

Assignment 4: Frequency Domain Analysis of LTI Systems and Sampling

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Book Questions

Problem 4.1

Using the course Matlab dtft function compute the magnitude, and phase for the following discrete-time signal:

$$\text{a)} x(n) = n(0.9)^n [u(n) - u(n - 21)]$$

The MATLAB code for computing and plotting the DTFT of $x(n)$ is below in the appendix. Figure 1 shows the plot of the spectrum.

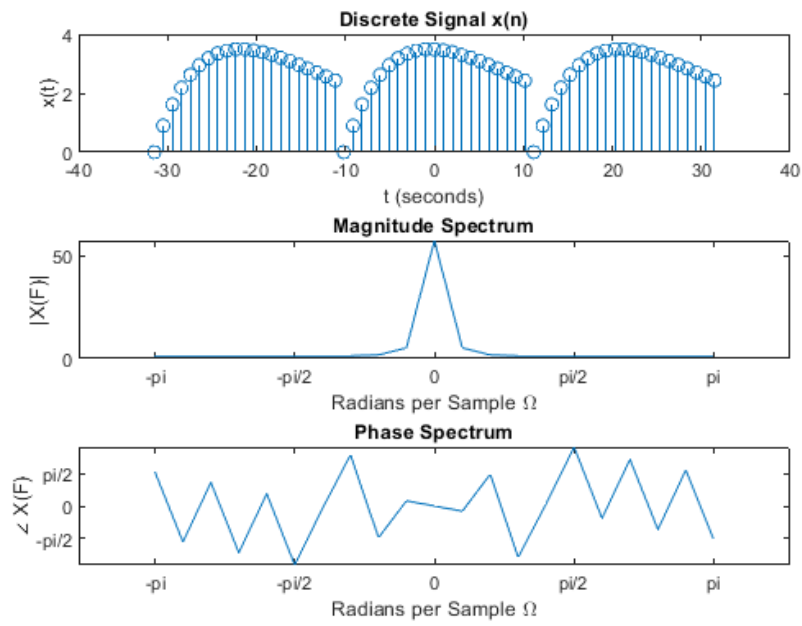


Figure 1: Plots of $x(n)$ and respective spectrum magnitude and phase plots.

$$b) x(n) = \cos\left(\frac{10}{\pi}n - \frac{\pi}{4}\right)[u(n) - u(n - 40)]$$

The MATLAB code for computing and plotting the DTFT of $x(n)$ is below in the appendix. Figure 2 shows the plot of the spectrum.

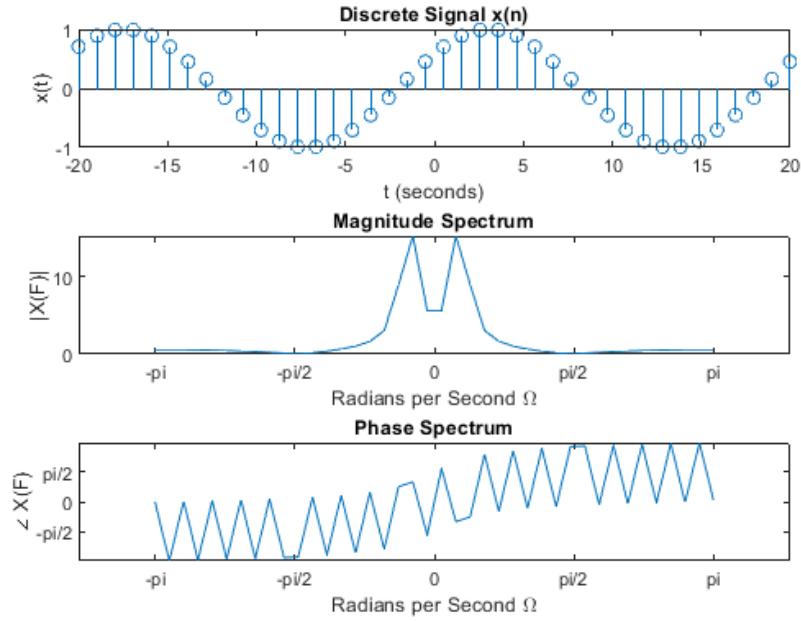


Figure 2: Plots of $x(n)$ and respective spectrum magnitude and phase plots.

Problem 4.2

The Rectangle window is a finite-duration sequence that is very useful in DSP:

$$\text{Rectangular: } R_m(n) = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise.} \end{cases}$$

a) for each of these windows, determine their DTFT for $M=10, 25, 50, 101$.

Let's first hand calculate the DTFT for $M = 10$, then plot the rest of the functions to see how the spectrum changes with M :

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Plug in $R_m(n)$ with $M = 10$ and we get:

$$X(\omega) = \sum_{n=0}^{10} e^{-j\omega n}$$

$$X(\omega) = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + \dots + e^{-10j\omega}$$

We have a geometric sum here, which could have alternatively be written as $X(\omega) = \sum_{n=0}^M (e^{-j\omega})^n$. Solving with M as a generic variable, this converges to:

$$X(\omega) = \frac{1 - (e^{-j\omega})^M}{1 - e^{-j\omega}}$$

$$X(\omega) = \frac{e^{-j\omega M/2}(2j)\sin(\omega M/2)}{e^{-j\omega}(2j)\sin(\omega/2)}$$

$$X(\omega) = \frac{\sin(\omega M/2)}{\sin(\omega/2)} e^{-j\omega(M-1)/2}$$

See figure 3 for a plot of the DTFT for each value of M . Notice how they scale as M increases.

b) Scale transform values so that the maximum value is equal to 1 then plot the magnitude of the normalized DTFT over $-\pi \leq \omega \leq \pi$.

Plots are shown in Figure 4, with included phase spectrum.

c) Study these plots and comment on their behavior as a function of M .

As M increases, the magnitude of the spectrum will increase and will result in a narrower bandwidth. One takeaway from this is that if we have a longer window we can produce a finer filter with narrow bandwidth. The phase spectrum also appears to be more linear as M increases.

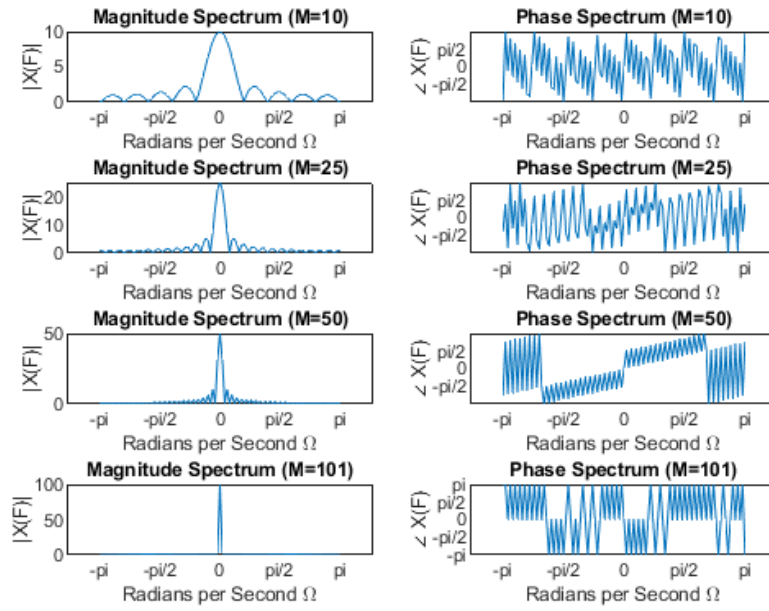


Figure 3: Plots of $X(\omega)$ for each M and respective spectrum magnitude and phase plots.

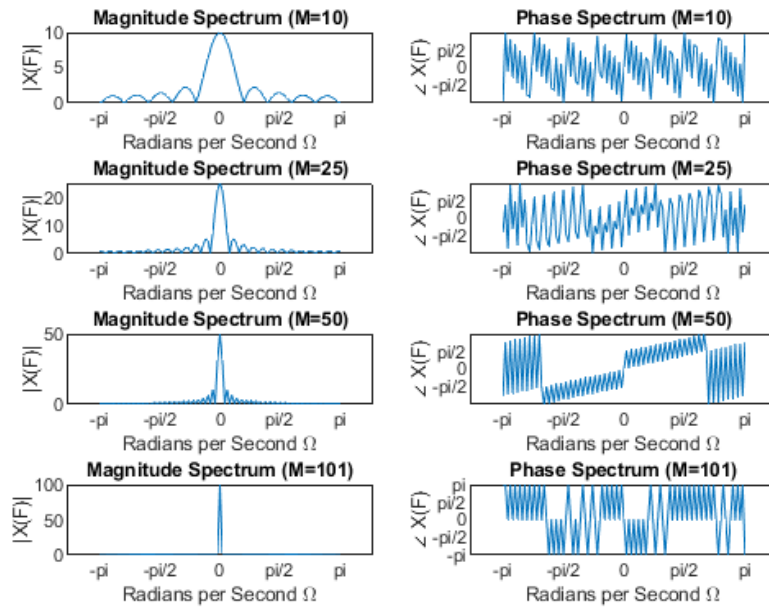


Figure 4: Spectrum plots of $x(n)$ for each M , with normalized magnitude spectrum.

Problem 4.3

Using the definition of the DTFT, determine the sequences corresponding to the following DTFT

a.) $X(\omega) = 3 + 2\cos(\omega) + 4\cos(2\omega)$

Using Euler's we can simplify the given spectrum $X(\omega)$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$X(\omega) = 3 + e^{j\omega} + e^{-j\omega} + 2e^{2j\omega} + 2e^{-2j\omega}$$

$$X(\omega) = 2e^{2j\omega} + e^{j\omega} + 3 + e^{-j\omega} + 2e^{-2j\omega}$$

$$X(\omega) = \sum_{n=-2}^2 x(n)e^{-jn\omega},$$

$$x(n) = [2, 1, \underset{\uparrow}{3}, 1, 2]$$

b.) $X(\omega) = [1 - 6\cos(3\omega) + 8\cos(5\omega)]e^{-j3\omega}$

$$X(\omega) = \left[1 - 6\frac{e^{3j\omega} + e^{-3j\omega}}{2} + 8\frac{e^{5j\omega} + e^{-5j\omega}}{2}\right]e^{-j3\omega}$$

$$X(\omega) = [1 - 3e^{3j\omega} + 3e^{-3j\omega} + 4e^{5j\omega} + 4e^{-5j\omega}]e^{-j3\omega}$$

$$X(\omega) = e^{-j3\omega} - 3e^{3j\omega-3j\omega} - 3e^{-3j\omega-3j\omega} + 4e^{5j\omega-3j\omega} + 4e^{-5j\omega-3j\omega}$$

$$X(\omega) = e^{-j3\omega} - 3 - 3e^{-6j\omega} + 4e^{2j\omega} + 4e^{-8j\omega}$$

$$X(\omega) = 4e^{2j\omega} - 3 + e^{-j3\omega} - 3e^{-6j\omega} + 4e^{-8j\omega}$$

$$x(n) = [4, 0, \underset{\uparrow}{-3}, 0, 0, 1, 0, 0, -3, 0, 4]$$

Problem 4.4

Determine the frequency response for an LTI system defined by the difference equation and plot its magnitude and phase as a function of ω :

$$y(n) = x(n) - x(n-1) + x(n-2) + 0.95y(n-1) - 0.9025y(n-2)$$

To find the frequency response of a system, we need to insert a sinusoid input for $x(n)$. Let $x(n) = Ae^{j\omega n}$:

$$y(n) = Ae^{j\omega n} - Ae^{j\omega(n-1)} + Ae^{j\omega(n-2)} + 0.95y(n-1) - 0.9025Ay(n-2)$$

$$y(n) = Ae^{j\omega n} - Ae^{j\omega n}e^{-j\omega} + Ae^{j\omega n}e^{-2j\omega} + 0.95y(n-1) - 0.9025y(n-2)$$

$$y(n) - 0.95y(n-1) + 0.9025y(n-2) = Ae^{j\omega n} - Ae^{j\omega n}e^{-j\omega} + Ae^{j\omega n}e^{-2j\omega}$$

Note that $e^{-j\omega}$ is a delay by one sample...

$$y(n)(1 - 0.95e^{-j\omega} + 0.9025e^{-2j\omega}) = Ae^{j\omega n}(1 - e^{-j\omega} + e^{-2j\omega})$$

$$y(n) = \frac{(1 - e^{-j\omega} + e^{-2j\omega})}{(1 - 0.95e^{-j\omega} + 0.9025e^{-2j\omega})} Ae^{j\omega n}$$

From this we can conclude that the frequency response $H(e^{j\omega n})$ is:

$$H(e^{j\omega n}) = \frac{(1 - e^{-j\omega} + e^{-2j\omega})}{(1 - 0.95e^{-j\omega} + 0.9025e^{-2j\omega})}$$

The plots of the magnitude and phase spectrum are below in Figure 5.

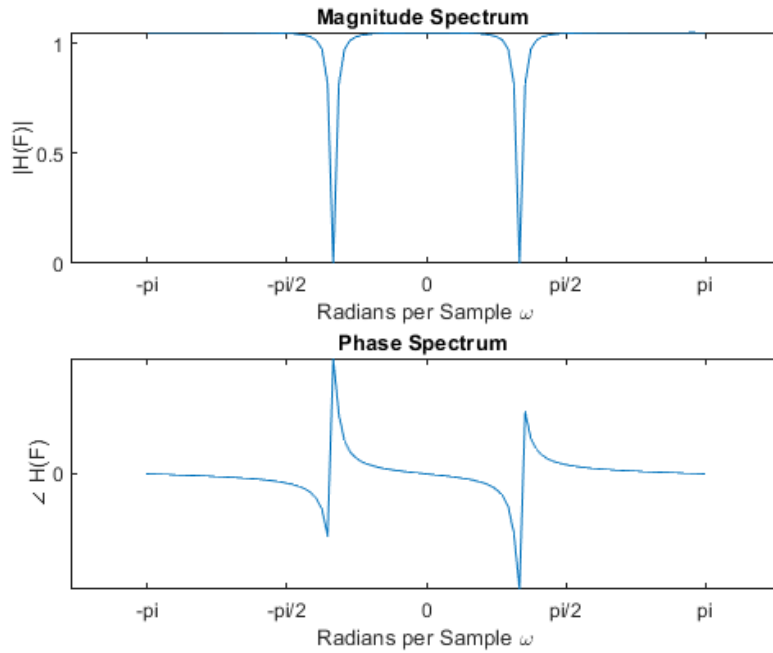


Figure 5: Spectrum plots of $H(e^{j\omega n})$.

Problem 4.5

A.) Use MATLAB to generate a pulsed sine wave signal having a frequency $F=10$ KHz and a pulse-width of $T=.5$ ms. The pulse repetition time is 1ms (pulse repetition rate is 1000 pulses/sec), i.e., the signal is zero between pulses. Model the signal to start at time $t=0$ and to end at a time $t=T_{stop}$. Use a sample time of .001ms to approximate an analog environment. Display the signal in MATLAB appropriate for an analog signal, first for $T_{stop}=1$ ms and then for

$T_{stop}=5ms$.

This was performed in MATLAB under the Problem 4.5 section in the code below. See Figures 6 and 7.

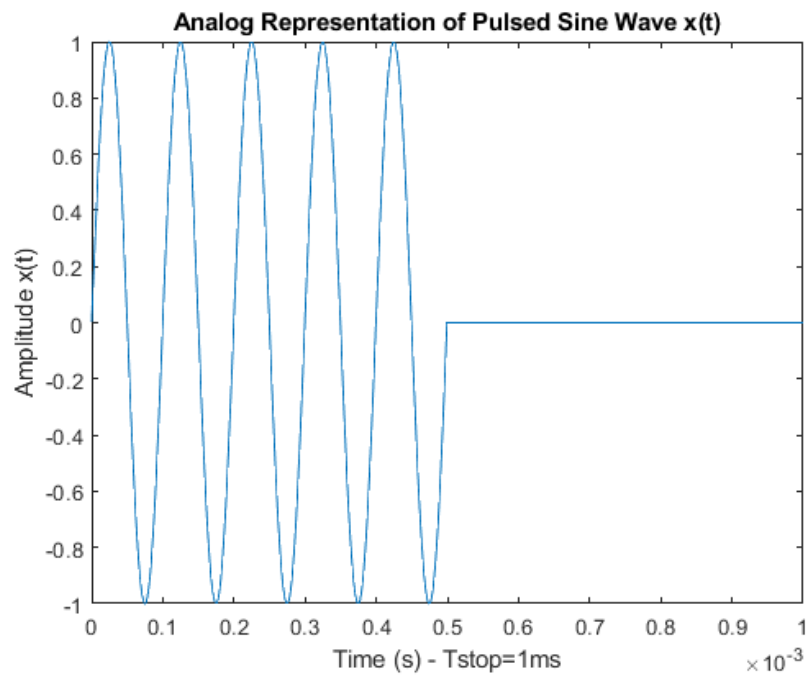


Figure 6: Plots of 10KHz sine pulse with pulse-width of 0.5ms and stop time at 1ms.

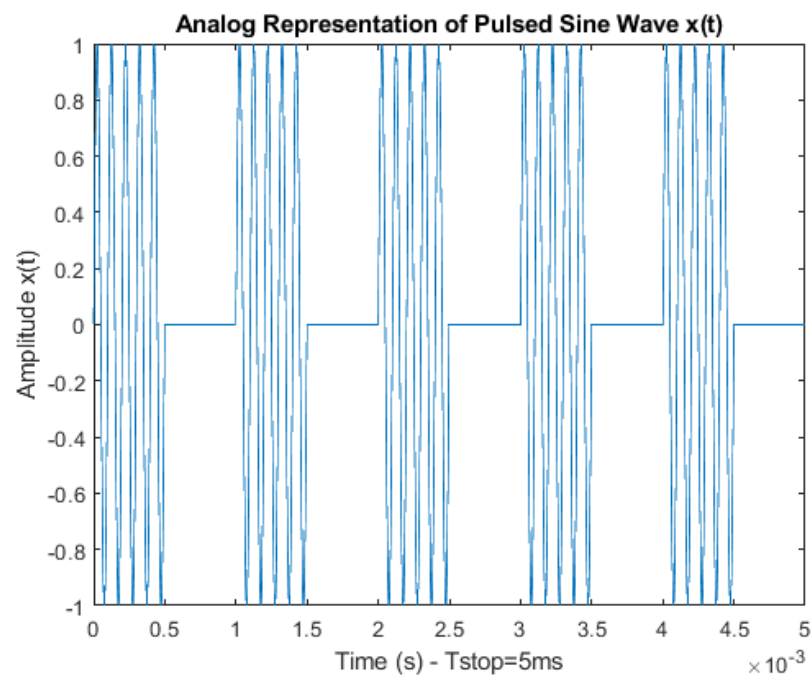


Figure 7: Plots of 10KHz sine pulse with pulse-width of 0.5ms and stop time at 5ms.

B.) In the MATLAB part (A) simulation sample the analog pulsed sine wave signal with sample times of .01ms. Display these sampled pulse sine wave in MATLAB appropriate for a discrete signal

Plots are given below in Figures 8 and 9.

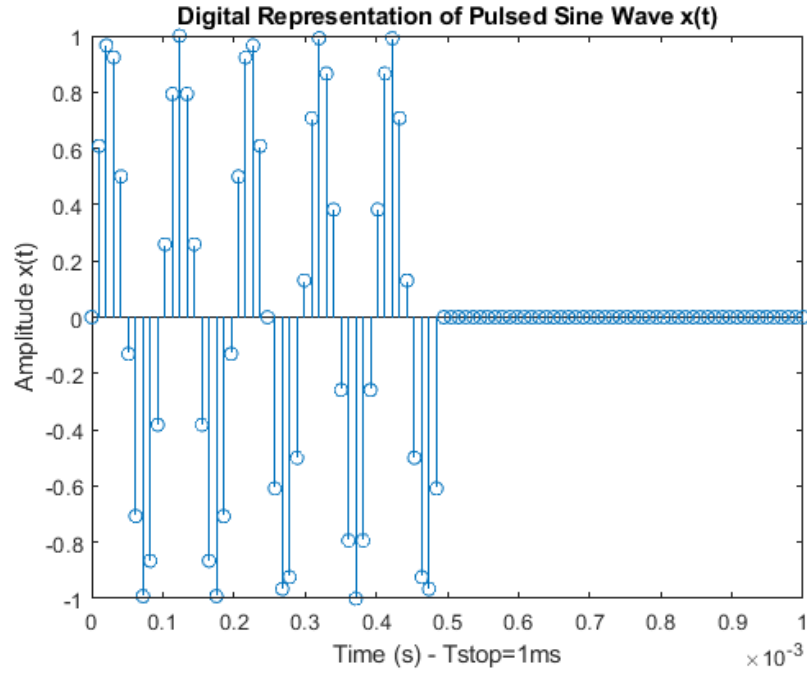


Figure 8: Plots of sampled 10KHz sine pulse with pulse-width of 0.1ms and stop time at 5ms.

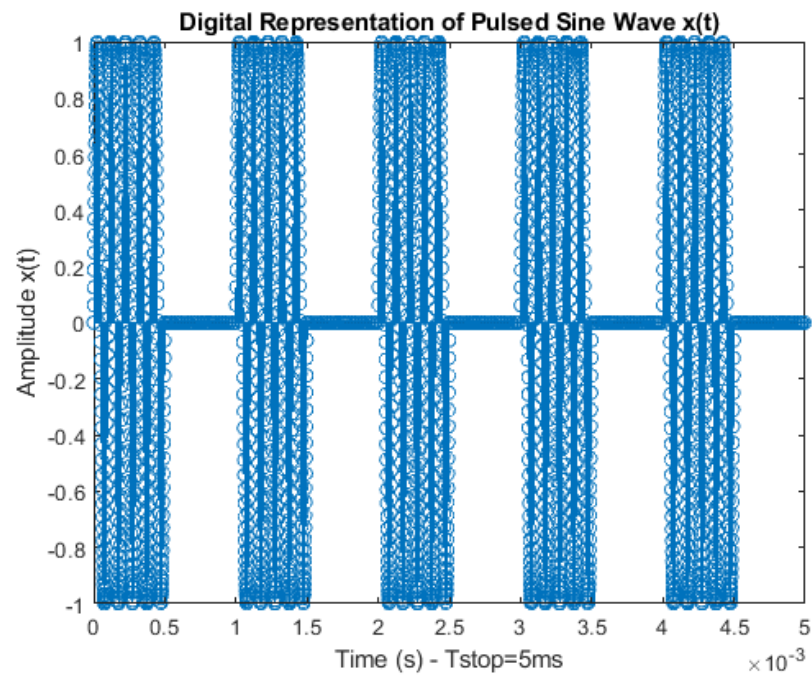


Figure 9: Plots of sampled 10KHz sine pulse with pulse-width of 0.5ms and stop time at 5ms.

C.) In MATLAB compute and display the magnitude of the DTFT of the pulse sin wave signals for w between 0 and π . Do this for each of the T_{stop} times given above. Determine the physical frequency associated with the largest peak in the magnitude spectrum. Estimate the frequency spacing of significant minor peaks away from the major peak. What comment can you make when comparing the DTFT's of the two signals?

Comparing the plots in Figures 10 and 11, we notice a few differences. First, as the number of pulses increase the magnitude increases. This is because there is more energy as the signal is prolonged. Second, the DTFT begins to look more like the ideal Fourier transform of a sinusoid (or Dirac Delta) as we approach infinite pulses.

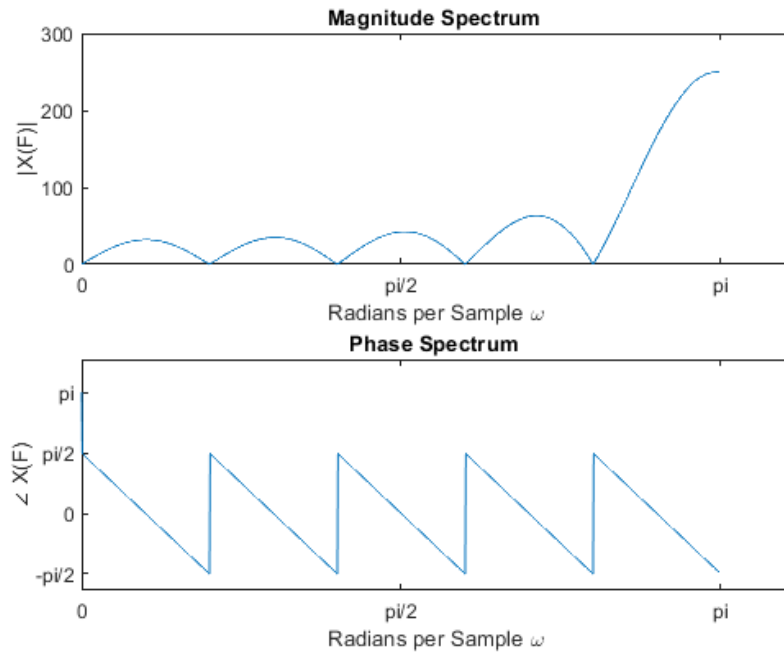


Figure 10: DTFT of sampled 10KHz sine pulse with pulse-width of 0.5ms and stop time at 1ms.

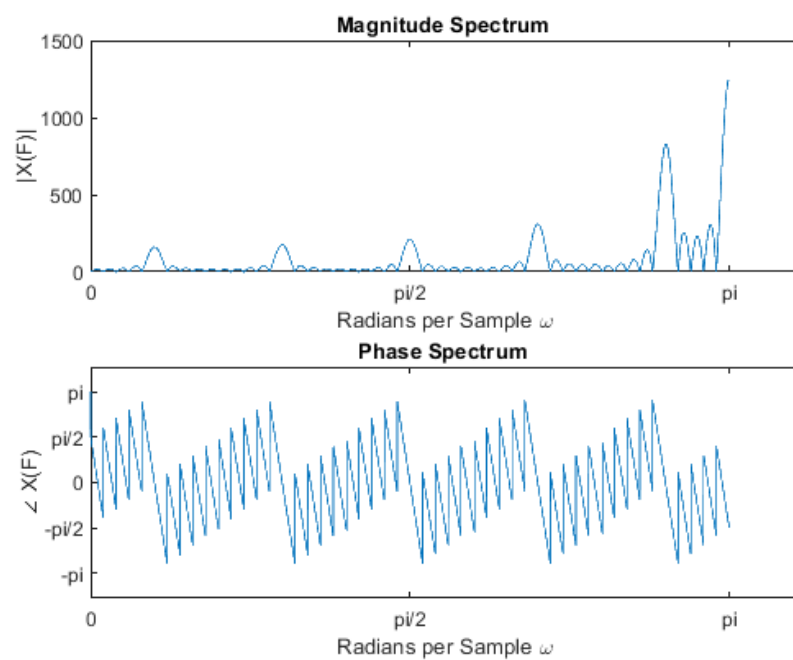


Figure 11: DTFT of sampled 10KHz sine pulse with pulse-width of 0.5ms and stop time at 5ms.

Matlab Code

```

1 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
2 %% Problem 4.1
3 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
4 % compute the magnitude and phase of the signals in parts
   A and B
5
6 % -- Part A:
7 xa = (0:20); % create x(n)=n*(0.9)^n
8 for n=0:20
9     xa(n+1) = xa(n)*(0.9)^n;
10 end
11
12 k = 10000; % number of points for refinement
13 n = linspace(-length(xa)/2,length(xa)/2,length(xa)); %
   the length() command is useful for dynamic code
14 w = linspace(-pi,pi,length(n));
15 X = dtft(xa,n,w);
16 Xmag = abs(X);
17 Xang = angle(X);
18
19 % plotting repeated x
20 P = 3; % The number of times we repeat the signal
21 xa = xa' * ones(1,P);
22 xa = xa(:); % long column vector
23 xa = xa'; % transpose to long row vector
24 n = linspace(-length(xa)/2,length(xa)/2,length(xa));
25 W = linspace(-pi,pi,length(X));
26
27
28 figure(1);
29 grid on;
30 subplot(3,1,1)
31 stem(n,xa);
32 title('Discrete Signal x(n)')
33 xlabel('t (seconds)')
34 ylabel('x(t)')
35
36
37 subplot(3,1,2)
38 plot(W,Xmag);
39 xlabel('Radians per Sample \Omega')
40 title('Magnitude Spectrum')
41 ylabel('|X(F)|')

```

```

42 set(gca, 'XTick', -pi:pi/2:pi)
43 set(gca, 'XTickLabel', {'-pi', '-pi/2', '0', 'pi/2', 'pi'})
44
45
46 subplot(3,1,3)
47 plot(W,Xang);
48 title('Phase Spectrum')
49 xlabel('Radians per Sample \Omega')
50 ylabel('\angle X(F)')
51 set(gca, 'YTick', -pi:pi/2:pi)
52 set(gca, 'YTickLabel', {'-pi', '-pi/2', '0', 'pi/2', 'pi'})
53 set(gca, 'XTick', -pi:pi/2:pi)
54 set(gca, 'XTickLabel', {'-pi', '-pi/2', '0', 'pi/2', 'pi'})
55
56 clear;
57 % -----
58 % - Part B:
59 % -----
60 n = (0:39);
61 xb = cos(pi/10.*n-pi/4);
62
63
64 k = 10000; % number of points for refinement
65 n = linspace(-length(xb)/2,length(xb)/2,length(xb)); %
    the length() command is useful for dynamic code
66 w = linspace(-pi,pi,length(n));
67 X = dtft(xb,n,w);
68 Xmag = abs(X);
69 Xang = angle(X);
70
71 % plotting repeated x
72 P = 1; % The number of times we repeat the signal
73 xb = xb' * ones(1,P);
74 xb = xb(:); % long column vector
75 xb = xb'; % transpose to long row vector
76 n = linspace(-length(xb)/2,length(xb)/2,length(xb));
77 W = linspace(-pi,pi,length(X));
78
79
80 figure(2);
81 grid on;
82 subplot(3,1,1)
83 stem(n,xb);
84 title('Discrete Signal x(n)')
85 xlabel('t (seconds)')
86 ylabel('x(t)')

```



```

87
88
89 subplot(3,1,2)
90 plot(W,Xmag);
91 xlabel('Radians per Second \Omega')
92 title('Magnitude Spectrum')
93 ylabel('|X(F)|')
94 set(gca,'XTick',-pi:pi/2:pi)
95 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
96
97
98 subplot(3,1,3)
99 plot(W,Xang);
100 title('Phase Spectrum')
101 xlabel('Radians per Second \Omega')
102 ylabel('\angle X(F)')
103 set(gca,'YTick',-pi:pi/2:pi)
104 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
105 set(gca,'XTick',-pi:pi/2:pi)
106 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
107
108 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
109 %% Problem 4.2
110 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
111
112 % -----
113 % - Part A
114 % -----
115 % declare the rectangle window function
116
117
118 clear
119 x_o=[ones(1,10),zeros(1,91); ...
120      ones(1,25),zeros(1,76); ...
121      ones(1,50),zeros(1,51);...
122      ones(1,101)...
123      ];
124
125
126 fig_idx = 0;
127 for m=1:4
128 x = x_o(m,:);
129 k = 10000; % number of points for refinement
130 n = linspace(-length(x)/2,length(x)/2,length(x)); % the
length() command is useful for dynamic code
131 w = linspace(-pi,pi,length(n));

```

```

132 X = dtft(x,n,w);
133 Xmag = abs(X);
134 Xang = angle(X);
135
136 % plotting repeated x
137 P = 1; % The number of times we repeat the signal
138 x = x' * ones(1,P);
139 x = x(:); % long column vector
140 x = x'; % transpose to long row vector
141 n = linspace(-length(x)/2,length(x)/2,length(x));
142 W = linspace(-pi,pi,length(X));
143
144
145
146 fig_idx = fig_idx+1
147 figure(3)
148 subplot(4,2,fig_idx)
149 plot(W,Xmag);
150 xlabel('Radians per Second \Omega')
151 title(strcat('Magnitude Spectrum (M=',int2str(sum(x)),')'))
152 ylabel('|X(F)|')
153 set(gca,'XTick',-pi:pi/2:pi)
154 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
155
156 fig_idx = fig_idx+1
157 subplot(4,2,fig_idx)
158 plot(W,Xang);
159 title(strcat('Phase Spectrum (M=',int2str(sum(x)),')'))
160 xlabel('Radians per Second \Omega')
161 ylabel('\angle X(F)')
162 set(gca,'YTick',-pi:pi/2:pi)
163 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
164 set(gca,'XTick',-pi:pi/2:pi)
165 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
166 end
167
168 % -----
169 % - Part B
170 % -----
171
172
173 fig_idx = 0;
174 for m=1:4
175 x = x_o(m,:);
176 k = 10000; % number of points for refinement

```

```

177 n = linspace(-length(x)/2,length(x)/2,length(x)); % the
    length() command is useful for dynamic code
178 w = linspace(-pi,pi,length(n));
179 X = dtft(x,n,w);
180
181
182 % normalize the spectrum
183 X = X./sum(x);
184
185 Xmag = abs(X);
186 Xang = angle(X);
187
188 % plotting repeated x
189 P = 1; % The number of times we repeat the signal
190 x = x' * ones(1,P);
191 x = x(:); % long column vector
192 x = x'; % transpose to long row vector
193 n = linspace(-length(x)/2,length(x)/2,length(x));
194 W = linspace(-pi,pi,length(X));
195
196
197
198 fig_idx = fig_idx+1;
199 figure(4)
200 subplot(4,2,fig_idx)
201 plot(W,Xmag);
202 xlabel('Radians per Second \Omega')
203 title(strcat('Magnitude Spectrum (M=',int2str(sum(x)),')'))
    )
204 ylabel('|X(F)|')
205 set(gca,'XTick',-pi:pi/2:pi)
206 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
207
208 fig_idx = fig_idx+1;
209 subplot(4,2,fig_idx)
210 plot(W,Xang);
211 title(strcat('Phase Spectrum (M=',int2str(sum(x)),')'))
212 xlabel('Radians per Second \Omega')
213 ylabel('\angle X(F)')
214 set(gca,'YTick',-pi:pi/2:pi)
215 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
216 set(gca,'XTick',-pi:pi/2:pi)
217 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
218 end
219
220 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

221 %% Problem 4.4
222 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
223 clear ;
224 k =100;
225 w = linspace(-pi,pi,k);
226 % for w=1:k
227 %  $H(w) = (1 - \exp(-1i * w) + \exp(-2i * w)) / (1 - 0.95 * \exp(-i * w) + 0.9025 * \exp(-2i * w))$ ;
228 % end
229 H = (1 - exp(-1i * w) + exp(-2i * w)) ./ (1 - 0.95 * exp(-i * w) + 0.9025 *
      exp(-2i * w)) ;
230 W = linspace(-pi,pi,length(H));
231 Hmag = abs(H);
232 Hang = angle(H);
233
234 figure(5)
235 subplot(2,1,1)
236 plot(W,Hmag);
237 xlabel('Radians per Sample \omega')
238 title('Magnitude Spectrum ')
239 ylabel('|H(F)|')
240 set(gca,'XTick',-pi:pi/2:pi)
241 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
242
243
244 subplot(2,1,2)
245 plot(W, Hang);
246 title('Phase Spectrum ')
247 xlabel('Radians per Sample \omega')
248 ylabel('\angle H(F)')
249 set(gca,'YTick',-pi:pi/2:pi)
250 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
251 set(gca,'XTick',-pi:pi/2:pi)
252 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
253
254 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
255 %% Problem 4.5
256 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
257 clear ;
258 F = 10e3;
259 pulseWidth = 0.5e-3; % each pulse
260 sampleTime = 0.001e-3; % in seconds
261 samp_per_sec = 1/sampleTime; % sample time of 0.001ms
262 T = linspace(0,pulseWidth,pulseWidth*samp_per_sec); %
      duration of pulse
263 x = sin(2*pi*F*T);

```

```

264
265 % Creation of repeated pulse
266 Tstop=5e-3; % 1 ms
267 pulseRepetitionTime = 2*pulseWidth;
268 P = floor(Tstop/pulseRepetitionTime); % repetitions
269 tmp = [1,0]' * ones(1,P); % this allows for zero spacing
    between pulses
270 x = x' * tmp(:)';
271 x = x(:); % long column vector
272 x = x'; % transpose to long row vector
273 n = linspace(0,P*pulseRepetitionTime,length(x)); % 2x
    pulse width

274
275 % analog representation
276 figure(6);
277 plot(n,x);
278 title('Analog Representation of Pulsed Sine Wave x(t)')
279 xlabel(strcat('Time (s) - Tstop=',num2str(Tstop*1e3),'ms'
    ));
280 ylabel('Amplitude x(t)')
281
282 % discrete sampled representation
283 figure(7)
284 stem(n,x);
285 title('Digital Representation of Pulsed Sine Wave x(t)')
286 xlabel(strcat('Time (s) - Tstop=',num2str(Tstop*1e3),'ms'
    ));
287 ylabel('Amplitude x(t)')
288
289
290
291 % DFT of x(n)
292 w = linspace(0,2*pi*F,length(n));
293 X = dtft(x,n,w);
294 Xmag = abs(X);
295 Xang = angle(X);
296 W = linspace(0,pi,length(X));
297
298 figure(8)
299 subplot(2,1,1)
300 plot(W,Xmag);
301 xlabel('Radians per Sample \omega')
302 title('Magnitude Spectrum ')
303 ylabel('|X(F)|')
304 set(gca,'XTick',-pi:pi/2:pi)
305 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})

```

```

306
307
308 subplot(2,1,2)
309 plot(W,Xang);
310 title('Phase Spectrum ')
311 xlabel('Radians per Sample \omega')
312 ylabel('\angle X(F)')
313 set(gca,'YTick',-pi:pi/2:pi)
314 set(gca,'YTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
315 set(gca,'XTick',-pi:pi/2:pi)
316 set(gca,'XTickLabel',{'-pi','-pi/2','0','pi/2','pi'})
317
318
319 %%
320 plot = 2
321
322 plot(n,x)

```