Module 2 - Homework 2

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Problem 1

We measure the resistance R of each resistor in a production line and we accept only the units where the resistance is between 96 and 104 ohms. Find the percentage of the accepted units...

(a) *If R is uniform between 90 and 110 ohms.*

We can model this as a probability density function $f_{\mathbf{R}}(x)$, where $\mathbf{R}$ is a continuous random variable that models the resistance of produced resistors. For this problem, assume that $\mathbf{R}$ is uniform. $f_{\mathbf{R}}(x)$ can be described as follows:

$$f_{\mathbf{R}}(x) = \begin{cases} 0 & x < 90 \\ 1/20 & 90 \leq x \leq 110 \\ 0 & x > 110 \end{cases}$$

Note that this is essentially a rect function with an area of 1 (or think of it as 100%). To find the percentage of accepted units we simply take the integral, or more formally (see page 90 in the book, eq 4-46):

$$P(\mathbf{X} \in B) = \int_B f_{\mathbf{R}}(x)dx$$

Let $B =$ numbers between 96 and 104. We get:

$$\int_{96}^{104} \frac{1}{20} dx = \boxed{0.4}$$

(B) *If R is normal with $\mu = 100$ and $\sigma = 2$ ohms.*

Using the definition listed on page 84 (eq 4-25) for a gaussian/normal distribution we have the following probability density function:

$$f_{\mathbf{R}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

Plug and chug, with $B =$ numbers between 96 and 104:

$$P(\mathbf{X} \in B) = \int_B f_{\mathbf{R}}(x)dx$$

$$P(\mathbf{X} \in B) = \frac{\sqrt{2}}{4\sqrt{\pi}} \int_{96}^{110} e^{-(1/8)(x-100)^2} dx$$

$$P(\mathbf{X} \in B) = \boxed{0.97725}$$

Problem 2

A fair coin is tossed until the first time that the same side appears twice in succession. Let the random variable N be the number of tosses required.

(a) *Determine the probability mass function for N*

A probability mass function can be defined as a summation of discrete probabilities (see pg 82 in the book):

$$f_x(x) = \sum_i p_i \delta(x - x_i)$$

With N as the random variable as described above, it will take at least two coin tosses to achieve the event where the same side appears twice in a row. We know by one of the axioms of probability that:

$$\sum_i p_x(x_i) = 1$$

The range of N is $[2, \infty)$, since it takes at least 2 coin tosses to achieve our desired outcome. We need to find a p_x that satisfies the following:

$$\sum_{i=2}^{\infty} p_x(x_i) = 1$$

Note that we have a fair coin. Let $p_x(2) = P((t,t) \text{ or } (h,h)) = 0.5$, where the set of all possible outcomes for $N = 2$ is $\{(t,t),(t,h),(h,t),(h,h)\}$. We have a type of geometric series for a random variable where:

$$\begin{cases} N_2 = 2 \text{ out of 4 outcomes} \\ N_3 = 2 \text{ out of 8 outcomes} \\ N_4 = 2 \text{ out of 16 outcomes} \\ N_5 = 2 \text{ out of 32 outcomes} \\ ... \\ N_i = 2^{i-1} \text{ out of } 2^i \text{ outcomes} \end{cases}$$

Remember that for a random variable outcome such as N_3, the outcomes $\{(t,t,t),(t,t,h),(h,h,h),(h,h,t)\}$ do not count as desired outcomes. Thus we obtain:

$$p_x(x_i) = \frac{2}{2^i} = \frac{1}{2^{i-1}}$$

Which is a geometric series, and fulfills

$$\boxed{f_x(x) = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \delta(x - x_i) = 1}$$

(b) *Let A be the event that N is even and B be the event that $N \leq 6$. Evaluate $P(A)$,$P(B)$, and $P(A \cap B)$.*

We have a probability mass function so we can represent some of these with ease using the definition of the probability density function (PDF) in terms of the probability mass function (PMF) (page 82 in the book):

$$F_x(x) = \int_{-\infty}^{x} p_x(x_i)dx$$

So we can obtain....

$$P(A) = P(N \leq x_i | i = 2k \text{ for all k}) = F_x(x_i) = \int_{-\infty}^{x} \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} \delta(x - x_i)dx$$

$$P(A) = \int_{-\infty}^{x} \frac{1}{2}\delta(x-2) + \frac{1}{8}\delta(x-4) + \frac{1}{32}\delta(x-6) + ...dx$$

$$\boxed{P(A) = \sum_{i=2k}^{\infty} \frac{1}{2^{i-1}}, 1 \leq k \leq \infty}$$

$$P(B) = P(N \leq 6) = F_x(6) = \int_{-\infty}^{6} \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} \delta(x - x_i)dx$$

$$P(B) = \int_{2}^{6} \frac{1}{2}\delta(x-2) + \frac{1}{4}\delta(x-3) + \frac{1}{8}\delta(x-4) + \frac{1}{16}\delta(x-5) + \frac{1}{32}\delta(x-6)dx$$

$$\boxed{P(B) = \frac{31}{32} = 0.968}$$

$$P(A \cap B) = P(N = \{2,4,6\}) = p_x(2) + p_x(4) + p_x(6)$$

$$\boxed{P(A \cap B) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} = 0.656}$$

Problem 3

A communication channel accepts an arbitrary voltage input v and outputs a voltage Y=v+N, where N is a Gaussian random variable with mean 0 and variance $\sigma^2 = 1$. Suppose that the channel is used to transmit binary information as follows:

$$\begin{cases} \text{to transmit 0} & \text{input -1} \\ \text{to transmit 1} & \text{input +1} \end{cases}$$

The receiver decides a 0 was sent if the voltage is negative and a1otherwise. Find the probability of the receiver making an error if:

(a) *a 0 was sent.*

We need to find the probability $P(Y < 0)$, since zero is the median value between our positive and negative voltages. In this case:

$$Y = -1 + N \sim (0, 1)$$

Now we need to put Y in terms of a random variable, specifically a Gaussian. Let's find the expected value (mean) and variance of Y:

$$E[Y] = E[-1] + E[N] = -1 + 0 = -1$$

$$Var[Y] = Var[-1 + N] = Var[N] = 1$$

Therefore:

$$Y \sim (-1, 1), P(Y < 0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-(x+1)^2/2} dx$$

$$P(Y < 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-(x+1)^2/2} dx$$

There are tables to solve the integral, or you can solve with some u substitution. I'm lazy so I'll just use my old TI-89:

$$P(Y < 0) = \frac{1}{2\pi}(2.109) = \boxed{0.841}$$

(b) *a 1 was sent.*

Very similar to the work above, we can represent Y in terms of a Gaussian random variable:

$$Y \sim (-1, 1), P(Y > 0)$$

$$P(Y > 0) = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x+1)^2/2} dx$$

$$P(Y > 0) = \frac{1}{2\pi}(2.109) = \boxed{0.841}$$

Problem 4

Simplified Optical Communication System:The number X of electrons counted by a receiver in an optical communication system is a Poisson random variable with rate λ_1 when a signal is present and with rate $\lambda_0 < \lambda_1$ when a signal is absent. Suppose that a signal is present with probability p

(a) *Find $P(signal\ present|X = k)$ and $P(signal\ absent|X = k)$*

Let A be the event that a signal is present such that $P(A) = p$.

$$P(A|X = k) = \frac{P(X = k \cap A)}{P(A)}$$

$$P(A|X = k) = \frac{P(X = k|A)P(A)}{P(X = k)}$$

Our current unknown above on the right hand side of the equation is $P(X = k)$. Note that A and $\overline{A}$ form two partitions of the sample space, and the event that X=k is also in the sample space. Using the total probability theorem (page 32 in the book)we have:

$$P(X = k) = P(X = k|A)P(A) + P(X = k|\overline{A})P(\overline{A})$$

Therefore, in our simplified notation, we have:

$$\boxed{P(A|X = k) = \frac{P(X = k|A)P(A)}{P(X = k|A)P(A) + P(X = k|\overline{A})P(\overline{A})}}$$

We similarly have the following for the case where a signal is absent:

$$\boxed{P(\overline{A}|X = k) = \frac{P(X = k|\overline{A})P(\overline{A})}{P(X = k|A)P(A) + P(X = k|\overline{A})P(\overline{A})}}$$

Where the following probabilities are broken down as:

$$P(X = k|A) = e^{-\lambda_1}\frac{\lambda_1^k}{k!}$$

$$P(X = k|\overline{A}) = e^{-\lambda_0}\frac{\lambda_0^k}{k!}$$

$$P(A) = p$$

$$P(\overline{A}) = 1 - p$$

(b) *The receiver uses the following decision rule:*

If $P(signal\ present|X = k) > P(signal\ absent|X = k)$, decide the signal is present; otherwise, decide the signal absent.

Show that this decision rule leads to the following threshold rule:

If $X > T$ decide the signal is present; otherwise the signal is absent.

From our last part, we found $P(A|X = k)$ and $P(\overline{A}|X = k)$. Let's solve the following inequality:

$$P(A|X = k) > P(\overline{A}|X = k)$$
$$e^{-\lambda_1}\frac{\lambda_1^k}{k!} > e^{-\lambda_0}\frac{\lambda_0^k}{k!}$$
$$e^{-\lambda_1}\lambda_1^k > e^{-\lambda_0}\lambda_0^k$$
$$ln(e^{-\lambda_1}\lambda_1^k) > ln(e^{-\lambda_0}\lambda_0^k)$$
$$ln(\lambda_1)k - \lambda_1 > ln(\lambda_0)k - \lambda_0$$
$$ln(\lambda_1)k - ln(\lambda_0)k > \lambda_1 - \lambda_0$$
$$\boxed{k > \frac{\lambda_1 - \lambda_0}{ln(\lambda_1) - ln(\lambda_0)}}$$

Note that k is a value that is produced by random variable X, and that the right hand of the expression is constant (and we can let it be T):

$$k > T$$

***(c)** What is the probability of error P_e for the above decision rule?*

We would find the probability that a signal was received given the decision rule, plus the probability that a signal was received given the decision rule. These are cases where we did not detect a signal.

$$P_e = P(A|X < T) + P(\overline{A}|X > T)$$

$$P_e = \boxed{\int_{-\infty}^{T} e^{-\lambda_1}\frac{\lambda_1^k}{k!}dk + \int_{T}^{\infty} e^{-\lambda_0}\frac{\lambda_0^k}{k!}dk}$$

Problem 5

Use the Matlab randn function to generate a large number of samples according to a Gaussian distribution. Let E be the event E=the sample is greater than1.5. Of those samples that are members of the event E, what proportion(relative frequency) is greater than 2. By computing this proportion you will have estimated the conditional probability $P(X > 2|X > 1.5)$. Calculate the exact *conditional probability analytically and compare it with the numerical results obtained through your Matlab program.*

Below is the analytical answer:

$$P(X > 2|X > 1.5) = \frac{P(X > 2 \cap X > 1.5)}{P(X > 1.5)}$$

$$P(X > 2|X > 1.5) = \frac{P(X > 2)}{P(X > 1.5)},$$

$$P(X > 1.5) = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x)^2/2} dx = 0.0668$$

$$P(X > 2) = \int_{2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x)^2/2} dx = 0.0228$$

Therefore:

$$P(X > 2|X > 1.5) = \frac{P(X > 2)}{P(X > 1.5)} = \frac{0.0228}{0.0668} \approx \boxed{0.341}$$

After a few repetitions at 100,000 samples we get about the exact value calculated shown in figure 1

Problem 6

PSK Digital Communication System:Complete the following exercises associated with this system

(a) *Derive an expression for the probability of error, which is given by*

$$P_e = P(A/2 + W \leq 0), \text{ where } W \sim N(0, 1).$$

Let X be a random variable such that X = A/2 + W, where A is a constant representing signal amplitude. Using the linear characteristics of the Gaussian operator, we can express X as a Gaussian $X \sim N(\frac{A}{2}, 1)$ (see problem 3 part A for details). We can represent P_e as:

$$\boxed{P_e = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-(x-A/2)^2/2} dx}$$

(b) *Make a plot of the estimated value of P_e versus A for values of A ranging from 0.1 to 5. Also, on this same plot, graph the true value of P_e using the theoretical expression you derived in part (a). You can utilize the code below for estimating P_e.*

Below is the plot:

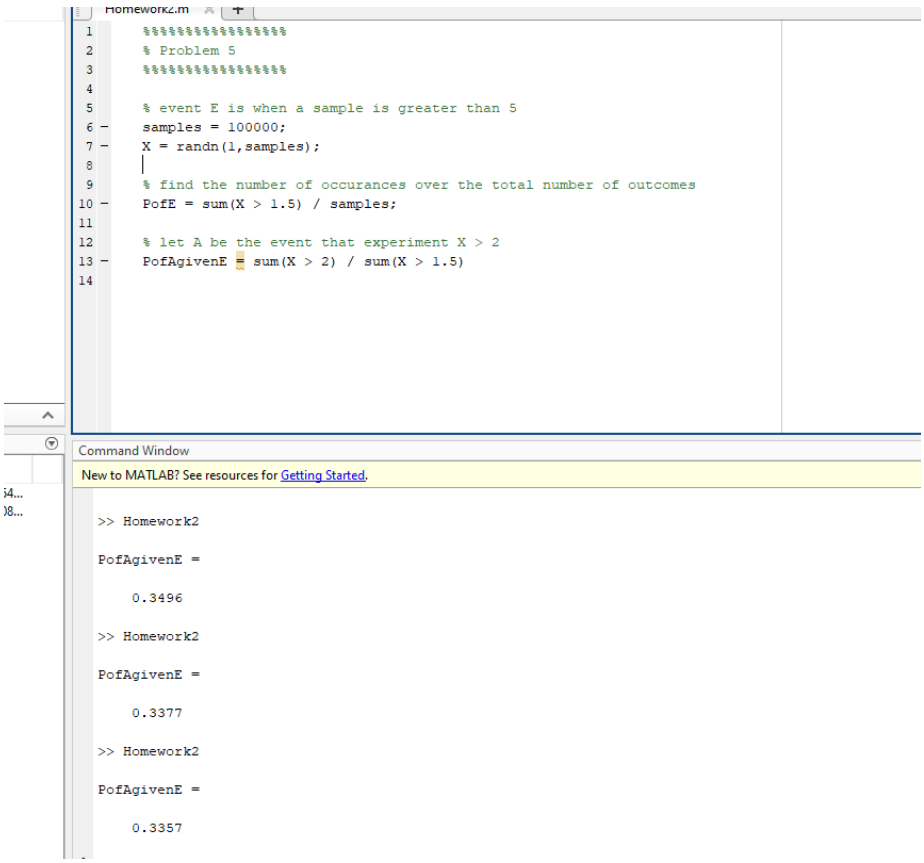
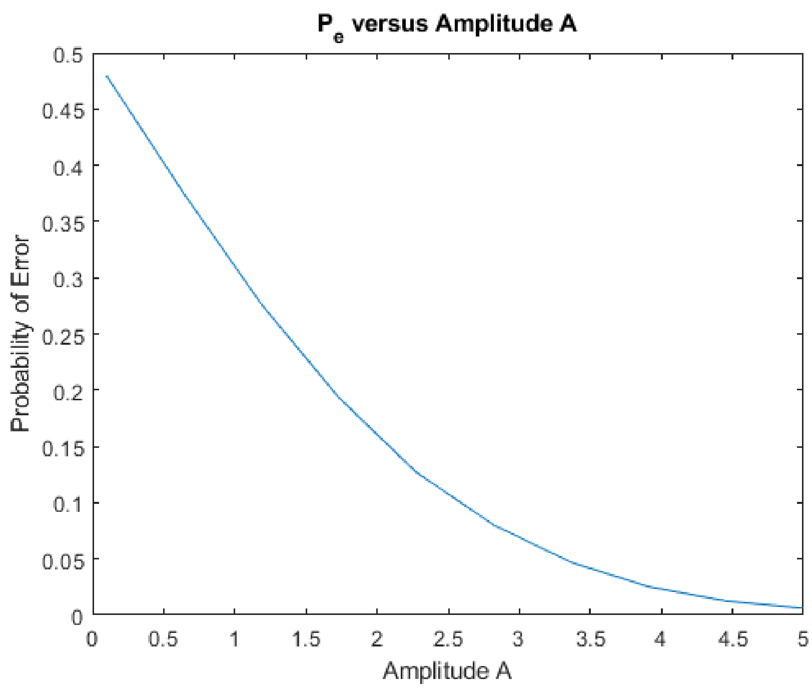


Figure 1: Simple script calculation of the above probability for problem 5. Notice how close the calculated values are with enough samples.

Listing 1: pskExp.m

```
function Pe = pskExp(A,n)
% A - vector of Amplitudes
% n - number of experiments
numAmps = numel(A);
Pe = zeros(numAmps,1);

% Simulate Pe for each amplitude value
for iAmp = 1:numAmps
    W = randn(n,1);
    error = sum(A(iAmp)/2 + W <= 0);
    Pe(iAmp) = error/n;
end
```



P_e versus Amplitude A
Probability of Error
Amplitude A
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Matlab Code

```matlab
%%%%%%%%%%%%%%%%%
% Problem 5
%%%%%%%%%%%%%%%%%

% event E is when a sample is greater than 5
samples = 100000;
X = randn(1,samples);

% find the number of occurances over the total number of
    outcomes
PofE = sum(X > 1.5) / samples;

% let A be the event that experiment X > 2
PofAgivenE = sum(X > 2) / sum(X > 1.5);

%%%%%%%%%%%%%%%%%
% Problem 6
%%%%%%%%%%%%%%%%%
% Plot the experiment
ampRange = linspace(0.1,5,10);
ProbE = zeros(1,length(ampRange));
experiments = 100000;
for k = 1:length(ampRange)
    ProbE(k) = pskExp(ampRange(k),experiments);
end

% Plot Pe vs A
x = 1:length(ampRange);

ProbE_True =  1 - 0.5*(1+erf((x-ampRange/2)/(sqrt(2))));

plot(ampRange,ProbE);
hold on;
plot(ampRange,ProbE_True);
title('P_e versus Amplitude A')
xlabel('Amplitude A');
ylabel('Probability of Error')
function Pe = pskExp(A,n)
% A - vector of Amplitudes
% n - number of experiments
% A = 5;
% n = 100;
numAmps = numel(A);
```

```
% simulate Pe for each amplitude value
for iAmp = 1:numAmps
    W = randn(n,1);
    error = sum(A(iAmp)/2 + W <= 0);
    Pe(iAmp) = error/n;
end

end
```