Module 8 - Homework 8

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Problem 7.1-6

In the transmission of a string of 15 bits, each bit is likely to be in error with probability of p=0.4 independently. Find the probabilities of the following events:

(A)

There are exactly 3 errors and 12 correct bits

The reception of each bit can be represented as a Bernoulli trial. There are $2^15 = 32768$ different bit combinations. Let $k = 12$ number of successes, and $n = 15$ be the number of Bernoulli trials. Let A be the event where we have exactly 3 errors and 12 correct bits. We get:

$$P(A) = \binom{n}{k}(1-p)^k(p)^{n-k}$$

$$P(A) = \binom{15}{12}(1-0.4)^{12}(0.4)^{15-12}$$

$$P(A) = (455)(0.6)^{12}(0.4)^3 = \boxed{0.063}$$

(B)

There are at least 4 errors

From our last problem we found the probability of having exactly 3 errors. If we calculate the same for 2,1, and 0 errors, we can sum them and find the complement of event B, where B is the event where we have at least 4 errors.

$$P(B) = 1 - P(\text{exactly 3 errors}) - P(\text{exactly 2 errors}) - P(\text{exactly 1 errors}) - P(\text{exactly 0 errors})$$

$$P(\text{exactly 2 errors}) = \binom{15}{13}(0.6)^{13}(0.4)^2 = 0.022$$

$$P(\text{exactly 1 errors}) = \binom{15}{14}(0.6)^{14}(0.4)^{1} = 0.0047$$

$$P(\text{exactly 0 errors}) = \binom{15}{15}(0.6)^{15}(0.4)^{0} = 0.00047$$

Combined we get:

$$P(B) = 1 - 0.063 - 0.022 - 0.0047 - 0.00047$$

$$P(B) = \boxed{0.909}$$

Problem 7.1-8

A network consists of ten links $s_1, s_2, , ..., s_{10}$ in cascade. If any one of the links fails, the entre system fails. All links are independent with equal probability of failure p.

(A)

The probability of a link falure is 0.03. What is the probability of failure of the network?

This is a Bernoulli trial problem very similar to the last problem we did. Let event A represent the failure of the network, and event B be the event that 0 out of 10 of the links fail. Let $n = 10$ trials representing our links, and $k = 10$ representing our successes of links:

$$P(A) = 1 - P(B) = 1 - \binom{10}{10}(1 - 0.03)^{10}(0.03)^{0} = \boxed{0.263}$$

(B)

The reliability of a network is the probability of not failing. If the system reliability is required to be 0.99, how small must be the failure probability of each link?

The reliability of the network is $1 - P(A)$ where $P(A)$ is the probability of failure. Let's solve for p with $1 - P(A) = 0.99$:

$$1 - P(A) = 0.99$$

$$1 - 1 + \binom{10}{10}(1 - p)^{10}(p)^{0} = 0.99$$

$$(1 - p)^{10} = 0.99$$

$$\boxed{p = 0.001}$$

(C)

Repeat part (a) if link s_1 has a probability of failure 0.03 while other links can fail with equal probability of 0.02.

We need to split this problem in two: Let B be the event that s_1 fails, and $P(C)$ be the event that any of the following links fail. Like part (A), let A be the event the network fails. We need to consider the situations where S_1 fails, and the remaining links fail. Using the Total Probability Theorem and the definition of Independent Events we can derive the expression for $P(A)$ as follows:

$$P(A) = P(B \cap C^c) + P(B \cap C) + P(B^c \cap C)$$
$$P(A) = P(B)P(C^c) + P(B)P(C) + P(B^c)P(C)$$

Notice we don't include $P(B^c \cap C^c) = P(B^c)P(C^c)$ because that is the scenario where no failure happens at all. Let's find $P(B)$ (already given) and $P(C)$:

$$P(B) = 0.03$$
$$P(B^c) = 1 - P(B) = 0.97$$
$$P(C) = 1 - \binom{9}{9}(1 - 0.02)^9(0.02)^0 = 0.166$$
$$P(C^c) = 1 - P(C) = 0.833$$

Plug these into our equations above to get:

$$P(A) = (0.03)(0.833) + (0.03)(0.166) + (0.97)(0.166) = \boxed{0.191}$$

This can be verified by finding $1 - P(B^c)P(C^c)$ since that is also the probability of failure, which gives the same answer.

Problem 7.1-15

In a binary communication channel, the receiver detects binary pulses with an error probability P_e. What is the probability that out of 100 received digits, no more than four digits are in error?

Another Bernoulli trial, let the question be event A. Just sum the outcomes of 0-4 "successes" out of $n = 100$ trials:

$$\boxed{P(A) = \sum_{k=0}^{4} \binom{100}{k}(1 - P_e)^{100-k}(P_e)^k}$$

Problem 7.2-2

A binary symmetric channel has an error probability P_e. The probability of transmitting 1 is Q. If the receiver detects an incoming digit as 1, what is the probability that the originally transmitted digit was 1? And 0?

See example 7.13 for full details. We are going to use the total probability theorem. Below are the probabilities:

$$P_{x|y}(0|1) = \frac{P_{y|x}(1|0)P_x(0)}{P_{y|x}(1|1)P_x(1) + P_{y|x}(1|0)P_x(0)}$$

$$P_{x|y}(0|1) = \frac{P_e(Q-1)}{(1-P_e)Q + P_e(1-Q)}$$

$$\boxed{P_{x|y}(0|1) = \frac{P_eQ - P_e}{Q - P_eQ + P_e - P_eQ}}$$

$$P_{x|y}(1|1) = \frac{P_{y|x}(1|1)P_x(1)}{P_{y|x}(1|1)P_x(1) + P_{y|x}(1|0)P_x(0)}$$

$$\boxed{P_{x|y}(1|1) = \frac{(1-P_e)Q}{(1-P_e)Q + P_e(1-Q)}}$$