Module 4 - Homework 4

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Problem 1

If $X \sim N(0, 3)$ and $Y = 3X^2$, find $\mu_y, \sigma_Y, M_Y(t)$, CF of Y, and $f_Y(y)$.

Let's first find μ_y

$$\mu_y = E[Y] = E[3X^2]$$
$$\mu_y = 3E[X^2]$$

To find $E[X^2]$, we can use the definition of variance (see page 144):

$$\sigma_X^2 = E[(X - \mu_X)^2] = E[(X - 0)^2] = E[(X - 0)^2] = 3$$

We then get

$$\boxed{\mu_y = 3\sigma_X^2 = 9}$$

For σ_Y we can relate it to the expected value of Y:

$$\sigma_Y = \sqrt{Var[Y]} = \sqrt{E[(Y - \mu_Y)^2]} = \sqrt{E[(Y^2 - 2Y\mu_Y + \mu_Y^2)]}$$

$$= \sqrt{E[Y^2] - E[2Y\mu_Y] + E[\mu_Y^2]]} = \sqrt{E[Y^2] - 18E[Y] + 81} = \sqrt{E[Y^2] - 81}$$

We need to find $E[Y^2]$, which is $E[Y^2] = 3^2E[X^4] = 9\sigma_X^4 = 48$

$$\sigma_Y^2 = E[(Y - \mu_Y)^2]$$
$$\sigma_Y^2 = E[(Y - 9)^2]$$

$$\sigma_Y^2 = E[Y^2 - 18Y + 81]$$
$$\sigma_Y^2 = E[Y^2] - E[18Y] + E[81]$$
$$\boxed{\sigma_Y^2 = 288}$$

This is where I'm running out of time so I'm going to hustle and not simplify too much:

The PDF is easy to find from here with $Y \sim N(9, 288)$:

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-(x-\mu_y)/(2\sigma_Y^2)}$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{576\pi}} e^{-(x-9)/(576)}}$$

The characteristic function would be:

$$\Phi_Y(w) = e^{j\mu_Y \omega^2 - \sigma_Y^2/2}$$

$$\boxed{\Phi_Y(w) = e^{j9\omega^2 - 144}}$$

For the moment generating function

$$\boxed{m_n = E[Y^n] = \int_{-\infty}^{\infty} y^n \frac{1}{\sqrt{576\pi}} e^{-(x-9)/(576)} dy}$$

Problem 2

Use the characteristic function (or the moment generating function or the probability generating function) to show that a Poisson probability mass function (pmf) is the limit of a binomial pmf with n approaching infinity and p approaching zero in such a way that $np = \lambda =$constant.

Below is some chicken scratch that I was doing for this problem before running out of time

The binomial pdf is:

$$f_b(x) = \binom{n}{k} p^k q^{n-k}$$

Characteristic function:

$$(pe^{j\omega} + q)^n$$

$$(pe^{j\omega} + q)^n = \int_{-\infty}^{\infty} \binom{n}{k} p^k q^{n-k} e^{j\omega x} dx$$

$$\sum_{k=0}^{n} \binom{n}{k} (pe^{j\omega})^k q^{n-k} = \int_{-\infty}^{\infty} \binom{n}{k} p^k q^{n-k} e^{j\omega x} dx$$

So let's take a limit of this with n approaching infinity and p approaching zero

$$f_p(x) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$$e^{-\lambda(1-e^{j\omega})} = \int_{-\infty}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} e^{j\omega x} dx$$

$$e^{-np(1-e^{j\omega})} = \int_{-\infty}^{\infty} e^{-np} \frac{np^k}{k!} e^{j\omega x} dx$$

$$e^{-np(1-e^{j\omega})} = \lim_{p->0} \lim_{n->\infty} \binom{n}{k} p^k q^{n-k}$$

Problem 3

For the joint probability mass function (PMF) given below for i= 0,1,...,10 and j= 0,1,...,11, are X and Y independent? What are the marginal PMFs?

$$p_{X,Y}(i,j) = \binom{10}{i}\binom{11}{j}\left(\frac{1}{2}\right)^{21}$$

X and Y are independent if the joint probability is the product of the marginal probabilities such that $p_{X,Y}(i,j) = p_X(i)p_Y(j)$. The marginal probabilities can be represented as a sum of all outcomes:

$$p_X(i) = \sum_{j=0}^{11}\binom{10}{i}\binom{11}{j}\left(\frac{1}{2}\right)^{21}$$

$$p_X(i) = \binom{10}{i}\sum_{j=0}^{11}\binom{11}{j}\left(\frac{1}{2}\right)^{21}$$

We can use the Binomial Theorem to break this down:

$$p_X(i) = \binom{10}{i}\left(\frac{1}{2}\right)^{21}\sum_{j=0}^{11}\binom{11}{j}1^j 1^{10-j}$$

$$p_X(i) = \binom{10}{i}\left(\frac{1}{2}\right)^{21}(1+1)^{11}$$

$$\boxed{p_X(i) = \binom{10}{i}\left(\frac{1}{2}\right)^{10}}$$

The marginal probability $p_Y(j)$ is:

$$p_Y(j) = \sum_{i=0}^{10}\binom{10}{i}\binom{11}{j}\left(\frac{1}{2}\right)^{21}$$

$$p_Y(j) = \binom{11}{j}\left(\frac{1}{2}\right)^{21}\sum_{i=0}^{10}\binom{10}{i}1^i 1^{10-i}$$

$$p_Y(j) = \binom{11}{j}\left(\frac{1}{2}\right)^{21}(2)^{10}$$

$$\boxed{p_Y(j) = \binom{11}{j}\left(\frac{1}{2}\right)^{11}}$$

Ok cool, so now we know X and Y are independent:

$$p_X(i)p_Y(j) = \binom{10}{i}\left(\frac{1}{2}\right)^{10}\binom{11}{j}\left(\frac{1}{2}\right)^{11}$$

$$\boxed{p_X(i)p_Y(j) = \binom{10}{i}\binom{11}{j}\left(\frac{1}{2}\right)^{21} = p_{X,Y}(i,j)}$$

Problem 4

The joint pdf of a2-dimensional Gaussian random vector is given by

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} exp\left\{ -\frac{1}{2(1-r^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2r\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

where r is the correlation coefficient and $-\infty < x_1 < \infty and - \infty < x_2 < \infty$.

Now suppose $r = 1/2, \mu_1 = 1, \mu_2 = 2, \sigma_1^2 = 1, \sigma_2^2 = 1$

(a)*Find the expression for $f_{X_1,X_2}(x_1,x_2)$*

Plug and chug:

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sqrt{1-(1/2)^2}} exp\left\{ -\frac{1}{2(1-(1/2)^2)}\left[\frac{(x_1-1)^2}{1} - 2(1/2)\frac{(x_1-1)(x_2-2)}{1} + \frac{(x_2-2)^2}{1}\right]\right\}$$

$$\boxed{f_{X_1,X_2}(x_1,x_2) = exp\left\{ -\frac{2}{3}x_1^2 + \frac{2}{3}x_1x_2 - \frac{2}{3}x_2^2 + 2x_2 - 2\right\}}$$

(b)*Find the marginal pdfs forX_1 and X_2*

We can do this by solving for $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$:

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)dx_2$$

$$\boxed{f_{X_1}(x_1) = \frac{\sqrt{3}}{3\pi}\int_{-\infty}^{\infty} exp\left\{-\frac{2}{3}x_1^2 + \frac{2}{3}x_1x_1 - \frac{2}{3}x_2^2 + 2x_2 - 2\right\}dx_2}$$

Which simplifies to something I don't have time to find. It's probably just the form of the usual Gaussian PDF with the mean and variance from above. And here's the other one:

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)dx_1$$

$$\boxed{f_{X_2}(x_2) = \frac{\sqrt{3}}{3\pi}\int_{-\infty}^{\infty} exp\left\{-\frac{2}{3}x_1^2 + \frac{2}{3}x_1x_1 - \frac{2}{3}x_2^2 + 2x_2 - 2\right\}dx_1}$$

(c)*For $c_2 = 1/16, 1/4, 1, 2, 4, 8$ and 16, plot contours of the ellipse*

$$\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2r\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right] = c^2$$

Sorry, out of time...

(a) *Plot the joint pdf.*

Sorry, out of time...

Problem 5

Suppose we have Gaussian random vector X with a mean vector and covariance matrix given below

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, C_X = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}, \mu_X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Also, suppose that,

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = AX + b$$

(a) *Find* $P(X_2 > 2)$

To find the mean and the variance given the matrices, use the columns of μ_X to find the mean for the respective random variable X and the diagonal of C_x to find the variance. Doing this we have the Gaussian RVs:

$$X_1 \sim N(1,4) \text{ and } X_2 \sim N(2,1)$$

From this information, we can then find $P(X_2 > 2)$:

$$P(X_2 > 2) = 1 - P(X_2 \leq 2)$$

$$P(X_2 > 2) = 1 - \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2} dx$$

$$\boxed{P(X_2 > 2) = 1 - 0.5 = 0.5}$$

Note that the mean of that Gaussian is also at 2, so you could have just guessed it. This is a good sanity check though.

(b) *Find* $P(Y_2 \leq 4)$

Lets first find the mean and covariance matrices:

$$E[Y] = AE[X] + b$$

$$\mu_Y = A\mu_X + b$$

$$\mu_Y = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mu_Y = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$C_Y = AC_XA^T$$

$$C_Y = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$C_Y = \begin{bmatrix} 21 & -6 \\ -6 & 3 \end{bmatrix}$$

Now that we have the mean and covariance matrices, we can determine that $Y_2 \sim N(2,3)$,and that:

$$P(Y_2 \leq 4) = \int_{-\infty}^{4} \frac{1}{\sqrt{2\pi(3)}} e^{-(y-2)^2/(2*3)} dy$$

$$P(Y_2 \leq 4) = \frac{1}{\sqrt{6\pi}} \int_{-\infty}^{4} e^{-(y-2)^2/(6)} dy$$

$$\boxed{P(Y_2 \leq 4) \approx 0.876}$$

(c) *Find* $f_Y(y)$ We are looking for the distribution of Y here. There is a really long equation that I got from office hours:

$$f_Y(y) = \frac{1}{(2\pi)^{3/2}|C_Y|^{1/2}} exp\Bigg[-\frac{1}{2}(Y-\mu_Y)^T C_Y^{-1}(Y-\mu_Y)\Bigg]$$

$$f_Y(y) = (0.0122) exp\Bigg[-\frac{1}{2}(Y-\mu_Y)^T C_Y^{-1}(Y-\mu_Y)\Bigg]$$

$$f_Y(y) = (0.0122) exp\Bigg[-\begin{bmatrix} y_1-3 & y_2-2 \end{bmatrix} \begin{bmatrix} 1/9 & 2/9 \\ 2/9 & 7/9 \end{bmatrix} \begin{bmatrix} y_1-3 \\ y_2-2 \end{bmatrix}\Bigg]$$

This is a bit of a beast to solve by hand, so I will use my TI-89 to simplify this.

$$\boxed{f_Y(y) = \frac{\sqrt{6}}{36\pi^{3/2}} exp\Bigg[\frac{-y_1^2}{18} - \frac{2y_1y_2}{9} + \frac{7y_1}{9} - \frac{7y_2^2}{18} + \frac{20y_2}{9} - \frac{61}{18}\Bigg]}$$