Assignment 10: Filter Implementations

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Problem 10.1

The Type-I linear-phase FIR filter is characterized by:

$$h(n) = h(M-1-n),\ 0 <= n <= M-1, \text{M odd}$$

Show that its amplitude response $H_r(\omega)$ *is given by:*

$$H_r(\omega) = \sum_{n=0}^{L} a(n) cos(\omega n)$$

where coefficients $a(n)$ *are obtained from* $h(n)$ *as:*

$$a(0) = h(\frac{M-1}{2}) \text{ (the middle sample)}$$

$$a(n) = 2h(\frac{M-1}{2} - n),\ 1 <= n <= \frac{M-1}{2}$$

In english, the characterization of a Type-I FIR filter is such that the impulse response is symmetric in the sample domain. Let's convert the impulse response into the z-domain:

$$H(z) = h(0) + h(1)z^{-1} + h()z^{-2} + ... + h(n)z^{-n}$$

$$H(z) = h(0)+h(1)z^{-1}+h()z^{-2}+...+h(M-2-n)z^{-(M-2-n)}+h(M-1-n)z^{-(M-1-n)}$$

Because of symmetric properties we can break down our z-transformation into:

$$H(z) = z^{-(M-1)/2}\left\{h\left(\frac{M-1}{2} + \sum_{n=0}^{(M-3)/2} h(n)\left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right]\right)\right\}$$

Now comes a trick: If we take the definition of the z-transformation and do some substitution, we can use the above equation to prove $H_r(\omega)$:

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-k}$$

Substitute z for z^{-1} and multiply both sides by $z^{-(M-1)}$:

$$z^{-(M-1)}H(z^{-1}) = \sum_{n=0}^{M-1} h(n)z^{k}z^{-(M-1)}$$

$$z^{-(M-1)}H(z^{-1}) = \sum_{n=0}^{M-1} h(n)z^{-(M-1-k)} \text{ *note the symmetry in the sum}$$

$$z^{-(M-1)}H(z^{-1}) = \pm H(z)$$

We can represent the frequency response as follows when $h(n) = h(M-1-n)$, M odd:

$$e^{j\omega(M-1)/2}H_r(\omega) = H(\omega)$$

And now we can substitute this into that long nasty equation above:

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)cos\left(\omega\left(\frac{M-1}{2} - n\right)\right)$$

Substitute in a(0) and a(n) from the problem statement and we simplify this to:

$$H_r(\omega) = a(0) + 2\sum_{n=(M-1)/2}^{(M-3)/2} h(\frac{M-1}{2} - n)cos(\omega(n))$$

$$H_r(\omega) = a(0) + 2\sum_{n=(M-1)/2}^{M-2} a(n)cos(\omega n)$$

Problem 10.2

The Type-2 linearphase FIR filter is characterized by

$$h(n) = h(M-1-n),\ 0 <= n <= M-1, \text{ M even}$$

Part A

Show that its amplitude response $H_r(\omega)$ *is given by*

$$H_r(\omega) = \sum_{n=1}^{\frac{M}{2}} b(n)cos(\omega(n - \frac{1}{2}))$$

where coefficients b(n) are obtained as defined as

$$b(n) = 2h(\frac{M}{2} - n),\ \ n = 1, 2, ...\frac{M}{2}$$

This will be very similar to the first problem except that h(n) is even instead. We begin with the following:

$$H(z) = h(0) + h(1)z^{-1} + h()z^{-2} + ... + h(n)z^{-n}$$

$$H(z) = h(0)+h(1)z^{-1}+h()z^{-2}+...+h(M-2-n)z^{-(M-2-n)}+h(M-1-n)z^{-(M-1-n)}$$

Because of symmetric properties we can break down our z-transformation into:

$$H(z) = z^{-(M-1)/2}\left(\sum_{n=0}^{(M/2)-1} h(n)\left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right]\right)$$

Now comes a trick: If we take the definition of the z-transformation and do some substitution, we can use the above equation to prove $H_r(\omega)$:

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-k}$$

Substitute z for z^{-1} and multiply both sides by $z^{-(M-1)}$:

$$z^{-(M-1)}H(z^{-1}) = \sum_{n=0}^{M-1} h(n)z^{k}z^{-(M-1)}$$

$$z^{-(M-1)}H(z^{-1}) = \sum_{n=0}^{M-1} h(n)z^{-(M-1-k)} \text{ *note the symmetry in the sum}$$

$$z^{-(M-1)}H(z^{-1}) = \pm H(z)$$

We can represent the frequency response as follows when $h(n) = h(M-1-n)$, M odd:

$$e^{j\omega(M-1)/2}H_r(\omega) = H(\omega)$$

And now we can substitute this into that long nasty equation above:

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)cos\left(\omega\left(\frac{M-1}{2} - n\right)\right)$$

Since h(n) is symmetric, we can flip the indices, then substitute in the b coefficients from the problem statement and we simplify this to:

$$H_r(\omega) = 2\sum_{n=1}^{M/2} h(\frac{M}{2} - n)cos(\omega(n - \frac{1}{2}))$$

$$H_r(\omega) = \sum_{n=1}^{M/2} b(n)cos(\omega(n - \frac{1}{2}))$$

Part B

Show that $H_r(\omega)$ can be further expressed as

$$H_r(\omega) = cos(\omega/2)\sum_{n=0}^{L}\bar{b}(n)cos(\omega n),\ L = \frac{M}{2} - 1$$

where coefficients b(n) are given by

$$b(1) = \bar{b}(0) + \frac{1}{2}\bar{b}(1),$$

$$b(n) = \frac{1}{2}[\bar{b}(n-1) + \bar{b}(n)],\ 2 <= n <= \frac{M}{2} - 1$$

$$b(\frac{M}{2}) = \frac{1}{2}[\bar{b}(\frac{M}{2} - 1)]$$

From the previous problem we determined that $H_r(\omega)$ for an even function could be expressed as:

$$H_r(\omega) = \sum_{n=0}^{M/2} b(n)cos(\omega(n - \frac{1}{2}))$$

Substitute $L = \frac{M}{2} - 1$ into the equation:

$$H_r(\omega) = \sum_{n=1}^{L+1} b(n)cos(\omega(n - \frac{1}{2}))$$

Next rearrange the b coefficients given above:

$$\sum_{n=1}^{L+1} b(k) = b(1) + b(2) + b(3) + ... + b(L+1)$$

$$\sum_{n=1}^{L+1} b(k) = [\bar{b}(0) + \frac{1}{2}\bar{b}(1)] + [\frac{1}{2}\bar{b}(1) + \frac{1}{2}\bar{b}(2)] + ... + \frac{1}{2}\bar{b}(L)$$

$$\sum_{n=1}^{L+1} b(k) = \bar{b}(0) + \bar{b}(1) + \bar{b}(2) + ... + \bar{b}(L)$$

$$\sum_{n=1}^{L+1} b(k) = \sum_{n=0}^{L} \bar{b}(k)$$

Now substitute into $H_r(\omega)$ from our last problem:

$$H_r(\omega) = \sum_{n=0}^{L} \bar{b}(k)cos(\omega(n - \frac{1}{2}))$$

And I guess there is some trick to get to this next step, but it is 12:21AM and I need to get to bed.

$$H_r(\omega) = cos(\omega/2)\sum_{n=0}^{L}\bar{b}(k)cos(\omega n)$$

Problem 10.3

Design a linear-phase bandpass filter using the Hann window design technique. The specifications are:

lower stopband edge: 0.2π
upper stopband edge: 0.75π
As = 40dB

lower passband edge: 0.35π
upper passband edge: 0.55π
Rp = 0.25 dB

Plot the impulse response and the magnitude response (in dB) of the designed filter. Do no use the fir1 function.

My approach to this problem is to design the ideal filter H_d using filterDesigner and then creating an $N = 64$ Hann window with the built-in Matlab function. My goal is to see how windowing can effect a filter performance. See the Matlab code below for implementation.

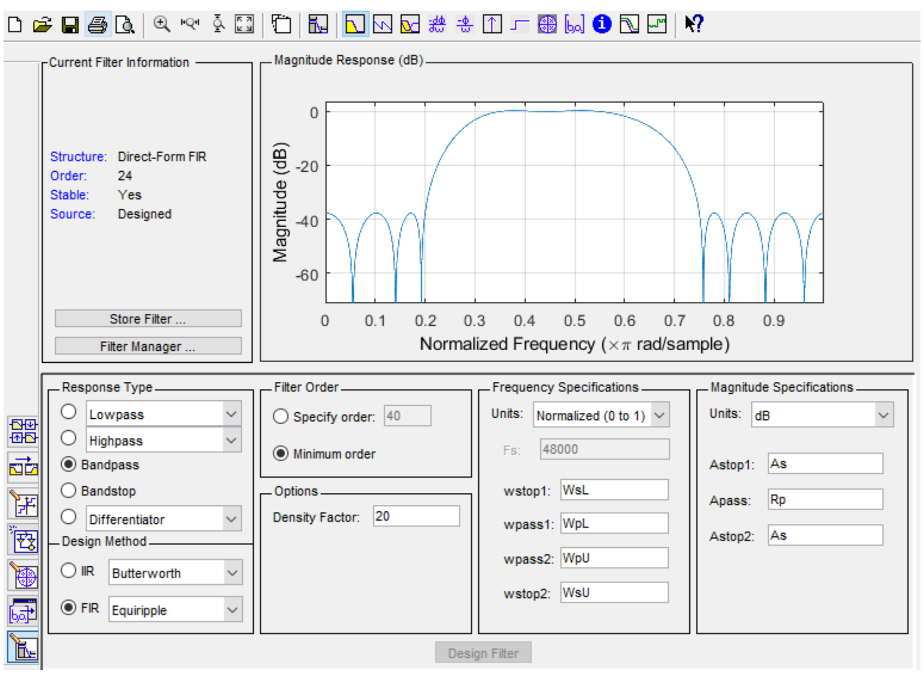

Figure 1: Filter Designer parameters and spectrum of our chosen filter. See above for requirements.

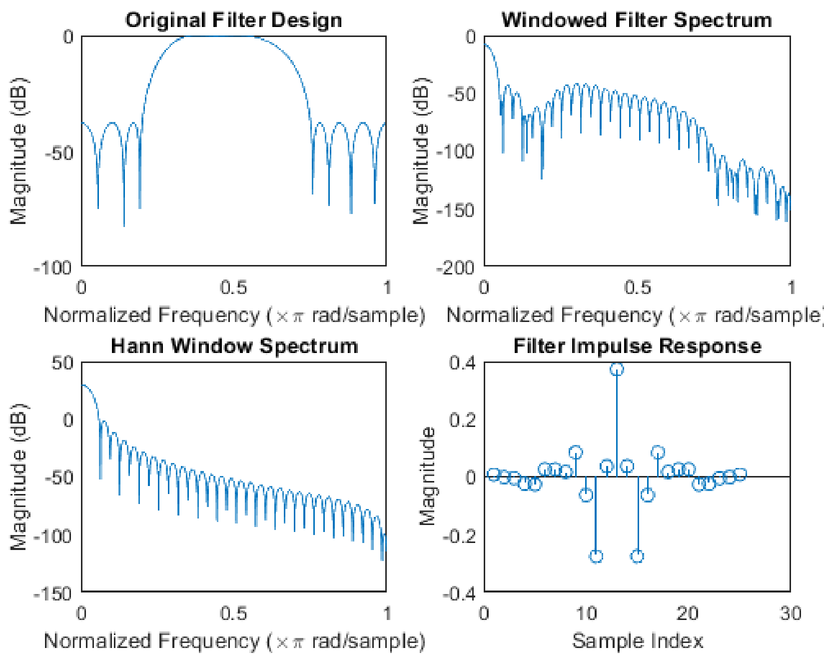

Figure 2: Comparison between the original filter and the window, with an output filter $H(\omega) = H_d(\omega)W(\omega)$, and the impulse response of $h_d(n)$

Problem 10.4

Design a bandstop filter using the Hamming window design technique. The specifications are:

lower stopband edge: 0.4π
upper stopband edge: 06π
As = 50 dB

lower passband edge: 0.3π
upper passband edge: 0.7π
Rp =0.2 dB

My approach to this problem is to design the ideal filter H_d using filterDesigner and then creating an $N = 64$ Hamming window with the built-in Matlab function. My goal is to see how windowing can effect a filter performance. See the Matlab code below for implementation.

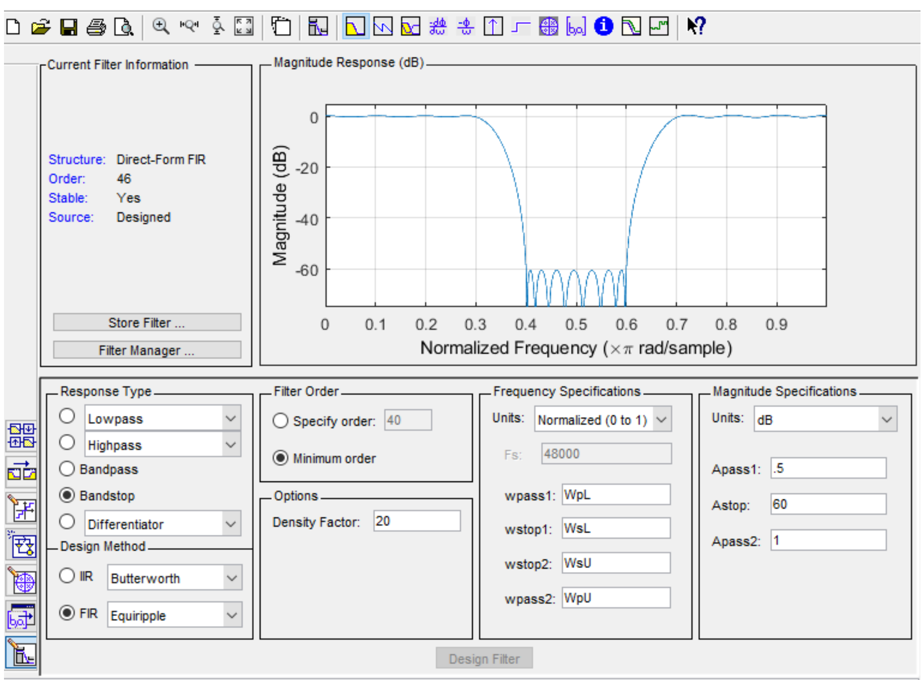

Figure 3: Filter Designer parameters and spectrum of our chosen filter. See above for requirements.

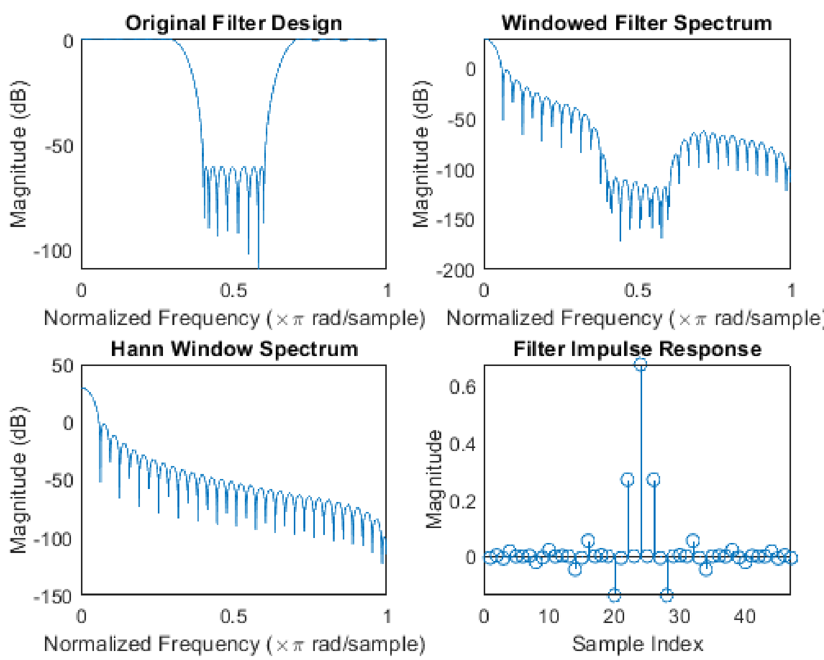

Figure 4: Comparison between the original filter and the window, with an output filter $H(\omega) = H_d(\omega)W(\omega)$, and the impulse response of $h_d(n)$

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%  Assignment 10
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%                              %%%
%%% NOTE: Load designFilter Sessions %%%
%%%                              %%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem 3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

WpL = 0.35; % Lower passband edge
WpU = 0.55; % Upper passband edge
WsL = 0.2  ; % Lower stopband edge
WsU = 0.75  ; % Upper stopband edge
As  = 40   ; % Stopband attenuation
Rp  = 0.25; % Passband ripple




% the freqz command returns an N point complex frequency
    response vector H
% given filter coefficients a and b
N = 1024;
WinN = 64;
Win3 = hann(WinN);
Ir = conv(Win3,Num3); % Impulse response of resultant
    filter
[Hd,OmegH] = freqz(Num3,1,N);
[W, OmegW] = freqz(Win3,1,N);


H = Hd.*W;


figure(1)
subplot(2,2,1);
plot(OmegH/pi,20*log10(abs(Hd)))
title('Original Filter Design')
xlabel('Normalized Frequency (\times\pi rad/sample)')
ylabel('Magnitude (dB)')
```

```
subplot(2,2,2);
plot(OmegH/pi,20*log10(abs(H)))
title('Windowed Filter Spectrum')
xlabel('Normalized Frequency (\times\pi rad/sample)')
ylabel('Magnitude (dB)')

subplot(2,2,3);
plot(OmegH/pi,20*log10(abs(W)))
title('Hann Window Spectrum')
xlabel('Normalized Frequency (\times\pi rad/sample)')
ylabel('Magnitude (dB)')

subplot(2,2,4);
stem(1:length(Num3),Num3)
title('Filter Impulse Response')
xlabel('Sample Index')
ylabel('Magnitude')


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem 4
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

WpL = 0.3; % Lower passband edge
WpU = 0.7; % Upper passband edge
WsL = 0.4; % Lower stopband edge
WsU = 0.6; % Upper stopband edge
As  = 50 ; % Stopband attenuation
Rp  = 0.2; % Passband ripple

% the freqz command returns an N point complex frequency response vector H
% given filter coefficients a and b
N = 1024;
WinN = 64;
Win4 = hann(WinN);
Ir = conv(Win4,Num4); % Impulse response of resultant filter
[Hd,OmegH] = freqz(Num4,1,N);
[W, OmegW] = freqz(Win4,1,N);


H = Hd.*W;


```

```
figure(2)
subplot(2,2,1);
plot(OmegH/pi,20*log10(abs(Hd)))
title('Original Filter Design')
xlabel('Normalized Frequency (\times\pi rad/sample)')
ylabel('Magnitude (dB)')

subplot(2,2,2);
plot(OmegH/pi,20*log10(abs(H)))
title('Windowed Filter Spectrum')
xlabel('Normalized Frequency (\times\pi rad/sample)')
ylabel('Magnitude (dB)')

subplot(2,2,3);
plot(OmegH/pi,20*log10(abs(W)))
title('Hann Window Spectrum')
xlabel('Normalized Frequency (\times\pi rad/sample)')
ylabel('Magnitude (dB)')

subplot(2,2,4);
stem(1:length(Num4),Num4)
title('Filter Impulse Response')
xlabel('Sample Index')
ylabel('Magnitude')
```