Module 2 - Homework 2

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Problem 3.1-1

Part A

Use direct integration to find the Fourier transform G(f) of signal

$$g_a(t) = \Pi(t-1) - 5exp(-2|t+3|)$$

The Fourier transform is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$

Let's apply the direct integration method:

$$G_a(f) = \int_{-\infty}^{\infty} (\Pi(t-1) - 5exp(-2|t+3|))e^{-j2\pi ft}dt$$

$$G_a(f) = \int_{-\infty}^{\infty} \Pi(t-1)e^{-j2\pi ft}dt - \int_{-\infty}^{\infty} 5exp(-2|t+3|)e^{-j2\pi ft}dt$$

Remember that the rect function has finite bounds:

$$\int_{-\infty}^{\infty} \Pi(t-1)e^{-j2\pi ft}dt = \int_{1/2}^{3/2} (1)e^{-j2\pi ft}dt$$

$$= \left[\frac{1}{-i2\pi f}e^{-j2\pi ft}\right]_{1/2}^{3/2}$$

$$= \frac{1}{-i2\pi f}\left[e^{-j2\pi f(3/2)} - e^{-j2\pi f(1/2)}\right]$$

$$= \frac{1}{-i2\pi f}\left[e^{-j3\pi f} - e^{-j\pi f}\right]$$

$$= \frac{1}{-i2\pi f}\left[e^{-j\pi f - j2\pi f} - e^{j\pi f - j2\pi f}\right]$$

$$= \frac{1}{-i2\pi f}\left[e^{-j\pi f}e^{-j2\pi f} - e^{j\pi f}e^{-j2\pi f}\right]$$

$$= \frac{-1}{\pi f} e^{-j2\pi f} \frac{e^{-j\pi f} - e^{j\pi f}}{i2}$$
$$= -e^{-j2\pi f} sinc(\pi f)$$

Remember that multiplying that exponential component translates to a time shift. Neat huh? Now we have the second integral to tackle...

$$\int_{-\infty}^{\infty} 5exp(-2|t+3|)e^{-j2\pi ft}dt = 5e^{j2\pi f(-3)} \int_{-\infty}^{\infty} e^{-2|t|}e^{-j2\pi ft}dt$$

$$= 5e^{j2\pi f(-3)} \left[\int_{-\infty}^{0} e^{2t}e^{-j2\pi ft}dt + \int_{0}^{\infty} e^{-2t}e^{-j2\pi ft}dt \right]$$

$$= 5e^{j2\pi f(-3)} \left[\frac{1}{2(\pi^2 f^2 + 1)} - \frac{j\pi f}{2(\pi^2 f^2 + 1)} \right]$$

$$= 10e^{j2\pi f(-3)} \left[\frac{4}{4 + (2\pi f)^2} \right]$$

Combine the monstrosity to get:

$$\boxed{G_a(f) = -e^{-j2\pi f} sinc(\pi f) - 10e^{j2\pi f(-3)} \left[\frac{4}{4 + (2\pi f)^2} \right]}$$

Part C

Use direct integration to find the inverse Fourier transform of:

$$G_c(f) = \delta(\pi f) - 2\delta(f - 0.5)$$

The inverse FT is defined as

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df$$

Solving we get...

$$g_c(t) = \int_{-\infty}^{\infty} \delta(\pi f)e^{j2\pi ft}df - \int_{-\infty}^{\infty} 2\delta(f - 0.5)e^{j2\pi ft}df$$

Use the sifting/sampling property:

$$g_c(t) = e^0 - e^{2\pi(0.5)t}$$

$$\boxed{g_c(t) = 1 - e^{\pi t}}$$

Problem 3.3-3

The fourier transform of the triangular pulse g(t) is given by:

$$G(f) = \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)$$

With this information, and the time-shifting and time scaling properties, find the Fourier transforms of the signals shown in Fig P3.3- b,c,d,e, and f

Figure B

The signal is time inverted. Example 3.9 proves a useful property that we are going to use:

$$g(-t) \Leftrightarrow G(-f)$$

$$\boxed{G(-f) = \frac{1}{(-2\pi f)^2}(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1)}$$

Figure C

There are two copies of the origninal signal; one is delayed by t=1, and the other is time inverted with a delay of t=1.

$$g_c(t) = g(t-1) + g(-t-1)$$

$$\boxed{G_c(f) = \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)(e^{-j2\pi f}) + \frac{1}{(-2\pi f)^2}(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1)(e^{-j2\pi f})}$$

Figure D

There are two copies of the original signal; one copy is time inverted and time advanced by t=-1, and the other is delayed by t=1.

$$g_d(t) = g(-t+1) + g(t-1)$$

$$\boxed{G_c(f) = \frac{1}{(-2\pi f)^2}(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1)(e^{j2\pi f}) + \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)(e^{-j2\pi f})}$$

Figure E

The signal is now a rect function from $t = (-1/2, 1/2)$ and amplitude of 1. This is a sum of two signals on top of each other (since they are triangular)

$$g_e(t) = g(t-1/2) + g(-t+1/2)$$

$$\boxed{G_e(f) = \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)(e^{j2\pi f}) + \frac{1}{(-2\pi f)^2}(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1)(e^{-j2\pi f})}$$

Figure F

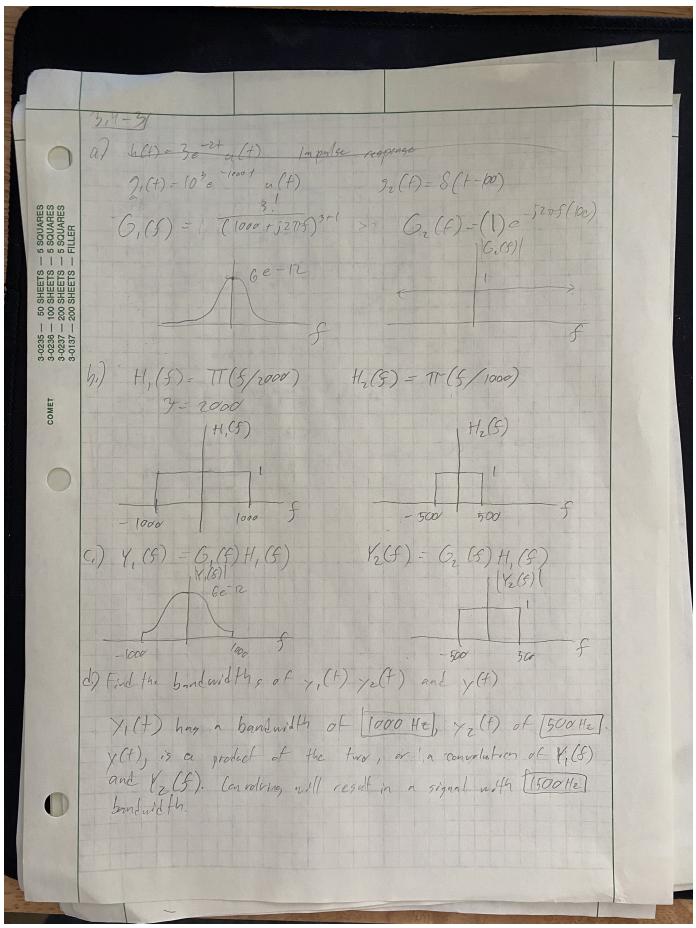
The signal is scaled by 1.5 in amplitude and 2 in time, and time delayed by t=2

$$g_f(t) = 1.5g(2(t-2))$$

$$G(f) = \frac{1.5}{(-2\pi f)^2}(e^{-j\pi f} + j\pi f e^{-j\pi f} - 1)(e^{j4\pi f})$$

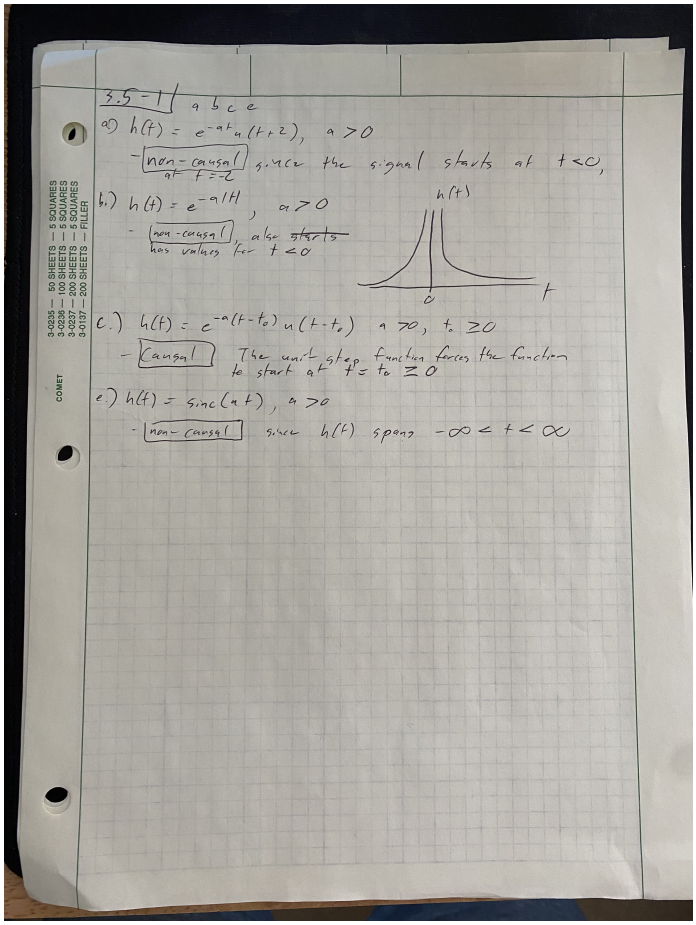
Problem 3.4-3

I did this problem by hand:



Problem 3.5-1

Also did this problem by hand:



3.5-1 a b c e

a) $h(t) = e^{-at}u(t+2)$, $a > 0$

- $\boxed{\text{non-causal}}$ since the signal starts at $t<0$, at $t=-2$

b.) $h(t) = e^{-a|t|}$, $a > 0$

- $\boxed{\text{non-causal}}$, also ~~starts~~ has values for $t < 0$

c.) $h(t) = e^{-a(t-t_0)}u(t-t_0)$ $a > 0$, $t_0 \geq 0$

- $\boxed{\text{causal}}$ The unit step function forces the function to start at $t = t_0 \geq 0$

e.) $h(t) = \text{sinc}(at)$, $a > 0$

- $\boxed{\text{non-causal}}$ since $h(t)$ spans $-\infty < t < \infty$

Problem 3.7-1 b

Use Parseval's theorem to solve the following integral:

$$\int_{-\infty}^{\infty} sinc^4(kt)dt$$

Remember that Parseval's theorem shows that the energy of a signal in the time domain is the same as a signal in the frequency domain. Here is Parseval's theorem:

$$E_g = \int_{-\infty}^{\infty} g(t)g^*(t)dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Lucky for us $sinc^4(kt)$ is not complex, so we don't have to find a complex conjugate. We can break the integral like this:

$$\int_{-\infty}^{\infty} sinc^4(kt)dt = \int_{-\infty}^{\infty} sinc^2(kt)sinc^2(kt)dt$$

Let $g(t) = sinc^2(kt)$, thus $G(f) = \frac{\pi}{k}\Delta(\frac{\pi f}{2k}) = \begin{cases} \frac{\pi}{k}(1 - \frac{\pi}{k}|f|) & |f| < \frac{k}{\pi} \\ 0 & |f| > \frac{k}{\pi} \end{cases}$

And then we get this:

$$\int_{-\infty}^{\infty} |\frac{\pi}{k}\Delta(\frac{\pi f}{2k})|^2 df$$

$$\int_{-k/\pi}^{k/\pi} |\frac{\pi}{k}(1 - \frac{\pi}{k}|f|)|^2 df$$

$$\int_{-k/\pi}^{0} (\frac{\pi}{k}(1 + \frac{\pi}{k}f))^2 df + \int_{0}^{k/\pi} (\frac{\pi}{k}(1 - \frac{\pi}{k}f))^2 df$$

$$\frac{\pi^2}{k^2}\int_{-k/\pi}^{0} ((1 + \frac{\pi}{k}f))^2 df + \frac{\pi^2}{k^2}\int_{0}^{k/\pi} ((1 - \frac{\pi}{k}f))^2 df$$

$$\frac{\pi^2}{k^2}(\frac{k}{3\pi}) + \frac{\pi^2}{k^2}(\frac{k}{3\pi})$$

$$= \boxed{\frac{2\pi}{3k}}$$