Assignment 4: Frequency Domain Analysis of LTI Systems and Sampling

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Book Questions

Problem 4.1

Using the course Matlab dtft function compute the magnitude, and phase for the following discrete-time signal:

$$\text{a)} x(n) = n(0.9)^n[u(n) - u(n-21)]$$

The MATLAB code for computing and plotting the DTFT of $x(n)$ is below in the appendix. Figure 1 shows the plot of the spectrum.

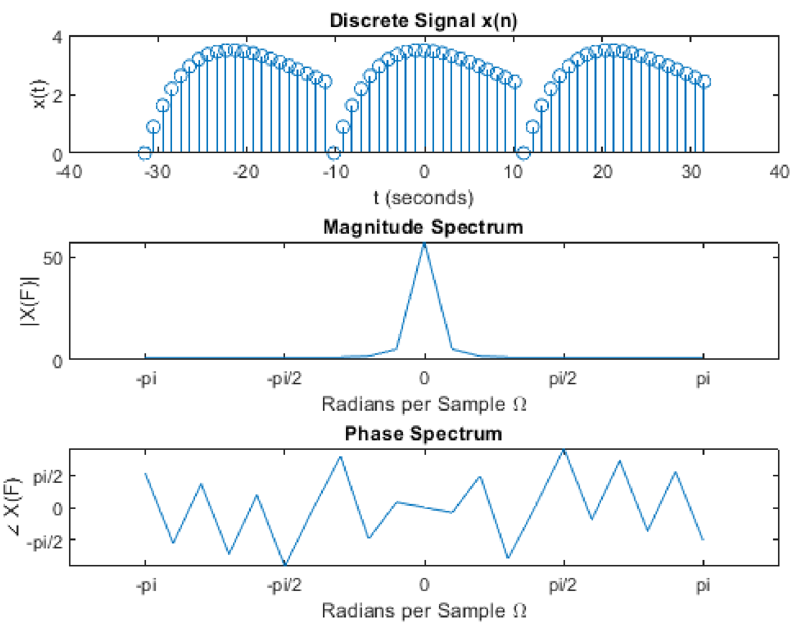

Figure 1: Plots of $x(n)$ and respective spectrum magnitude and phase plots.

$$b) x(n) = cos(\frac{10}{\pi}n - \frac{\pi}{4})[u(n) - u(n-40)]$$

The MATLAB code for computing and plotting the DTFT of $x(n)$ is below in the appendix. Figure 2 shows the plot of the spectrum.

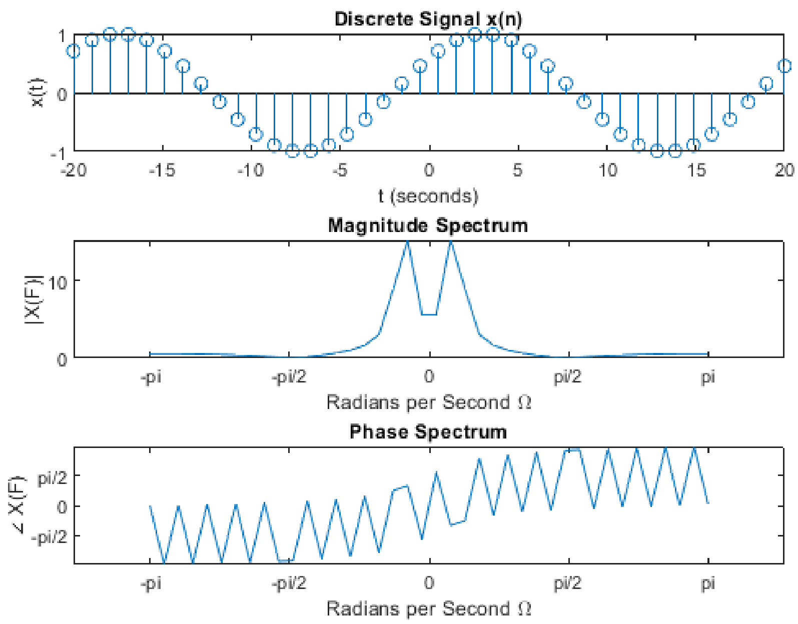

Figure 2: Plots of $x(n)$ and respective spectrum magnitude and phase plots.

Problem 4.2

The Rectangle window is a finite-duration sequence that is very useful in DSP:

$$\text{Rectangular: } R_m(n) = \begin{cases} 1, & 0 <= n <= M \\ 0, & \text{otherwise.} \end{cases}$$

a) for each of these windows, determine their DTFT for M=10,25,50,101.

Let's first hand calculate the DTFT for $M = 10$, then plot the rest of the functions to see how the spectrum changes with M:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Plug in $R_m(n)$ with $M = 10$ and we get:

$$X(\omega) = \sum_{n=0}^{10} e^{-j\omega n}$$

$$X(\omega) = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + ... + e^{-10j\omega}$$

We have a geometric sum here, which could have alternatively be written as $X(\omega) = \sum_{n=0}^{M}(e^{-j\omega})^n$. Solving with M as a generic variable, this converges to:

$$X(\omega) = \frac{1 - (e^{-j\omega})^M}{1 - e^-j\omega}$$

$$X(\omega) = \frac{e^{-j\omega M/2}(2j)sin(\omega M/2)}{e^{-j\omega}(2j)sin(\omega/2)}$$

$$X(\omega) = \frac{sin(\omega M/2)}{sin(\omega/2)}e^{-j\omega(M-1)/2}$$

See figure 3 for a plot of the DTFT for each value of M. Notice how they scale as M increases.

b) Scale transform values so that the maximum value is equal to 1 then plot the magnitude of the normalized DTFT over $-\pi <= \omega <= \pi$.

Plots are shown in Figure 4, with included phase spectrum.

c) Study these plots and comment on their behavior as a function of M.

As M increases, the magnitude of the spectrum will increase and will result in a narrower bandwidth. One takeaway from this is that if we have a longer window we can produce a finer filter with narrow bandwidth. The phase spectrum also appears to be more linear as M increases.

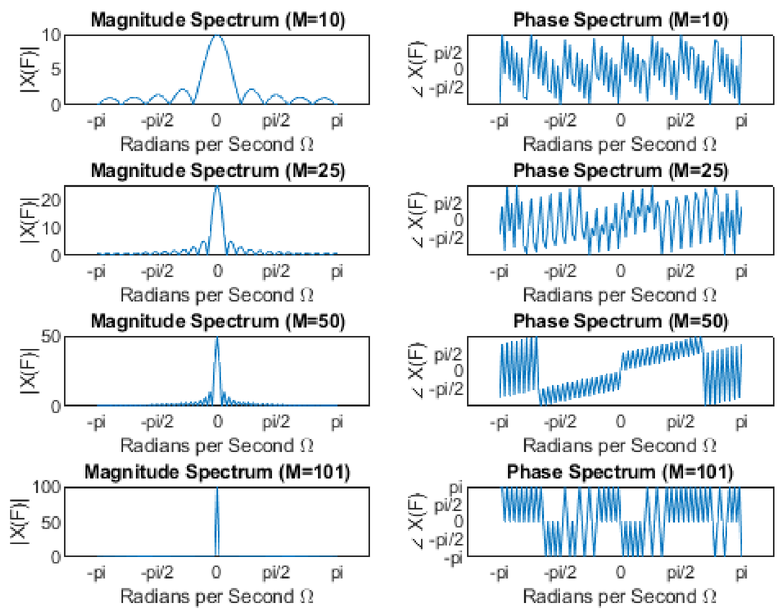

Figure 3: Plots of $X(\omega)$ for each M and respective spectrum magnitude and phase plots.

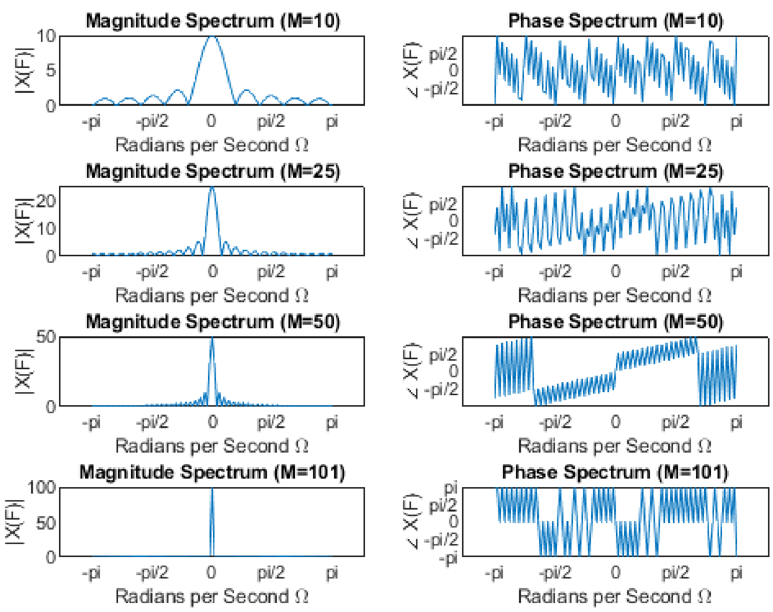

Figure 4: Spectrum plots of $x(n)$ for each M, with normalized magnitude spectrum.

Problem 4.3

Using the definition of the DTFT, determine the sequences corresponding to the following DTFT
a.) $X(\omega) = 3 + 2cos(\omega) + 4cos(2\omega)$

Using Euler's we can simplify the given spectrum $X(\omega)$

$$cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$X(\omega) = 3 + e^{j\omega} + e^{-j\omega} + 2e^{2j\omega} + 2e^{-2j\omega}$$

$$X(\omega) = 2e^{2j\omega} + e^{j\omega} + 3 + e^{-j\omega} + 2e^{-2j\omega}$$

$$X(\omega) = \sum_{n=-2}^{2} x(n)e^{-j\omega},$$

$$x(n) = [2, 1, \underset{\uparrow}{3}, 1, 2]$$

b.) $X(\omega) = [1 - 6cos(3\omega) + 8cos(5\omega)]e^{-j3\omega}$

$$X(\omega) = \big[1 - 6\frac{e^{3j\omega} + e^{-3j\omega}}{2} + 8\frac{e^{5j\omega} + e^{-5j\omega}}{2}\big]e^{-j3\omega}$$

$$X(\omega) = \big[1 - 3e^{3j\omega} + 3e^{-3j\omega} + 4e^{5j\omega} + 4e^{-5j\omega}\big]e^{-j3\omega}$$

$$X(\omega) = e^{-j3\omega} - 3e^{3j\omega-3j\omega} - 3e^{-3j\omega-3j\omega} + 4e^{5j\omega-3j\omega} + 4e^{-5j\omega-3j\omega}$$

$$X(\omega) = e^{-j3\omega} - 3 - 3e^{-6j\omega} + 4e^{2j\omega} + 4e^{-8j\omega}$$

$$X(\omega) = 4e^{2j\omega} - 3 + e^{-j3\omega} - 3e^{-6j\omega} + 4e^{-8j\omega}$$

$$x(n) = [4, 0, \underset{\uparrow}{-3}, 0, 0, 1, 0, 0, -3, 0, 4]$$

Problem 4.4

Determine the frequency response for an LTI system defined by the difference equation and plot its magnitude and phase as a function of ω*:*

$$y(n) = x(n) - x(n-1) + x(n-2) + 0.95y(n-1) - 0.9025y(n-2)$$

To find the frequency response of a system, we need to insert a sinusoid input for $x(n)$. Let $x(n) = Ae^{j\omega n}$:

$$y(n) = Ae^{j\omega n} - Ae^{j\omega(n-1)} + Ae^{j\omega(n-2)} + 0.95y(n-1) - 0.9025Ay(n-2)$$

$$y(n) = Ae^{j\omega n} - Ae^{j\omega n}e^{-j\omega} + Ae^{j\omega n}e^{-2j\omega} + 0.95y(n-1) - 0.9025y(n-2)$$

$$y(n) - 0.95y(n-1) + 0.9025y(n-2) = Ae^{j\omega n} - Ae^{j\omega n}e^{-j\omega} + Ae^{j\omega n}e^{-2j\omega}$$

Note that $e^{-j\omega}$ is a delay by one sample...

$$y(n)(1 - 0.95e^{-j\omega} + 0.9025e^{-2j\omega}) = Ae^{j\omega n}(1 - e^{-j\omega} + e^{-2j\omega})$$

$$y(n) = \frac{(1 - e^{-j\omega} + e^{-2j\omega})}{(1 - 0.95e^{-j\omega} + 0.9025e^{-2j\omega})} Ae^{j\omega n}$$

From this we can conclude that the frequency response $H(e^{j\omega n})$ is:

$$H(e^{j\omega n}) = \frac{(1 - e^{-j\omega} + e^{-2j\omega})}{(1 - 0.95e^{-j\omega} + 0.9025e^{-2j\omega})}$$

The plots of the magnitude and phase spectrum are below in Figure 5.

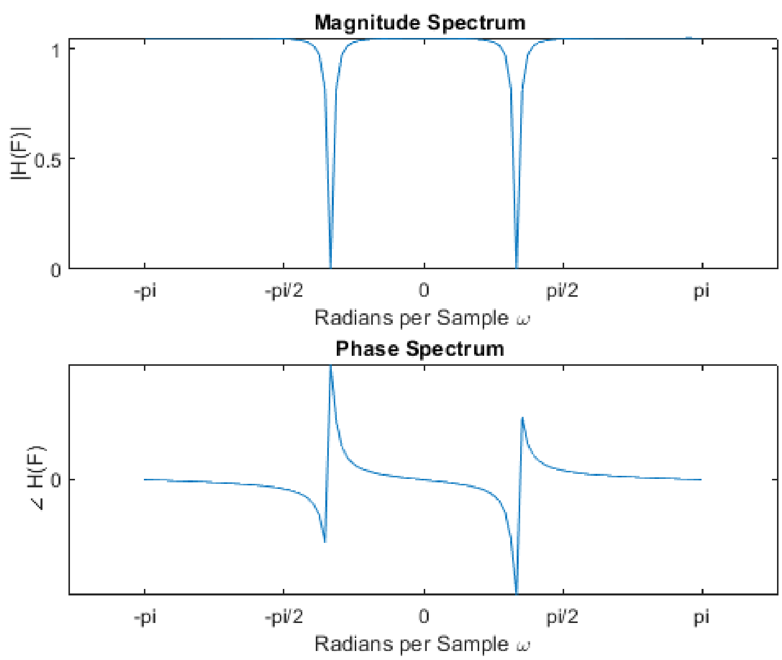

Figure 5: Spectrum plots of $H(e^{j\omega n})$.

Problem 4.5

A.) Use MATLAB to generate a pulsed sine wave signal having a frequency F=10 KHz and a pulse-width of T=.5ms. The pulse repetition time is 1ms (pulse repetition rate is 1000 pulses/sec), i.e., the signal is zero between pulses. Model the signal to start at time t=0 and to end at a time t=Tstop. Use a sample time of .001ms to approximate an analog environment.Display the signal in MATLAB appropriate for an analog signal, first for Tstop=1ms and then for

Tstop=5ms.

This was performed in MATLAB under the Problem 4.5 section in the code below. See Figures 6 and 7.

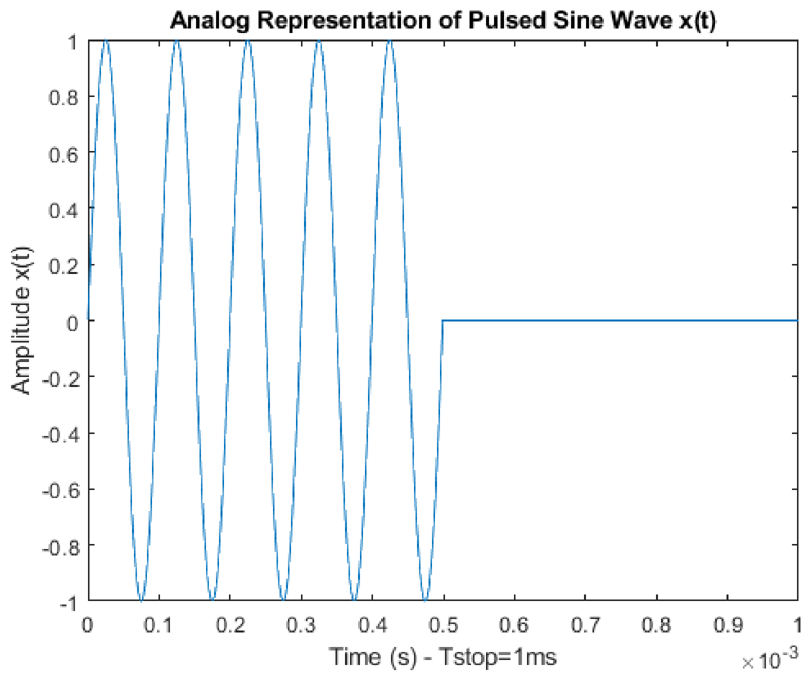

Figure 6: Plots of 10KHz sine pulse with pulse-width of 0.5ms and stop time at 1ms.

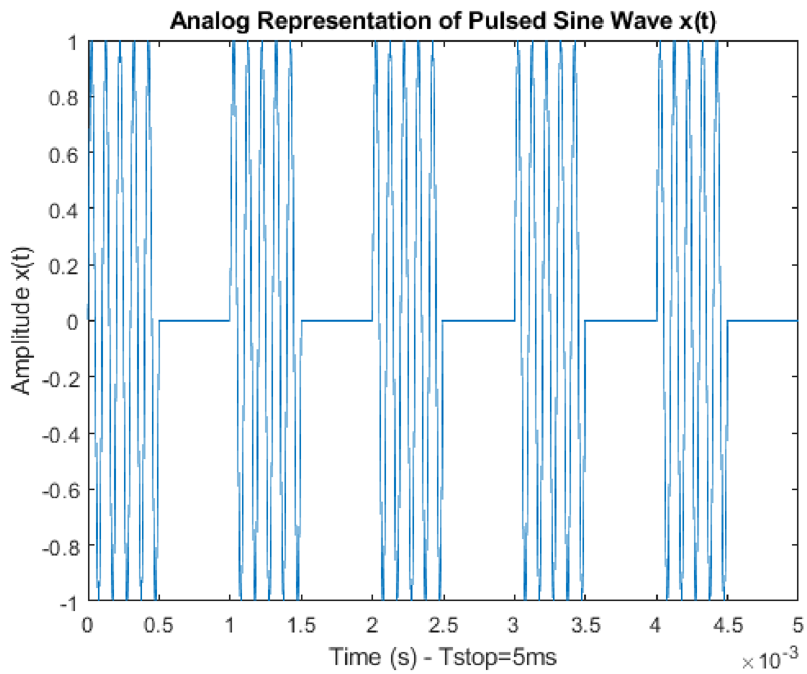

Figure 7: Plots of 10KHz sine pulse with pulse-width of 0.5ms and stop time at 5ms.

B.) In the MATLAB part (A) simulation sample the analog pulsed sine wave signal with sample times of.01ms. Display these sampled pulse sine wave sin MATLAB appropriate for a discrete signal

Plots are given below in Figures 8 and 9.

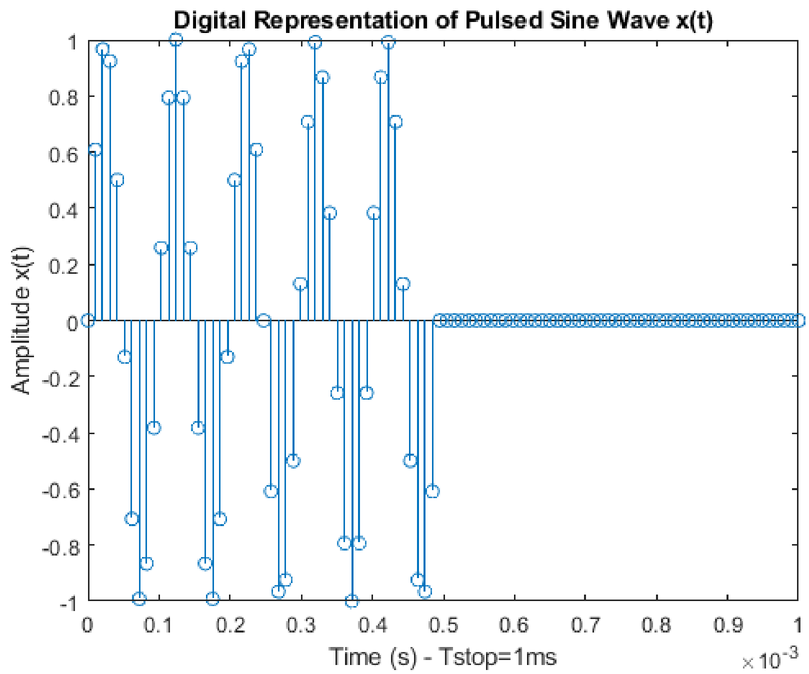

Figure 8: Plots of sampled 10KHz sine pulse with pulse-width of 0.1ms and stop time at 5ms.

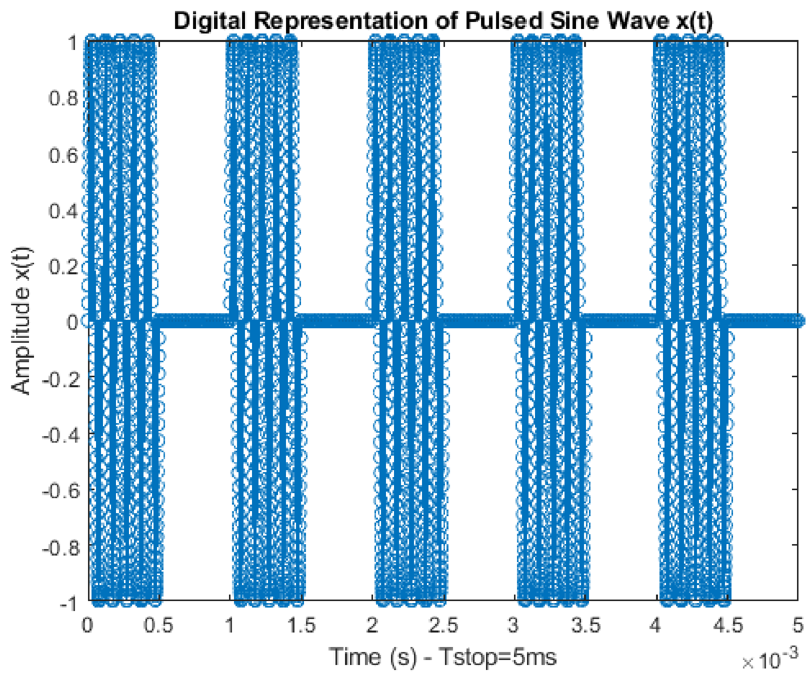

Figure 9: Plots of sampled 10KHz sine pulse with pulse-width of 0.5ms and stop time at 5ms.

C.) In MATLAB compute and display the magnitude of the DTFT of the pulse sin wave signals for w between 0 and pi. Do this for each of the Tstop times given above. Determine the physical frequency associated with the largest peak in the magnitude spectrum. Estimate the frequency spacing of significant minor peaks away from the major peak. What comment can you make when comparing the DTFT's of the two signals?

Comparing the plots in Figures 10 and 11, we notice a few differences. First, as the number of pulses increase the magnitude increases. This is because there is more energy as the signal is prolonged. Second, the DTFT begins to look more like the ideal Fourier transform of a sinusoid (or Dirac Delta) as we approach infinite pulses.

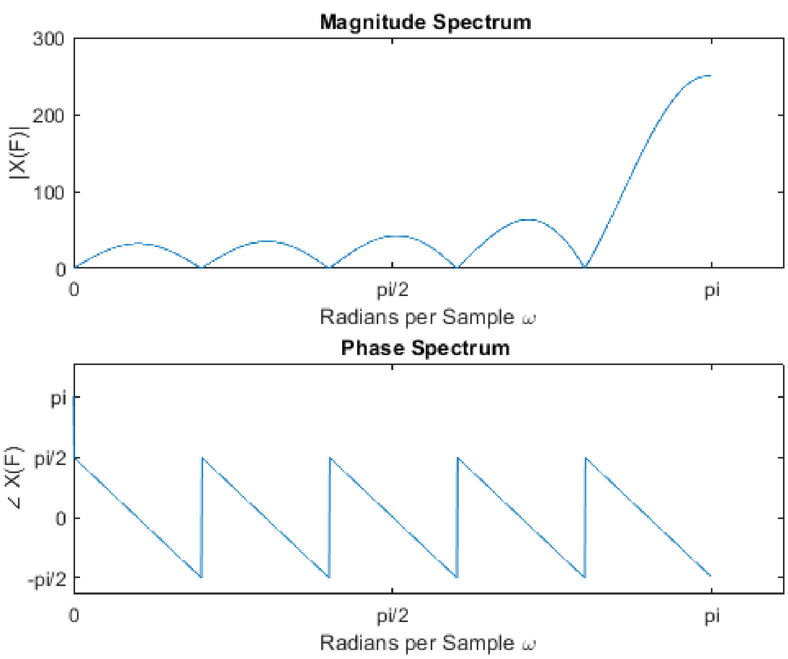

Figure 10: DTFT of sampled 10KHz sine pulse with pulse-width of 0.5ms and stop time at 1ms.

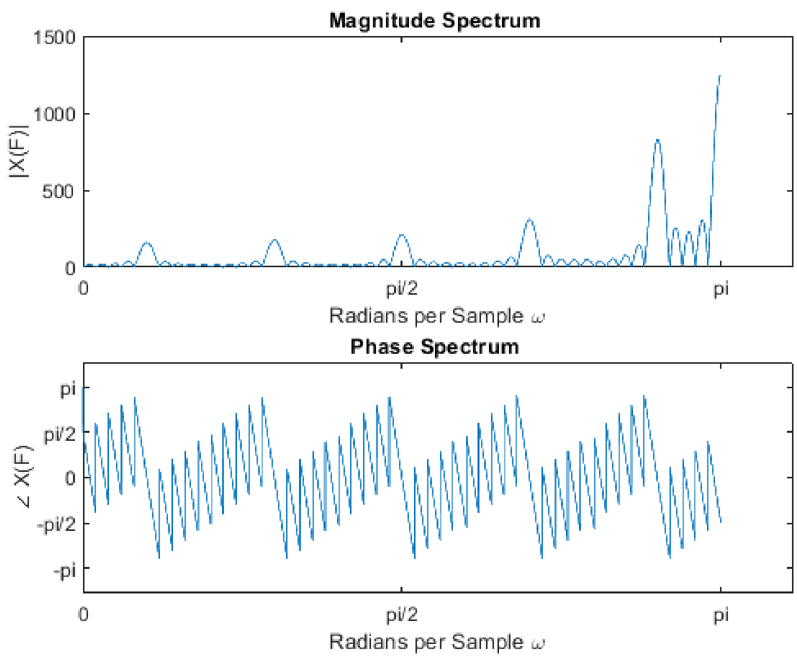

Figure 11: DTFT of sampled 10KHz sine pulse with pulse-width of 0.5ms and stop time at 5ms.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem 4.1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% compute the magnitude and phase of the signals in parts
    A and B

% - Part A:
xa = (0:20); % create x(n)=n*(0.9)^n
for n=0:20
   xa(n+1) = xa(n+1)*(0.9)^n;
end

k = 10000; % number of points for refinement
n = linspace(-length(xa)/2,length(xa)/2,length(xa)); %
    the length() command is useful for dynamic code
w = linspace(-pi,pi,length(n));
X = dtft(xa,n,w);
Xmag = abs(X);
Xang = angle(X);

% plotting repeated x
P = 3; % The number of times we repeat the signal
xa = xa' * ones(1,P);
xa = xa(:); % long column vector
xa = xa'; % transpose to long row vector
n = linspace(-length(xa)/2,length(xa)/2,length(xa));
W = linspace(-pi,pi,length(X));


figure(1);
grid on;
subplot(3,1,1)
stem(n,xa);
title('Discrete Signal x(n)')
xlabel('t (seconds)')
ylabel('x(t)')


subplot(3,1,2)
plot(W,Xmag);
xlabel('Radians per Sample \Omega')
title('Magnitude Spectrum')
ylabel('|X(F)|')
```

```
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


subplot(3,1,3)
plot(W,Xang);
title('Phase Spectrum')
xlabel('Radians per Sample \Omega')
ylabel('\angle X(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})

 clear;
 %-----------------------------------------
 %- Part B:
 %-----------------------------------------
 n = (0:39);
 xb = cos(pi/10.*n-pi/4);


 k = 10000; % number of points for refinement
n = linspace(-length(xb)/2,length(xb)/2,length(xb)); % the length() command is useful for dynamic code
w = linspace(-pi,pi,length(n));
X = dtft(xb,n,w);
Xmag = abs(X);
Xang = angle(X);

% plotting repeated x
P = 1; % The number of times we repeat the signal
xb = xb' * ones(1,P);
xb = xb(:); % long column vector
xb = xb'; % transpose to long row vector
n = linspace(-length(xb)/2,length(xb)/2,length(xb));
W = linspace(-pi,pi,length(X));


figure(2);
grid on;
subplot(3,1,1)
stem(n,xb);
title('Discrete Signal x(n)')
xlabel('t (seconds)')
ylabel('x(t)')
```

```
subplot(3,1,2)
plot(W,Xmag);
xlabel('Radians per Second \Omega')
title('Magnitude Spectrum')
ylabel('|X(F)|')
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


subplot(3,1,3)
plot(W,Xang);
title('Phase Spectrum')
xlabel('Radians per Second \Omega')
ylabel('\angle X(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})

 %%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem 4.2
%%%%%%%%%%%%%%%%%%%%%%%%%%

% ---------------
% - Part A
% ---------------
% declare the rectangle window function


clear
x_o=[ones(1,10),zeros(1,91); ...
     ones(1,25),zeros(1,76); ...
     ones(1,50),zeros(1,51);...
     ones(1,101)...
     ];


fig_idx = 0;
for m=1:4
x = x_o(m,:);
k = 10000; % number of points for refinement
n = linspace(-length(x)/2,length(x)/2,length(x)); % the length() command is useful for dynamic code
w = linspace(-pi,pi,length(n));
```

```
X = dtft(x,n,w);
Xmag = abs(X);
Xang = angle(X);

% plotting repeated x
P = 1; % The number of times we repeat the signal
x = x' * ones(1,P);
x = x(:); % long column vector
x = x'; % transpose to long row vector
n = linspace(-length(x)/2,length(x)/2,length(x));
W = linspace(-pi,pi,length(X));



 fig_idx = fig_idx+1
 figure(3)
 subplot(4,2,fig_idx)
 plot(W,Xmag);
 xlabel('Radians per Second \Omega')
 title(strcat('Magnitude Spectrum (M=',int2str(sum(x)),')'))
 ylabel('|X(F)|')
 set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})

 fig_idx = fig_idx+1
 subplot(4,2,fig_idx)
 plot(W,Xang);
 title(strcat('Phase Spectrum (M=',int2str(sum(x)),')'))
 xlabel('Radians per Second \Omega')
 ylabel('\angle X(F)')
  set(gca,'YTick',-pi:pi/2:pi)
  set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
   set(gca,'XTick',-pi:pi/2:pi)
  set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
end

% ----------------
% - Part B
% ----------------


 fig_idx = 0;
for m=1:4
x = x_o(m,:);
k = 10000; % number of points for refinement
```

```
n = linspace(-length(x)/2,length(x)/2,length(x)); % the
    length() command is useful for dynamic code
w = linspace(-pi,pi,length(n));
X = dtft(x,n,w);


% normalize the spectrum
X = X./sum(x);

Xmag = abs(X);
Xang = angle(X);

% plotting repeated x
P = 1; % The number of times we repeat the signal
x = x' * ones(1,P);
x = x(:); % long column vector
x = x'; % transpose to long row vector
n = linspace(-length(x)/2,length(x)/2,length(x));
W = linspace(-pi,pi,length(X));



fig_idx = fig_idx+1;
figure(4)
subplot(4,2,fig_idx)
plot(W,Xmag);
xlabel('Radians per Second \Omega')
title(strcat('Magnitude Spectrum (M=',int2str(sum(x)),')'
    ))
ylabel('|X(F)|')
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})

fig_idx = fig_idx+1;
subplot(4,2,fig_idx)
plot(W,Xang);
title(strcat('Phase Spectrum (M=',int2str(sum(x)),')'))
xlabel('Radians per Second \Omega')
ylabel('\angle X(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
end

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%% Problem 4.4
%%%%%%%%%%%%%%%%%%%%%%%%%%
clear;
k =100;
w = linspace(-pi,pi,k);
% for w=1:k
% H(w) = (1-exp(-1i*w)+exp(-2i*w))/(1-0.95*exp(-i*w)+0.9025*exp(-2i*w));
% end
H = (1-exp(-1i*w)+exp(-2i*w))./(1-0.95*exp(-i*w)+0.9025*exp(-2i*w)) ;
W = linspace(-pi,pi,length(H));
Hmag = abs(H);
Hang = angle(H);

figure(5)
subplot(2,1,1)
plot(W,Hmag);
xlabel('Radians per Sample \omega')
title('Magnitude Spectrum ')
ylabel('|H(F)|')
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


subplot(2,1,2)
plot(W,Hang);
title('Phase Spectrum ')
xlabel('Radians per Sample \omega')
ylabel('\angle H(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})

%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Problem 4.5
%%%%%%%%%%%%%%%%%%%%%%%%%%
clear;
F = 10e3;
pulseWidth = 0.5e-3; % each pulse
sampleTime = 0.001e-3; % in seconds
samp_per_sec = 1/sampleTime; % sample time of 0.001ms
T = linspace(0,pulseWidth,pulseWidth*samp_per_sec); % duration of pulse
x = sin(2*pi*F*T);
```

```

% Creation of repeated pulse
Tstop=5e-3; % 1 ms
pulseRepetitionTime = 2*pulseWidth;
P = floor(Tstop/pulseRepetitionTime); % repetitions
tmp = [1,0]'* ones(1,P); % this allows for zero spacing
    between pulses
x = x' * tmp(:)';
x = x(:); % long column vector
x = x'; % transpose to long row vector
n = linspace(0,P*pulseRepetitionTime,length(x)); % 2x
    pulse width

% analog representation
figure(6);
plot(n,x);
title('Analog Representation of Pulsed Sine Wave x(t)')
xlabel(strcat('Time (s) - Tstop=',num2str(Tstop*1e3),'ms'
    ));
ylabel('Amplitude x(t)')

% discrete sampled representation
figure(7)
stem(n,x);
title('Digital Representation of Pulsed Sine Wave x(t)')
xlabel(strcat('Time (s) - Tstop=',num2str(Tstop*1e3),'ms'
    ));
ylabel('Amplitude x(t)')



% DTFT of x(n)
w = linspace(0,2*pi*F,length(n));
X = dtft(x,n,w);
Xmag = abs(X);
Xang = angle(X);
W = linspace(0,pi,length(X));

figure(8)
subplot(2,1,1)
plot(W,Xmag);
xlabel('Radians per Sample \omega')
title('Magnitude Spectrum ')
ylabel('|X(F)|')
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
```

```
subplot(2,1,2)
plot(W,Xang);
title('Phase Spectrum ')
xlabel('Radians per Sample \omega')
ylabel('\angle X(F)')
 set(gca,'YTick',-pi:pi/2:pi)
 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
  set(gca,'XTick',-pi:pi/2:pi)
 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})


 %%
 plot = 2

 plot(n,x)
```