^2]E[Y^2] + E[X]^2E[Y]^2)$$

$$Var(Z) = Var(X^2) + Var(Y^2) - 2(\sigma_X^2\sigma_Y^2 + 0)$$

Now we simplify the variance expression:

$$Var(X^2) = E[X^4] - E[X^2]^2 = 3\sigma_X^4 - \sigma_X^4 = 2\sigma_X^4$$

Proceed to plug and chug...

$$\boxed{Var(Z) = Var(W) = 2\sigma_X^4 + 2\sigma_Y^4 - 2(\sigma_X^2\sigma_Y^2)}$$

Note that because X and Y are independent, W and Z are as well (see pg 174 of the book). We can conclude that the joint distribution $f_{Z,W}(z, w) = f_Z(z)f_W(w)$, which is:

$$f_{Z,W}(z, w) = f_Z(z)f_W(w) = \left(\frac{1}{\sqrt{2\pi\sigma_Z^2}}\right)e^{-(z-\mu_Z)^2/(2\sigma_Z^2)}\left(\frac{1}{\sqrt{2\pi\sigma_w^2}}\right)e^{-(w-\mu_W)^2/(2\sigma_W^2)}$$

$$\boxed{f_{Z,W}(z, w) = \left(\frac{1}{2\pi\sigma_Z\sigma_W}\right)e^{-(z-\mu_Z)^2/(2\sigma_Z^2)}e^{-(w-\mu_W)^2/(2\sigma_W^2)}}$$

Where $\mu_Z = E[Z]$, $\mu_W = E[W]$, and $\sigma_W^2 = Var(W)$, $\sigma_Z^2 = Var(Z)$

Problem 3

Write a Matlab program that generates independent jointly Gaussian random variables X and Y that are both zero-mean, unit variance by using the following procedure:

1. Generate U_1 and U_2, two independent random variables uniformly (use **rand**) distributed in the unit interval.
2. Set $R^2 = -2\log(U_1)$ and $\Theta = 2\pi U_2$.
3. Set $X = R\cos(\Theta) = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$.
4. Set $Y = R\sin(\Theta) = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$.

Make one histogram plot of X and another histogram plot of Y using 1000 samples in both cases. Note that R2 is an exponential random variable with parameter1/2and Θ is a random variable uniformly distributed in the interval $(0, 2\pi)$.

Figure 2 shows a plot of the histograms generated by following the steps below. Code is inserted in the appendix with appropriate comments.

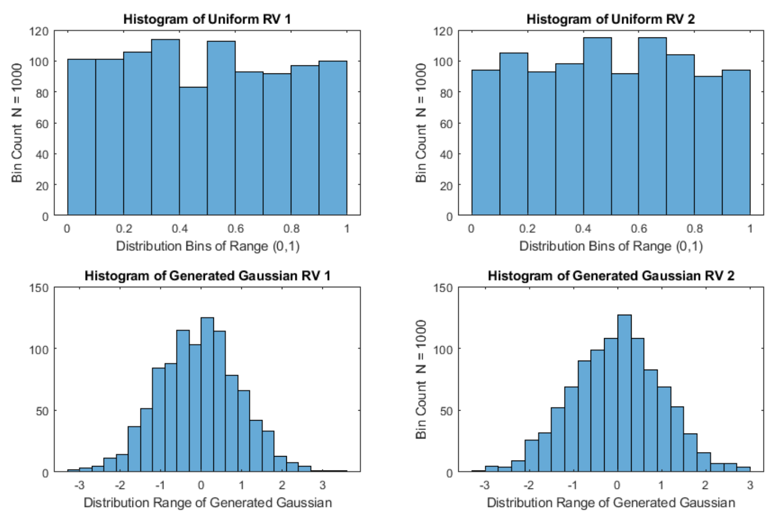

Figure 2: Histograms of the generated uniform random variables, and the respective Gaussian conversions performed in Matlab.

Problem 4

(A) Use the auxiliary variable method to find the PDF of

$$Z = \frac{X}{X+Y}$$

Let $Y = W$, and solve for X:

$$Z = \frac{X}{X+Y}$$

$$X = \frac{-WZ}{Z-1}$$

Next find the Jacobian. Note that we are finding $J(z,w)$, so we can find $f_{Z,W}(z,w) = f_{X,Y}|J(z,w)|$

$$J(z,w) = det \begin{bmatrix} \frac{dx}{dz} & \frac{dx}{dw} \\ \frac{dy}{dz} & \frac{dy}{dw} \end{bmatrix}$$

$$J(z,w) = det \begin{bmatrix} \frac{w}{(z-1)^2} & \frac{-z}{z-1} \\ 0 & 1 \end{bmatrix}$$

$$J(z,w) = \frac{w}{(z-1)^2}$$

Now we can find the pdf of Z:

$$f_{Z,W}(z,w) = \frac{|w|}{|(z-1)^2|} f_{X,Y}(\frac{-wz}{z-1}, w)dw$$

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} \frac{|w|}{(z-1)^2} f_{X,Y}(\frac{-wz}{z-1}, w)dw}$$

(B) Find the PDF of Z if X and Y are independent exponential random variables with the same parameter a.

X and Y being independent means $f_{X,Y}(x,y,) = f_X(x)f_Y(y)$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{|w|}{(z-1)^2} f_X(\frac{-wz}{z-1}) f_Y(w)dw$$

The PDF of an exponential random variable is $f_X(x) = \lambda e^{\lambda x}$. Assume $\lambda = a$ for the parameter.

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{|w|}{(z-1)^2} \lambda e^{-\lambda \frac{-wz}{z-1}} \lambda e^{-\lambda w} dw$$

$$f_Z(z) = \frac{1}{(z-1)^2} \int_0^{\infty} w\lambda^2 e^{\frac{\lambda w}{z-1}} dw$$

$$f_Z(z) = \frac{\lambda^2}{(z-1)^2}\int_0^\infty we^{\frac{\lambda w}{z-1}}dw$$

Let $u = \frac{-\lambda w}{z-1}$ and $du = \frac{-\lambda}{z-1}dw$

$$f_Z(z) = \frac{\lambda^2}{(z-1)^2}\int_0^\infty w(\frac{z-1}{-\lambda})e^{-u}du$$

$$f_Z(z) = \frac{-\lambda}{(z-1)}\int_0^\infty we^{-u}du$$

$$\boxed{f_Z(z) = \int_0^\infty ue^{-u}du = 1}$$

Notice that this is can be represented as a uniform distribution $U(a,b) \sim f_X(x) = \frac{1}{b-a}a < x < b$ where $a = 0$ and $b = 1$, so $\boxed{Z \sim U(0,1)}$

Problem 5

Write a function that randomly generates a length 50 sequence of A,B,C, and D letters using the rand function in Matlab. The function should also return the probability mass function associated with this sequence. Execute this function once and print out the sequence generated and the resulting pmf. Also, calculate the entropy of the pmf.

For this problem, we are going to use a uniform random variable. The chars will be mapped by using the rand function, and scaling the $(0,1)$ values to $(1,4)$ to represent all 4 chars. I took the liberty to add the integer value 65 for readability of the generated sequence, since 'A' = 65 on the ascii table. Below is a screenshot of our calculated results, in addition to the code added to the appendix.

```
>> Homework5
Calculated entropy: 12.358976
Calculated PMF: 0.200000,0.180000,0.300000,0.320000
Generated sequence: AAAAAAAAAABBBBBBBBBCCCCCCCCCCCCCCCDDDDDDDDDDDDDDDD
>>
```

Matlab Code

```
%%%%%%%%%%%%%%%%
% Problem 3
%%%%%%%%%%%%%%%%

% generate the two independent random variables uniformly distributed from
% (0,1).
samples = 1000;
U_1 = rand(1,samples);
U_2 = rand(1,samples);

R = sqrt(-2*log(U_1));
Theta = 2*pi*U_2;

X = R.*cos(Theta);
Y = R.*sin(Theta);

% Histogram plots of X and Y
figure(1)
subplot(2,2,1);
histogram(U_1);
title("Histogram of Uniform RV 1");
xlabel("Distribution Bins of Range (0,1)");
ylabel("Bin Count  N = "+samples);
subplot(2,2,2);
histogram(U_2);
title("Histogram of Uniform RV 2");
xlabel("Distribution Bins of Range (0,1)");
ylabel("Bin Count  N = "+samples);
subplot(2,2,3);
histogram(X);
title("Histogram of Generated Gaussian RV 1");
xlabel("Distribution Range of Generated Gaussian");
subplot(2,2,4);
histogram(Y);
title("Histogram of Generated Gaussian RV 2");
xlabel("Distribution Range of Generated Gaussian");
ylabel("Bin Count  N = "+samples);
%%%%%%%%%%%%%%%%
% Problem 5
%%%%%%%%%%%%%%%%


```

```
%% Generate a 50 length sequence of A,B, C, and D, letters

% We need a random variable to produce values from 0 to 1
RV = rand(1,50); % uniform random variable

% calculate entropy before scaling
Entropy = -sum(RV.*log2(RV));

fprintf("Calculated entropy: %f\n",Entropy);
% note that A,B,C and D have ascii values of 65-68. We can scale the
% RV to generate values from 0-4, then add 65
RV = RV.*4 + 65; % scale values to be from 0-4

% round down, so we are left with integers that we can cast to chars
RV = floor(RV);

% Histcounts can take your data and bin the values. You pass in the RV,
% the edges (Inf allows us to bin the letter D at the edge), the remaining
% parameters scale the values back down so the weights sum to 1.
PMF = histcounts(RV,[unique(RV) Inf],'Normalization','probability');
fprintf("Calculated PMF: %f,%f,%f,%f \n",PMF(1),PMF(2),PMF(3),PMF(4));
% sorting this makes it easier to compare the pmf to the results
RV = sort(char(RV));
fprintf("Generated sequence: "+RV+"\n");
```