# Assignment 3: Fourier Series & Transforms of Continuous & Discrete Time Signals

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## Book Questions

### Problem 4.1

*Consider the full-wave rectified sinusoid in Figure 1*
*(a) - Determine its spectrum $X_a(F)$.*

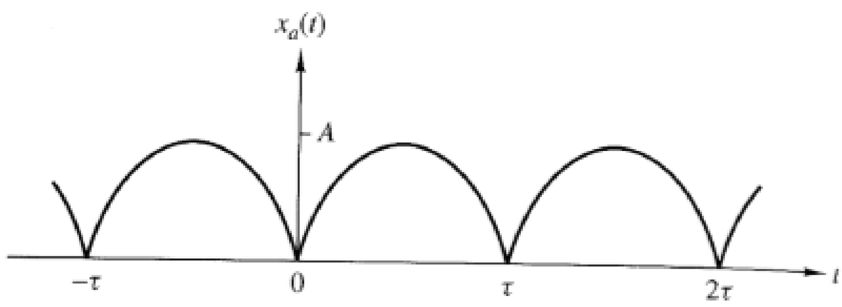


Figure 1: A full-wave rectified sinusoid.

We have a continuous, periodic function $x_a(t) = |Asin(\frac{\pi}{\tau}t)|$, where $N = 2\tau, f = \frac{1}{2\tau}$. To determine the spectrum, we must find the Fourier Series of the signal. There are three methods to find this: The trigonometric, harmonic and exponential Continous-Time Fourier Series. For this problem, we will use the exponential Fourier Series:

$$c_k = 1/T_p \int_0^{T_p} x(t)e^{-j\Omega_k t}dt, \Omega_k = 2\pi k F_p, F_p = 1/T_p$$

let $T_p = \tau$, or $\Omega_k = \frac{2\pi k}{\tau}$ I'm choosing this value because if you look at the graph, the pattern repeats every $\tau$ rather than every $2\tau$. If we had $sin()$ instead of $|sin()|$ then our $T_p = 2\tau$. The math is easier with a period of $\tau$ since we can have $x(t) = Asin(\frac{2\pi}{\tau})$ integrated from $0 <= t <= \tau$, and we can toss out the absolute value operator.

Before computing the Fourier Series coefficients, we need to determine a value $k$, which is the number of series (or the $k$th harmonic). The higher our k value, the more refined our spectrum will turn out. Let's start with $k = 0$ first:

$$c_o = \frac{1}{\tau}\int_0^{\tau} Asin(\frac{2\pi}{\tau}t)e^0 dt$$

$$c_o = \frac{A}{\tau}\int_0^{\tau} sin(\frac{2\pi}{\tau}t) dt$$

$$c_o = \frac{A}{\tau}\big[\frac{-\tau}{2\pi}cos(\frac{2\pi}{\tau}(\tau)) - \frac{-\tau}{2\pi}cos(\frac{2\pi}{\tau}(0))\big]$$

$$c_o = \frac{A}{\tau}\frac{-\tau}{2\pi}\big[cos(2\pi) - cos(0)\big]$$

Notice here: had we let $T_p = 2\tau$ instead of $\tau$ we would end up with the same result in the next step, since $cos(2\pi) - cos(0) = cos(4\pi) - cos(0) = 0$.

$$c_o = 0$$

The next coefficient is similar. We will still use the exponential CTFS method to find $c_1$:

$$c_1 = \frac{1}{\tau}\int_0^{\tau} Asin(\frac{2\pi}{\tau}t)e^{-j\frac{2\pi}{\tau}t} dt$$

In this case, I did use a calculator to solve the integral and simplified the expression to the following:

$$c_1 = \frac{1}{\tau}(-\frac{j\tau}{2})$$

$$c_1 = -\frac{j}{2}$$

For $c_2$ I'll use the trigonometric CTFS and find coefficients $a_2$ and $b_2$, where $c_2 = \frac{1}{2}(a_2 - jb_2)$

$$a_2 = \frac{2}{\tau}\int_0^{\tau} x(t)cos(\Omega_k t) dt$$

$$a_2 = \frac{2}{\tau}\int_0^{\tau} Asin(\frac{2\pi}{\tau}t)cos(\frac{2\pi}{\tau}(2)t) dt$$

$$a_2 = \frac{2A}{\tau}\int_0^{\tau} sin(\frac{2\pi}{\tau}t)cos(\frac{4\pi}{\tau}t) dt$$

I could use a calculator, but I need to brush up on integration by parts. Let $u = sin(\frac{2\pi}{\tau}t)$ and $du = -\frac{\tau}{2\pi}cos(\frac{2\pi}{\tau}t)dt$, then let $dv = cos(\frac{4\pi}{\tau}t)dt$ and $v = \frac{-\tau}{4\pi}sin(\frac{4\pi}{\tau}t)$

$$\int_0^\tau udv = uv - \int_0^\tau vdu$$

$$\int_0^\tau sin(\frac{2\pi}{\tau}t)cos(\frac{4\pi}{\tau}t)dt = \frac{-\tau}{4\pi}sin(\frac{2\pi}{\tau}t)sin(\frac{4\pi}{\tau}t) - \int_0^\tau \frac{-\tau}{4\pi}sin(\frac{4\pi}{\tau}t)\frac{-\tau}{2\pi}cos(\frac{2\pi}{\tau}t)$$

Ok, maybe this is way too much for the assignment. Next time, convert the sinusoidal functions into exponentials using Euler's rules. But long story short, you would do integration by parts again on the rhs of the equation. From there you subtract the integrals out of the equation and stuff... but the calculator shows:

$$a_2 = 0$$

For $b_2$ we will also use the calculator:

$$b_2 = \frac{2}{\tau}\int_0^\tau x(t)sin(\Omega_k t)dt$$

$$b_2 = 0$$

Solve for $c_2$, even though we know the answer:

$$c_2 = \frac{1}{2}(a_2 - jb_2) = 0$$

Now that we have stepped through the process we will just plug in our respective $c_k$ values to the calculator. I'm using the TI-89 calculator and using the following command:

1/x*integral(A*sin(2pi/x*t)*e^(-j2pi/x*k*t),t,0,x)

Where $x = \tau$, and I insert k for each $c_k$. Our Fourier Series coefficients are:

$$c_k = [0, 0, 0, 0, 0, \frac{jA}{2}, 0, \frac{-jA}{2}, 0, 0, 0, 0, 0]$$

This may seem really simple, but let's plot this and maybe this will look familiar:

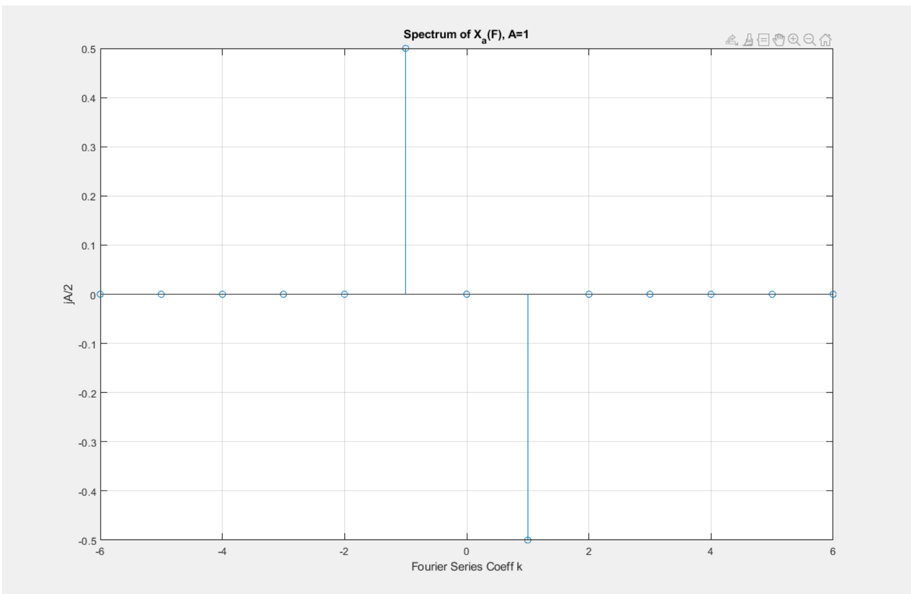


Figure 2: Spectrum plot of $X_a(F)$. Note that this looks like the Fourier transform of a sine wave.

*(b) - Compute the power of the signal.*

$$P_x = \frac{1}{T_p} \int_0^{\tau} |Asin(\frac{2\pi}{\tau} t)|^2 dt$$

$$P_x = \frac{|A|^2}{T_p} \int_0^{\tau} |sin(\frac{2\pi}{\tau} t)|^2 dt$$

$x(t)$ is real with positive magnitude for interval $0 <= t <= \tau$

$$P_x = \frac{A^2}{T_p} \int_0^{\tau} sin(\frac{2\pi}{\tau} t)^2 dt$$

Using integration by parts we eventually end up with:

$$P_x = \frac{A^2}{2}$$

*(c) - Plot the power spectral density*

To do this we take our Fourier Series Coefficients and create a plot of $|c_k|^2$. See Figure 3 for the plot, and the MATLAB section in the appendix for the code.

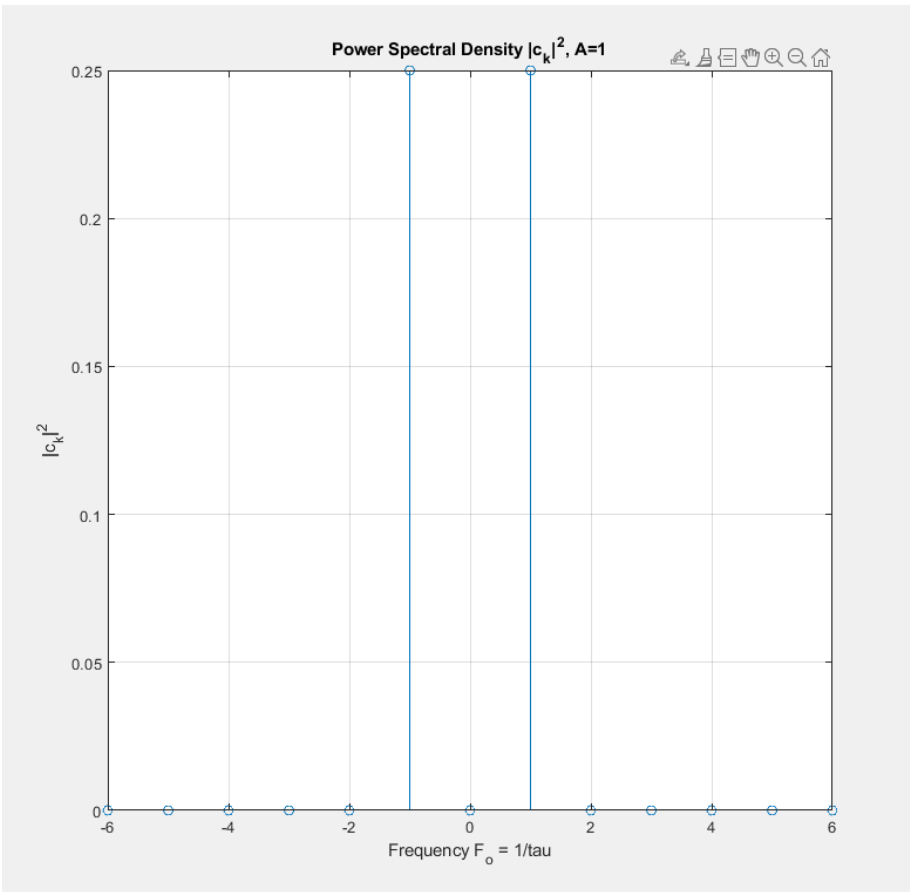


Figure 3: Power Spectral Density $|c_k|^2$.

*(d) - Check the validity of Parseval's relation for this signal.*
Parseval's relation is defined as follows:

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

We obtained $P_x = \frac{A^2}{2}$, and the sum of our coefficients is:

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = |\frac{jA}{2}|^2 + |\frac{-jA}{2}|^2$$

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{A^2}{2} = P_x$$

## Problem 4.2

*Compute and sketch the magnitude and phase spectra for the following signals $(a > 0)$:*

*(a) -* $x_a(t) = \begin{cases} Ae^{-at}, & \text{for } t >= 0 \\ 0, & \text{for } t < 0. \end{cases}$

This is an aperiodic, continuous signal. Remember that in order for there to be a Fourier transform of a signal $x_a(t)$, Dirichlet conditions must be satisfied. These are (1) the signal has a finite number of finite discontinuities, (2) the signal has a finite number of maxima and minima, and (3) the signal is absolutely integrable. This exponential equation satisfies those conditions.
To find the magnitude and phase spectra, we need to find the spectrum $X(F)$ of $x_a(t)$, which could very likely be complex in value such that:

$$X(F) = |X(F)|e^{j\Theta(F)}$$

Where $|X(F)|$ is the magnitude spectra and $\Theta(F)$ is the phase spectrum. Let's find $X(F)$:

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$$

$$X(F) = \int_{0}^{\infty} Ae^{-at}e^{-j2\pi Ft}dt$$

$$X(F) = A\int_{0}^{\infty} e^{-j2\pi Ft-at}dt$$

$$X(F) = A\int_{0}^{\infty} e^{(-j2\pi F-a)(t)}dt$$

$$X(F) = A\Big[\frac{1}{(-j2\pi F-a)}\big(e^{(-j2\pi F-a)(\infty)} - e^{(-j2\pi F-a)(0)}\big)\Big]$$

$$X(F) = A\Big[\frac{-1}{(j2\pi F + a)}\big(0 - 1\big)\Big]$$

$$X(F) = \frac{A}{(j2\pi F + a)}$$

$$X(F) = \frac{A}{(j2\pi F + a)} \frac{(-j2\pi F + a)}{(-j2\pi F + a)}$$

$$X(F) = \frac{A}{(2\pi F)^2 + a^2}(a - j2\pi F)$$

From here we can extract the magnitude and phase of the spectrum:

$$|X(F)| = \sqrt{real^2 + imag^2}$$

$$|X(F)| = \frac{A}{(2\pi F)^2 + a^2}\sqrt{(a)^2 + (2\pi F)^2}$$

$$|X(F)| = \frac{A}{(2\pi F)^2 + a^2}\sqrt{(a)^2 + (2\pi F)^2}\frac{\sqrt{(a)^2 + (2\pi F)^2}}{\sqrt{(a)^2 + (2\pi F)^2}}$$

$$|X(F)| = \frac{A}{\sqrt{(a)^2 + (2\pi F)^2}}$$

$$\Theta(F) = tan^{-1}(imag/real)$$

$$\Theta(F) = -tan^{-1}(-2\pi F/a)$$

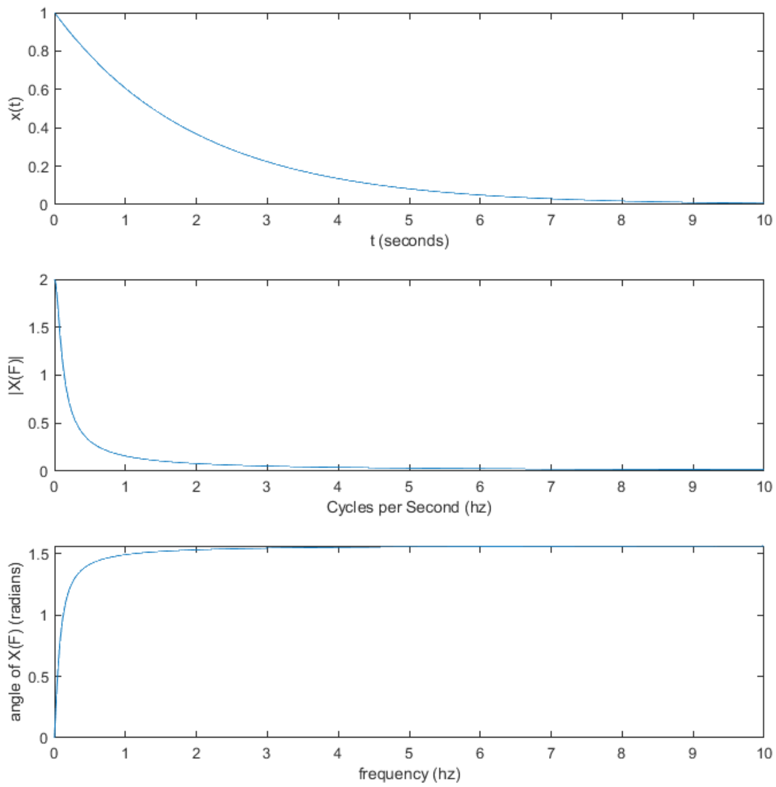


Figure 4: Plots of $x(t)$ and respective spectrum magnitude and phase plots.

*(b)* - $x_a(t) = Ae^{-a|t|}$

This is a decaying exponential function that is symmetric across the $x_a(t)$ axis. It satisfies the Dirichlet conditions just like the last problem. Because $Ae^{-a|t|}$ is symmetric about the axis, the spectrum can be represented as follows:

$$X(F) = \int_{-\infty}^{0} Ae^{-at}e^{-j2\pi Ft}dt + \int_{0}^{\infty} Ae^{-at}e^{-j2\pi Ft}dt$$

$$X(F) = 2\int_{0}^{\infty} Ae^{-at}e^{-j2\pi Ft}dt$$

$$X(F) = \frac{2A}{(j2\pi F + a)}$$

It's kinda convenient that this problem is similar to the last part. The magnitude and phase are as follows:

$$|X(F)| = \frac{2A}{\sqrt{(a)^2 + (2\pi F)^2}}$$

$$\Theta(F) = -tan^{-1}(-2\pi F/a)$$

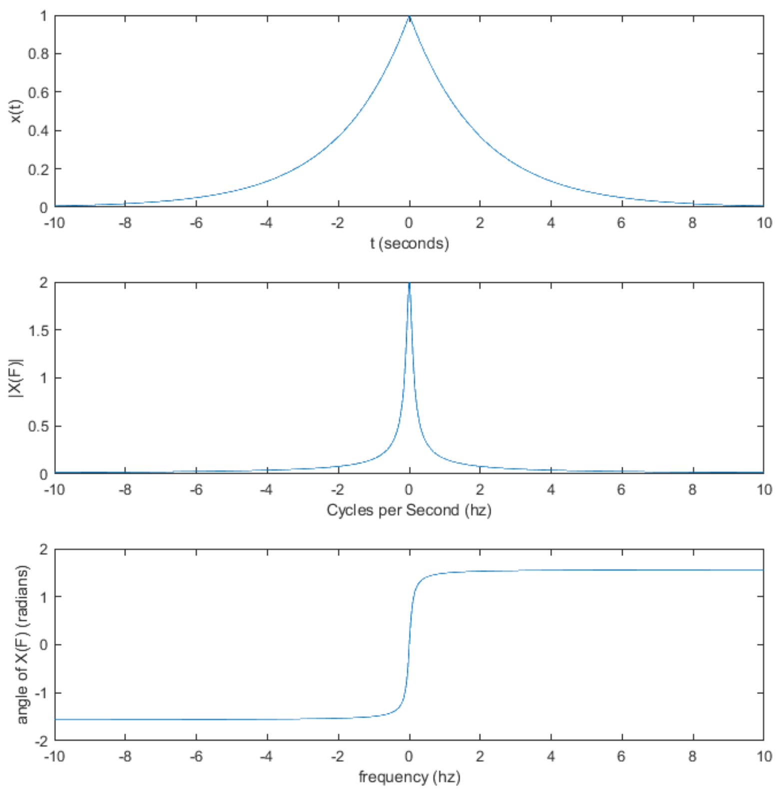


Figure 5: Plots of $x(t)$ and respective spectrum magnitude and phase plots.

## Problem 4.4

*Consider the following periodic signal:*

$$x(n) = ..., 1, 0, 1, 2, 3, 2, 1, 0, 1, ...$$

*(a) - Sketch the signal x(n) and its magnitude and phase spectra.*

Since we are dealing with a discrete signal, we can do this all in MATLAB. See the MATLAB section for the code used on this problem. Our plots are below in Figure 6:

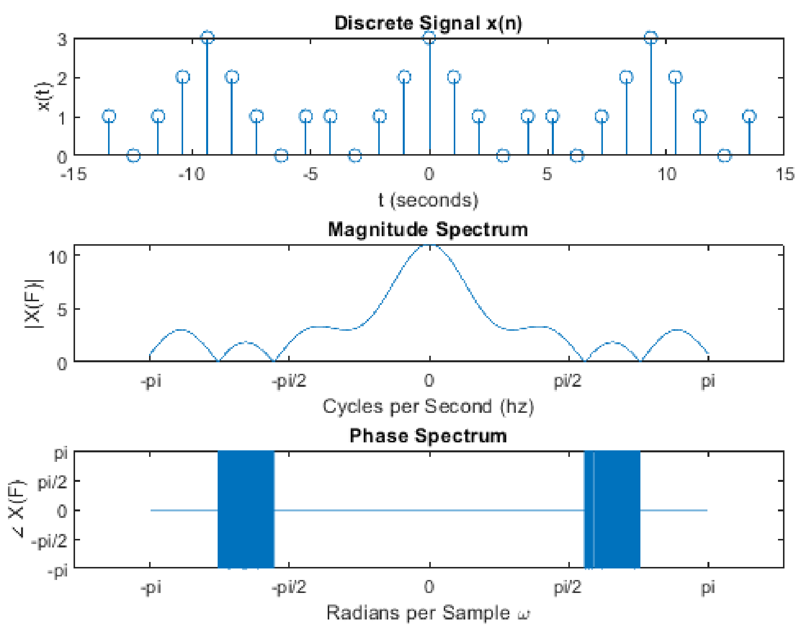


Figure 6: Plots of $x(n)$ and respective spectrum magnitude and phase plots.

*(b) - Using the results in part (a), verify Perseval's relation by computing the power in the time and frequency domains.*

Parseval's relation in discrete time is as follows (from eq 4.2.1 in the book):

$$\sum_{k=0}^{N-1} |c_k|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |x(n)|^2$$

In the MATLAB code below I found the series coefficients and found the power. I obtained an approximate value $P_x = 2.333$

## Problem 4.6 (d)

*Determine and sketch the magnitude and phase spectra of the following periodic signals:*
*(d)* - $x(n) = \{..., -2, -1, 0, 1, 2, -2, -1, 0, 1, 2, ...\}$

This is very similar to the last problem, so I created a MATLAB section specifically for this problem. Below in Figure

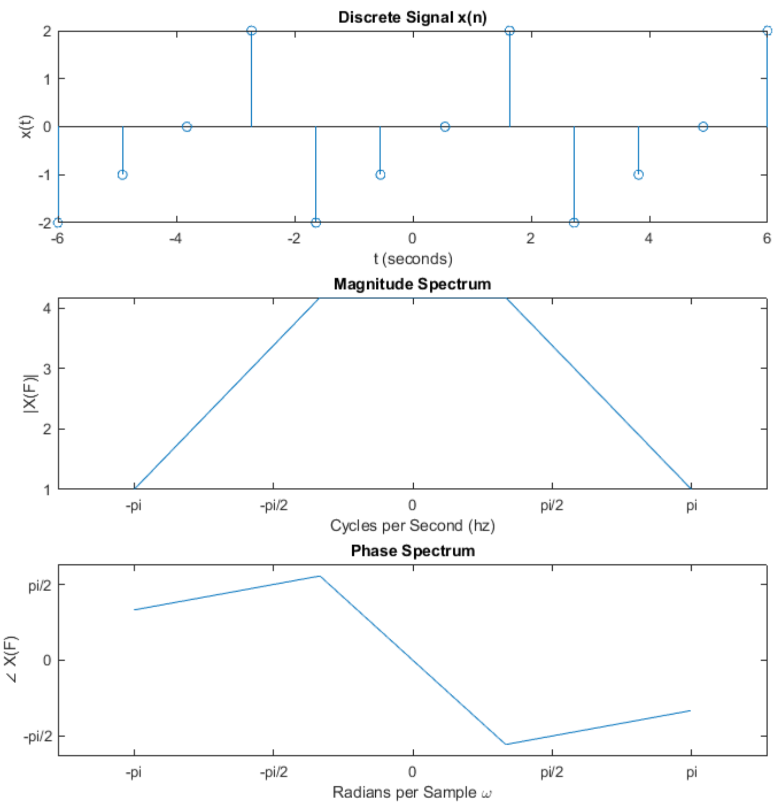


Figure 7: Plots of $x(n)$ and respective spectrum magnitude and phase plots.

## Problem 4.9 (d)

*Compute the Fourier transform of the following signals:*
*(d)* - $x(n) = (\alpha^n sin\omega_o n)u(n)$, $|\alpha| < 1$ Since $x(n)$ approaches 0 as $n = \infty$, and the signal is aperiodic, we can use the analysis equation given below:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Where $\omega$ is our radians per sample ranging from $[-\pi, \pi]$. Note that this is the DFT instead of the DTFT. We can insert our function and get the following:

$$X(\omega) = \sum_{n=-\infty}^{\infty} (\alpha^n sin\omega_o n)u(n)e^{-j\omega n}$$

$$X(\omega) = \sum_{n=0}^{\infty} (\alpha^n sin\omega_o n)e^{-j\omega n}$$

It would be easiest to compute this in MATLAB and then plot $X(\omega)$ since this is a discrete Fourier transform. Plotting the results will yield a visual of the Fourier transform (see Figure 8). Another trick we can use is the convolution property, where multiplication in the time domain is convolution in the frequency domain. Let's divide $x(n)$ such that:

$$x_1(n) = \alpha^n u(n), |\alpha| < 1 \text{ and } x_2(n) = sin(\omega_o n)$$

$$X_2(\omega) = \sum_{n=-\infty}^{\infty} sin(\omega_o n)e^{-j\omega n}$$

There is an identity for this, but also look at the CTFT of $sin(ax)$:

$$X_2(\omega) = \frac{\pi}{j}[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

$$X_1(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$X_1(\omega) = 1 + a^1 e^{-j\omega} + a^2 e^{-j\omega 2} + a^3 e^{-j\omega 3} + ... + 0$$

$$X_1(\omega) = \frac{1}{1 - \alpha e^{-j\omega}} \text{ (see pg 291)}$$

Now we convolve $X_0$ with $X_1$:

$$X(\omega) = X_0(\omega) * X_1(\omega)$$

$$X(\omega) = \frac{\pi}{j}\left[\frac{1}{1 - \alpha e^{-j(-\omega_o)}} + \frac{1}{1 - \alpha e^{-j(\omega_o)}}\right]$$

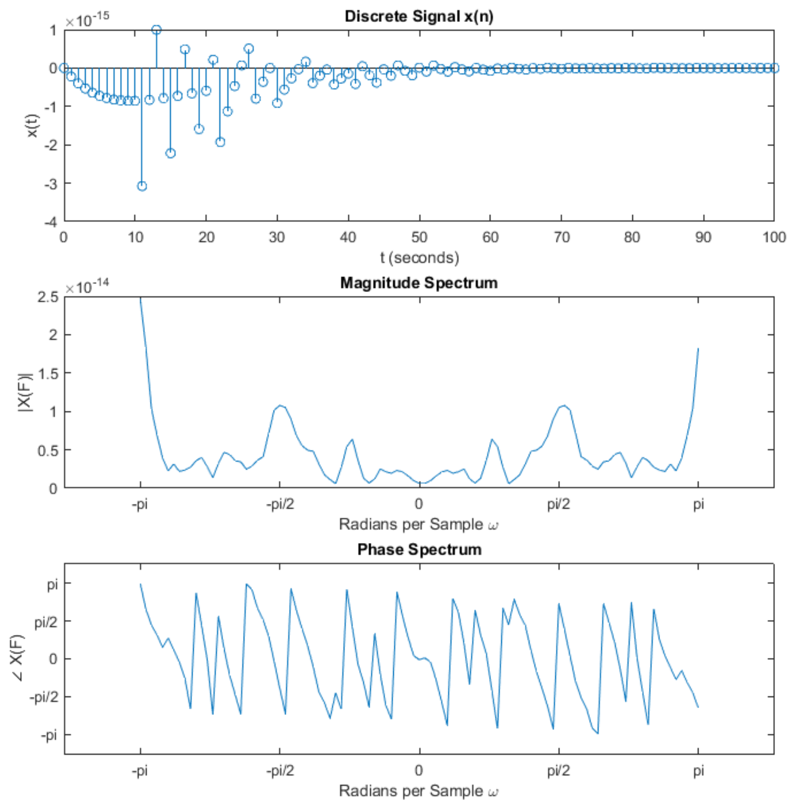


Figure 8: Plots of $x(n)$ and respective spectrum magnitude and phase plots.

## Matlab Code - Problem 1

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem 4.1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
A = 1;
c_k = [0,0,0,0,0,A*1/2,0,A*(-1/2),0,0,0,0,0];
n = (-6:6);
stem(n,c_k);
grid on;
title('Spectrum of X_a(F), A=1');
xlabel('Fourier Series Coeff k');
ylabel('jA/2');

% part c - power spectral density
power_spectral = abs(c_k).^2;
figure(2);
stem(n,power_spectral);
grid on;
title('Power Spectral Density |c_k|^2, A=1');
xlabel('Frequency F_o = 1/tau');
ylabel('|c_k|^2');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem 4.2 - Aperiodic Fourier Txfm
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
N = 10;
k = 0.01; % sets how refined the plot is
F = (0:k:N);
A = 1;
a = 0.5;
duration = 10;


% part A functions
t = (0:k:duration);
x = A*exp(-a.*t);
X = A ./ (a^2+(F.*(2*pi)).^2).^(0.5);
TH = -atan((-2*pi).*F./a);
% TH = -atan2((-2*pi).*F./a);

% part B functions
% t = (-duration:k:duration);
% F = (-N:k:N);
% x = A*exp(-a.*abs(t));
```

```
% X = A ./ (a^2+(F.*(2*pi)).^2).^(0.5);
% TH = -atan((-2*pi).*F./a);

subplot(3,1,1)
plot(t,x);
xlabel('t (seconds)')
ylabel('x(t)')


subplot(3,1,2)
plot(F,X);
xlabel('Cycles per Second (hz)')
ylabel('|X(F)|')

subplot(3,1,3)
plot(F,TH);
xlabel('frequency (hz)')
ylabel('angle of X(F) (radians)')
 set(gca,'YTick',0:pi/8:pi/2)
 set(gca,'YTickLabel',{'0','pi/8','pi/4','3pi/8','pi/2'})

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%
 %% Problem 4.4
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%

% ----------------------------------------
% part a - Magnitude and Phase Spectrum
% ----------------------------------------
x = [1,0,1,2,3,2,1,0,1];
k = 10000; % number of points for refinement
n = linspace(-length(x)/2,length(x)/2,length(x)); % the length() command is useful for dynamic code
w = linspace(-pi,pi,length(n));
% find the specrtum by calculating the dtft of periodic x
X = dtft(x,n,w);
N = (0:length(X)-1);
W = linspace(-pi,pi,length(X)); % use for even spacing of n items
Xmag = abs(X);
Xang = angle(X);

% ---------------------------------
%  part b - Power and Parseval's
% ---------------------------------
% our Spectrum X is a sum of our coefficients c_k
% finding c_k
```

```
N_c = length(x);
for k=1:N_c
    ck(k) = 0;
    for m=1:N_c
        ck(k) = ck(k) + 1/N_c * x(m)*exp((-1i*2*pi*k/N_c)
            *n(m));
    end
end
P_ck = sum(abs(ck).^2);
P_x = sum(x.*conj(x))/length(x);



% ---------------------
% Plots of part a
% ---------------------




% plotting repeated x
P = 3; % The number of times we repeat the signal
x = x' * ones(1,P);
x = x(:); % long column vector
x = x'; % transpose to long row vector
n = linspace(-length(x)/2,length(x)/2,length(x));
W = linspace(-pi,pi,length(X));


subplot(3,1,1)
stem(n,x);
title('Discrete Signal x(n)')
xlabel('t (seconds)')
ylabel('x(t)')


subplot(3,1,2)
plot(W,Xmag);
xlabel('Cycles per Second (hz)')
title('Magnitude Spectrum')
ylabel('|X(F)|')
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


subplot(3,1,3)
```

```
plot(W,Xang);
title('Phase Spectrum')
xlabel('Radians per Sample \omega')
ylabel('\angle X(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 %% Problem 4.6 (d)
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 % ---------------------------------------------
% part a - Magnitude and Phase Spectrum
% ---------------------------------------------
x = [-2,-1,0,2];
k = 10000; % number of points for refinement
n = linspace(-length(x)/2,length(x)/2,length(x)); % the length() command is useful for dynamic code
% n = (-5:5);
% n = n-length(x);
w = linspace(-pi,pi,length(n));
% find the specrtum by calculating the dtft of periodic x
X = dtft(x,n,w);
N = (0:length(X)-1);
W = linspace(-pi,pi,length(X)); % use for even spacing of n items
Xmag = abs(X);
Xang = angle(X);

% ------------------------------------
%  part b - Power and Parseval's
% ------------------------------------
% our Spectrum X is a sum of our coefficients c_k
% P_X = sum(Xmag.^2);
% P_x = sum(abs(x).^2)/length(x);



% --------------------
% Plots of part a
% --------------------



```

```
% plotting repeated x
P = 3; % The number of times we repeat the signal
x = x' * ones(1,P);
x = x(:); % long column vector
x = x'; % transpose to long row vector
n = linspace(-length(x)/2,length(x)/2,length(x));
W = linspace(-pi,pi,length(X));


subplot(3,1,1)
stem(n,x);
title('Discrete Signal x(n)')
xlabel('t (seconds)')
ylabel('x(t)')


subplot(3,1,2)
plot(W,Xmag);
xlabel('Cycles per Second (hz)')
title('Magnitude Spectrum')
ylabel('|X(F)|')
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


subplot(3,1,3)
plot(W,Xang);
title('Phase Spectrum')
xlabel('Radians per Sample \omega')
ylabel('\angle X(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})



 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 %% Problem 4.9 (d)
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 a = 0.9;
 n=(0:100); % remember that x(n) has a unit step function
     component, hence 0
 f = 1; % cycles per sample
 w = 2*pi*f; % radians per sample conversion
```

```
 x = a.^n.*sin(w.*n);
 X = DFT(x,length(n));
W = linspace(-pi,pi,length(X));

subplot(3,1,1)
stem(n,x);
title('Discrete Signal x(n)')
xlabel('t (seconds)')
ylabel('x(t)')


subplot(3,1,2)
plot(W,abs(X));
xlabel('Radians per Sample \omega')
title('Magnitude Spectrum')
ylabel('|X(F)|')
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


subplot(3,1,3)
plot(W,angle(X));
title('Phase Spectrum')
xlabel('Radians per Sample \omega')
ylabel('\angle X(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
```