Module 3 - Homework 3

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Problem 1

The random variable X is uniform in the interval $(-2\pi, 2\pi)$. *Find* $f_Y(y)$ *if:*

(a)$Y = X^3$

To find f_Y, we first need to define $f_X(x)$ for the uniform variable X. Note that $\int_{-2\pi}^{2\pi} f_X(x)dx = 1$

$$f_X(x) = \frac{1}{4\pi}, |x| < 2\pi$$

Next we need to state $f_Y(y)$ in terms of $f_X(x)$.

$$f_Y(y) = f_X(x)\frac{1}{|\frac{dy}{dx}|}\Big|_{x=y}$$

Solve for $\frac{dy}{dx}$ when $y = x^3$, put in terms of y, then find $\frac{1}{|\frac{dy}{dx}|}\big|_{x=y}$

$$\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = 3(\sqrt[3]{y})^2$$

$$\frac{1}{|\frac{dy}{dx}|}\Big|_{x=y} = \frac{1}{|3(\sqrt[3]{y})^2|}$$

From that we get:

$$f_Y(y) = \frac{1}{4\pi}\frac{1}{|3(\sqrt[3]{y})^2|}$$

$$f_Y(y) = \frac{1}{12\pi(\sqrt[3]{y})^2}$$

Don't forget we need to find the range of this too, since $|x| < 2\pi$:

$$|x| < 2\pi$$

$$|\sqrt[3]{y}| < 2\pi$$
$$|y| < 8\pi^3$$

Therefore:

$$\boxed{f_Y(y) = \frac{1}{12\pi(\sqrt[3]{y})^2}, |y| < 8\pi^3}$$

(b)$Y = X^4$

This will be a little different from our last problem since we need to be mindful of the domain and range of the function:

$$y = x^4$$
$$\pm\sqrt[4]{y} = x$$

We have $f_X(x)$ from our previous problem, but we still need to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 4x^3$$

Next we find $f_Y(y)$

$$f_Y(y) = f_X(x)\frac{1}{|\frac{dy}{dx}|}\bigg|_{x=y}$$

Note that when putting x in terms of y, we get:

$$f_Y(y) = [f_X(-\sqrt[4]{y}) + f_X(-\sqrt[4]{y})]\frac{1}{|\frac{dy}{dx}|}\bigg|_{x=y}$$

This is because we need to account for all values of x, and the way our initial equation is, we need to have an expression of multiple functions. Plugging in values we get:

$$f_Y(y) = \left[\frac{1}{4\pi} + \frac{1}{4\pi}\right]\frac{1}{|4x^3|}\bigg|_{x=y}$$
$$f_Y(y) = \frac{2}{4\pi}\frac{1}{|4(\sqrt[4]{y})^3|}$$
$$\boxed{f_Y(y) = \frac{1}{8\pi(\sqrt[4]{y})^3}, 0 < y < 16\pi^4}$$

(c)$Y = 2sin(3X + 40^o)$

This one is a bit more tricky, but remember this: X can only have values from $(-2\pi, 2\pi)$ and for Y, $-2 < y < 2$ due to the nature of the sinusoidal function. In (a), Y was not constrained itself but (b) $Y > 0$ due to the nature of $y = x^4$. Using the domain and range that we laid out, we can find $\frac{dy}{dx}$ first:

$$\frac{dy}{dx} = 6cos(3x + \frac{2\pi}{9})$$

Next we find $f_Y(y)$:

$$f_Y(y) = f_X(x)\frac{1}{|\frac{dy}{dx}|}\Bigg|_{x=y}$$

$$f_Y(y) = f_X(x)\frac{1}{|6cos(3x + \frac{2\pi}{9})|}\Bigg|_{x=y}$$

Y in terms of X will require some trig. Remember that $sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$, and that $y^2 + \text{opposite}^2 = 1$:

$$y = 2sin(3X + 40^o)$$

$$y = 2$$

$$x = \frac{1}{3}\sin^{-1}(y/2) - \frac{2\pi}{27}$$

Plug this in:

$$f_Y(y) = f_X(x)\frac{1}{|6cos(3(\frac{1}{3}\sin^{-1}(y/2) - \frac{2\pi}{27}) + \frac{2\pi}{9})|}$$

$$f_Y(y) = f_X(x)\frac{1}{|6cos((\sin^{-1}(y/2)))|}$$

I discovered an identity: $cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

$$f_Y(y) = f_X(x)\frac{1}{|6\sqrt{1-(y/2)^2}|}$$

$$f_Y(y) = f_X(x)\frac{1}{6\pi\sqrt{1-(y^2/4)}}$$

$$f_Y(y) = f_X(x)\frac{1}{6\pi\sqrt{\frac{1}{4}(4-y^2)}}$$

$$f_Y(y) = f_X(x)\frac{1}{3\pi\sqrt{(4-y^2)}}$$

$f_X(x)$ is actually broken into a few parts, since this problem has several roots. If you do the horizontal line test, the line will intersect the function Y 12 times. One way to find this is by finding the period of the sinusoid:

$$Y = 2sin(3X + 40^o)$$

With frequency $f = 3$, period $T = \frac{1}{f} = \frac{2\pi}{3}$. In interval $(-2\pi, 2\pi)$, there are $\frac{4\pi}{\frac{2\pi}{3}} = 6$ periods. Note that in one period of a sinusoid there are two x-axis crossings when you perform the horizontal line test, totaling 12 roots for Y. We could be explicit about the roots, but since they are all equivalent to each other (yet shifted in phase), we can represent them generally as:

$$f_X(x) = \sum_{k=11}^{12} \frac{1}{4\pi} = \frac{12}{4\pi},$$

$$f_Y(y) = f_X(x)\frac{1}{3\pi\sqrt{(4-y^2)}}$$

$$f_Y(y) = \frac{12}{4\pi}\frac{1}{3\pi\sqrt{(4-y^2)}}$$

$$f_Y(y) = \frac{12}{12\pi\sqrt{(4-y^2)}} = \boxed{\frac{1}{\pi\sqrt{(4-y^2)}}}$$

Problem 2

Generating Continuous Random Variables*:One can generate samples of any random variable for a given probability density function $f_X(x)$ (where X is a random variable) by using a uniform distribution defined over [0,1]. The procedure for doing this consists of the following three steps:*

1. *Determine the function $u = \int_{-\infty}^{x} f_X(t)dt$*
2. *Determine its inverse $x = g(u)$*
3. *If U is a sample uniformly distributed on [0,1], then $X = g(U)$ is a sample whose distribution is given by $f_X(x)$*

The random variable X has an Rayleigh distribution if its probability density function is given by

$$f_X(x) = \frac{x}{\sigma^2}e^{-x^2/2\sigma^2}, x \in [0, \infty)$$

(a) *Determine a function X=g(U)such that X has a Rayleigh distribution using the procedure described above*

$$u = \int_0^x \frac{t}{\sigma^2}e^{-t^2/2\sigma^2}dt$$

$$u = \frac{1}{\sigma^2}\int_0^x te^{-t^2/2\sigma^2}dt$$

$$u = \frac{1}{\sigma^2}\left[\sigma^2 e^{-t^2/2\sigma^2}\right]_{t=x}$$

$$u = \left[e^{-x^2/2\sigma^2} - e^0\right]$$

$$u = 1 - e^{-x^2/2\sigma^2}$$

Now we find the inverse:

$$e^{-x^2/2\sigma^2} = 1 - u$$

$$-x^2/2\sigma^2 = ln(1-u)$$

$$-x^2 = 2\sigma^2 ln(1-u)$$

$$\boxed{x = i\sqrt{2}\sigma\sqrt{ln(1-u)}}$$

(b) *Write a Matlab program that simulates this distribution and compares it to the theoretical Rayleigh distribution.You can use the rand,hist, and bar functions.*

Below is a plot with the theoretical distribution compared to the simulated distribution using our equation we found in part a. Notice how closely the plots line up for N=10000.

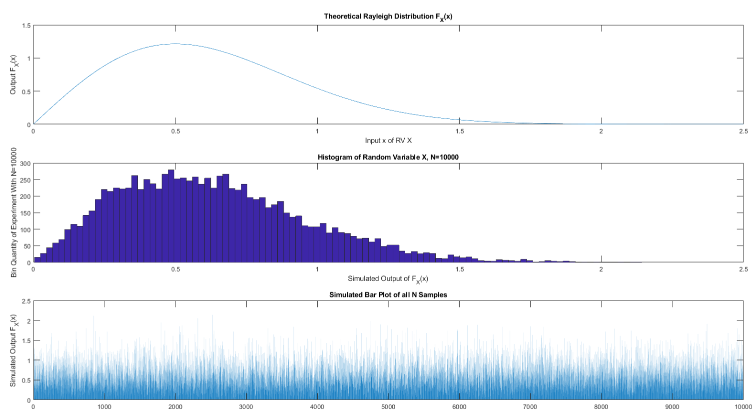

Figure 1: The Rayleigh theoretical distribution compared to a simulated one using the method introduced in this problem.

Problem 3

See the appendix for the Matlab code.

(a) *Show that if* $Y = aX + b$*, then* $\sigma_Y = |a|\sigma_x$

$$Var[Y] = Var[aX + b]$$

Note that the variance of constant b is zero, and a scalar multiple constant a can be pulled out of the variance operator as follows:

$$Var[Y] = a^2 Var[X]$$

σ_Y is the standard deviation of Y, which can be substituted into the equation with the relation $\sigma_Y = \sqrt{Var[Y]}$

$$\sigma_Y^2 = a^2 \sigma_X^2$$

$$\boxed{\sigma_Y = |a|\sigma_X}$$

(b) *Find* μ_Y *and* σ_Y *if* $Y = (X - \mu_x)/\sigma_X$

$$E[Y] = E\left[\frac{X - \mu_X}{\sigma_X}\right]$$

Remember that the expectation operator is linear:

$$\mu_Y = \frac{E[X - \mu_X]}{E[\sigma_X]}$$

$$\mu_Y = \frac{E[X] - E[\mu_X]}{\sigma_X}$$

$$\mu_Y = \frac{\mu_X - \mu_X}{\sigma_X}$$

$$\boxed{\mu_Y = 0}$$

For σ_Y:

$$Var[Y] = Var\left[\frac{X - \mu_X}{\sigma_X}\right]$$

$$\sigma_Y = Var\left[\frac{X - \mu_X}{\sigma_X}\right]$$

$$\sigma_Y = \frac{1}{\sigma_X^2} Var\left[X - \frac{\mu_X}{\sigma_X}\right]$$

$$\sigma_Y = \frac{1}{\sigma_X^2} Var[X]$$

$$\sigma_Y = \frac{1}{\sigma_X^2} \sigma_X^2$$

$$\boxed{\sigma_Y = 1}$$

Problem 4

Show that if $E[X] = \mu$, then:

$$E[e^{sX}] = e^{s\mu} \sum_{n=0}^{\infty} \gamma_n \frac{s^n}{n!}$$

$$\gamma_n = E[(X - \mu_x)^n]$$

This looks similar to the moment generating function (discrete case), which is defined as:

$$M_X(s) = E[e^{sX}] = \sum_n e^{sx_n} p_X(x_n) = \sum_{n=0}^{\infty} E[X^n] \frac{s^n}{n!}$$

We want to figure out how to use the MGF to prove the problem. Let $Y = X - \mu_x$

$$M_Y(s) = E[e^{sY}] = E[e^{s(X-\mu)}] = e^{-s\mu} E[e^{sX}] = e^{-s\mu} M_x(s)$$

Therefore:

$$e^{s\mu} \sum_{n=0}^{\infty} \gamma_n \frac{s^n}{n!} = e^{s\mu} \sum_{n=0}^{\infty} E[Y^n] \frac{s^n}{n!} = e^{s\mu} M_Y(s)$$

$$e^{s\mu} M_Y(s) = e^{s\mu} \left[e^{-s\mu} E[e^{sX}] \right] = \boxed{E[e^{sX}]}$$

Problem 5

Three types of customers arrive at a service station. The times required to service type1and type 2 customers are exponential random variables with respective means 1 and 10 seconds. Type3 customers require a constant service time of2 seconds. Suppose that the proportion of type 1, 2, and 3 customers is 1/2, 1/8, and 3/8, respectively. Find the probability that an arbitrary customer requires more than 15 seconds of service time. Compare the above probability to the bound provided by the Markov inequality.

The probability density function of an exponential random variable can be defined as follows:

$$f_x(x) = \lambda e^{-\lambda x}, x \geq 0, \lambda \geq 0$$

The pdf has a mean of $1/\lambda$. Types 1 and 2 customers can be modeled as:

$$f_1(t) = e^{-t}$$

$$f_2(t) = \frac{1}{10} e^{-\frac{1}{10}t}$$

Type 3 customers can be modeled as a uniform random variable:

$$f_3(t) = \begin{cases} 1/2 & 0 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $\int_0^\infty f_1(t)dt = 1$, $\int_0^\infty f_2(t)dt = 1$, and $\int_0^\infty f_3(t)dt = 1$, and are thus valid PDFs, and that the proportion of all 3 types of customers sum to 1. We can model our probability function as follows:

$$f(t) = \frac{1}{2} f_1(t) + \frac{1}{8} f_2(t) + \frac{3}{8} f_3(t)$$

To find the probability that $t > 15$, we can find $1 - P(t \leq 15)$

$$P(t \leq 15) = \int_0^{15} \frac{1}{2} e^{-t} dt + \int_0^{15} \frac{1}{8} \frac{1}{10} e^{-\frac{1}{10}t} dt + \int_0^{15} \frac{3}{8} u(t) - u(t-2) dt$$

$$P(t \leq 15) = 0.5 + 0.097 + 0.375 = 0.972$$

$$1 - P(t \leq 15) = P(t > 15) = \boxed{0.0279}$$

Matlab Code

```matlab
%%%%%%%%%%%%%%%%
% Problem 2
%%%%%%%%%%%%%%%%

n = 10000; % sample size


%% Default Rayleigh distribution
sigma = 0.5;
x_lim = 2.5;
y = linspace(0,x_lim,n);
f_X = raylpdf(y,sigma); % Matlab's built in Rayleigh PDF

%% Model our inverse function x = g(u)
U = rand(n,1);

% result from x=g(u) where g(u) is the inverse of the CDF of X
x = i*sqrt(2)*sigma*sqrt(log(1-U));

%% Create plots of the distribution
figure(1)
subplot(3,1,1)
plot(y,f_X)
title('Theoretical Rayleigh Distribution F_X(x)')
xlabel('Input x of RV X')
ylabel('Output F_X(x)')

subplot(3,1,2)
hist(-x,100)
title(['Histogram of Random Variable X, N=',num2str(n)])
xlabel('Simulated Output of F_X(x)')
ylabel(['Bin Quantity of Experiment With N=',num2str(n)])

subplot(3,1,3)
bar(-x)
title('Simulated Bar Plot of all N Samples')
xlabel(['Samples 1 through N=',num2str(n)])
ylabel('Simulated Output F_X(x)')
```