Module 8 - Homework 8

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Problem 1

Let s be selected at random from the interval S = [0,1]and let the probability that s is in a subinterval of S be given by the length of the subinterval. Define the following sequences of random variables for $n \geq 1$*:*

$$X_n(s) = s^n, Y_n(s) = cos^2(2\pi s), Z_n(s) = cos^n(2\pi s)$$

Do the sequences converge, and if so, in what sense and to what limiting random variable?

The RV of selecting s from S can be described by a uniform distribution. Sequences of random variables can converge in distribution, in probability, in mean, and by almost sure convergence. See chapter seven at http://www.probabilitycourse.com for additional information. See figure 1 for an illustration of the convergences and how they relate. For this problem, we will check the RVs to see if they follow the strongest convergences and work our way to the weaker ones. Work will be shown for the convergences that work, so I don't clutter the assignment with failed attempts.

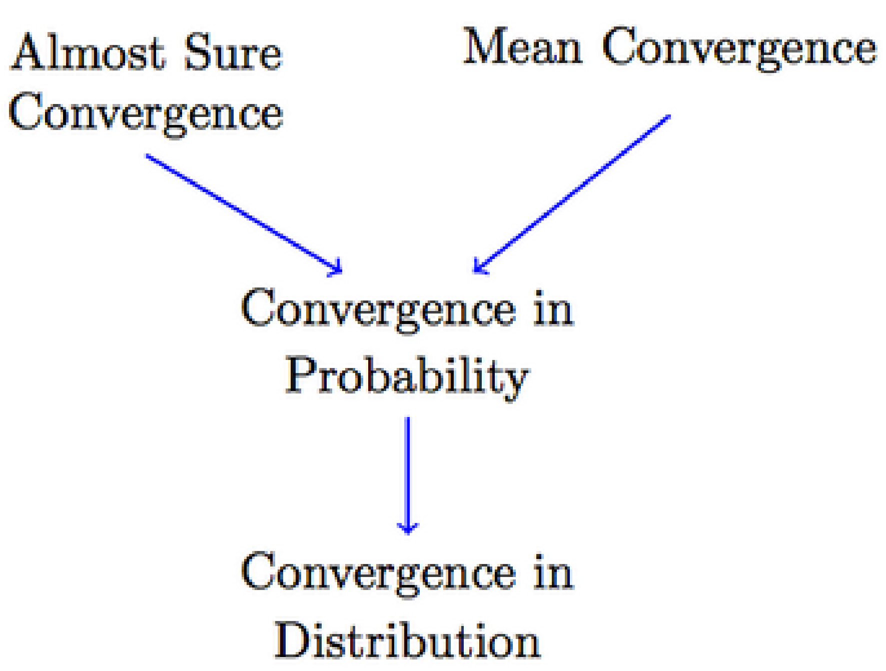

Figure 1: Types of convergence, with the strongest types being on top. A stronger convergence also implies the weaker convergences.

(A) $X_n(s) = s^n$

We have $\lim_{n\to\infty} X_n(s) = 0$ since $s < 1$. Looking at the definition of Almost Sure Convergence, if we define RV $X(s) = 0$, then

$$P(\lim_{n\to\infty} X_n(s) = X(s))$$

$$\boxed{P(\lim_{n\to\infty} 0 = 0) = 1}$$

.

Now we will check for a convergence in mean to zero.

$$\lim_{n\to\infty} E[|X_n - X|^r]$$

$$\lim_{n\to\infty} E[|s^n - 0|^r]$$

$$\lim_{n\to\infty} \int_0^1 s^{rn} ds, \text{ note the bounds are from } s \in (0,1)$$

$$\boxed{\lim_{n\to\infty} \frac{s^{rn+1}}{rn+1} = 0}$$

From above, we know that $X_n(s) \xrightarrow{a.s} 0$ and $X_n(s) \xrightarrow{L^r} 0$ for $r \geq 1$. This implies that it also converges in probability and distribution.

(B) $Y_n(s) = cos^2(2\pi s)$

Notice that $Y_n(s)$ does not depend on n, so $\lim_{n\to\infty} Y_n(s) = Y(s)$ if we let $\boxed{Y(s) = cos^2(2\pi s)}$. Thus $Y_n(s) \xrightarrow{a.s} Y(s)$, and $Y_n(s) \xrightarrow{L^r} Y(s)$, implying that it also converges in probability and distribution to the same $Y(s)$.

(C) $Z_n(s) = cos^n(2\pi s)$

This problem is a bit more interesting because like $Y_n(s)$, $Z_n(s)$ will converge to 1 for $s = \{0, 1\}$. For the interval $s = (0, 1)$, $cos(2\pi s)$ will range from $(1, -1]$. When $s = 0.5$, $Z_n(0.5) = 1^n = 1$ and converges for that boundary case. For all other s, $|cos(2\pi s)| < 1$ and $\lim_{n->\infty} cos^n(2\pi s) = 0$ The limiting RV can be defined as:

$$\boxed{Z(s) = \begin{cases} 1 & s = 0, 0.5, 1 \\ 0 & otherwise \end{cases}}$$

Since it converges for all s, it converges surely.

Problem 2

Consider a sequence $X_n, n = 1, 2, 3...$ such that

$$X_n = \begin{cases} n \text{ with probability } \frac{1}{n^2} \\ 0 \text{ with probability } 1 - \frac{1}{n^2} \end{cases}$$

Show that

(a) X_n *converges in probability to 0*

A sequence of random variables $X_1, X_2, X_3, ...$ converges in probability to a random variable X, shown by $X_n \xrightarrow{p} X$,if

$$\lim_{n->\infty} P(|X_n - X| \geq \epsilon) = 0$$

In our case, $X_n \xrightarrow{p} 0$ (X_n converges in probability to 0), so we get:

$$\lim_{n->\infty} P(|X_n - 0| \geq \epsilon) = 0$$

As n approaches infinity, $X_n = 0$ with a probability of $\lim_{n\to\infty} 1 - \frac{1}{n^2} = 1$, resulting in:

$$P(|0 - 0| \geq \epsilon) = 0$$

$$\boxed{P(0 \geq \epsilon) = 0}$$

By definition of convergence in probability, $\epsilon > 0$, thus proving that $X_n \xrightarrow{p} 0$.

(b) X_n *converges in mean for* $r < 2$

The definition of convergence in mean can be stated as follows: let $r \geq 1$ be a fixed number.A sequence of random variables $X_1, X_2, X_3, ...$ converges in the rth mean or the L^r norm to a random variable X, shown by $X_n \xrightarrow{L^r} X$, if

$$\lim_{n\to\infty} E(|X_n - X|^r) = 0$$

If r = 2, it is called the mean-square convergence, and is shown by $X_n \xrightarrow{m.s} X$.

From part a $X_n \xrightarrow{p} 0$. Also remember that $E[X] = \sum_x x p_X(x)$

$$E(|X_n - 0|^r)$$

$$\lim_{n\to\infty} E(|X_n|^r) = \lim_{n\to\infty} \sum_x x p_X(x)$$

$$\lim_{n\to\infty} E(|X_n|^r) = \lim_{n\to\infty} ((|n|^r)(\frac{1}{n^2}) + (|0|^r)(1 - \frac{1}{n^2}))$$

$$\lim_{n\to\infty} E(|X_n|^r) = \lim_{n\to\infty}\left(\frac{n^r}{n^2}\right)$$

Note that the absolute value is removed since $n \to \infty$ and not $-\infty$. When $r = 2$, $\lim\limits_{n\to\infty} \frac{n^r}{n^2} \to 1 \neq 0$. For $r = 1$:

$$\boxed{\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0}$$

Which means that X_n does converge in mean for $r < 2$. Note that by definition of convergence in mean, r cannot be less than 1.

(c) X_n *does not converge to 0 in the r-th mean for any* $r \geq 2$

I kinda showed this last problem but let me do this a different way: Let c be some constant such that $X_n \xrightarrow{p} c$. Starting with the definition of convergence in mean we have:

$$\lim_{n\to\infty} E(|X_n - c|^r) = \lim_{n\to\infty} \sum_x x p_X(x)$$

$$\lim_{n\to\infty} E(|X_n - c|^r) = \lim_{n\to\infty} ((|n-c|^r)(\frac{1}{n^2}) + (|0-c|^r)(1-\frac{1}{n^2}))$$

$$\lim_{n\to\infty} E(|X_n - c|^r) = \lim_{n\to\infty} ((|n-c|^r)(\frac{1}{n^2}) + (c^r)(1-\frac{1}{n^2}))$$

$$\lim_{n\to\infty} E(|X_n - c|^r) = \lim_{n\to\infty} ((\frac{|n-c|^r}{n^2}) + (c^r - \frac{c^r}{n^2}))$$

$$\lim_{n\to\infty} E(|X_n - c|^r) = \lim_{n\to\infty} (\frac{|n-c|^r}{n^2} + \frac{c^r n^2}{n^2} - \frac{c^r}{n^2})$$

$$\lim_{n\to\infty} E(|X_n - c|^r) = \lim_{n\to\infty} (\frac{|n-c|^r + c^r n^2 - c^r}{n^2})$$

Let $c = 0$. Knowing that $n > 0$, we end up with:

$$\lim_{n\to\infty} E(|X_n - c|^r) = \lim_{n\to\infty} (\frac{n^r}{n^2})$$

When $r = 2$, $\lim\limits_{n\to\infty}(1) \to 1 \neq 0$, and any $r > 2$, $\lim\limits_{n\to\infty}(\frac{n^r}{n^2}) \to \infty \neq 0$. Therefore X_n does not converge for $r \geq 2$.

(d) X_n *converges almost surely to 0*

A sequence of random variables $X_1, X_2, X_3, ...$ converges almost surely to a random variable X, show by $X_n \xrightarrow{a.s} X, if$

$$P\left(\left\{s \in S : \lim_{n\to\infty} X_n(s) = X(s)\right\}\right) = 1$$

Let X converge almost surely such that $X_n \xrightarrow{a.s} 0$. We want to show that the probability of $X_n(s)$ converges to $X(s)$ is equal to 1. Let $A = \{s \in S : \lim_{n\to\infty} X_n(s) = X(s)\}$ $P(A) = 1$ when $\lim_{n\to\infty} X_n(s) = X(s)$. We have already shown this above, but let's formally show this:

$$\lim_{n\to\infty} X_n(s) = 0$$

Remember that his holds true since as $n \to \infty$, the probability $P_X(X = n) = 0$ and $P_X(X = 0) = 1$. With that we can show:

$$\boxed{P(A) = P\left(\left\{s \in S : (0) = (0)\right\}\right) = 1}$$

Problem 3

The number of accidents in a certain city is modeled by a Poisson random variable with an average rate of 10 accidents per day. Suppose that the number of accidents on different days are independent. Use the central limit theorem to find the probability that there will be more than 3800 accidents in a certain year. Assume that there are 365 days in a year.

The Central Limit Theorem is defined as follows: Let $X_1, X_2, ...X_n$ be independent and identically distributed random variables with expected value $E[X_i] = \mu < \infty$ and variance $0 < Var(X_i) = \sigma^2 < \infty$. Then, the random variable

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + X_2 + ... + X_n - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to the standard normal random variable as n goes to infinity, that is:

$$\lim_{n\to\infty} P(Z_n \leq x) = \Phi(x), \text{ for all } x \in \mathbf{R}$$

where $\Phi(x)$ is the standard normal CDF.

For our problem, let X_n be a Poisson Random Variable representing the number of accidents in a certain city. Let $\lambda = 10$, and n be the n-th day of the year. We know that $\mu = E[X_i] = \lambda$, and $\sigma = \sqrt{Var(X_i)} = \sqrt{\lambda}$. $n = 365$ to represent the number of days in a year. We can use the central limit theorem to find:

$$Z_{365} = \frac{X_1 + X_2 + ...X_{365} - (365)(10)}{\sqrt{365}\sqrt{10}}$$

Remember that 3800 is just a realization of our series of X, so to find the probability that there will be less than or equal to 3800 accidents in a year, we write:

$$P(Z_{365} \leq \frac{3800-(365)(10)}{\sqrt{365}\sqrt{10}}) = \Phi(\frac{3800-(365)(10)}{\sqrt{365}\sqrt{10}}) = \Phi(2.483)$$

Which leads us to find the probability that there will be more than 3800 accidents to be:

$$1 - P(Z_{365} \leq 2.483) = 1 - 0.9935 = \boxed{0.0065}$$

Pro Tip: the Matlab function normcdf(x) will return $\Phi(x)$

Problem 4

Use the Matlab script cltDemo.m on Blackboard to display the repeated convolution of the PDF of a uniform random variable on the interval(0,1). Next modify the program to display the repeated convolution of the PDF

$$f_X(x) = \begin{cases} |2-4x| & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Which PDF results in a faster convergence of a Gaussian PDF and why?

The Matlab code for this problem is listed in the appendix, and respective figures are shown below. The uniform PDF converges faster; note that the plot of this PDF is a rect function. We see a triangular function for N=1, which results from convolving two rectangular functions. Recursively convolving the result will quickly yield a Gaussian PDF.

Notice that the PDF f_X given takes longer to converge than the uniform PDF and an exponential PDF that I did for comparison. The uniform and exponential PDFs converge faster since they more closely resemble a Gaussian PDF than our f_X. Another reason why our version of f_X takes longer to converge is because the PDF has maximum values at both 0 and 1, with a mean of 1. If you look at figure 5, the expected value is nowhere close to the median of the distribution like we see in the uniform and exponential RVs.

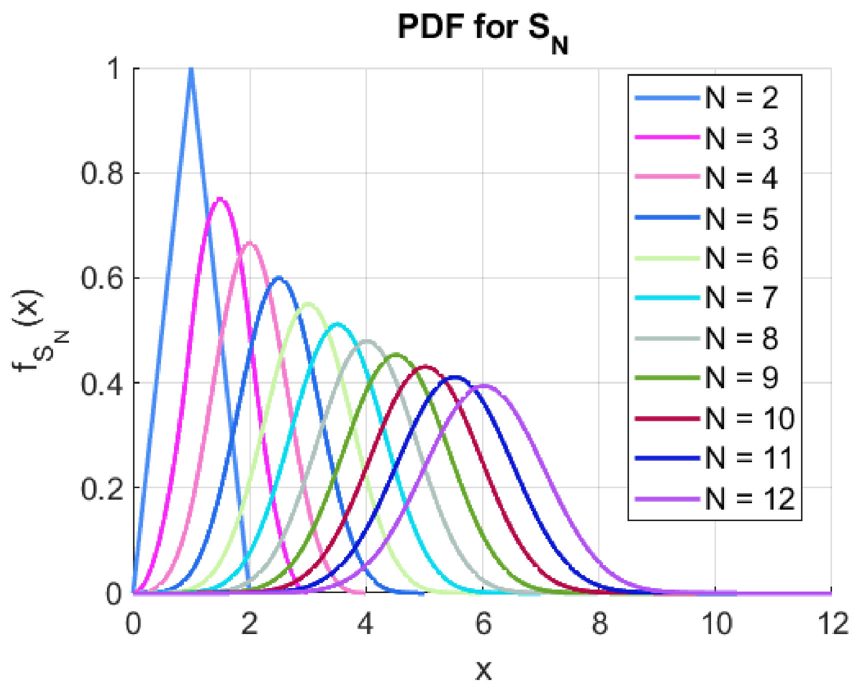

Figure 2: Central limit convergence of a uniform random variable. Notice that it converges to a Gaussian PDF by the third iteration.

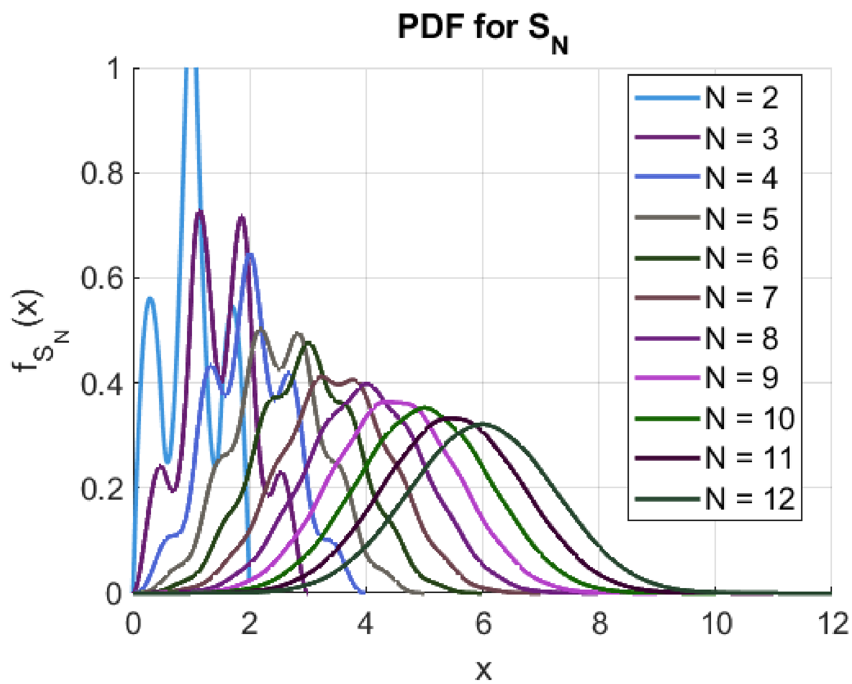

Figure 3: Central limit convergence of the given random variable. Notice that it takes about 10-12 iterations before it gets close to a Gaussian distribution.

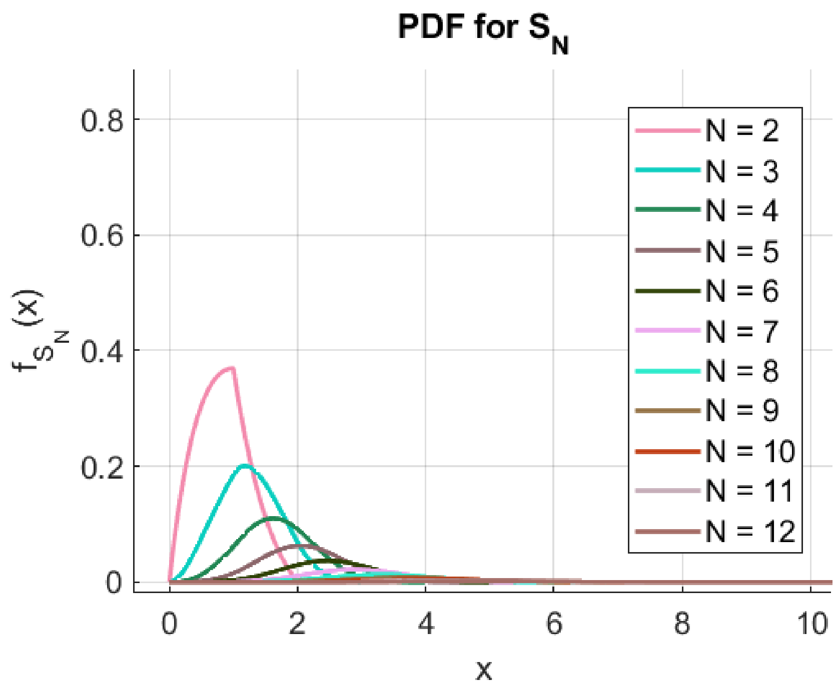

Figure 4: Central limit convergence of an exponential random variable. Notice that it takes 4 iterations before it converges to a Gaussian distribution.

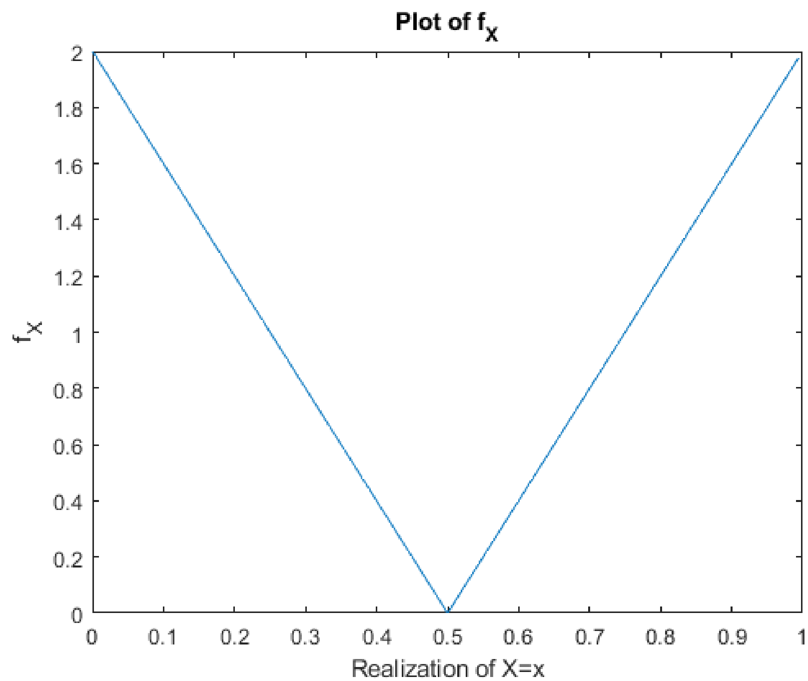

Figure 5: A rough plot of f_X. Compared to the uniform and exponential PDFs, this one least resembles a Gaussian RV. This is why it takes longer to converge.

Matlab Code

```matlab
%% %%%%%%%%%%%%%%
% Problem 4
%%%%%%%%%%%%%%%%

%% Part 1 - display a repeated convolution of a uniform random variable
X = rand(10); % generate a uniform RV on interval (0,1)

% generate a PDF of the RV
delu = 0.005; % PDF resolution
u = [0:delu:1-delu]'; % f_X defined on interval [0,1)

% f_X = ones(length(u),1); % A uniform PDF
f_X = abs(2-4*u); % The given PDF for this problem
% f_X = exppdf(u); % also try an exponential PDF
N = 12; % Convergence iterations

plotName = 'hw8prob4sp20_uniform';
cltDemo(delu,u,f_X,N,plotName);
```

```matlab
function [xArr,f_S_arr] = cltDemo(delu,u,f_X,N,plotName)
% This program demonstrates the central limit theorem. It determines the
% PDF for the sum S_N of N IID (Independent and Identically Distributed)
% random variables. Each marginal PDF is assumed to be nonzero over the
% interval (0,1). The repeated convolution integral is implemented using
% a discrete convolution. The plots of the PDF of S_N as N increases are
% all plotted on the same graph
%
% delu represents the increment over interval u=[0:delu:1-delu]
%         where delu is very small
% u - represents u=[0:delu:1-delu]
% f_X - the pdf defined on u (default is a uniform random variable)
% N - number of random variables summed
% plotName - name of resulting plot file name

if (nargin == 0)
    delu = 0.005;
```

```
    u = [0:delu:1-delu]'; % f_X defined on interval [0,1)
    f_X = ones(length(u),1); % try f_X = abs(2-4*u) for really strange PDF
    N = 12;
    plotName = 'hw8prob4sp20_uniform';
end

x = [u;u+1]; % Increase abcissa values since repeated convolution
             % increases nonzero width of output
f_S = zeros(length(x),1);

% Start discrete convolution approximation to continuous convolution
for jj = 1:length(x)
    for ii=1:length(u)
        if (jj-ii > 0) & (jj-ii <= length(f_X))
            f_S(jj) = f_S(jj) + f_X(ii)*f_X(jj-ii)*delu;
        end
    end
end

xArr{1} = x;
f_S_arr{1} = f_S;

% plot results for N = 2
for n = 3:N
    x = [x;u+n-1]; % increase abcissa values since repeated convolution
                   % increases nonzero width of output
    f_S = [f_S; zeros(length(u),1)];
    g = zeros(length(f_S),1);
    for jj = 1:length(x) % Start discrete convolution
        for ii = 1:length(u)
            if (jj - ii > 0)
                g(jj,1) = g(jj,1) + f_X(ii)*f_S(jj-ii)*delu;
            end
        end
    end
    f_S = g;
    xArr{n-1} = x;
    f_S_arr{n-1} = f_S;
end
plotCLTpdf(xArr,f_S_arr,plotName);

```

```
%————— Helper Function ——————————
function plotCLTpdf(xArr,f_S_arr,plotName)
figure
axisFont = 14;
titleFont = 16;
numPlots = numel(xArr);
N = numPlots + 1;
legendArr = cell(1,numPlots);

% Plot the PDFs
hold on
for iPlot = 1:numPlots
    plot(xArr{iPlot},f_S_arr{iPlot},'Color',[rand rand
        rand],...
        'LineWidth',2);
    legendArr{iPlot} = ['N = ',num2str(iPlot+1)];
end
hold off
grid
axis([0 N 0 1])
legend(legendArr,'Location','Best','FontSize',axisFont);
xlabel('x','FontSize',axisFont);
ylabel('f_{S_N}(x)','FontSize',axisFont);
title('PDF for S_N','FontSize',titleFont);

% Adjust the axes font
fig = gcf;
ax = get(fig,'CurrentAxes');
set(ax,'FontSize',axisFont);

% Save the figure
fig = gcf;
fig.PaperPositionMode = 'auto';
print(plotName,'-dpng','-r0');
```