Module 7 - Homework 7

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Problem 1

Suppose the conditional random variable X—N has a binomial distribution with parameters p and N, where N has a Poisson distribution with mean λ . What is the marginal distribution for X?

The marginal distribution is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,n)dn$$

We are given:

$$p_{X|N}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1...N$$

$$p_N(n) = e^{-\lambda}\frac{\lambda^n}{n!}, n = 0, 1, ...$$

We can find the marginal distribution of X by using the total probability theorem:

$$P(X = k) = \sum_{n=0}^{N} p_{X|N}(k,n)p_N(n)$$

Note that the lower bound is zero because the Poisson and binomial distributions aren't defined for those regions.

$$P(X = k) = \sum_{n=0}^{\infty} \left(\binom{n}{k} p^k (1-p)^{n-k} \right) e^{-\lambda}\frac{\lambda^n}{n!}$$

$$P(X = k) = \sum_{n=0}^{\infty} \left(\frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \right) e^{-\lambda}\frac{\lambda^n}{n!}$$

$$P(X = k) = e^{-\lambda}p^k \sum_{n=0}^{\infty} \left(\frac{\lambda^n}{k!(n-k)!} (1-p)^{n-k} \right)$$

Notice that because $(n-k)!$ can't have negative terms, we need to adjust the lower bound of the summation:

$$P(X=k) = \frac{e^{-\lambda}p^k}{k!}\sum_{n=k}^{N}\left(\frac{\lambda^n}{(n-k)!}(1-p)^{n-k}\right)$$

I think there is a way to simplify this... let me come back to it.

Problem 2

The input X to a communication channel is a zero-mean, unit-variance Gaussian random variable. The channel output Y is the sum of X and a noise signal N, where N is a zero-mean Gaussian random variable with variance σ_N^2.X and N are independent random variables.

(A)

Find the conditional pdf of Y given X=x. Hint: $Y = N + x$ is a linear function of N Let's find the mean and variance of Y. Note that x is constant:

$$E[Y] = E[x] + E[N] = x + 0 = x$$

$$Var[Y] = Var[x+N] = Var[N] = \sigma_N^2$$

$$\boxed{f_{Y|X}(y) \sim N(x, \sigma_N^2)}$$

(B)

Find the joint pdf of X and Y

$$f_{X,Y}(x,y) =$$

The joint pdf of a 2-dimensional Gaussian random vector is given by

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2r\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

where r is the correlation coefficient and $-\infty < x_1 < \infty and -\infty < x_2 < \infty$. For our case, we need to first find the correlation between X and Y, then substitute the respective variables.

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$Cov(X,Y) = E[X(X+N)] - 0$$

$$Cov(X,Y) = E[X^2 + NX] = E[X^2] + E[X]E[N]$$

$$Cov(X,Y) = E[X^2] + 0 = 1$$

This leaves us with a correlation coefficient:

$$r = \frac{Cov(x,y)}{\sigma_X \sigma_Y} = \frac{1}{\sigma_N}$$

Now we can go back to our equation with $x_1 = x$, $\sigma_1 = \sigma_X = 1$, $x_2 = y$, $\sigma_2 = \sigma_Y = \sigma_N$, $\mu_1 = \mu_X$, $\mu_2 = \mu_Y$:

$$f_{X,Y}(x,y) = \frac{1}{2\pi(1)(\sigma_y)\sqrt{1-(\frac{1}{\sigma_N})^2}} exp\left\{-\frac{1}{2(1-(\frac{1}{\sigma_N})^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2(\frac{1}{\sigma_N})\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$$\boxed{f_{X,Y}(x,y) = \frac{1}{2\pi(\sigma_y)\sqrt{1-(\frac{1}{\sigma_N})^2}} exp\left\{-\frac{1}{2(1-(\frac{1}{\sigma_N})^2)}\left[\frac{(x)^2}{1} - 2\frac{(x)(y)}{\sigma_N^2} + \frac{(y)^2}{\sigma_N^2}\right]\right\}}$$

(C)

Find the conditional pdf of the input X given the observation $Y = y$ We have $X = Y - N$ when we solve for X.

$$E[X] = E[y] - E[N] = 0 - 0 = 0$$

$$Var[X] = Var[y - N] = Var[-N] = (-1)^2 Var[N] = \sigma_N^2$$

$$f_{X|Y}(x) \sim N(0, \sigma_N^2)$$

(D)

Suppose that when $Y = y$ we estimate the input X by the value $x_0 = g(y)$ that maximizes the probability $P[x_0 < X < x_0 + dx|Y = y]$. Find x_0. What happens to $g(y)$ as σ_N^2 approaches zero? As σ_N^2 approaches infinity? Evaluate $E[(X - g(Y))^2]$,the mean square error of the estimate

First find x_0 by finding the max value of $P(X|Y = y)$

$$0 = \frac{d}{dx}P(X|Y = y)$$

Let's now find the mean-square error of the estimate:

$$E[(X - g(Y))^2]$$

Problem 3

Let X, Y, and Z be independent zero-mean, unit-variance Gaussian random variables and let $R = \sqrt{X^2 + Y^2 + Z^2}$.Find the PDF of R

First note that the joint density function is (remember that they are independent):

$$f_{X,Y,Z}(x, y, z) = \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2)} \frac{1}{\sqrt{2\pi}} e^{-1/2(y^2)} \frac{1}{\sqrt{2\pi}} e^{-1/2(z^2)}$$

$$f_{X,Y,Z}(x, y, z) = \frac{1}{2\pi\sqrt{2\pi}} e^{-1/2(x+y+z)^2}$$

Let $x = rcos(\theta), y = rsin(\theta), z = 0$, and substitute into our distribution:

$$f_{X,Y,Z}(rcos(\theta), rsin(\theta), 0) = \frac{1}{2\pi\sqrt{2\pi}} e^{-1/2(rcos(\theta)+rsin(\theta)+0)^2}$$

$$f_{X,Y,Z}(rcos(\theta), rsin(\theta), 0) = \frac{1}{2\pi\sqrt{2\pi}} e^{-1/2(r^2(cos(\theta)+sin(\theta))^2)}$$

$$f_{X,Y,Z}(rcos(\theta), rsin(\theta), 0) = \frac{1}{2\pi\sqrt{2\pi}} e^{-1/2(r^2(cos^2(\theta)+2cos(\theta)sin(\theta)+sin^2(\theta)))}$$

$$f_{X,Y,Z}(rcos(\theta), rsin(\theta), 0) = \frac{1}{2\pi\sqrt{2\pi}} e^{-1/2(r^2(1+2cos(\theta)sin(\theta)))}$$

$$f_{X,Y,Z}(rcos(\theta), rsin(\theta), 0) = \frac{1}{2\pi\sqrt{2\pi}} e^{-1/2(r^2(1+sin(2\theta)))}$$

The reason why we set $z = 0$ is so that we only have to do a transformation involving 2 variables. To find the distribution of R, we will need to solve for the following:

$$f_{R,\theta}(r, \theta) = f_{X,Y}(rcos(\theta), rsin(\theta), 0))|J|$$

The jacobian is:

$$J = \begin{bmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} \end{bmatrix} \big]\!\!= \big[$$

Problem 4

Let X and Y be independent exponential random variables with parameters α and β, respectively. LetZ=X+Y

(A)

Find the characteristic function of Z. Because X and Y are independent we can use the following property:

$$C_{X+Y}(t) = E[e^{jt(X+Y)}] = E[e^{jtX}]E[e^{jtY}] = C_X(t)C_Y(t)$$

Thus we get the following characteristic function for Z:

$$C_Z(t) = C_X(t)C_Y(t) = \boxed{\frac{\alpha}{\alpha - jt}\frac{\beta}{\beta - jt}}$$

(B)

Find the PDF of Z from the characteristic function found in part (a).Hint:Use partial fractions.

$$f_Z(z) = \int_{-\infty}^{\infty} C_Z(t) e^{-jtz} dt$$

Above is the definition of the pdf in terms of the characteristic function. Note that we are dealing with exponential RVs so the lower bound is t=0.

$$f_Z(z) = \int_{0}^{\infty} C_X(t) C_Y(t) e^{-jtz} dt$$

$$f_Z(z) = \int_{0}^{\infty} \frac{\alpha}{\alpha - jt} \frac{\beta}{\beta - jt} e^{-jtz} dt$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{-\alpha\beta}{(t + j\alpha)(t + j\beta)} e^{-jtz} dt$$

Use partial fraction decomposition to break this up:

$$\frac{-\alpha\beta}{(t + j\alpha)(t + j\beta)} = \frac{\theta_1}{(t + j\alpha)} + \frac{\theta_2}{(t + j\beta)}$$

$$-\alpha\beta = \theta_1(t + j\beta) + \theta_2(t + j\alpha)$$

If you let $t = -bj$ and $t = -aj$, then we can find the thetas... and get:

$$\theta_1 = \frac{-\alpha\beta j}{a - b}, \theta_2 = \frac{\alpha\beta j}{a - b}$$

Plug it into our initial calculations:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{\frac{-\alpha\beta j}{a-b}}{t + j\beta} + \frac{\frac{\alpha\beta j}{a-b}}{t + j\alpha} e^{-jtz} dt$$

$$f_Z(z) = \frac{\alpha}{\alpha - \beta} \int_{0}^{\infty} \frac{\beta}{\beta - jt} e^{-jtz} dt + \frac{\beta}{\alpha - \beta} \int_{0}^{\infty} \frac{\alpha}{\alpha - jt} e^{-jtz} dt$$

Notice that we have the sum of two scaled exponential pdfs. There is a trick here to simplify it but I haven't figured it out yet... I'll come back to it.

Problem 5

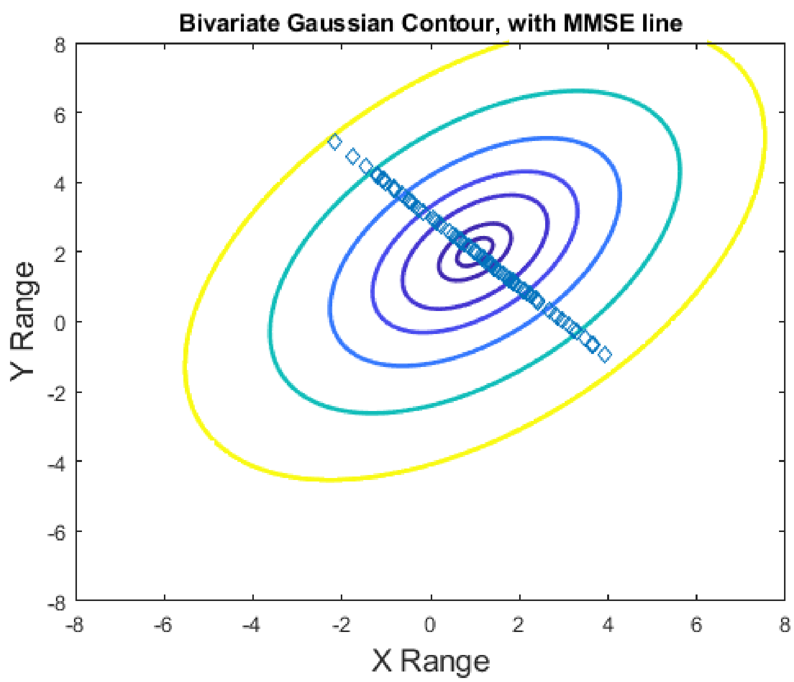
If a bivariate Gaussian PDF has a mean vector and a covariance matrix given below. plot the contours of constant PDF.Next find the minimum mean square error (MMSE) prediction of Y given X=x and plot it on top of the contourplot. Explain the significance of the plot

$$\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Recall that the MMSE in this case is given by:

$$E[Y|x] = \mu_y + \frac{\rho_{X,Y}\sigma_Y}{\sigma_X}(x - \mu_X)$$

Below is a plot for this problem. I took the liberty to plot several MMSE points to show the pattern and deduct the significance. Notice that is forms a line that intersects the origin, and how it crosses the contour lines. The MMSE line crosses the countour along a path such that the Y realization is most likely, $y = -x + 3$. E[Y]=2 and E[1], and plugging in x=E[X] will yield E[Y] and produce the resulting MMSE line.



Matlab Code

```
%% %%%%%%%%%%%%%%%
% Problem 5
%%%%%%%%%%%%%%%%%

% the goal of this plot is to show the regression line. See slide 19 of
% this week's lecture notes.

% Bivariate gaussian mean and covariance matrix
mu = [1;2];
C = [2,-1;-1,2];

% plot the contours of the constant bivariate gaussian PDF
gridVec = -8:0.1:8;
muVec = mu';
varVec = [C(1) C(4)]; % the diag of the covariance matrix is the var
r = 1/2;
[X,Y,Z,f] = hw4prob4SP21ContourPDF(gridVec,muVec,varVec,r);

figure(1)
csq = [1/16 1/4 1 2 4 8 16];
[cmat,h] = contour(X,Y,Z,csq,'LineWidth',2);
title("Bivariate Gaussian Contour, with MMSE line")
xlabel('X Range','FontSize',14);
ylabel('Y Range','FontSize',14);

% Find the minimum mean square error prediction Y given X=x
mu_y = mu(2);
mu_x = mu(1);
std_dev_X = sqrt(C(1));
std_dev_Y = sqrt(C(4));

% remember that the covariance of X and Y are the bordering matrix values
% of their respective variance values in said matrix
cov_XY = C(2); % also C(3)

% Realization of X
x = std_dev_X.*randn(1,100)+mu_x;
```

```
E_Y = mu_y + (cov_XY*std_dev_Y)/std_dev_X * (x-mu_x);

hold on
plot(x,E_Y,'d');
```