Module 10 - Homework 10

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Problem 1

Let W(t) be the standard Brownian Motion. Note that W(t)is a Wiener process.

Part (a)

Find $P(-3 < W(3) < 3)$ A Brownian motion can be described as the following (see the probabilitycourse.com website for details on the Brownian Motion):

$$W(t) = W(n\delta) = \sum_{i=1}^{n} X_i$$

Where δ is a time interval of the process, and X_i is a random variable independent from it's other iterations, with the following properties:

$$E[X_i] = 0$$

$$Var(X_i) = \delta$$

Since W(t) is a sum of IID RVs, we can represent W(3) as a Gaussian:

$$E[W(3)] = \sum_{i=1}^{n} E[X_i] = 0$$

$$Var(W(3)) = \sum_{i=1}^{n} Var(X_i)$$

$$Var(W(3)) = nVar(X_i)$$

$$Var(W(3)) = n\delta$$

Note that $t = n\delta$, where n is the number of time interval steps.

$$Var(W(3)) = 3$$

Now, we can solve the problem:

$$P(-3 < W(3) < 3) = \Phi(\frac{3-0}{3}) - \Phi(\frac{-3-0}{3})$$

$$P(-3 < W(3) < 3) = \Phi(1) - \Phi(-1)$$

$$\boxed{P(-3 < W(3) < 3) = 0.6827}$$

Part (b)

Find $P(1 < W(3) + W(7) < 10)$

$$E[W(3) + W(7)] = E[W(3)] + E[W(7)] = 0$$

$$Var(W(3) + W(7)) = Var(W(3)) + Var(W(7)) + 2Cov(W(3), W(7))$$

$$Var(W(3) + W(7)) = 3 + 7 + 0 = 10$$

$$P(1 < W(3) + W(7) < 10) = \Phi(\frac{10 - 0}{10}) - \Phi(\frac{1 - 0}{10})$$

$$\boxed{P(1 < W(3) + W(7) < 10) = \Phi(1) - \Phi(0.1) = 0.3015}$$

Part (c)

Find $P(W(3) > 3|W(4) = 1)$

$$W(3) \sim N(0, 3) \text{ and } W(4) \sim N(0, 4)$$

We can find the conditional expected value and variance for $W(3)$ and $W(4)$ (see class notes slide 12):

$$E[W(3)|W(4) = 1] = \mu_{W(3)} + \frac{r_{W(4)W(3)}\sigma_{W(3)}}{\sigma_{W(4)}}(1 - \mu_{W(4)})$$

$$E[W(3)|W(4) = 1] = 0 + \frac{\sqrt{3/4}(3)}{(4)}(1 - 0) = \frac{3\sqrt{3}}{8}$$

$$Var(W(3)|W(4) = 1) = (1 - r^2_{W(4)W(3)})\sigma_{W(3)}$$

$$Var(W(3)|W(4) = 1) = (1 - 3/4)(3) = \frac{3}{4}$$

With this, we can find the probability:

$$P(W(3) > 3|W(4) = 1) = 1 - \Phi(\frac{3 - \frac{3\sqrt{3}}{8}}{3/4})$$

$$\boxed{P(W(3) > 3|W(4) = 1) = 1 - \Phi(3.134) = 8.622 \times 10^{-4}}$$

Problem 2

Suppose that for the random process shown below, all of the ω_n *are constants, and the* θ_n *are IID random variables,each uniformly distributed over* $[0, 2\pi)$.

$$X(t) = \sum_{n=1}^{N} a_n cos(\omega_n t + \theta_n)$$

Part (a)

Determine the autocorrelation function of X(t)

The autocorrelation is $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$. Remember that when random variables θ_n are IID, $\phi_n = G(\theta_n)$ will also be independent from each other.

$$R_X(t_1, t_2) = E[\sum_{n=1}^{N} a_n cos(\omega_n t_1 + \theta_n) \sum_{m=1}^{N} a_m cos(\omega_n t_2 + \theta_m)]$$

$$R_X(t_1, t_2) = E[\sum_{n=1}^{N} a_n cos(\omega_n t_1 + \theta_n) \sum_{m=1}^{N} a_m cos(\omega_n t_2 + \theta_m)]$$

$$R_X(t_1, t_2) = E[\sum_{n=1}^{N} a_n cos(\omega_n t_1 + \theta_n)]E[\sum_{n=1}^{N} a_n cos(\omega_n t_2 + \theta_n)]$$

$$R_X(t_1, t_2) = \sum_{n=1}^{N} a_n E[cos(\omega_n t_1 + \theta_n)] \sum_{n=1}^{N} a_n E[cos(\omega_n t_2 + \theta_n)]$$

For this, we will need to use the definition of the expected value and integrate the sum of sinusoids. Remember that $E[G(X)] = \int_{-\infty}^{\infty} G(x) f_X(x)dx$, and $f_X(x) = 1/2\pi$ for the uniform distribution.

$$R_X(t_1, t_2) = \sum_{n=1}^{N} a_n \int_0^{2\pi} cos(\omega_n t_1 + \theta_n)\frac{1}{2\pi} d\theta_n \sum_{n=1}^{N} a_n \int_0^{2\pi} cos(\omega_n t_2 + \theta_n)\frac{1}{2\pi} d\theta_n$$

$$R_X(t_1, t_2) = \sum_{n=1}^{N} a_n(0) \sum_{n=1}^{N} a_n(0)$$

$$\boxed{R_X(t_1, t_2) = 0}$$

Part (b)

Determine the power spectral density of X(t)
The power spectral density of $X(t)$ is the fourier transform of the autocorrelation function. This makes the math easy for us:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-2j\pi f \tau} d\tau$$

$$S_X(f) = \int_{-\infty}^{\infty} (0) e^{-2j\pi f \tau} d\tau$$

$$\boxed{S_X(f) = 0}$$

Problem 3

For the transition probability matrix

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

Part (a)

Find the eigendecomposition so that $P = V \wedge V^{-1}$, where $\wedge$ is a diagonal matrix of the eigenvalues of P, and V is the matrix whose columns are the corresponding eigenvectors.You can use the Matlab function $[V, D] = eig(P)$. Running the Matlab command produced the following matrices:

$$V = \begin{bmatrix} 0.5774 + 0.0000i & 0.4630 + 0.2765i & 0.4630 - 0.2765i \\ 0.5774 + 0.0000i & -0.6893 + 0.0000i & -0.6893 + 0.0000i \\ 0.5774 + 0.0000i & -0.1749 - 0.4511i & -0.1749 + 0.4511i \end{bmatrix}$$

$$\wedge = \begin{bmatrix} 1.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.2000 + 0.1414i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.2000 - 0.1414i \end{bmatrix}$$

Part (b)

Determine P^{100} We can easily calculate this in Matlab by creating Matrix P and evaluating it to the 100th power:

$$\boxed{P^{100} = \begin{bmatrix} 0.4697 & 0.2424 & 0.2879 \\ 0.4697 & 0.2424 & 0.2879 \\ 0.4697 & 0.2424 & 0.2879 \end{bmatrix}}$$

Part (c)

Find the steady state distribution of this Markov Chain and compare it to your answer in (b).

calculate ergocicity. Is it recurrent, communicate and aperiodic?

We need to solve for $\pi P = \pi$.

$$\pi P = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

We get the following system of equations:

$$0.6\pi_1 + 0.3\pi_2 + 0.1\pi_3 = \pi_1$$

$$0.3\pi_1 + 0.3\pi_2 + 0.4\pi_3 = \pi_2$$

$$0.4\pi_1 + 0.1\pi_2 + 0.5\pi_3 = \pi_3$$

Remember that the following equation applies to all Markov chain problems:

$$\pi_1 + \pi_2 + \pi_3 = 1$$

I'm going to pick the first two of the three equations and solve using matrix form:

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$0.3\pi_1 - 0.7\pi_2 + 0.4\pi_3 = 0$$

$$0.4\pi_1 + 0.1\pi_2 - 0.5\pi_3 = 0$$

Convert to matrix form and solve for π:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.3 & -0.7 & 0.4 \\ 0.4 & 0.1 & -0.5 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0.3333 & 0.6452 & 1.1828 \\ 0.3333 & -0.9677 & -0.1075 \\ 0.3333 & 0.3226 & -1.0753 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0.3333 \\ 0.3333 \\ 0.3333 \end{bmatrix}}$$

With this we can produce the stable matrix P^n as $n \to \infty$

$$P^n = \begin{bmatrix} 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix}$$

Notice that P^{100} is approaching this matrix, it just doesn't converge yet.

Problem 4

Two gamblers play the following game. A fair coin is flipped; if the outcome is heads, player A pays player B \$1, and if the outcome is tails player B pays player A \$1. The game is continued until one of the players goes broke. Suppose that initially player A has \$1 and player B has \$2, so a total of\$3is up for grabs. Let X_n denote the number of dollars held by player A after n trials.

Part (a)

Show that X_n is a Markov chain. In other words, show that X_n satisfies the Markov property.

X_n is a Markov process if the future of the process given the present is independent of the past. Let k be the amount of dollars that a player has in state n. We can express the future state X_{n+1} as:

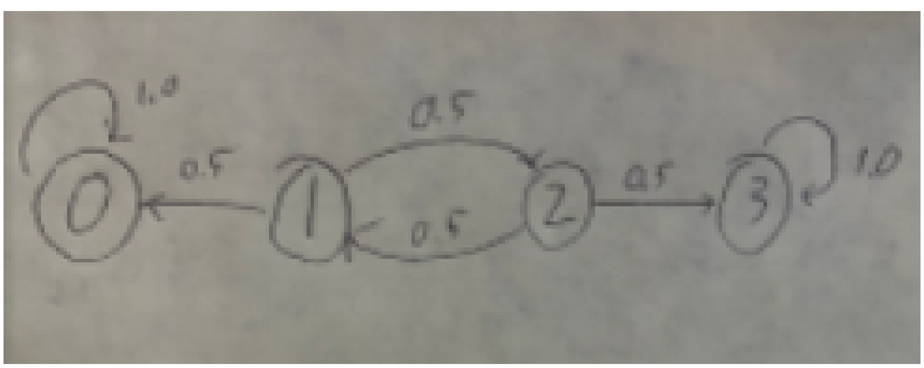
$$\boxed{(X_{n+1}|X_n = k) = \begin{cases} k+1 & \text{heads} \\ k-1 & \text{tails} \end{cases}}$$

Note that this does not depend on $n-1$, the past state, but rather the present state n.

Part (b)

Sketch the state transition diagram for X_n and give the one-step transition probability matrix P. I did a hand sketch of the transition diagram in Figure . As for the matrix P, think of the rows being the current state, and the columns being the potential next state:

$$\boxed{P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$



Part (c)

Find the limit of P^n as $n \to \infty$

The quick way to check this is to calculate something like P^{100} in Matlab and see what the matrix approaches. In our case we get:

$$\boxed{\lim_{n\to\infty} P^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

The better way to find this is to solve for $\pi P = \pi$. The problem I had with this method is that we get three equations with 4 unknowns so I can't solve using this method. I'm assuming that P probably does not satisfy some condition (Ergodicity?):

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$
$$0.5\pi_1 - \pi_2 + 0.5\pi_3 = 0$$
$$0.5\pi_2 - \pi_3 + 0.5\pi_4 = 0$$

Part (d)

Find the probability that player A eventually wins. Since player A starts with \$1, we can look at the matrix from part C (specifically $P_{3,1}$)and determine that they have a $\boxed{1/3}$ chance of winning.